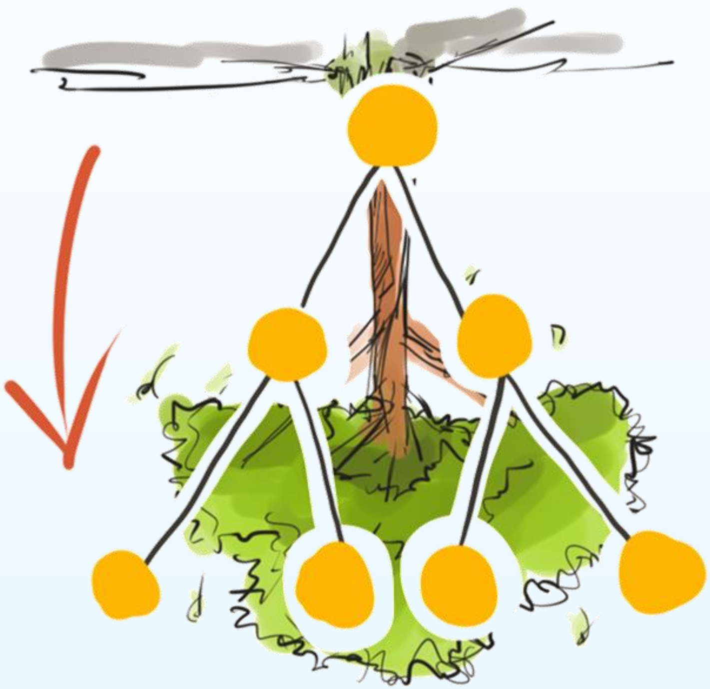

Lecture 13: Graph ADT and Theory

01204212 Abstract Data Types and Problem Solving

Department of Computer Engineering
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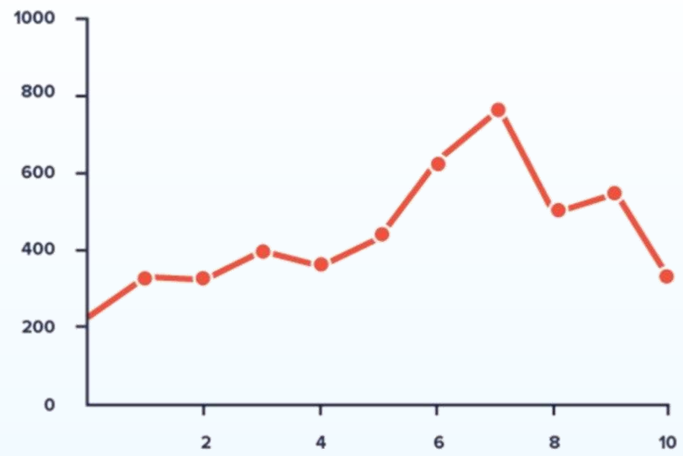
Bangkok, Thailand.

Outline

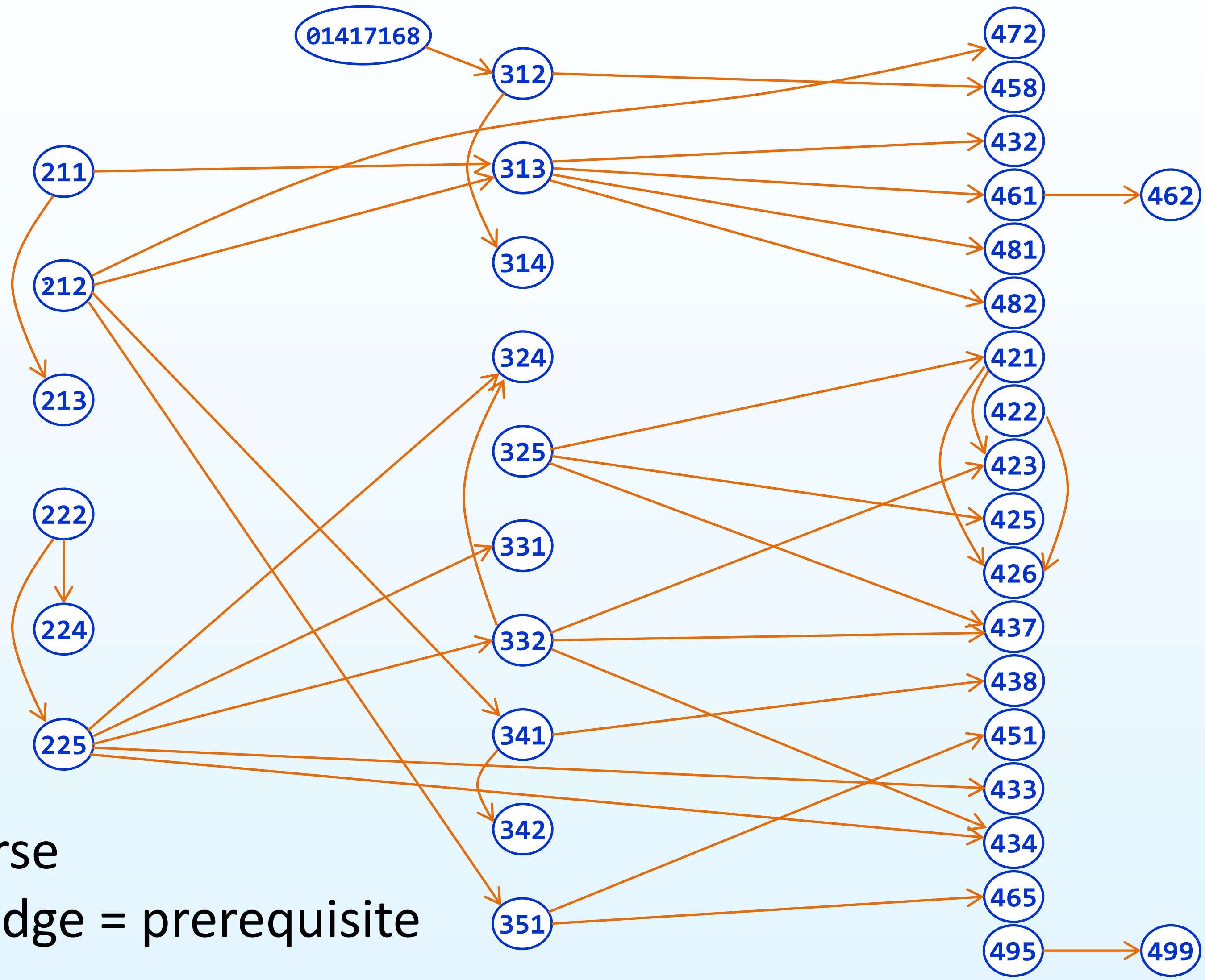
- Graph ADT and Representations
- Graph Traversals
- Connectivity
- Bipartite Graphs
- Topological Sort

