# Application of Discontinuous Galerkin methods to the unsteady, compressible Navier-Stokes equations

## A review of the state of the art

Björn Müller
Seminar Strömungsmechanik, Kontinuumsmechanik und geophysikalische Mechanik
Fachgebiet für Strömungsdynamik (FDY)
TU Darmstadt

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# Outline



Introduction to the Discontinuous Galerkin framework

Application of the DG approach to the compressible Navier-Stokes equations

Time discretization

Outlook on some advanced features

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  $\rightarrow$ Discontinuous Galerkin (DG, sometimes also called DG-FEM)

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| Method | High-order accuracy | hp-adaptivity | Conservativity |
|---|---|---|---|
| FVM | ($\checkmark$) | x | $\checkmark$ |
| FEM | $\checkmark$ | ($\checkmark$) | ($\checkmark$) |
| DG | $\checkmark$ | $\checkmark$ | $\checkmark$ |

Figure: Comparison of the different approaches

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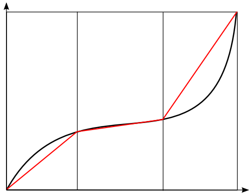

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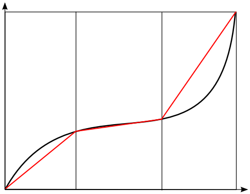

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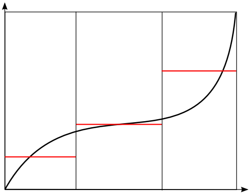

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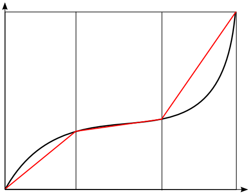

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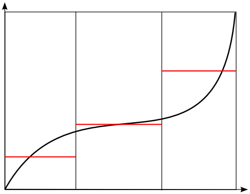

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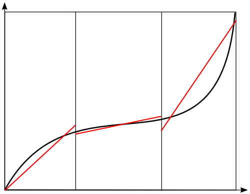

(c) First order DG

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  4. Specification of a *numerical flux* so that a unique value at interfaces can be recovered
- For the sake of simplicity, this procedure will be introduced by means of a scalar consveration law in the following

## Example:
## Application to a scalar conservation law



- Considered equation:

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = 0 \tag{1}$$

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- Multiplication by the test function $\Phi$ and integration over the cell $K$ yields

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- After application of Gauss' theorem:

$$\int_K \frac{\partial u}{\partial t} \Phi d\vec{x} = - \int_{\partial K} \left( \vec{f}(u) \cdot \vec{n} \right) \Phi ds + \int_K \vec{f}(u) \cdot \nabla \Phi d\vec{x} \tag{3}$$

# Example continued: Approximation of the solution

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- (Modal) Approximation of order $p$ of the exact solution in cell $K$:

$$u(\vec{x}, t) \approx \tilde{u}(\vec{x}, t) = \sum_{i=1}^{N} \tilde{u}_i^K(t) \cdot \varphi_i(\vec{x}) \tag{4}$$

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$$\text{degree}(\varphi_i) \leq p \quad \forall i \tag{5}$$

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- Usually, the polynomials $\varphi$ are also chosen as test functions $\phi$

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- The choice of $g$ defines the numerical properties of the whole scheme!

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- Note: Another possibility of deriving a DG scheme is the so called *strong formulation*. For details see [Gassner2009b] or [Hesthaven2007]

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# Notation



| Symbol ($i, j = 1, 2, 3$) | Physical interpretation | Unit |
|---|---|---|
| $x_i$ | Cartesian coordinates | $m$ |
| $u_i$ | Velocity components | $\frac{m}{s}$ |
| $p$ | Pressure | $\frac{N}{m^2}$ |
| $\tau_{ij}$ | Components of the stress tensor | $\frac{N}{m^2}$ |
| $\rho$ | Mass density | $\frac{kg}{m^3}$ |
| $e$ | Specific inner energy | $\frac{J}{kg}$ |
| $q_i$ | Components of the heat flow | $\frac{J}{m^2}$ |
| $\mu$ | Dynamic viscosity | $\frac{Ns}{m^2}$ |
| $T$ | Absolute temperature | $K$ |
| $\lambda$ | Thermal conductivity | $\frac{W}{Km}$ |

