Numerical Simulation of Compressible Flows with Immersed Boundaries Using Discontinuous Galerkin Methods

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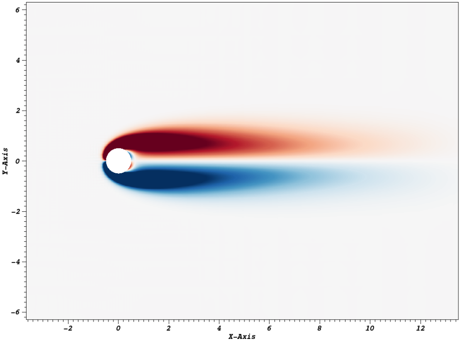

Outline

Introduction

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- Aim of this thesis:
  - Verification of CNS extension for BoSSS for inviscid flows
  - Validation for viscid flows
  - Both using immersed boundaries
- CNS extension of BoSSS (Bounded Support Spectral Solver) numerically solves the Compressible Navier-Stokes equations using a Discontinuous Galerkin method
- CNS has already been verified and validated for:
  - Inviscid flows using immersed boundaries by [1, Müller 2014], though not for the flow around a cylinder
  - Viscid flows using curved elements by [2, Ayers 2015]

Flow Properties

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- Compressible flow
- Ideal gas
  - Heat capacity ratio $\gamma = \dfrac{c_p}{c_v} = 1.4$
- Newtonian fluid
  - Stress tensor $\tau_{ij} = \mu \left[\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} \right) - \dfrac{2}{3} \dfrac{\partial u_k}{\partial x_k} \delta_{ij} \right]$
- Mach number $\mathrm{Ma} = \dfrac{v_\infty}{a_\infty} = 0.2$
- Reynolds number $\mathrm{Re} = \dfrac{\rho_\infty V_\infty L}{\mu_\infty} \propto \dfrac{\text{inertia forces}}{\text{viscous forces}}$
- Prandtl number $\mathrm{Pr} = \dfrac{\mu_\infty c_p}{k_\infty} \propto \dfrac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}$

Non-dimensional 2D Compressible Navier-Stokes Equations

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- 2D
- Non-dimensional conserved flow variables: density ρ, momentum ρu, ρv, energy ρE

$$\boxed{\frac{\partial U}{\partial t}} + \boxed{\left(\frac{\partial F_c^x(U)}{\partial x} + \frac{\partial F_c^y(U)}{\partial y}\right)} - \boxed{\left(\frac{\partial F_v^x(U,\nabla U)}{\partial x} + \frac{\partial F_v^y(U,\nabla U)}{\partial y}\right)} = 0$$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix} \quad F_c^x = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho E + p) \end{pmatrix} \quad F_c^y = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix}$$

- Temporal derivative
- Convective fluxes
- Viscous fluxes

$$F_v^x = \frac{1}{\mathrm{Re}} \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xx} u + \tau_{xy} v + \frac{\gamma}{\mathrm{Pr}(\gamma - 1)} \kappa \frac{\partial T}{\partial x} \end{pmatrix} \quad F_v^y = \frac{1}{\mathrm{Re}} \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{xy} u + \tau_{yy} v + \frac{\gamma}{\mathrm{Pr}(\gamma - 1)} \kappa \frac{\partial T}{\partial y} \end{pmatrix}$$

The Discontinuous Galerkin Space Discretisation

- Discrete weak formulation of $\dfrac{\partial c}{\partial t} + \nabla \cdot \mathbf{f}(c) = 0$
- Discretisation Ω_h of Ω
- Multiplication of PDE by set of cell-local test functions $\Phi_{i,j}$
- Integration over cell $\mathcal{K}_i$ and integration by parts
- Modal approximation for $c(\mathbf{x}, t) \mid_{\mathcal{K}_i} \approx \sum_{k=0}^{M} c_{i,k}(t)\Phi_{i,k}(\mathbf{x})$ with Galerkin approach (identical Ansatz and test functions)
- Discontinuous approach ➔ flux function $f = f(c^-, c^+, \mathbf{n})$

$$\Rightarrow \int\limits_{\mathcal{K}_i} \frac{\partial c_i}{\partial t}\Phi_{i,j}\, dV + \sum_{e=1}^{E_i} \int\limits_{\mathcal{E}_{i,e}} f\left(c^-, c^+, \mathbf{n}\right)\Phi_{i,j}\, dA - \int\limits_{\mathcal{K}_i} \mathbf{f}\left(c_i\right) \cdot \nabla\Phi_{i,j}\, dV = 0$$

