

# M/STAT 501: Weighted Least Squares Example

## Solutions

### Professor Ratings

Bleske-Rechek and Fritsch (2011) analyzed a data set of the ratings of 366 instructors at one large campus in the Midwest. Each instructor in the data had at least 10 ratings over a several year period. Students provided ratings from 1 (worst) to 5 (best). These data are built into R in the `alr4` library.

```
library(pracma) # eye function
library(alr4) # data set
data(Rateprof)
```

Let  $Y_{ij}$  be the quality rating of the  $i$ th instructor by the  $j$ th student,  $j = 1, \dots, n_i$ , and  $\bar{Y}_i = \sum_{j=1}^{n_i} Y_{ij}/n_i$  be the mean quality rating for the  $i$ th instructor. Similarly, let  $x_{1ij}$  and  $x_{2ij}$  be the easiness and helpfulness ratings, respectively, of the  $i$ th instructor by the  $j$ th student, with mean easiness and mean helpfulness for the  $i$ th instructor denoted by  $\bar{x}_{1i}$  and  $\bar{x}_{2i}$ . Note that the data set only reports mean ratings, not individual student's ratings.

Do in class:

1. Assume  $E(Y_{ij}|X) = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij}$  and  $Var(Y_{ij}|X) = \sigma^2$ . Derive the expression for  $E(\bar{Y}_i|X)$  and  $Var(\bar{Y}_i|X)$ .

$$\begin{aligned} E(\bar{Y}_i|X) &= \frac{1}{n_i} \sum_{j=1}^{n_i} E(Y_{ij}|X) \\ &= \frac{1}{n_i} \sum_{j=1}^{n_i} (\beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij}) \\ &= \beta_0 + \beta_1 \bar{x}_{1i} + \beta_2 \bar{x}_{2i} \end{aligned}$$

Assuming  $Y_{ij}$  is independent of  $Y_{ik}$  for  $j \neq k$ ,

$$Var(\bar{Y}_i|X) = \frac{1}{n_i^2} \sum_{j=1}^{n_i} Var(Y_{ij}|X) = \frac{\sigma^2}{n_i}.$$

2. If we fit a linear model to  $E(\bar{Y}_i|X)$ , would it meet the constant variance assumption? Explain why or why not.

No. The variance of  $Y_i$  depends on  $n_i$  (see previous derivation).

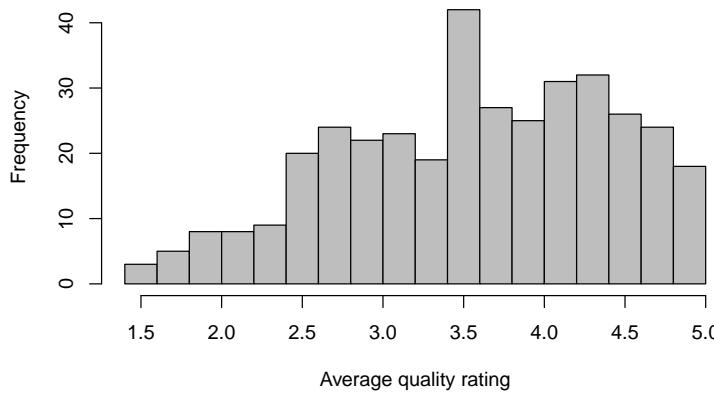
3. Let  $\mathbf{Y} = (\bar{Y}_1 \ \bar{Y}_2 \ \dots \ \bar{Y}_{366})'$  be the response vector for this data set with variance-covariance matrix  $Var(\mathbf{Y}) = \sigma^2 \boldsymbol{\Omega}$ . Write out the elements in the first four rows and first four columns of  $\boldsymbol{\Omega}$ , i.e., report the  $4 \times 4$  matrix that consists of elements in rows 1-4 and columns 1-4.

$$\Omega = \begin{pmatrix} \frac{1}{n_1} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{n_2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{n_3} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{n_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

4. Generate plots to investigate the distributions and relationships between the three variables of interest.

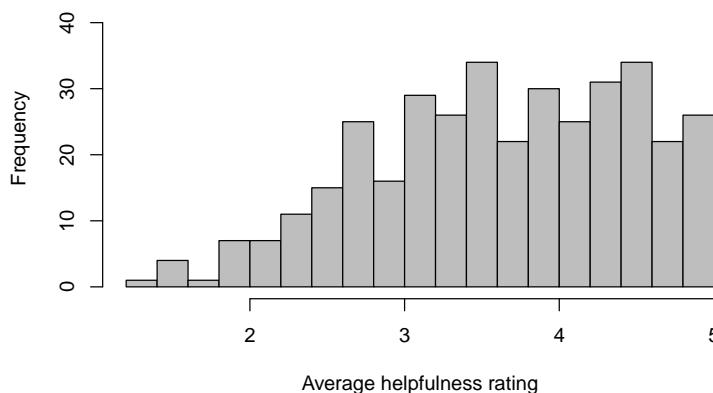
```
hist(Rateprof$quality, breaks=15, col="gray",
     xlab="Average quality rating",
     main ="Histogram of Average quality rating", ylim=c(0,45))
```

