LECTURE NOTES

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DUAL LINEAR PROGRAMMING PROBLEMS

Introduction

For every linear programming problem there is a corresponding linear programming problem called the **dual**. If the original problem is a maximization problem then the dual problem is minimization problem and if the original problem is a minimization problem then the dual problem is maximization problem. In either case the final table of the dual problem will contain both the solution to the dual problem and the solution to the original problem.

The solution of the dual problem is readily obtained from the original problem solution if the simplex method is used.

The formulation of the dual problem also sometimes referred as the concept of duality is helpful for the understanding of the linear programming. The variable of the dual problem is known as the dual variables or shadow price of the various resources. The dual problem is easier to solve than the original problem. The dual problem solution leads to the solution of the original problem and thus efficient computational techniques can be developed through the concept of duality. Finally, in the competitive strategy problem solution of both the original and dual problem is necessary to understand the complete problem.

Dual Problem Formulation

If the original problem is in the standard form then the dual problem can be formulated using the following rules:

- The number of constraints in the original problem is equal to the number of dual variables. The number of constraints in the dual problem is equal to the number of variables in the original problem.
- The original problem profit coefficients appear on the right hand side of the dual problem constraints.
- If the original problem is a maximization problem then the dual problem is a minimization problem. Similarly, if the original problem is a minimization problem then the dual problem is a maximization problem.
- The original problem has less than or equal to ($\leq$) type of constraints while the dual problem has greater than or equal to ($\geq$) type constraints.
- The coefficients of the constraints of the original problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

The Dual Linear Programming Problem is explained with the help of the following Example.

Example

Consider the following product mix problem:

Three machine shops A, B, C produces three types of products X, Y, Z respectively. Each product involves operation of each of the machine shops. The time required for each operation on various products is given as follows:

Machine Shops

Products	A	B	C	Profit per unit
X	10	7	2	$12
Y	2	3	4	$3
Z	1	2	1	$1
Available Hours	100	77	80	

The available hours at the machine shops A, B, C are 100, 77, and 80 only. The profit per unit of products X, Y, and Z is \$12, \$3, and \$1 respectively.

Solution:

The formulation of Linear Programming (original problem) is as follows:

Maximize

$$12x_1 + 3x_2 + x_3$$

Subject to:

$$10x_1 + 2x_2 + x_3 \leq 100$$

$$7x_1 + 3x_2 + 2x_3 \leq 77$$

$$2x_1 + 4x_2 + x_3 \leq 80$$

$$x_1, x_2, x_3 \geq 0$$

We introduce the slack variables s_4, s_5 and s_6 then the equalities becomes as:

Maximize

$$12x_1 + 3x_2 + x_3$$

Subject to:

$$10x_1 + 2x_2 + x_3 \quad + s_4 \quad = 100$$

$$7x_1 + 3x_2 + 2x_3 \quad + s_5 = 77$$

$$2x_1 + 4x_2 + x_3 \quad + s_6 \quad = 80$$

$$x_1, x_2, x_3, s_4, s_5, s_6 \geq 0$$

Form the above equations, the first simplex table is obtained is as follows:

CB	Basic Variable	C_j XB	12 x_1	3 x_2	1 x_3	0 s_4	0 s_5	0 s_6
0	s_4	100	10	2	1	1	0	0
0	s_5	77	7	3	2	0	1	0
0	s_6	80	2	4	1	0	0	1
	z_j-c_j		-12	-3	-1	0	0	0

Note that the basic variables are s_4, s_5 and s_6. Therefore $CB_1 = 0$, $CB_2 = 0$, $CB_3 = 0$.

The smallest negative element in the above table of $z_1 - c_1$ is -12. Hence, x_1 becomes a basic variable in the next iteration.

Determine the minimum ratios

$$\text{Min} \left[\frac{100, \ 72, \ 80}{10 \quad 7 \quad 2} \right] = 10$$

Here the minimum value is s_4, which is made as a non-basic variable.

The next Table 2 is calculated using the following rules:

The revised basic variables are x_1, s_5, s_6. Accordingly we make $CB_1=22$, $CB_2=0$ and $CB_3=0$. Since x_1 is the incoming variable we make x_1 coefficient one by dividing each element of row 1 by 10. Thus the numerical value of the element corresponding to x_2 is 2/10, corresponding to x_3 is 1/10, corresponding to s_4 is 1/10, corresponding to s_5 is 0/10 and corresponding to s_6 is 0/10 in Table 2.

The incoming basic variable should only appear in the first row. So we multiply first row of Table 2 by 7 and subtract if from the second row of Table 1 element by element.

Thus,

The element corresponding to x_1 in the second row of Table 2 is zero

The element corresponding to x_2 is $3 - 7 * \dfrac{2}{10} = \dfrac{16}{10}$

By using this way we get the elements of the second and the third row in Table 2.

Similarly, the calculation of numerical values of basic variables in Table 2 is done

CB	Basic	C_j	22	6	2	0	0	0
	Variable	XB	x_1	x_2	x_3	s_4	s_5	s_6
12	x_1	10	1	2/10	1/10	1/10	0	0
0	s_5	7	0	16/10	13/10	-7/10	1	0
0	s_6	60	0	18/5	4/5	-1/5	0	1
	z_j-c_j		0	-3/5	1/5	6/5	0	0

z_2-c_2 = -3/5. So x_2 becomes a basic variable in the next iteration.

Determine the minimum of the ratios

$$\text{Min} \left[\frac{10, \quad 7, \quad 60}{\dfrac{2}{10} \quad \dfrac{16}{10} \quad \dfrac{18}{5}} \right] = \text{Min} \left[50, \ \frac{70, \ 300}{16 \quad 18} \right] = \frac{70}{16}$$

So that the variable s_5 will be a non basic variable in the next iteration.

6. From Table 2, the Table 3 is calculated using the rules (i), (ii) and (iii) mentioned above.

CB	Basic	C_j	12	3	1	0	0	0
	Variable	XB	x_1	x_2	x_3	s_4	s_5	s_6

12	x_1	73/8	1	0	-1/16	3/16	-1/8	0
3	s_5	35/8	0	1	13/16	-7/16	5/8	0
0	s_6	177/4	0	0	-17/8	11/8	-9/4	1
	z_j-c_j		0	0	11/16	15/16	3/8	0

Since all the $z_i - c_j \geq 0$, the optimum solution is as:

$$x_1 = 73/8 \text{ and } x_2 = 35/8 \text{ and}$$

The Maximum Profit is: $981/8 = \$122.625$
Suppose an investor is deciding to purchase the resources A, B, C. What offers is he going to produce?

Let, assume that W_1, W_2 and W_3 are the offers made per hour of machine time A, B and C respectively.

Then these prices W_1, W_2 and W_3 must satisfy the conditions given below:

W_1, W_2, $W_3 \geq 0$

Assume that the investor is behaving in a rational manner; he would try to bargain as much as possible so that the total annual payable to the produces would be as little as possible. This leads to the following condition:

Minimize

$$100W_1 + 77W_2 + 80W_3$$

The total amount offer by the investor to the three resources viz. A, B and C required to produce one unit of each product must be at least as high as the profit gained by the producer per unit.

Since, these resources enable the producer to earn the specified profit corresponding to the product he would not like to sell it for anything less assuming he is behaving rationally. This leads to the following conditions:

$$10w_1 + 7w_2 + 2w_3 \geq 12$$
$$2w_1 + 3w_2 + 4w_3 \geq 3$$
$$w_1 + 2w_2 + w_3 \geq 1$$

Thus, in this case we have a linear problem to ascertain the values of the variable w_1, w_2, w_3. The variables w_1, w_2 and w_3 are called as **dual variables**.

Note:

The original (primal) problem illustrated in this example
considers the objective function maximization
contains $\leq$ type constraints
has non-negative constraints
This original problem is called as primal problem in the standard form.

Dual Problem Properties
The following are the different properties of dual programming problem:

If the original problem is in the standard form, then the dual problem solution is obtained from the $z_j - c_j$ values of slack variables.

For example: In the Example 4.1, the variables s4, s4 and s6 are the slack variables. Hence the dual problem solution is $w_1 = z_4 - c_4 = 15/16$, $w_2 = z_5 - c_5 = 3/8$ and $w_3 = z_6 - c_6 = 0$.

The original problem objective function maximum value is the minimum value of the dual problem objective function.

For example:

From the above Example 4.1 we know that the original problem maximum values is $981/8 = 122.625$. So that the minimum value of the dual problem objective function is

$$100*15/16 + 77*3/8 + 80*0 = 981/8$$

Here the result has an important practical implication. If both producer and investor analyzed the problem then neither of the two can outmanoeuvre the other Shadow Price: A resource shadow price is its unit cost, which is equal to the increase in profit to be realized by one additional unit of the resource. For example:

Let the minimum objective function value is expressed as:

$$100*15/16 + 77*3/8 + 80*0$$

If the first resource is increased by one unit the maximum profit also increases by 15/16, which is the first dual variable of the optimum solution. Therefore, the dual variables are also referred as the resource shadow price or imputed price. Note that in the previous example the shadow price of the third resource is zero because there is already an unutilized amount, so that profit is not increased by more of it until the current supply is totally exhausted.

In the originals problem, if the number of constraints and variables is m and n then the constraint and variables in the dual problem is n and m respectively. Suppose the slack variables in the original problem is represented by $y_1, y_1, ..., y_n$ and the surplus variables are represented by $z_1, z_2, ..., z_n$ in the dual problem.

Suppose, the original problem is not in a standard form, then the dual problem structure is unchanged. However, if a constraint is greater than or equal to type, the corresponding dual variable is negative or zero. Similarly, if a constraint in the original problem is equal to type, then the corresponding dual variables is unrestricted in sign.

Example 4.2

Consider the following linear programming problem

Maximize

$$22x1 + 25x2 + 19x3$$

Subject to:

$$18x1 + 26x2 + 22x3 \leq 350$$
$$14x1 + 18x2 + 20x3 \geq 180$$
$$17x1 + 19x2 + 18x3 = 205$$
$$x1, x2, x3 \geq 0$$

Note that this is a primal or original problem.

The corresponding dual problem for this problem is as follows:

Minimize

$$250w1 + 80w2 + 105w3$$

Subject to:

$$18w1 + 4w2 + 7w3 \geq 22$$
$$26w1 + 18w2 + 19w3 \geq 25$$
$$22w1 + 20w2 + 18w3 \geq 19$$
$$w1 \geq 0, w2 \leq, \text{ and } w3 \text{ is unrestricted in sign } (+ \text{ or } -).$$

Now, we can solve this using simplex method as usual.

Simple Way of Solving Dual Problem

Solving of dual problem is simple; this is illustrated with the help of the following Example.

Example :

Minimize

$$P = x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \geq 8$$
$$2x_1 + y \geq 12$$
$$x_1 \geq 1$$

Solution:

Step 1: Set up the P-matrix and its transpose

$$P = \begin{bmatrix} 1 & 1 & 8 \\ 2 & 1 & 12 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Pivoting

Pivoting about the **2** we get:

w_1	w_2	w_3	s_1	s_2	g	z
1	2	1	1	0	0	1
1/2	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

w_1	w_2	w_3	s_1	s_2	g	z
1/2	**1**	1/2	1/2	0	0	1/2
1	1	0	0	1	0	2
-8	-12	-1	0	0	1	0

w_1	w_2	w_3	s_1	s_2	g	z
1/2	**1**	1/2	1/2	0	0	1/2
1/2	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

The most negative entry in the bottom row to the left of the last column is **-2**. The smallest non-negative ratio is the **1/2** in the first row. This is the next pivot element.

w_1	w_2	w_3	s_1	s_2	g	z
1/2	1	1/2	1/2	0	0	1/2
1/2	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

Pivoting about the (table above) **1/2**:

Since there are no negative entries in the bottom row and to the left of the last column, the process is complete. The solutions are at the feet of the slack variable columns.

Therefore,

w_1	w_2	w_3	s_1	s_2	g	z
1	2	1	1	0	0	1
0	-1	-1	-1	1	0	1
0	4	7	8	0	1	8

The optimum solution provided by $x_1 = 8$ and $x_2 = 0$

The Minimum Value is: 8

TRANSPORTATION PROBLEM

Introduction
   A special class of linear programming problem is **Transportation Problem**, where the objective is to minimize the cost of distributing a product from a number of **sources** (e.g. factories) to a number of **destinations** (e.g. warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given rout is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

   The transportation problem special feature is illustrated here with the help of following Example.

Example:
   Suppose a manufacturing company owns three factories (sources) and distribute his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Usually the above table is referred as Transportation Table, which provides the basic information regarding the transportation problem. The quantities inside the table are known as transportation cost per unit of product. The capacity of the factories 1, 2, 3 is 50, 100 and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100,60,50,50, and 40 respectively.

In this case, the transportation cost of one unit
     from factory 1 to retail agency 1 is 1,
     from factory 1 to retail agency 2 is 9,
     from factory 1 to retail agency 3 is 13, and so on.

   A transportation problem can be formulated as linear programming problem using variables with two subscripts.

   Let

    x_{11}=Amount to be transported from factory 1 to retail agency 1

    x_{12}= Amount to be transported from factory 1 to retail agency 2

    ……..

    ……..

    ……..

    ……..

    x_{35}= Amount to be transported from factory 3 to retail agency 5.

Let the transportation cost per unit be represented by $C_{11}, C_{12}, …..C_{35}$ that is $C_{11}=1, C_{12}=9$, and so on.

Let the capacities of the three factories be represented by $a_1=50, a_2=100, a_3=150$.

Let the requirement of the retail agencies are $b_1=100, b_2=60, b_3=50, b_4=50,$ and $b_5=40$. Thus, the problem can be formulated as

   Minimize

    $C_{11}x_{11}+C_{12}x_{12}+……………+C_{35}x_{35}$

   Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a2$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a3$$

$$x_{11} + x_{21} + x_{31} = b_1$$

$$x_{12} + x_{22} + x_{32} = b_2$$

$$x_{13} + x_{23} + x_{33} = b_3$$

$$x_{14} + x_{24} + x_{34} = b_4$$

$$x_{15} + x_{25} + x_{35} = b_5$$

$$x_{11}, x_{12}, \ldots\ldots, x_{35} \geq 0.$$

Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem. There are varieties of procedures, which are described in the next section.

Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

       North West Corner Method

       Least Cost Method

       Vogel Approximation Method

Step 2: Determine the optimal solution using the following method

      1.MODI (Modified Distribution Method) or UV Method.

Basic Feasible Solution of a Transportation Problem

The computation of an initial feasible solution is illustrated in this section with the help of the example discussed in the previous section. The problem in the example 1.1 has 8 constraints and 15 variables we can eliminate one of the constraints since $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4 + b_5$. Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most 7 non-zero X_{ij}. Generally, any basic feasible solution with m sources (such as factories) and n destination (such as retail agency) has at most m + n -1 non-zero X_{ij}.

