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A simple expression for the general oscillator bracket

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Abstract

It is shown that there exists a simple generating function for the three-dimensional oscillator wave functions which are simultaneous eigenstates of angular momentum. This function is used to derive a new formula for the oscillator transformation brackets in the most general case of two particles with different masses moving in oscillator potentials with different frequencies.

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1. Introduction

A well-known problem in nuclear structure calculations is the evaluation of matrix elements of a two-body interaction which depends on the relative coordinates of the two particles. The basis states are usually taken to be products of single-particle wave functions appropriate to some potential well. The calculations can be greatly simplified if the single-particle functions are expanded in harmonic oscillator eigenfunctions $\phi_{n\ell m}(\boldsymbol{r})$ so that the basis functions are linear combinations of states with the form

$$|n_1\ell_1(\boldsymbol{r}_1), n_2\ell_2(\boldsymbol{r}_2) : \Lambda\lambda\rangle = \sum_{m_1 m_2} \langle \ell_1 m_1 \ell_2 m_2 | \Lambda\lambda\rangle \phi_{n_1\ell_1 m_1}(\boldsymbol{r}_1)\phi_{n_2\ell_2 m_2}(\boldsymbol{r}_2), \tag{1}$$

where the quantities $\langle \ell_1 m_1 \ell_2 m_2 | \Lambda\lambda\rangle$ are vector coupling coefficients. In many structure calculations the oscillator eigenstates are used directly as the basis functions. One is left with the problem of evaluating matrix elements of the type

$$\langle n_3\ell_3(\boldsymbol{r}_1), n_4\ell_4(\boldsymbol{r}_2) : \Lambda\lambda | V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) | n_1\ell_1(\boldsymbol{r}_1), n_2\ell_2(\boldsymbol{r}_2) : \Lambda\lambda\rangle. \tag{2}$$

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The general technique which has been developed for handling the above integral involving a central force can also be used, without essential complications, for interactions which are noncentral or non-local, or which depend on spin and isospin. The only requirement is that the basis states be harmonic oscillator eigenfunctions. If both particles have the same mass and move in the same oscillator potential the most general method for evaluating expression (2) is the Talmi transformation [1]. This represents the two-particle oscillator wave function (1) as a finite linear superposition of similar products of two oscillator functions, one with argument proportional to the true relative coordinate, the other describing the motion of the centre of mass, i.e.

$$|n_1\ell_1(\boldsymbol{r}_1), n_2\ell_2(\boldsymbol{r}_2) : \Lambda\lambda\rangle = \sum_{NLn\ell} \langle NL, n\ell : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle |NL(\boldsymbol{R}), n\ell(\boldsymbol{r}) : \Lambda\lambda\rangle\,, \tag{3}$$

where the usual definitions of the new coordinates are

$$\boldsymbol{R} = \frac{(\boldsymbol{r}_1 + \boldsymbol{r}_2)}{\sqrt{2}}\,, \qquad \boldsymbol{r} = \frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2)}{\sqrt{2}}. \tag{4}$$

When this transformation is performed on both sides of the matrix element (2) it is clear that the centre of mass functions integrate out and the calculation involves only a finite number of radial integrals of the form

$$\langle n'\ell(r)|V(\sqrt{2}r)|n\ell(r)\rangle. \tag{5}$$

The coefficients in Eq. (3) are known as the Moshinsky brackets and they are independent of the total angular momentum z-component λ, the particle mass μ and the frequency ω of the oscillator. The sums over N, L, n and ℓ are finite because of the condition

$$2N + L + 2n + \ell = 2n_1 + \ell_1 + 2n_2 + \ell_2. \tag{6}$$

The oscillator brackets have been considered by a number of authors. Moshinsky [2] has given an explicit formula for the case $n_1 = n_2 = 0$ and a recursion relation which enables the calculation of the other brackets. Subsequently, a tabulation was published by Brody and Moshinsky [3]. Since this table was not sufficient for Hartree–Fock calculations the problem was reconsidered by Baranger and Davies [4]. They derived another type of recursion relation, similar to one established earlier by Arima and Terasawa [5], which was especially suitable for calculating the coefficients with small values of the relative angular momentum quantum number ℓ. They also gave a formula for the case $\ell = 0$ and several explicit expressions for the general oscillator bracket.

The transformation coefficients were generalised by Smirnov [6,7] to the case of two particles with different masses, each moving in an oscillator potential with frequency ω, and he obtained a closed form expression for the general bracket which now depends on the ratio of the masses. His results were considerably simplified by Bakri [8,9] who noticed that, apart from a normalization factor, exactly the same brackets are involved in the transformation of the functions $r_1^{2n_1+\ell_1} Y_{\ell_1}^{m_1}(\hat{r}_1) r_2^{2n_2+\ell_2} Y_{\ell_2}^{m_2}(\hat{r}_2)$ to relative and centre of mass coordinates. Finally, it has been shown by Gal [10] that the oscillator brackets can easily be generalised further to the case of two particles with different masses moving in oscillator potentials with different frequencies. Gal's approach differs from the others in that he makes extensive use of group theory to obtain a complete formula.

However, all the cited results are excessively complicated and lead to expressions whose structure is far from transparent. It is the purpose of this paper to present a new approach to the problem based on a generating function for the three-dimensional harmonic oscillator wave functions. This allows the derivation of an expression for the most general oscillator bracket involving different masses and oscillator frequencies. The final explicit formula has a simple and symmetrical structure which is well adapted for automatic computation.

The main body of the following derivation was presented in 1968 [11]. However, it was not published at that time because none of the component ideas were, in themselves, original. The generating function had been introduced by Kumar [12]. Numerical values and computational schemes were already available [3,4,6–9] and even a generalization to arbitrary mass and oscillator frequency was published shortly thereafter [10] (although this latter result appeared in very unwieldy form since Gal [10] was unable to evaluate analytically the sums over magnetic quantum numbers). Nevertheless, the combination of all these features (generating function, arbitrary mass and frequency, numerical evaluation) has still only been carried out by Buck [11]. Because it leads to a highly elegant and readily comprehensible derivation of a transparent formula for the general Brody–Moshinsky bracket we feel that its inherent simplicity makes it worth presenting to a wider audience. In addition, we have extended the original work to include an explicit demonstration that the final formula is identical to that of Bakri [8,9] in the case of equal oscillator frequencies, and a Fortran code has been written to calculate numerical values for the brackets. A small sample of values is compared with values from the tabulation of Brody and Moshinsky [3] in Table 1.

For the more general case of unequal masses and/or unequal frequencies, the Brody–Moshinsky bracket, $\langle NLn\ell : \Lambda | n_1\ell_1 n_2\ell_2 : \Lambda \rangle$, reduces to an algebraic expression involving the oscillator length parameters $b_i^2 = \hbar/(\mu_i\omega_i)$ of the two particles $i = 1, 2$. A few of the simplest results for this situation are listed in Table 2.