Figure: The notation used for the formulation of the Navier-Stokes equations

# Assumptions

- Newtonian fluid ($i, j = 1, 2, 3$)

$$\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \tag{9}$$

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- Fourier's law of heat conduction ($i = 1, 2, 3$):

$$\frac{\partial q_i}{\partial t} = -\lambda \frac{\partial T}{\partial x_i} \tag{10}$$

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- Energy

$$\rho \left( \frac{\partial e}{\partial t} + u_i \frac{\partial e}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( \lambda \frac{\partial T}{\partial x_i} \right) - p \frac{\partial u_i}{\partial x_i} + 2\mu \left( \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} - \frac{1}{3} \left( \frac{\partial u_i}{\partial x_i} \right)^2 \right) \tag{13}$$

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- We still need relations for $T$ and $p$

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- For example for ideal gases usually *Sutherland's law*

$$\mu = \mu_0 \left( \frac{T}{T_0} \right)^{\frac{3}{2}} \frac{T_0 + S}{T + S} \tag{16}$$

with the *reference temperature* $T_0$, the *reference viscosity* $\mu_0 = \mu(T_0)$ and the *Sutherland temperature S* is used

# Dimensionless formulation



| Physical quantity | Dimensionless formulation |
|---|---|
| Cartesian coordinates | $\hat{x}_i = \frac{x_i}{L}$ |
| Time | $\hat{t} = \frac{t u_\infty}{L}$ |
| Mass density | $\hat{\rho} = \frac{\rho}{\rho_\infty}$ |
| Components of the momentum | $\hat{\rho}\hat{u}_i = \frac{\rho u_i}{\rho_\infty u_\infty}$ |
| Pressure | $\hat{p} = \frac{p}{\rho_\infty u_\infty^2}$ |
| Energy per volume | $\hat{\rho}\hat{E} = \frac{\rho e + u_i u_i}{\rho_\infty u_\infty^2}$ |
| Absolute temperature | $\hat{T} = \frac{T}{T_\infty}$ |
| Thermal conductivity | Depends on the law for $p$ and $T$ |
| Components of the stress tensor | Depends on the law for $\hat{\mu}$ |

Figure: Dimensionless quantities (for $i = 1, 2, 3$) for the characteristic quantities $L$, $\rho_\infty$, $u_\infty$, $T_\infty$

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- Abstract (dimensionless) formulation of a second order conservation law:

$$\frac{\partial U}{\partial \hat{t}} + \frac{\partial F_i(U)}{\partial \hat{x}_i} - \frac{1}{Re} \frac{\partial G_i(U, \nabla U)}{\partial \hat{x}_i} = 0 \tag{17}$$

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- The $F_i$ represent the *convectice fluxes* while the $G_i$ denote the *dissipative fluxes*
- For the application of the DG approach, only weak assumptions for concerning the structure of $F_i$ and $G_i$ are needed

# Navier Stokes in conservation form



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## Navier Stokes in conservation form

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  - Vector of unknowns

$$U(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{t}) = \begin{pmatrix} \hat{\rho} \\ \hat{\rho}\hat{u}_1 \\ \hat{\rho}\hat{u}_2 \\ \hat{\rho}\hat{u}_3 \\ \hat{\rho}\hat{E} \end{pmatrix} \tag{18}$$

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  - Convective and diffusive fluxes:

$$F_i(U) = \begin{pmatrix} \hat{\rho}\hat{u}_i \\ \hat{\rho}\hat{u}_i\hat{u}_1 + \delta_{1i}\hat{p} \\ \hat{\rho}\hat{u}_i\hat{u}_2 + \delta_{2i}\hat{p} \\ \hat{\rho}\hat{u}_i\hat{u}_3 + \delta_{3i}\hat{p} \\ \hat{u}_i(\hat{\rho}\hat{E} + \hat{p}) \end{pmatrix} \quad G_i(U, \nabla U) = \begin{pmatrix} 0 \\ \hat{\tau}_{i1} \\ \hat{\tau}_{i2} \\ \hat{\tau}_{i3} \\ \hat{u}_j\hat{\tau}_{ji} - \hat{\lambda}\frac{\partial \hat{T}}{\partial \hat{x}_i} \end{pmatrix} \tag{19}$$

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# Second order systems: Naive approach

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- Main difference compared to the scalar example: *Second order terms*

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# Overview

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    - Main advantage: Low computational cost per time-step

