

From Finite-Valued Nondeterministic Transducers to Deterministic Two-Tape Automata

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Abstract

The question whether $P = NP$ revolves around the discrepancy between active production and mere verification by Turing machines. In this paper, we examine the analogous problem for finite transducers and automata. Every nondeterministic finite transducer defines a binary relation associating each input word with all output words that the transducer can successfully produce on the given input. Finite-valued transducers are those for which the relation takes only finitely many values for each input word. We characterize finite-valued, functional, and unambiguous nondeterministic transducers whose relations can be verified by a deterministic two-tape automaton, show how to construct such an automaton if one exists, and prove the undecidability of the criterion.

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Related Version Many results presented in this paper were also formally proved in Isabelle/HOL, a theorem prover using higher-order logic, ensuring their correctness beyond any reasonable doubt. The formalization is available at <https://github.com/stacs21/automata>. A full, non-anonymized version of this paper, including the appendices, is published on arXiv and freely available at <https://arxiv.org/abs/2005.13710>.

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1 Introduction

Automata. One of the simplest computation models is a finite automaton reading any given input word from left to right while deterministically updating its state according to a transition function that depends on the current state and symbol that is currently read; once the input word has been read completely, it is either accepted or rejected depending on the state in which the automaton ends up. We refer to such an automaton as a deterministic finite automaton (DFA). For nondeterministic finite automaton (NFA), in contrast, the transition function may permit an arbitrary number of transitions from the current state instead of exactly one; the input word is then accepted if it is accepted for any of the possible choices of transitions. The set of words accepted by an automaton A is its recognized language $L(A)$. It is well-known that deterministic and nondeterministic finite automata both recognize the same class of *regular* languages [13]; that is, nondeterminism does not add any computational power to the finite automaton model.

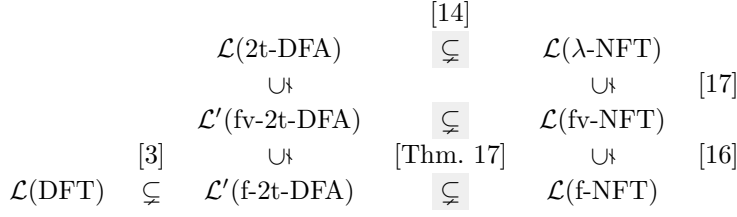
Two-Tape Automata. One way to generalize finite automata is the multi-tape model. Here, an automaton has multiple tapes, each with one head that can read the word written on the tape from left to right. There are several different but equivalent ways to model multi-tape automata, most notably the Rabin-Scott model and the Turing machine model. In the Rabin-Scott model, only one of the heads is reading a symbol and advancing to the next one in each step; the current state determines on which tape this happens. The computation is either accepted or rejected once all heads have reached the end of the tape. In the Turing machine model, in contrast, the multi-tape automaton reads all symbols at the current head positions simultaneously and can then move forward any subset of the heads in each step. In this paper, we only examine deterministic finite automata with two tapes (2t-DFAs).

Transducers. *Transducers* were introduced by Elgot and Mezei [8] as a generalization of automata. A transducer reads the given input word symbol by symbol, just like an automaton, but additionally produces a separate output word while doing so. During each transition reading one input symbol, the transducer produces a possibly empty sequence of output symbols. The final output word is obtained by concatenating the outputs of all transitions. After transducing the entire input word in an output word, the transducer decides whether the computation is accepting or not depending on the current state.

Functionality and Finite-Valuedness. A deterministic transducer associates any given input word with the output word produced on it if the corresponding computation is accepting. A nondeterministic transducer can have multiple accepting computations on an input word, associating it with all corresponding output words. Denoting the input and output alphabet by Σ and Γ , respectively, a transducer's language is a relation between Σ^* and Γ^* . For two-tape automata, we analogously consider the tapes as containing an input over Σ and an output over Γ , respectively. Thus, the language of a two-tape automaton is a relation $L \subseteq \Sigma^* \times \Gamma^*$ as well.

Given such a relation, we say that an input $a \in \Sigma^*$ is *associated* with an output $u \in \Gamma^*$ if $(a, u) \in L$. The relation L is *functional* if any input is associated with at most one output. Note that such an L describes a function that may be only partial. Given a $k \in \mathbb{N}$, we say that L is *k-valued* if it associates at most k outputs with every input. If there is a $k \in \mathbb{N}$ such that L is k -valued, we call L *finite-valued*. We call two-tape automata and transducers functional or finite-valued if their languages are. We denote the restriction of machine model M to functional or finite-valued ones by f-M and fv-M , respectively. Moreover, we denote by $\mathcal{L}(M)$ the family of languages recognizable by machine of model M .

See Figure 1 for an overview of the hierarchy of languages that are of interest to us.



■ **Figure 1** The strict hierarchy of languages recognized by relevant machine models. Most of the strict inclusions are trivial; they are briefly discussed in the introduction. The decision problem associated with the shaded inclusions are exactly the undecidable ones. References are given for the nontrivial proofs of (un)decidability. Let $\mathcal{L}'(\text{fv-2t-DFA}) = \{L \setminus (\{\lambda\} \times \Gamma^*) \mid L \in \mathcal{L}\}$ denote the set of fv-2t-DFA-recognizable relations with all pairs (λ, x) associating the empty input λ to any output removed; $\mathcal{L}'(\text{f-2t-DFA})$ is defined analogously. Writing \mathcal{L}' instead of \mathcal{L} is necessary in the two cases only for the inclusions to their right. The reasons for this are technical and discussed in the introduction.

The vertical inclusions in it are trivial since functionality implies finite-valuedness. The strictness follows from relations such as $\{(0, 0), (0, 00), \dots, (0, 0^k)\}$ —which is k -valued but not functional for any $k > 1$ —and $\{0, (0, 0^i) \mid i \in \mathbb{N}\}$ —which is not even finite-valued—since they are trivially recognizable by 2t-DFA and as well as NFTs.

It is worth pointing out that determinism implies functionality for transducers but not for two-tape automata.

Motivation. The main motivation underlying all computation models is to study their expressive power, that is, the set of languages that machines can recognize in each model. For instance, DFTs are strictly less expressive than functional NFTs. Characterizing the languages that can be recognized in one given model but not the other is also of interest. For example, Béal and Carton show that the so-called *twinning property*, which has been introduced by Choffrut [6] and is decidable in polynomial time [18], is both a necessary and sufficient condition for a functional NFT to have an equivalent DFT [3].

Inclusion Hierarchy. We prove all inclusions in Figure 1, going from left to right. The inclusion $\mathcal{L}(\text{DFT}) \subseteq \mathcal{L}(2t\text{-DFA})$ has been formally proved by Aho et al. [1, Thms. 3.1 and 3.2]. The intuition is simple: Given a DFT, we can construct an equivalent 2t-DFA that simply simulates the DFT on the first tape and checks whether the output of the DFT matches the content of the second tape. It is easy to construct an example showing that the inclusion is strict, see Appendix A.

The inclusion $\mathcal{L}(2t\text{-DFA}) \subseteq \mathcal{L}(\lambda\text{-NFT})$, where $\lambda\text{-NFT}$ denotes an NFT that may have λ -transitions, is relatively trivial. Intuitively, one can take a transducer that just guesses the output and then verifies it by simulating the automaton. We need to consider transducers with λ -transitions—i.e., transducers that can produce output without reading any input symbol—for the inclusion to hold because a general two-tape automaton can advance its output head arbitrarily many times while its input head is standing still.

In the two inclusions $\mathcal{L}(\text{f-2t-DFA}) \subseteq \mathcal{L}(\text{f-NFT})$ and $\mathcal{L}(\text{fv-2t-DFA}) \subseteq \mathcal{L}(\text{fv-NFT})$ intuitively the transducer just guesses the output and then verifies it by simulating the automaton. This works because the output head of a finite-valued (or functional) automaton can only advance by at most n positions, where n denotes the number of the automaton's states, while the input head stands still. Otherwise, one could pump in a non-empty output subword on which the automaton loops without changing the input word. However, one needs to exclude computations on the empty input word λ since the transducer cannot produce (thus guess) any output without reading an input symbol.

Question of Decidability. We now turn our attention to the corresponding decidability questions. Whether a machine is finite-valued is decidable even in the case of nondeterministic transducers [17]. We can also decide whether the given fv-NFT is functional or not [16]. It follows from these that we can decide the finite-valuedness and functionality for two-tape automata as well.