The special structure of the transportation problem allows securing a non artificial basic feasible solution using one the following three methods.

       North West Corner Method

       Least Cost Method

       Vogel Approximation Method

The difference among these three methods is the **quality** of the initial basic feasible solution they produce, in the sense that a better that a better initial solution yields a smaller objective value. Generally the Vogel Approximation Method produces the **best** initial basic feasible solution, and the North West Corner Method produces the **worst**, but the North West Corner Method involves least computations.

North West Corner Method:

The method starts at the North West (upper left) corner cell of the tableau (variable x_{11}).

Step -1: Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

Step -2: Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

Step -3: If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step -1.

Example:

Consider the problem discussed in Example to illustrate the North West Corner Method of determining basic feasible solution.

Factories	Retail Agency					
	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

The allocation is shown in the following tableau:

						Capacity		
	1	9	13	36	51	50		
50								
	24	12	16	20	1	100 50		
50	50							
	14	33	1	23	26			
	10	50	50	40		150 140	90	40
Requirement	100	60	50	50	40			
	50	10						

The arrows show the order in █████ all██████ **old** ██████nts are generated. The starting basic solution is given as

$$x_{11} = 50,$$
$$x_{21} = 50, \ x_{22} = 50$$
$$x_{32} = 10, \ x_{33} = 50, \ x_{34} = 50, \ x_{35} = 40$$

The corresponding transportation cost is

$50 * 1 + 50 * 24 + 50 * 12 + 10 * 33 + 50 * 1 + 50 * 23 + 40 * 26 = 4420$

It is clear that as soon as a value of X_{ij} is determined, a row (column) is eliminated from further consideration. The last value of X_{ij} eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most $m + n - 1$ positive X_{ij} if the transportation problem has m sources and n destinations.

Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which C_{ij} is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly. If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

Example:

The least cost method of determining initial basic feasible solution is illustrated with the help of problem presented in the section 1.1.

Capacity

1 50	9	13	36	51	~~50~~
24	12 60	16	20	1 40	~~100 60~~
14 50	33	1 50	23 50	26	~~— — —~~ ~~150 100 50~~

Requirement   ~~100~~   60   50   50   40

                   50

The Least Cost method is applied in the following manner:

We observe that $C_{11}=1$ is the minimum unit cost in the table. Hence $X_{11}=50$ and the first row is crossed out since the row has no more capacity. Then the minimum unit cost in the uncrossed-out row and column is $C_{25}=1$, hence $X_{25}=40$ and the fifth column is crossed out. Next $C_{33}=1$ is the minimum unit cost, hence $X_{33}=50$ and the third column is crossed out. Next $C_{22}=12$ is the minimum unit cost, hence $X_{22}=60$ and the second column is crossed out. Next we look for the uncrossed-out row and column now $C_{31}=14$ is the minimum unit cost, hence $X_{31}=50$ and crossed out the first column since it was satisfied. Finally $C_{34}=23$ is the minimum unit cost, hence $X_{34}=50$ and the fourth column is crossed out.

So that the basic feasible solution developed by the Least Cost Method has transportation cost is

$$1 * 50 + 12 * 60 + 1 * 40 + 14 * 50 + 1 * 50 + 23 * 50 = 2710$$

Note that the minimum transportation cost obtained by the least cost method is much lower than the corresponding cost of the solution developed by using the north-west corner method.

Vogel Approximation Method (VAM):

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

Step 1: For each row (column) with strictly positive capacity (requirement), determine a **penalty** by subtracting the **smallest** unit cost element in the row (column) from the next **smallest** unit cost element in the same row (column).

Step 2: Identify the row or column with the **largest penalty** among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

Step 3: We select X_{ij} as a basic variable if C_{ij} is the **minimum cost** in the row or column with **largest penalty**. We choose the numerical value of X_{ij} as high as possible subject to the row and the column constraints. Depending upon whether a_i or b_j is the smaller of the two i^{th} row or j^{th} column is crossed out

Step 4: The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

Example 1.4:

Consider the following transportation problem

Origin	Destination				a_i
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240

Note: a_i=capacity (supply)

b_j=requirement (demand)

Now, compute the penalty for various rows and columns which is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

Look for the highest penalty in the row or column, the highest penalty occurs in the **second column** and the minimum unit cost i.e. c_{ij} in this column is $c_{12}=22$. Hence assign 40 to this cell i.e. $x_{12}=40$ and cross out the second column (since second column was satisfied. This is shown in the following table:

Origin	Destinatio		a_i	Colum

	n					n
	1	2	3	4		Penalty
1	20	22 **40**	17	4	80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	

Row Penalty	4	15	8	3	

The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the **first row** and the minimum unit cost in this row is $c_{14}=4$, hence $x_{14}=80$ and cross out the first row. The modified table is as follows:

	Destination					
Origin	1	2	3	4	a_i	Column Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 8 which occurs in the **third column** and the **minimum cost** in this column is $c_{23}=9$, hence $x_{23}=30$ and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows:

	Destination					
Origin	1	2	3	4	a_i	Column Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7	40	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	

Row Penalty	8	15	8	8	

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **second row** and the **smallest cost** in this row is $c_{24}=15$, hence $x_{24}=30$ and cross out the fourth column with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				a_i Column Penalty	
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7 **30**	10	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	

Row Penalty	8	15	8	8	

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **second row** and the **smallest cost** in this row is $c_{21}=24$, hence $xi_{21}=10$ and cross out the second row with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				a_i Column Penalty	
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24 **10**	37	9 **30**	7 **30**	0	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **third row** and the **smallest cost** in this row is $c_{31}=32$, hence $xi_{31}=50$ and cross out the third row or first column. The modified table is as follows:

Origin	Destination	a_i Column

	1	2	3	4		Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24 **10**	37	9 **30**	7 **30**	0	17
3	32 **50**	37	20	15	0	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The transportation cost corresponding to this choice of basic variables is

22 * 40 + 4 * 80 + 9 * 30 + 7 * 30 + 24 * 10 + 32 * 50 = 3520

Modified Distribution Method

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method.

Step 1: Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

Step 2: Introduce dual variables corresponding to the row constraints and the column constraints. If there are m origins and n destinations then there will be m+n dual variables. The dual variables corresponding to the row constraints are represented by u_i, i=1,2,…..m where as the dual variables corresponding to the column constraints are represented by v_j, j=1,2,…..n. The values of the dual variables are calculated from the equation given

below $u_i + v_j = c_{ij}$ if $x_{ij} > 0$

Step 3: Any basic feasible solution has m + n -1 $x_{ij} > 0$. Thus, there will be m + n -1 equation to determine m + n dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

Step 4: If x_{ij}=0, the dual variables calculated in Step 3 are compared with the c_{ij} values of this allocation as $c_{ij} - u_i - v_j$. If al $c_{ij} - u_i - v_j \geq 0$, then by the ***theorem of complementary slackness*** it can be shown that the corresponding solution of the transportation problem is optimum. If one or more $c_{ij} - u_i - v_j <$ 0, we select the cell with the least value of $c_{ij} - u_i - v_j$ and allocate as much as possible subject to the row and column constraints. The allocations of the number of adjacent cell are adjusted so that a basic variable becomes non-basic.

Step 5: A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

Example:

For example consider the transportation problem given below:

Suppl y

1	9	13	36	51	**50**
24	12	16	20	1	**100**
14	33	1	23	26	**150**

Demand	**100**	**70**	**50**	**40**	**40**	**300**

Step 1: First we have to determine the basic feasible solution. The basic feasible solution using least cost method is

$x_{11}=50$, $x_{22}=60$, $x_{25}=40$, $x_{31}=50$, $x_{32}=10$, $x_{33}=50$ and $x_{34}=40$

Step 2: The dual variables u_1, u_2, u_3 and v_1, v_2, v_3, v_4, v_5 can be calculated from the corresponding c_{ij} values, that is

$u_1+v_1=1$ $u_2+v_2=12$ $u_2+v_5=1$ $u_3+v_1=14$

$u_3+v_2=33$ $u_3+v_3=1$ $u_3+v_4=23$

Step 3: Choose one of the dual variables arbitrarily is zero that is $u_3=0$ as it occurs most often in the above equations. The values of the variables calculated are

$u_1= -13$, $u_2= -21$, $u_3=0$

$v_1=14$, $v_2=33$, $v_3=1$, $v_4=23$, $v_5=22$

Step 4: Now we calculate $c_{ij} – u_i – v_j$ values for all the cells where $x_{ij}=0$ (.e. unallocated cell by the basic feasible solution)

 That is

Cell(1,2)= $c_{12}-u_1-v_2$ = 9+13-33 = -11

Cell(1,3)= $c_{13}-u_1-v_3$ = 13+13-1 = 25

Cell(1,4)= $c_{14}-u_1-v_4$ = 36+13-23 = 26

Cell(1,5)= $c_{15}-u_1-v_5$ = 51+13-22 = 42

Cell(2,1)= $c_{21}-u_2-v_1$ = 24+21-14 = 31

Cell(2,3)= $c_{23}-u_2-v_3$ = 16+21-1 = 36

Cell(2,4)= $c_{24}-u_2-v_4$ = 20+21-23 = 18

Cell(3,5)= $c_{35}-u_3-v_5$ = 26-0-22 = 4

Note that in the above calculation all the $c_{ij} – u_i – v_j \geq 0$ except for cell (1, 2) where $c_{12} – u_1 – v_2 = $ 9+13-33 = -11.

Thus in the next iteration x_{12} will be a basic variable changing one of the present basic variables non-basic. We also observe that for allocating one unit in cell (1, 2) we have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net transportation cost for each unit of such reallocation is

-33 -1 + 9 +14 = -11

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in the cell (3, 2) will be negative. Thus, the revised basic feasible solution is

$x_{11}=40$, $x_{12}=10$, $x_{22}=60$, $x_{25}=40$, $x_{31}=60$, $x_{33}=50$, $x_{34}=40$

Unbalanced Transportation Problem

       In the previous section we discussed about the balanced transportation problem i.e. the total supply(capacity) at the origins is equal to the total demand (requirement) at the destination. In this section we are going to discuss about the unbalanced transportation problems i.e. when the total supply is not equal to the total demand, which are called as **unbalanced transportation problem**.

       In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost.

Example:

       Consider the following unbalanced transportation problem

Warehouses

Plant	W_1	W_2	W_3	Supply
X	20	17	25	400
Y	10	10	20	500
Demand	400	400	500	

       In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

Warehouses

Plant	W_1	W_2	W_3	Supply
X	20	17	25	400
Y	10	10	20	500
Unsatisfied Demand	0	0	0	400

Demand   400   400   500   1300
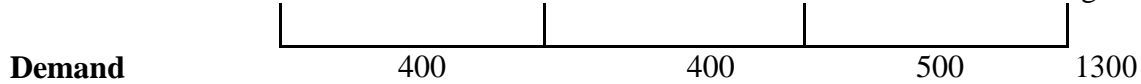
Now we can solve as balanced problem discussed as in the previous sections.

Degenerate Transportation Problem

In a transportation problem, if a basic feasible solution with m origins and n destinations has less than m+n -1 positive X_{ij} i.e. occupied cells, then the problem is said to be a **degenerate transportation problem**. The degeneracy problem does not cause any serious difficulty, but it can cause computational problem wile determining the optimal minimum solution.

There fore it is important to identify a degenerate problem as early as beginning and take the necessary action to avoid any computational difficulty. The degeneracy can be identified through the following results:

"In a transportation problem, a degenerate basic feasible solution exists if and only if some partial sum of supply (row) is equal to a partial sum of demand (column). For example the following transportation problem is degenerate. Because in this problem

$a_1 = 400 = b_1$

$a_2 + a_3 = 900 = b_2 + b_3$

Warehouses

Plant	w_1	w_2	w_3	**Supply** (a_i)
X	20	17	25	400
Y	10	10	20	500
Unsatisfied demand	0	0	0	400
Demand (b_j)	400	400	500	1300

There is a technique called perturbation, which helps to solve the degenerate problems.

Perturbation Technique:

The degeneracy of the transportation problem can be avoided if we ensure that no partial sum of ai (supply) and bj (demand) is equal. We set up a new problem where

$a_i = a_i + d$ i = 1, 2, ……, m

$b_j = b_j$ j = 1, 2, ……, n -1

$b_n = b_n + m_d$ d > 0

This modified problem is constructed in such a way that no partial sum of a_i is equal to the b_j. Once the problem is solved, we substitute d = 0 leading to optimum solution of the original problem.

ASSIGNMENT PROBLEM

Introduction

The Assignment Problem can define as follows:

Given n facilities, n jobs and the effectiveness of each facility to each job, here the problem is to assign each facility to one and only one job so that the measure of effectiveness if optimized. Here the optimization means Maximized or Minimized. There are many management problems has a assignment

problem structure. For example, the head of the department may have 6 people available for assignment and 6 jobs to fill. Here the head may like to know which job should be assigned to which person so that all tasks can be accomplished in the shortest time possible. Another example a container company may have an empty container in each of the location 1, 2,3,4,5 and requires an empty container in each of the locations 6, 7, 8,9,10. It would like to ascertain the assignments of containers to various locations so as to minimize the total distance. The third example here is, a marketing set up by making an estimate of sales performance for different salesmen as well as for different cities one could assign a particular salesman to a particular city with a view to maximize the overall sales.

Note that with n facilities and n jobs there are n! possible assignments. The simplest way of finding an optimum assignment is to write all the n! possible arrangements, evaluate their total cost and select the assignment with minimum cost. Bust this method leads to a calculation problem of formidable size even when the value of n is moderate. For n=10 the possible number of arrangements is 3268800.

Assignment Problem Structure and Solution

The structure of the Assignment problem is similar to a transportation problem, is as follows:

	1	2	…	n	
1	c_{11}	c_{12}	…	c_{1n}	1
2	c_{21}	c_{21}	…	c_{2n}	1
	.	.	.	.	.
Workers					
	.	.	.	.	
	.	.	.	.	
n	c_{n1}	c_{n2}	…	c_{nn}	1
	1	1	…	1	

The element c_{ij} represents the measure of effectiveness when i^{th} person is assigned j^{th} job. Assume that the overall measure of effectiveness is to be minimized. The element x_{ij} represents the number of ith individuals assigned to the jth job. Since i^{th} person can be assigned only one job and j^{th} job can be assigned to only one person we have the following

$x_{i1} + x_{i2} + ……………. + x_{in} = 1$, where i = 1, 2, ... , n

$x_{1j} + x_{2j} + ……………. + x_{nj} = 1$, where j = 1, 2, ... , n

and the objective function is formulated as

Minimize $c_{11}x_{11} + c_{12}x_{12} + ……….. + c_{nn}x_{nn}$

$x_{ij} \geq 0$

The assignment problem is actually a special case of the transportation problem where m = n and $a_i = b_j = 1$. However, it may be easily noted that any basic feasible solution of an assignment problem contains $(2_n – 1)$ variables of which $(n – 1)$ variables are zero. Because of this high degree of degeneracy the usual computation techniques of a transportation problem become very inefficient. So, hat a separate computation technique is necessary for the assignment problem.

