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Few-Body Systems

Spectrum and Static Properties of Heavy Baryons

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Abstract. Using several realistic interquark potentials, the ground-state energies of all the baryons containing one, two or three heavy quarks of type *c* or *b* are studied within a non-relativistic quark model. The three-body problem is rigorously solved using the Faddeev formalism. Various static properties, such as mass radii, charge radii, magnetic moments, and wave functions at the origin are calculated as well. The complete spectrum for all these baryons is computed using a harmonic-oscillator basis with states up to 8 quanta. Emphasis is put on the levels lying below the thresholds corresponding to quark-pair creation.

1 Introduction

Baryon spectroscopy has already a long history (see ref. [1]). Most of the known resonances have been excited with hadronic or electromagnetic probes. The baryon sector is a good laboratory for exploring strong interactions and consequently for testing quantum chromodynamics (QCD), the present theory of strong interactions. The nucleon resonances have been studied with special care and facility because of the modest energies involved and because the target is very abundant. However, strange matter is very important too, and a lot of work has also been developed in that direction.

As soon as new quark flavours were discovered and confirmed, it became evident that baryons containing heavy flavors *c* or *b* also play an important role for understanding QCD more deeply [2]. However, the production of such exotic systems is seriously inhibited by energy and cross-section considerations. Nevertheless, with the advent of new facilities, such as the Tevatron at Fermilab or the LHC at CERN, energy and luminosity necessary for studying those baryons have become available, and a new field of interesting physics has opened up [3].

Besides the exciting hope of discovering new states of matter, the interest in studying such heavy systems relies on a specific feature: When the masses of the quarks increase, the spectrum becomes more and more dense and there exist more and more excited states with relatively low energies. One can imagine that there exist some states below the threshold of pion production. In that case such resonances are fully stable under strong interactions and must decay through the electromagnetic interaction, giving them an appreciable lifetime. The situation is even more favourable because for baryons that do not contain ordinary quarks one-pion production is strictly forbidden because of isospin conservation. In such a situation the lowest threshold is usually a two-pion decay, with a double-excitation energy and with a strong OZI suppression. In practice, the important threshold corresponds to the creation of an ordinary quark pair shared between the baryon and the meson of the final state; this threshold can be as high as several hundreds MeV and there may exist several baryonic resonances below it. From the experimental point of view, this situation is ideal for discriminating and determining them unambiguously. A lot of new states can be discovered in the region of 7 to 15 GeV.

Of course, the production of such interesting baryons needs the creation of two or three heavy quarks in a single baryon. Besides the penalty due to the large energy involved, the alternative heavy-meson production should disfavour the corresponding mechanism. Nevertheless, with the future high-energy facilities, there exists some hope to partly reach this objective. In previous works, only some selected samples of heavy-baryon resonances were considered. An important aim of this paper is to perform a systematic study of the spectra (up to 800 MeV excitation energy) of *all* heavy baryons in order to guess and predict *all* the candidates for such a stability under strong interactions. Even a rough estimate of their masses should help the experimentalists in their investigations.

From the theoretical viewpoint, heavy baryons are also of special interest. They allow to check the flavour independence of the confinement potential, and hence the validity of QCD itself. One of the first studies on this subject was proposed by Mathur et al. [4] using very crude mass formulae. Later on, more microscopic calculations were performed using a harmonic-oscillator basis within a nonrelativistic approach by Copley et al. [5] and by Maltman and Isgur [6]. Since then, other methods were proposed and further studies on this subject have already been carried out using potential models [7–9], relativized quark models [10, 11] or QCD sum rules [12, 13].

Strictly speaking, a nonrelativistic treatment is fully justified only for baryons built with three heavy quarks. Indeed relativistic treatments of two-body, three-body and more complicated systems have been proposed in the past and they are worthwhile to be studied and developed. Although a fully covariant formalism seems very difficult to be applied in practice for the three-body problems, an interesting approach based on a so-called relativistic Hamiltonian dynamics (very similar to the light-cone dynamics) was used with success to give an explanation of the missing binding energy for three-nucleon systems [14]. In this study, relativity effects are taken into account directly at the level of the three-body systems. Other approaches incorporate relativity at the level of the two-body potential by introducing relativistic corrections at order $(v/c)^2$. Such a formalism was proposed by Tsuge et al. [11] and applied to heavy baryons to show that the confinement should be predominantly of vector type. Thus, it seems that relativistic effects might have some importance.

On the other side, the nonrelativistic potential model has been applied with success to domains where its validity could *a priori* be questioned. A number of works was devoted to understand why it behaves so nicely. The answer is far from definite, but it was shown [15], in the meson sector, that the spectrum coming from a relativistic treatment could be simulated very accurately by a nonrelativistic scheme provided the parameters of the model (masses, coupling constants, ...) are renormalized correctly. This does not justify the nonrelativistic method itself, but tries to explain how our ignorance of a precise mechanism can be accounted for by a phenomenological one. The nonrelativistic approach is certainly the method that allows to explain the most important bulk of hadronic spectroscopic properties. Moreover, it is the only one that can deal properly with the centre-of-mass motion, a crucial ingredient for few-body problems. This is why I shall stick to the nonrelativistic approach in the following study.

Even in such a scheme, there can be discrepancy between theory and experiment because the underlying physics is not understood or because the resulting mathematical equations are solved only approximately. Now, after the intense work on three-body problems in both nuclear and atomic physics, we have at our disposal very powerful tools to solve the three-body problem very efficiently and with high accuracy. In particular, the Faddeev formalism is very well suited to this job [16, 17]. Since we have the ability to perform an exact numerical solution of the mathematical problem, one possible source of uncertainty between experiment and theory is eliminated. This allows us to focus our studies on the dynamical aspect of the problem. In my opinion, I see two important questions to be answered, (i) the dependence of the results on the interquark potentials, (ii) the importance, in the wave function, of components more complicated than three quark ones.

In this paper, I shall investigate in great detail the first point. Another aim of this paper is to see whether the results depend significantly on the potential under consideration. One can learn about some physical implications from a careful inspection of such a comparison. I shall apply the Faddeev formalism to *all* baryons containing at least one heavy quark c or b, with different quark-quark potentials. This represents 18 baryons; explicitly there exist 8 systems with one heavy quark: $\Lambda_c(udc)$, $\Sigma_c(uuc)$, $\Lambda_b(udb)$, $\Sigma_b(uub)$, $\Xi_c(usc)$, $\Omega_c(ssc)$, $\Xi_b(usb)$, $\Omega_b(ssb)$; 6 systems with two heavy quarks: $\Xi_{cc}(ucc)$, $\Xi_{cb}(ucb)$, $\Xi_{bb}(ubb)$, $\Omega_{cc}(scc)$, $\Omega_{cb}(scb)$, $\Omega_{bb}(sbb)$; and 4 systems with three heavy quarks: $\Omega_{ccc}(ccc)$, $\Omega_{ccb}(ccb)$, $\Omega_{cbb}(cbb)$, $\Omega_{bbb}(bbb)$.

To have reliable conclusions, the potentials employed must have been tested seriously and with success in the meson sector. Moreover, I also include some three-body forces in a phenomenological way to improve the values of the resulting masses. I discuss this point in the next section. I think that energies are not sufficient to discriminate fully between the various potentials. To have a better idea of the wave functions, I calculate also several observables: mass and charge mean-square radii, magnetic moments, and wave functions at the origin. It is my experience that this last observable is especially sensitive to the potentials as well as to the numerical method used to solve the three-body problem. The third section is devoted to the discussion of the Faddeev formalism and to the definition of the various observables. In the fourth section, I present the results obtained along the previous philosophy. I also report the lowest resonances for every system and emphasize those which lie below the threshold corresponding to an ordinary quark-pair creation. I hope that my results may serve as a guide for future experiments on these exotic systems. I draw some conclusions in the last section.

2 The Quark-Quark Potential

Strictly speaking, QCD does not imply the existence of interquark potentials; one may for instance describe the interactions between the quarks in terms of flux tubes generated by gluon fields. However, within a nonrelativistic description, quark-quark potentials are widely used and their success in hadron spectroscopy is *a posteriori* a clear justification that most of the underlying physical dynamics can be included phenomenologically through the form of potentials. Indeed, at high energy or at short range, perturbative QCD leads to the so-called one-gluon-exchange potential of Coulomb type, corrected by hyperfine, tensor, and spin-orbit terms. What is less clear is the behaviour of QCD at small energies or at long range. Here again, lattice gauge theories show that the static interaction between two quarks (or one quark, one antiquark) implies linear confinement at large distance. These two characteristics are the essence of the familiar "Coulomb + linear" potentials that are of common use in the literature [18–22].

It appears that the tensor and spin-orbit terms are not essential for hadron spectroscopy and are usually neglected. As a consequence, the orbital angular momentum L and the total spin S are separately good quantum numbers. In contrast, the hyperfine term is crucial; a contact form, as it results from a Fermi-Breit approximation of the central part, is clearly not adequate for a proper treatment that goes beyond perturbative approaches because it leads to a collapse in relative s-waves. Several other forms, with short range, have been proposed in the past.

Most of the potentials presented in the literature differ essentially by the values of the parameters and by the treatment of the hyperfine term. Among them, the one proposed by Bhaduri and collaborators [23] works quite well in both the meson and baryon sectors. It will be named *BD* hereafter and it has the form

$$V_{ij}^{q\bar{q}}(r) = -\frac{\kappa}{r} + \lambda r - \Lambda + \frac{\kappa}{m_i m_j}\frac{\exp(-r/r_0)}{r r_0^2}\vec{\sigma}_i\vec{\sigma}_j \tag{1}$$

with

$\kappa = 0.52$; $\lambda = 0.186\,\mathrm{GeV}^2$; $\Lambda = 0.9135\,\mathrm{GeV}$; $r_0 = 2.305\,\mathrm{GeV}^{-1} = \text{constant}$;

$m_u = m_d = 0.337\,\mathrm{GeV}$; $m_s = 0.600\,\mathrm{GeV}$; $m_c = 1.870\,\mathrm{GeV}$; $m_b = 5.259\,\mathrm{GeV}$.