What is a Graph?



- Yes, this is a graph
- But we are interested in a different kind of graph
- Consider graphs representing the following problems …

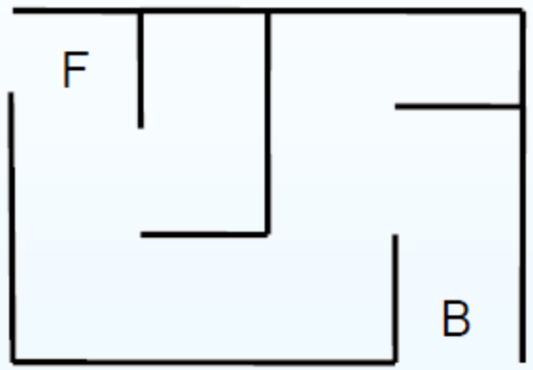
CPE Course Prerequisites

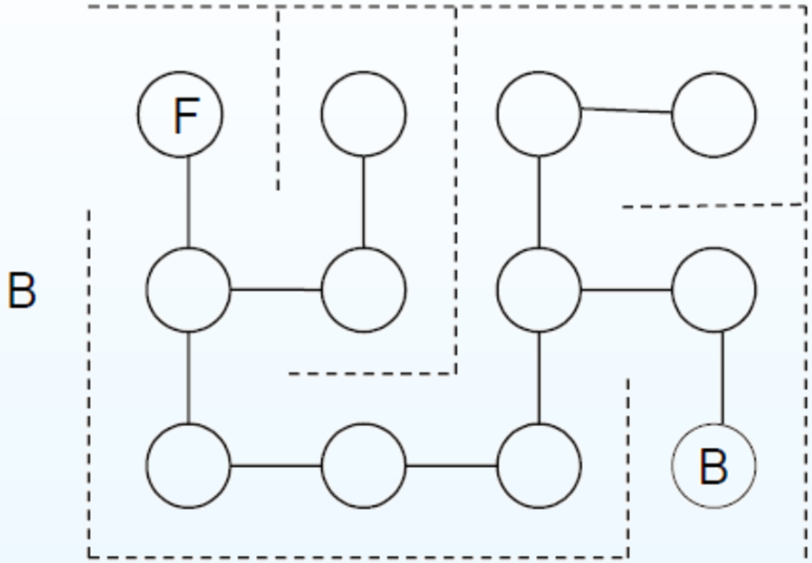


node = course

(directed) edge = prerequisite

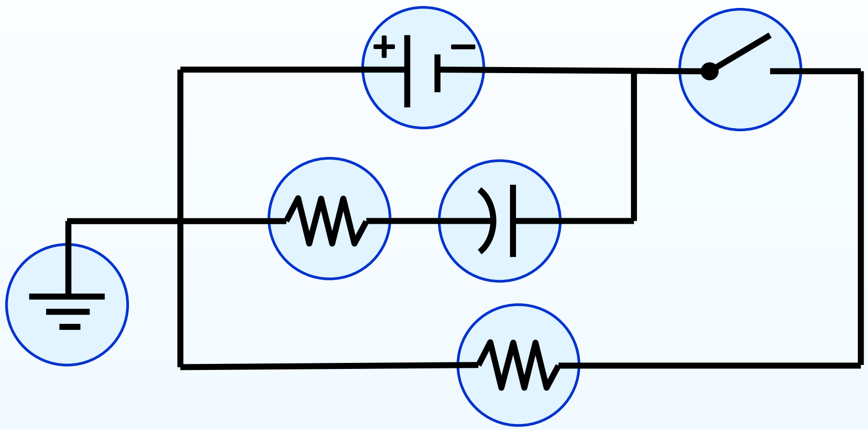
Maze Representation