In the intervening years several alternative formulae and computation schemes for the transformation brackets have been presented. Talman [13] has used the generating function method to obtain a formula for arbitrary masses, but equal frequencies, while Talman and Lande [14] have proposed a direct diagonalization scheme to obtain numerical values. Trlifaj [15] has also studied the arbitrary mass and equal frequency case, and obtained a closed form expression involving a sum over five variables. (Both Talman [13] and Trlifaj [15] refer to the original report of Buck [11]). A computer code for equal mass and equal frequency brackets has been published by Feng and Tamura [16]. Finally, we should mention that more recent work has centred on the extension to three or more particles with arbitrary masses [17,18], and a computer code for the evaluation of such brackets has been published by Gan et al. [19].

Table 1
Results obtained by programming Eq. (56) for $\langle NL, n\ell : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle$ with $\mu_1 = \mu_2$ and $\omega_1 = \omega_2$, compared with Brody and Moshinsky [3] for a few typical cases

n_1	l_1	n_2	l_2	N	L	n	l	Λ	Eq. (56)	Ref. [3]
0	0	0	0	0	0	0	0	0	0.1000000000D+01	1.00000000
0	1	0	3	0	2	1	0	2	−0.4183300003D+00	−0.41833000
0	1	0	5	0	1	0	5	6	0.5000000090D−01	0.50000001
0	2	0	2	0	1	0	3	4	−0.5551115123D−15	−0.00000001
0	2	0	4	1	3	0	1	3	−0.2988071431D+00 [a]	0.29880716
0	2	0	5	0	5	0	2	4	−0.1178511400D+00 [a]	0.11785111
2	2	1	3	0	3	1	6	4	0.1939477687D+00 [a]	−0.19394780
2	2	1	3	1	0	2	5	5	−0.7097762015D−01	−0.07097762
2	2	1	4	0	2	4	2	2	−0.1332892965D+00	−0.13328930
2	2	1	4	3	2	0	4	4	0.9959470961D−01	0.09959471

[a] Our phase convention differs from that of Moshinsky [2] by the factor $(-1)^{L+\ell+\Lambda}$.

2. The oscillator equations

The wave equation for two independent particles with masses μ_1 and μ_2 moving in oscillator potentials with frequencies ω_1 and ω_2, respectively, is

$$\left[-\frac{\hbar^2}{2\mu_1}\nabla_1^2 + \tfrac{1}{2}\mu_1\omega_1^2 r_1^2 - \frac{\hbar^2}{2\mu_2}\nabla_2^2 + \tfrac{1}{2}\mu_2\omega_2^2 r_2^2\right]\psi = E\psi. \tag{7}$$

Defining the dimensionless coordinates

$$\boldsymbol{\rho}_1 = \left(\frac{\mu_1\omega_1}{\hbar}\right)^{1/2}\boldsymbol{r}_1, \qquad \boldsymbol{\rho}_2 = \left(\frac{\mu_2\omega_2}{\hbar}\right)^{1/2}\boldsymbol{r}_2 \tag{8}$$

Table 2
Some of the simplest Brody-Moshinsky brackets, $\langle NLn\ell : \Lambda | n_1\ell_1 n_2\ell_2 : \Lambda\rangle$, for the general case of unequal masses and/or unequal frequencies in terms of the oscillator length parameters $b_i^2 = \hbar/(\mu_i\omega_i)$ for particles $i = 1,2$

n_1	l_1	n_2	l_2	N	L	n	l	Λ	Eq. (56)
0	0	0	0	0	0	0	0	0	1
0	0	0	1	0	1	0	0	1	$b_1/\sqrt{b_1^2+b_2^2}$
0	0	0	1	0	0	0	1	1	$-b_2/\sqrt{b_1^2+b_2^2}$
0	0	0	2	0	2	0	0	2	$b_1^2/(b_1^2+b_2^2)$
0	0	0	2	0	1	0	1	2	$-\sqrt{2}b_1b_2/(b_1^2+b_2^2)$
0	0	0	2	0	0	0	2	2	$b_2^2/(b_1^2+b_2^2)$
0	1	0	1	1	0	0	0	0	$\sqrt{2}b_1b_2/(b_1^2+b_2^2)$
0	1	0	1	0	1	0	1	0	$(b_1^2-b_2^2)/(b_1^2+b_2^2)$
0	1	0	1	0	0	1	0	0	$-\sqrt{2}b_1b_2/(b_1^2+b_2^2)$
0	1	0	1	0	1	0	1	1	-1 [a]
0	1	0	1	0	2	0	0	2	$\sqrt{2}b_1b_2/(b_1^2+b_2^2)$
0	1	0	1	0	1	0	1	2	$(b_1^2-b_2^2)/(b_1^2+b_2^2)$
0	1	0	1	0	0	0	2	2	$-\sqrt{2}b_1b_2/(b_1^2+b_2^2)$

[a] Our phase convention differs from that of Moshinsky [2] by the factor $(-1)^{L+\ell+\Lambda}$.

and writing

$$E = \epsilon_1 \hbar\omega_1 + \epsilon_2 \hbar\omega_2 \tag{9}$$

Eq. (7) takes the form

$$\left[\hbar\omega_1 \left\{\tfrac{1}{2}\nabla^2_{\rho_1} - \tfrac{1}{2}\rho_1^2 + \epsilon_1\right\} + \hbar\omega_2 \left\{\tfrac{1}{2}\nabla^2_{\rho_2} - \tfrac{1}{2}\rho_2^2 + \epsilon_2\right\}\right] \psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = 0. \tag{10}$$

The product eigenfunctions of Eq. (10), with uncoupled angular momenta, can be written as

$$\psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \phi_{n_1\ell_1m_1}(\boldsymbol{\rho}_1)\phi_{n_2\ell_2m_2}(\boldsymbol{\rho}_2) \tag{11}$$

corresponding to

$$\epsilon_1 = 2n_1 + \ell_1 + \tfrac{3}{2}, \quad \epsilon_2 = 2n_2 + \ell_2 + \tfrac{3}{2}. \tag{12}$$

It is required to represent the function (11) as a linear combination of similar products of two oscillator functions with arguments $\boldsymbol{\rho}$ and $\boldsymbol{\Re}$, where

$$\boldsymbol{\rho} \propto \boldsymbol{r}_1 - \boldsymbol{r}_2 = \boldsymbol{r} \tag{13}$$

and $\boldsymbol{\Re}$ is an independent linear function of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. For this purpose it would be convenient to be able to define the new coordinates so that the operator in Eq. (10) is transformed into a similar uncoupled sum of two oscillator Hamiltonians, one for each of the new variables. However, it is easy to convince oneself that when $\omega_1 \neq \omega_2$ such a transformation is not possible. This difficulty is overcome by noting that the eigenfunctions (11) also satisfy the modified equation

$$\left[\left\{\tfrac{1}{2}\nabla^2_{\rho_1} - \tfrac{1}{2}\rho_1^2 + \epsilon_1\right\} + \left\{\tfrac{1}{2}\nabla^2_{\rho_2} - \tfrac{1}{2}\rho_2^2 + \epsilon_2\right\}\right] \psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = 0. \tag{14}$$

for which the desired type of transformation *is* possible.