The parameters were essentially fitted in the charmonium system.

In order to improve the quality of the meson spectra, I recently proposed, in collaboration with C. Semay, new types of potentials [24]. We chose the form

$$\begin{aligned} V_{ij}^{q\bar{q}}(r) = &-\frac{\kappa(1-\exp(-r/r_c))}{r} + \lambda r^p - \Lambda \\ &+ \frac{2\pi}{3m_i m_j}\kappa'(1-\exp(-r/r_c))\frac{\exp(-r^2/r_0^2)}{\pi^{3/2} r_0^3}\vec{\sigma}_i\vec{\sigma}_j \end{aligned} \tag{2}$$

with

$$r_0(m_i, m_j) = A\left(\frac{2m_i m_j}{m_i + m_j}\right)^{-B}.$$

Several comments are in order concerning this expression.

1. The Coulomb potential is affected by an r-dependent form factor; this is intended to simulate asymptotic freedom: $\kappa(0) = 0$ with finite $\kappa'(0)$. This form is purely phenomenological and is proposed only for simplicity. A pure Coulomb $\kappa =$ constant is obtained by setting $r_c = 0^+$; the same r-dependence is kept for the hyperfine term $\kappa'(r)$.

2. We considered two different powers for the confining term; $p = 1$ corresponding to the linear behaviour suggested by lattice gauge theories; $p = \frac{2}{3}$ corresponding to the value that gives, in a nonrelativistic treatment of mesons, the correct asymptotic (for large angular momenta) Regge trajectories [25].

3. We preferred a Gaussian form factor for the hyperfine term rather than a Yukawa one, because of better numerical convergence, and also because a Gaussian form is very well suited for getting closed analytical formulae in some specific approximations.

4. The range r_0 of the hyperfine term is mass dependent. This leads to very strong improvement in the resulting spectra (see also ref. [26]).

The parameters of the potentials were determined through a best fit on a large sample of meson states in every flavour sector. In this way, we proposed four new potentials, suitable for all types of mesons.

Potential $AL1$:

$$p = 1; \quad r_c = 0;$$
$$m_u = m_d = 0.315\,\text{GeV}; \quad m_s = 0.577\,\text{GeV}; \quad m_c = 1.836\,\text{GeV}; \quad m_b = 5.227\,\text{GeV};$$
$$\kappa = 0.5069; \quad \kappa' = 1.8609; \quad \lambda = 0.1653\,\text{GeV}^2; \quad \Lambda = 0.8321\,\text{GeV};$$
$$B = 0.2204; \quad A = 1.6553\,\text{GeV}^{B-1}.$$

Potential $AL2$:

$$p = 1; \quad r_c \neq 0;$$
$$m_u = m_d = 0.320\,\text{GeV}; \quad m_s = 0.587\,\text{GeV}; \quad m_c = 1.851\,\text{GeV}; \quad m_b = 5.231\,\text{GeV};$$
$$\kappa = 0.5871; \quad \kappa' = 1.8475; \quad \lambda = 0.1673\,\text{GeV}^2; \quad \Lambda = 0.8182\,\text{GeV};$$
$$B = 0.2132; \quad A = 1.6560\,\text{GeV}^{B-1}; \quad r_c = 0.1844\,\text{GeV}^{-1}.$$

Potential $AP1$:

$$p = \tfrac{2}{3}; \quad r_c = 0;$$
$$m_u = m_d = 0.277\,\text{GeV}; \quad m_s = 0.553\,\text{GeV}; \quad m_c = 1.819\,\text{GeV}; \quad m_b = 5.206\,\text{GeV};$$
$$\kappa = 0.4242; \quad \kappa' = 1.8025; \quad \lambda = 0.3898\,\text{GeV}^{5/3}; \quad \Lambda = 1.1313\,\text{GeV};$$
$$B = 0.3263; \quad A = 1.5296\,\text{GeV}^{B-1}.$$

Potential $AP2$:

$$p = \tfrac{2}{3}; \quad r_c \neq 0;$$
$$m_u = m_d = 0.280\,\text{GeV}; \quad m_s = 0.569\,\text{GeV}; \quad m_c = 1.840\,\text{GeV}; \quad m_b = 5.213\,\text{GeV};$$

$$\kappa = 0.5743; \quad \kappa' = 1.8993; \quad \lambda = 0.3978\,\text{GeV}^{5/3}; \quad \Lambda = 1.1146\,\text{GeV};$$
$$B = 0.3478; \quad A = 1.5321\,\text{GeV}^{B-1}; \quad r_c = 0.3466\,\text{GeV}^{-1}.$$

More details on the determination of the parameters can be found in ref. [24]. For the light-quark mesons, all these potentials are as good as BD but they are definitely better for heavy-quark sectors. The potentials with $r_c \neq 0$ are slightly better than those with $r_c = 0$. For the $BD, AL1, AL2, AP1, AP2$ potentials, a χ^2-fit based on 36 different mesons gives the values 34.3, 20.5, 12.7, 24.8, 16.4 (in arbitrary units), respectively (see ref. [24]).

These potentials have also been tested on light-flavour baryons [24]. In that case, the usual $V_{ij}^{qq}(r) = \frac{1}{2} V_{ij}^{q\bar{q}}(r)$ prescription, coming from a $\vec{\lambda}_i \vec{\lambda}_j$ colour dependence, has been used throughout. We have several reasons to believe that it is quite a good approximation. About fifty resonances have been calculated and compared to the experimental values. Here too, all these new potentials give much better results than BD. In this case, potentials with $p = \frac{2}{3}$ confinement give slightly better agreement than those with $p = 1$ (essentially because a number of resonances with high angular momenta were considered). Explicitly, the χ^2-fit based on 56 baryons gives the values 16.1, 10.8, 12.3, 8.8, 9.1 (in arbitrary units), respectively.

Lastly, they have been compared in the sector composed of 2 quarks and 2 antiquarks (still with the $\vec{\lambda}_i \vec{\lambda}_j$ colour dependence) and it was shown that they give very similar results [27]. Thus we are fairly confident in the physical energies coming from a nonrelativistic treatment using those potentials. As explained previously the absence of some relativistic corrections is not dramatic since the other parameters have been fitted in order to take them partially into account.

Non-perturbative QCD probably gives rise to many-body forces for complicated systems. We have no real idea of what the form of such forces may be nor of their effects. In the case of baryons, if one compares the results obtained using only two-body forces with experimental values, it appears that three-body forces decrease in magnitude as the masses of the quarks increase. In the following, I shall take care of three-body forces, only at the level of the energies, by subtracting from the results obtained with two-body forces a phenomenological three-body contribution of the form

$$V_{123} = \frac{C}{m_1 m_2 m_3}. \tag{3}$$

The only justification is simplicity and the fact that it leads to very good results. Such a term was also considered in the work of Bhaduri et al. [23]. The effect of such a term is shown in a subsequent section.

3 The Three-Body Problem

A correct treatment of three-body problems is a long-standing problem in atomic and nuclear physics. It seems that most studies of baryonic properties rely on quite crude approximations. From the possible methods to solve the three-body problem I mention Monte-Carlo techniques [28], the hyperspherical formalism [29–31], harmonic-oscillator (HO) expansions [32], and Faddeev equations [16, 33]. Some properties, such as energies or mean square radii, can be obtained quite accurately whatever method is employed. On the contrary, correlations at short range, such as the wave function at the origin, require a very precise and correct treatment. In this sense, the Faddeev equations are the ideal tool. To my knowledge, I was the first, in collaboration with C. Gignoux, to apply them to the description of the baryons in the (u, d, s) sector [33]. Since the formalism is now well known, I shall skip a lot of technical details, and focus my attention to the main philosophy.

In a baryon, the singlet colour wave function is completely antisymmetric, so that the Pauli principle requires a complete symmetry for the rest of the wave function; this part includes isospin, spin and space degrees of freedom. I shall adopt a scheme of type [1(23)] for the various couplings occurring naturally. Then there exist three different sets of Jacobi coordinates for the system. They may be written as

$$\begin{aligned}
\vec{x}_i &= \left[\frac{m_1(m_2+m_3)}{m_i(m_j+m_k)}\right]^{1/2}(\vec{r}_j-\vec{r}_k),\\
\vec{y}_i &= \left[\frac{m_1(m_2+m_3)(m_j+m_k)}{m_j m_k M}\right]^{1/2}\left(\frac{m_j\vec{r}_j+m_k\vec{r}_k}{m_j+m_k}-\vec{r}_i\right),\\
\vec{R} &= \frac{m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3}{M},
\end{aligned} \tag{4}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and $M = m_1 + m_2 + m_3$ is the total mass of the system. The mass-dependent coefficients are introduced to simplify the kinetic-energy term and to produce some useful properties under permutation of particles.