node = room
edge = door or passage

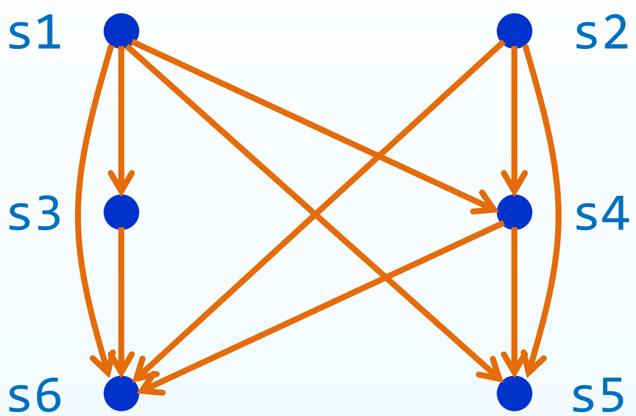
Electrical Circuit



node = battery, switch, resistor, etc.
edge = connection

Precedence

```
a = 0;     // s1
b = 1;     // s2
c = a+1;   // s3
d = b+a;   // s4
e = d+1;   // s5
e = c+d;   // s6
```

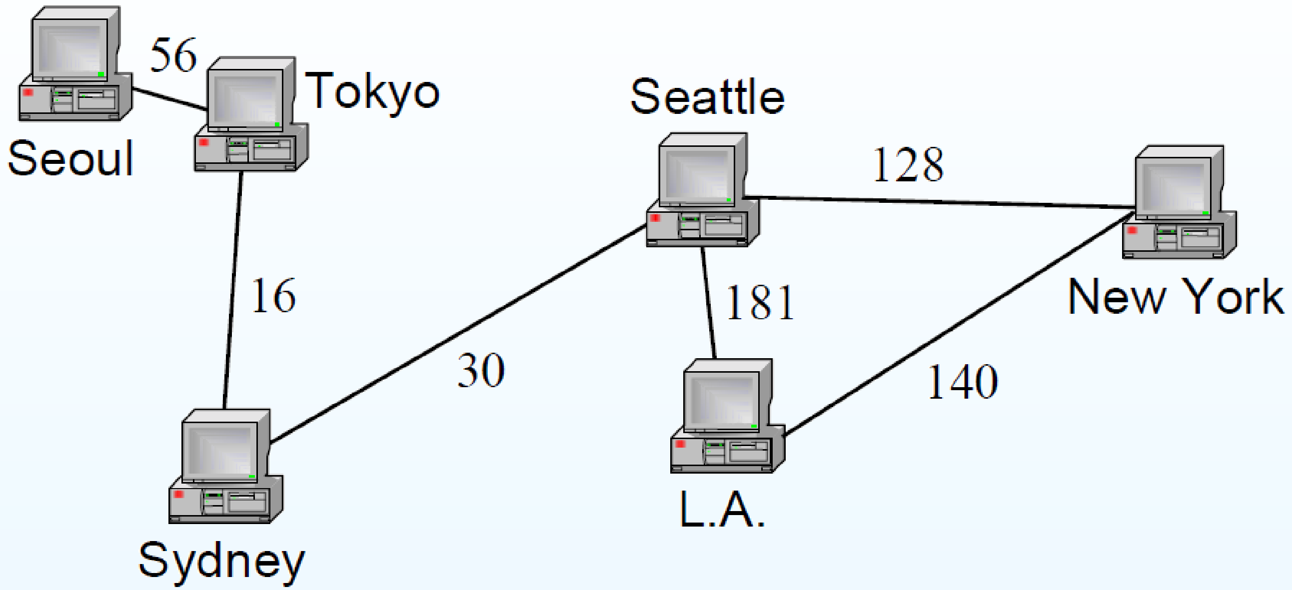


Which statements must execute before s6?