The new coordinates are most conveniently introduced by means of the orthogonal transformation

$$\begin{aligned}\boldsymbol{\rho} &= \boldsymbol{\rho}_1 \cos\beta - \boldsymbol{\rho}_2 \sin\beta\,,\\ \boldsymbol{\Re} &= \boldsymbol{\rho}_1 \sin\beta + \boldsymbol{\rho}_2 \cos\beta\,,\end{aligned} \tag{15}$$

which has a Jacobian equal to unity and which transforms Eq. (14) to

$$\left[\left\{\tfrac{1}{2}\nabla^2_{\rho} - \tfrac{1}{2}\rho^2 + \epsilon\right\} + \left\{\tfrac{1}{2}\nabla^2_{\Re} - \tfrac{1}{2}\Re^2 + \mathcal{E}\right\}\right] \psi(\boldsymbol{\rho}, \boldsymbol{\Re}) = 0\,, \tag{16}$$

where

$$\epsilon + \mathcal{E} = \epsilon_1 + \epsilon_2. \tag{17}$$

The angle β in Eqs. (15) is fixed by the condition (13), i.e.

$$\boldsymbol{\rho} = \left(\frac{\mu_1\omega_1}{\hbar}\right)^{1/2} \boldsymbol{r}_1 \cos\beta - \left(\frac{\mu_2\omega_2}{\hbar}\right)^{1/2} \boldsymbol{r}_2 \sin\beta$$
$$= \left(\frac{\mu_1\omega_1\mu_2\omega_2}{\hbar(\mu_1\omega_1+\mu_2\omega_2)}\right)^{1/2} (\boldsymbol{r}_1 - \boldsymbol{r}_2) , \tag{18}$$

which implies

$$\tan\beta = \left(\frac{\mu_1\omega_1}{\mu_2\omega_2}\right)^{1/2} \quad \text{or} \quad \cos 2\beta = \frac{\mu_2\omega_2 - \mu_1\omega_1}{\mu_2\omega_2 + \mu_1\omega_1}, \qquad 0 \leqslant \beta \leqslant \frac{\pi}{2}. \tag{19}$$

Thus $\mathfrak{R}$ is proportional to the centre of mass coordinate of the two particles only when $\omega_1 = \omega_2$; *but this does not matter*, since the functions of $\mathfrak{R}$ always integrate out in the two-body matrix element.

Clearly, the eigenfunctions of Eq. (16) may be written in the uncoupled form

$$\psi(\boldsymbol{\rho}, \mathfrak{R}) = \phi_{n\ell m}(\boldsymbol{\rho})\phi_{NLM}(\mathfrak{R}) , \tag{20}$$

corresponding to

$$\epsilon = 2n + \ell + \tfrac{3}{2}, \qquad \mathcal{E} = 2N + L + \tfrac{3}{2} , \tag{21}$$

where the new ϕ's are again oscillator functions. Hence the product function (11) can be represented as a finite linear combination of all the product functions (20) which satisfy the condition

$$2N + L + 2n + \ell = \chi = 2n_1 + \ell_1 + 2n_2 + \ell_2. \tag{22}$$

The expansion coefficients are just the overlap integrals

$$\langle NLM, n\ell m | n_1\ell_1 m_1, n_2\ell_2 m_2\rangle = \int d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \phi^*_{NLM}(\mathfrak{R})\phi^*_{n\ell m}(\boldsymbol{\rho})\phi_{n_1\ell_1 m_1}(\boldsymbol{\rho}_1)\phi_{n_2\ell_2 m_2}(\boldsymbol{\rho}_2). \tag{23}$$

The Moshinsky brackets are obtained from the brackets (23) by coupling the angular momenta on both sides to good total angular momentum, i.e.

$$\langle NL, n\ell : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle = \sum_{\lambda=\text{constant}} \langle \ell_1 m_1 \ell_2 m_2 | \Lambda\lambda\rangle \langle LM\ell m | \Lambda\lambda\rangle \times \langle NLM, n\ell m | n_1\ell_1 m_1, n_2\ell_2 m_2\rangle. \tag{24}$$

3. A generating function

The basic problem is to find an algebraic expression for the overlap integral (23). The new approach developed here is to use the method of generating functions. It will now be shown that there exists a simple generating function for the complete set of three-dimensional oscillator eigenstates that are simultaneous eigenfunctions of the angular momentum. This function can be used to advantage in many problems which involve integrals containing such eigenstates.

The wave equation for a particle of mass μ in an oscillator potential of frequency ω may be written in the dimensionless form

$$\left[\tfrac{1}{2}\nabla_\rho^2 - \tfrac{1}{2}\rho^2 + \epsilon\right]\phi(\boldsymbol{\rho}) = 0\,, \tag{25}$$

where $\boldsymbol{\rho} = \sqrt{(\mu\omega/\hbar)}\boldsymbol{r}$ is a dimensionless position vector and $\epsilon = E/\hbar\omega$.

The properly orthonormalized eigenfunctions of Eq. (25) with good angular momenta and with eigenvalues

$$\epsilon = 2n + \ell + \tfrac{3}{2} \tag{26}$$

are given by the expression

$$\phi_{n\ell m}(\boldsymbol{\rho}) = \left[\frac{2(n!)}{\Gamma(n+\ell+\frac{3}{2})}\right]^{1/2} \exp(-\rho^2/2)\rho^\ell L_n^{(\ell+1/2)}(\rho^2)Y_\ell^m(\hat{\boldsymbol{\rho}})\,, \tag{27}$$

where $n, \ell = 0, 1, 2, 3, \ldots$

In this formula $Y_\ell^m(\hat{\boldsymbol{\rho}})$ is a spherical harmonic and the functions of the form $L_p^{(\alpha)}(x)$ are the associated Laguerre polynomials as defined by Magnus and Oberhettinger [20], i.e.

$$L_p^{(\alpha)}(x) = \frac{\exp(x)x^{-\alpha}}{p!}\frac{d^p}{dx^p}\left[\exp(-x)x^{p+\alpha}\right]. \tag{28}$$

The above authors also give a formula which may be regarded as defining a generating function for the associated Laguerre polynomials, i.e.

$$\exp(z)(xz)^{-\alpha/2}J_\alpha(2\sqrt{xz}) = \sum_{p=0}^{\infty}\frac{L_p^{(\alpha)}(x)}{\Gamma(p+\alpha+1)}z^p, \quad \alpha > -1\,, \tag{29}$$

where $J_\alpha(y)$ is a Bessel function. Writing

$$\begin{aligned} x &= \rho^2\,, & z &= s^2\,, \\ p &= n\,, & \alpha &= \ell + \tfrac{1}{2}\,, \end{aligned} \tag{30}$$

Eq. (29) becomes

$$\exp(s^2)(s\rho)^{-\ell}\sqrt{\frac{1}{s\rho}}J_{\ell+1/2}(2s\rho) = \sum_{n=0}^{\infty}\frac{L_n^{(\ell+1/2)}(\rho^2)}{\Gamma(n+\ell+\frac{3}{2})}s^{2n}. \tag{31}$$