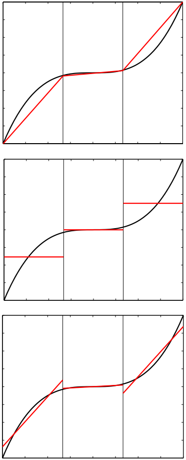

Figure: Adapted from [1]

The Immersed Boundary Method

- Division into
  - the physical region: $\mathcal{A} = \{\vec{x} \in \Omega_h : \varphi(\vec{x}) > 0\}$,
  - the void region: $\mathcal{B} = \{\vec{x} \in \Omega_h : \varphi(\vec{x}) < 0\}$,
  - the immersed boundary: $\mathcal{I} = \{\vec{x} \in \Omega_h : \varphi(\vec{x}) = 0\}$

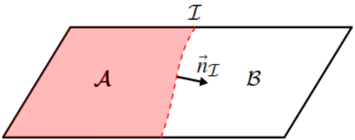

Figure: Cut cell with physical (red) and void region (white) [3]

- Restrict problem domain to physical region:

$$\int_{\mathcal{A}_i} \frac{\partial c_i}{\partial t} \Phi_{i,j} \, dV + \sum_{e=1}^{E_i} \int_{\mathcal{E}_{i,e}^{\mathcal{A}}} f\left(c^-, c^+, \mathbf{n}\right) \Phi_{i,j} \, dA + \int_{\mathcal{I}_i} f\left(c^-, c^+, \mathbf{n}_{\mathcal{I}}\right) \Phi_{i,j} \, dA - \int_{\mathcal{A}_i} \mathbf{f}\left(c_i\right) \cdot \nabla \Phi_{i,j} \, dV = 0$$

- Solution via explicit Euler time discretisation
  - Time step size depends on cell with smallest volume

    ➔ Cell agglomeration

Cell Agglomeration

- Agglomeration threshold $0 \leq \alpha \leq 1$
- Cells $\mathcal{K}_s^{\text{src}}$ with $\text{frac}(\mathcal{A}_i) = \frac{\text{meas}(\mathcal{A}_i)}{\text{meas}(\mathcal{K}_i)} \leq \alpha$ get agglomerated to neighbouring cell $\mathcal{K}_s^{\text{tar}}$
- Neighbouring cells are weakly coupled via fluxes
  ➔ basis $\vec{\Phi}_i$ can be extended from the target cell into the source cell

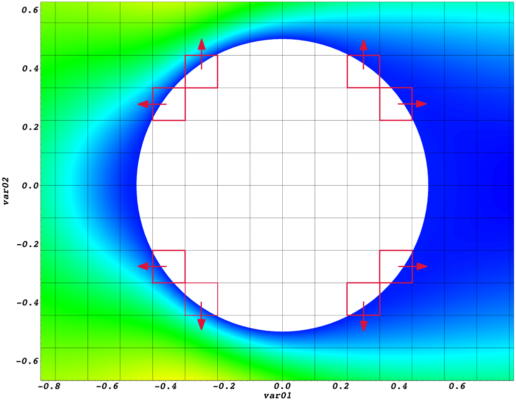

Figure: Cell agglomeration for α = 0.3

Problem Specification

- Supersonic inlet
- Adiabatic slip wall

- Flow properties:
  - Symmetrical flow
  - Isentropic inviscid flow with $\frac{p}{\rho^\gamma} = \text{const}$
    ➔ Entropy $s = 0$

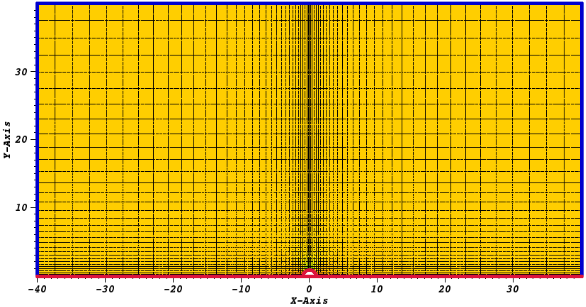

Figure: Mesh with 64 Cells per Direction

- Domain:
  - $0 \leq y \leq 40$
  - $-40 \leq x \leq 40$
  - Cylinder with radius $r = 1$ at $(0, 0)$
    ➔ Level set $\varphi = x^2 + y^2 - 1$ set as adiabatic slip wall