➡ s1, s2, s3, s4

node = statement
edge = precedence requirement

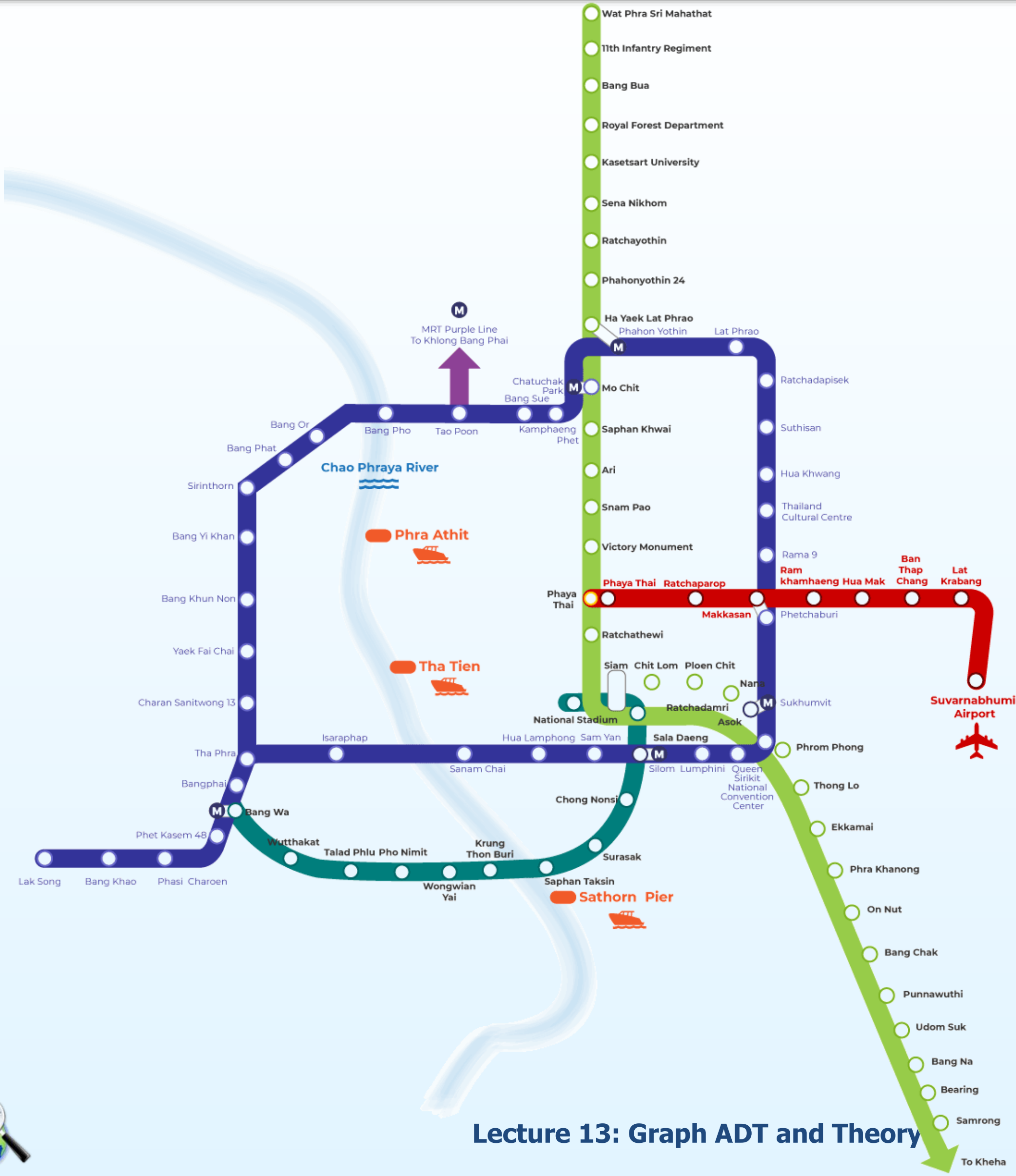
Information Transmission in a Computer Network



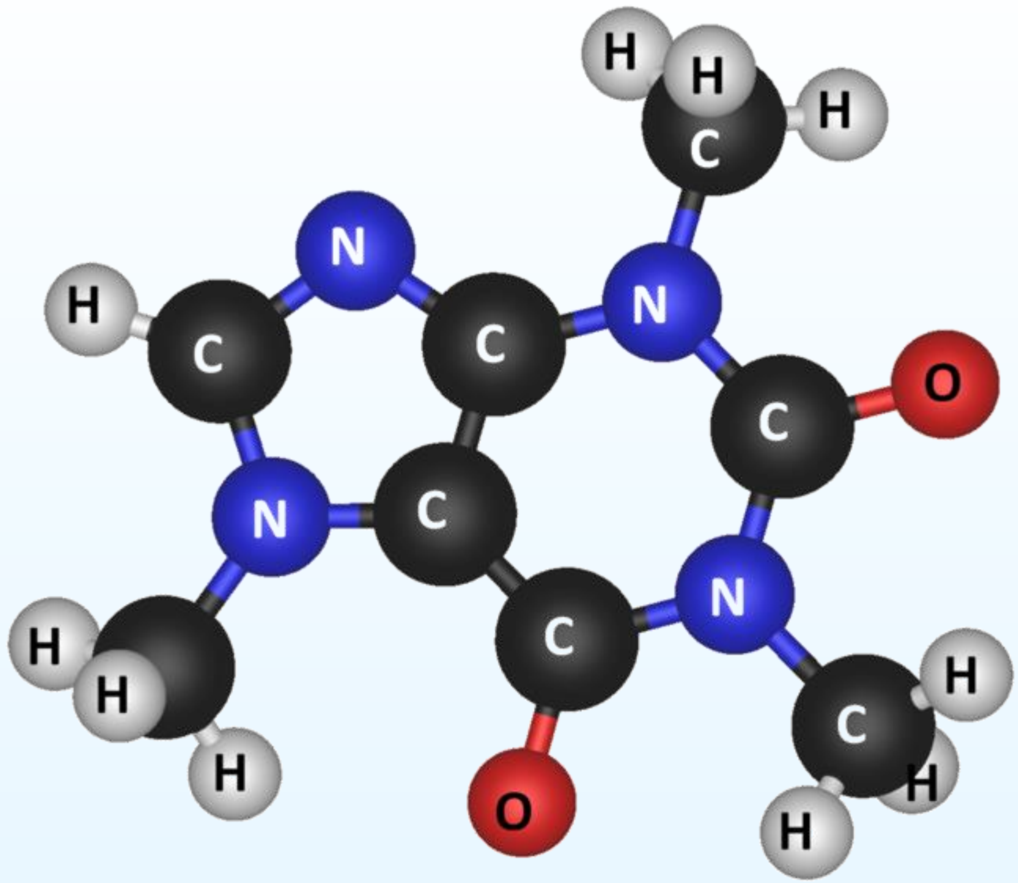
node = computer
edge = connection
weight = transmission rate