The solution of the assignment problem is based on the following results:

"If a constant is added to every element of a row/column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original assignment problem and vice versa". – This result may be used in two different methods to solve the assignment problem. If in an assignment problem some cost elements c_{ij} are negative, we may have to convert them into an equivalent assignment problem where all the cost elements are non-negative by adding a suitable large constant to the cost elements of the relevant row or column, and then we look for a feasible solution which has zero assignment cost after adding suitable constants to the cost elements of the

various rows and columns. Since it has been assumed that all the cost elements are non-negative, this assignment must be optimum. On the basis of this principle a computational technique known as Hungarian Method is developed. The Hungarian Method is discussed as follows.

Hungarian Method:

The Hungarian Method is discussed in the form of a series of computational steps as follows, when the objective function is that of minimization type.

Step 1:

From the given problem, find out the cost table. Note that if the number of origins is not equal to the number of destinations then a dummy origin or destination must be added.

Step 2:

In each row of the table find out the smallest cost element, subtract this smallest cost element from each element in that row. So, that there will be at least one zero in each row of the new table. This new table is known as First Reduced Cost Table.

Step 3:

In each column of the table find out the smallest cost element, subtract this smallest cost element from each element in that column. As a result of this, each row and column has at least one zero element. This new table is known as Second Reduced Cost Table.

Step 4:

Now determine an assignment as follows:

For each row or column with a single zero element cell that has not be assigned or eliminated, box that zero element as an assigned cell.

For every zero that becomes assigned, cross out all other zeros in the same row and for column.

If for a row and for a column there are two or more zero and one can't be chosen by inspection, choose the assigned zero cell arbitrarily.

The above procedures may be repeated until every zero element cell is either assigned (boxed) or crossed out.

Step 5:

An optimum assignment is found, if the number of assigned cells is equal to the number of rows (and columns). In case we had chosen a zero cell arbitrarily, there may be an alternate optimum. If no optimum solution is found i.e. some rows or columns without an assignment then go to **Step 6**.

Step 6:

Draw a set of lines equal to the number of assignments which has been made in **Step 4**, covering all the zeros in the following manner

Mark check (√) to those rows where no assignment has been made.

Examine the checked (√) rows. If any zero element cell occurs in those rows, check (√) the respective columns that contains those zeros.

Examine the checked (√) columns. If any assigned zero element occurs in those columns, check (√) the respective rows that contain those assigned zeros.

The process may be repeated until now more rows or column can be checked.

Draw lines through all unchecked rows and through all checked columns.

Step 7:

Examine those elements that are not covered by a line. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them.

Add this smallest element to every element that lies at the intersection of two lines. Then the resulting matrix is a new revised cost table.

Example:

Problem

A work shop contains four persons available for work on the four jobs. Only one person can work on any one job. The following table shows the cost of assigning each person to each job. The objective is to assign person to jobs such that the total assignment cost is a minimum.

Jobs

Persons	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D	25	23	24	24

Solution

As per the Hungarian Method
Step 1: The cost Table

Jobs

Persons	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D	25	23	24	24

Step 2: Find the First Reduced Cost Table

Jobs

Persons		1	2	3	4
	A	0	5	2	8
	B	0	3	8	2
	C	2	0	4	7
	D	2	0	1	1

Step 3: Find the Second Reduced Cost Table

Jobs

Persons		1	2	3	4
	A	0	5	1	7
	B	0	3	7	1
	C	2	0	3	6
	D	2	0	0	0

Step 4: Determine an Assignment

By examine row A of the table in Step 3, we find that it has only one zero (cell A1) box this zero and cross out all other zeros in the boxed column. In this way we can eliminate cell B1.
Now examine row C, we find that it has one zero (cell C2) box this zero and cross out (eliminate) the zeros in the boxed column. This is how cell D2 gets eliminated.

There is one zero in the column 3. Therefore, cell D3 gets boxed and this enables us to eliminate cell D4.

Therefore, we can box (assign) or cross out (eliminate) all zeros.

The resultant table is shown below:

Jobs

		1	2	3	4
Persons	**A**	0	5	1	7
	B	0	3	7	1
	C	2		3	6
	D	2	0		0

Step 5:   0

The solution obtained in Step 4 is not optimal. Because we were able to make three assignments when four were required.

Step 6:

Cover all the zeros of the table shown in the Step 4 with three lines (since already we made three assignments).

Check row B since it has no assignment. Note that row B has a zero in column 1, therefore check column1. Then we check row A since it has a zero in column 1. Note that no other rows and columns are checked. Now we may draw three lines through unchecked rows (row C and D) and the checked column (column 1). This is shown in the table given below:

Jobs

		1	2	3	4
Persons	**A**	0	5	1	7
	B	0	3	7	1
	C	2		3	6
	D	2	0		0

Step 7:

Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (C1 and D1) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

Jobs

		1	**2**	**3**	**4**
	A	0	4	0	6
Persons	**B**	0	2	6	0
	C	3	0	3	6
	D	3	0	0	0

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:

Determine an assignment

Examine each of the four rows in the table given in Step 7, we may find that it is only row C which has only one zero box this cell C2 and cross out D2.

Note that all the remaining rows and columns have two zeros. Choose a zero arbitrarily, say A1 and box this cell so that the cells A3 and B1 get eliminated.

Now row B (cell B4) and column 3 (cell D4) has one zero box these cells so that cell D4 is eliminated. Thus, all the zeros are either boxed or eliminated. This is shown in the following table

Jobs

	1	**2**	**3**	**4**
	___	4	0	6

Persons		1	2	3
	A	0		
	B	0	2	6
	C	3		3	06
	D	3	0		0

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

0

The total cost of assignment is: 78 that is A1 + B4 + C2 + D3

$$20 + 17 + 17 + 24 = 78$$

2.3 Unbalanced Assignment Problem

0

In the previous section we assumed that the number of persons to be assigned and the number of jobs were same. Such kind of assignment problem is called as **balanced assignment problem**. Suppose if the number of person is different from the number of jobs then the assignment problem is called as **unbalanced**.

If the number of jobs is less than the number of persons, some of them can't be assigned any job. So that we have to introduce on or more dummy jobs of zero duration to make the unbalanced assignment problem into balanced assignment problem. This balanced assignment problem can be solved by using the Hungarian Method as discussed in the previous section. The persons to whom the dummy jobs are assigned are left out of assignment.

Similarly, if the number of persons is less than number of jobs then we have introduce one or more dummy persons with zero duration to modify the unbalanced into balanced and then the problem is solved using the Hungarian Method. Here the jobs assigned to the dummy persons are left out.

Example:

Solve the following unbalanced assignment problem of minimizing the total time for performing all the jobs.

Jobs

		1	2	3	4	5
	A	5	2	4	2	5
	B	2	4	7	6	6

Workers						
C	6	7	5	8	7	
D	5	2	3	3	4	
E	8	3	7	8	6	
F	3	6	3	5	7	

Solution

In this problem the number of jobs is less than the number of workers so we have to introduce a dummy job with zero duration.

The revised assignment problem is as follows:

Jobs

Workers		1	2	3	4	5	6
	A	5	2	4	2	5	0
	B	2	4	7	6	6	0
	C	6	7	5	8	7	0
	D	5	2	3	3	4	0

E	8	3	7	8	6	0
F	3	6	3	5	7	0

Now the problem becomes balanced one since the number of workers is equal to the number jobs. So that the problem can be solved using Hungarian Method.

Step 1: The cost Table

		Jobs					
		1	**2**	**3**	**4**	**5**	**6**
	A	5	2	4	2	5	0
Workers	**B**	2	4	7	6	6	0
	C	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Step 2: Find the First Reduced Cost Table

Jobs

		1	2	3	4	5	6
	A	5	2	4	2	5	0
	B	2	4	7	6	6	0
Workers	**C**	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Step 3: Find the Second Reduced Cost Table

		Jobs					
		1	**2**	**3**	**4**	**5**	**6**
	A	3	0	1	0	1	0
	B	0	2	4	4	2	0
Workers	**C**	4	5	2	6	3	0
	D	3	0	0	1	0	0
	E	6	1	4	6	2	0
	F	1	4	0	3	3	0

Step 4: Determine an Assignment
By using the Hungarian Method the assignment is made as follows:

Jobs

	1	2	3	4	5	6
A	3	0	1	0	1	0
B		2	4	4	2	0
C	4 / 0	5	2	6	3	
D	3	0	0	1		0
E	6	1	4	6	2	0
F	1	4		3	3	0

Workers

0

Step 5:

The solution obtained in Step 4 is not optimal. Because we were able to make five assignments when six were required.

Step 6:

Cover all the zeros of the table shown in the Step 4 with five lines (since already we made five assignments).

Check row E since it has no assignment. Note 0 that row B has a zero in column 6, therefore check column6. Then we check row C since it has a zero in column 6. Note that no other rows and columns are

checked. Now we may draw five lines through unchecked rows (row A, B, D and F) and the checked column (column 6). This is shown in the table given below:

Jobs

Workers		1	2	3	4	5	6
	A	3	0	1	[0]	1	0
	B		2	4	4	2	0
	C	[4] (0)	5	2	6	3	
	D	3	0	0	1		0
	E	6	1	4	6	2	[0]
	F	1	4		3	3	0

[0]

Step 7:

Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (A6, B6, D6 and F6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

1	2	3	4	5	6

[0]

A	3	0	1	0	1	1
B	0	2	4	4	2	1
Workers C	3	4	1	5	2	0
D	3	0	0	1	0	1
E	5	0	3	5	1	0
F	1	4	0	3	3	1

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:

Determine an assignment

Jobs

	1	2	3	4	5	6
A	3	0	1	0	1	1
B		2	4	4	2	1

Workers	C	30	4	1	5	2	
	D	3	0	0	1		1
	E	5		3	5	1	0 0
	F	1	4		3	3	1

0

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

Thus, the worker A is assigned to Job4, worker B is assigned to job 1, worker C is assigned to job 6, worker D is assigned to job 5, worker E is assigned to job 2, and worker F is assigned to job 3. Since the Job 6 is dummy so that worker C can't be assigned.

The total minimum time is: 14 that is A4 + B1 + D5 + E2 + F3

$$2 + 2 + 4 + 3 + 3 = 14$$

DYNAMIC PROGRAMMING

NETWORK PROBLEMS
Introduction
A network consists of several destinations or jobs which are linked with one another. A manager will have occasions to deal with some network or other. Certain problems pertaining to networks are taken up for consideration in this unit.

SHORTEST PATH PROBLEM
Imagine a salesman or a milk vendor or a post man who has to cover certain previously earmarked places to perform his daily routines. It is assumed that all the places to be visited by him are connected well for a suitable mode of transport. He has to cover all the locations. While doing so, if he visits the same place again and again on the same day, it will be a loss of several resources such as time, money, etc. Therefore he shall place a constraint upon himself not to visit the same place again and again on the same day. He shall be in a position to determine a route which would enable him to cover all the locations, fulfilling the constraint.

The shortest route method aims to find how a person can travel from one location to another, keeping the total distance traveled to the minimum. In other words, it seeks to identify the shortest route to a series of destinations.

EXAMPLE
Let us consider a real life situation involving a shortest route problem.

A leather manufacturing company has to transport the finished goods from the factory to the store house. The path from the factory to the store house is through certain intermediate stations as indicated in the following diagram. The company executive wants to identify the path with the shortest distance so as to minimize the transportation cost. The problem is to achieve this objective.
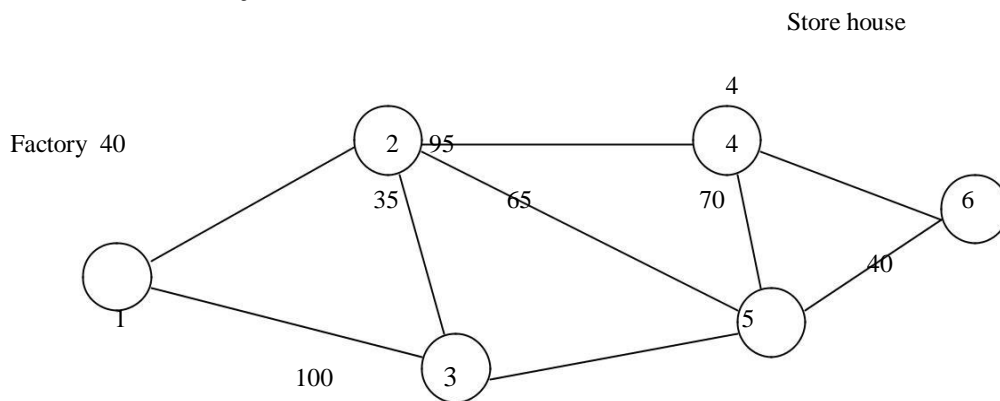
The shortest route technique can be used to minimize the total distance from a node designated as the **starting node or origin** to another node designated as the **final node**.

In the example under consideration, the origin is the factory and the final node is the store house.

STEPS IN THE SHORTEST ROUTE TECHNIQUE
The procedure consists of starting with a set containing a node and enlarging the set by choosing a node in each subsequent step.

Step 1:

First, locate the origin. Then, find the node nearest to the origin. Mark the distance between the origin and the nearest node in a box by the side of that node.

In some cases, it may be necessary to check several paths to find the nearest node.

Step 2:

Repeat the above process until the nodes in the entire network have been accounted for. The last distance placed in a box by the side of the ending node will be the distance of the shortest route. We note that the distances indicated in the boxes by each node constitute the

shortest route to that node. These distances are used as intermediate results in determining the next nearest node.

SOLUTION FOR THE EXAMPLE PROBLEM

Looking at the diagram, we see that node 1 is the origin and the nodes 2 and 3 are neighbours to the origin. Among the two nodes, we see that node 2 is at a distance of 40 units from node 1 whereas node 3 is at a distance of 100 units from node 1. The minimum of {40, 100} is 40. Thus, the node nearest to the origin is node 2, with a distance of 40 units. So, out of the two nodes 2 and 3, we select node 2. We form a set of nodes {1, 2} and construct a path connecting the node 2 with node 1 by a thick line and mark the distance of 40 in a box by the side of node 2. This first iteration is shown in the following diagram.