Robustness Study – Preparation

- Variation of polynomial degree $1 \leq P \leq 3$
- Variation of position of the cylinder's centre point from -0.075 to $+0.075$ with step size 0.015
  ➔ Level set $\varphi = (x - \text{shift})^2 + y^2 - 1$
- Mesh as shown before (64 cells per direction)
- Constant agglomeration threshold $\alpha = 0.5$
  ➔ several different cell agglomerations occur

Aim: Proving the robustness of the solver

Robustness Study – Evaluation

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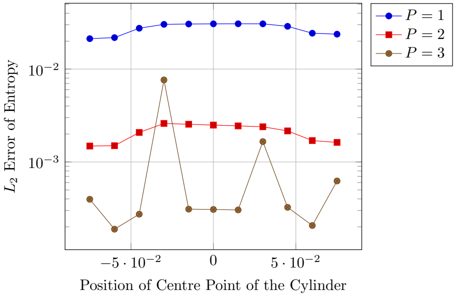
- Constant error of entropy for degrees 1 and 2
  ➔ Only slight influence by cell agglomeration
- Large error differences for degree 3
  - For some cases huge effort in getting calculations to work (early breakup)
  - Discordant cases should be examined more closely



➔ Sufficiently robust for lower degrees, higher degree calculations strongly influenced by cell agglomeration

Comvergence Study – Preparation

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- Variation of polynomial degree $0 \leq P \leq 4$
- Variation of mesh size: 32, 64 and 128 cells per direction
- Constant level set $\varphi = x^2 + y^2 - 1$
- Constant agglomeration threshold $\alpha = 0.3$

Aim: Proving convergence rate of $\mathcal{O}(h^{P+1})$

Convergence Study – Evaluation

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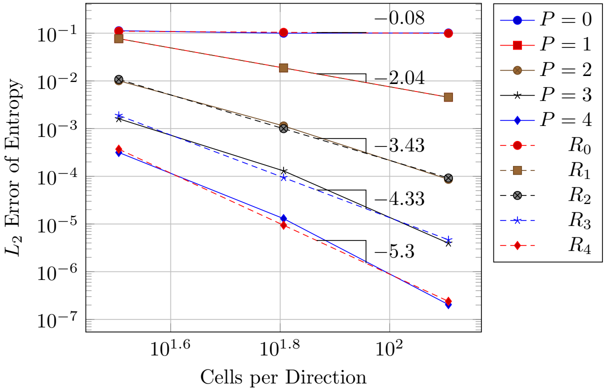
- Bad convergence rate for $P = 0$
- Convergence of roughly $\mathcal{O}(h^{P+1})$ for $1 \leq P \leq 4$



Convergence rate for higher degrees as expected

➔ Convergence verified

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Theory – Viscid Flow Around a Cylinder

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- $Re \leq 40 - 50$: laminar steady regime
- $40 - 50 \leq Re \leq 190$: laminar vortex shedding (Kármán vortex street)
- $190 \leq Re$: increasing 3D effects
- Characteristic values:
  - Coefficient of drag C_D
  - Coefficient of lift C_L
  - Wake separation length W^*
  - Strouhal number (frequency of vortex shedding)

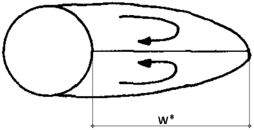

Figure: Laminar steady regime [4]

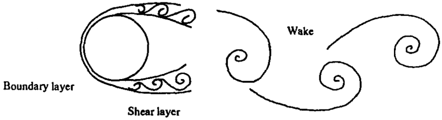

Figure: Kármán Vortex Street [4]

Simulation Properties

- Domain:
  - $-20 \leq x \leq 20$
  - $-20 \leq y \leq 20$
  - Cylinder with radius $r = 0.5$ at $(0, 0)$
    ➔ Level set $\varphi = x^2 + y^2 - 0.25$ set as isothermal wall
- Variation of mesh size: 40, 60 and 80 cells per direction
- Variation of polynomial degree $1 \leq P \leq 3$
- Constant agglomeration threshold $\alpha = 0.3$