Now the Bessel functions of half integral index are simply related to the standard spherical Bessel functions [21], i.e.

$$J_{\ell+1/2}(2s\rho) = \sqrt{\frac{4s\rho}{\pi}}j_\ell(2s\rho). \tag{32}$$

Thus Eq. (31) may be rewritten in the form

$$\exp(s^2) j_\ell(2s\rho) = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{\rho^\ell L_n^{(\ell+1/2)}(\rho^2)}{\Gamma(n+\ell+\frac{3}{2})} s^{2n+\ell}. \tag{33}$$

Multiplying both sides of this equation by $4\pi i^\ell Y_\ell^m(\hat{\boldsymbol{\rho}}) Y_\ell^{m*}(\hat{\boldsymbol{s}})$ and summing over ℓ and m gives

$$\exp(s^2) \sum_{\ell m} 4\pi i^\ell j_\ell(2s\rho) Y_\ell^m(\hat{\boldsymbol{\rho}}) Y_\ell^{m*}(\hat{\boldsymbol{s}})$$
$$= \sum_{n\ell m} \frac{2\sqrt{\pi^3} i^\ell}{\Gamma(n+\ell+\frac{3}{2})} \left[\rho^\ell L_n^{(\ell+1/2)}(\rho^2) Y_\ell^m(\hat{\boldsymbol{\rho}})\right] s^{2n+\ell} Y_\ell^{m*}(\hat{\boldsymbol{s}}). \tag{34}$$

The summation on the left-hand side of this equation is easily recognised as the multipole expansion of a plane wave $\exp(2i\boldsymbol{s}\cdot\boldsymbol{\rho})$. Hence, on multiplying both sides by $\exp(-\rho^2/2)$ and using expression (27), one arrives at the result

$$G(\boldsymbol{s},\boldsymbol{\rho}) = \exp(s^2 + 2i\boldsymbol{s}\cdot\boldsymbol{\rho} - \rho^2/2) = \sum_{n\ell m} A_{n\ell} s^{2n+\ell} Y_\ell^{m*}(\hat{\boldsymbol{s}}) \phi_{n\ell m}(\boldsymbol{\rho})$$
$$\text{or} \quad = \sum_{n\ell m} A_{n\ell} s^{2n+\ell} Y_\ell^{m}(\hat{\boldsymbol{s}}) \phi^*_{n\ell m}(\boldsymbol{\rho}), \tag{35}$$

where

$$A_{n\ell} = i^\ell \left[\frac{2\pi^3}{n!\,\Gamma(n+\ell+\frac{3}{2})}\right]^{1/2}. \tag{36}$$

The alternative expression given in Eq. (35) is obtained by noting that the scalar product of spherical harmonics is a real quantity.

The existence of this generating function and the simple form for its expansion are clearly connected with the result of Bakri [8,9] mentioned in the introduction. A typical oscillator eigenfunction may be represented in terms of the generating function by means of the operational equation

$$\phi_{n\ell m}(\boldsymbol{\rho}) = \left[\frac{1}{A_{n\ell}(2n+\ell)!}\right] \times \left[\frac{\partial^{2n+\ell}}{\partial s^{2n+\ell}} \int Y_\ell^m(\hat{\boldsymbol{s}}) G(\boldsymbol{s},\boldsymbol{\rho}) d\hat{\boldsymbol{s}}\right]_{s=0}. \tag{37}$$

The complex conjugate function $\phi^*_{n\ell m}(\boldsymbol{\rho})$ is given by an equation similar to (37) with $Y_\ell^m(\hat{\boldsymbol{s}})$ replaced by $Y_\ell^{m*}(\hat{\boldsymbol{s}})$.

4. The overlap integral

It is now possible to write the overlap integral (23) in a form which allows the integrations over $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_2$ to be performed easily. With the coordinate vectors $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$, $\mathfrak{R}$ and $\boldsymbol{\rho}$ associate the expansion vectors $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. Then replace each of the oscillator functions in Eq. (23) by its representation in the form (37) and substitute the result in Eq. (24). Thus one obtains the following operational expression for the general oscillator bracket:

$$
\begin{aligned}
&\langle NL, n\ell : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle \\
&= \frac{1}{A_{n_1\ell_1}A_{n_2\ell_2}A_{NL}A_{n\ell}(2n_1+\ell_1)!(2n_2+\ell_2)!(2N+L)!(2n+\ell)!} \\
&\times \sum_{\lambda=\text{constant}} \langle \ell_1 m_1 \ell_2 m_2 | \Lambda\lambda\rangle \langle LM\ell m|\Lambda\lambda\rangle \left[\frac{\partial^{2n_1+\ell_1}}{\partial s^{2n_1+\ell_1}} \frac{\partial^{2n_2+\ell_2}}{\partial t^{2n_2+\ell_2}} \frac{\partial^{2N+L}}{\partial u^{2N+L}} \frac{\partial^{2n+\ell}}{\partial v^{2n+\ell}} \right. \\
&\times \left. \int \left\{ Y_{\ell_1}^{m_1}(\hat{s}) Y_{\ell_2}^{m_2}(\hat{t}) Y_L^{M*}(\hat{u}) Y_\ell^{m*}(\hat{v})\; J(s,t,u,v) \right\} d\hat{s}\, d\hat{t}\, d\hat{u}\, d\hat{v} \right]_{s,t,u,v=0}, \qquad (38)
\end{aligned}
$$

where

$$
J(s,t,u,v) = \int d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \{ G(s,\boldsymbol{\rho}_1) G(t,\boldsymbol{\rho}_2) G(u,\boldsymbol{\Re}) G(v,\boldsymbol{\rho}) \}. \qquad (39)
$$

The integral in Eq. (39) may be evaluated straightforwardly, as shown below, and the explicit result is

$$
J(s,t,u,v) = \pi^3 \exp\left[-2\{ \boldsymbol{s}\cdot\boldsymbol{u}\sin\beta + \boldsymbol{s}\cdot\boldsymbol{v}\cos\beta + \boldsymbol{t}\cdot\boldsymbol{u}\cos\beta - \boldsymbol{t}\cdot\boldsymbol{v}\sin\beta \} \right], \qquad (40)
$$

where β is given by Eq. (19).

