2.1) The production function is concave in K_t and L_t and everywhere increasing, so taking first-order conditions for the static labor problem yields an interior solution, and the firm pays competitive wages so takes them as given. This follows the notation of the slides.

The firm's cash flow is given by

$$\Pi(K_t, L_t, I_t, p_t^k, w_t) = p_t^y F(K_t, L_t, w_t) - p_t^k I_t - w_t L_t.$$

$p_t^y = 1$ is numeraire.

The firm's first-order conditions for labor are

$$\frac{\partial \Pi}{\partial L_t} = 0 \rightarrow \quad F_L(K_t, L_t, w_t, p_t^k) = w_t \quad \text{, that is,}$$

the firm hires until the nominal wage equals the value of marginal product of labor, and the subscript denotes a partial derivative.

Here $F_L = \alpha K_t^\alpha L_t^{\alpha-1}$, so the firm hires labor such that

$$\alpha K_t^\alpha L_t^{\alpha-1} = w_t \text{ , or}$$

$$L_t^{*\,\alpha-1} = \frac{1}{\alpha} w_t K_t^{-\alpha}$$

$$\boxed{L_t^* = \left[\frac{w_t}{K_t^\alpha \alpha}\right]^{\frac{1}{\alpha-1}} \qquad (1)}$$

Replacing (1) into the cash flow equation above, we have

$$\Pi(K_t, L_t^*, w_t, p_t^k) = F(K_t, L_t^*(K, w, p_t^k), w_t) - w_t L_t^* - p_t^k I_t$$

replacing $I_t = K_{t+1} - K_t(1-\delta)$

$$= K_t^\alpha L_t^{*\alpha} - w_t L_t^* - p_t^k (K_{t+1} - K_t(1-\delta))$$

and inserting L_t^*,

$$\boxed{\Pi = K_t^\alpha \left(\frac{1}{\alpha} w_t K_t^{-\alpha}\right)^{\frac{\alpha}{\alpha-1}} - w_t \left(\frac{1}{\alpha} w_t K_t^{-\alpha}\right)^{\frac{1}{\alpha-1}} - p_t^k K_{t+1} + p_t^k K_t(1-\delta)}$$

2.2 Letting $\Pi(K_t, w_t, p_t^k)$ denote flow revenue net of labor costs, the firm's sequential problem becomes:

$$V(k_0, x_0) = \max_{\{K_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \; \Pi(K_t, L_t^*, p_t^k, w_t) - p_t K_{t+1} - p_t^k K_t (1-\delta) \right]$$

The Bellman equation, given the recursive nature of the problem and the firm's decreasing returns to scale in production, can be written as below, where $\Pi^*(K_t, w_t, p_t^k, L_t^*)$ denotes cash flows pre-investment given the optimal labor choice, or

$$\Pi^*(K_t, w_t, p_t^k) = F(K_t, L_t^*, w_t, p_t^k) - w_t L_t^* :$$

$$V(K_t, w_t, p_t^k) = \max_{I_t} \left[\Pi^*(K_t, w_t, p_t^k, L_t^*) - p_t I_t + \beta E_t V(K_{t+1}, w_{t+1}, p_{t+1}^k) \right.$$

Again using that $K_{t+1} = (1-\delta) K_t + I_t$ (and assuming I_t becomes operational only in the next period),

$$\boxed{V(K_t, w_t, p_t^k) = \max_{I_t} \left[\Pi^*(K_t, L_t^*, w_t, p_t^k) - p_t I_t + \beta E_t V(K_t(1-\delta) + I_t, w_{t+1}, p_{t+1}^k) \right] (2^*)}$$

In our context with our particular functional form, this yields

$$V(K_t, w_t, p_t^k) = \max_{I_t} K_t^{\alpha} \alpha^{\frac{\alpha}{1-\alpha}} w_t^{\frac{-\alpha}{1-\alpha}} K_t^{\frac{\alpha^2}{1-\alpha}} - w_t^{\frac{-\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} K_t^{\frac{\alpha}{1-\alpha}} - p_t^k K_{t+1} + p_t^k K_t (1-\delta) + \beta E_t V(K_{t+1}, w_{t+1}, p_{t+1}^k)$$

2.3) a The production function is concave in capital and smooth, so the first-order conditions will give us an interior solution:

The optimal investment will be

$$\frac{\partial V(K_t, w_t, p_t^k)}{\partial I_t} = 0 \rightarrow p_t^k = \beta E_t V_K(K_t(1-\delta) + I_t, w_{t+1}, p_{t+1}^k) \qquad (3)$$