**Histogram of Average quality rating**



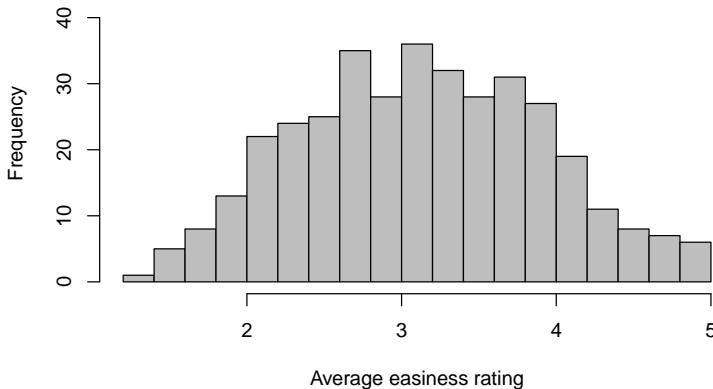
```
hist(Rateprof$helpfulness, breaks=15, col="gray",
     xlab="Average helpfulness rating",
     main ="Histogram of Average helpfulness rating", ylim=c(0,45))
```

**Histogram of Average helpfulness rating**

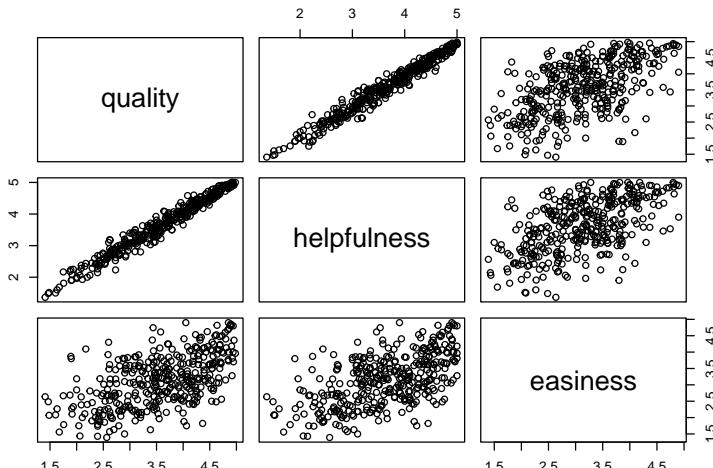


```
hist(Rateprof$easiness, breaks=15, col="gray",
     xlab="Average easiness rating",
     main ="Histogram of Average easiness rating", ylim=c(0,45))
```

### Histogram of Average easiness rating



```
plot(Rateprof[,c(8,9,11)])
```



```
cor(Rateprof[,c(8,9,11)])
```

```
##           quality helpfulness  easiness
## quality      1.0000000  0.9810314 0.5651154
## helpfulness   0.9810314  1.0000000 0.5635184
## easiness     0.5651154  0.5635184 1.0000000
```

- Fit the weighted least squares model. Write the equation of the fitted model, and interpret each of the three fitted coefficients in context of the problem.

```
mod <- lm(quality ~ easiness + helpfulness, weights = numRaters, data = Rateprof)
summary(mod)
```

```
##
## Call:
## lm(formula = quality ~ easiness + helpfulness, data = Rateprof,
##      weights = numRaters)
##
## Weighted Residuals:
##       Min     1Q   Median     3Q    Max
## -2.88556 -0.50405  0.07072  0.49018  2.53349
##
## Coefficients:
```

```

##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.05189    0.04010 -1.294   0.197
## easiness     0.01287    0.01288  0.999   0.318
## helpfulness  0.98674    0.01188 83.093 <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8417 on 363 degrees of freedom
## Multiple R-squared:  0.965, Adjusted R-squared:  0.9648
## F-statistic:  5001 on 2 and 363 DF, p-value: < 2.2e-16

# Check with matrix algebra
y <- Rateprof$quality
n <- length(y)
X <- matrix(cbind(rep(1, n), Rateprof$easiness, Rateprof$helpfulness), nrow = n, ncol = 3)
Omega <- diag(1/Rateprof$numRaters)
P <- X %*% solve(t(X)) %*% solve(Omega) %*% X %*% t(X) %*% solve(Omega)
beta_hat <- P %*% y
res <- (eye(n) - P) %*% y
SSE <- (1/(n-3))*t(y) %*% solve(Omega) %*% (eye(n) - P) %*% y
res_SE <- sqrt(SSE)

```

6. Fit the ordinary least squares fit to these data. How do the coefficients change? How does the residual standard error change? Why?

```

mod.OLS <- lm(quality ~ easiness + helpfulness, data = Rateprof)
summary(mod.OLS)

##
## Call:
## lm(formula = quality ~ easiness + helpfulness, data = Rateprof)
##
## Residuals:
##      Min       1Q   Median       3Q      Max 
## -0.51900 -0.09828  0.01194  0.09409  0.50101 
## 
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 0.009125  0.041047  0.222   0.824    
## easiness    0.019387  0.013224  1.466   0.143    
## helpfulness 0.965372  0.012210 79.063 <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.1623 on 363 degrees of freedom
## Multiple R-squared:  0.9626, Adjusted R-squared:  0.9624
## F-statistic:  4677 on 2 and 363 DF, p-value: < 2.2e-16

```