➤ Supersonic inlet

➤ Isothermal wall

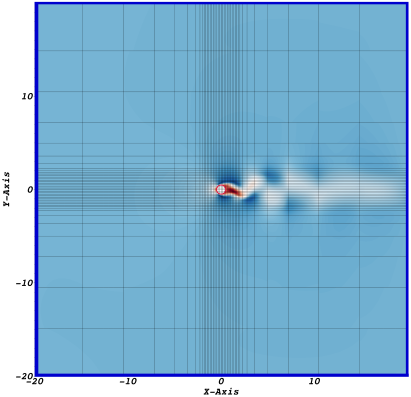

Figure: Coarsest mesh with 40×40 cells

Compared Values

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- Steady flow simulations for Re = 20 and Re = 40:
  - Coefficient of drag C_D
  - Wake separation length W^*

    ➔ found from evaluating x at $y = 0$ where x-velocity $u = 0$
- Unsteady flow simulations for Re = 100 and Re = 200:
  - Coefficient of drag C_D
  - Coefficient of lift C_L
  - St found from evaluating frequency of C_L

$$C_D = \frac{d}{q_\infty L_\infty}$$

$$C_L = \frac{l}{q_\infty L_\infty}$$

$$\mathrm{St} = \frac{fL_\infty}{V_\infty}$$

d : drag force

l : lift force

q_∞ : dynamic pressure

L_∞ : cylinder diameter

V_∞ : flow velocity

Simulation at Re = 20 I

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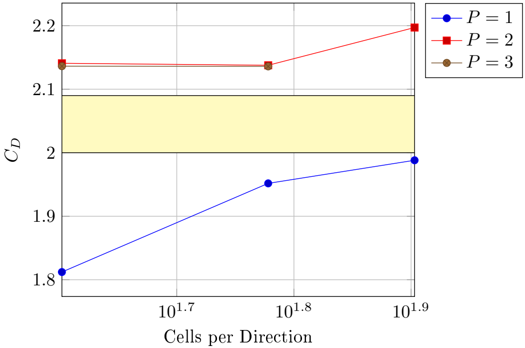
Re = 20	Source	2D/3D	W^*	C_D
Numerical – Incompressible	Dennis and Chang	2D	0.94	2.05
	Fornberg	2D	0.91	2.00
	Linnick and Fasel	2D	0.93	2.06
Experimental	Coutanceau and Bouard	-	0.93	-
	Tritton	-	-	2.09
Numerical – Compressible	Brehm, Hader and Fasel (Ma = 0.1)	3D	0.96	2.02
	Ayers	2D	0.975	2.06
	Present Results:	**2D**	**0.928**	**2.136**

- W^* coincides with literature, C_D much too high

Simulation at Re = 20 II

- None of the values lies in the range found in literature
- Simulation for P = 2, CpD = 80 should be examined again
  ➔ perhaps mistake during simulation (e.g. at restart)
- Overall, P = 1 behaviour as expected, P = 2 and 3 yield much too high results



Simulation at Re = 40 I

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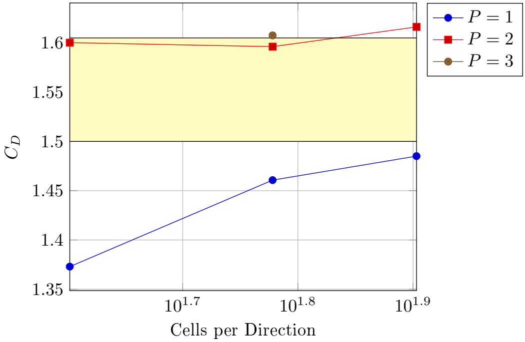
Re = 40	Source	2D/3D	W^*	C_D
Numerical – Incompressible	Dennis and Chang	2D	2.35	1.52
	Fornberg	2D	2.24	1.50
	Linnick and Fasel	2D	2.28	1.54
Experimental	Coutanceau and Bouard	-	2.13	-
	Tritton	-	-	1.59
Numerical – Compressible	Brehm, Hader and Fasel (Ma = 0.1)	3D	2.26	1.51
	Ayers	2D	2.250	1.605
	Present Results:	**2D**	**2.201**	**1.608**