To obtain V_K, we note that for the value function to give a true optimum, both sides of equation (2^*) must hold identically, and thus we can ~~derive~~ take the derivative of both sides:

$$\frac{\partial V(K_t, w_t, p_t^k)}{\partial K} = \Pi_K^*(K_t, w_t, p_t^k) - p_t^k \frac{\partial I_t}{\partial K_t} + (1-\delta)\beta E_t V_K(K_t(1-\delta) + I_t, w_{t+1}, p_{t+1}^k)$$

$$+ \beta E_t V_K(K_t(1-\delta) + I_t, w_{t+1}, p_{t+1}^k)\frac{\partial I_t}{\partial K_t}$$

$$= \Pi_K^* - \frac{\partial I}{\partial K_t}\underbrace{\left(-p_t^k + \beta E_t V_K(K_t(1-\delta) + I_t, w_{t+1}, p_{t+1}^k)\right)}_{=0 \text{ by } (3)}$$

$$+ (1-\delta)\underbrace{\beta E_t V_K(K_{t+1}, w_{t+1}, p_{t+1}^k)}_{= p_t^k \text{ by } (3)}$$

So that $V_K(K_t, w_t, p_t^k) = \Pi_K^*(K_t, w_t, p_t^k) + (1-\delta)\beta E_t p_t^k. \qquad (4)$

Furthermore, as $\pi^*(K_t, w_t, p_t^k) \overset{=1}{=} p_t^y F(K_t, L_t^*(K_t, w_t, p_t^k), w_t, p_t^k) - w_t L_t^*(K_t, w_t, p_t^k)$, then the total derivative $\frac{d\pi^*}{dK}$ can be obtained by the chain rule:

$$\frac{d\pi^*}{dK} = \Pi_K^* = \frac{\partial \pi^*}{\partial K_t} + \frac{\partial \pi^*}{\partial L}\frac{\partial L}{\partial K} + \frac{\partial \pi^*}{\partial w}\frac{\partial w}{\partial K} + \frac{\partial \pi^*}{\partial p_t^k}\frac{\partial p_t^k}{\partial K} \qquad (5)$$

As Π^* already is at the maximum labor choice, $\frac{\partial \pi^*}{\partial L} = 0$.

As p_t^k, w_t are stochastic, $\frac{\partial w_t}{\partial K} = \frac{\partial p_t^k}{\partial K} = 0$, so (5) simplifies to

$$\Pi_K^*(K_t, w_t, p_t^k) = \frac{\partial \pi^*}{\partial K} = F_K(K_t, L_t^*, w_t, p_t^k) \qquad (6).$$

Inserting (6) back into (4) yields

$$V_K(K_t, w_t, p_t^k) = F_K(K_t, L_t^*, w_t, p_t^k) + (1-\delta)p_t^k \qquad (7).$$

2.3b) Iterating forward one year and plugging that function into (3) yields the optimality condition,

$$(0c) \qquad \boxed{p_t^k = \beta E_t \left[p_{t+1}^y F_K \left(k_{t+1}, L_{t+1}^* (p_{t+1}, k_{t+1}, w_{t+1}), w_{t+1}, p_{t+1}\right) + (1-\delta) p_{t+1}^K \right]}$$

Given our functional form, with

$F_K(k_t, L_t) = \alpha K_t^{\alpha-1} L_t^{\alpha}$,

and $F_K(k_t, L_t^*) = \alpha (K_t^{\alpha-1}) \alpha^{\frac{\alpha}{1-\alpha}} w_t^{\frac{-\alpha}{1-\alpha}} K_t^{\frac{\alpha^2}{1-\alpha}}$, the above becomes

$$p_t^k = \beta E[\alpha K_{t+1}^{*\,\alpha-1} \alpha^{\frac{\alpha}{1-\alpha}} [w_t(1-g)]^{\frac{-\alpha}{1-\alpha}} K_{t+1}^{*\,\frac{\alpha^2}{1-\alpha}}] + (1-\delta)\beta E[p_{t+1}^k],$$

using that $w_{t+1} = w_t(1+g)$.

Solving for K_{t+1}^*, we have

$$\boxed{K_{t+1}^* = \left[\frac{p_t^k - (1-\delta) p_t^k \beta E[\varepsilon_{t+1}]}{\beta E\,[w_t(1+g)]^{\frac{-\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}}} \right]^{\frac{1-\alpha}{2\alpha-1}} \qquad (8)}$$

2.4) We work first with the transformed variable: let $X = \ln \varepsilon_t$, so that $X \sim N(\mu, \sigma^2)$.