To derive Eq. (40), substitute the generating functions defined in Eq. (35) into the integral (39). This gives

$$
\begin{aligned}
J(s,t,u,v) = {} & \exp(s^2+t^2+u^2+v^2) \int d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \exp\left[-(\rho_1^2+\rho_2^2+\Re^2+\rho^2)/2 \right] \\
& \times \exp\left[2i(\boldsymbol{s}\cdot\boldsymbol{\rho}_1 + \boldsymbol{t}\cdot\boldsymbol{\rho}_2 + \boldsymbol{u}\cdot\boldsymbol{\Re} + \boldsymbol{v}\cdot\boldsymbol{\rho}) \right]. \qquad (41)
\end{aligned}
$$

Using the transformation (15), which implies $\Re^2+\rho^2 = \rho_1^2+\rho_2^2$, yields the result

$$
J = \exp(s^2+t^2+u^2+v^2) K_1 K_2, \qquad (42)
$$

where

$$
K_j = \int d\boldsymbol{\rho}_j \exp(-\rho_j^2) \exp(2i\boldsymbol{\gamma}_j\cdot\boldsymbol{\rho}_j), \quad j=1,2 \qquad (43)
$$

and the vectors $\boldsymbol{\gamma}_j$ are given by

$$
\begin{aligned}
\boldsymbol{\gamma}_1 &= \boldsymbol{s} + \boldsymbol{u}\sin\beta + \boldsymbol{v}\cos\beta, \\
\boldsymbol{\gamma}_2 &= \boldsymbol{t} + \boldsymbol{u}\cos\beta - \boldsymbol{v}\sin\beta. \qquad (44)
\end{aligned}
$$

The integrals (43) are essentially the three-dimensional Fourier transforms of a Gaussian function and have the values

$$
K_j = \sqrt{\pi^3}\exp(-\gamma_j^2). \qquad (45)
$$

Substituting the results (45) and (44) into Eq. (42) and multiplying out the arguments of the exponentials leads easily to the expression (40).

5. Evaluation

From the general formulae (38) and (40) it is possible to derive several different algebraic expressions for the oscillator brackets, depending on the order in which the indicated operations are performed. If, for instance, the angular integrations over $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ are performed first, followed by the differentiations with respect to u and v, then on setting $u = v = 0$ the resulting expression is algebraically equivalent to the starting point of the work of Baranger and Davies [4]. This approach requires some difficult manipulations in order to arrive at an explicit formula.

It turns out to be much easier and more symmetrical to do all the angular integrations first. The result is then expanded as a power series in s, t, u, and v, and operations of the form $[1/(2n_1+\ell_1)!]\partial^{2n_1+\ell_1}/\partial s^{2n_1+\ell_1}$, etc., followed by setting $s = t = u = v = 0$ are equivalent to finding the coefficient of $s^{2n_1+\ell_1}t^{2n_2+\ell_2}u^{2N+L}v^{2n+\ell}$ in this expansion. The result of summing over the angular momentum z-components can be written directly as a $9j$ coefficient and the final explicit expression is obtained after a few trivial substitutions.

The first step is to write Eq. (40) in the form

$$J(s,t,u,v) = \pi^3 \exp\left(i[2i\{\boldsymbol{s}\cdot\boldsymbol{u}\sin\beta + \boldsymbol{s}\cdot\boldsymbol{v}\cos\beta + \boldsymbol{t}\cdot\boldsymbol{u}\cos\beta - \boldsymbol{t}\cdot\boldsymbol{v}\sin\beta\}]\right) \tag{46}$$

and to represent each exponential factor by the usual expansion for a plane wave. Hence,

$$\begin{aligned} J(s,t,u,v) &= (4\pi)^4\pi^3 \\ &\times \sum_{\ell_a m_a} i^{\ell_a} j_{\ell_a}(2isu\sin\beta) Y_{\ell_a}^{m_a}(\hat{\boldsymbol{u}}) Y_{\ell_a}^{m_a *}(\hat{\boldsymbol{s}}) \sum_{\ell_b m_b} i^{\ell_b} j_{\ell_b}(2isv\cos\beta) Y_{\ell_b}^{m_b}(\hat{\boldsymbol{v}}) Y_{\ell_b}^{m_b *}(\hat{\boldsymbol{s}}) \\ &\times \sum_{\ell_c m_c} i^{\ell_c} j_{\ell_c}(2itu\cos\beta) Y_{\ell_c}^{m_c}(\hat{\boldsymbol{u}}) Y_{\ell_c}^{m_c *}(\hat{\boldsymbol{t}}) \sum_{\ell_d m_d} i^{-\ell_d} j_{\ell_d}(2itv\sin\beta) Y_{\ell_d}^{m_d}(\hat{\boldsymbol{v}}) Y_{\ell_d}^{m_d *}(\hat{\boldsymbol{t}}) , \end{aligned} \tag{47}$$

where $j_\ell(ix)$ is a spherical Bessel function of imaginary argument. Now use the well-known formula (see, for example, Ref. [22])

$$Y_{\ell_1}^{m_1}(\hat{\boldsymbol{x}}) Y_{\ell_2}^{m_2}(\hat{\boldsymbol{x}}) = \sum_\ell \left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}\right]^{1/2} \langle \ell_1 m_1 \ell_2 m_2 | \ell m\rangle \langle \ell_1 0 \ell_2 0 | \ell 0\rangle Y_\ell^m(\hat{\boldsymbol{x}}) \tag{48}$$

to couple together the pairs of angular functions in Eq. (47) with the same arguments.

When this is done the angular integrations in Eq. (38) become trivial and the result is

$$\begin{aligned} &\int \left\{ Y_{\ell_1}^{m_1}(\hat{\boldsymbol{s}}) Y_{\ell_2}^{m_2}(\hat{\boldsymbol{t}}) Y_L^{M*}(\hat{\boldsymbol{u}}) Y_\ell^{m*}(\hat{\boldsymbol{v}}) J(s,t,u,v) \right\} d\hat{\boldsymbol{s}}\, d\hat{\boldsymbol{t}}\, d\hat{\boldsymbol{u}}\, d\hat{\boldsymbol{v}} \\ &= \frac{(4\pi)^2\pi^3}{\sqrt{(2\ell_1+1)(2\ell_2+1)(2L+1)(2\ell+1)}} \end{aligned}$$

$$\times \sum_{\substack{\ell_a\ell_b\ell_c\ell_d \\ m_am_bm_cm_d}} i^{\ell_a+\ell_b+\ell_c-\ell_d}[(2\ell_a+1)(2\ell_b+1)(2\ell_c+1)(2\ell_d+1)]$$
$$\times j_{\ell_a}(2isu\sin\beta)j_{\ell_b}(2isv\cos\beta)j_{\ell_c}(2itu\cos\beta)j_{\ell_d}(2itv\sin\beta)$$
$$\times\langle\ell_am_a\ell_bm_b|\ell_1m_1\rangle\langle\ell_cm_c\ell_dm_d|\ell_2m_2\rangle\langle\ell_am_a\ell_cm_c|LM\rangle\langle\ell_bm_b\ell_dm_d|\ell m\rangle$$
$$\times\langle\ell_a0\ell_b0|\ell_10\rangle\langle\ell_c0\ell_d0|\ell_20\rangle\langle\ell_a0\ell_c0|L0\rangle\langle\ell_b0\ell_d0|\ell0\rangle\,. \tag{49}$$

It now remains only to substitute this expression into Eq. (38), expand the spherical Bessel functions in a power series and extract the coefficient of $s^{2n_1+\ell_1}t^{2n_2+\ell_2}u^{2N+L}v^{2n+\ell}$ as explained earlier.

