M/STAT 501 Fall 2025: Homework 8

Complete by Friday, Dec 5

Directions: This homework only consists of *Practice Problems*. Solutions will be posted in Canvas, and you should check and compare your work to these problems by Friday, Dec. 5.

Practice Problems (do not turn in, solutions posted in Canvas)

1. Let $\mathbf{x} = (X_1, X_2)'$ be bivariate normal with $\boldsymbol{\mu}_X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{\Sigma}_X = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and let

$$\mathbf{y}|\mathbf{x} \sim N_2\left(\boldsymbol{\mu}_{\mathbf{y}|\mathbf{x}} = \begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix}, \boldsymbol{\Sigma}_{\mathbf{y}|\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

   a. Denoting $\mathbf{y} = (Y_1, Y_2)'$, what is the distribution of $Y_2|Y_1 = y_1$? (Note that this is a univariate distribution.)
   b. What is the distribution of $\mathbf{w} = \mathbf{x} - \mathbf{y}$? (Note that this is a bivariate distribution.)

2. Let $\mathbf{y}_i \sim Multinoulli(\mathbf{p})$, $i = 1, \cdots, n$, where $\sum_{j=1}^K p_j = 1$. The Multinoulli pmf is given as:

$$f(\mathbf{y}) = \begin{cases} \prod_{j=1}^K p_j^{y_j} & \mathbf{y} \in \left\{\mathbf{v} \in \{0,1\}^K : \sum_{i=1}^K v_i = 1\right\} \\ 0 & \text{else} \end{cases}$$

   and let $\mathbf{x} = \mathbf{y}_1 + \cdots + \mathbf{y}_n$.

   a. Show that $\mathbf{x} \sim Multinomial(n, \mathbf{p})$.
   b. Find the joint moment generating function of $\mathbf{x}$.

3. Consider the vector form of a multivariate normal pdf with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$:

$$f(\mathbf{y}) = \frac{\exp\left\{-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu})\right\}}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}}.$$

   Show that, for $n = 2$, this pdf reduces to the form given in Definition 4.5.10 on p. 175 of our textbook.

4. Let Y be a positive (univariate) random variable. (Assume that the expectations below exist). Show that:

   a. $E[Y^\alpha] \leq (E[Y])^\alpha$ when $0 \leq \alpha \leq 1$
   b. $\frac{E[Y^{\alpha-1}]}{E[Y^\alpha]} \leq E[1/Y]$ for $0 < \alpha < 1$