Embedding equitable (s,p)-edge-colorings of K_n

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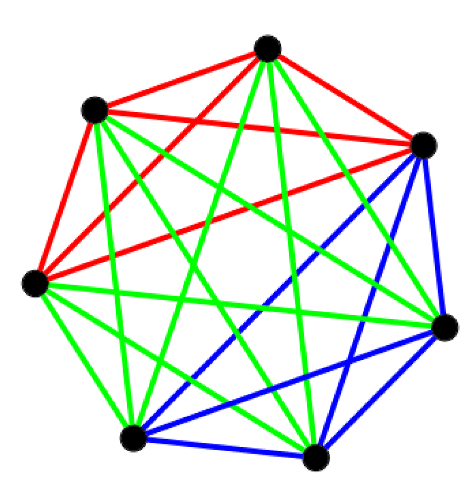
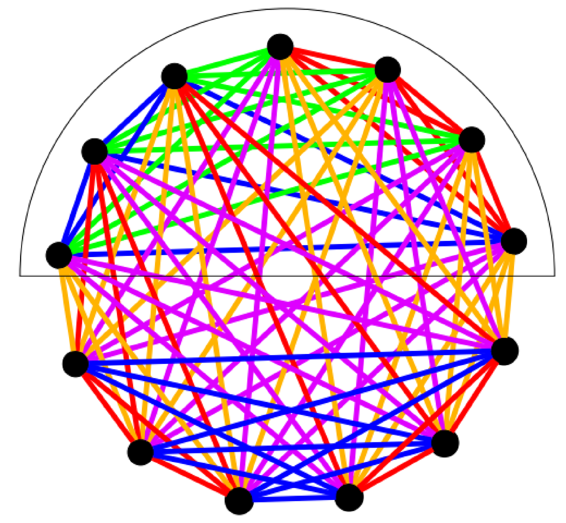
Scheduling Problem

- There is a $s_1 = 3$ hour conference with $n_1 = 7$ guests. Each guest gets a 1 hour break and has $p_1 = 2$ hours to meet with the 6 other guests, so they meet with $b(n) = 6/2 = 3$ other guests every hour.
- The conference expands to $n_2 = 13$ guests and $s_2 = 5$ hours. Can we make a new schedule keeping the old where guests still meet with 3 people per hour?
- We represent this problem with a graph: the guests are vertices, meetings are edges, and hours are colors.
- This problem is an application of embedding an equitable (s_1, p_1)-edge coloring of K_{n_1} in an equitable (s_2, p_2)-edge-coloring of K_{n_2}.



Equitable $(3,2)$-edge-coloring of K_7



Equitable $(5,4)$-edge coloring of K_{13} after detachment

Key Definitions

- An **equitable (s,p)-edge-coloring** is a coloring of the edges of a graph with s total colors so that each vertex has p different colors incident to it, and the number of edges of each color incident to a vertex differ by at most 1.
- An **amalgamation** of a graph G is defined by the function, ϕ, from the vertices of G to the vertices of H where if there is an edge $\{x,y\}$ in G, then there is an edge $\{\phi(x), \phi(y)\}$ in H.
- We say an equitable (s_1, p_1) edge-coloring of G_1, say W_1, can be **embedded** into an equitable (s_2, p_2)-edge coloring of G_2, say W_2, if there exists an equitable (s_2, p_2)-edge coloring of G_2 with a proper subgraph of G_1 with coloring W_1.
- If H is embedded in G, we call the colors used in the coloring of H **old colors**, and call colors used in G but not H **new colors**.
- If a color appears at a vertex, define $b(n)$ to be the number of edges of that color incident to the vertex.
- $\chi'_p(G)$ is the smallest integer s for which there exists an equitable (s,p)-edge-coloring of G.