With these conventions, the Faddeev amplitudes $|\Psi_{\alpha_i}\rangle$ are expressed as (in coordinate representation $\langle \vec{x}_i, \vec{y}_i | \Psi_{\alpha_i}\rangle = \Psi_{\alpha_i}(\vec{x}_i, \vec{y}_i)$)

$$\Psi_{\alpha_i}(\vec{x}_i,\vec{y}_i) = \frac{d_{\alpha_i}(x_i,y_i)}{x_i y_i}\,\eta_\tau^{TT_z}(i,jk)\Big(\Big[Y_{l_y}(\hat{y}_i)\,Y_{l_x}(\hat{x}_i)\Big]_L \chi_\sigma^S(i,jk)\Big)_{JM}. \tag{5}$$

Here, $\eta_\tau^{TT_z}(i, jk)$ is the isospin wave function in which the (j, k) pair is coupled to intermediate isospin τ, the total isospin being T. $\chi_\sigma^S(i, jk)$ is the spin wave function with an analogous meaning of the quantum numbers. The index α_i labelling the various amplitudes denotes $\alpha_i = (\tau, \sigma, l_y, l_x)$.

The Faddeev amplitudes are coupled through a set of integro-differential equations [16]. In this paper, for solving the Faddeev equations, I consider only two-body interquark potentials with the usual prescription $V_{ij}^{qq}(r) = \frac{1}{2} V_{ij}^{q\bar{q}}(r)$. Technically, one must adopt a mesh in the (x_i, y_i) plane and solve the resulting linear system to get the eigenenergies E and the values of $d_{\alpha_i}(x_i, y_i)$ at each point.

In the case of 3 identical particles, only Faddeev amplitudes of type $\Psi_{\alpha_1}(\vec{x}_1, \vec{y}_1)$ enter into the game. For 2 identical particles one needs $\Psi_{\alpha_1}(\vec{x}_1, \vec{y}_1)$ and $\Psi_{\alpha_2}(\vec{x}_2, \vec{y}_2)$, while for 3 different particles $\Psi_{\alpha_1}(\vec{x}_1, \vec{y}_1)$, $\Psi_{\alpha_2}(\vec{x}_2, \vec{y}_2)$, and $\Psi_{\alpha_3}(\vec{x}_3, \vec{y}_3)$ are necessary. The total wave function is

$$|\Psi\rangle = \sum_{i=1}^{3}\sum_{\alpha}|\Psi_{\alpha_i}\rangle. \tag{6}$$

If one prefers using a harmonic-oscillator approach, one expands $d_{\alpha_i}(x_i, y_i)$ in radial HO wave functions u_{nl}. More precisely,

$$d_{\alpha_i}(x_i, y_i) = c_{\alpha_i} u_{n_y l_y}(y_i) u_{n_x l_x}(x_i), \tag{7}$$

where the labelling of the states is now $\alpha_i = (\tau, \sigma, n_y, l_y, n_x, l_x)$. In this case, the Hamiltonian is diagonalized in the corresponding basis Ψ_{α_i} to obtain the eigenenergies E and the eigenfunctions c_{α_i}.

In the following, I shall calculate some static properties of the baryons in their ground states $L = 0$. More precisely, I shall study the following observables:

1. The mass mean-square radius,

$$\langle R_m^2 \rangle = \langle \Psi | \sum_{i=1}^{3} \frac{m_i}{M} (\vec{r}_i - \vec{R})^2 | \Psi \rangle. \tag{8}$$

The relative coordinate $\vec{r}_i - \vec{R}$ is just proportional to $\vec{y}_i$, so that essentially one needs to compute the integral $\int d_{\alpha_i}^2(x_i, y_i) y_i^2 \, dx_i \, dy_i$. Strictly speaking, $\langle R_m^2 \rangle$ is not an observable, but it is nevertheless a very interesting quantity, which gives an idea of the actual physical size of the baryon.

2. The charge mean-square radius,

$$\langle R_c^2 \rangle_{T_z} = \langle \Psi_{T_z} | \sum_{i=1}^{3} e_i (\vec{r}_i - \vec{R})^2 | \Psi_{T_z} \rangle. \tag{9}$$

Since $e_i = \frac{1}{6}(1 + 3\tau_z^i)$ is composed of isoscalar and isovector parts, one needs to calculate the charge radius for each T_z component of the isospin multiplet. For the space wave function, I use the same integral as the one introduced for $\langle R_m^2 \rangle$. The charge mean-square radii are deduced from form factors at $q^2 = 0$ in lepton-baryon scattering.

3. The wave function at the origin for the j-k pair,

$$|\Psi_i(0)|^2 = \langle \Psi | \delta(\vec{r}_j - \vec{r}_k) | \Psi \rangle. \tag{10}$$

It is interesting and natural to introduce a correlation function,

$$\rho_i(x_i) = \int \langle \Psi(\vec{x}_i, \vec{y}_i) | \Psi(\vec{x}_i, \vec{y}_i) \rangle \, d\vec{y}_i \, d\Omega_{x_i}, \tag{11}$$

which is normalized to $\int \rho_i(x_i) x_i^2 \, dx_i = 1$.

With this convention,

$$|\Psi_i(0)|^2 = A_i \rho_i(x_i = 0), \tag{12}$$

where A_i is a kinematical constant depending on the masses through the definition (4) of the Jacobi coordinates. For simplicity, I shall denote $\rho_i(0)$ as the wave function at the origin for the (jk) pair. This observable is intimately related to leptonic and semi-leptonic decays of baryons.

4. The magnetic moments: If one does not include mesonic exchange currents and relativistic effects, the magnetic moment operator is simply the sum of the magnetic moments of each quark, with orbital and spin contributions. Thus, the

magnetic moment is given traditionally as

$$\mu = \langle \Psi_{J,M=J} | \sum_{i=1}^{3} \frac{e_i}{2m_i} (l_z^i + 2s_z^i) | \Psi_{J,M=J} \rangle. \qquad (13)$$

In this expression, the quarks are considered as elementary Dirac particles, so that the gyromagnetic factor is set equal to 2 exactly. In the naive quark model, the orbital contribution is always neglected. This is justified if, within the baryons, all the quark pairs are in relative *s* waves. I shall discuss this point in the following section.

4 Results

4.1 Numerical Procedure

Solving the three-body problem in the best way possible is a heavy task. Of course, the Faddeev formalism is intended to do that job, but one must be very cautious in the numerical procedure. Essentially two sources of uncertainties arise: (i) the choice and the number of Faddeev amplitudes and (ii) the grid in the (x_i, y_i) plane. Let me comment on these two points, starting with the last one. In fact, for technical reasons, it is much better, at least in the case of bound states, to use polar coordinates (ρ, θ_i) instead of Cartesian coordinates (x_i, y_i). The point is that the hyperradius ρ, with the definition of the Jacobi coordinates (4), is an invariant: $\rho = (x_1^2 + y_1^2)^{1/2} = (x_2^2 + y_2^2)^{1/2} = (x_3^2 + y_3^2)^{1/2}$. Moreover, the choice of ρ allows a partial decoupling in the resulting equations. I chose a mesh with n_ρ values in the ρ direction and n_θ values for the angles $\theta_i = \tan^{-1} y_i/x_i$. I used equally-spaced values for θ_1, but it is better to adopt a non-regular mesh in the ρ direction, in order to have more points for the interesting short-range part of the wave function. This is realized using a starting value ρ_1 and computing the further points with some accelerator acr such that $\rho_{i+1} - \rho_i = \mathrm{acr}(\rho_i - \rho_{i-1})$.

I always took $n_\theta = 12$; I checked that with $n_\theta = 16$ or $n_\theta = 24$ the variation in the results is negligible. When integration over the angle θ is necessary, the Faddeev amplitudes for some intermediate values are computed with the help of spline functions.

It is my experience that a value of n_ρ between 60 and 100 is enough. Here again, I checked that pushing the method to $n_\rho = 150$ or 200 does not change the results in any significant way. Actually, I adopted the value $n_\rho = 100$ for baryons with 2 or 3 identical particles, and $n_\rho = 60$ for those with 3 different particles. I found that acr = 1.01 is a good compromise in any case. The starting value ρ_1 is essentially a function of the type of baryon under consideration, since a baryon with many heavy quarks is evidently a more compact object than a baryon containing few of them. In practice, I took $\rho_1 = 0.02$ fm for baryons with a single heavy quark (giving an extension of $\rho_m = 3.4$ fm for the wave function with $n_\rho = 100$) and $\rho_1 = 0.008$ fm for baryons with 2 or 3 heavy quarks (giving an extension of $\rho_m = 1.36$ fm for the wave function with $n_\rho = 100$).

For the Faddeev amplitudes, I need a limitation of their number, which is, in principle, infinite. The baryon ground states always have a zero orbital angular momentum $L = 0$ so that $l_x = l_y$. It is clear that the most important waves are those with the lowest values of l_x. Practically, I have chosen all the Faddeev amplitudes with $l_x = l_y < 2$. It appears that going from inclusion of amplitudes with $l_x = 0$ only to inclusion of all the amplitudes $l_x = 0$ *and* $l_x = 1$ yields about 10 MeV in the binding energies whereas pushing to amplitudes with $l_x > 1$ adds only 1 to 3 MeV. These considerations allow me to take $n_\alpha = 6$ amplitudes for the baryons with two identical quarks, and $n_\alpha = 12$ amplitudes for the baryons with three different quarks.

The dimension of the matrices to be diagonalized to obtain the eigenenergies is $N = n_\alpha \times n_\theta \times n_\rho$; typically we have $5000 < N < 15000$, but, with the use of the hyperradius ρ, these matrices appear with a band structure, the dimension of the band being of the order of a few hundred. The eigenenergies E are obtained by an inverse iteration principle that does not imply a complete diagonalization but only the iterative solution of a linear system.

Due to all the precautions taken in the numerical procedure, I estimate that the values resulting from my Faddeev treatment of the ground states are exact to less than 5 MeV.