- W^* coincides with literature, C_D slightly high

Simulation at Re = 40 II

- Better results than for Re=20
- Simulation for $P = 2$, CpD = 80 once again yields discordant values
- $P = 2$ and $P = 3$ quite high compared to literature



Comparison of Re = 20 and Re = 40

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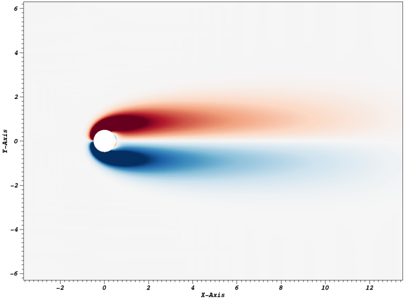
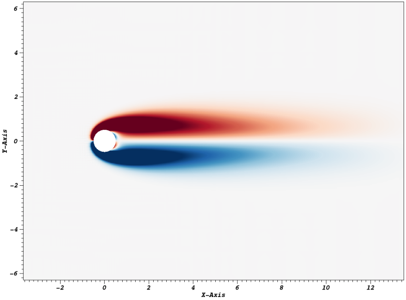
	W^*	C_D
Re = 20	0.928	2.136
Re = 40	2.201	1.608



(a) Re = 20



(b) Re = 40

Simulation at Re = 100 I

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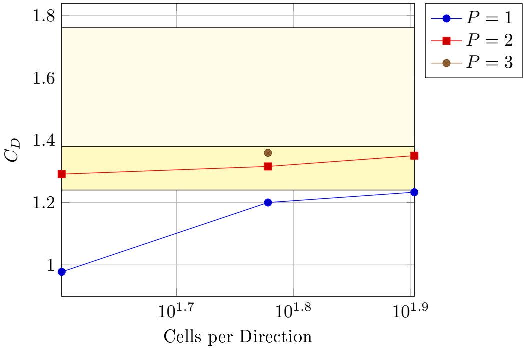
Re = 100	Source	2D/3D	St	C_D	C_L
Numerical – Incompressible	Gresho, Chan, Lee, et al.	2D	0.18	1.76	-
	Linnick and Fasel (λ = 0.056)	2D	0.169	1.38 ± 0.010	± 0.337
	Linnick ad Fasel (λ = 0.023)	2D	0.1696	1.34 ± 0.009	± 0.333
	Persillon and Braza	2D	0.165	1.253	-
	Saiki and Biringen	2D	0.171	1.26	-
	Persillon and Braza	3D	0.164	1.240	-
	Liu, Zheng and Sung	3D	0.165	1.35 ± 0.012	± 0.339
Experimental	Berger and Wille	-	$0.16 - 0.17$	-	-
	Clift, Grace and Weber	-	-	1.24	-
	Williamson	-	0.164	-	-
Numerical – Compressible	Brehm, Hader and Fasel (Ma = 0.1)	3D	0.165	1.32 ± 0.01	± 0.32
	Ayers	2D	0.167	1.371 ± 0.011	± 0.333
	Present Results:	**2D**	**0.1669**	$\mathbf{1.3593 \pm 0.00805}$	$\mathbf{\pm 0.3291}$

- All values coincides with literature

Simulation at Re = 100 II

- Higher degree values lie in expected range
- Values which should be most accurate (P = 2, CpD = 80 and P = 3, CpD = 60) produce rather high results
- P = 2 seems to converge against P = 3, CpD = 60 value