Interesting Case

Theorem[2, 3, 4]: For $1 \le p \le n-1$,

$$\chi'_p(K_n) = \begin{cases} p+1 & \text{if } p \text{ even and } n \equiv p+1 \pmod{2p} \\ p & \text{otherwise.} \end{cases}$$

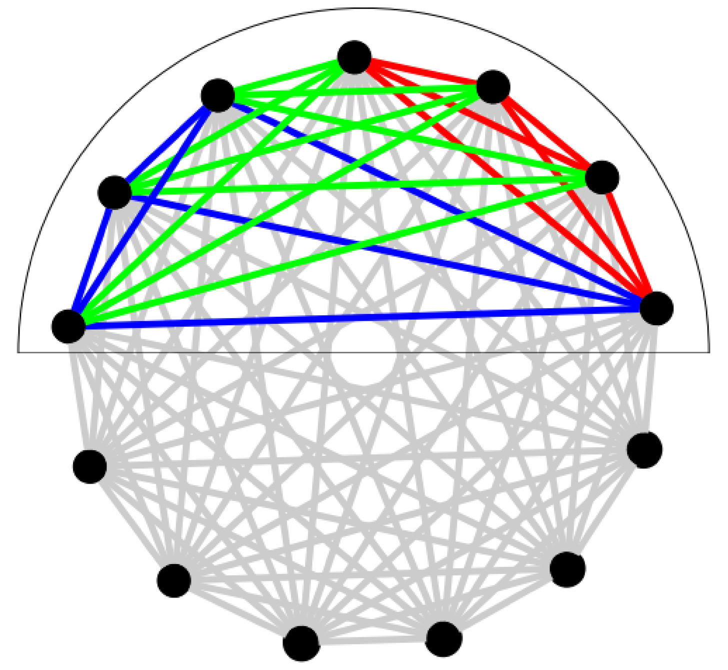
For the remainder of this poster, assume we are in the first case. We let $n_1 = 2t_1(b(n))+1$ and $n_2 = 2t_2(b(n))+1$ where $b(n)$ is an odd positive integer and t_1 and t_2 are positive integers.

Embedding when $t_2 = 2t_1$

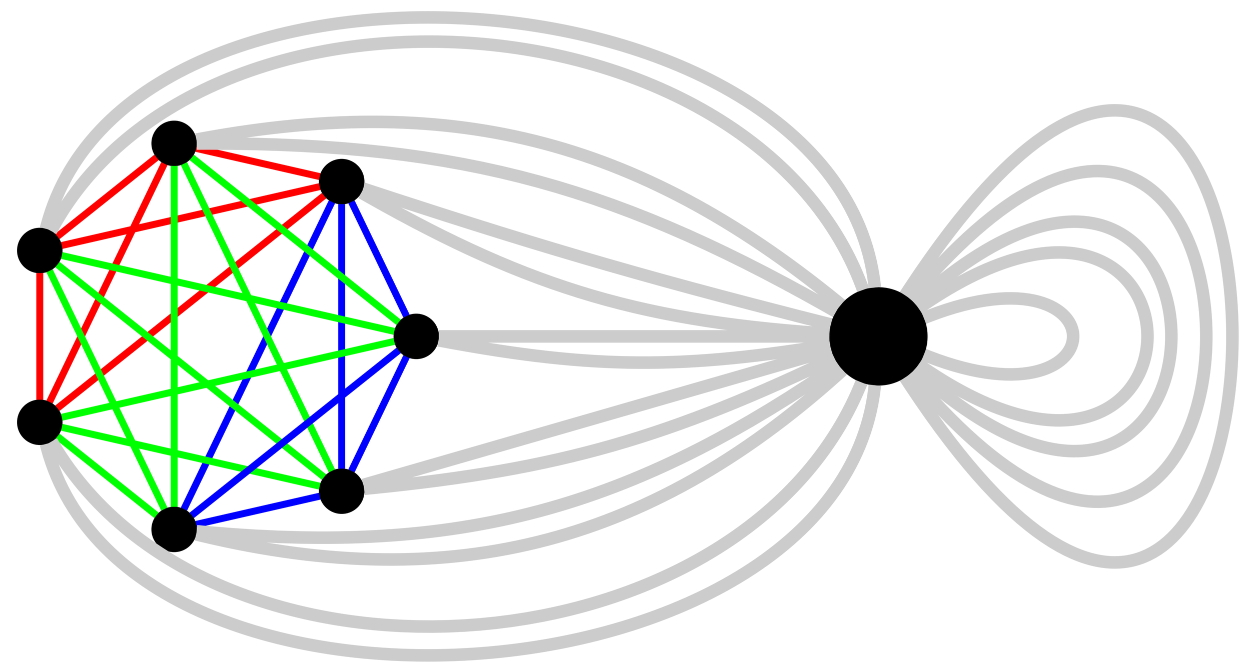
Theorem: An equitable $(2t_1+1, 2t_1)$-edge-coloring of K_{n_1} can be embedded in an equitable $(2t_2+1, 2t_2)$-edge-coloring of K_{n_2} when $t_2 = 2t_1$.