Via a reduction from the Post correspondence problem, Fischer and Rosenberg proved that, for any $k \geq 2$, it is undecidable whether a k -tape NFA has any equivalent k -tape DFA [14]. In particular, given a 2t-NFA, it is undecidable whether there is an equivalent 2t-DFA. This can be reformulated with λ -NFTs, as they are equivalent to 2t-NFAs. Their proof relies on relations that are not finite-valued, however. In this paper, we extend Fischer and Rosenberg’s undecidability result [14] to the finite-valued, functional, and even unambiguous case. The undecidability also implies the strictness of the corresponding inclusions.

For the remaining decision problem, we can construct from the given f-2t-DFA an equivalent NFT, as long as we consider automata with accepting relations with a non-empty input word, as described above. Finally, we can check whether the described NFT, which is equivalent to the given fv-2t-DFA, has an equivalent DFT via the decidable twinning property [3].

Further Model Extensions. We remark that both multi-tape automata and transducers can be extended to the *two-way* model that allows the heads to move not only forward but also back, from right to left [4, 5]. Filiot et al. [10] showed that it is decidable whether a two-way DFT can be transformed into an equivalent one-way DFT and provide a characterization of the two-way DFTs for which this is possible. It is known that two-way DFTs compute exactly those relations that are expressible as monadic second-order logic string transductions [9]. Rabin and Scott [13] proved that two-way automata are as powerful as one-way automata for a single tape but recognize strictly more languages with multiple tapes.

Yao and Rivest have shown that increasing the number of heads on a single tape yields a strict hierarchy of languages [19]; see also the survey by Holzer et al. [11]. A more recent result by Raszyk et al. [15] proves that DFTs with multiple heads are strictly more expressive than f-NFTs. Asking whether multi-head two-way DFAs can simulate two-head one-way NFAs is equivalent to the famous question whether $L = NL$ [12].

In the present paper, we exclusively consider the one-way model with one head per tape.

1.1 Our Contributions

We compare the computational power of 2t-DFAs and fv-NFTs. To this end, we relax the notion of bounded variation from [6], and show the resulting property characterizes the fv-NFTs whose relations are computable by a 2t-DFA, that is, given an fv-NFT, there is an equivalent 2t-DFA if and only if the fv-NFT lies in said subclass. Finally, we show this membership question to be undecidable.

We have formally proved many results of this paper (Lemma 7, Theorem 9, Claim 15, Claim 16, and Theorem 10 for functional NFTs) as well as Examples 18 and 19 in the proof assistant Isabelle [2] to ensure their correctness. We also provide pen-and-paper proofs for all of our results, some of which had to be deferred to the appendices.

2 Preliminaries

We formally describe the automaton model and transducer model considered in this paper and then introduce some useful terminology, in particular our core notion of bounded trailing.

There are many equivalent ways of extending the one-tape automaton model to multiple tapes. We base our definition on the Turing machine model.

► **Definition 1.** A two-tape deterministic finite automaton (*2t-DFA*), is a septuple $A = (Q, \Sigma, \Gamma, \sqcup, \delta, q_0, F)$ consisting of

- finite, nonempty sets Q , Σ , and Γ , the set of states and the input and output alphabet,
- a blank symbol $\sqcup \notin \Sigma \cup \Gamma$,
- an initial state $q_0 \in Q$ and a set of accepting states $F \subseteq Q$, and
- a transition function δ satisfying the following properties:
 - $\delta: Q \times (\Sigma \cup \{\sqcup\}) \times (\Gamma \cup \{\sqcup\}) \rightarrow Q \times \{\text{Stay}, \text{Advance}\} \times \{\text{Stay}, \text{Advance}\}$,
 - (q, \sigma, \gamma) \mapsto (q', m_1, m_2),
 - $\sigma = \sqcup \implies m_1 = \text{Stay}$ (i.e., the first head stops at the end delimiter \sqcup),
 - $\gamma = \sqcup \implies m_2 = \text{Stay}$ (i.e., the second head stops at the end delimiter \sqcup), and
 - $m_1 = \text{Advance} \vee m_2 = \text{Advance}$ (i.e., at least one head advances in every step).

The computation of a 2t-DFA proceeds as follows. There are two tapes that we call the *input* and *output* tape; the former contains a word $a \in \Sigma^*$, the latter a word $u \in \Gamma^*$. Both words are delimited by the blank symbol \sqcup at the end. The two tapes have one reading head each, which we refer to as the *input* and *output* head. The input and output head are initially positioned on the first symbol of a and u , respectively. Depending on the current state of the automaton and what symbols the two heads are reading, either the input or output head or both advance by one symbol in each step. As soon as a head reaches the blank symbol delimiting a word, it cannot move any further. The computation ends when both heads have reached the end of the input. A word pair (a, u) is in the language accepted by the automaton if and only if the computation on this pair of words ends in an accepting state. A *configuration* consists of the positions of the two heads and the current state. We write $q \xrightarrow{a/u} q'$ if the automaton can start in a state q and end up in a state q' by reading a word a with the input head and u with the output head.

We now formally define a nondeterministic finite transducer.

► **Definition 2.** A nondeterministic finite transducer (*NFT*) is a sextuple $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q , Σ , Γ , q_0 , and F are defined as for a 2t-DFA in Definition 1, but the transition function δ now is a function that maps each pair $(q, \sigma) \in Q \times \Sigma$ of a state and input symbol to a finite subset of $Q \times \Gamma^*$, describing the nondeterministic transition options.

The computation of an NFT proceeds like the computation of an NFA except that the NFT produces in each step a possibly empty sequence of output symbols. The outputs from the single steps are concatenated to obtain the final output of the complete computation. Using the visualization of a two-tape machine, we can describe the computation of an NFT as reading the word on the input tape symbol-wise while writing to the initially empty output tape, appending each step's output. We write $q \xrightarrow{a/u} q'$ if the transducer transitions from state q to state q' with the input head reading $a \in \Sigma^*$ and the output head producing $u \in \Gamma^*$.

The computation ends once the entire input word has been read, that is, when the input head has reached the blank symbol. If the transducer is in an accepting state at this moment, then the word pair (a, u) on the two tapes is in the relation $L(T)$ computed by T . If the transition function does not offer any option for a step and thus forcibly ends the computation before the blank symbol on the input tape is reached, then this computation does not contribute to the relation $L(T)$.

A binary relation $R \subseteq \Sigma^* \times \Gamma^*$ is *finite-valued* if a constant bounds the number of outputs $u \in \Gamma^*$ per input $a \in \Sigma^*$, that is, $\exists k \in \mathbb{N} \forall a \in \Sigma^*: |\{u \in \Gamma^* \mid (a, u) \in R\}| \leq k$.

198 ► **Definition 3.** An NFT T is a finite-valued nondeterministic finite transducer (fv-NFT)
 199 if $L(T)$ is a finite-valued relation. If $L(T)$ is even functional, then T is a functional
 200 nondeterministic finite transducer (f-NFT).

201 We will use the following two notions related to NFTs many times in our proofs.

202 ► **Definition 4** (Shortcut guarantee g). Let an NFT $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be given. Let
 203 $Q' \subseteq Q$ denote the set of all useful states, that is, states from which an accepting state can
 204 be reached on some input. For every $q \in Q'$, let $g_q = \min\{|x| \mid \exists u \in \Gamma^*, f \in F: q \xrightarrow{u} f\}$
 205 denote the length of a shortest word x that leads from q into an accepting state. We call
 206 $g(T) = \max_{q \in Q'} g_q$ the shortcut guarantee of T .

207 ► **Definition 5** (Output speed s). Let an NFT $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be given. We call
 208 $s(T) = \max\{|\gamma| \mid \exists q, q' \in Q, \sigma \in \Sigma: (q', \gamma) \in \delta(q, \sigma)\}$ the output speed of T . Note that $s(T)$
 209 is well-defined since δ only maps to finite subsets of $Q \times \Gamma^*$.

210 Bounded Trailing

211 We now introduce our notion of *bounded trailing* that characterizes fv-NFTs having an
 212 equivalent fv-2t-DFA. We begin by recapitulating the notion of *bounded variation* introduced
 213 by Choffrut [6], which characterizes f-NFTs having an equivalent DFT.

214 For any two words u, v , we denote by $u \wedge v$ their longest common prefix and define the
 215 distance $d(u, v) = |u| + |v| - 2|u \wedge v|$.