4.2 Determination of the Three-Body Forces

The three-body forces, parametrized by Eq. (3), are introduced only at the level of the energies and are intended only to get calculated ground-state energies closer to the experimental ones. We have only one parameter, the constant C to do that job. This value must take into account most of the known energies; moreover to have a better determination it is natural to retain in our comparison the systems with very well determined masses, i.e. those stable under strong interactions. The small electromagnetic corrections, if any, are included by taking the centroid of the isospin multiplets. With this in mind, I retained in my list only the stable ground states of the (u, d, s) sector; this means in practice the $N, \Lambda, \Sigma, \Xi, \Omega$ systems. Since I am not interested in a very precise value of the mass, I did not try to make a best fit of the experimental data but I determined the value of C so as to yield good overall values.

In Table 1, I report the values of C for each of the studied potentials as well as the energies E_2 obtained before three-body-force corrections (upper row) and E_3 after corrections (lower row). One can see that our four new potentials give very similar results for E_2 whereas the *BD* potential gives slightly higher values. As a consequence, *BD* needs a larger three-body correction; actually it needs a C value twice as large as the other potentials. Once the three-body forces are included, the mass values are in quite good agreement with the experimental data; in particular, the *AL*1 potential is very good since it reproduces each ground-state energy within less than 9 MeV.

One observes also that the contribution of the three-body-force term is around 100 MeV in the case of the nucleon; it is of the same order or greater than the two-body binding energies. For the Ω system, the three-body forces are of the order of 10 to 20 MeV and represent only a small part of the binding energies. For the systems under consideration in this paper, the three-body-force contribution should be even weaker due to its mass dependence. Nevertheless,

Table 1. Masses (in MeV) of the various stable (under strong interaction) baryons obtained with our five different potentials. The values result from a Faddeev treatment with the numerical procedure described in the text. In the first row are shown the masses due to two-body forces only. My estimate is that they are exact to within 5 MeV. The masses obtained after correction due to three-body forces are displayed in the second row; the value of the constant C (see Eq. (3)) used to calculate them is also specified there

System	N	Λ	Σ	Ξ	Ω
Experiment	939	1115	1192	1318	1672
BD	1022	1178	1261	1369	1685
$C = 444 \cdot 10^7$	906	1113	1196	1332	1664
$AL1$	998	1154	1231	1343	1674
$C = 202 \cdot 10^7$	933	1119	1196	1324	1663
$AL2$	1001	1159	1237	1349	1684
$C = 268 \cdot 10^7$	919	1114	1192	1325	1671
$AP1$	997	1156	1238	1347	1674
$C = 170 \cdot 10^7$	917	1116	1198	1327	1664
$AP2$	993	1156	1241	1351	1684
$C = 211 \cdot 10^7$	897	1109	1194	1328	1673

they have been fully included in order to have a better estimate of the ground-state masses.

4.3 Ground-State Masses

In this section, I study the ground-state masses of *all* the systems containing 1, 2 or 3 heavy quarks of type c or b; this represents the 18 baryons specified in the Introduction. The nomenclature is the usually accepted one and it is recalled in Table 2. The calculation is performed along the Faddeev formalism with the numerical procedure described above and by adding a three-body-force contribution with the values of the constant C as reported in Table 1. I estimate that these results are the exact to less than 5 MeV. The BD potential as well as the new proposed potentials were used and compared. The corresponding values are displayed in Table 2. One can see that all the considered potentials give very similar results, since the maximum deviation between them does not exceed 30 MeV. Of course, we have very few points of comparison with experiment, but it is remarkable that the $AL1$ potential reproduces the experimental data for the $\Lambda_c, \Sigma_c, \Xi_c$, and Λ_b systems with less than 5 MeV deviation. This fact, along with the good results presented in Table 1, gives rise to a strong confidence in these results.

Moreover, my calculations give mass values in very good agreement with those obtained in previous works [7, 9, 10, 12, 13]; however, the number of systems

Table 2. Masses (in MeV) of the ground states of the various heavy baryons studied in this paper, obtained with our five different potentials. The values result from a Faddeev treatment with the numerical procedure described in the text. The masses presented here include three-body-force corrections using the value of the constant C reported in Table 1. The names for the baryons are the conventional ones but the quark content is also shown explicitly to avoid confusion. The baryons with only one heavy quark c or b appear in the first part, whereas those with two and three heavy quarks are shown in the second and third parts of the table

	BD	$AL1$	$AL2$	$AP1$	$AP2$
$\Lambda_c(udc)$	2300	2285	2288	2299	2298
$\Sigma_c(uuc)$	2473	2455	2460	2473	2476
$\Lambda_b(udb)$	5653	5638	5632	5655	5638
$\Sigma_b(uub)$	5858	5845	5839	5864	5854
$\Xi_c(usc)$	2490	2467	2472	2479	2481
$\Omega_c(ssc)$	2700	2675	2682	2675	2678
$\Xi_b(usb)$	5826	5806	5800	5817	5802
$\Omega_b(ssb)$	6046	6034	6028	6032	6019
$\Xi_{cc}(ucc)$	3631	3607	3614	3623	3626
$\Xi_{cb}(ucb)$	6932	6915	6912	6927	6906
$\Xi_{bb}(ubb)$	10197	10194	10175	10204	10176
$\Omega_{cc}(scc)$	3739	3710	3717	3709	3708
$\Omega_{cb}(scb)$	7023	7003	6996	6996	6971
$\Omega_{bb}(sbb)$	10271	10267	10246	10258	10224
$\Omega_{ccc}(ccc)$	4806	4799	4803	4798	4796
$\Omega_{ccb}(ccb)$	8032	8019	8009	8003	7984
$\Omega_{cbb}(cbb)$	11220	11217	11195	11194	11163
$\Omega_{bbb}(bbb)$	14370	14398	14370	14381	14348

studied in this paper is far more important and the numerical calculation is done with a high accuracy. Moreover, the spin-orbit and tensor forces are essentially inoperative in that case and their absence in the used potential should not affect the theoretical ground-state energies. By comparing the results obtained with all the potentials and the success to reproduce all the known baryons, I expect the mass values given by the $AL1$ potential (for example) to reproduce the experimental values with less than 20 MeV error.

4.4 Mass Mean-Square Radii

Following the same philosophy as the one described in the previous subsection, I calculated the mass mean-square radii for all systems and for all potentials. The Faddeev wave functions have been used. The corresponding results (in fm^2) are collected in Table 3. Here again, all the potentials give essentially the same contribution. In fact, this observable is not really very sensitive to the potential or to the three-body numerical treatment; I checked that using a harmonic-oscillator basis including up to 8 quanta with the same potentials gives practically

Table 3. Mass mean-square radii (in fm^2, as obtained from formula (8)) for the baryons studied in Table 2. The conventions are identical to the ones used in Table 2. Wave functions resulting from a Faddeev treatment have been employed

	BD	$AL1$	$AL2$	$AP1$	$AP2$
$\Lambda_c(udc)$	0.097	0.104	0.101	0.109	0.103
$\Sigma_c(uuc)$	0.111	0.121	0.117	0.128	0.121
$\Lambda_b(udb)$	0.043	0.045	0.044	0.046	0.044
$\Sigma_b(uub)$	0.051	0.054	0.053	0.056	0.053
$\Xi_c(usc)$	0.097	0.104	0.100	0.107	0.101
$\Omega_c(ssc)$	0.100	0.108	0.103	0.110	0.102
$\Xi_b(usb)$	0.045	0.048	0.046	0.048	0.046
$\Omega_b(ssb)$	0.050	0.054	0.052	0.055	0.051
$\Xi_{cc}(ucc)$	0.076	0.083	0.079	0.085	0.079
$\Xi_{cb}(ucb)$	0.043	0.046	0.044	0.047	0.044
$\Xi_{bb}(ubb)$	0.031	0.033	0.032	0.033	0.033
$\Omega_{cc}(scc)$	0.073	0.078	0.074	0.077	0.073
$\Omega_{cb}(scb)$	0.043	0.045	0.044	0.045	0.042
$\Omega_{bb}(sbb)$	0.030	0.032	0.032	0.032	0.032
$\Omega_{ccc}(ccc)$	0.062	0.069	0.066	0.069	0.064
$\Omega_{ccb}(ccb)$	0.038	0.040	0.039	0.039	0.038
$\Omega_{cbb}(cbb)$	0.026	0.028	0.028	0.027	0.028
$\Omega_{bbb}(bbb)$	0.019	0.021	0.021	0.021	0.023

the same results. For systems containing only one heavy quark, the results lie between 0.05 and 0.1 fm^2; for those with two heavy quarks the range is rather 0.03 to 0.08 fm^2 while for those with three heavy quarks it is of the order 0.02 to 0.06 fm^2. This means that the spatial extension of such baryons is between 0.1 to 0.3 fm. This must be compared to the value 0.47 fm for the nucleon. Probably, the contribution due to meson exchange would enhance these values substantially [34–36].