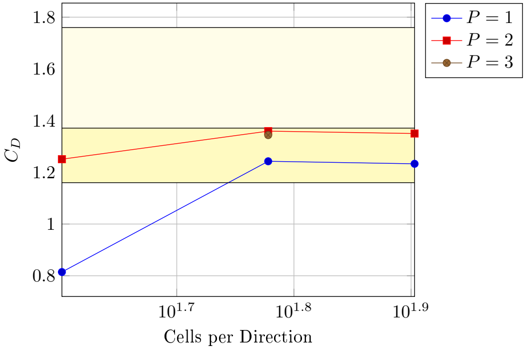


Simulation at Re = 200 I

Re = 200	Source	2D/3D	St	C_D	C_L
Numerical – Incompressible	Belov, Martinelli and Jameson	2D	0.193	1.19 ± 0.042	± 0.64
	Gresho, Chan, Lee et al.	2D	0.21	1.76	-
	Linnick and Fasel (λ = 0.056)	2D	0.199	1.37 ± 0.046	± 0.70
	Linnick and Fasel (λ = 0.023)	2D	0.197	1.34 ± 0.044	± 0.69
	Miyake, Sakamoto, Tokunaga et al.	2D	0.196	1.34 ± 0.043	± 0.67
	Persillon and Braza	2D	0.198	1.321	-
	Saiki and Biringen	2D	0.197	1.18	-
	Persillon and Braza	3D	0.181	1.306	-
	Liu, Zheng and Sung	3D	0.192	1.31 ± 0.049	± 0.69
Experimental	Berger and Wille	-	$0.18 - 0.19$	-	-
	Clift, Grace and Weber	-	-	1.16	-
	Williamson	-	0.181	-	-
Numerical – Compressible	Brehm, Hader and Fasel (Ma = 0.1)	3D	0.192	1.3 ± 0.04	± 0.66
	Ayers	2D	0.201	1.371 ± 0.011	± 0.70
	Present Results:	**2D**	**0.2002**	$\mathbf{1.344 \pm 0.0462}$	$\mathbf{\pm 0.6887}$

Simulation at Re = 200 II

- All values seen in table coincide with literature
- Compared to 3D results by [5, Brehm et al.] all values slightly higher
- All C_D values (except for the least accurate one) lie in expected range
- $P = 2$ seems to converge against higher value than $P = 1$



Comparison of Re = 100 and Re = 200 I

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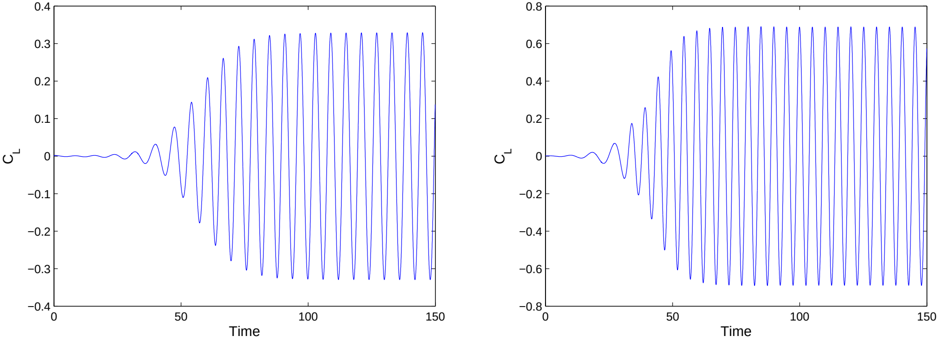

- Higher amplitude and frequency for C_L at Re = 200 compared to Re = 100
- Steady state is reached earlier

Comparison of Re = 100 and Re = 200 II

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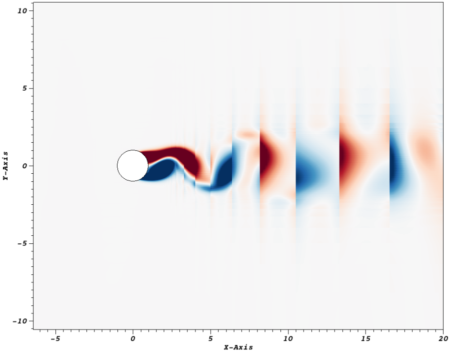

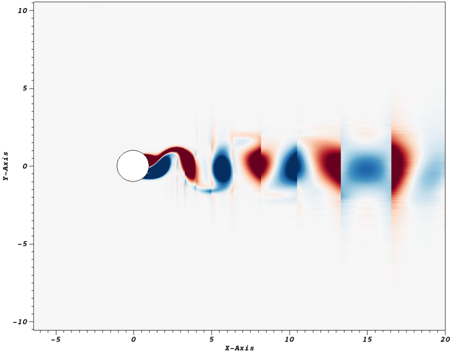

	St	C_D	C_L
Re = 100	0.1669	1.359 ± 0.00805	± 0.3291
Re = 200	0.2002	1.344 ± 0.0462	± 0.6887

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Summary

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- Inviscid flow – Robustness:
  - Satisfying for lower degrees
  - Improvable for high-order degrees
- Inviscid flow – Convergence:
  - Convergence rate of $\mathcal{O}(h^{P+1})$ proven
- Viscid flow – Laminar steady regime
  - Wake separation length accurate
  - Strange behaviour for C_D values at $P = 2$ simulations
  - Re = 40 has yielded better results than Re = 20
  - Comparison showed physically correct behaviour
- Viscid flow – Laminar vortex shedding
  - Accurate results
  - $P = 1$ and $P = 2$ seem to converge against different values
  - Comparison showed physically correct behaviour