Thus, by using the expansions of the typical form

$$j_{\ell_a}(2ix)=\frac{i^{\ell_a}\sqrt{\pi}}{2}\sum_{a=0}^{\infty}\frac{(x)^{2a+\ell_a}}{a!\Gamma(a+\ell_a+\frac{3}{2})}\,, \tag{50}$$

Eq. (38) for the general oscillator bracket may be written as an explicit algebraic expression:

$$\langle NL,n\ell:\Lambda|n_1\ell_1,n_2\ell_2:\Lambda\rangle$$
$$=\frac{\pi^7}{A_{n_1\ell_1}A_{n_2\ell_2}A_{NL}A_{n\ell}\sqrt{(2\ell_1+1)(2\ell_2+1)(2L+1)(2\ell+1)}}$$
$$\times\sum_{\substack{abcd\\ \ell_a\ell_b\ell_c\ell_d\\ m_am_bm_cm_d\\ \lambda=\text{constant}}}\frac{(-1)^{\ell_a+\ell_b+\ell_c}[(2\ell_a+1)(2\ell_b+1)(2\ell_c+1)(2\ell_d+1)]}{a!b!c!d!\Gamma(a+\ell_a+\frac{3}{2})\Gamma(b+\ell_b+\frac{3}{2})\Gamma(c+\ell_c+\frac{3}{2})\Gamma(d+\ell_d+\frac{3}{2})}$$
$$\times\langle\ell_1m_1\ell_2m_2|\Lambda\lambda\rangle\langle LM\ell m|\Lambda\lambda\rangle(\sin\beta)^{2a+\ell_a+2d+\ell_d}\,(\cos\beta)^{2b+\ell_b+2c+\ell_c}$$
$$\times\langle\ell_am_a\ell_bm_b|\ell_1m_1\rangle\langle\ell_cm_c\ell_dm_d|\ell_2m_2\rangle\langle\ell_am_a\ell_cm_c|LM\rangle\langle\ell_bm_b\ell_dm_d|\ell m\rangle$$
$$\times\langle\ell_a0\ell_b0|\ell_10\rangle\langle\ell_c0\ell_d0|\ell_20\rangle\langle\ell_a0\ell_c0|L0\rangle\langle\ell_b0\ell_d0|\ell0\rangle\,, \tag{51}$$

where the extraction of the correct coefficient imposes the following restrictions on the summation indices in Eq. (51):

$$2a+\ell_a+2b+\ell_b=2n_1+\ell_1=f_1\,,$$
$$2c+\ell_c+2d+\ell_d=2n_2+\ell_2=f_2\,,$$
$$2a+\ell_a+2c+\ell_c=2N+L=F\,,$$
$$2b+\ell_b+2d+\ell_d=2n+\ell=f. \tag{52}$$

These restrictions automatically ensure the condition

$$f_1+f_2=\chi=F+f\,, \tag{53}$$

in agreement with Eq. (22). Further, the last four vector coupling coefficients in Eq. (51) imply additional restrictions on the allowed values of ℓ_a, ℓ_b, ℓ_c and ℓ_d. For example, $\ell_a+\ell_b+\ell_1=$ even.

The sum over magnetic quantum numbers may be performed easily, with the result

$$\sum_{\substack{m_a m_b m_c m_d \\ \lambda=\text{constant}}} \langle \ell_1 m_1 \ell_2 m_2 | \Lambda\lambda\rangle \langle \ell_a m_a \ell_b m_b | \ell_1 m_1\rangle \langle \ell_c m_c \ell_d m_d | \ell_2 m_2\rangle$$
$$\times \langle LM\ell m|\Lambda\lambda\rangle \langle \ell_a m_a \ell_c m_c | LM\rangle \langle \ell_b m_b \ell_d m_d | \ell m\rangle$$
$$= \sqrt{(2\ell_1+1)(2\ell_2+1)(2L+1)(2\ell+1)} \begin{Bmatrix} \ell_a & \ell_b & \ell_1 \\ \ell_c & \ell_d & \ell_2 \\ L & \ell & \Lambda \end{Bmatrix} . \tag{54}$$

The last step is to substitute the values of the A coefficients as defined in Eq. (36) and to replace the Γ functions by their algebraic forms

$$\Gamma(x+\tfrac{3}{2}) = \frac{\sqrt{\pi}(2x+1)!!}{2^{x+1}} , \tag{55}$$

where $(2x+1)!!$ means $1\cdot 3\cdot 5\cdot \ldots \cdot (2x+1)$.

Thus the final expression is found to be

$$\langle NL, n\ell : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle = i^{-(\ell_1+\ell_2+L+\ell)} \times 2^{-(\ell_1+\ell_2+L+\ell)/4}$$
$$\times \sqrt{(n_1)!(n_2)!(N)!(n)![2(n_1+\ell_1)+1]!!\ [2(n_2+\ell_2)+1]!!}$$
$$\times \sqrt{[2(N+L)+1]!!\ [2(n+\ell)+1]!!}$$
$$\times \sum_{\substack{abcd \\ \ell_a\ell_b\ell_c\ell_d}} \Bigg[(-1)^{\ell_a+\ell_b+\ell_c} 2^{(\ell_a+\ell_b+\ell_c+\ell_d)/2} (\sin\beta)^{2a+\ell_a+2d+\ell_d}$$
$$\times (\cos\beta)^{2b+\ell_b+2c+\ell_c} \begin{Bmatrix} \ell_a & \ell_b & \ell_1 \\ \ell_c & \ell_d & \ell_2 \\ L & \ell & \Lambda \end{Bmatrix}$$
$$\times \frac{[(2\ell_a+1)(2\ell_b+1)(2\ell_c+1)(2\ell_d+1)]}{a!b!c!d![2(a+\ell_a)+1]!!\ [2(b+\ell_b)+1]!!}$$
$$\times \frac{1}{[2(c+\ell_c)+1]!!\ [2(d+\ell_d)+1]!!}$$
$$\times \langle \ell_a 0 \ell_c 0 | L0\rangle \langle \ell_b 0 \ell_d 0 | \ell 0\rangle \langle \ell_a 0 \ell_b 0 | \ell_1 0\rangle \langle \ell_c 0 \ell_d 0 | \ell_2 0\rangle \Bigg] , \tag{56}$$

where β is defined in terms of the particle masses and oscillator frequencies by Eq. (19) and the summations are restricted by the conditions (52). Although this summation over eight indices (with three independent constraints amongst them) can, in principle, be rewritten as an unconstrained sum over five independent indices, we do not feel that such a manipulation represents a real simplification since the manifest symmetry of the expression is lost from the final form. The final formula (56) can be made to appear less formidable by writing $N = n_3$, $L = \ell_3$, $n = n_4$, $\ell = \ell_4$ and adopting product notation, so that

$$\langle n_3\ell_3, n_4\ell_4 : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle = \prod_{k=1}^{4} i^{-l_k} \sqrt{\frac{(n_k)![2(n_k+\ell_k)+1]!!}{\sqrt{2^k}}}$$

$$\times \sum_{\substack{abcd \\ \ell_a\ell_b\ell_c\ell_d}} \left[(-1)^{\ell_d} (\sin\beta)^{2a+\ell_a+2d+\ell_d} \, (\cos\beta)^{2b+\ell_b+2c+\ell_c} \begin{Bmatrix} \ell_a & \ell_b & \ell_1 \\ \ell_c & \ell_d & \ell_2 \\ \ell_3 & \ell_4 & \Lambda \end{Bmatrix} \right.$$

$$\left. \times \prod_{p=a}^{d} \frac{(-1)^{\ell_p}\sqrt{2^{\ell_p}}(2\ell_p+1)}{p![2(p+\ell_p)+1]!!} \langle \ell_a 0 \ell_b 0 | \ell_1 0\rangle \langle \ell_c 0 \ell_d 0 | \ell_2 0\rangle \langle \ell_a 0 \ell_c 0 | \ell_3 0\rangle \langle \ell_b 0 \ell_d 0 | \ell_4 0\rangle \right]. \tag{57}$$