ITERATION No. 1

Now we search for the next node nearest to the set of nodes {1, 2}. For this purpose, consider those nodes which are neighbours of either node 1 or node 2. The nodes 3, 4 and 5 fulfill this condition. We calculate the following distances.

The distance between nodes 1 and 3 = 100.
The distance between nodes 2 and 3 = 35.
The distance between nodes 2 and 4 = 95.
The distance between nodes 2 and 5 = 65.
Minimum of {100, 35, 95, 65} = 35.

Therefore, node 3 is the nearest one to the set {1, 2}. In view of this observation, the set of nodes is enlarged from {1, 2} to {1, 2, 3}. For the set {1, 2, 3}, there are two possible paths, viz. Path 1 → 2 → 3 and Path 1 → 3 → 2. The Path 1 → 2 → 3 has a distance of 40 + 35 = 75 units while the Path 1 → 3 → 2 has a distance of 100 + 35 = 135 units.

Minimum of {75, 135} = 75. Hence we select the path 1 → 2 → 3 and display this path by thick edges. The distance 75 is marked in a box by the side of node 3. We obtain the following diagram at the end of Iteration No. 2.

ITERATION No. 2
REPEATING THE PROCESS

We repeat the process. The next node nearest to the set {1, 2, 3} is either node 4 or node 5. Node 4 is at a distance of 95 units from node 2 while node 2 is at a distance of 40 units from node 1. Thus, node 4 is at a distance of 95 + 40 = 135 units from the origin.

As regards node 5, there are two paths viz. 2 → 5 and 3 → 5, providing a link to the origin. We already know the shortest routes from nodes 2 and 3 to the origin. The minimum distances have been indicated in boxes near these nodes. The path 3 → 5 involves the shortest distance. Thus, the distance between nodes 1 and 5 is 95 units (20 units between nodes 5 and 3 + 75 units between node 3 and the origin). Therefore, we select node 5 and enlarge the set from {1, 2, 3} to {1, 2, 3, 5}. The distance 95 is marked in a box by the side of node 5. The following diagram is obtained at the end of Iteration No. 3.

ITERATION No. 3

Now 2 nodes remain, viz., nodes 4 and 6. Among them, node 4 is at a distance of 135 units from the origin (95 units from node 4 to node 2 + 40 units from node 2 to the origin). Node 6 is at a distance of 135 units from the origin (40 + 95 units). Therefore, nodes 4 and 6 are at equal distances from the origin. If we choose node 4, then travelling from node 4 to node 6 will involve an additional distance of 40 units. However, node 6 is the ending node. Therefore, we select node 6 instead of node 4. Thus the set is enlarged from {1, 2, 3, 5} to {1, 2, 3, 5, 6}. The distance 135 is marked in a box by the side of node 6. Since we have got a path beginning from the start node and terminating with the stop node, we see that the solution to the given problem has been obtained. We have the following diagram at the end of Iteration No. 4.

ITERATION No. 4
MINIMUM DISTANCE

Referring to the above diagram, we see that **the shortest route** is provided by the path
1 → 2 ⟶ 3 → 5 → 6 with a minimum distance of 135 units.

Replacement models

Introduction:

The replacement problems are concerned with the situations that arise when some items such as men, machines and usable things etc need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

If a firm wants to survive the competition it has to decide on whether to replace the out dated equipment or to retain it, by taking the cost of maintenance and operation into account. There are two basic reasons for considering the replacement of an equipment. They are Physical impairment or malfunctioning of various parts. Obsolescence of the equipment.

The physical impairment refers only to changes in the physical condition of the equipment itself. This will lead to decline in the value of service rendered by the equipment, increased operating cost of the equipments, increased maintenance cost of the equipment or the combination of these costs. Obsolescence is caused due to improvement in the existing Tools and machinery mainly when the technology becomes advanced therefore, it becomes uneconomical to continue production with the same equipment under any of the above situations. Hence the equipments are to be periodically replaced. Some times, the capacity of existing facilities may be in adequate to meet the current demand. Under such cases, the following two alternatives will be considered. Replacement of the existing equipment with a new one Argument the existing one with an additional equipments.
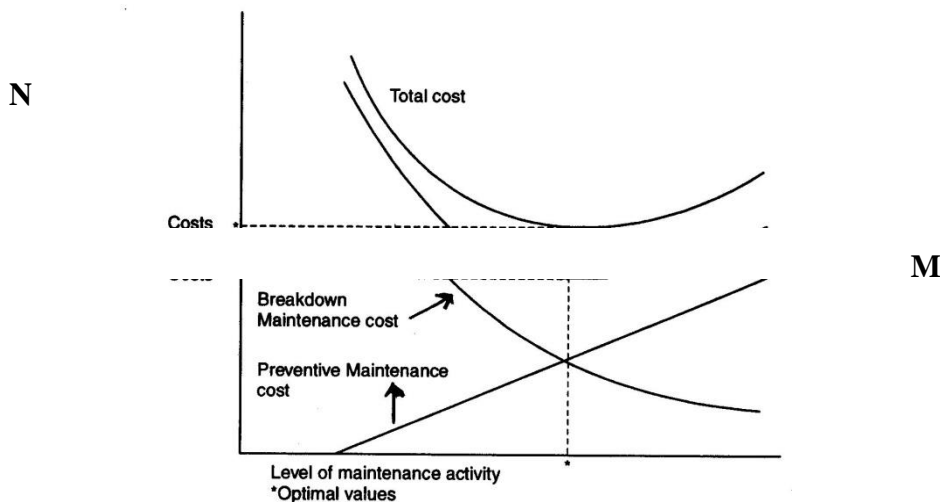
Type of Maintenance

Maintenance activity can be classified into two types
          Preventive Maintenance
          Breakdown Maintenance

Preventive maintenance (PN) is the periodical inspection and service which are aimed to detect potential failures and perform minor adjustments a requires which will prevent major operating problem in future. Breakdown maintenance is the repair which is generally done after the equipment breaks down. It is offer an emergency which will have an associated penalty in terms of increasing the cost of maintenance and downtime cost of equipment, Preventive maintenance will reduce such costs up-to a certain extent. Beyond that the cost of preventive maintenance will be more when compared to the cost of the breakdown maintenance.

Total cost = Preventive maintenance cost + Breakdown maintenance cost.

This total cost will go on decreasing up-to P with an increase in the level of maintenance up-to a point, beyond which the total cost will start increasing from P. The level of maintenance corresponding to the minimum total cost at P is the Optional level of maintenance this concept is illustrated in the follows diagram

The points M and N denote optimal level of maintenance and optimal cost respectively **Types of replacement problem**

The replacement problem can be classified into two categories. Replacement of assets that deteriorate with time (replacement due to gradual failure, due to wear and tear of the components of the machines) this can be further classified into the following types. Determination of economic type of an asset. Replacement of an existing asset with a new asset. Simple probabilistic model for assets which will fail completely (replacement due to sudden failure).

Determination of Economic Life of an asset

Any asset will have the following cost components

Capital recovery cost (average first cost), Computed form the first cost (Purchase price) of the asset.

Average operating and maintenance cost.

Total cost which is the sum of capital recovery cost (average first cost) and average operating and maintenance cost.

A typical shape of each of the above cost with respect to life of the asset is shown below
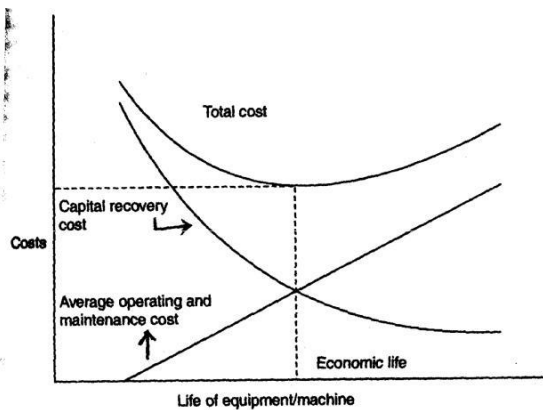
Fig. 13.2 Chart showing economic life.

From figure, when the life of the machine increases, it is clear that the capital recovery cost (average first cost) goes on decreasing and the average operating and maintenance cost goes on increasing. From the beginning the total cost goes on decreasing upto a particular life of the asset and then it starts increasing. The point P were the total cost in the minimum is called the Economic life of the asset. To solve problems under replacement, we consider the basics of interest formula. Present worth factor denoted by (P/F, i,n). If an amount P is invested now with amount earning interest at the rate i per year, then the future sum (F) accumulated after n years can be obtained.

P   -   Principal sum at year Zero

F   -   Future sum of P at the end of the nth year

i   -   Annual interest rate

     - Number of interest periods. Then

the formula for future sum $F = P (1 + i)^n$

$P = F/(1 + i)^n$ = Fx (present worth factor)

If A is the annual equivalent amount which occurs at the end of every year from year one through n years is given by

$$A = \frac{P \times i (1 + i)^n}{(1 + i)^n - 1}$$

$$= P (A / P, i, n)$$

$$= P \times \text{equal payment series capital recovery factor}$$

Example:

A firm is considering replacement of an equipment whose first cost is Rs. 1750 and the scrap value is negligible at any year. Based on experience, it is found that maintenance cost is zero during the first year and it increases by Rs. 100 every year thereafter. When should be the equipment replaced if

i = 0%

i = 12%

Solution :

Given the first cost = Rs 1750 and the maintenance cost is Rs. Zero during the first years and then increases by Rs. 100 every year thereafter. Then the following table shows the calculation.

Calculations to determine Economic life

(a) First cost Rs. 1750   Interest rate = 0%

End of year (n)	Maintenance cost at end of year	Summation of maintenance	Average cost of maintenance through the	Average first cost if replaced at the given	Average total cost through the given year

A	B (Rs)	Cost C (Rs)	given year D (in Rs)	year and E (Rs)	F (Rs)
		C = B	C/A	$\dfrac{1750}{A}$	D + E
1	0	0	0	1750	1750
2	100	100	50	875	925
3	200	300	100	583	683
4	300	600	150	438	588
5	400	1000	200	350	550
6	**500**	**1500**	**250**	**292**	**542**
7	600	2100	300	250	550
8	700	2800	350	219	569

The value corresponding to any end-of-year (n) in Column F represents the average total cost of using the equipment till the end of that particulars year.

In this problem, the average total cost decreases till the end of the year 6 and then it increases. Hence the optimal replacement period is 6 years ie   the economic life of the equipment is 6 years.

(e) When interest rate i = 12%

When the interest rate is more than 0% the steps to get the economic life are summarized in the following table.

Calculation to determine Economic life
First Cost = Rs. 1750   Interest rate = 12%

End of year (n)	Maintenance cost at end of years	(P/F,12v,n)	Present worth as beginning of years 1 of maintenance costs	Summation of present worth of maintenance costs through the given year	Present simulator maintenance cost and first cost	(A/P, 12%,n) $=\dfrac{i\,(1+i)^n}{(1+i)^n -1}$ G	Annual equipment total cost through the giver year
A	B	C	D	E	F	G	H
	B (iR)	$C = \dfrac{1}{(1+12/100)^n}$	BxC	D	E+ Rs. 1750		FxG
1	0	0.8929	0	0	1750	1.1200	1960
2	100	0.7972	79.72	79.72	1829.72	0.5917	1082.6
3	200	0.7118	142.36	222.08	1972.08	0.4163	820.9
4	300	0.6355	190.65	412.73	2162.73	0.3292	711.9
5	400	0.5674	226.96	639.69	2389.69	0.2774	662.9
6	500	0.5066	253.30	892.99	2642.99	0.2432	642.7
7	600	0.4524	271.44	1164.43	2914.430	0.2191	638.5
8	700	0.4039	282.73	1447.16	3197.16	0.2013	680.7

Identify the end of year for which the annual equivalent total cost is minimum in column. In this problem the annual equivalent total cost is minimum at the end of year hence the economics life of the equipment is 7 years.

Simple probabilistic model for items which completely fail

Electronic items like bulbs, resistors, tube lights etc. generally fail all of a sudden, instead of gradual failure. The sudden failure of the item results in complete breakdown of the system. The system may contain a collection of such items or just an item like a single tube-light. Hence we use some replacement policy for such items which would minimize the possibility of complete breakdown. The following are the replacement policies which are applicable in these cases.

i) Individual replacement policy :

Under this policy, each item is replaced immediately after failure.

ii) Group replacement policy :

Under group replacement policy, a decision is made with regard the replacement at what equal internals,

all the item are to be replaced simultaneously with a provision to replace the items individually which

fail during the fixed group replacement period. Among the two types of replacement polices, we have to decide which replacement policy we have to follow. Whether individual replacement policy is better than group replacement policy. With regard to economic point of view. To decide this, each of the replacement policy is calculated and the most economic one is selected for implementation.

Exercise :

List and explain different types of maintenance

Discuss the reasons for maintenance.

Distinguish between breakdown maintenance and preventive maintenance.

Distinguish between individual and group replacement polices.

A firm is considering replacement of an equipment whose first cost is Rs.4000 and the scrap value is negligible at the end of any year. Based on experience, it has been found that the maintenance cost is zero during the first year and it is Rs.1000 for the second year. It increase by Rs.300 every years thereafter. When should the equipment be replace if i = 0% When should the equipment be replace if i = 12%

Ans . a) 5 years   b) 5 years

A company is planning to replace an equipment whose first cost is Rs.1,00,000. The operating and maintenance cost of the equipment during its first year of operation is Rs.10,000 and it increases by Rs. 2,000 every year thereafter. The release value of the equipment at the end of the first year of its operation is Rs.65,000 and it decreases by Rs.10,000 every year thereafter. Find the economic life of the equipment by assuming the interest rate as 12%.

[Ans : Economic life = 13 years and the corresponding annual equivalent cost = Rs. 34,510]

The following table gives the operation cost, maintenance cost and salvage value at the end of every year of machine whose purchase value is Rs. 12,000. Find the economic life of the machine assuming.

The interest rate as 0%

The interest rate as 15%

End of year	Operation cost at the end of year (Rs)	Maintenance cost at the end of year (Rs)	Salvage value at the end of year (Rs)
1	2000	2500	8000
2	3000	3000	7000
3	4000	3500	6000
4	5000	4000	5000
5	6000	4500	4000
6	7000	5000	3000
7	8000	5500	2000
8	9000	6000	1000

Ans :

Economic life of the machine = 2 years
Economic life of the machine = 2 years

PROJECT NETWORK

KEY CONCEPTS

Certain key concepts pertaining to a project network are described below:

1. Activity

An activity means a work. A project consists of several activities. An activity takes time. It is represented by an arrow in a diagram of the network. For example, an activity in house construction can be flooring. This is represented as follows:
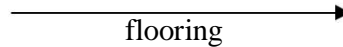
flooring

Construction of a house involves various activities. Flooring is an activity in this project. We can say that a project is completed only when all the activities in the project are completed.