Outlook for Future Works

- Improvement of robustness
- Examination of agglomeration error
- Detailed convergence study for viscid flows
- Examination of 3D influences at higher Reynolds numbers

The End

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videos ende, fragen

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Appendix

Robustness Study

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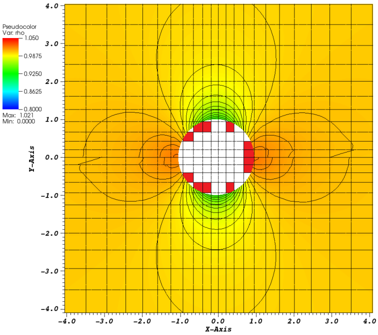
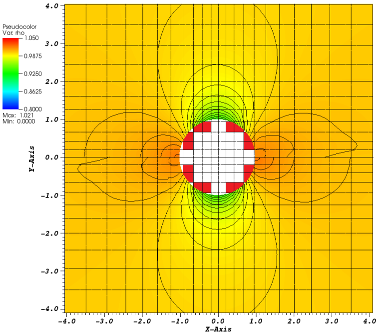


(c) Degree 2, shift -0.06



(d) Degree 2, shift -0.03

- Visualised agglomerated cells

Degrees of Freedom for Viscid Simulations

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DoFs		CpD		
		40	60	80
DG	1	4800	10800	19200
	2	9600	21600	38400
	3	16000	36000	64000

Table: Degrees of Freedom for Different Simulation Properties

Values for Re = 20 I

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C_D		CpD		
		40	60	80
DG	1	1.812	1.952	1.988
	2	2.141	2.138	2.197
	3	2.136	2.136	-

Table: C_D Values for Each Simulation (Re = 20)

W^*		CpD		
		40	60	80
DG	1	0.956	1.044	0.927
	2	0.887	0.943	0.916
	3	0.921	0.928	-

Table: Wake Separation Lengths for Each Simulation (Re = 20)

Values for Re = 20 II

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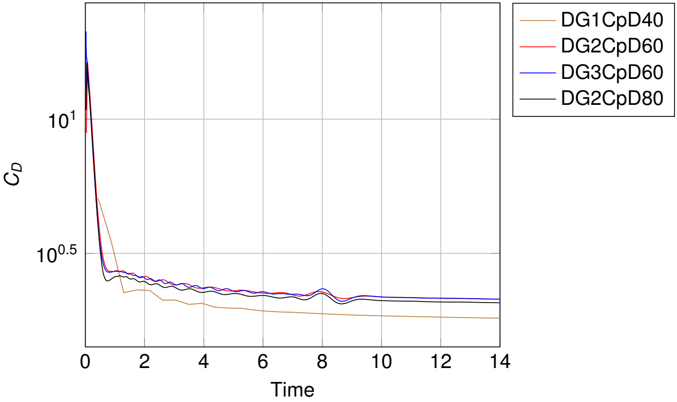

Figure: Coefficient of Drag over Time (Re = 20)

Values for Re = 40 I

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C_D		CpD		
		40	60	80
DG	1	1.373	1.461	1.485
	2	1.600	1.596	1.616
	3	-	1.608	-

Table: C_D Values for Each Simulation (Re = 40)

W^*		CpD		
		40	60	80
DG	1	2.342	2.338	2.236
	2	2.115	2.182	2.182
	3	-	2.201	-

Table: Wake Separation Lengths for Each Simulation (Re = 40)

Values for Re = 40 II

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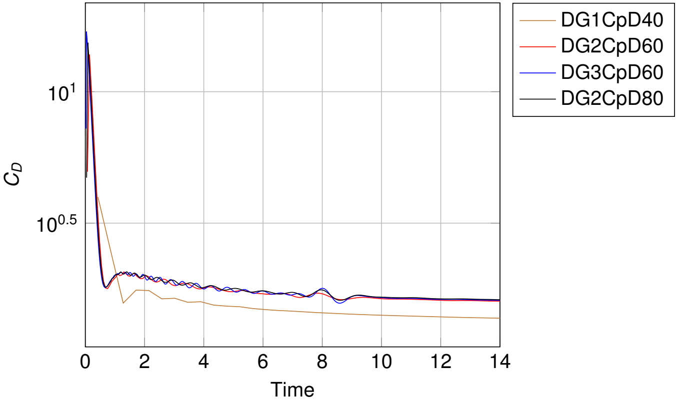