Finally, it should be noted that the coupling conventions used here agree with those of Ref. [4]. The above bracket differs from that defined by Moshinsky [2] through the phase factor $(-1)^{L+\ell+\Lambda}$. The expression (56) has been programmed and checked against the tabulation of Brody and Moshinsky [3] for the special case $\mu_1 = \mu_2$, $\omega_1 = \omega_2$. A few examples are shown in Table 1. In addition, we show in Appendix A that our expression for the special case $\omega_1 = \omega_2$ is identical in form to the expression derived by Bakri in Eqs. (2.5) and (2.9) of Ref. [8], (where unequal masses are permissible). To see this, our definition of $\beta = \tan^{-1}\sqrt{(\mu_1\omega_1/\mu_2\omega_2)}$ must be replaced by $\beta_{\mathrm{Bakri}} = 2\tan^{-1}\sqrt{(\mu_2/\mu_1)}$. Indeed, if one writes $\beta = \tan^{-1}(b_2/b_1)$ in terms of the oscillator length parameters $b_i^2 = \hbar/(\mu_i\omega_i)$ and generalizes $\beta_{\mathrm{Bakri}} = 2\tan^{-1}(b_1/b_2)$ accordingly, then Bakri's formula can be applied directly to the general case.

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Appendix

Our expression (56) is actually identical to that of Bakri (Eqs. (2.5) and (2.9) of Ref. [8]) when $\omega_1 = \omega_2$ and our angle $\beta = \tan^{-1}\sqrt{(\mu_1\omega_1/\mu_2\omega_2)}$ is replaced by $\beta_{\mathrm{Bakri}} = 2\tan^{-1}\sqrt{(\mu_2/\mu_1)}$. To see this most readily take Eq. (51) and substitute for the normalization constants $A_{n\ell}$, etc., from Eq. (36) and use the result of Eq. (54) for the sum over the magnetic quantum numbers. Most of the work now involves modifying our notation so that it looks like Bakri's. To this end write $n_3, \ell_3 = N, L$ and $n_4, \ell_4 = n, \ell$. Then

$$\langle n_3\ell_3, n_4\ell_4 : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle = \frac{\pi}{4}\prod_{i=1}^{4}\left\{ i^{-\ell_i}\sqrt{n_i!\,\Gamma(n_i+\ell_i+\tfrac{3}{2})}\right\} \sum_{\substack{abcd \\ \ell_a\ell_b\ell_c\ell_d}} \left[(-1)^{\ell_a+\ell_b+\ell_c} \right.$$

$$\times \frac{[\ell_a][\ell_b][\ell_c][\ell_d](\sin\beta)^{2a+\ell_a+2d+\ell_d}\,(\cos\beta)^{2b+\ell_b+2c+\ell_c}}{a!b!c!d!\,\Gamma(a+\ell_a+\tfrac{3}{2})\;\Gamma(b+\ell_b+\tfrac{3}{2})\;\Gamma(c+\ell_c+\tfrac{3}{2})\;\Gamma(d+\ell_d+\tfrac{3}{2})}$$

$$\times \left\{ \begin{matrix} \ell_a & \ell_b & \ell_1 \\ \ell_c & \ell_d & \ell_2 \\ \ell_3 & \ell_4 & \Lambda \end{matrix} \right\} \langle \ell_a 0 \ell_c 0 | \ell_3 0 \rangle \langle \ell_b 0 \ell_d 0 | \ell_4 0 \rangle \langle \ell_a 0 \ell_b 0 | \ell_1 0 \rangle \langle \ell_c 0 \ell_d 0 | \ell_2 0 \rangle \Bigg] , \tag{A.1}$$

where we adopt Bakri's notation that $[\ell_a] = 2\ell_a + 1$, etc. We now take the factor $\cos\beta^{2n_1+\ell_1} \sin\beta^{2n_2+\ell_2}$ outside the sum, write the Clebsch–Gordan coefficients in terms of $3j$ symbols according to

$$\langle \ell_a m_a \ell_b m_b | \ell_c m_c \rangle = (-1)^{\ell_a - \ell_b + m_c} \sqrt{(2\ell_c + 1)} \begin{pmatrix} \ell_a & \ell_b & \ell_c \\ m_a & m_b & -m_c \end{pmatrix} , \tag{A.2}$$

which enables us to take a factor $\sqrt{[\ell_i]}$ outside the sum and into the product, and use the constraints of Eq. (52) to rewrite the powers of cos and sin remaining inside the sum as $\pm(2a + \ell_a - 2c - \ell_c)$ so that we obtain a tangent (or cotangent). Now

$$\begin{aligned} &\langle n_3\ell_3, n_4\ell_4 : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda \rangle \\ &= \frac{\pi}{4} \prod_{i=1}^{4} \left\{ i^{-\ell_i} \sqrt{[\ell_i] n_i! \Gamma(n_i + \ell_i + \tfrac{3}{2})} \right\} (\cos\beta)^{2n_1+\ell_1} (\sin\beta)^{2n_2+\ell_2} \\ &\times \sum_{\substack{abcd \\ \ell_a\ell_b\ell_c\ell_d}} \Bigg[(-1)^{\ell_a+\ell_b+\ell_c} \frac{[\ell_a][\ell_b][\ell_c][\ell_d]}{a!b!c!d!} \\ &\times \begin{pmatrix} \ell_a & \ell_b & \ell_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_c & \ell_d & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_a & \ell_c & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_b & \ell_d & \ell_4 \\ 0 & 0 & 0 \end{pmatrix} \\ &\times \frac{(\tan\beta)^{2a+\ell_a-2c-\ell_c}}{\Gamma(a+\ell_a+\tfrac{3}{2})\ \Gamma(b+\ell_b+\tfrac{3}{2})\ \Gamma(c+\ell_c+\tfrac{3}{2})\ \Gamma(d+\ell_d+\tfrac{3}{2})} \\ &\times \left\{ \begin{matrix} \ell_a & \ell_b & \ell_1 \\ \ell_c & \ell_d & \ell_2 \\ \ell_3 & \ell_4 & \Lambda \end{matrix} \right\} \Bigg] . \end{aligned} \tag{A.3}$$