216 ► **Definition 6** (Bounded variation). An NFT $T = (Q, \Sigma, \Gamma, q_0, F, \delta)$ has bounded variation if

$$217 \quad \forall k \in \mathbb{N}: \exists t \in \mathbb{N}: \quad \forall q_1, q_2 \in Q, \quad f_1, f_2 \in F, \quad a, b_1, b_2 \in \Sigma^*, \quad u, v_1, v_2, w_1, w_2 \in \Gamma^*:$$

$$218 \quad q_0 \xrightarrow{a} uv_1 \rightarrow q_1 \xrightarrow{b_1} w_1 \rightarrow f_1 \quad \wedge \quad q_0 \xrightarrow{a} uv_2 \rightarrow q_2 \xrightarrow{b_2} w_2 \rightarrow f_2 \quad \wedge \quad |b_1| + |b_2| \leq k \quad \implies \quad d(v_1 w_1, v_2 w_2) \leq t$$

220 We will first simplify Definition 6. It follows from the definition of d that $d(v_1, v_2) -$
 221 $||w_1| - |w_2|| \leq d(v_1 w_1, v_2 w_2) \leq d(v_1, v_2) + |w_1| + |w_2|$. (These bounds are tight; consider
 222 $v_1 = 000$, $v_2 = 0$, $w_1 = 0$ and either $w_2 = 000$ or $w_2 = 100$.) Using the bounded output
 223 speed s of T (Definition 5), we obtain $|w_1| + |w_2| \leq s(|b_1| + |b_2|) \leq sk$. Hence, the difference
 224 between $d(v_1, v_2)$ and $d(v_1 w_1, v_2 w_2)$ only depends on T and k and can be hidden within
 225 the choice of t . We can therefore simplify $d(v_1 w_1, v_2 w_2)$ to $d(v_1, v_2)$. Using the shortcut
 226 guarantee g of T (Definition 4) for b_1 and b_2 , we can now assume $|b_1| + |b_2| \leq 2g(T)$ without
 227 loss of generality, allowing us to drop the universally quantified variable $k \in \mathbb{N}$ altogether.
 228 Finally, we have $d(uv_1, uv_2) = d(v_1, v_2)$ and can thus eliminate u by adding as a prefix to v_1
 229 and v_2 . All in all, we obtain the following equivalent formulation of bounded variation.

230 ► **Lemma 7.** An NFT $T = (Q, \Sigma, \Gamma, q_0, F, \delta)$ has bounded variation if

$$231 \quad \exists t \in \mathbb{N}: \quad \forall f_1, f_2 \in F, \quad q_1, q_2 \in Q, \quad a, b_1, b_2 \in \Sigma^*, \quad v_1, v_2, w_1, w_2 \in \Gamma^*:$$

$$232 \quad q_0 \xrightarrow{a} v_1 \rightarrow q_1 \xrightarrow{b_1} w_1 \rightarrow f_1 \quad \wedge \quad q_0 \xrightarrow{a} v_2 \rightarrow q_2 \xrightarrow{b_2} w_2 \rightarrow f_2 \quad \implies \quad d(v_1, v_2) \leq t$$

234 Let us call a computation *useful* if it can be extended to an accepting computation.
 235 Intuitively, an NFT has bounded variation if the outputs of any two useful computations on
 236 the same input word only differ on suffixes of bounded length.

237 Our notion of bounded trailing relaxes the notion of bounded variation by only looking
 238 at pairs of useful computations on the same input word such that the one output word u is a
 239 prefix of the other output word uv and such that both computations can be extended to be
 240 accepting and have uv as a common prefix of the output. In particular, bounded variation
 241 implies bounded trailing. We now formally define our notion of bounded trailing.

242 ► **Definition 8** (Bounded trailing). An NFT $T = (Q, \Sigma, \Gamma, q_0, F, \delta)$ has bounded trailing if

243 $\exists t \in \mathbb{N}: \quad \forall f_1, f_2 \in F, \quad q_1, q_2 \in Q, \quad a, b_1, b_2 \in \Sigma^*, \quad u, v, w_1, w_2 \in \Gamma^*:$

$$244 \quad q_0 \xrightarrow{uv} q_1 \xrightarrow{w_1} f_1 \quad \wedge \quad q_0 \xrightarrow{u} q_2 \xrightarrow{vw_2} f_2 \quad \implies \quad |v| \leq t$$

245

246 Example 18 in Appendix C describes a concrete transducer with bounded trailing.

247 The following three sections prove bounded trailing to be a necessary and sufficient condition for an fv-NFT to have an equivalent 2t-DFA and that bounded trailing is undecidable.
 248 In contrast, bounded variation is decidable since it is equivalent to the so-called *twinning property*, which is decidable in polynomial time [18]. We also point out that bounded trailing
 249 does not characterizes having an equivalent 2t-DFA for NFTs that are not finite-valued.
 250 Constructing a counterexample is straightforward; we provide one in Appendix B.
 251
 252

253 3 Bounded Trailing is Sufficient

254 In this section, we show that any NFT that has bounded trailing can be transformed into
 255 an equivalent 2t-DFA. Let T be an NFT with a trailing bound $t \in \mathbb{N}$. We construct an
 256 equivalent 2t-DFA A that simulates all nondeterministic computations of transducer T that
 257 are compatible with the output seen so far. Automaton A uses its states to maintain a
 258 subword z of the output word with the following property. For the currently read prefix
 259 p of the input word, there is a prefix x of the output word such that for any accepting
 260 computation of T that starts in the initial state q_0 of T , reads p , reaches a state q of T , and
 261 produces an output w consistent with the given output tape, we can write w as xy for a
 262 prefix y of z . In other words: Whatever prefix p of its input word automaton A has read at
 263 any point, it has always stored a z such that for some x every accepting computation of T
 264 on p consistent with A 's output word has the form $q_0 \xrightarrow{p} q$ for some prefix y of z .

265 Automaton A stores in its current state a representation of each such computation of T ,
 266 namely the pair $(q, |y|)$. We denote the set of these pairs by P . We show that storing P is
 267 feasible with a finite set of states by maintaining a subword z of length at most $r = s + t$,
 268 where s and t are T 's output speed and trailing bound, respectively. Initially, z is empty and
 269 the set P of pairs (q, n) stored in A 's state contains only a single pair, $(q_0, 0)$, where q_0 is
 270 T 's initial state. This reflects the fact that the only computation of T on the empty prefix
 271 of the input tape keeps T in its initial state and produces no output. If $L(T) = \emptyset$, then A
 272 immediately transitions into a rejecting sink state.

273 Automaton A now proceeds as follows. As long as the length of the subsequence z
 274 stays below r and the output tape has not been fully read, the next symbol on the output
 275 tape is read and appended to z . Moreover, A removes from the set P all representations
 276 of computations that cannot be extended to an accepting computation with the extended
 277 subsequence z of the output tape. If the set P becomes empty, then automaton A transitions
 278 into a rejecting sink state, because there is no accepting computation of T consistent with the
 279 given input and output tape. Otherwise, A determines the minimum m such that $(q, m) \in P$
 280 for some q and drops the first m output symbols from the subsequence z . This corresponds
 281 to cutting off from z a prefix of length m and appending it to x . Note that this is sound
 282 because none of the stored computations end before outputting the new x . This way, A
 283 maintains the invariant that there is some state q of T such that $(q, 0) \in P$.

284 Once the length of the subsequence z becomes r or the output tape has been fully read,
 285 automaton A reads in the next symbol from the input tape and then simulates every single
 286 step that is nondeterministically possible for every single stored computation and updates

the set P accordingly to a new P' . The mentioned invariant guarantees that $(q, 0) \in P$ for some state q of T . The fact that T has bounded trailing with a trailing bound t implies that $n \leq t$ holds for every pair $(q, n) \in P$. Hence, performing one further nondeterministically possible step continuing T 's computation represented by $(q, n) \in P$ yields a pair $(q', n') \in P'$ that satisfies $n' \leq t + s$ since s is the longest output that T can produce while reading a single symbol. This is just within the length limit $r = s + t$ that we set for z . As before, automaton A rejects if the set P' becomes empty, because this means there is no accepting computation of T consistent with the given input and output tape. Otherwise, automaton A performs on P' the normalization described in the previous paragraph to obtain a new set P that maintains the invariant that there is some q' for which $(q', 0) \in P$.

