Method card 5

Course: statistics
Topic: Common discrete distributions 2024-2025

For each distribution, you will find

- **some model example**
- **the PMF**
- **expectation and variance**

You will also find important and useful formula.

Bernoulli distribution: $X \rightsquigarrow \mathcal{B}(p)$**,** $p \in [0,1]$

Examples of use: 2 outcomes, Head and Tails, success and fail, yes or no, male or female, etc.
State space: $\{0,1\}$
PMF: $p_X(1) = p$, $p_X(0) = 1-p$
Moments: $\mathbb{E}[X] = p$, $\mathbb{V}[X] = p(1-p)$

Rademacher distribution: $X \rightsquigarrow \mathcal{R}(p)$**,** $p \in [0,1]$

Examples of use: same as Bernoulli but with a different parametrization
State space: $\{-1,1\}$
PMF: $p_X(1) = p$, $p_X(-1) = 1-p$
Moments: $\mathbb{E}[X] = 2p-1$, $\mathbb{V}[X] = 4\,p\,(1-p)$

Uniform distribution: $X \rightsquigarrow \mathcal{U}(\{1,...,n\})$**,** $n \in \mathbb{N}^*$

Example of use: selection of 1 element among n with equal probability
State space: $\{1,...,n\}$
PMF: $p_X(k) = \dfrac{1}{n}$ for $k \in \{1,...,n\}$
Moments: $\mathbb{E}[X] = \dfrac{n+1}{2}$, $\mathbb{V}[X] = \dfrac{n^2-1}{12}$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

Hypergeometric distribution: $X \rightsquigarrow \mathcal{H}(n, N_1, N_2)$**,** $n, N_1,\ N_2 \in \mathbb{N}^*$**,** $n \leq N_1 + N_2$

Example of use: Counting the red tokens when drawing **WITHOUT** replacement n tokens out of an urn which has respectivelly N_1 and N_2 red and blue tokens.
State space: $\{0,...,n\}$
PMF: For $k \in \{0,...,n\}$, $p_X(k) = \dfrac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N_1+N_2}{n}}$ with the convention $\binom{N}{k} = 0$ if $k > N$
Moments: $\mathbb{E}[X] = n\dfrac{N_1}{N_1+N_2}$, $\mathbb{V}[X] = n\dfrac{N_1}{N_1+N_2}\dfrac{N_2}{N_1+N_2}\dfrac{N_1+N_2-n}{N_1+N_2-1}$

$$\boxed{\sum_{k=0}^{n} \binom{N_1}{k}\binom{N_2}{n-k} = \binom{N_1+N_2}{n}, \quad \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}}$$

Binomial distribution: $X \rightsquigarrow \mathcal{B}(n,p)$**,** $n \in \mathbb{N}^*$**,** $p \in [0,1]$

Example of use: Counting the red tokens when drawing **WITH** replacement n tokens out of an urn which has a proportion p of red tokens.
State space: $\{0, ..., n\}$
PMF: For $k \in \{0, ..., n\}$, $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$
Moments: $\mathbb{E}[X] = np$, $\mathbb{V}[X] = np(1-p)$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n}$$

Geometric distribution: $X \rightsquigarrow \mathcal{G}(p)$**,** $p \in [0,1]$

Example of use: First occurrence of a success when trying infinitely many times the same two-outcomes experiment.
State space: $\mathbb{N}^*$
PMF: For $k \in \mathbb{N}^*$, $p_X(k) = (1-p)^{k-1} p$
Moments: $\mathbb{E}[X] = \dfrac{1}{p}$, $\mathbb{V}[X] = \dfrac{1-p}{p^2}$

$$\boxed{\sum_{k=0}^{+\infty} \gamma^k = \frac{1}{1-\gamma}, \quad \sum_{k=1}^{+\infty} k\gamma^{k-1} = \frac{1}{(1-\gamma)^2}, \quad \sum_{k=2}^{+\infty} k(k-1)\gamma^{k-2} = \frac{2}{(1-\gamma)^3}}$$

Poisson distribution: $X \rightsquigarrow \mathcal{P}(\lambda)$**,** $\lambda > 0$

Example of use: Number of occurrence of an event whose average frequency is λ per unit.
State space: $\mathbb{N}$
PMF: For $k \in \mathbb{N}$, $p_X(k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$
Moments: $\mathbb{E}[X] = \lambda$, $\mathbb{V}[X] = \lambda$

$$\boxed{\sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} = \sum_{k=1}^{+\infty} k \frac{\lambda^{k-1}}{k!} = \sum_{k=2}^{+\infty} k(k-1) \frac{\lambda^{k-2}}{k!} = \mathrm{e}^{\lambda}}$$