2. Event

It is the beginning or the end of an activity. Events are represented by circles in a project network diagram. The events in a network are called the **nodes**.

Example:

Stop   **Start**   **Stop**
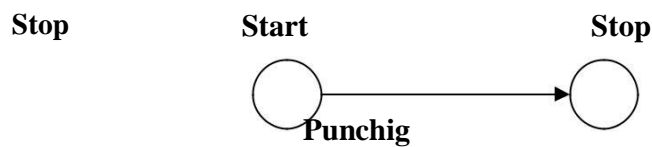
Punchig

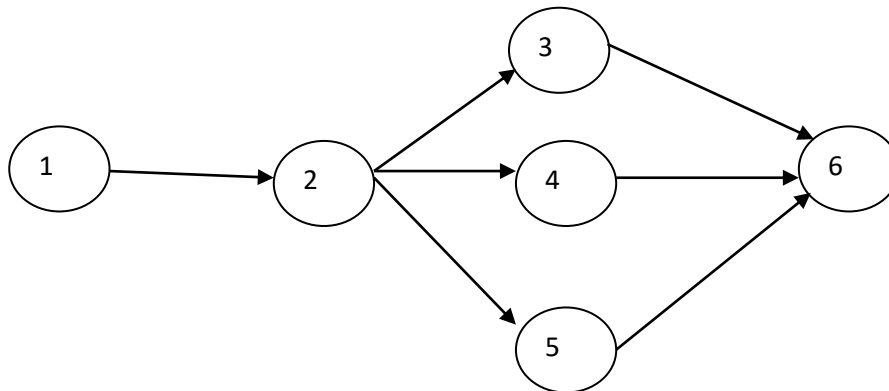Starting a punching machine is an activity. Stopping the punching machine is another activity.

3. Predecessor Event

The event just before another event is called the predecessor event.

4. Successor Event

The event just following another event is called the successor event.

Example: Consider the following.

In this diagram, event 1 is predecessor for the event 2.
Event 2 is successor to event 1.
Event 2 is predecessor for the events 3, 4 and 5.
Event 4 is predecessor for the event 6.
Event 6 is successor to events 3, 4 and 5.

5. Network

A network is a series of related activities and events which result in an end product or service. The activities shall follow a prescribed sequence. For example, while constructing a house, laying the foundation should take place before the construction of walls. Fitting water tapes will be done towards the completion of the construction. Such a sequence cannot be altered.

6. Dummy Activity

A dummy activity is an activity which does not consume any time. Sometimes, it may be necessary to introduce a dummy activity in order to provide connectivity to a network or for the preservation of the logical sequence of the nodes and edges.

7. Construction of a Project Network

A project network consists of a finite number of events and activities, by adhering to a certain specified sequence. There shall be a **start event** and an **end event (or stop event)**. All the other events shall be between the start and the end events. The activities shall be marked by directed arrows. An activity takes the project from one event to another event.

An event takes place at a point of time whereas an activity takes place from one point of time to another point of time.

CONSTRUCTION OF PROJECT NETWORK DIAGRAMS

Problem 1:

Construct the network diagram for a project with the following activities:

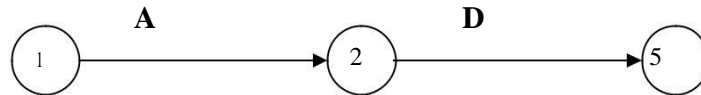| Event Event	Activity	Predecessor Activity
1 2	A	-
1 3	B	-
1 4	C	-
2 5	D	A
3 6	E	B
4 6	F	C
5 6	G	D

Solution:

The start event is node 1.

The activities A, B, C start from node 1 and none of them has a predecessor activity. A joins nodes1 and 2; B joins nodes 1 and 3; C joins nodes 1 and 4. So we get the following:
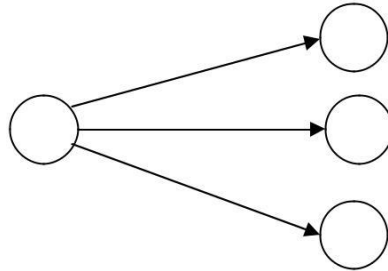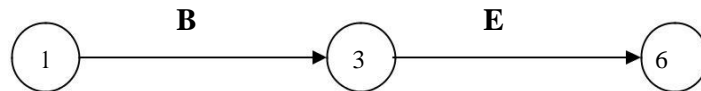

This is a part of the network diagram that is being constructed.
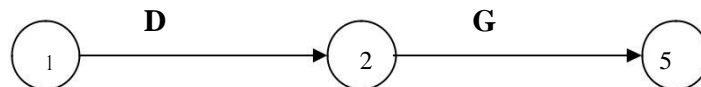
Next, activity D has A as the predecessor activity. D joins nodes 2 and 5. So we get
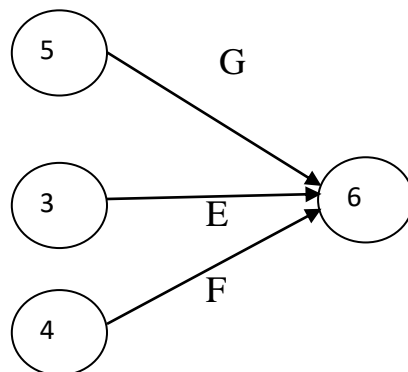

Next, activity E has B as the predecessor activity. E joins nodes 3 and 6. So we get

Next, activity G has D as the predecessor activity. G joins nodes 5 and 6. Thus we obtain
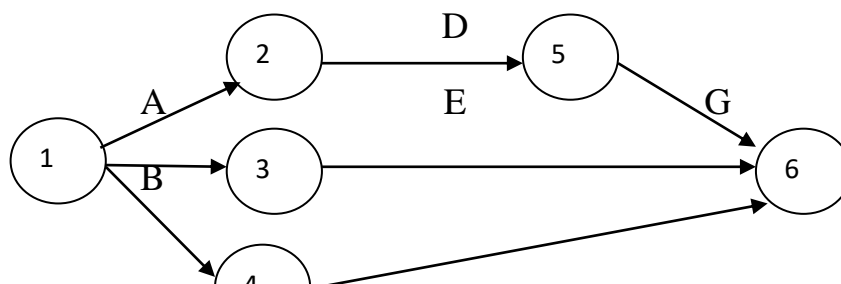

Since activities E, F, G terminate in node 6, we get

6 is the end event.

Combining all the pieces together, the following network diagram is obtained for the given project:

We validate the diagram by checking with the given data.

CRITICAL PATH METHOD (CPM)
INTRODUCTION
The critical path method (CPM) aims at the determination of the time to complete a project and the important activities on which a manager shall focus attention.
ASSUMPTION FOR CPM
In CPM, it is assumed that precise time estimate is available for each activity.
PROJECT COMPLETION TIME
From the start event to the end event, the time required to complete all the activities of the project in the specified sequence is known as the project completion time.
PATH IN A PROJECT
A continuous sequence, consisting of nodes and activities alternatively, beginning with the start event and stopping at the end event of a network is called a path in the network.
CRITICAL PATH AND CRTICAL ACTIVITIES
Consider all the paths in a project, beginning with the start event and stopping at the end event. For each path, calculate the time of execution, by adding the time for the individual activities in that path.

The path with the largest time is called the **critical path** and the activities along this path are called the **critical activities** or **bottleneck activities**. The activities are called critical because they cannot be delayed. However, a non-critical activity may be delayed to a certain extent. Any delay in a critical activity will delay the completion of the whole project. However, a certain permissible delay in a non –critical activity will not delay the completion of the whole project. It shall be noted that delay in a non-critical activity beyond a limit would certainly delay the completion the whole project. Sometimes, there may be several critical paths for a project. A project manager shall pay special attention to critical activities.

Problem 1:
The following details are available regarding a project:

Activity	Predecessor Activity	Duration (Weeks)
A	-	3
B	A	5
C	A	7
D	B	10
E	C	5

F	D,E	4

Determine the critical path, the critical activities and the project completion time.

Solution:

First let us construct the network diagram for the given project. We mark the time estimates along the arrows representing the activities. We obtain the following diagram:

Start event   **B**   **D**   **End event**
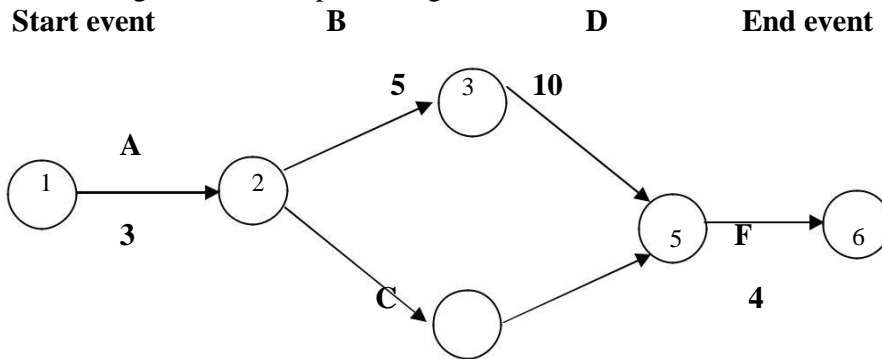
Consider the paths, beginning with the start node and stopping with the end node. There are two such paths for the given project. They are as follows:

Activity	Predecessor	Duration	EST	EFT	LST	LFT
A	-	3	0	3	0	3
B	A	7	3	10	3	10
C	A	5	3	8	10	15
D	B	10	10	20	10	20
E	C	5	8	13	15	20
F	D,E	4	20	24	20	24

Critical Path is A $\longrightarrow$ B $\longrightarrow$ D $\longrightarrow$ F

Project duration is 24 weeks

Programme Evaluation Review Technique

INTRODUCTION

Programme Evaluation and Review Technique (PERT) is a tool that would help a project manager in project planning and control. It would enable him in continuously monitoring a project and taking corrective measures wherever necessary. This technique involves statistical methods.

ASSUMPTIONS FOR PERT

Note that in CPM, the assumption is that precise time estimate is available for each activity in a project. However, one finds most of the times that this is not practically possible.

In PERT, we assume that it is not possible to have precise time estimate for each activity and instead, probabilistic estimates of time alone are possible. A multiple time estimate approach is followed here. In probabilistic time estimate, the following 3 types of estimate are possible:

Pessimistic time estimate (t_p)

Optimistic time estimate (t_o)

Most likely time estimate (t_m)

The optimistic estimate of time is based on the assumption that an activity will not involve any difficulty during execution and it can be completed within a short period. On the other hand, a pessimistic estimate is made on the assumption that there would be unexpected problems during the execution of an activity and hence it would consume more time. The most likely time estimate is made in between the optimistic and the pessimistic estimates of time. Thus the three estimates of time have the relationship

$$t_o \quad t_m \quad t_p .$$

Practically speaking, neither the pessimistic nor the optimistic estimate may hold in reality and it is the most likely time estimate that is expected to prevail in almost all cases. Therefore, it is preferable to give more weight to the most likely time estimate.

We give a weight of 4 to most likely time estimate and a weight of 1 each to the pessimistic and optimistic time estimates. We arrive at a time estimate (t_e) as the weighted average of these estimates as follows:

$$t_e \quad \frac{t_o \quad 4 t_m \quad t_p}{6}$$

Since we have taken 6 units (1 for t_p , 4 for t_m and 1 for t_o), we divide the sum by 6. With this time estimate, we can determine the project completion time as applicable for CPM.

Since PERT involves the average of three estimates of time for each activity, this method is very practical and the results from PERT will be have a reasonable amount of reliability.

MEASURE OF CERTAINTY

The 3 estimates of time are such that

$$t_o \quad t_m \quad t_p .$$

Therefore the range for the time estimate is $t_p \quad t_o .$

The time taken by an activity in a project network follows a distribution with a standard deviation of one sixth of the range, approximately.

i.e., The standard deviation = $\dfrac{t_p \quad t_o}{6}$

and the variance = $^2 \quad \left(\dfrac{t_p \quad t_o}{6}\right)^2$

The certainty of the time estimate of an activity can be analysed with the help of the variance. The greater the variance, the more uncertainty in the time estimate of an activity.

Problem 1:

Two experts A and B examined an activity and arrived at the following time estimates.

Expert	Time Estimate

	t_o	t_m	t_p
A	4	6	8
B	4	7	10

Determine which expert is more certain about his estimates of time:

Solution:

Variance ( ) in time estimates = $\dfrac{t_p - t_o}{6}^2$

In the case of expert A, the variance = $\dfrac{8-4}{6}^2 = \dfrac{4}{9}$

As regards expert B, the variance = $\dfrac{10-4}{6}^2 = 1$

So, the variance is less in the case of A. Hence, it is concluded that the expert A is more certain about his estimates of time.

Determination of Project Completion Time in PERT

Problem 2:

Find out the time required to complete the following project and the critical activities:

Activity	Predecessor Activity	Optimistic time estimate (t_o days)	Most likely time estimate (t_m days)	Pessimistic time estimate (t_p days)
A	-	2	4	6
B	A	3	6	9
C	A	8	10	12
D	B	9	12	15
E	C	8	9	10

F	D, E	16	21	26
G	D, E	19	22	25
H	F	2	5	8
I	G	1	3	5

Solution:

From the three time estimates t_p , t_m and t_o , calculate t_e for each activity. We obtain the following table:

Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	$t_o + 4t_m + t_p$	Time estimate $t_e = \dfrac{t_o + 4\,t_m + t_p}{6}$
A	2	16	6	24	4
B	3	24	9	36	6
C	8	40	12	60	10
D	9	48	15	72	12
E	8	36	10	54	9
F	16	84	26	126	21

G	19	88	25	132	22
H	2	20	8	30	5
I	1	12	5	18	3

Time for the path: 4+10+9+ 6+3 = 32 days.
Compare the times for the four paths.
Maximum of {48, 31, 49, 32} = 49.
We see that Path III has the maximum time.
Therefore the critical path is Path III. i.e., 1   2 4 5 6   8.
The critical activities are A, C, E, F and H.
The non-critical activities are B, D, G and I.
Project time (Also called project length) = 49 days.

UNIT I

INTRODUCTION TO OPERATIONS RESEARCH

Introduction

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" - which is the term we will use.

Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "OR/MS" or "ORMS". Yet other terms sometimes used are "industrial engineering" ("IE") and "decision science" ("DS"). In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

Operation Research is a relatively new discipline. The contents and the boundaries of the OR are not yet fixed. Therefore, to give a formal definition of the term Operations Research is a difficult task. The OR starts when mathematical and quantitative techniques are used to substantiate the decision being taken. The main activity of a manager is the decision making. In our daily life we make the decisions even without noticing them. The decisions are taken simply by common sense, judgment and expertise without using any mathematical or any other model in simple situations. But the decision we are concerned here with are complex and heavily responsible. Examples are public transportation network planning in a city having its own layout of factories, residential blocks or finding the appropriate product mix when there exists a large number of products with different profit contributions and production requirement etc.

Operations Research tools are not from any one discipline. Operations Research takes tools from different discipline such as mathematics, statistics, economics, psychology, engineering etc. and combines these tools to make a new set of knowledge for decision making. Today, O.R. became a professional discipline which deals with the application of scientific methods for making decision, and especially to the allocation of scarce resources. The main purpose of O.R. is to provide a rational basis for decisions making in the absence of complete information, because the systems composed of human, machine, and procedures may do not have complete information.

Operations Research can also be treated as science in the sense it describing, understanding and predicting the systems behavior, especially man-machine system. Thus O.R. specialists are involved in three classical aspect of science, they are as follows:

     i) Determining the systems behavior
     ii) Analyzing the systems behavior by developing appropriate models
     iii) Predict the future behavior using these models

The emphasis on analysis of operations as a whole distinguishes the O.R. from other research and engineering. O.R. is an interdisciplinary discipline which provided solutions to problems of military operations during World War II, and also successful in other operations.

Stages of Development of Operations Research

The stages of development of O.R. are also known as phases and process of O.R, which has six important steps. These six steps are arranged in the following order:

Step I: Observe the problem environment
Step II: Analyze and define the problem
Step III: Develop a model
Step IV: Select appropriate data input
Step V: Provide a solution and test its reasonableness
Step VI: Implement the solution

O.R. Tools and Techniques

Operations Research uses any suitable tools or techniques available. The common frequently used tools/techniques are mathematical procedures, cost analysis, electronic computation. However, operations researchers given special importance to the development and the use of techniques like linear programming, game theory, decision theory, queuing theory, inventory models and simulation. In addition to the above techniques, some other common tools are non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling (PERT/CPM), symbolic Model, information theory, and value theory. There is many other Operations Research tools/techniques also exists. The brief explanations of some of the above techniques/tools are as follows:

Linear Programming:

This is a constrained optimization technique, which optimize some criterion within some constraints. In Linear programming the objective function (profit, loss or return on investment) and constraints are linear. There are different methods available to solve linear programming.

Game Theory:

This is used for making decisions under conflicting situations where there are one or more players/opponents. In this the motive of the players are dichotomized. The success of one player tends to be at the cost of other players and hence they are in conflict.

Decision Theory:

Decision theory is concerned with making decisions under conditions of complete certainty about the future outcomes and under conditions such that we can make some probability about what will happen in future.

Queuing Theory:

This is used in situations where the queue is formed (for example customers waiting for service, aircrafts waiting for landing, jobs waiting for processing in the computer system, etc). The objective here is minimizing the cost of waiting without increasing the cost of servicing.

Inventory Models:

Inventory model make a decisions that minimize total inventory cost. This model successfully reduces the total cost of purchasing, carrying, and out of stock inventory.

Simulation:

Simulation is a procedure that studies a problem by creating a model of the process involved in the problem and then through a series of organized trials and error solutions attempt to determine the best solution. Some times this is a difficult/time consuming procedure. Simulation is used when actual experimentation is not feasible or solution of model is not possible.

Non-linear Programming:

This is used when the objective function and the constraints are not linear in nature. Linear relationships may be applied to approximate non-linear constraints but limited to some range, because approximation becomes poorer as the range is extended. Thus, the non-linear programming is used to determine the approximation in which a solution lies and then the solution is obtained using linear methods.

Dynamic Programming:

Dynamic programming is a method of analyzing multistage decision processes. In this each elementary decision depends on those preceding decisions and as well as external factors.

Integer Programming:

If one or more variables of the problem take integral values only then dynamic programming method is used. For example number or motor in an organization, number of passenger in an aircraft, number of generators in a power generating plant, etc.

Markov Process:

Markov process permits to predict changes over time information about the behavior of a system is known. This is used in decision making in situations where the various states are defined. The probability from one state to another state is known and depends on the current state and is independent of how we have arrived at that particular state.

Network Scheduling:

This technique is used extensively to plan, schedule, and monitor large projects (for example computer system installation, R & D design, construction, maintenance, etc.). The aim of this technique is minimize trouble spots (such as delays, interruption, production bottlenecks, etc.) by identifying the critical factors. The different activities and their relationships of the entire project are represented diagrammatically with the help of networks and arrows, which is used for identifying critical activities and path. There are two main types of technique in network scheduling, they are:

Program Evaluation and Review Technique (PERT) – is used when activities time is not known accurately/ only probabilistic estimate of time is available.

Critical Path Method (CPM) – is used when activities time is know accurately.

Information Theory:

This analytical process is transferred from the electrical communication field to O.R. field. The objective of this theory is to evaluate the effectiveness of flow of information with a given system. This is used mainly in communication networks but also has indirect influence in simulating the examination of business organizational structure with a view of enhancing flow of information.

Applications of Operations Research

Today, almost all fields of business and government utilizing the benefits of Operations Research. There are voluminous of applications of Operations Research. Although it is not feasible to cover all applications of O.R. in brief. The following are the abbreviated set of typical operations research applications to show how widely these techniques are used today:

Accounting:

- Assigning audit teams effectively
- Credit policy analysis
- Cash flow planning
- Developing standard costs
- Establishing costs for byproducts
- Planning of delinquent account strategy

Construction:
- Project scheduling, monitoring and control
- Determination of proper work force
- Deployment of work force
- Allocation of resources to projects

Facilities Planning:
- Factory location and size decision
- Estimation of number of facilities required
- Hospital planning
- International logistic system design
- Transportation loading and unloading
- Warehouse location decision

Finance:
- Building cash management models
- Allocating capital among various alternatives
- Building financial planning modelsInvestment analysis
- Portfolio analysis
- Dividend policy making

Manufacturing:
- Inventory control
- Marketing balance projection
- Production scheduling
- Production smoothing

Marketing:

- Advertising budget allocation
- Product introduction timing
- Selection of Product mix
- Deciding most effective packaging alternative

Organizational Behavior / Human Resources:
- Personnel planning
- Recruitment of employees
- Skill balancing
- Training program scheduling
- Designing organizational structure more effectively

Purchasing:
- Optimal buying
- Optimal reordering
- Materials transfer

Research and Development:
- R & D Projects control
- R & D Budget allocation
- Planning of Product introduction

Limitations of Operations Research

Operations Research has number of applications; similarly it also has certain limitations. These limitations are mostly related to the model building and money and time factors problems involved in its application. Some of them are as given below:

Operations Researchers job needs a mathematician or statistician, who might not be aware of the business problems. Similarly, a manager is unable to understand the complex nature of Operations Research. Thus there is a big gap between the two personnel.

Magnitude of Calculations

The aim of the O.R. is to find out optimal solution taking into consideration all the factors. In this modern world these factors are enormous and expressing them in quantitative model and establishing relationships among these require voluminous calculations, which can be handled only by machines.

Money and Time Costs

The basic data are subjected to frequent changes, incorporating these changes into the operations research models is very expensive. However, a fairly good solution at present may be more desirable than a perfect operations research solution available in future or after some time.

Non-quantifiable Factors

When all the factors related to a problem can be quantifiable only then operations research provides solution otherwise not. The non-quantifiable factors are not incorporated in O.R. models. Importantly O.R. models do not take into account emotional factors or qualitative factors.

Implementation

Once the decision has been taken it should be implemented. The implementation of decisions is a delicate task. This task must take into account the complexities of human relations and behaviour and in some times only the psychological factors.

LINEAR PROGRAMMING –GRAPHICAL METHOD

Introduction to Linear Programming

Linear Programming is a special and versatile technique which can be applied to a variety of management problems viz. Advertising, Distribution, Investment, Production, Refinery Operations, and Transportation analysis. The linear programming is useful not only in industry and business but also in non-profit sectors such as Education, Government, Hospital, and Libraries. The linear programming method is applicable in problems characterized by the presence of decision variables. The objective function and the constraints can be expressed as **linear functions** of the **decision variables**. The decision variables represent quantities that are, in some sense, controllable inputs to the system being modeled. An objective function represents some principal objective criterion or goal that measures the effectiveness of the system such as maximizing profits or productivity, or minimizing cost or consumption. There is always some practical limitation on the availability of resources viz. man, material, machine, or time for the system. These constraints are expressed as linear equations involving the decision variables. Solving a

linear programming problem means determining actual values of the decision variables that optimize the objective function subject to the limitation imposed by the constraints.

The main important feature of linear programming model is the presence of linearity in the problem. The use of linear programming model arises in a wide variety of applications. Some model may not be strictly linear, but can be made linear by applying appropriate mathematical transformations. Still some applications are not at all linear, but can be effectively approximated by linear models. The ease with which linear programming models can usually be solved makes an attractive means of dealing with otherwise intractable nonlinear models.

Linear Programming Problem Formulation

The linear programming problem formulation is illustrated through a product mix problem. The product mix problem occurs in an industry where it is possible to manufacture a variety of products. A product has a certain margin of profit per unit, and uses a common pool of limited resources. In this case the linear programming technique identifies the products combination which will maximize the profit subject to the availability of limited resource constraints.

Example:

Suppose an industry is manufacturing two types of products P1 and P2. The profits per Kg of the two products are Rs.30 and Rs.40 respectively. These two products require processing in three types of machines. The following table shows the available machine hours per day and the time required on each machine to produce one Kg of P1 and P2. Formulate the problem in the form of linear programming model.

Profit/Kg	P1 Rs.30	P2 Rs.40	Total available Machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

Solution:

The procedure for linear programming problem formulation is as follows:

Introduce the decision variable as follows:

Let x_1 = amount of P1

x_2 = amount of P2

In order to maximize profits, we establish the objective function as

$$30x_1 + 40x_2$$

Since one Kg of P1 requires 3 hours of processing time in machine 1 while the corresponding requirement of P2 is 2 hours. So, the first constraint can be expressed as

$$3x_1 + 2x_2 \leq 600$$

Similarly, corresponding to machine 2 and 3 the constraints are

$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

In addition to the above there is no negative production, which may be represented algebraically as

$$x_1 \geq 0 \qquad ; \qquad x_2 \geq 0$$

Thus, the product mix problem in the linear programming model is as follows:

Maximize
$$30x_1 + 40x_2$$
Subject to:
$$3x_1 + 2x_2 \leq 600$$
$$3x_1 + 5x_2 \leq 800$$
$$5x_1 + 6x_2 \leq 1100$$
$$x_1 \geq 0, x_2 \geq 0$$

Formulation with Different Types of Constraints

The constraints in the previous example 2.1 are of "less than or equal to" type. In this section we are going to discuss the linear programming problem with different constraints, which is illustrated in the following Example.

Example :

A company owns two flour mills viz. A and B, which have different production capacities for high, medium and low quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs.2000 and Rs.1500 per day to run mill A and B respectively. On a day, Mill A produces 6, 2 and 4 quintals of high, medium and low quality flour, Mill B produces 2, 4 and 12 quintals of high, medium and low quality flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically.

Solution:

Let us define x1 and x2 are the mills A and B. Here the objective is to minimize the cost of the machine runs and to satisfy the contract order. The linear programming problem is given by

Minimize
$$2000x_1 + 1500x_2$$
Subject to:
$$6x_1 + 2x_2 \geq 8$$
$$2x_1 + 4x_2 \geq 12$$
$$4x_1 + 12x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0$$

Graphical Analysis of Linear Programming

This section shows how a two-variable linear programming problem is solved graphically, which is illustrated as follows:

Example :

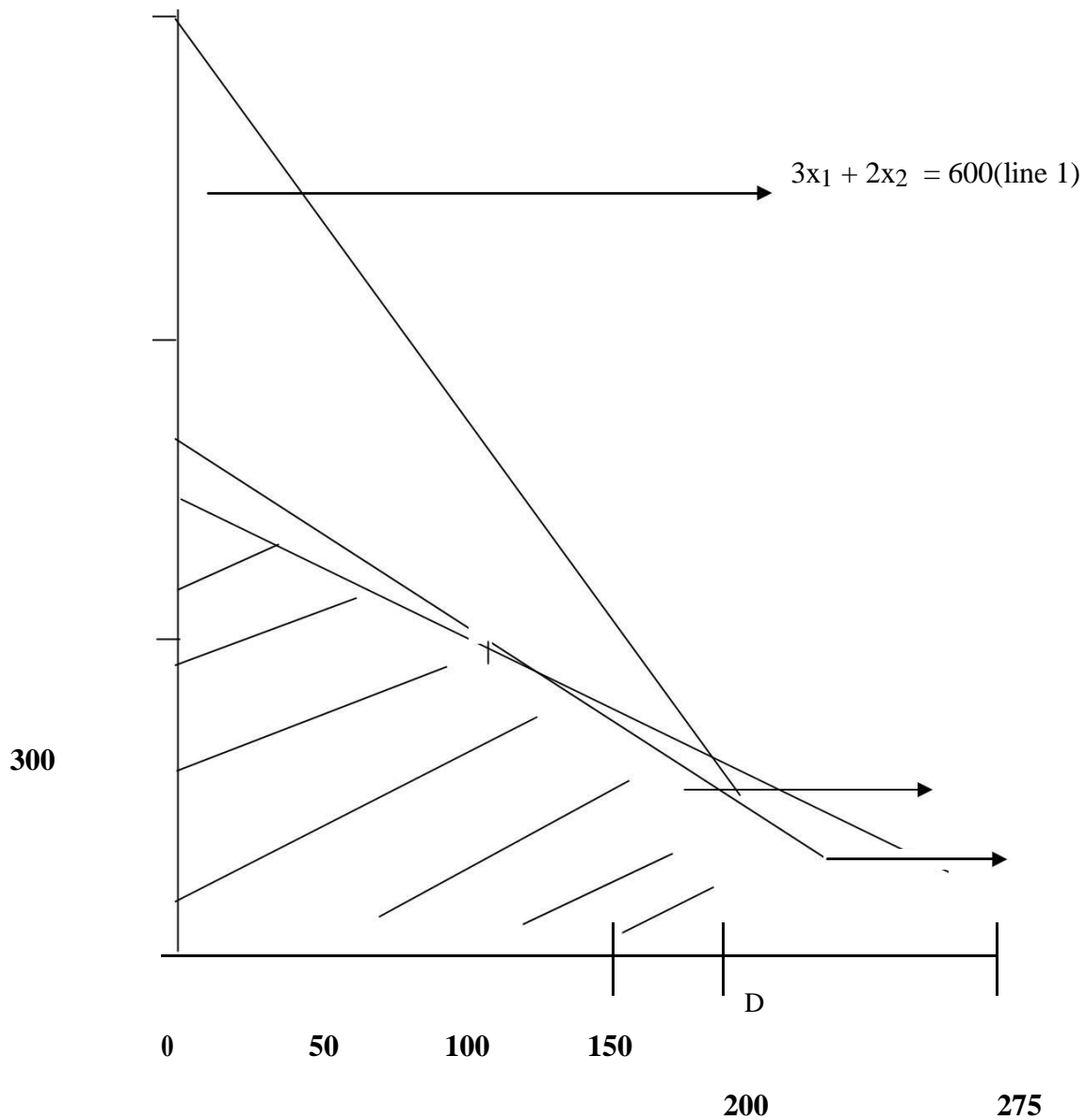Consider the problem discussed

Maximize

$30x_1 + 40x_2$

Subject to:

$3x_1 + 2x_2 \leq 600$

$3x_1 + 5x_2 \leq 800$

$5x_1 + 6x_2 \leq 1100$

$x_1 \geq 0, x_2 \geq 0$

From the first constraints $3x_1 + 2x_2 \leq 600$, draw the line $3x_1 + 2x_2 = 600$ which passes through the point (200, 0) and (0, 300). This is shown in the following graph as line 1.