To obtain Bakri's formulae (i.e. Eqs. (2.5) and (2.9) of Ref. [8]) we can take the constraining equations (52), which provide three *independent* relations, to define b and d. Then our eightfold sum (with relations (52) between the indices) becomes a sixfold sum over the indices $a, c, \ell_a, \ell_b, \ell_c, \ell_d$ with the single constraint $2n_3+\ell_3 = 2a+\ell_a+2c+\ell_c$. To get closer to Bakri's notation [8] make the identifications

$$\ell_a \to k_1 , \quad \ell_b \to j_1 , \quad \ell_c \to k_2 , \quad \ell_d \to j_2 , \quad a \to \nu_1 , \quad c \to \nu_2 . \tag{A.4}$$

These relations define the six summation indices in Bakri's notation. The constraining relation $2n_3 + \ell_3 = 2a + \ell_a + 2c + \ell_c$ is identical to Bakri's restriction $k + 2\nu = 2n_3 + \ell_3$, where $k = k_1 + k_2$ and $\nu = \nu_1 + \nu_2$.

It only remains to identify correctly Bakri's factor $f^{N_1 N_2 L_1 L_2}_{\nu_1,\nu_2}$ and to show that his phase factors have the same effect as ours. To this end let us follow Bakri in defining

$$N_i = n_i - (k_i + j_i - \ell_i)/2 , \qquad L_i = (\ell_i + j_i - k_i)/2 \quad \text{for } i = 1,2 . \tag{A.5}$$

Now, by definition, we have immediately

$$a! = \nu_1!\,, \quad c! = \nu_2!\,, \quad a + \ell_a + \tfrac{3}{2} = k_1 + \nu_1 + \tfrac{3}{2}\,, \quad c + \ell_c + \tfrac{3}{2} = k_2 + \nu_2 + \tfrac{3}{2}\,. \tag{A.6}$$

Using the definitions of N_i and L_i from Eq. (A.5) together with Eqs. (52) we easily see that

$$b! = (N_1 - \nu_1)!\,, \quad d! = (N_2 - \nu_2)!\,, \quad b + \ell_b + \tfrac{3}{2} = n_1 + L_1 - \nu_1 + \tfrac{3}{2}\,,$$
$$d + \ell_d + \tfrac{3}{2} = n_2 + L_2 - \nu_2 + \tfrac{3}{2}\,. \tag{A.7}$$

We can therefore write

$$\begin{aligned}\frac{1}{f^{N_1N_2L_1L_2}_{\nu_1,\nu_2}} &= a!b!c!d!\Gamma(a + \ell_a + \tfrac{3}{2})\Gamma(b + \ell_b + \tfrac{3}{2})\Gamma(c + \ell_c + \tfrac{3}{2})\Gamma(d + \ell_d + \tfrac{3}{2}) \\ &= \nu_1!\nu_2!(N_1 - \nu_1)!(N_2 - \nu_2)!\Gamma(n_1 + L_1 + \tfrac{3}{2} - \nu_1) \\ &\quad\times \Gamma(n_2 + L_2 + \tfrac{3}{2} - \nu_2)\Gamma(k_1 + \tfrac{3}{2} + \nu_1)\Gamma(k_2 + \tfrac{3}{2} + \nu_2)\end{aligned} \tag{A.8}$$

in accord with Eq. (2.5) of Bakri [8].

Finally we must check the phase factors. Bakri has $(-1)^{n_1+n_2+n_3+n_4+j_2}$ (where $j_2 = \ell_d$ in our notation) and in Eq. (A.1) we have $(-1)^{\ell_a+\ell_b+\ell_c-(\ell_1+\ell_2+\ell_3+\ell_4)/2}$. To see that these are equivalent, take the four relations of Eq. (52), add them all together, and divide by 2. Since we are now writing $N = n_3$ and $n = n_4$ this yields

$$n_1 + n_2 + n_3 + n_4 + \tfrac{1}{2}(\ell_1 + \ell_2 + \ell_3 + \ell_4) = 2(a + b + c + d) + (\ell_a + \ell_b + \ell_c + \ell_d). \tag{A.9}$$

Since a, b, c and d are all integers, the term $2(a + b + c + d)$ contributes $+1$ to the overall phase. Also, since ℓ_d is an integer, the contribution to the phase of $+\ell_d$ is identical to that of $-\ell_d$. We can therefore deduce that the phase due to the argument $n_1 + n_2 + n_3 + n_4 + \ell_d$ (Bakri, Eq. (2.8) [8]) is identical to that due to the argument $\ell_a + \ell_b + \ell_c - \tfrac{1}{2}(\ell_1 + \ell_2 + \ell_3 + \ell_4)$.

We obtain a final result identical to that of Bakri if we set $\omega_1 = \omega_2$ so that we can now write

$$\tan\beta = \sqrt{\frac{\mu_1}{\mu_2}} = \cot\left(\frac{\beta_{\text{Bakri}}}{2}\right). \tag{A.10}$$

In addition, this means that $\sin\beta = \cos(\beta_{\text{Bakri}}/2)$ and $\cos\beta = \sin(\beta_{\text{Bakri}}/2)$. Our final result can thus be written

$$\begin{aligned}&\langle n_3\ell_3, n_4\ell_4 : \Lambda | n_1\ell_1, n_2\ell_2 : \Lambda\rangle \\ &= \frac{\pi}{4}\prod_{i=1}^{4}\left\{(-1)^{n_i}\sqrt{[\ell_i]n_i!\Gamma(n_i + \ell_i + \tfrac{3}{2})}\right\}[\sin(\beta_{\text{Bakri}}/2)]^{2n_1+\ell_1} \\ &\times [\cos(\beta_{\text{Bakri}}/2)]^{2n_2+\ell_2} \sum_{\substack{k_1,k_2,\nu_1,\nu_2,j_1,j_2 \\ 2\nu_1+2\nu_2+k_1+k_2=2n_3+\ell_3}} (-1)^{j_2}[j_1][j_2][k_1][k_2]\end{aligned}$$

$$\times[\cot(\beta_{\text{Bakri}}/2)]^{k_1-k_2+2\nu_1-2\nu_2}\begin{pmatrix} k_1 & k_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 & \ell_1 & j_1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times\begin{pmatrix} k_2 & \ell_2 & j_2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & \ell_4 \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} k_1 & j_1 & \ell_1 \\ k_2 & j_2 & \ell_2 \\ \ell_3 & \ell_4 & \Lambda \end{Bmatrix} f^{N_1N_2L_1L_2}_{\nu_1,\nu_2}, \tag{A.11}$$

identical to Eq. (2.9) of Bakri [8]. Since the only explicit dependence on the frequencies ω_i and masses μ_i in this equation comes from the angle β, which can be related to the oscillator length parameters $b_i^2 = \hbar/(\mu_i\omega_i)$, it is only necessary to generalize $\cot(\beta_{\text{Bakri}}/2)$ to our expression for $\tan\beta$, Eq. (19), for Bakri's formula to be applicable to the general case.

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