Embedding Equitable (s, p)-Edge-Colorings of K_n

Stafford Yerger

Joint work with Mika Olufemi
Mentor: Dr. Stacie Baumann

College of Charleston

10/18/24

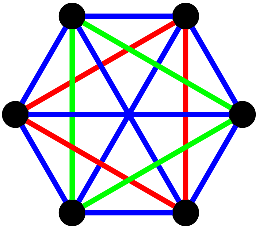
Acknowledgments

- Stafford Yerger
  - CofC SSME summer research grant funded this research project
- Mika Olufemi
  - The research reported in this presentation was supported by CofC SURF (Grant No. SU2024-21) and the College of Charleston Honors College
- Dr. Stacie Baumann
  - Supported by both of the SURF and SSME grants listed above

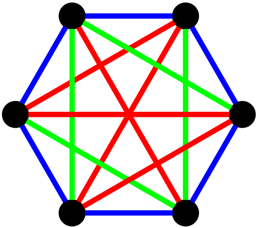
Definitions

A graph G is said to have an $\boldsymbol{(s,p)}$ **-edge-coloring** $E : E(G) \mapsto C = \{1,2,\ldots,s\}$ if:

1. the edges in $E(G)$ are colored with exactly s colors, and
2. for each vertex $u \in V(G)$, the edges incident to u are colored using exactly p colors.



A (3,2)-edge-coloring of K_6



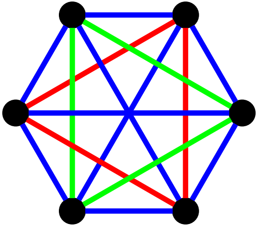
A (3,3)-edge-coloring of K_6

Definitions

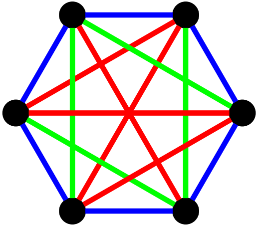
The (s,p)-edge-coloring, is said to be **equitable** if

3. for each vertex $u \in V(G)$ and for each $\{i,j\} \subset C(u)$, $|b(u,i) - b(u,j)| \le 1$,

where $C(u) = \{i \mid \text{there is an edge incident to } u \text{ colored } i\}$, and $b(u,i)$ is the number of edges incident to u that are colored i.



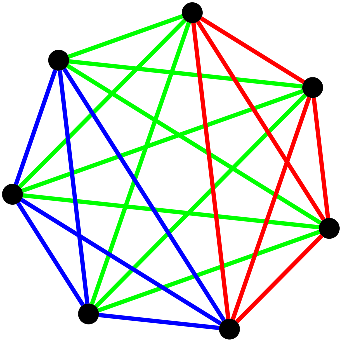
An equitable $(3,2)$-edge coloring of K_6



An equitable $(3,3)$-edge coloring of K_6

Scheduling Problem

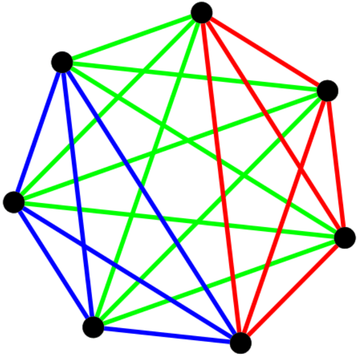
- There is a $s_1 = 3$ hour conference with $n_1 = 7$ guests. Each guest gets a 1 hour break and has $p_1 = 2$ hours to meet with the 6 other guests, so they meet with $b(n) = 6/2 = 3$ other guests every hour.



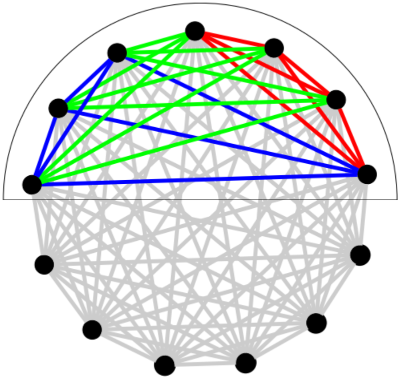
Equitable (3,2) *edge-coloring of* K_7

Scheduling Problem

- The conference expands to $n_2 = 13$ guests and $s_2 = 5$ hours. Can we make a new schedule keeping the old where guests still meet with 3 people per hour?



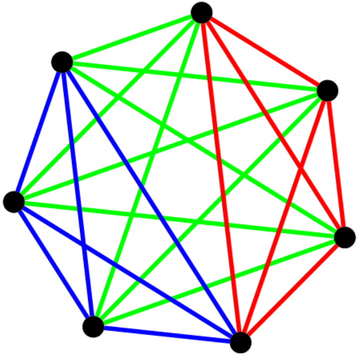
Equitable (3,2) edge-coloring of K_7



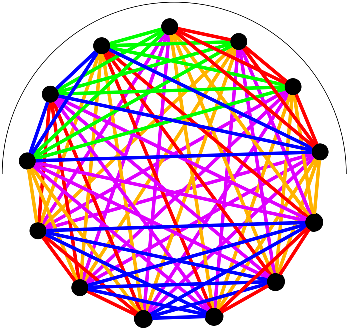
K_{13}

Embedding

We say an equitable (s_1, p_1)-edge-coloring of G_1 say W_1 can be *embedded* into an equitable (s_2, p_2)-edge-coloring of G_2 say W_2 if there exists an equitable (s_2, p_2) edge-coloring of G_2 with a proper subgraph G_1 with coloring W_1.



