Homework 2

1 Directions:

- **Due: Friday Feb. 16, 2018, at 8pm, uploaded on Blackboard**. Any submission uploaded to blackboard after 11:59:59pm will be marked late.
- You must turn in the homework via blackboard **as a single pdf**. The images (scans/photographs) must be sufficiently high quality that the writing is legible. If writing is illegible, if the image is too dark, or if the contrast is too low then you will receive no or only partial credit. The responses must be in order. *Make sure to check the file you upload on blackboard before finishing.* Scanners are available in the libraries.
- The deadline is hard. If you do not make it, you may submit your homework until Saturday Feb. 17 at noon, with a 25 points penalty. No homework will be accepted later than noon of Jan 27.
- Motivate all your answers. When you are asked to use Matlab, include the code as text in your submission. (preferably "print" the code in the editor as a pdf file).
- Any questions about material taught in class or guidance on homework must be asked in office hours, not over email. If you are unable to attend office hours due to time-conflicts, email the staff and we will try to accomodate you. We may request your schedule to confirm the time conflicts. **No questions will be answered the day before the HW deadline (Feb. 15 and Feb. 16)**.

2 Problems

Problem 1 (25 points)**.** With reference to Lecture 8, prove the equivalence between Problem (P) and Problem (Q) defined in Slide 3.

Hint: Prove Step 1 and Step 2, as discussed in Class. For Step 1, it is ok to prove only the "left-to-right" implication.

Bonus [10 points]: Prove the "right-to-left" implication in the statement of Step 1.

Problem 2. [10 points] Convert the following problem,

$$\begin{aligned} \min \quad & 2x_1 + 5x_2 + 2x_3 - x_4 \\ \text{s.t.} \quad & 4x_2 - 2x_4 \geq 7 + x_3 \\ & x_2 \leq 3x_1 \\ & 2x_2 - x_1 \geq x_3 + 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

into vector/matrix format,

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Problem 3. [10 points] Consider the following feasible region:

$$\begin{aligned} \frac{1}{2}x + y &\leq 4 \\ x &\leq 6 \\ x, y &\geq 0 \end{aligned}$$

a. Graph the feasible region.
b. Give a *maximizing* objective function such that $(6, 0)$ and $(0, 0)$ are optimal in the feasible region above.
c. Give a *maximizing* objective function such that $(6, 1)$ is uniquely optimal in the feasible region above.
d. Give a *maximizing* objective function such that $(6, 1)$ and $(0, 0)$ are optimal in the feasible region above.
e. Add a constraint using x and y that makes the current feasible region infeasible.

Problem 4. [15 points]
Part I (9 points) Find ALL *local* and *global* optimal solutions of the following *minimization* problems, whose objective function is denoted by f and feasible set by $\mathcal{X}$ (do NOT use Matlab or any other solver):

(a). $f(x) = \sin x$ and $\mathcal{X} = [0, 3\pi]$;

(b). $f(x) = \sin x$ and $\mathcal{X} = [0, 4\pi]$;

(c). $f(x) = \sin x$ and $\mathcal{X} = [-\pi/4, (11/4)\pi]$;

(d). $f(x) = \max(0, x^2 - 1)$ and $\mathcal{X} = (-\infty, \infty)$;

(e). $f(x) = \begin{cases} \frac{1}{x} \cdot \sin(x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0; \end{cases}$ and $\mathcal{X} = [0, (2/3)\pi]$;

Part II (6 points) Provide an example of optimization problem

(a). that does not have any GLOBAL optimal solution but it does have at least one LOCAL optimal solution;

(b). whose set of local optimal solutions coincides with that of the global optimal solutions.

Problem 5. [15 points] Consider the following data fitting problem. We are given m data points of the form

$$(a_1^i, \ldots, a_n^i, b^i) \qquad i = 1, \ldots, m.$$

The data $(a_1^i, \ldots, a_n^i)$ for $i = 1, \ldots, m$ represent n explanatory factors, and the data b^i for $i = 1, \ldots, m$ represent the response. For example, the data might come from m different people, where $(a_1^i, \ldots, a_n^i)$ represents various types of demographic and educational attainment information for person i, and b^i represents person i's salary.

We wish to use these data points to estimate a linear predictive model between the explanatory factors and the response: we want to estimate parameters $(x_1, \ldots, x_n)$ such that

$$b \approx \sum_{j=1}^{n} a_j x_j$$

for explanatory factors $(a_1, \ldots, a_n)$ and response b.

Given a particular parameter vector $(x_1, \ldots, x_n)$, the *residual*, or prediction error, at the ith data point is defined as

$$\left| b^i - \sum_{j=1}^{n} a_j^i x_j \right|.$$

Given a choice between alternative predictive models, one typically chooses a model that "explains" the available data as best as possible, i.e., a model that results in small residuals.

(a). One possibility is to minimize the total prediction error

$$\min \ \sum_{i=1}^{m} \left| b^i - \sum_{j=1}^{n} a_j^i x_j \right|,$$

with respect to $(x_1, \ldots, x_n)$, subject to no constraints. Formulate this problem as a linear program.

(b). In an alternative formulation, one could minimize the largest residual

$$\min \max_{i=1,\dots,m} \left| b^i - \sum_{j=1}^{n} a_j^i x_j \right|,$$

with respect to $(x_1, \dots, x_n)$, subject to no constraints. Formulate this problem as a linear program.

Problem 6. (25 total points) Professor Scutari is so proud of your grade on the first midterm that he gets you a job with Purdue Facilities. They are rebuilding our beloved Grissom Hall. For the next three months, they need to store extra equipment, building materials, etc. A local company offers warehouse space based on monthly usage. You need to set up an LP to find the cheapest way to store what is needed. Below is a chart of the space you need for each month. Next to it is a chart of the leasing options−the total price per square foot leased and the corresponding duration. Note that you may have multiple, overlapping leases (e.g. two one-month leases for March and April and a two-month lease starting in April). All leases should end by May. You may lease more space than required.

Month	Space Required (sq. ft.)
March	30,000
April	20,000
May	40,000

Duration (months)	Cost per Sq. ft. leased
1	\$65
2	\$100
3	\$135