Finally, if both the input and output tape have been fully read, automaton A accepts if and only if there is some $q \in F$ with $(q, |z|) \in P$, that is, an accepting state q of T that some computation of T arrives at after producing an output that matches xz until the very end, meaning that the output is equal to the content of the output tape.

See Appendix C for a concrete example of the construction described here. We conclude this section by formally stating its main result.

► **Theorem 9.** *Any NFT with bounded trailing has an equivalent 2t-DFA.*

4 Bounded Trailing is Necessary

In this section, we prove the reverse of Theorem 9 for the case of finite-valuedness.

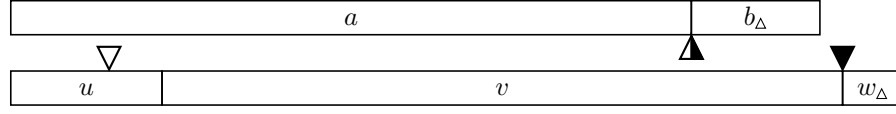
► **Theorem 10.** *Any fv-NFT with an equivalent 2t-DFA has bounded trailing.*

Proof. Let an fv-NFT T and a 2t-DFA A with $L(T) = L(A)$ be given. (We may assume that T and A use a common input alphabet Σ and a common output alphabet Γ .) Let k be an arbitrary integer such that T is k -valued. We assume that T has unbounded trailing and derive from this a contradiction to the k -valuedness of T , thus proving the theorem. We may assume without loss of generality that A moves exactly one head in each step because any step moving both heads at once can be simulated by two steps moving only one head at a time using one additional intermediate state.

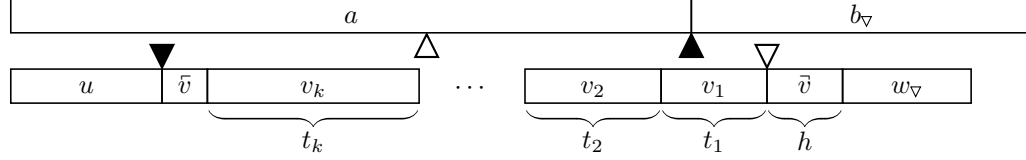
Denote the state sets of transducer T and automaton A by Q_T , Q_A , respectively. Let g and s be the transducer's shortcut guarantee and output speed, respectively. Finally, we define a *homestretch length* $h = (g + 1) \cdot (|Q_A| + 1)$ and a *trail length* minimum $t = h + t_1 + \dots + t_k$, where t_i is recursively defined by $t_i = 2s + s(1 + i(t_0 + t_1 + \dots + t_{i-1})) \cdot (1 + |Q_T| \cdot |Q_A|^i)$ with the base case $t_0 = h + (g + 1)s$. The reason for choosing exactly these values for h and t will become clear during the proof. For now, we note that they depend only on the given transducer T and the automaton A , hence they are well-defined and fixed within this proof.

Since T has unbounded trailing, it has two accepting computations $q_0 \xrightarrow{a} q_\Delta \xrightarrow{b_\Delta} f_\Delta$ and $q_0 \xrightarrow{a} q_\nabla \xrightarrow{b_\nabla} f_\nabla$ with $|v| > t$, where q_0 is the initial state and f_Δ and f_∇ are accepting states of T . We decompose the *trail* v of length at least t as $\bar{v}v_kv_{k-1} \dots v_2v_1\bar{v}$ with $|v_k| = t_k, \dots, |v_1| = t_1$ and $|\bar{v}| = h$. We call \bar{v} the *homestretch*. We consider the input-output prefix (a, uv) common to both computations of automaton A . The two-tape automaton model ensures that the two heads of A will eventually reach the end of a and v_1 , respectively. Depending on which one does so first, we distinguish two cases.

Case Δ : Input head is first In this case, we consider the input-output pair (ab_Δ, uvw_Δ) ; see Figure 2a. We begin by showing why we may assume without loss of generality that $|b_\Delta| \leq g$. For this, we consider transducer T 's configuration after the computation $q_0 \xrightarrow{a} q_\Delta$;



(a) Case Δ : Automaton A 's head positions when its input head has just reached the end of a and transducer T 's head positions after its computation $q_0 \xrightarrow{a} q_\Delta$.



(b) Case ∇ : Automaton A 's head positions when the output head has just reached the end of v_1 and transducer T 's head positions after computing $q_0 \xrightarrow{a} q_\nabla$.

■ **Figure 2** Two different input-output pairs in the relation $L(A) = L(T)$. The black and white triangles represent the heads of transducer T and automaton A , respectively. The second subfigure shows a decomposition $\bar{v}v_kv_{k-1} \dots v_2v_1\bar{v}$ of v .

the corresponding head positions are indicated by the black triangles in Figure 2a. The shortcut guarantee g ensures the existence of a word pair $(b, w) \in \Sigma^* \times \Gamma^*$ and an accepting state $f \in F$ such that $q_\Delta \xrightarrow{b} f$ and $|b| \leq g$; we can therefore substitute b , w , and f for b_Δ , w_Δ , and f_Δ if necessary and thus assume b_Δ .

Now, we consider automaton A 's computation on the same input-output pair (ab_Δ, uvw_Δ) . Let x denote the output word's suffix that has not yet been read by A when its input head has just reached the end of a ; see the white triangles in Figure 2a for A 's head positions. The input head has only b_Δ left to read but the output head all of x left to read. We have already established that $|b_\Delta| \leq g$ using the shortcut guarantee, and we know that $|x| \geq |\bar{v}| = h$ since x contains the homestretch \bar{v} . In each step, A advances exactly one head by exactly one symbol, and a head does not move anymore once it has reached the end of the word written on its tape. Thus at most g movements remain for the input head but at least h for the output head. We call a step in which the input head moves an *input step* and a step in which the output head moves an *output step*. The input steps split the remaining computation into at most $g + 1$ sequences of consecutive output steps. Since there are at least h output steps, there is at least one sequence of $h/(g + 1)$ uninterrupted output steps. Because of $h/(g + 1) > |Q_A|$, this sequence contains at least two different output steps leading A into the same state. Choosing any two such steps, we can repeat a nonempty part of the output word arbitrarily often, namely, the part that starts at the position of the output head immediately after the first step and ends with the symbol at the position of the output head just before the second step. This results in arbitrarily many accepting computations, with the word on the input tape unmodified. Hence A associates infinitely many different output words with the same input word ab_Δ , contradicting the k -valuedness of $L(A) = L(T)$.

Case ∇ : Output head is first In this case, the output head reaches the end of v_1 before the input head has finished reading a . This remains true for A 's computation on the input-output pair (ab_∇, uvw_∇) ; see Figure 2b.

In what follows we establish an upper bound on the length of the remaining output

$\bar{v}w_\nabla$. The homestretch length $|\bar{v}| = h$ is already fixed. The length of w_∇ can be bounded by combining the shortcut guarantee g and the output speed s as follows. Consider the transducer's computation $q_\nabla \xrightarrow{b_\nabla} f_\nabla$. Since the output head can write at most s symbols per step, we can cut off from b_∇ a prefix \bar{b}_∇ such that $q_\nabla \xrightarrow{\bar{b}_\nabla} q'$ for some prefix \bar{w}_∇ of w_∇ with length $|\bar{w}_\nabla| < s$ and some state q' . Denote by \bar{b}_∇ and \bar{w}_∇ the remaining suffixes such that $b_\nabla = \bar{b}_\nabla \bar{b}_\nabla$ and $w_\nabla = \bar{w}_\nabla \bar{w}_\nabla$. The state q' is useful as evidenced by the accepting computation

$$q_0 \xrightarrow{a} q_\nabla \xrightarrow{\bar{b}_\nabla} q' \xrightarrow{\bar{b}_\nabla} f_\nabla.$$

Using the definition of the shortcut guarantee g , we can therefore assume $|\bar{b}_\nabla| \leq g$ by substituting a sufficiently short b' and some appropriate w' and f' for \bar{b}_∇ , \bar{w}_∇ , and f_∇ if necessary. From this, we then obtain $|\bar{w}_\nabla| \leq g \cdot s$ since the output speed s tells us how many symbols the transducer can output at most when reading one input symbol. Replacing b_∇ by $\bar{b}_\nabla \bar{b}_\nabla$ and w_∇ by $\bar{w}_\nabla \bar{w}_\nabla$ if necessary, we can thus assume that $|w_\nabla| \leq |\bar{w}_\nabla \bar{w}_\nabla| < s + g \cdot s = (g+1)s$ without loss of generality. This finally yields the desired upper bound $|\bar{v}w_\nabla| < h + (g+1)s$.