4.5 Charge Mean-Square Radii

The charge mean-square radii for all the isospin components of all the systems and for all the potentials have been calculated using the Faddeev wave functions. The results (in $e\,\mathrm{fm}^2$) are reported in Table 4. In this case, the various potentials give again results of the same order but the deviations between them is more pronounced. The potentials with a linear confinement ($BD, AL1, AL2$) always give smaller absolute values than the potentials with power $\frac{2}{3}$ confinement ($AP1, AP2$). However, the new potentials ($AL1, AL2$) always give results closer to ($AP1, AP2$) than to BD. The spread in the different values for the different systems is much more pronounced than in the case of the mass mean-square radii because, in addition to the mass, the charge also plays a very important role. Here again, meson exchanges can contribute significantly to the results but they have

Table 4. Charge mean-square radii (in $e\,\mathrm{fm}^2$, as obtained from formula (9)) for the baryons studied in Table 2. Each isospin component is explicitly displayed and the conventions are the same as in Table 2. Wave functions resulting from a Faddeev treatment have been used

	BD	$AL1$	$AL2$	$AP1$	$AP2$
Λ_c^+	0.117	0.129	0.124	0.147	0.139
Σ_c^0	−0.224	−0.256	−0.245	−0.304	−0.287
Σ_c^+	0.134	0.151	0.145	0.174	0.164
Σ_c^{++}	0.494	0.557	0.535	0.652	0.615
Λ_b^0	0.115	0.128	0.123	0.148	0.140
Σ_b^-	−0.280	−0.318	−0.305	−0.369	−0.347
Σ_b^0	0.138	0.157	0.151	0.183	0.172
Σ_b^+	0.555	0.633	0.607	0.734	0.692
Ξ_c^0	−0.145	−0.161	−0.154	−0.183	−0.171
Ξ_c^+	0.160	0.177	0.170	0.206	0.196
Ω_c^0	−0.111	−0.124	−0.117	−0.131	−0.120
Ξ_b^-	−0.193	−0.212	−0.203	−0.234	−0.219
Ξ_b^0	0.151	0.168	0.161	0.197	0.187
Ω_b^-	−0.164	−0.183	−0.173	−0.191	−0.174
Ξ_{cc}^+	−0.034	−0.038	−0.037	−0.054	−0.052
Ξ_{cc}^{++}	0.285	0.315	0.301	0.350	0.329
Ξ_{cb}^0	−0.064	−0.072	−0.070	−0.089	−0.083
Ξ_{cb}^+	0.275	0.306	0.295	0.341	0.317
Ξ_{bb}^-	−0.128	−0.143	−0.137	−0.161	−0.152
Ξ_{bb}^0	0.215	0.242	0.231	0.277	0.259
Ω_{cc}^+	0.008	0.009	0.009	0.006	0.007
Ω_{cb}^0	−0.023	−0.025	−0.024	−0.028	−0.024
Ω_{bb}^-	−0.083	−0.090	−0.086	−0.092	−0.086
Ω_{ccc}^{++}	0.124	0.138	0.131	0.138	0.128
Ω_{ccb}^+	0.089	0.097	0.092	0.095	0.090
Ω_{cbb}^0	0.032	0.034	0.033	0.033	0.032
Ω_{bbb}^-	−0.019	−0.021	−0.021	−0.021	−0.023

been omitted. Relativistic effects in the case of systems containing one light quark may also have some relevance [37] but in my opinion meson-exchange effects are much more important. I think that only relative quantities among the systems make sense with this calculation.

4.6 Magnetic Moments

The magnetic moments (in nuclear magnetons) for all the systems and all the potentials have been calculated using Faddeev wave functions. They are presented in Table 5. In this case too, the results are very similar whatever potential is employed. This is due to the fact that the components $l_x = 1 = l_y$ in the wave function are not important and have negligible influence on the magnetic moment.

Table 5. Magnetic moments (in nuclear magnetons) for all the baryons listed in Table 4. Wave functions resulting from a Faddeev treatment have been employed

	BD	$AL1$	$AL2$	$AP1$	$AP2$
Λ_c^+	0.335	0.341	0.338	0.345	0.341
Σ_c^0	−1.348	−1.435	−1.414	−1.617	−1.600
Σ_c^+	0.507	0.548	0.539	0.638	0.631
Σ_c^{++}	2.363	2.532	2.492	2.893	2.863
Λ_b^0	−0.059	−0.060	−0.060	−0.060	−0.060
Σ_b^-	−1.218	−1.305	−1.284	−1.486	−1.470
Σ_b^0	0.639	0.682	0.672	0.773	0.765
Σ_b^+	2.496	2.669	2.627	3.032	3.000
Ξ_c^0	0.357	0.360	0.357	0.361	0.359
Ξ_c^+	0.166	0.211	0.205	0.241	0.229
Ω_c^0	−0.806	−0.835	−0.822	−0.867	−0.845
Ξ_b^-	−0.052	−0.055	−0.055	−0.058	−0.057
Ξ_b^0	−0.106	−0.086	−0.087	−0.073	−0.075
Ω_b^-	−0.676	−0.703	−0.691	−0.734	−0.713
Ξ_{cc}^+	0.755	0.784	0.775	0.834	0.825
Ξ_{cc}^{++}	−0.172	−0.206	−0.200	−0.292	−0.290
Ξ_{cb}^0	0.117	0.058	0.063	0.004	0.025
Ξ_{cb}^+	−0.254	−0.198	−0.203	−0.141	−0.167
Ξ_{bb}^-	0.230	0.251	0.246	0.297	0.292
Ξ_{bb}^0	−0.698	−0.742	−0.732	−0.834	−0.825
Ω_{cc}^+	0.620	0.635	0.629	0.646	0.636
Ω_{cb}^0	0.047	0.009	0.014	−0.020	−0.007
Ω_{bb}^-	0.095	0.101	0.098	0.108	0.103
Ω_{ccc}^{++}	1.004	1.023	1.015	1.033	1.021
Ω_{ccb}^+	0.466	0.475	0.471	0.479	0.473
Ω_{cbb}^0	−0.191	−0.193	−0.192	−0.195	−0.193
Ω_{bbb}^-	−0.178	−0.180	−0.180	−0.180	−0.180

Thus, the calculated value is entirely due to the spin contribution in the operator and it is very close to the value expected from a naive quark-model treatment of the baryons. The magnetic moments are usually small; this is a direct consequence of the large quark masses entering the game. In that case also, I expect that contributions coming from meson exchange may not be negligible. Here again relativistic corrections may have some influence but the constituent masses take partially care of them on a phenomenological level; since the naive quark model gives a rather good overall agreement with experimental data, I believe that the values given in this paper may have the right order of magnitude.

4.7 Wave Functions at the Origin

The wave function at the origin $\rho_i(0)$, as defined by Eq. (11), is a quantity that is very sensitive to the numerical treatment of the three-body problem and to the

Table 6. Wave function at the origin $\rho_i(0)$ (in fm^{-3}) as defined by Eq. (11) for baryons with one heavy quark. For each baryon, the value for every possible pair is displayed in separate rows

	BD	$AL1$	$AL2$	$AP1$	$AP2$
$\Lambda_c(udc)$ ud pair	23.1	14.8	15.0	12.1	11.9
uc pair	8.8	7.4	7.7	6.1	6.4
$\Sigma_c(uuc)$ uu pair	6.9	6.1	6.6	5.0	5.6
uc pair	8.2	6.5	6.7	5.3	5.5
$\Lambda_b(udb)$ ud pair	23.3	14.9	15.2	12.2	12.1
ub pair	8.8	7.4	7.7	6.1	6.4
$\Sigma_b(uub)$ uu pair	6.8	6.0	6.4	4.9	5.4
ub pair	7.1	5.7	5.9	4.7	4.9
$\Xi_c(usc)$ us pair	70.5	53.5	54.6	51.0	52.1
uc pair	40.0	35.5	37.0	34.4	37.0
sc pair	30.8	27.1	27.7	25.3	26.4
$\Omega_c(ssc)$ ss pair	16.5	14.6	15.5	13.8	15.3
sc pair	17.8	15.1	15.4	14.4	15.0
$\Xi_b(usb)$ us pair	93.1	70.2	72.0	66.8	68.4
ub pair	51.4	46.1	48.1	44.6	48.0
sb pair	39.9	34.9	35.5	32.4	33.4
$\Omega_b(ssb)$ ss pair	16.7	14.6	15.6	13.7	15.3
sb pair	16.0	13.5	13.8	12.8	13.3

potential in use. This observable is of special importance for leptonic and semi-leptonic transitions from the exotic baryon. I want to stress that, before any relativistic corrections are employed, the absolute value of this quantity depends dramatically on the numerical procedure used to solve the three-body problem. The wave function at the origin is especially sensitive to short-range correlations which are not precisely taken into account by approximate treatments. I have some experience with the HO basis. If one restricts oneself to the 0 quantum expansion, the resulting value is underestimated by one order of magnitude. Even with a basis pushed to 4 quanta the net result is still off by a factor 4 as compared to the exact value [33]. Thus a Faddeev approach is an unavoidable requirement if one desires to get reliable conclusions. This point is one of the major motivations for performing a proper three-body Faddeev calculation. In Table 6, I show the values obtained from the Faddeev treatment for all the systems containing only one heavy quark (in fm^{-3}). Each quark pair is analyzed separately, summing nevertheless over the two possible $\sigma = 0$ and $\sigma = 1$ spin couplings for the pair. One sees that the various potentials can give rise to different values but the discrepancy does not exceed a factor of 2 (this is less than what one would obtain by an

Table 7. Same as Table 6 for the baryons with two heavy quarks

	BD	$AL1$	$AL2$	$AP1$	$AP2$
$\Xi_{cc}(ucc)$ cc pair	75.4	65.9	65.5	62.3	59.7
uc pair	136	119	123	121	126
$\Xi_{cb}(ucb)$ uc pair	268	230	237	224	245
ub pair	225	206	212	208	222
cb pair	143	127	113	121	106
$\Xi_{bb}(ubb)$ bb pair	446	398	290	356	238
ub pair	647	582	592	592	608
$\Omega_{cc}(scc)$ cc pair	79.9	70.9	68.4	68.9	65.8
sc pair	115	102	103	105	106
$\Omega_{cb}(scb)$ sc pair	229	201	202	208	209
sb pair	193	176	177	179	182
cb pair	151	135	121	131	115
$\Omega_{bb}(sbb)$ bb pair	467	418	306	379	254
sb pair	579	524	517	536	518