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- E.g. in [Gassner2009c] similar formulations for our case are shown and can be used to determine the time-step restriction

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- Local time-stepping
- Hp-adaptivity

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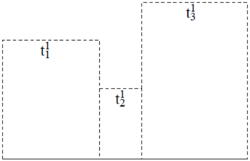


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Figure: Visualization of the local-time-stepping procedure (taken from [Gassner2009c])

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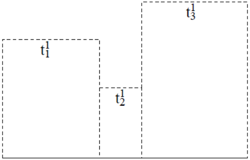


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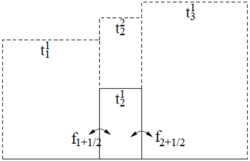


(b) Local advancement of the cell with the sharpest restriction

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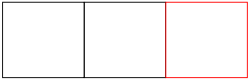

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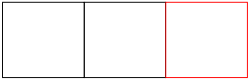

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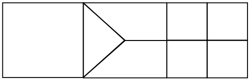

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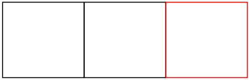

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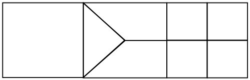

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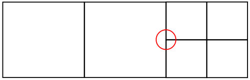

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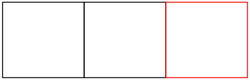

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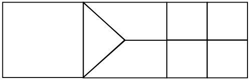

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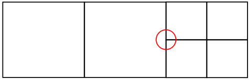

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- Due to the simple interaction between cells ($\rightarrow$ fluxes), DG schemes can (relatively) easily cope with so called *hanging nodes*

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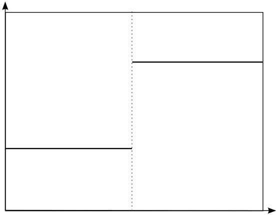

(a) Example of a discontinuous exact solution

# Local order adaptation: Order reduction near shocks



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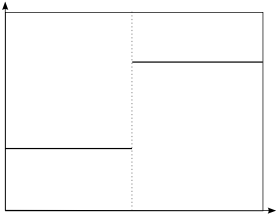
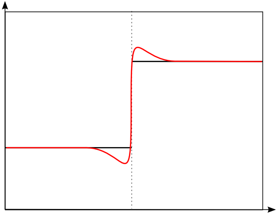


(a) Example of a discontinuous exact solution



(b) Gibbs phenomenon for a higher order approxmation

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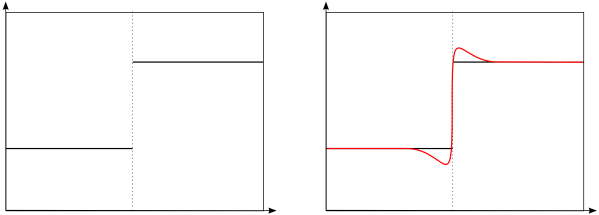

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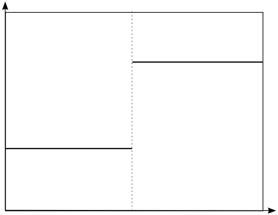

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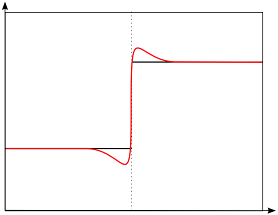

(b) Gibbs phenomenon for a higher order approxmation

- In FVM, for example, usually *slope limiters* are applied to avoid this behaviour
- The simple order reduction in DG is a much “cleaner” approach since these limiters decrease the local order of exactness anyway

# Local order adaptation: Order elevation



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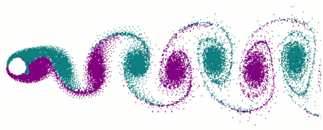
- Example: Kárman vortex street



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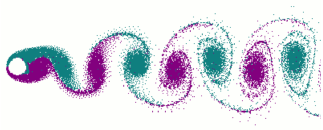

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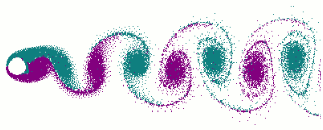

  - Whirls should be resolved very accurately but they move through the domain!
  - DG approach: Dynamically increase the local order near whirls and decrease it again if the whirl has moved on

## The end

Thank you for your attention!

Any questions?

# References I

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