Bangkok BTS and MRT Map

node = station
edge = linkage



Molecules



node = atom
edge = bond

Graph Definition

- A graph is a collection of nodes and edges
- Formally,
  - A graph $G = (V, E)$ where
  - V is a set of vertices or nodes
  - $E \subseteq V \times V$ is a set of edges that connect vertices

Graph ADT

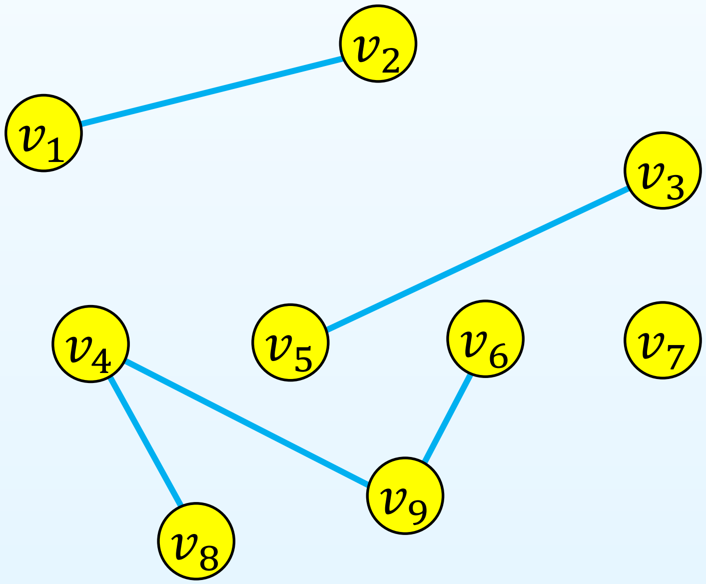
- Graph ADT describes a container storing an adjacency relation
- Operations include:
  - Inserting or removing a vertex (and all edges connecting that vertex)
  - Inserting or removing an edge
  - Querying.
    - The number of vertices
    - The number of edges
    - The vertices adjacent to a given vertex
    - The question that "are two vertices adjacent?"
    - The question that "are two vertices connected?"
- The running time of these operations will depend on the representation

Graph Example

- Consider a collection of vertices $V = \{v_1, v_2, \ldots, v_9\}$ where $|V| = 9$
- Associated with these vertices are $|E| = 5$ edges

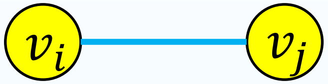
$$E = \{(v_1, v_2), (v_3, v_5), (v_4, v_8), (v_4, v_9), (v_6, v_9)\}$$

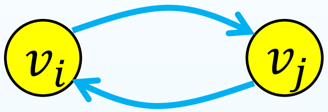
  - The pair (v_i, v_j) indicates that two vertices v_i and v_j are adjacent



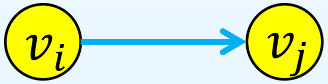
Undirected and Directed Graphs

- If the order of edge pairs (v_i, v_j) does not matter, the graph is an undirected graph: $(v_i, v_j) = (v_j, v_i)$





- If the order of edge pairs (v_i, v_j) matters, the graph is a directed graph (or called a digraph): $(v_i, v_j) \neq (v_j, v_i)$



