Mutual Independence

Recall the definition of *mutual independence* among a set of n random variables:

DEF 4.6.5 (Mutual Independence) Let $\mathbf{X} = (X_1, ..., X_n)'$ be an n-dimensional random vector with joint pdf or pmf $f_{\mathbf{X}}(x_1, ..., x_n)$ and marginal pdfs or pmfs for each X_i denoted by $f_{X_i}(x_i)$. Then $X_1, ..., X_n$ are ***mutually* independent random variables** if, for every $(x_1, ..., x_n)' \in \mathbb{R}^n$,

$$f(x_1, x_2, ..., x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n) = \prod_{i=1}^{n} f_{X_i}(x_i).$$

Since the equation above holds for all $(x_1, ..., x_n)' \in \mathbb{R}^n$, this definition implies that any subset of these random variables are also mutually independent. For example, let $k < n$ and consider the joint pdf/pmf of $(X_1, ..., X_k)'$:

$$\begin{aligned}
f(x_1, x_2, ..., x_k) &= \int_{\infty}^{\infty} \cdots \int_{\infty}^{\infty} f(x_1, x_2, ..., x_n) dx_{k+1} \cdots dx_n \\
&= \int_{\infty}^{\infty} \cdots \int_{\infty}^{\infty} f_{X_1}(x_1) \cdots f_{X_n}(x_n) dx_{k+1} \cdots dx_n \quad \text{by independence} \\
&= f_{X_1}(x_1) \cdots f_{X_k}(x_k) \int_{\infty}^{\infty} \cdots \int_{\infty}^{\infty} f_{X_{k+1}}(x_{k+1}) \cdots f_{X_n}(x_n) dx_{k+1} \cdots dx_n \\
&= f_{X_1}(x_1) \cdots f_{X_k}(x_k) \int_{\infty}^{\infty} f_{X_n}(x_n) \left(\int_{\infty}^{\infty} f_{X_{n-1}}(x_{n-1}) \cdots \left(\int_{\infty}^{\infty} f_{X_{k+1}}(x_{k+1}) dx_{k+1} \right) \cdots dx_{n-1} \right) dx_n \\
&= f_{X_1}(x_1) \cdots f_{X_k}(x_k) \prod_{i=k+1}^{n} \left(\int_{\infty}^{\infty} f_{X_i}(x_i) dx_i \right) \\
&= f_{X_1}(x_1) \cdots f_{X_k}(x_k) \quad \text{since each pdf integrates to 1}
\end{aligned}$$

where integrals are replaced by sums in the discrete case.

Therefore, if $X_1, ..., X_n$ are mutually independent, then any subset $\{X_{j_1}, \ldots, X_{j_m}\}$ of random variables are also mutually independent, where $m \in \{2, \ldots, n\}$ and $j_i \in \{1, \ldots, n\}$ for all $i = 1, \ldots, m$.