Recall that $\bar{v}v_kv_{k-1} \dots v_2v_1\bar{v}$ is the unique decomposition of v with $|v_k| = t_k, \dots, |v_1| = t_1$ and $|\bar{v}| = h$. Since $t_0 = h + (g+1)s$, we immediately obtain $|v_kv_{k-1} \dots v_1\bar{v}w_\nabla| \leq t_0 + t_1 + \dots + t_i$. Let \bar{a} be a 's suffix that is still unread at the moment when the output head of the automaton reaches the start of \bar{v} . Denote automaton A 's initial state by \hat{q}_0 and choose any decomposition of a into $\bar{a}a_ka_{k-1} \dots a_2a_1\bar{a}$ such that A 's computation $\hat{q}_0 \xrightarrow{ab_\nabla} f_\nabla$ splits into

$$\hat{q}_0 \xrightarrow{\bar{a}} \hat{q}_{k+1} \xrightarrow{a_k} \hat{q}_k \xrightarrow{a_{k-1}} \dots \xrightarrow{a_2} \hat{q}_2 \xrightarrow{a_1} \hat{q}_1 \xrightarrow{\bar{a}b_\nabla} f_\nabla.$$

We will now prove the following claim by induction over $i \in \{0, 1, \dots, k\}$:

▷ **Claim 11.** For every $i \in \{0, 1, \dots, k\}$, there is an input word $b_{i+1} \in \Sigma^*$ and, for every $j \in \{1, 2, \dots, i\}$, an output word $\tilde{v}_j \in \Gamma^*$ with $|\tilde{v}_j| \leq |v_j|$, such that transducer T has the accepting computation $q_0 \xrightarrow{a} q_\nabla \xrightarrow{\bar{v}v_kv_{k-1} \dots v_{i+1}\tilde{v}_i\tilde{v}_{i-1} \dots \tilde{v}_2\tilde{v}_1\bar{v}w_\nabla} f_\nabla$ and such that automaton A has $i+1$ computations that all read the same input word ab_{i+1} overall, all begin with $\hat{q}_0 \xrightarrow{\bar{a}a_ka_{k-1} \dots a_{i+1}} \hat{q}_{i+1}$, and from \hat{q}_{i+1} lead to accepting states while reading the outputs $v_i \dots v_2v_1, v_i \dots v_2\tilde{v}_1, v_i \dots \tilde{v}_2\tilde{v}_1, \dots$, and $\tilde{v}_i \dots \tilde{v}_2\tilde{v}_1$, respectively.

Instantiated for $i = k$, this claim shows that the relation $\mathcal{L}(A) = \mathcal{L}(T)$ associates a single input word ab_{k+1} with $k+1$ distinct output words of pairwise different lengths, contradicting the k -valuedness and thus concluding the proof of Theorem 10. Proving the induction basis is trivial: For $i = 0$, the claim coincides precisely with the situation shown in Figure 2b when setting $b_1 = b_\nabla$. It remains to prove the induction step. Due to the space constraints, we have to restrict ourselves to a brief outline. Assuming the claim for $i-1$ as our induction hypothesis, our goal is to find nonempty subwords \tilde{b}_i of b_i and \tilde{v}_i of v_i such that not only transducer T 's accepting computation loops on them, but all of A 's pairwise distinct accepting computations on the i different input-output pairs loop on the common input subword \tilde{b}_i without advancing the output head at all. This is achievable since the length t_i of the subword v_i in the output is recursively defined to be sufficiently large—allowing us to force transducer T into producing plenty of output while reading the input b_i —and simultaneously small enough—ensuring that automaton A has very few steps that advance the output head while reading b_i . The full proof including all technical details is deferred to Appendix D. ◀

In Appendix E, we provide an illustrative example of an fv-NFT with unbounded trailing. According to Theorem 10, it has no equivalent fv-2t-DFA, implying the strictness of the inclusion $\mathcal{L}(\text{fv-2t-DFA}) \subseteq \mathcal{L}(\text{fv-NFT})$; see Figure 1.

5 Bounded Trailing is Undecidable

In this section, we prove that determining whether an fv-NFT has bounded trailing is undecidable. This is achieved by reducing the halting problem on the empty input, which is known to be undecidable, to the problem of determining whether an fv-NFT has bounded trailing. We present a reduction via a third problem, namely determining whether a Turing machine reaches infinitely many configurations on the empty input.

We begin by formally defining the standard model of a deterministic Turing machine with a single tape that is unbounded in both directions.

► **Definition 12.** A Turing machine is a sextuple $M = (Q, \Gamma, \sqcup, q_0, F, \delta)$, where

- Q and Γ are finite, nonempty sets, the set of states and the alphabet, respectively,
- $\sqcup \notin \Gamma$ is the blank symbol,
- $q_0 \in Q$ is the initial state,
- $F \subseteq Q$ is the set of accepting states, and
- $\delta : Q \times (\Gamma \cup \{\sqcup\}) \rightarrow Q \times \Gamma \times \{\text{Left}, \text{Right}\}$ is a (partial) transition function.

A *configuration* of a Turing machine consists of its current state, the content of the tape, and the position of the head on the tape. We will only consider the computations of a Turing machine on the empty input, the initial configuration thus always consists of the initial state q_0 , a tape containing only blank symbols, and the head scanning one of them. A configuration is called *accepting* if its current state q is accepting, i.e., if $q \in F$. A configuration is called *halting* if it is accepting or the transition function is undefined for the current state $q \in Q$ and the symbol $a \in \Gamma$ currently scanned by the head. If a configuration is not halting, the next configuration reached in one step of the Turing machine's computation is obtained by updating the current state, writing a non-blank symbol to the tape's cell scanned by the head, and moving the head either one cell to the left or one cell to the right.

Any configuration reached during the Turing machine's computation on the empty input consists of a finite contiguous sequence of non-blank symbols and the position of the head scanning either any symbol within this sequence or one of the two blank symbols delimiting it. Hence, we can represent a configuration of this machine as a finite sequence of cells of two types: Every configuration contains exactly one cell of the first type, namely the one currently scanned by the head. In our representation, this type of cell contains some potentially blank symbol and the current state. The second type contains a non-blank symbol only. The initial configuration is represented by a single cell containing the blank symbol and the initial state—recall that we consider only the computation on the empty input.

A Turing machine halts on the empty input if it reaches a halting configuration during its computation starting in the initial configuration. The undecidability of determining whether a halting configuration can be reached is well known. A straightforward reduction, found in Appendix F, shows that it is also undecidable whether a given Turing machine reaches infinitely many different configurations during its computation on the empty input.

► **Lemma 13.** The problem of determining whether a Turing machine reaches infinitely many different configurations during its computation on the empty input is undecidable.

Because both the set of states and the alphabet are finite, the set of all configurations of a bounded length is necessarily finite. It follows that a Turing machine M reaches infinitely many configurations if and only if it reaches configurations of arbitrary length.

► **Corollary 14.** The problem of determining whether a Turing machine reaches configurations of arbitrary length on the empty input is undecidable.

446 We now show how to construct from a given deterministic Turing machine M an fv-NFT
 447 T that has unbounded trailing if and only if M reaches configurations of arbitrary length.

448 A valid input for T is a sequence of M 's configurations followed by one of two special
 449 symbols that we call *mode indicators*. The configurations are represented by finite sequences
 450 of cells as described in the previous section and separated from each other by a dedicated
 451 symbol not occurring anywhere else. The two mode indicators are represented by a cell
 452 containing either of the two words in $\{\text{copy}, \text{step}\}$. The transducer T starts its computation
 453 by nondeterministically guessing the mode indicator and then operates in the corresponding
 454 mode described below. If the guess turns out to be wrong or if the input is invalid in any way,
 455 the computation is aborted and the transducer rejects the input. The two possible types of
 456 input-output pairs after an accepting computation are depicted in Figure 4a in Appendix G.

457 **Copy Mode** The input is output symbol by symbol, omitting the mode indicator in the end.

458 **Step Mode** In the first step, output M 's entire fixed initial configuration c_0 while reading
 459 and remembering the first input symbol. Then read from the sequence in the input
 460 one configuration c after the other. While reading c , compute and output the successor
 461 configuration c' that M reaches from c in one computation step. Of course, this all
 462 assumes that such a configuration c' exists; otherwise, T aborts the computation as it
 463 does for invalid inputs.