The MGF of a standard normal is, for $Z \sim N(0,1)$

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz$$

completing the square gives

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2tz + t^2 - t^2)} dz$$

$$= e^{\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} dz$$, as $e^{t^2/2}$ doesn't depend on Z, then we can pull it out.

But now the integral is just a $N(t,1)$ pdf, so equals 1:

$$M_Z(t) = e^{t^2/2}.$$

Then $M_X(t) = E[e^{t(\mu + \sigma Z)}]$ for $X = \mu + \sigma Z$, or $\frac{X-\mu}{\sigma} = Z$

$= E(e^{\mu t} e^{\sigma Z})$, and $e^{\mu t}$ is now known

$= e^{\mu t} E(e^{\sigma Z})$, and replacing $t = \sigma z$ in the above calculation, we have

$M_X(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$,

2.4) Now for $\varepsilon_t = e^x$, then

$$E(\varepsilon_t^m) = E(e^{mx}) = M_x(m) = e^{m\mu + \frac{1}{2}\sigma^2 m^2}.$$

That is, the mth moment of log-normal ε_t is the MGF of the normal, evaluated at $t = m$.

Then to obtain the mean, we take $m=1$, and get

$$E[\varepsilon_t] = M_x(1) = e^{\mu + \frac{1}{2}\sigma^2} \qquad (9),$$

which doesn't depend on the time subscript, so that

$$E[\varepsilon_{t+1}] = e^{\mu + \frac{1}{2}\sigma^2} \text{ as well.}$$

Now, inserting the value for $E(\varepsilon_{t+1})$ into (8) and taking logs, we have:

$$\ln K_{t+1}^* = \frac{1-\alpha}{2\alpha-1}\left[\ln\left(p_t^k - (1-\delta)p_t^k \beta E(\varepsilon_{t+1})\right) - \ln \beta E_t\left[w_t(1+g)\right]^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}}\right]$$

using log rules: $\ln(ab) = \ln(a) + \ln(b)$, and $\ln a^x = x \ln a$, we can simplify:

$$= \frac{1-\alpha}{2\alpha-1}\left[\ln\left(p_t^k\left(1-(1-\delta)\beta E(\varepsilon_{t+1})\right)\right) - \ln\beta - \ln E_t(w_t(1+g))^{\frac{-\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}}\right]$$

$$= \frac{1-\alpha}{2\alpha-1}\left[\ln p_t^k + \ln\left(1-(1-\delta)\beta E(\varepsilon_{t+1})\right) - \ln\beta - \frac{1}{1-\alpha}\ln\alpha + \frac{\alpha}{1-\alpha}\ln w_t + \frac{\alpha}{1-\alpha}\ln(1+g)\right]$$

, as $w_{t+1} \doteq (1+g)w_t$ is deterministic. & pulling out the (-1) from the $\frac{1-\alpha}{(-1)(1-2\alpha)}$ and distributing it inside the [ ]:

$$= \frac{1-\alpha}{1-2\alpha}\left[\frac{1}{1-\alpha}\left[\ln\alpha - \alpha[\ln(w_t) + \ln(1+g)]\right] + \ln\beta - \ln p_t^k - \ln\left(1-(1-\delta)\beta E(\varepsilon_{t+1})\right)\right]$$

and letting $\beta = \frac{1}{1+r} = (1+r)^{-1}$, so $\ln(1+r)^{-1} = -\ln(1+r)$, and now substituting $E(\varepsilon_{t+1}) = e^{\frac{1}{2}\sigma^2+\mu}$:

$$\ln K_{t+1}^* = \frac{1-\alpha}{1-2\alpha}\left[\frac{1}{1-\alpha}\left[\ln\alpha - \alpha[\ln(w_t) + \ln(1+g)]\right] - \ln(1+r) - \ln p_t^k - \ln\left(1 - \frac{1-\delta}{1+r}e^{\frac{1}{2}\sigma^2+\mu}\right)\right],$$

as desired. The current stock of capital doesn't appear here because in this model of investment with no adjustment costs, the next-period capital stock is simply the current plus optimal investment: the choice of next-period stock is not dependent on today's since it costs nothing additional to bridge the difference between optimal capital today and optimal capital tomorrow.