Equitable $(3, 2)$ edge-coloring of K_7



Equitable $(5, 4)$ edge-coloring of K_{13} after detachment

Interesting Case

Let $\chi'_p(n)$ be the smallest integer s such that there exists an equitable (s,p)-edge-coloring of K_n.

Theorem (Li, Matson, Rodger, 2018)

For any edge-coloring (K_2-decomposition) of K_n,

$$\chi'_p(n) = \begin{cases} p+1 & \text{if } n \equiv p+1 \pmod{2p} \\ p & \text{otherwise.} \end{cases}$$

This case is interesting because when $s > p$ interchanging along a 2-edge-colored path is not a viable approach.

For the remainder of this talk, assume we are in the first case. We let $n_1 = 2t_1(b(n))+1$ and $n_2 = 2t_2(b(n))+1$ where $b(n)$ is an odd positive integer and t_1 and t_2 are positive integers where $p_1 = 2t_1$ and $p_2 = 2t_2$.

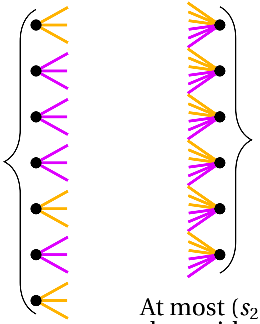
Main Theorem

Theorem (Baumann, Olufemi, Y.)

An equitable $(2t_1+1, 2t_1)$*-edge-coloring of* K_{n_1} *can be embedded in an equitable* $(2t_2+1, 2t_2)$*-edge-coloring of* K_{n_2} *if and only if* $t_2 \geq 2t_1$*.*

- $t_2 \geq 2t_1$ is necessary
- When $2t_1 = t_2$ the embedding is possible by the single amalgamation method.
- When $2t_1 < t_2 \leq 4t_1$ we use another more complicated amalgamation technique to complete the embedding.
- When $4t_1 < t_2$ we use an algorithm that utilizes repeated applications of the two techniques.

Necessary Condition



So, $n_1(s_2 - s_1 - 1)b(n) \leq (n_2 - n_1)(s_2 - s_1)b(n)$. However, the maximum and minimum can not be achieved at the same time, so we are left with the following inequality. If we have an embedding, then

$$n_1(s_2 - s_1 - 1) \leq (n_2 - n_1)(s_2 - s_1) - 1.$$

For positive integers, this reduces to $t_2 \geq 2t_1$.

Main Theorem

Theorem (Baumann, Olufemi, Y.)

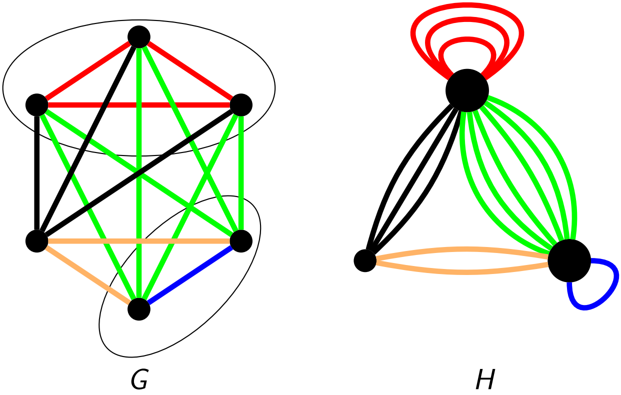
An equitable $(2t_1+1, 2t_1)$-edge-coloring of K_{n_1} can be embedded in an equitable $(2t_2+1, 2t_2)$-edge-coloring of K_{n_2} if and only if $t_2 \geq 2t_1$.

✓ $t_2 \geq 2t_1$ is necessary

- When $2t_1 = t_2$ the embedding is possible by the single amalgamation method.
- When $2t_1 < t_2 \leq 4t_1$ we use another more complicated amalgamation technique to complete the embedding.
- When $4t_1 < t_2$ we use an algorithm that utilizes repeated applications of the two techniques.