464 To see that T can in fact realize these computations, observe that the changes necessary
 465 to turn a configuration c into its successor configuration c' do, on the one hand, only depend
 466 on the single type-one cell of c containing the currently scanned symbol and the current state
 467 and, on the other hand, only affect the immediate proximity of this cell.

468 Hence, the successor configuration c' can indeed be computed from c by a finite transducer
 469 that essentially is still copying each configuration symbol by symbol, but letting the input
 470 head run ahead by one cell while keeping the contents of the three most recent input cells in
 471 a buffer. This allows the transducer to modify the configuration in the right place to produce
 472 the successor configuration even for configuration changes of the Turing machine that move
 473 its only head to the left. Transducer T can be effectively computed from any given Turing
 474 machine M ; we omit the technicalities.

475 The described transducer T is *unambiguous*; that is, there is at most one accepting
 476 computation for every input word. This is easily checked as follows. On the one hand,
 477 T rejects all invalid inputs anyway. On a valid input, on the other hand, T takes only a
 478 single nondeterministic decision to choose the operating mode and then the computation
 479 proceeds deterministically, being accepting for only one of the two choices, depending on the
 480 mode indicator at the end of the input word. Transducer T 's being unambiguous implies
 481 that the undecidability stated in Theorem 17 remains valid even when we are restricted to
 482 unambiguous NFTs instead of fv-NFTs. Namely, unambiguity implies functionality and thus
 483 finite-valuedness—but of course not vice versa—which trivially means that Theorems 9 and 10
 484 hold for functional and unambiguous NFTs too. Nevertheless, unambiguous transducers are
 485 just as expressive as functional ones [7]. We state T 's relevant properties formally.

486 \triangleright **Claim 15.** T is unambiguous and thus both functional and finite-valued NFT.

487 We will now sketch the proof of the crucial connection between the length of M 's
 488 configurations and T 's trailing.

489 \triangleright **Claim 16.** T has unbounded trailing if and only if Turing machine M reaches configurations
 490 of arbitrary length during its computation on the empty input.

The transducer T takes only one nondeterministic decision during any computation, namely to operate in either copy or step mode, the rest of the computation is deterministic. According to Definition 8, the only way for any trailing to occur is therefore a pair of two computations, one in copy mode and one in step mode, producing consistent output words. One of these two output words must be a prefix of the other; we call it the common prefix of the two computations. See Figure 4b in Appendix G for an example of the arising situation. In the next paragraph we argue why the configurations within the common prefix represent a valid computation of M and why the current trail length is equal to the length of the configuration currently read, up to a constant.

Since transducer T always starts by outputting the initial configuration c_0 in step mode, this has to be the first configuration on the output tape in copy mode as well. In copy mode, T can only write this configuration c_0 to the beginning of the output tape if c_0 is also the first configuration on the input tape. The step mode behavior now ensures that the second configuration on the output tape is the successor of c_0 , call it c_1 . Iterating this argument, we see that the common prefix indeed contains a valid computation c_0, c_1, \dots, c_k , where each configuration is the successor of the previous one under one computation step of M . Finally, the trail is always as long as the configuration that is currently being read, up to the size of the one cell by which the input head is leading. Thus the current trail length is indeed always, up to constant, equal to the length of the current configuration in the simulation of the computation of M on the empty input word.

Finally, we obtain the main result of this section by combining Corollary 14 and Claims 15 and 16. The straightforward proof is deferred to Appendix F.

► **Theorem 17.** *Whether a given fv-NFT has bounded trailing is an undecidable problem.*

6 Conclusion

We have shown fv-NFTs to be more powerful than fv-2t-DFAs, characterized fv-NFTs having an equivalent fv-2t-DFA as those with bounded trailing, proved that it is undecidable whether an fv-NFT has bounded trailing, and showed how to construct the fv-2t-DFA if it is possible. Furthermore, all of these results hold true for functional and unambiguous transducers because Theorems 9 and 10 proves the characterization for any fv-NFT—which includes the functional and unambiguous transducers—and because we have shown our criterion’s undecidability—which in turn implies the strict difference in expressive power—using an unambiguous transducer, which is most restrictive.

Fischer and Rosenberg [14] have shown already in the sixties that deciding whether an 2t-NFA (or equivalently, a λ -NFT) can be transformed into an equivalent 2t-DFA is an undecidable problem; however, their proof requires relations that are not finite-valued and thus does not yield any result for fv-NFTs. Moreover, neither a characterization of transducers with an equivalent deterministic automaton nor a method to construct an equivalent deterministic automaton if it exists has been described so far. The present paper addresses all these issues for the case of fv-NFTs.

We emphasize again that the bounded trailing property defined in this paper does not characterize NFTs having an equivalent 2t-DFA unless the relation is finite-valued; a corresponding counterexample of a transducer with both an equivalent 2t-DFA and unbounded trailing is found in Appendix B. Finding a reasonable characterization of NFTs with an equivalent 2t-DFA and a method to transform the former into the latter if possible remains as an open research opportunity.

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575 **A** Example Mentioned in the Introduction

576 As mentioned in the paragraph Inclusion Hierarchy in the introduction, we present the
577 concrete example of a language that can be recognized by an f-2t-DFA but not by a DFT.

578 Consider the relation that maps any binary string w to $0^{|w|}$ if the last symbol of w is 0
579 and to $1^{|w|}$ otherwise. It is easy to construct a functional 2t-DFA that verifies this relation
580 The automaton advances its two heads synchronously, always comparing the current output
581 symbol with the previous one. If there is a discrepancy at any point, the automaton rejects;
582 otherwise, the output word is guaranteed to be uniform. If the two heads reach the end
583 of the input and output word at the same time and the last two symbols match, then the
584 automaton accepts; otherwise, it rejects. A DFT cannot compute this relation, however,

because it cannot output any symbol until it knows the last symbol of the word on the first tape, and by that time, the transducer cannot recall the length of the input word, which equals the number of copies of the last symbol that the transducer needs to produce.

B Non-Characterization for Infinite-Valued Transducer

The following describes an infinite-valued transducer that has an equivalent 2t-DFA but nevertheless unbounded trailing.

Consider the relation $L_{iv} = \{(0^n, 0^m) \mid n \in \mathbb{N} \wedge m \in \mathbb{N} \wedge 0 \leq m \leq 2n\}$, which is clearly not finite-valued. It is easy for a 2t-DFA to recognize L_{iv} by repeatedly advancing first the output head twice and then the input head once. Any input-output pair is in the language if and only if the end of the output is reached before the end of the input and all read symbols before the end markers are 0. It is also simple to build an equivalent NFT with unbounded trailing as follows. Whenever reading one input symbol, the transducer writes a 0 to the output tape either zero, one, or two times.

This transducer does indeed have unbounded trailing since for every even $t \in \mathbb{N}$, it has the two computations

$$q_0 \xrightarrow{\lambda 0^{t/2}} q_\Delta \xrightarrow{\lambda} f_\Delta \quad \text{and} \quad q_0 \xrightarrow{\lambda} q_\nabla \xrightarrow{0^t \lambda} f_\nabla,$$

where λ is the empty word and $a = b_1 = b_2 = 0^{t/2}$, $u = w_1 = w_2 = \lambda$, and $v = 0^t$ in Definition 8.

C Example for Section 3

We provide a concrete example of the general construction described Section 3; that is, we use a given NFT with bounded trailing to construct an equivalent 2t-DFA.

► **Example 18.** We consider the NFT T with the set of states $Q = \{q_0, q_1, q_2, q_3\}$, initial state $q_0 \in Q$, the set of accepting states $F = \{q_0\}$, and the transition function δ given in Figure 3a. Let us define the language $L_0 = \{(aa, ababa), (aa, ababab), (ab, ababaa)\}$. One can check that the language computed by T is the Kleene closure of L_0 (i.e., $L(T) = L_0^*$) and that the trailing of T is bounded by $t = 1$.

The computation of automaton A on the input-output pair $(aa, ababab)$ is summarized in Figure 3b. We now describe this computation in detail. Because $L(T) \neq \emptyset$, the initial state of automaton A is storing $z = \lambda$, where λ denotes the empty word, and $P = \{(q_0, 0)\}$. The output speed of T —that is, the length of a longest output that T can produce while reading one input symbol—is $s = 4$. Hence, the maximum length of the subsequence z maintained in A 's state is $r = s + t = 5$.