Table 8. Same as Table 6 for the baryons with three heavy quarks

	BD	$AL1$	$AL2$	$AP1$	$AP2$
$\Omega_{ccc}(ccc)$ cc pair	88.3	75.5	73.5	73.0	68.8
$\Omega_{ccb}(ccb)$ cc pair	101	89.2	85.6	83.3	81.7
cb pair	98.1	86.6	77.6	86.3	73.1
$\Omega_{cbb}(cbb)$ bb pair	537	485	353	445	293
cb pair	565	513	442	511	406
$\Omega_{bbb}(bbb)$ bb pair	686	612	423	538	330

approximate treatment!). Indeed, among the five tested potentials, BD gives results that deviate much more from those given by the others. On the other hand, all potentials agree on the relative contribution for each of the pairs (ratio of the value for a (ij) pair to the value for the (jk) pair) to within a few per cent. When a baryon contains a quark with a large mass, the wave function at the origin tends to be larger as required by the Heisenberg principle, what tells us that this object should be more compact. Moreover, when a quark pair is coupled to spin 0 (as the ud pair in Λ_c), it contributes much more than in the case of a spin-1 coupling (as the uu pair in Σ_c); this is the consequence of the hyperfine force, which is attractive for a singlet pair and repulsive for a triplet one. Similar conclusions can be drawn concerning the baryons with two heavy quarks in Table 7 and the baryons with three heavy quarks in Table 8.

To grasp the importance of performing an exact Faddeev treatment in such a case, I have calculated the Ω_{bb} ground state with a HO-basis treatment up to 8 quanta and with the $AL1$ potential. Instead of the values 418 and 524, one finds in that case 269 and 416; the difference is appreciable. Thus a very good three-body treatment is crucial for getting precise values of the wave function at the origin.

4.8 Complete Spectrum

For the reasons explained in the Introduction, I find it very interesting to calculate the complete spectrum for all baryons under consideration. However, this represents so many states that it is hardly conceivable to use the Faddeev treatment to compute all of them because it costs a lot of computer time and also because the technical method (inverse iterative process) requires as many runs as the number of studied states. Since the energies are rather insensitive to the numerical procedure, I decided to use a harmonic-oscillator basis up to 8 quanta to evaluate the spectrum; I checked on some samples that the error introduced is indeed small. As already stated, the potentials are such that L and S are good quantum numbers and consequently all the states with $|L-S| \leq J \leq L+S$ are degenerate. In practice, I restricted myself to all states with $L = 0$, 1, 2, 3, 4 and $S = \frac{1}{2}, \frac{3}{2}$ with positive and negative parities. I made the calculations with all the five different potentials. It is remarkable that they all give very similar results as they differ for each state by less than 50 MeV, the BD potential showing the most pronounced difference.

The amount of resulting data is far too abundant to be reported here. I decided to present the values obtained with the $AL1$ potential only and to restrict myself to states with angular momenta lower than $\frac{9}{2}$ and with maximum excitation energy. For each baryon, I also displayed the energy of the various thresholds corresponding to an ordinary quark-pair creation: All levels lying below the lowest one are presumably stable under strong interactions. The thresholds are calculated with the same potential and with the same numerical procedure (HO up to 8 quanta) as the three-body systems. However, since the potential does not include annihilation effects, the π and η particles are degenerate; this situation is strongly broken experimentally (with all potentials the mass of the π is correctly reproduced and consequently the mass of the η is too low by 410 MeV). Moreover, for $L \neq 0$ states, the neglected contributions of the spin-orbit or tensor forces, although expected small, can reach a value (50 MeV) comparable to the uncertainty introduced by the potentials. For a precise determination of the resonance masses they should be included. However, my purpose is more modest; I just want to examine the number and the spin characteristics of the presumably stable candidates. Given the fact that in the very heavy baryons (it is only in that case that a $L \neq 0$ level can be stable) the corresponding thresholds are lying at several hundreds MeV excitation energy, introduction of spin-orbit and tensor forces could not change the conclusions (except for the states very close to the threshold).

In Fig. 1, the baryon levels of type (udQ) (Q is a heavy quark of type c or b) with both isospins $I = 0$ and $I = 1$ are shown. The maximum excitation energy considered in this case is 1 GeV. The spectrum of the Λ_Q baryons is not very dense; this is due

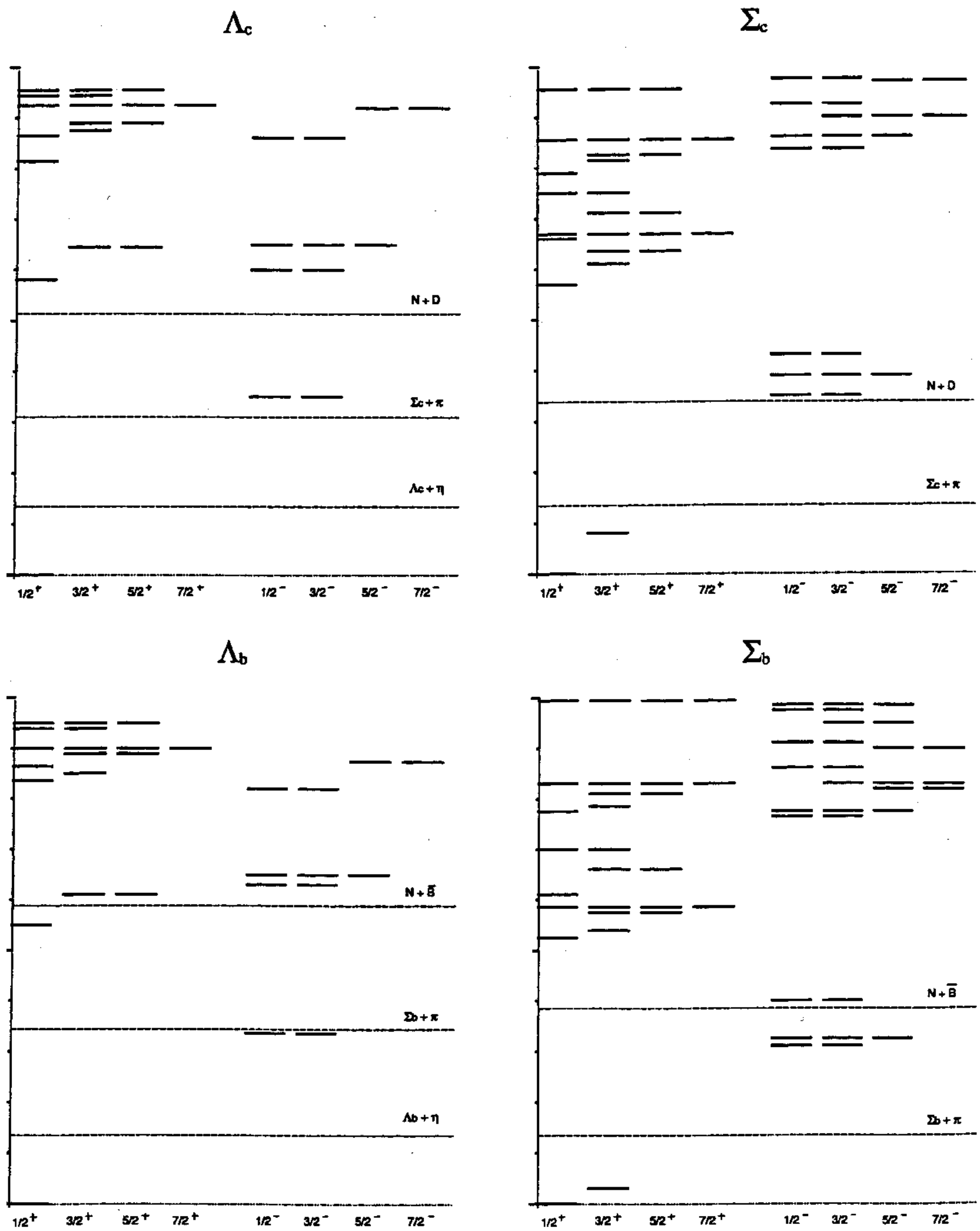


Fig. 1. Spectrum of the baryons of type (udQ), where $Q = c$ or b is a heavy quark, for both values of the isospin $I = 0$ and 1. To obtain the theoretical values symbolized by a solid horizontal bar, a diagonalization of the Hamiltonian with the $AL1$ potential has been performed in a harmonic-oscillator basis up to 8 quanta. Only states with $J \leq \frac{7}{2}$ have been reported; on the left-hand side of the figure the positive-parity states have been collected whereas the negative-parity states appear on the right-hand side. All states below 1 GeV excitation energy are displayed. Each tick on the vertical axis corresponds to 100 MeV. The thresholds coming from a quark-pair creation are indicated by a dashed line. In the special case of Σ_Q systems, a $\Lambda_Q + \pi$ threshold exists below the ground state itself (not shown in the figure)