Figure: Coefficient of Drag over Time (Re = 40)

Values for Re = 100 I

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C_D		CpD 40	CpD 60	CpD 80
DG	1	0.9777 ± 0.0003	1.2 ± 0.0834	1.233 ± 0.0118
DG	2	1.291 ± 0.0082	1.3156 ± 0.0089	1.3501 ± 0.0079
DG	3	-	1.3593 ± 0.00805	-

Table: C_D Values for Each Simulation

C_L		CpD 40	CpD 60	CpD 80
DG	1	± 0.00155	± 0.291	± 0.2789
DG	2	± 0.2672	± 0.3154	± 0.3135
DG	3	-	± 0.3291	-

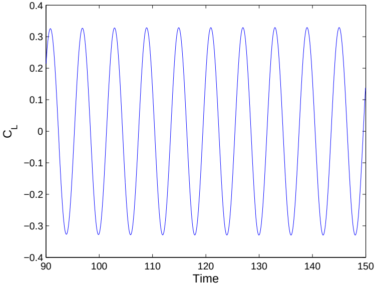
Table: C_L Values for Each Simulation

St		CpD 40	CpD 60	CpD 80
DG	1	0.1001	0.1506	0.1502
DG	2	0.1669	0.1669	0.1670
DG	3	-	0.1669	-

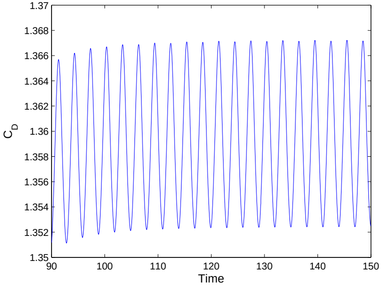
Table: Strouhal Numbers for Each Simulation

fdy

Values for Re = 100 II



(a) Lift Coefficient over Time for $90\,\mathrm{s} < t < 150\,\mathrm{s}$



(b) Drag Coefficient over Time for $90\,\mathrm{s} < t < 150\,\mathrm{s}$

Values for Re = 200 I

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C_D		CpD		
		40	60	80
DG	1	0.8144 ± 0.0028	1.2427 ± 0.0281	1.2256 ± 0.0309
	2	1.2508 ± 0.0339	1.3593 ± 0.0080	1.3501 ± 0.0079
	3	-	1.344 ± 0.0462	-

Table: C_D Values for Each Simulation

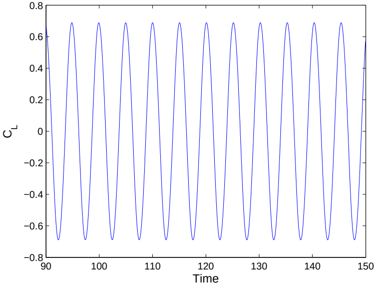
C_L		CpD		
		40	60	80
DG	1	$\pm 3.2629 \cdot 10^{-5}$	± 0.5304	± 0.2789
	2	± 0.5653	± 0.6433	± 0.6376
	3	-	± 0.6887	-

Table: C_L Values for Each Simulation

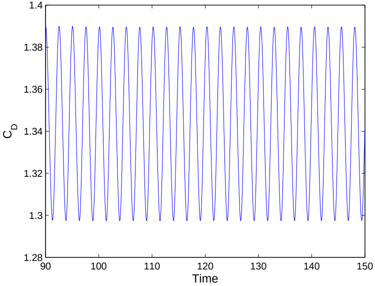
St		CpD		
		40	60	80
DG	1	0	0.1836	0.1838
	2	0.2003	0.2002	0.2002
	3	-	0.2002	-

Table: Strouhal Numbers for Each Simulation

Values for Re = 200 II



(c) Lift Coefficient over Time for $90\,\mathrm{s} < t < 150\,\mathrm{s}$



(d) Drag Coefficient over Time for $90\,\mathrm{s} < t < 150\,\mathrm{s}$