Half Plane - A linear inequality in two variables is called as a half plane.

Boundary   - The corresponding equality (line) is called as the boundary of the half plane.

Close Half Plane – Half plane with its boundary is called as a closed half plane.

In this case we must decide in which side of the line $3x_1 + 2x_2 = 600$ the half plane is located. The easiest way to solve the inequality for x2 is

$$3x_1 \leq 600 - 2x_2$$

And for the fixed x1, the coordinates satisfy this inequality are smaller than the corresponding ordinate on the line and thus the inequality is satisfied for all the points below the line 1.

Similarly, we have to determine the closed half planes for the inequalities $3x_1 + 5x_2 \leq 800$ and

$5x_1 + 6x_2 \leq 1100$ (line 2 and line 3 in the graph). Since all the three constraints must be satisfied simultaneously we have consider the intersection of these three closed half planes. The complete intersection of these three closed half planes is shown in the above graph as ABCD. The region ABCD is called the feasible region, which is shaded in the graph.

Feasible Solution:

Any non-negative value of x_1, x_2 that is $x_1 \geq 0$ and $x_2 \geq 0$ is known as feasible solution of the linear programming problem if it satisfies all the existing constraints.

Feasible Region:

The collection of all the feasible solution is called as the feasible region.

LINEAR PROGRAMMING – SIMPLEX METHOD

Introduction

The Linear Programming with two variables can be solved graphically. The graphical method of solving linear programming problem is of limited application in the business problems as the number of variables is substantially large. If the linear programming problem has larger number of variables, the suitable method for solving is Simplex Method. The simplex method is an iterative process, through which it reaches ultimately to the minimum or maximum value of the objective function.

The simplex method also helps the decision maker/manager to identify the following:

Redundant   Constraints
Multiple Solutions
Unbounded   Solution
Infeasible Problem

Basics of Simplex Method

The basic of simplex method is explained with the following linear programming problem.

Example:

Maximize

$60x_1 + 70x_2$

Subject to:

$2x_1 + x_2 \leq 300$

$3x_1 + 4x_2 \leq 509$

$4x_1 + 7x_2 \leq 812$

$x_1, x_2 \geq 0$

Solution

First we introduce the variables

$$s_3, s_4, s_5 \geq 0$$

So that the constraints becomes equations, thus

$$2x_1 + x_2 + s_3 \qquad = 300$$
$$3x_1 + 4x_2 \qquad + s_4 = 509$$
$$4x_1 + 7x_2 \qquad\quad + s_5 = 812$$

Corresponding to the three constraints, the variables s_3, s_4, s_5 are called as slack variables. Now, the system of equation has three equations and five variables.

There are two types of solutions they are basic and basic feasible, which are discussed as follows:

Basic Solution

We may equate any two variables to zero in the above system of equations, and then the system will have three variables. Thus, if this system of three equations with three variables is solvable such a solution is called as basic solution.

For example suppose we take $x_1=0$ and $x_2=0$, the solution of the system with remaining three variables is $s_3=300$, $s_4=509$ and $s_5=812$, this is a basic solution and the variables s_3, s_4, and s_5 are known as basic variables where as the variables x_1, x_2 are known as non-basic variables.

The number of basic solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is m and the number of variables including the slack variables is n then there are at most ${}^nC_{n-m} = {}^nC_m$ basic solutions.

Basic Feasible Solution

A basic solution of a linear programming problem is called as basic feasible solutions if it is feasible it means all the variables are non-negative. The solution $s_3=300$, $s_4=509$ and $s_5=812$ is a basic feasible solution.

The number of basic feasible solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is m and the number of variables including the slack variables is n then there are at most ${}^nC_{n-m} = {}^nC_m$ basic feasible solutions.

Every basic feasible solution is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of given constraints. It is impossible to identify the extreme points geometrically if the problem has several variables but the extreme points can be identified using basic feasible solutions. Since one the basic feasible solution will maximize or minimize the objective function, the searching of extreme points can be carry out starting from one basic feasible solution to another.

The Simplex Method provides a systematic search so that the objective function increases in the cases of maximization progressively until the basic feasible solution has been identified where the objective function is maximized.

Simplex Method Computation

This section describes the computational aspect of simplex method. Consider the following linear programming problem

Maximize
$$60x_1 + 70x_2$$
Subject to:
$$2x_1 + x_2 + s_3 \qquad = 300$$
$$3x_1 + 4x_2 \qquad + s_4 = 509$$
$$4x_1 + 7x_2 \qquad\quad + s_5 = 812$$
$$x_1, x_2, s_3, s_4, s_5 \geq 0$$

The profit $Z=60x_1 + 70x_2$ i.e. Maximize $60x_1 + 70x_2$

The standard form can be summarized in a compact table form as

In this problem the slack variables s_3, s_4, and s_5 provide a basic feasible solution from which the simplex computation starts. That is $s_3==300$, $s_4=509$ and $s_5=812$. This result follows because of the special structure of the columns associated with the slacks.

If z represents profit then $z=0$ corresponding to this basic feasible solution. We represent by C_B the coefficient of the basic variables in the objective function and by XB the numerical values of the basic variable.

So that the numerical values of the basic variables are: $XB_1=300$, $XB_2=509$, $XB_3=812$. The profit $z=60x_1+70x_2$ can also expressed as $z-60x_1-70x_2=0$. The simplex computation starts with the first compact standard simplex table as given below:

C_B	Basic	C_j	60	70	0	0	0
	Variables	XB	x_1	x_2	s_3	s_4	s_5
0	s_3	300	2	1	1	0	0
0	s_4	509	3	4	0	1	0
0	s_5	812	4	7	0	0	1
	z		-60	-70	0	0	0

In the objective function the coefficients of the variables are $CB_1=CB_2=CB_3=0$. The topmost row of the Table 1 denotes the coefficient of the variables x_1, x_2, s_3, s_4, s_5 of the objective function respectively. The column under x_1 indicates the coefficient of x_1 in the three equations respectively. Similarly the remaining column also formed.

On seeing the equation $z=60x_1+70x_2$ we may observe that if either x_1 or x_2, which is currently non-basic is included as a basic variable so that the profit will increase. Since the coefficient of x_2 is higher we choose x_2 to be included as a basic variable in the next iteration. An equivalent criterion of choosing a new basic variable can be obtained the last row of Table 1 i.e. corresponding to z.

Since the entry corresponding to x_2 is smaller between the two negative values, x_2 will be included as a basic variable in the next iteration. However with three constraints there can be only three basic variables.

Thus, by bringing x_2 a basic variable one of the existing basic variables becomes non-basic. The question here is How to identify this variable? The following statements give the solution to this question.

Consider the first equation i.e. $2x_1 + x_2 + s_3 = 300$

From this equation

$$2x_1+s_3=300-x_2$$

But $x_1=0$. Hence, in order that $s_3 \geq 0$

$$300-x_2 \geq 0$$

$$i.e.\ x_2 \leq 300$$

Similarly consider the second equation i.e. $3x_1 + 4x_2 + s_4 = 509$ From this equation

$$3x_1+s_4=509-4x_2$$

But, $x_1=0$. Hence, in order that $s_4 \geq 0$

$$509-4x_2 \geq 0$$

i.e. $x_2 \leq 509/9$

Similarly consider the third equation i.e. $4x_1 + 7x_2 + s_5 = 812$

From this equation

$$4x_1+s_5=812-7x_2$$

But $x_1=0$. Hence, in order that $s_5 \geq 0$

$$812-7x_2 \geq 0$$

i.e. $x_2 \leq 812/7$

Therefore the three equation lead to

$x_2 \leq 300$, $x_2 \leq 509/9$, $x_2 \leq 812/7$

Thus $x_2 = \text{Min } (x_2 \leq 300, x_2 \leq 509/9, x_2 \leq 812/7)$ it means

$x_2 = \text{Min } (x_2 \leq 300/1, x_2 \leq 509/9, x_2 \leq 812/7) = 116$

Therefore $x_2 = 116$

If $x_2 = 116$, you may be note from the third equation

$$7x_2+s_5=812$$

i.e. $s_5 = 0$

Thus, the variable s_5 becomes non-basic in the next iteration.

So that the revised values of the other two basic variables are

$$S_3=300-x_2=184$$
$$S_4=509-4*116=45$$

Refer to Table 1, we obtain the elements of the next Table i.e. Table using the following rules:
        We allocate the quantities which are negative in the z-row. Suppose if all the quantities are positive, the inclusion of any non-basic variable will not increase the value of the objective function. Hence the present solution maximizes the objective function. If there are more than one negative values we choose the variable as a basic variable corresponding to which the z value is least as this is likely to increase the more profit.

Let x_j be the incoming basic variable and the corresponding elements of the jth row column be denoted by $Y1_j$, $Y2_j$ and $Y3_j$ respectively. If the present values of the basic variables are XB_1, XB_2 and XB_3 respectively, then we can compute.

Min $[XB_1/Y1_j, XB_2/Y2_j, XB_3/Y3j]$ for $Y1j, Y2j, Y3j>0$.

Note that if any $Y_{ij} \leq 0$, this need not be included in the comparison. If the minimum occurs corresponding to XBr/Yrj then the rth basic variable will become non-basic in the next iteration.

Using the following rules the Table 2 is computed from the Table

The revised basic variables are s_3, s_4 and x_2. Accordingly, we make $CB_1=0$, $CB_2=0$ and $CB_3=70$.

As x_2 is the incoming basic variable we make the coefficient of x_2 one by dividing each element of row-3 by 7. Thus the numerical value of the element corresponding to x_1 is 4/7, corresponding to s_5 is 1/7 in Table 2.

The incoming basic variable should appear only in the third row. So we multiply the third-row of Table 2 by 1 and subtract it from the first-row of Table 1 element by element. Thus the element corresponding to x_2 in the first-row of Table 2 is 0.

Therefore the element corresponding to x_1 is 2-1*4/7=10/7 and the element corresponding to s_5 is 0-1*1/7=-1/7

In this way we obtain the elements of the first and the second row in Table 2. In Table 2 the numerical values can also be calculated in a similar way.

C_B	Basic Variables	C_j X_B	60 x_1	70 x_2	0 s_3	0 s_4	0 s_5
0	s_3	184	10/7	0	1	0	-1/7
0	s_4	45	5/7	0	0	1	-4/7
70	x_2	116	4/7	1	0	0	1/7
	zj-cj		-140/7	0	0	0	70/7

Let CB_1, CB_2, Cb_3 be the coefficients of the basic variables in the objective function. For example in Table 2 $CB_1=0$, $CB_2=0$ and $CB_3=70$. Suppose corresponding to a variable J, the quantity z_j is

defined as $z_j=CB_1, Y_1+CB_2, Y_{2j}+CB_3Y_{3j}$. Then the z-row can also be represented as Z_j-C_j.

For example:

$z_1 - c_1 = 10/7*0+5/7*0+70*4/7-60 = -140/7$

$z_5 - c_5 = -1/7*0-4/7*0+1/7*70-0 = 70/7$

Now we apply rule (1) to Table 2. Here the only negative z_j-c_j is $z_1-c_1 = -140/7$ Hence x_1 should become a basic variable at the next iteration.

2. We compute the minimum of the ratio

$$\text{Min} \begin{pmatrix} \dfrac{184}{\dfrac{10}{7}}, \dfrac{45}{\dfrac{5}{7}}, \dfrac{116}{\dfrac{4}{7}} \end{pmatrix} = \text{Min} \begin{pmatrix} \dfrac{644, 63, 203}{5} \end{pmatrix} = 63$$

This minimum occurs corresponding to s_4, it becomes a non basic variable in next iteration.

3. Like Table 2, the Table 3 is computed sing the rules (i), (ii), (iii) as described above.

C_B	Basic Variables	C_j X_B	60 x_1	70 x_2	0 s_3	0 s_4	0 s_5
0	s3	94	0	0	1	-2	1
60	x1	63	1	0	0	7/5	-4/5
70	x2	80	0	1	0	-4/5	3/5
	z_j-c_j		0	0	0	28	-6

$z_5 - c_5 < 0$ should be made a basic variable in the next iteration.

Now compute the minimum ratios

$$\text{Min} \begin{pmatrix} \dfrac{94,}{1} & \dfrac{80}{\frac{3}{5}} \end{pmatrix} = 94$$

Note: Since $y_{25} = -4/5 < 0$, the corresponding ratio is not taken for comparison.
The variable s_3 becomes non basic in the next iteration.

3. From the Table 3, Table 4 is calculated following the usual steps.

C_B	Basic Variables	C_j X_B	60 x_1	70 x_2	0 s_3	0 s_4	0 s_5
0	s5	94	0	0	1	-2	1
60	x1	691/5	1	0	4/5	-1/5	0
70	x2	118/5	0	1	-3/5	2/5	0
	z_j-c_j		0	0	6	16	0

Note that $z_j - c_j \geq 0$ for all j, so that the objective function can't be improved any further.
Thus, the objective function is maximized for $x_1 = 691/5$ and $x_2 = 118/5$ and
The maximum value of the objective function is 9944.

Tow Phase and M-Method

In the last two section we discussed the simplex method was applied to linear programming problems with less than or equal to ($\leq$) type constraints. Thus, there we could introduce slack variables which provide an initial basic feasible solution of the problem.

Generally, the linear programming problem can also be characterized by the presence of both 'less than or equal to' type or 'greater than or equal to ($\geq$)' type constraints. In such case it is not always possible to obtain an initial basic feasible solution using slack variables.