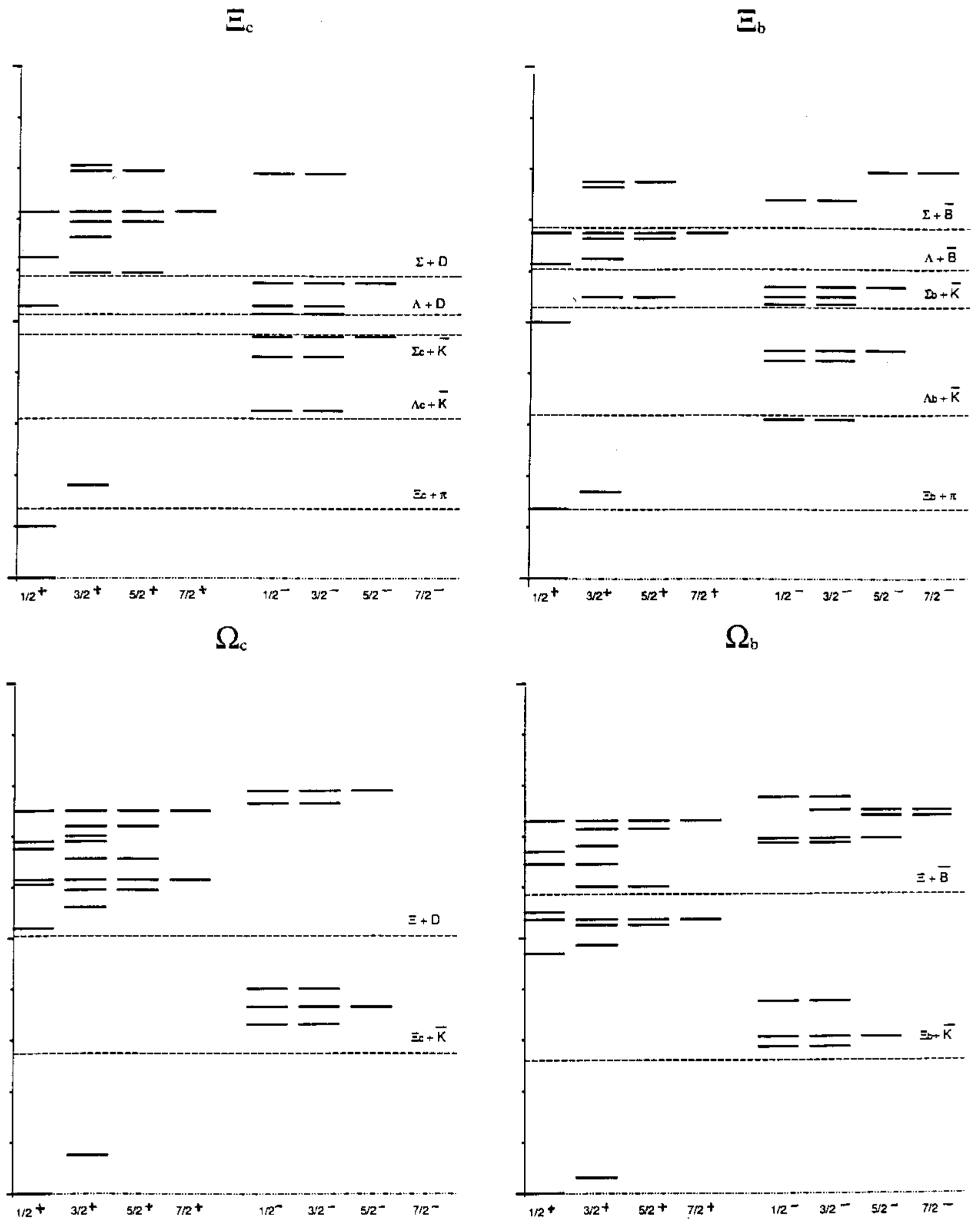
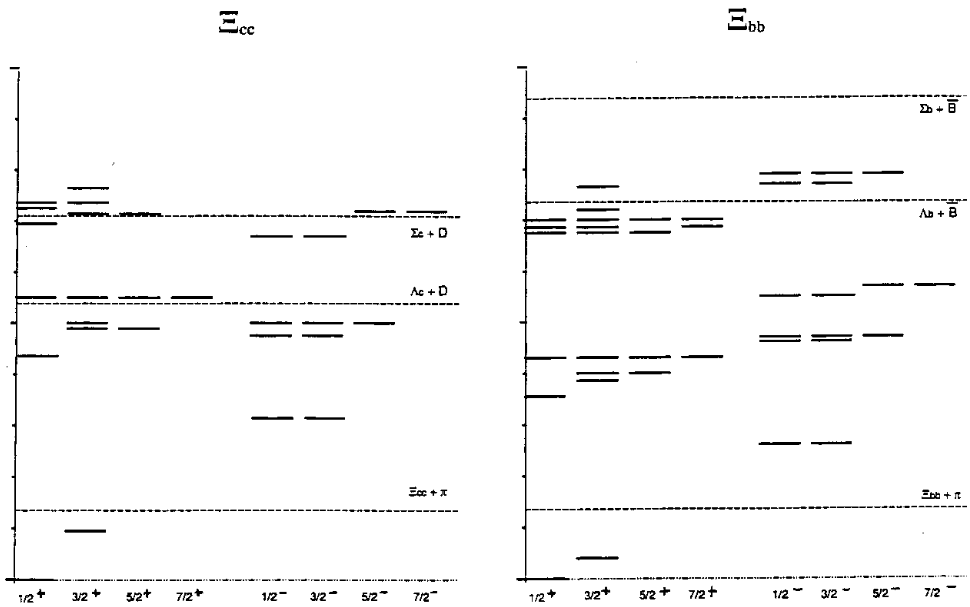


Fig. 2. Same as in Fig. 1, for the baryons of type (usQ) (in the upper part) and (ssQ) (in the lower part), with Q standing for a heavy quark c or b. Only the states with an excitation energy less than 800 MeV are reported

to the antisymmetry of the space wave function. Only the ground state is stable. On the contrary, the spectrum of the Σ_Q systems is richer but paradoxically all the resonances (including the ground state) can decay by strong interaction because the threshold $\Lambda_Q + \pi$ is lower than the ground state (threshold not shown in the figure). This situation is completely different from the usual Σ and Λ baryons.

In Fig. 2, the baryons $\Xi_Q(usQ)$ and $\Omega_Q(ssQ)$ are displayed. In this case, the maximum excitation energy is 800 MeV. For the Ξ_Q systems, all quarks are



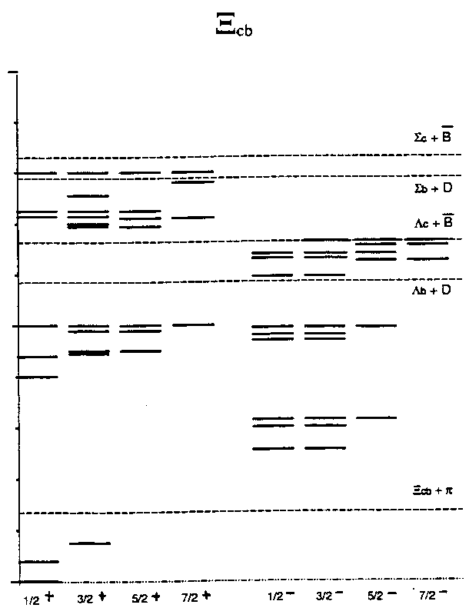


Fig. 3. Same as in Fig. 2 for the baryons of type (uQ_1Q_2), with Q_i standing for a heavy quark c or b

different and only the ground state and its first excitation with the same quantum numbers are stable. Indeed, this resonance is not really a radial excitation of the ground state; these two levels have been denoted in the past as the Ξ_Q and Ξ'_Q particles: The first one corresponds to a pair (us) in a singlet spin state, while the second one corresponds to the same pair in a triplet spin state. The interquark potential depending on spin degrees of freedom, these configurations are not pure but mix together and give rise to the two stable states reported in the figure. The Ω_Q