2.5) using the approximation that $\cancel{\epsilon_g}$ $x \approx \ln(1+x)$, and that $i_t := \frac{I_t - \delta K_t}{K_t}$ (using the notation of I_t for investment

that only becomes valid in the next period) we have:

$$1 + i_t = \frac{K_t}{K_t} + \frac{I_t - \delta K_t}{K_t} = \frac{K_t(1-\delta) + I_t}{K_t} = \frac{K_{t+1}}{K_t}.$$

Then $i_t \approx \ln(1+i_t)$

$$= \ln\left(\frac{K_{t+1}}{K_t}\right) = \ln(K^*_{t+1}) - \ln(K^*_t) \text{ by log rules.}$$

Now, investment in period $t-1$ ~~do~~ lead to optimal capital stock in period t, and likewise optimal investment in period t lead to optimal capital in time $t+1$.

Therefore we may utilize the formula we found in (2.4), with adaptations for the difference in time subscripts. This yields, breaking apart the pieces in (2.4)'s equation into the parts that are time-independent

$$\ln(K^*_{t+1}) = \underbrace{\left[\frac{1-\alpha}{1-2\alpha}\left[\frac{1}{1-\alpha}\ln\alpha - \frac{\alpha}{1-\alpha}\ln(1+g) - \ln(1+r) - \ln(1) + \frac{1+\delta}{1+r}e^{\frac{1}{2}\sigma^2+\mu}\right.\right.}_{=\Gamma(\alpha, g, r, \sigma, \mu, \delta)}$$

$$+ \frac{1-\alpha}{1-2\alpha}\left[\frac{-\alpha}{1-\alpha}\ln w_t - \ln p^k_t\right]$$

Then $\ln(K^*_t) = \Gamma(\alpha, g, r, \sigma, \mu, \delta) + \left[\frac{1-\alpha}{1-2\alpha}\left[\frac{-\alpha}{1-\alpha}\ln\left(\frac{w_t}{1+g}\right) - \ln\left(\frac{p^k_t}{\varepsilon_t}\right)\right]\right]$

So that

$$\ln(K^*_{t+1}) - \ln(K^*_t) = \frac{1-\alpha}{1-2\alpha}\left[\frac{-\alpha}{1-\alpha}\ln w_t - \ln p^k_t + \frac{\alpha}{1-\alpha}\ln\left(\frac{w_t}{1+g}\right) + \ln\left(\frac{p^k_t}{\varepsilon_t}\right)\right]$$

$$= \frac{1-\alpha}{1-2\alpha}\left[\frac{-\alpha}{1-\alpha}\ln(1+g) - \ln\varepsilon_t\right],$$ and using again that $\ln(1+g) \approx g$ for small g,

$$\approx \frac{1-\alpha}{1-2\alpha}\left[\frac{-\alpha}{1-\alpha}g - \ln\varepsilon_t\right]$$

$$i_t \approx \frac{-\alpha}{1-2\alpha}g - \frac{1-\alpha}{1-2\alpha}\ln\varepsilon_t,$$ as we wished to show.

Since $\alpha \in (0, \frac{1}{2})$, ~~g~~ ~~g~~ an increase in g (wage-growth is positive) causes a decrease in investment. As ε_t provides the shock behind the price of capital, ↘

2.5b) An increase in the shock (say, if μ increased so that the expected value of the shock increased) to capital prices causes a decrease in the net investment rate.

Interestingly, the interest rate does not affect the net investment rate.

2.6) Continuing to use our log approximation, and assuming g is known, we have:

$E[i_t] \approx E\left[\frac{-\alpha}{1-2\alpha} g - \frac{1-\alpha}{1-2\alpha} \ln \varepsilon_t\right].$

$= \frac{-\alpha}{1-2\alpha} g - \frac{1-\alpha}{1-2\alpha} E[\ln \varepsilon_t]$. As $\ln \varepsilon_t \sim N(\mu, \sigma^2)$,

by the linearity of expectation, $E[\ln \varepsilon_t] = \mu$

$\boxed{E[i_t] = \frac{-\alpha}{1-2\alpha} g - \frac{1-\alpha}{1-2\alpha} \mu.}$, as we wished to show.

$\text{cov}(i_t, i_{t-1}) = E(i_t i_{t-1}) - E(i_t)E(i_{t-1})$ by the definition of covariance

We'll show this is zero, and then $\text{corr}(i_t, i_{t-1}) = \frac{\text{cov}(i_t, i_{t-1})}{SD(i_t)\, SD(i_{t-1})}$ will be clearly zero as well.

Then note $E[i_{t-1}] = E[i_t]$, as the expectation did not depend on time t, and $\ln \varepsilon_t$ are iid, so that $\ln \varepsilon_{t-1} \sim N(\mu, \sigma^2)$, and $E(i_{t-1})$ has the same expectation as above.