Maximum Number of Edges

- For an undirected graph, the maximum number of edges is

$$|E| \leq \binom{|V|}{2} = \frac{|V|(|V|-1)}{2} = O(|V|^2)$$

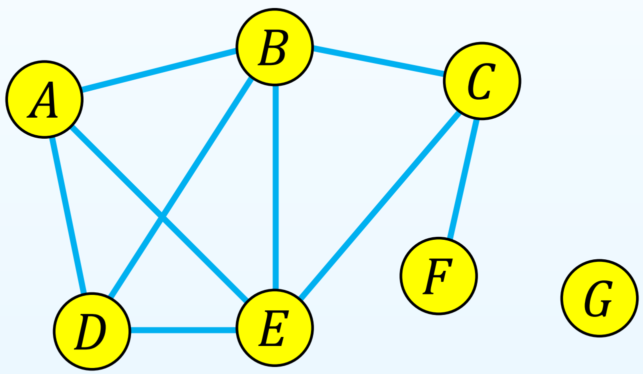
- For a directed graph, the maximum number of edges is

$$|E| \leq 2\binom{|V|}{2} = 2\frac{|V|(|V|-1)}{2} = O(|V|^2)$$

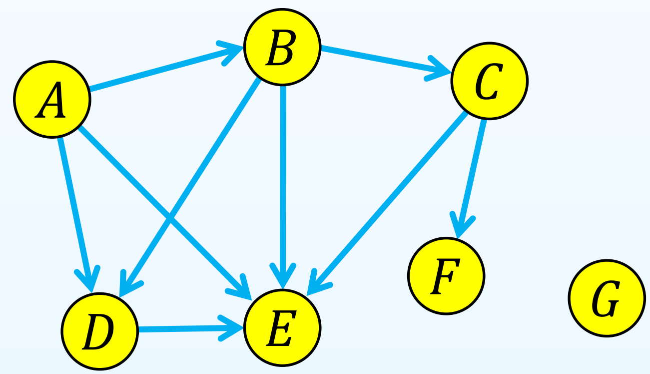
Graph Example

Given $G = (V, E)$ where $V = \{A, B, C, D, E, F, G\}$ and $E = \{(A,B), (A,D), (A,E), (B,C), (B,D), (B,E), (C,E), (C,F), (D,E)\}$

- If G is an undirected graph



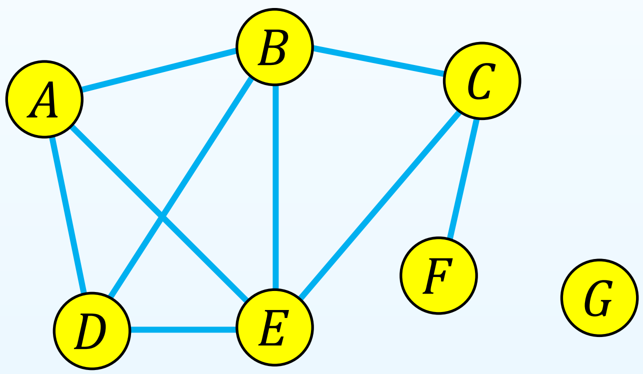
- If G is a directed graph



Degrees

- If G is an undirected graph

Degree: # of edges incident with



$\deg(A) = \deg(C) = \deg(D) = 3$

$\deg(B) = \deg(E) = 4$

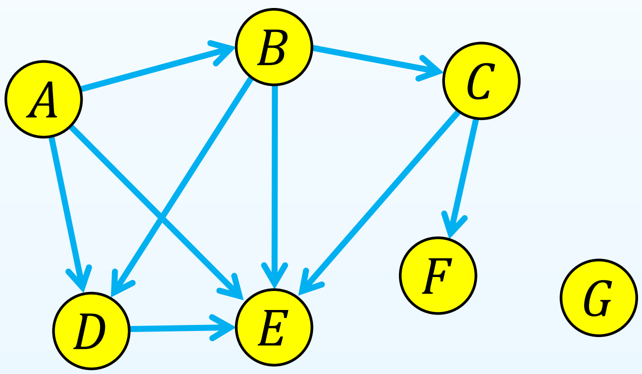
$\deg(F) = 1$

$\deg(G) = 0$

- If G is a directed graph

In-degree: # of edges with the terminal vertex

Out-degree: # of edges with the initial vertex



$\text{in\_deg}(A) = 0 \quad \text{out\_deg}(A) = 3$

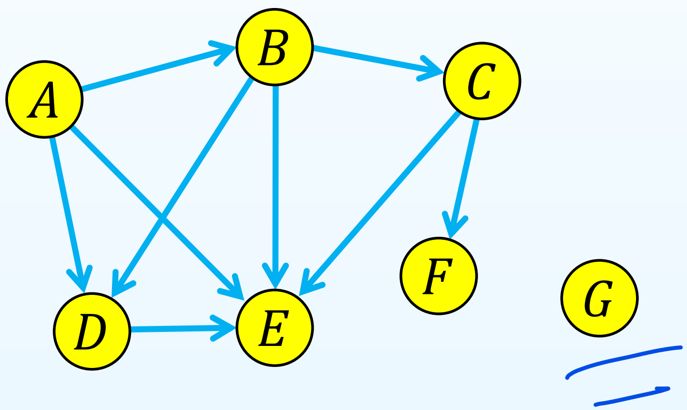
$\text{in\_deg}(B) = 1 \quad \text{out\_deg}(B) = 3$