Because the output tape consists of six symbols, the first $r = 5$ of them are read and appended to z . The nonempty output words in $L(T)$ all start with the same five symbols, and the only pair in P , namely $(q_0, 0)$, can be extended to an accepting computation consistent with next five output symbols. These symbols are therefore successively output and appended to z , while $(q_0, 0)$ is being kept in P . After this, A 's state is storing $z = ababa$ and $P = \{(q_0, 0)\}$.

Now that the length of z is $5 = r$, the first symbol a from the input tape is read. All nondeterministic steps from $\delta(q_0, a) = \{(q_1, aba), (q_2, abab), (q_3, abab)\}$ are consistent with z , which leads to $P' = \{(q_1, 3), (q_2, 4), (q_3, 4)\}$. The minimum m for which $(q, m) \in P'$ for some q is $m = 3$. After performing the normalization of P' with this m , the state of automaton A is storing $z = ba$ and $P = \{(q_1, 0), (q_2, 1), (q_3, 1)\}$.

Since the length of z is now $2 < r$, automaton A reads the next output symbol b and appends it to z , which thus becomes $z = bab$. Then A removes the pair $(q_3, 1)$ from the set P because this pair can no longer be extended to an accepting computation consistent with the extended subsequence $z = bab$ since the only possible transition from q_3 produces the output aa , which is inconsistent with the suffix ab of $z = bab$. The remaining two pairs $(q_1, 0)$ and $(q_2, 1)$ are still consistent with the extended $z = bab$. As $(q_1, 0)$ stays in P , no normalization need be performed.

Now that the end of the output tape has been reached, A reads in the next symbol a from the input tape despite $|z| = 3 < r$. Performing one step that reads a on every pair in P yields $P' = \{(q_0, 3)\}$. Note that $(q_0, 2) \notin P'$ because each transition from q_0 starts with a b , the last symbol of z . After one more normalization, A 's state is storing $z = \lambda$ and $P = \{(q_0, 0)\}$.

Finally, A has reached the end of both the input and output tape, thus it checks whether $(q, |z|) \in P$ for some $q \in F$. As $(q_0, 0) \in P$, $q_0 \in F$, $|z| = 0$, A accepts the input-output pair $(aa, ababab) \in L(T)$.

| | q_0 | q_1 | q_2 | q_3 |
|-----|--|-----------------------------|-----------------|-----------------|
| a | $\{(q_1, aba), (q_2, abab), (q_3, abab)\}$ | $\{(q_0, ba), (q_0, bab)\}$ | $\{(q_0, ab)\}$ | \emptyset |
| b | \emptyset | \emptyset | \emptyset | $\{(q_0, aa)\}$ |

(a) The transition function δ of the given NFT T .

| | | | |
|--|---|--|--|
| | $P = \{(q_0, 0)\}$ $x = \lambda$ $z = \lambda$ | | $P = \{(q_0, 0)\}$ $x = \lambda$ $z = ababa$ |
| | $P = \{(q_1, 0), (q_2, 1), (q_3, 1)\}$ $x = aba, z = ba$ | | $P = \{(q_1, 0), (q_2, 1)\}$ $x = aba$ $z = bab$ |
| | $P = \{(q_0, 0)\}$ $x = ababab$ $z = \lambda$ | | |

(b) The computation of the 2t-DFA A on the input $(aa, ababab)$. The positions of the heads are marked in black, the subsequence z of the output tape is shaded gray, and x consists of the white cells before z .

Figure 3 Illustrations pertaining to Example 18.

D Proof of Claim 11

For convenience, we restate Claim 11 before providing its full proof.

Claim 11. For every $i \in \{0, 1, \dots, k\}$, there is an input word $b_{i+1} \in \Sigma^*$ and, for every $j \in \{1, 2, \dots, i\}$, an output word $\tilde{v}_j \in \Gamma^*$ with $|\tilde{v}_j| \leq |v_j|$, such that transducer T has the accepting computation $q_0 \xrightarrow{a} q_\nabla \xrightarrow{\tilde{v}_k v_{k-1} \dots v_{i+1} \tilde{v}_i \tilde{v}_{i-1} \dots \tilde{v}_2 \tilde{v}_1 \tilde{v} w_\nabla} f_\nabla$ and such that automaton A has $i+1$ computations that all read the same input word ab_{i+1} overall, all begin with $\hat{q}_0 \xrightarrow{\tilde{a} a_k a_{k-1} \dots a_{i+1}} \hat{q}_{i+1}$, and from \hat{q}_{i+1} lead to accepting states while reading the outputs $v_i \dots v_2 v_1$, $v_i \dots v_2 \tilde{v}_1$, $v_i \dots \tilde{v}_2 \tilde{v}_1$, \dots , and $\tilde{v}_i \dots \tilde{v}_2 \tilde{v}_1$, respectively.

Proof. Proving the induction basis is trivial: For $i = 0$, the claim coincides precisely with the situation shown in Figure 2b when setting $b_1 = b_\nabla$. It remains to prove the induction step. We fix an arbitrary $i \in \{1, \dots, k\}$, assume the claim for $i-1$ as our induction hypothesis, and prove it for i . First consider T 's computation given by the induction hypothesis for

656 $i - 1$. Because T produces at most s output symbols when reading one input symbol, there
 657 are decompositions $b_i = \bar{b}_i \bar{b}_i \bar{b}_i$ and $v_i = \bar{v}_i \bar{v}_i \bar{v}_i$ with $|\bar{v}_i| < s$ and $|\bar{v}_i| < s$ such that T has a
 658 computation

$$659 \quad q_0 \xrightarrow{a} q_\nabla \xrightarrow{\bar{b}_i / \bar{v} v_k v_{k-1} \dots v_{i+1} \bar{v}_i} \bar{q}_i \xrightarrow{\bar{b}_i / \bar{v}_i} \bar{q}_i \xrightarrow{\bar{b}_i / \bar{v}_i \bar{v}_{i-1} \dots \bar{v}_1 \bar{v} w_\nabla} f_\nabla.$$

660 Note that $|\bar{v}_i| > t_i - 2s = s \cdot (1 + i(t_0 + t_1 + \dots + t_{i-1})) \cdot (1 + |Q_T| \cdot |Q_A|^i)$. Thus $\bar{q}_i \xrightarrow{\bar{b}_i / \bar{v}_i} \bar{q}_i$
 661 contains more than $(1 + i(t_0 + t_1 + \dots + t_{i-1})) \cdot (1 + |Q_T| \cdot |Q_A|^i)$ steps during which T
 662 produces nonempty output. We call the positions of \bar{b}_i at which T produces nonempty output
 663 during its computation T -productive.

664 Now consider the i different computations of A given by the induction hypothesis for $i - 1$.
 665 They all start with a common prefix that splits into $\hat{q}_0 \xrightarrow{\bar{a} a_k a_{k-1} \dots a_{i+1} / u \bar{v} v_k v_{k-1} \dots v_{i+1}} \hat{q}_{i+1} \xrightarrow{a_i / v_i} \hat{q}_i$.
 666 Note that in the remainder of these computations, automaton A always scans the same
 667 input word $a_{i-1} \dots a_2 a_1 b_i$ but i different output words. These output words are indeed
 668 pairwise different since their lengths are $|\bar{v}_{i-1} \dots \bar{v}_2 \bar{v}_1 \bar{v} w_\nabla| < \dots < |v_{i-1} \dots v_2 v_1 \bar{v} w_\nabla| <$
 669 $|v_{i-1} \dots v_2 v_1 \bar{v} w_\nabla| \leq t_0 + t_1 + \dots + t_{i-1}$ are pairwise different due to the fact that $|\bar{v}_j| < |v_j|$
 670 for all $j \in \{1, \dots, i - 1\}$.