The greater than or equal to type of linear programming problem can be solved by using the following methods:

Two Phase Method

M Method

In this section we will discuss these two methods.

Two Phase Method

We discuss the Two Phase Method with the help of the following Example 3.3.

Example

Minimize

$12.5x_1 + 14.5x_2$

Subject to:

$x_1 + x_2 \geq 2000$

$0.4x_1 + 0.75x_2 \geq 1000$

$0.075x_1 + 0.1x_2 \leq 200$

$x_1, x_2 \geq 0$

Solution

Here the objective function is to be minimized; the values of x_1 and x_2 which minimized this objective function are also the values which maximize the revised objective function i.e.

Maximize

$-12.5x_1 - 14.5x_2$

We can multiply the second and the third constraints by 100 and 1000 respectively for the convenience of calculation.

Thus, the revised linear programming problem is:

Maximize

$-12.5x_1 - 14.5x_2$

Subject to:

$x_1 + x_2 \geq 2000$

$40x_1 + 75x_2 \geq 100000$

$75x_1 + 100x_2 \leq 200000$

$x_1, x_2 \geq 0$

Now we convert the two inequalities by introducing **surplus variables** s_3 and s_4 respectively.

The third constraint is changed into an equation by introducing a **slack variable** s_5.

Thus, the linear programming problem becomes as

Maximize

$-12.5x_1 - 14.5x_2 = -25/2x_1 - 29/2x_2$

Subject to:

$x_1 + x_2 \ -s_3 \qquad\quad = 2000$

$40x_1 + 75x_2 \qquad -s_4 = 100000$

$75x_1 + 100x_2 \qquad\quad +s_5 = 200000$

$x_1, x_2, s_3, s_4, s_5 \geq 0$

Even though the surplus variables can convert greater than or equal to type constraints into

equations they are unable to provide initial basic variables to start the simplex method calculation. So we may have to introduce two more additional variables a_6 and a_7 called as **artificial variable** to facilitate the calculation of an initial basic feasible solution.

In this method the calculation is carried out in two phases hence tow phase method.

Phase I

In this phase we will consider the following linear programming problem

Maximize

$$-a_6 - a_7$$

Subject to:

$$x_1 + x_2 - s_3 \quad + a_6 = 2000$$
$$40x_1 + 75x_2 \quad - s_4 + a_7 = 100000$$
$$75x_1 + 100x_2 \quad + s_5 = 200000$$

$$x_1, x_2.s_3, s_4, s_5, a_6, a_7 \geq 0$$

The initial basic feasible solution of the problem is

$$a_6 = 2000, a_7 = 100000 \text{ and } s_5 = 200000.$$

As the minimum value of the objective function of the Phase I is zero at the end of the Phase I calculation both a_6 and a_7 become zero.

CB	Basic variables	C_j	0	0	0	0	0	-1	-1
		XB	x_1	x_2	s_3	s_4	s_5	a_6	a_7
-1	a_6	2000	1	1	-1	0	0	1	0
-1	a_7	100000	40	75	0	-1	0	0	1
0	s_5	200000	75	100	0	0	1	0	0
		z_j-c_j	-41	-76	1	1	0	0	0

CB	Basic variables	C_j	0	0	0	0	0	-1
		XB	x_1	x_2	s_3	s_4	s_5	a_6
-1	a_6	2000/3	7/15	0	-1	1/75	0	1
0	x_2	4000/3	8/15	1	0	-1/75	0	0
0	s_5	200000/3	65/3	0	0	4/3	1	0
		z_j-c_j	-1/15	0	1	-1/75	0	0

Here x_2 becomes a basic variable and a_7 becomes non basic variable in the next iteration. It is no longer considered for re-entry in the table.

Table 2
Then x1 becomes a basic variable and a6 becomes a non basic variable in the next iteration.

CB	Basic variables	C_j X_B	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5
0	x_1	10000/7	1	0	-15/7	1/35	0
0	x_2	4000/7	0	1	8/7	-1/35	0
0	s_5	250000/7	0	0	325/7	16/21	1
		z_j-c_j	0	0	0	0	0

The calculation of Phase I end at this stage. Note that, both the artificial variable have been removed and also found a basic feasible solution of the problem.
        The basic feasible solution is:
$$x_1 = 10000/7, \; x_2 = 4000/2, \; s_5 = 250000/7.$$

Phase II

The initial basic feasible solution obtained at the end of the Phase I calculation is used as the initial basic feasible solution of the problem. In this Phase II calculation the original objective function is introduced and the usual simplex procedure is applied to solve the linear programming problem.

CB	Basic variables	C_j X_B	-25/2 x_1	-29/2 x_2	0 s_3	0 s_4	0 s_5
-25/2	x_1	10000/7	1	0	-15/7	1/35	0
-29/2	x_2	4000/7	0	1	8/7	-1/35	0
0	s_5	250000/7	0	0	325/7	5/7	1
		z_j-c_j	0	0	143/14	2/35	0

In this Table 1 all $z_j-c_j \geq 0$ the current solution maximizes the revised objective function.
Thus, the solution of the problem is:
$$x_1 = 10000/7 = 1428 \text{ and } x_2 = 4000/7 = 571.4 \text{ and}$$
The Minimum Value of the objective function is: 26135.3

M Method

In this method also we need artificial variables for determining the initial basic feasible solution. The M method is explained in the next Example 3.4 with the help of the previous Example.

Example

Maximize

$$-12.5x_1 - 14.5x_2$$

Subject to:

$$x_1 + x_2 - s_3 = 2000$$
$$40x_1 + 75x_2 - s_4 = 100000$$
$$75x_1 + 100x_2 + s_5 = 200000$$
$$x_1, x_2, s_3, s_4, s_5 \geq 0.$$

Introduce the artificial variables a_6 and a_7 in order to provide basic feasible solution in the second and third constraints. The objective function is revised using a **large positive number** say **M**.

Thus, instead of the original problem, consider the following problem i.e.

Maximize

$$-12.5x_1 - 14.5x_2 - M(a_6 + a_7)$$

Subject to:

$$x_1 + x_2 - s_3 + a_6 = 2000$$
$$40x_1 + 75x_2 - s_4 + a_7 = 100000$$
$$75x_1 + 100x_2 + s_5 = 200000$$
$$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0.$$

The coefficient of a_6 and a_7 are large negative number in the objective function. Since the objective function is to be maximized in the optimum solution, the artificial variables will be zero. Therefore, the basic variable of the optimum solution are variable other than the artificial variables and hence is a basic feasible solution of the original problem.

The successive calculation of simplex tables is as follows:

CB	Basic variables	C_j XB	-12.5 x_1	-14.5 x_2	0 s_3	0 s_4	0 s_5	-M a_6	-M a_7
-M	a_6	2000	1	1	-1	0	0	1	0
-M	a_7	100000	40	75	0	-1	0	0	1
0	s_5	200000	75	100	0	0	1	0	0
		z_j-c_j	-41M +12.5	-76M +14.5	M	M	0	0	0

Since M is a large positive number, the coefficient of M in the $z_j - c_j$ row would decide the entering basic variable. As -76M < -41M, x_2 becomes a basic variable in the next iteration replacing a_7. The artificial variable a_7 can't be re-entering as basic variable.

CB	Basic variables	C_j XB	-12.5 x_1	-14.5 x_2	0 s_3	0 s_4	0 s_5	-M a_6
-M	a_6	2000/3	7/15	0	-1	1/75	0	1

-14.5	x_2	4000/3	8/15	1	0	-1/75	0	0
0	s_5	200000/3	65/3	0	0	4/3	1	0
	z_j-c_j		-7/15M +143/30	0	M	-M/75 +29/150	0	0

Now x_1 becomes a basic variable replacing a_6. Like a_7 the variable a_6 also artificial variable so it can't be re-entering in the table.

CB	Basic variables	C_j XB	-12.5 x_1	-14.5 x_2	0 s_3	0 s_4	0 s_5
-12.5	x_1	10000/7	1	0	-15/7	1/35	0
-14.5	x_2	4000/7	0	1	8/7	-1/35	0
0	s_5	250000/7	0	0	325/7	16/21	1
		z_j-c_j	0	0	143/14	2/35	0

Hence

The optimum solution of the problem is $x_1 = 10000/7$, $x_2 = 4000/7$ and
The Minimum Value of the Objective Function is: 26135.3

Multiple Solutions

The simplex method also helps in identifying multiple solutions of a linear programming problem. This is explained with the help of the following Example.

Example

Consider the following linear programming problem.

Maximize

$$2000x_1 + 3000x_2$$

Subject to:

$$6x_1 + 9x_2 \leq 100$$
$$2x_1 + x_2 \leq 20$$
$$x_1, x_2 \geq 0.$$

Solution

       Introduce the slack variables s_3 and s_4, so that the inequalities can be converted in to equation as follows:

$$6x_1 + 9x_2 + s_3 \quad = 100$$
$$2x_1 + x_2 \quad + s_4 = 20$$
$$x_1, x_2, s_3, s_4 \geq 0.$$

The computation of simple procedure and tables are as follows:

CB	Basic variables	C_j XB	2000 x_1	3000 x_2	0 s_3	0 s_4
0	s_3	100	6	9	1	0
0	s_4	20	2	1	0	1
		z_j-c_j	-2000	-3000	0	0

CB	Basic variables	Cj XB	2000 x_1	3000 x_2	0 s_3	0 s_4
0	x_2	100/9	2/3	1	1/9	0
0	s_4	80/9	4/3	0	-1/9	1
		z_j-c_j	0	0	3000/9	0

       Here z_j-$c_j \geq 0$ for all the variables so that we can't improve the simplex table any more. Hence it is optimum.

       The optimum solution is $x_1 = 0$, $x_2 = 100/9$ and

       The maximum value of the objective function is: $100000/3 = 33333.33$.

       However, the z_j-c_j value corresponding to the non basic variable x_1 is also zero. This indicates that there is more than one optimum solution for the problem exists.

       In order to calculate the value of the alternate optimum solution we have to introduce x_1 as a basic variable replacing s_4. The next Table 3 shows the computation of this.

CB	Basic variables	C_j XB	2000 x_1	3000 x_2	0 s_3	0 s_4
3000	x_2	20/3	0	1	1/6	1/2
2000	x_1	20/3	1	0	-1/12	3/4
		z_j-c_j	0	0	1000/3	3000

Thus,

$x_1 = 20/3$, $x_2 = 20/3$ also maximize the objective function and

The Maximum value of the objective function is: $100000/3 = 33333.33$

Thus, the problem has multiple solutions.

Unbounded Solution

In this section we will discuss how the simplex method is used to identify the unbounded solution. This is explained with the help of the following Example.

Example

Consider the following linear programming problem.

Maximize

$$5x_1 + 4x_2$$

Subject to:

$$x_1 - x_2 \le 8$$
$$x_1 \le 7$$
$$x_1, x_2 \ge 0.$$

Solution :

Introduce the slack variables s3 and s4, so that the inequalities becomes as equation as follows:

$$x_1 + s_3 = 7$$
$$x_1 - x_2 + s_4 = 8$$
$$x_1, x_2, s_3, s_4 \ge 0.$$

The calculation of simplex procedures and tables are as follows:

CB	Basic variables	C_j XB	5 x_1	4 x_2	0 s_3	0 s_4
0	s_3	7	1	0	1	0

0	s_4	8	1	-1	0	1
		z_j-c_j	-5	-4	0	0

CB	Basic variables	C_j XB	5 x_1	4 x_2	0 s_3	0 s_4
5	x_1	7	1	0	1	0
0	s_4	1	0	-1	-1	1
		z_j-c_j	0	-4	5	0

Note that $z_2-c_2 < 0$ which indicates x_2 should be introduced as a basic variable in the next iteration. However, both $y_{12} \leq 0$, $y_{22} \leq 0$.

Thus, it is not possible to proceed with the simplex method of calculation any further as we cannot decide which variable will be non basic at the next iteration. This is the criterion for unbounded solution.

NOTE: If in the course of simplex computation $z_j-c_j < 0$ but $y_{ij} \leq 0$ for all i then the problem has no finite solution.

But in this case we may observe that the variable x_2 is unconstrained and can be increased arbitrarily. This is why the solution is unbounded.

3.8 Infeasible Solution

This section illustrates how to identify the infeasible solution using simplex method. This is explained with the help of the following Example.

Example
Consider the following problem.

Minimize

$$200x_1 + 300x_2$$

Subject to:

$$2x_1 + 3x_2 \geq 1200$$
$$x_1 + x_2 \leq 400$$
$$2x_1 + 3/2x_2 \geq 900$$
$$x_1, x_2 \geq 0$$

Solution

Since it is a minimization problem we have to convert it into maximization problem and introduce the slack, surplus and artificial variables. The problem appears in the following manner after doing all these procedure.

Maximize

$-200x_1 - 300x_2$

Subject to:

$2x_1 + 3x_2 \quad -s_3 + a_6 = 1200$

$x_1 + x_2 \quad +s_4 = 400$

$2x_1 + 3/2x_2 - s_5 + a_7 = 900$

$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0$

Here the a_6 and a_7 are artificial variables. We use two phase method to solve this problem.

Phase I

Maximize

$-a_6 - a_7$

Subject to:

$2x_1 + 3x_2 \quad -s_3 + a_6 = 1200$

$x_1 + x_2 \quad +s_4 = 400$

$2x_1 + 3/2x_2 - s_5 + a_7 = 900$

$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0$

The calculation of simplex procedures and tables are as follows:

CB	Basic variables	C_j XB	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5	-1 a_6	-1 a_7
-1	a_6	1200	2	3	-1	0	0	1	0
0	s_4	400	1	1	0	1	0	0	1
-1	a_7	900	2	3/2	0	0	-1	0	0
		z_j-c_j	-4	-9/2	1	0	1	0	0

CB	Basic	C_j	0	0	0	0	0	-1
	variables	XB	x_1	x_2	s_3	s_4	s_5	a_7
0	x_2	400	2/3	1	-1/3	0	0	0
0	s_4	0	1/3	0	1/3	1	0	0
-1	a_7	300	1	0	1/2	0	-1	1
		z_j-c_j	-1	0	-1/2	0	1	0

Table 2

CB	Basic	C_j	0	0	0	0	0	-1
	variables	XB	x_1	x_2	s_3	s_4	s_5	a_7
0	x_2	400	0	1	-1	-2	0	0
0	x_1	0	1	0	1	3	0	0
-1	a_7	300	0	0	-1/2	-3	-1	1
		z_j-c_j	0	0	1/2	3	1	0

Note that z_j-$c_j \geq 0$ for all the variables but the artificial variable a_7 is still a basic variable. This situation indicates that the problem has no feasible solution.