$\text{in\_deg}(C) = 1 \quad \text{out\_deg}(C) = 2$

$\text{in\_deg}(D) = 2 \quad \text{out\_deg}(D) = 1$

Sources and Sinks

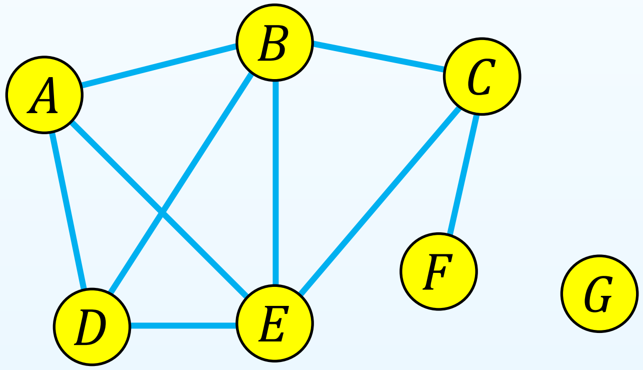
- For a directed graph,
  - Vertices with an in-degree of zero are described as sources
  - Vertices with an out-degree of zero are described as sinks



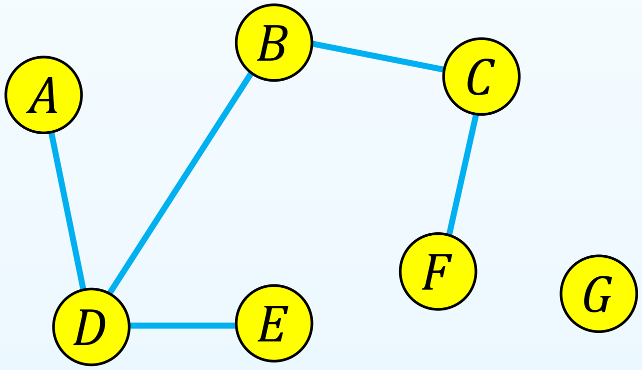
Sources: A and G
Sinks: E, F, and G

Subgraphs

A subgraph is a graph consisting of a subset of vertices and a subset of edges that connected those vertices in the original graph



the original graph



a subgraph

Subgraphs

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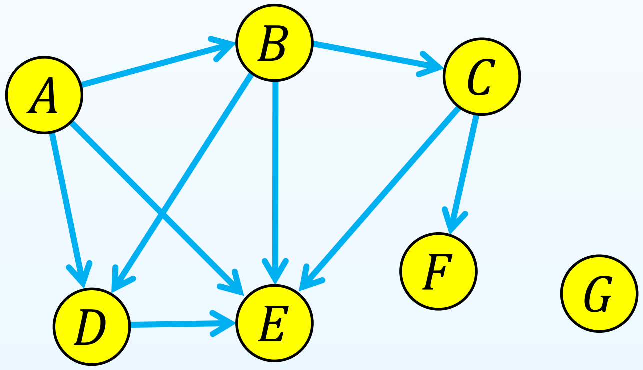
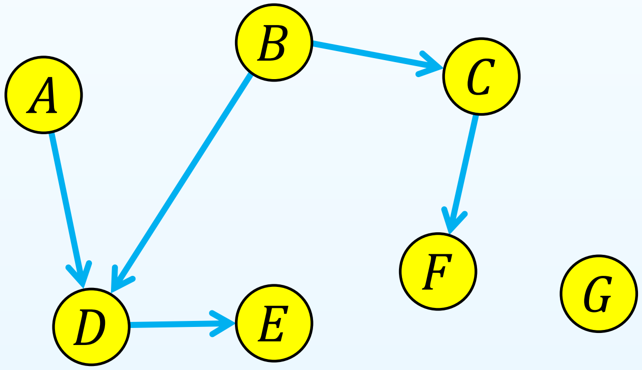


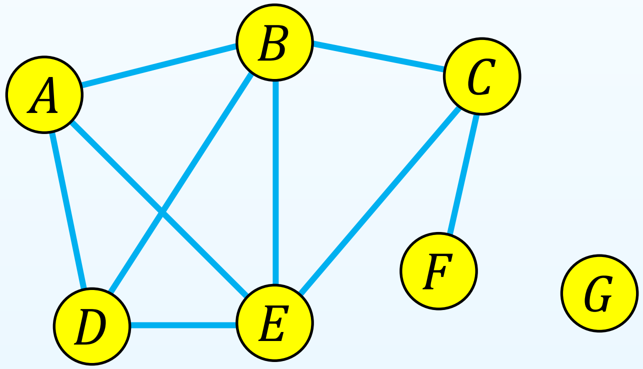
the original graph



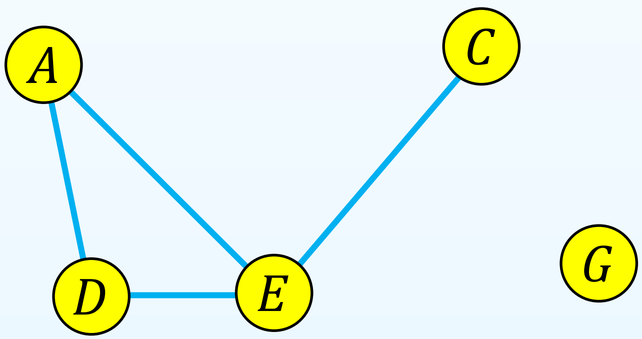
a subgraph

Vertex-Induced Subgraphs

A vertex-induced subgraph is a graph consisting of a subset of vertices where the edges are all edges in the original graph