When $t_2 = 2t_1$, we can use the **Single Amalgamation Method** to create an edge coloring that satisfies our conditions. The general steps are:

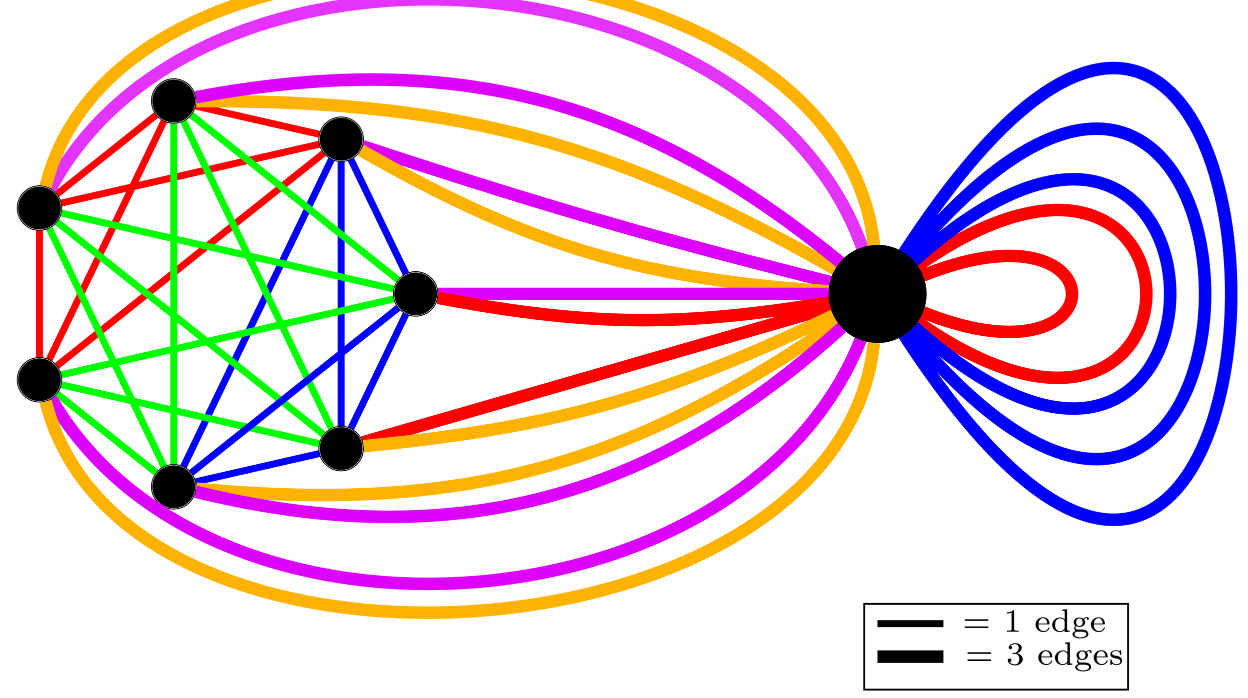
1. Start with an equitable (s_1, p_1)-edge coloring of K_{n_1} and add $n_2 - n_1$ vertices so we have a total of n_2 vertices. We will demonstrate this general process with the specific example: $n_1 = 7$, $s_1 = 3$, $p_1 = 2$, $n_2 = 13$, $s_2 = 5$, $p_2 = 4$, $b(n) = 3$.