671 Since these are i different computations, each of which has at most $t_0 + t_1 + \dots + t_{i-1}$ output
 672 symbols, there are within the remaining $a_{i-1} \dots a_2 a_1 b_i$ input suffix at most $i(t_0 + t_1 + \dots + t_{i-1})$
 673 positions such that, for at least one of the i computations by A , the input head of A stays
 674 put in said position while its output head advances. We call these positions A -productive
 675 and the others A -unproductive. It follows in particular that the subword \bar{b}_i in $b_i = \bar{b}_i \bar{b}_i \bar{b}_i$
 676 contains at most $i(t_0 + t_1 + \dots + t_{i-1})$ positions that are A -productive. They delimit at most
 677 $1 + i(t_0 + t_1 + \dots + t_{i-1})$ sequences of consecutive A -unproductive positions in \bar{b}_i . Using the
 678 lower bound on the number of T -productive positions in \bar{b}_i derived above, we conclude that
 679 at least one of these A -unproductive sequences contains more than

$$680 \quad \frac{(1 + i(t_0 + t_1 + \dots + t_{i-1})) \cdot (1 + |Q_T| \cdot |Q_A|^i) - i(t_0 + t_1 + \dots + t_{i-1})}{1 + i(t_0 + t_1 + \dots + t_{i-1})} > |Q_T| \cdot |Q_A|^i$$

681 T -productive positions. Thus \bar{b}_i has a decomposition $\bar{b}_i \bar{b}_i \bar{b}_i$ such that, on the one hand, all i
 682 computations of A given by the induction hypothesis for $i - 1$ loop through \bar{b}_i simultaneously
 683 without advancing the output head at all and, on the other hand, T 's partial computation
 684 on \bar{b}_i decomposes into

$$685 \quad \bar{q}_i \xrightarrow{\bar{b}_i' / \bar{v}_i'} \bar{q}_i \xrightarrow{\bar{b}_i / \bar{v}_i} \bar{q}_i \xrightarrow{\bar{b}_i' / \bar{v}_i'} \bar{q}_i$$

686 with a loop that starts at some state \bar{q}_i and, while reading the input sequence \bar{b}_i , produces
 687 some output \bar{v}_i that is nonempty due to at least one T -productive position in \bar{b}_i . Removing
 688 the nonproductive loop \bar{b}_i from all i of A 's computations given by the induction hypothesis
 689 for $i - 1$, we obtain all but the last of A 's computations in the claim for i . Moreover, deleting
 690 the loop on \bar{b}_i from T 's computation in the induction hypothesis for $i - 1$ and defining
 691 $b_{i+1} = \bar{b}_i \bar{b}_i \bar{b}_i$ and $\bar{v}_i = \bar{v}_i \bar{v}_i' \bar{v}_i$ yields the computation for T in the claim for i . Finally, we
 692 use $\mathcal{L}(A) = \mathcal{L}(T)$ to obtain from this new computation of T the missing last computation of
 693 A . All mentioned computations of A start with the same prefix $\hat{q}_0 \xrightarrow{\bar{a} a_k a_{k-1} \dots a_{i+1} / u \bar{v} v_k v_{k-1} \dots v_{i+1}} \hat{q}_{i+1}$
 694 since A is deterministic and all input and output in this prefix has remained unchanged.
 695 This concludes the induction step and thus the proof of Claim 11. ◀

E A finite-valued transducer with unbounded trailing

We provide an illustrative example of an fv-NFT with unbounded trailing, as mentioned at the end of Section 4. In fact, the described transducer is also functional and even unambiguous.

► **Example 19.** The transducer’s alphabets are $\Sigma = \Gamma = \{0, 1\}$ and its relation is $\{(0^i 10^j 10, 0^i) \mid i, j \in \mathbb{N}\} \cup \{(0^i 10^j 11, 0^j) \mid i, j \in \mathbb{N}\}$. This union contains exactly the pairs of the form $(0^i 10^j 1\beta, 0^k)$, where β is a bit valued 0 or 1 and $k = i$ if $\beta = 0$ and $k = j$ if $\beta = 1$. This relation is easily computed by an fv-NFT that nondeterministically guesses β , then copies either 0^i or 0^j to the output tape, and finally accepts or rejects depending on whether the guess was correct.

No 2t-DFA can compute this relation, however, for the following intuitive reason. Consider an input-output pair $(0^i 10^j 1\beta, 0^k)$ with sufficiently large i, j , and k . By the time the input head reaches the first 1, the output head has either read most of 0^k already or has a large part of it still lying ahead. In the first case, the automaton may have potentially checked whether $k = i$, but cannot remember how many zeroes of 0^k the output head has already passed making it impossible to check whether $k = j$ if the input ends with $\beta = 1$. In the second case, the automaton can potentially still check whether $k = j$, but cannot remember how many zeroes the input head has already passed, that is, the value of i . This makes it impossible to check whether $k = i$ if the input ends with $\beta = 0$. Therefore, the automaton fails on inputs with $\beta = 1$ in the first case and on inputs with $\beta = 0$ in the second case.

To show formally that the described transducer has unbounded trailing we use the variable names of Definition 8. For any given $t \in \mathbb{N}$, we can choose $u = w_1 = w_2 = \lambda$, where λ denotes the empty word, $a = v = 0^t$, $b_1 = 10^t 10$, and $b_2 = 10^t 11$. This yields two computations showing that the trailing bound must be at least t , namely

$$q_0 \xrightarrow{\lambda 0^t} q_\Delta \xrightarrow{10^t 10} f_\Delta \text{ and } q_0 \xrightarrow{0^t} q_\nabla \xrightarrow{10^t 11} f_\nabla.$$

F Proof of Lemma 13

For convenience, we restate Lemma 13 before providing its proof.

► **Lemma 13.** *The problem of determining whether a Turing machine reaches infinitely many different configurations during its computation on the empty input is undecidable.*

Proof. We prove the lemma by contradiction. Suppose that there is an algorithm A_∞ that decides for every given Turing machine M whether it reaches infinitely many different configurations on the empty input. We will use A_∞ to design an algorithm A_H that decides for any Turing machine M whether it reaches a halting configuration on the empty input. The latter problem is known to be undecidable, yielding the desired contradiction.

Given a Turing machine M , algorithm A_H first invokes A_∞ to decide whether M reaches infinitely many different configurations on the empty input. If it does, then A_H outputs “No” because reaching a halting configuration implies reaching only finitely many configurations in total. Otherwise, A_H simulates M ’s deterministic computation on the empty input step by step, remembering all configurations, until either a halting or a previously encountered configuration is reached. Then A_H outputs “Yes” in the former case and “No” in the latter. ◀

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 Illustration for Section 5

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This appendix provides with Figure 4 an illustration of the computations by the transducer that is constructed in the proof of Theorem 17.

| | | | | | | | | | | | | |
|-------|-------|-------|----------|-------|------|--|-------|--------|--------|----------|------------|--------|
| c_1 | c_2 | c_3 | \cdots | c_k | copy | | c_1 | c_2 | c_3 | \cdots | c_k | step |
| c_1 | c_2 | c_3 | \cdots | c_k | | | c_0 | c'_1 | c'_2 | \cdots | c'_{k-1} | c'_k |

(a) The two types of valid inputs for T and the corresponding accepted outputs. Each c_i with $i > 0$ encodes an arbitrary configuration of M , c_0 is its initial configuration, and c'_i is c_i 's successor configuration.

| | | | | | | | | | | | | |
|-------|-------|-------|----------|-------|------|--|-------|-------|-------|----------|-------|-----------|
| c_0 | c_1 | c_2 | \cdots | c_k | copy | | c_0 | c_1 | c_2 | \cdots | c_k | step |
| c_0 | c_1 | c_2 | \cdots | c_k | | | c_0 | c_1 | c_2 | \cdots | c_k | c_{k+1} |

(b) Two accepting computation patterns of T that make unbounded trailing inevitable if M reaches arbitrarily long configurations. Here, c_0 is still the initial configuration of M on the empty input, but c_{i+1} is now the successor configuration of c_i , under a single computation step of M .

Figure 4 Accepting computation patterns of transducer T , which depends on Turing machine M .

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 Proof of Theorem 17

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For convenience, we restate Theorem 17 before providing its proof.

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► **Theorem 17.** *Whether a given fv-NFT has bounded trailing is an undecidable problem.*

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Proof. We prove the theorem by contradiction. Suppose that there is an algorithm A deciding whether an fv-NFT has bounded trailing. We construct an algorithm A_∞ deciding whether a Turing machine M reaches configurations of arbitrary length. The latter problem is undecidable by Corollary 14, yielding a contradiction.

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Given a Turing machine M , algorithm A_∞ constructs the fv-NFT T and uses algorithm A to decide whether T has bounded trailing. If it does, then A_∞ outputs “No,” otherwise it outputs “Yes.” The correctness of algorithm A_∞ follows from Claim 16. ◀

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