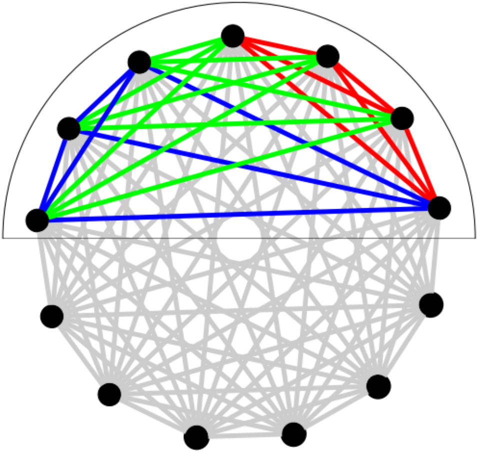
Amalgamation Definitions

The amalgamation of a graph G defined by the amalgamation function $\phi : V(G) \to V(H)$ is the graph H, possibly with multiple edges and loops, with vertex set $V(H)$ and the multiset of edges $E(H) = \{\{\phi(a), \phi(b)\} \mid \{a, b\} \in E(G)\}$, where $\{\phi(a), \phi(a)\}$ denotes a loop on vertex $\phi(a)$.

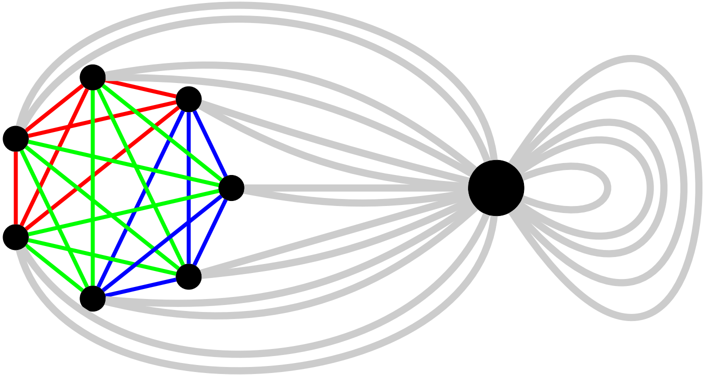


Step One

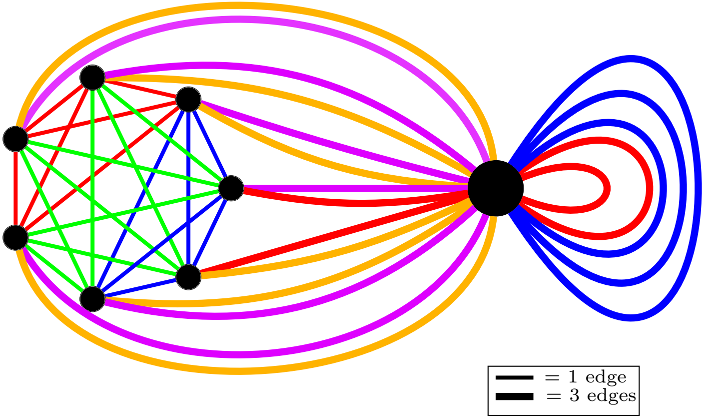
Start with an equitable (s_1, p_1)-edge coloring of K_{n_1} and add $n_2 - n_1$ vertices so we have a total of n_2 vertices. We will demonstrate this general process with the specific example: $n_1 = 7,\ s_1 = 3,\ p_1 = 2,\ n_2 = 13,\ s_2 = 5,\ p_2 = 4,\ b(n) = 3.$



Step 2: Amalgamate the new vertices

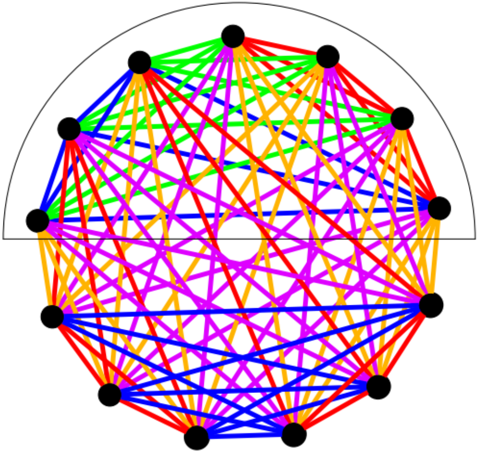


Step 3: Color the edges of the new graph



Final Step

We can use a theorem (Bahmanian, Rodger) to detach the amalgamated vertices, which leaves us with an equitable (s_2, p_2)-edge-coloring of K_{n_2}.



Equitable (5,4) edge-coloring of K_{13} after detachment

Main Theorem

Theorem (Baumann, Olufemi, Y.)

An equitable $(2t_1+1, 2t_1)$-edge-coloring of K_{n_1} can be embedded in an equitable $(2t_2+1, 2t_2)$-edge-coloring of K_{n_2} if and only if $t_2 \geq 2t_1$.

- ✓ $t_2 \geq 2t_1$ is necessary
- ✓ When $2t_1 = t_2$ the embedding is possible by the single amalgamation method.
- When $2t_1 < t_2 \leq 4t_1$ we use another more complicated amalgamation technique to complete the embedding.
- When $4t_1 < t_2$ we use an algorithm that utilizes repeated applications of the two techniques.

Future Work

- Consider $n \not\equiv p+1 \pmod{2p}$
- Consider embeddings where p is fixed instead of $b(n)$
- Look at the structure of the colorings