2. Amalgamate the new vertices.

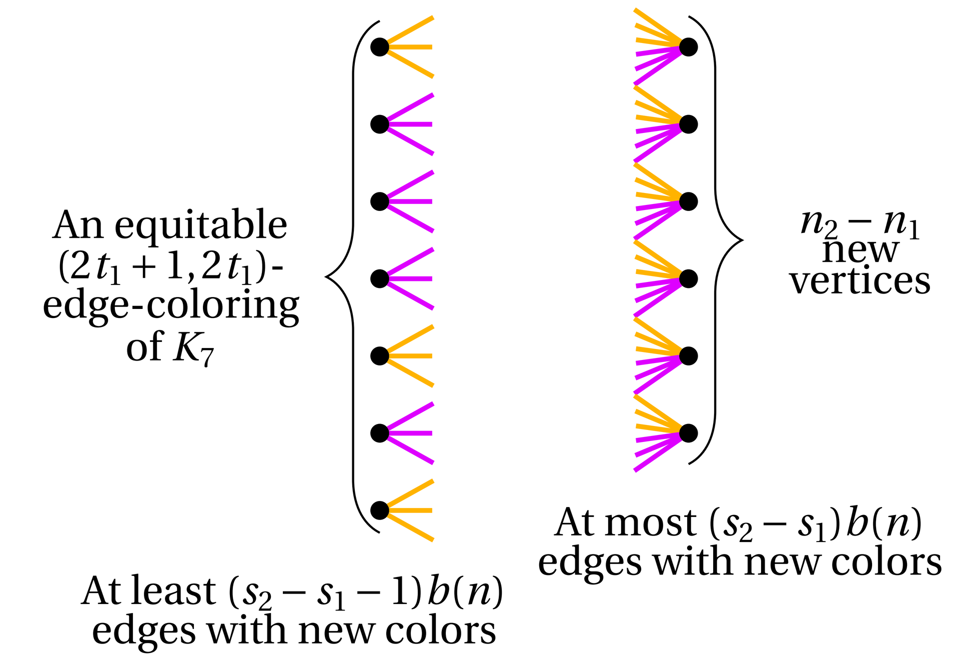


3. Color the edges of the new graph.



4. We can use a theorem in [1] to detach the amalgamated vertices, which leaves us with an equitable (s_2, p_2)-edge-coloring of K_{n_2}.

Necessary Condition



So, $n_1(s_2 - s_1 - 1)b(n) \le (n_2 - n_1)(s_2 - s_1)b(n)$. However, the maximum and minimum can not be achieved at the same time, so we are left with the following inequality. If we have an embedding, then

$$n_1(s_2 - s_1 - 1) \le (n_2 - n_1)(s_2 - s_1) - 1.$$

For positive integers, this reduces to $t_2 \ge 2t_1$.

Main Theorem

Theorem: An equitable $(2t_1+1, 2t_1)$-edge-coloring of K_{n_1} can be embedded in an equitable $(2t_2+1, 2t_2)$-edge-coloring of K_{n_2} if and only if $t_2 \ge 2t_1$.

- When $2t_1 < t_2 \le 4t_1$ we use another more complicated amalgamation technique to complete the embedding.
- When $4t_1 < t_2$ we use an algorithm that utilizes repeated applications of the two techniques.

Future Work

- Consider $n \not\equiv p+1 \pmod{2p}$
- Consider embeddings where p is fixed instead of $b(n)$
- Look at the structure of the colorings

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References

[1] M. A. Bahmanian and C. A. Rodger, *Multiply balanced edge colorings of multigraphs*, J. Graph Theory, 70 (2012), pp. 297–317.

[2] A. Hilton and D. de Werra, *A sufficient condition for equitable edge-colourings of simple graphs*, Discrete Math., 128 (1994), pp. 179–201.

[3] S. Li, E. B. Matson, and C. A. Rodger, *Extreme equitable block-colorings of C_4-decompositions of $K_v - F$*, Australas. J. Combin., 71 (2018), pp. 92–103.

[4] S. Li and C. A. Rodger, *Equitable block-colorings of C_4-decompositions of $K_v - F$*, Discrete Math., 339 (2016), pp. 1519–1524.

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