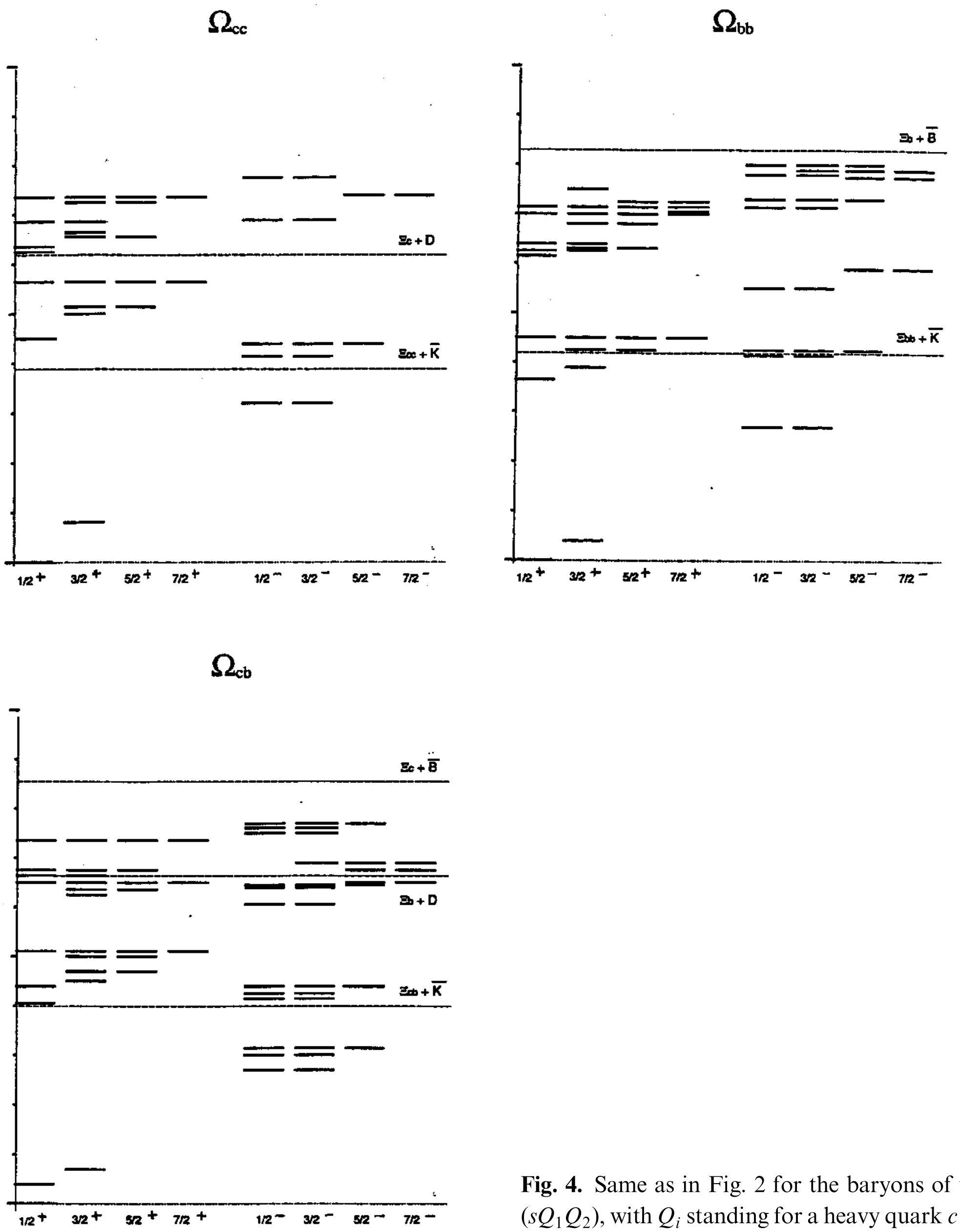


Fig. 4. Same as in Fig. 2 for the baryons of type (sQ_1Q_2), with Q_i standing for a heavy quark c or b

systems do not share this property because the Pauli principle forces the (ss) pair to be in a triplet state. But now, with a very favourable symmetric space function there can exist two states with low energy: The ground state with $S = \frac{1}{2}$ (similar to the Σ or the nucleon) and its spin excitation $S = \frac{3}{2}$ (as the Σ^* or the Δ). However, due to the weakness of the hyperfine term in heavy baryons (the spin resonance in the Ω_b particle is only at 30 MeV excitation energy to be compared to the 300 MeV in the case of the nucleon), these two levels are stable.

In Fig. 3, the resonances for the baryons of type $\Xi_{Q_1Q_2}$ up to 800 MeV excitation energy are shown. For the Ξ_{QQ} systems the situation is rather similar to the Ω_Q, with the

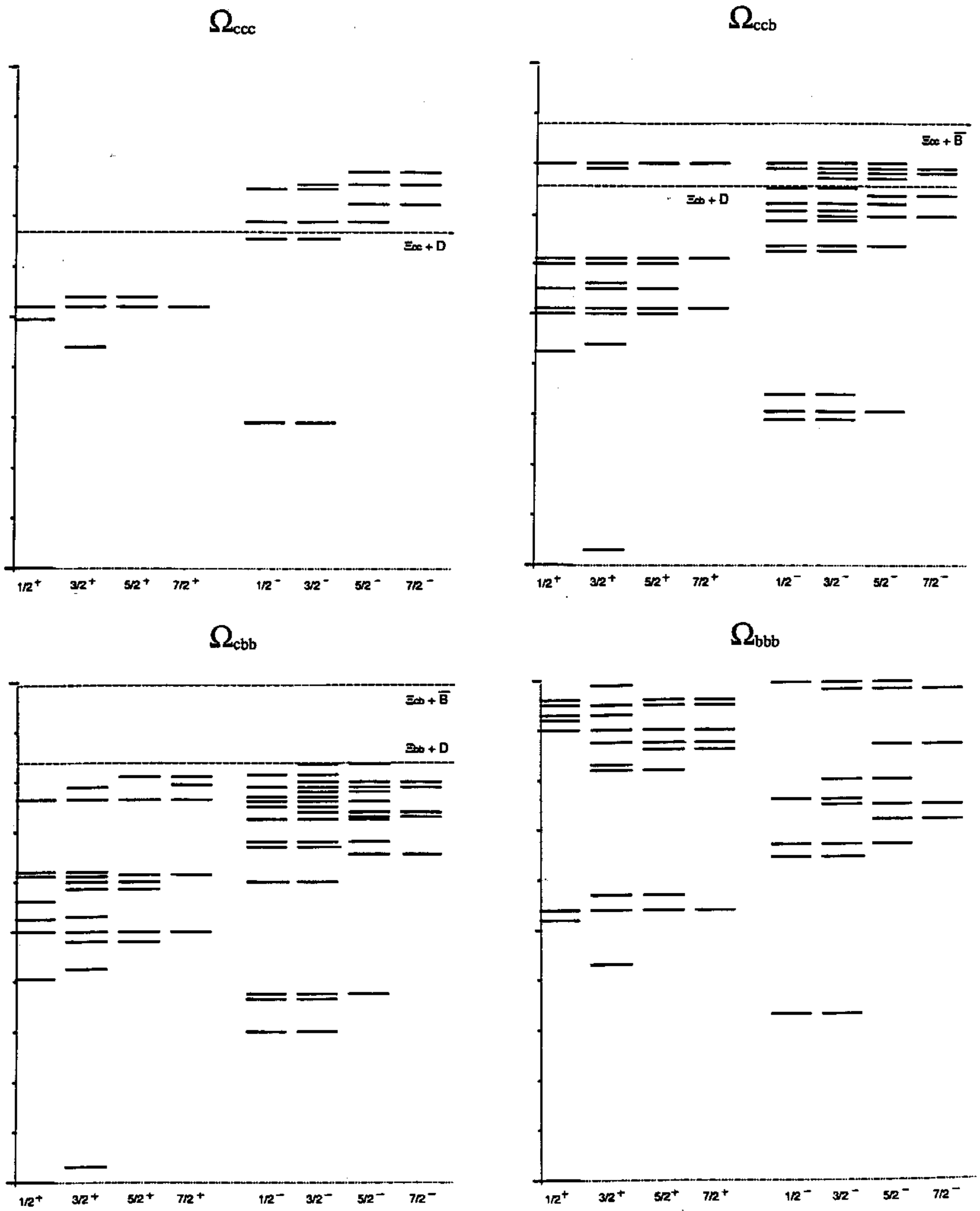


Fig. 5. Same as in Fig. 2 for the baryons of type $(Q_1Q_2Q_3)$, with Q_i standing for a heavy quark c or b. For the particular case of Ω_{bbb}, the threshold $\Xi_{bb}+\bar{B}$ lies slightly above 1 GeV and is not shown in the figure; in this case the maximum excitation energy displayed is 1 GeV

(QQ) pair replacing the (ss) pair; as a consequence only the ground state and its spin excitation are stable. The Ξ_{cb} particle contains three different quarks and is more similar to the Ξ_Q systems but with the fundamental difference that the hyperfine interaction is now so weak that it can bind not only the "Ξ'_{cb}" but also the spin resonance $S=\frac{3}{2}$.

Fig. 4 contains the results corresponding to the $\Omega_{Q_1Q_2}$ systems. The situation is similar to the $\Xi_{Q_1Q_2}$ but is even more favourable due to two factors: (i) because of the heavier s quark, this new object is more compact and it takes more benefit of the short-range attractive Coulomb term giving rise to more binding and (ii) the π cannot be present in the thresholds which are now much higher in energy (400 MeV instead of 140 MeV). In that case, not only the previous particular excitations are strongly bound, but even the lowest negative-parity resonances with $L = 1$ become stable and, for the Ω_{bb}, the first radial $L = 0$ resonances become stable as well.

Fig. 5 shows the baryon levels containing three heavy quarks up to 800 MeV excitation energy. The situation explained in the preceding case is still valid but it is amplified by the large mass of the participating quarks. The thresholds are now at around 700 MeV excitation energy, more than 1 GeV in the case of Ω_{bbb} (this is why I have calculated the resonances up to 1 GeV for this particular baryon). As a consequence, plenty of resonances with both spin, orbital and radial excitations should be stable in those systems, opening the door of a very rich spectroscopy.

5 Conclusions

In this paper, I have studied *all* the baryons containing at least one heavy quark of type c or b. The ground-state energies have been computed along the Faddeev formalism using five different potentials. A phenomenological contribution due to three-body forces has been added as well. The solution of the three-body problem was carefully investigated in order to eliminate a possible source of uncertainty. The aim was to compare the sensitivity of the results on the various potentials. The conclusion is that realistic potentials that are successful in describing the meson sector give also very similar masses in the baryon sector despite the fact that they have different structures. This is very encouraging and one may hope that they in general provide results very close to experimental data; indeed this is already the case for all known systems.

In addition to the mass, I have also calculated a number of static properties. Among them, the mass mean-square radii and the magnetic moments are rather insensitive to the three-body treatment and to the potentials. In contrast, the charge mean-square radii are more dependent on the model but all the potentials give a satisfactory agreement. The most dramatic observable is the wave function at the origin. It is very dependent on the numerical procedure: A very precise solution of the three-body problem is clearly a necessity for getting the right answer. Moreover, the various potentials can give results differing by a factor 2 in the case of very heavy baryons (the agreement being much better for light baryons). Thus, this observable seems to be a very good test for discriminating between various potentials.

I have also calculated all the excited resonances up to 1 GeV (or 800 MeV in some cases) for all the baryons considered here. The remarkable fact is that all the potentials give very similar results. The interesting feature concerning these heavy baryons is that a considerable number of excited states is bound under strong interactions. Unfortunately, this phenomenon is most pronounced for the baryons containing three heavy quarks, which are beyond experimental feasibility nowadays. Nevertheless, this evidence encourages further intense experimental activities in this domain. I hope that my work will be an incitement in that direction and will serve as a guide to future experiments.

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