Then $E[i_t i_{t-1}] = E\left[\left(\frac{-\alpha}{1-2\alpha} g - \frac{1-\alpha}{1-2\alpha} \ln \varepsilon_t\right)\left(\frac{-\alpha}{1-2\alpha} g - \frac{1-\alpha}{1-2\alpha} \ln \varepsilon_{t-1}\right)\right]$

$= E\left[\left(\frac{-\alpha}{1-2\alpha} g\right)^2 + \left(\frac{-\alpha}{1-2\alpha} g\right)\left(\frac{-(1-\alpha)}{1-2\alpha}\right) \ln \varepsilon_{t-1} + \left(\frac{-(1-\alpha)}{1-2\alpha} \ln \varepsilon_t\right)\left(\frac{-\alpha}{1-2\alpha}\right) g + \left(\frac{1-\alpha}{1-2\alpha}\right)^2 \ln \varepsilon_t \ln \varepsilon_{t-1}\right]$

by expanding terms. By the linearity of expectation and taking out what's known, we have

$= \left(\frac{-\alpha}{1-2\alpha} g\right)^2 + \frac{\alpha(1-\alpha)}{(1-2\alpha)^2} g E[\ln \varepsilon_{t-1}] + g \frac{\alpha(1-\alpha)}{(1-2\alpha)^2} E[\ln \varepsilon_t] + \left(\frac{1-\alpha}{1-2\alpha}\right)^2 E(\ln \varepsilon_t \ln \varepsilon_{t-1})\}$

As the $\ln \varepsilon_t$ are iid, thus independent across time, their expectation factors, so $E(\ln \varepsilon_t \ln \varepsilon_{t-1}) = E(\ln \varepsilon_t) E(\ln \varepsilon_{t-1})$

2.6) Then replacing $E[\ln \xi_t] = E[\ln \xi_{t-1}] = \mu$, we have

$$E(i_t i_{t-1}) = \left(\frac{-\alpha}{1-2\alpha}g\right)^2 + \frac{2\alpha(1-\alpha)}{(1-2\alpha)^2} g\mu + \left(\frac{1-\alpha}{1-2\alpha}\right)^2 \mu^2,$$

$$= E(i_t)E(i_{t-1}),$$

So that $\mathrm{cov}(i_t, i_{t-1}) = E(i_t i_{t-1}) - E(i_t)E(i_{t-1}) = 0$, and thus $\mathrm{corr}(i_t, i_{t-1}) = 0$, as desired.

~~Finally,~~ $\cancel{V[i_t] = E[i_t^2] - E(i_t)^2}$ (Sorry, not needed)

Note $V(a+bX)$ for a, b degenerate is

$$V(a+bX) = b^2 V(X).$$

~~We have~~ $\cancel{E(i_t)}$

Then $V(i_t) = V\left(\frac{-\alpha}{1-2\alpha}g - \frac{1-\alpha}{1-2\alpha}\ln \xi_t\right)$

$= \left(\frac{1-\alpha}{1-2\alpha}\right)^2 V(\ln \xi_t)$, and as $\ln \xi_t \sim N(\mu, \sigma^2)$,

$$V(i_t) = \left(\frac{1-\alpha}{1-2\alpha}\right)^2 \sigma^2,$$ as desired.

These relationships, most strikingly the absence of any variable dependent on time, indicate that there's no persistence in investment in this model, which was a key fact discussed in lecture. Because there's no cost to rapid changes in investment, once a shock to the price of capital occurs, investment responds immediately to the change, and then in the next period with that shock (in t+1), responds to the (t+1) shock: there's no relationship over time, and the investment rate's variance is a linear function in the variance of the price shock: when the variance of $\ln \xi_t$ changes, the variance of net investment moves in perfect lockstep. As Giuseppe said, "garbage in, garbage out": there are no underlying mechanisms acting on the model here, we get precisely what we'd put in by assumption.

2.7) The interest rate r never shows up!
The only value that changes with the growth rate of wages is the expected value of the net investment rate, $E(i_t)$.

That is, serial autocorrelation, the expected value, and variance on the net investment rate are entirely independent of the investment rate.

Why is this? It's ~~likely~~ possibly due to there being no costs of adjustment in this model: if it doesn't cost the firm to delay or bring forward investment, there's no additional burden for discounting: the firm simply "makes investment happen" in the period in which it's optimal to do so.

Additionally, the price of capital is independent of the interest rate: typically, one ~~would~~ could think that interest rates might be persistent over time, affecting the future costs of capital. However, since capital adjusts in one period, and the interest rate doesn't change, these considerations are moot.

I'd be interested in unpacking this more in Office Hours, I'm not particularly satisfied with this answer (explanation).