the original graph



a vertex-induced subgraph

Vertex-Induced Subgraphs

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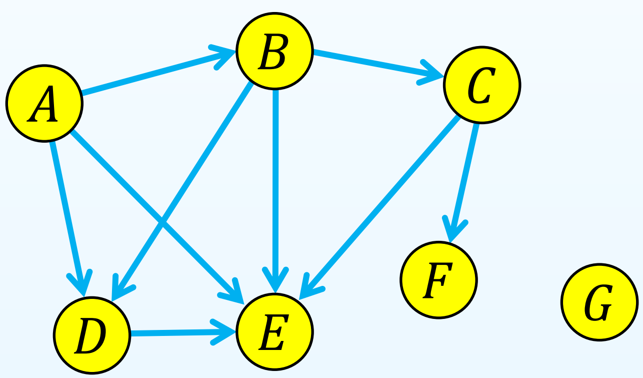
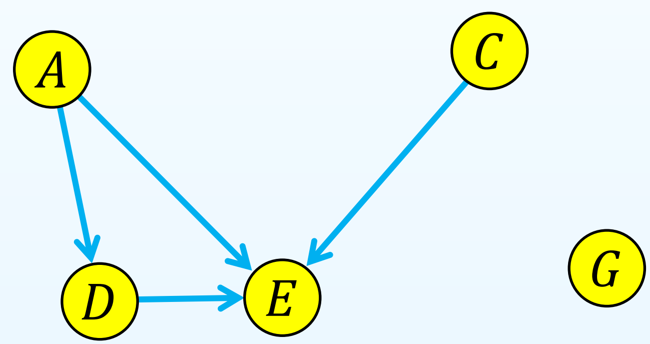


the original graph

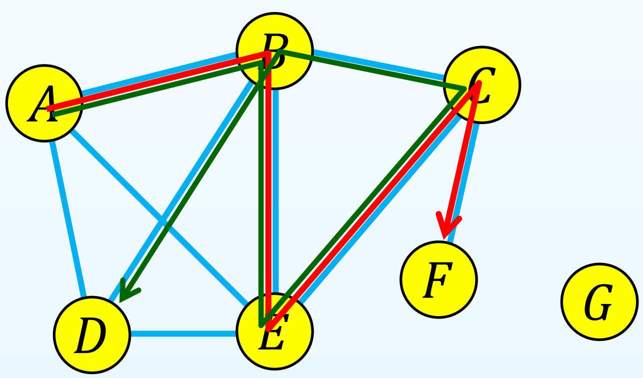


a vertex-induced subgraph

Paths

A path is and ordered sequence of vertices $\langle v_i, v_{i+1}, \ldots, v_k \rangle$ where (v_j, v_{j+1}) is an edge for $j = i, \ldots, k-1$

- If G is an undirected graph

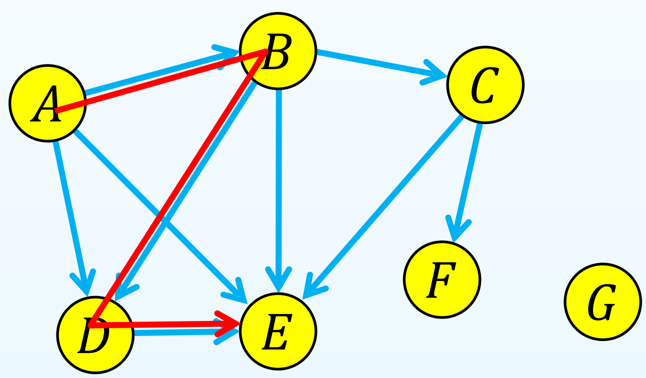


$\langle A, B, E, C, F \rangle$ is a path of length 4

$\langle A, B, E, C, B, D \rangle$ is a path of length 5

$\langle A \rangle$ is a path of length 0

- If G is a directed graph

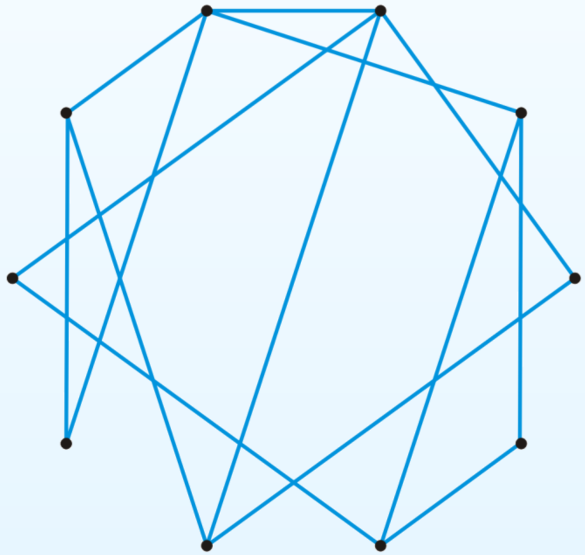


$\langle A, B, D, E \rangle$ is a path of length 3

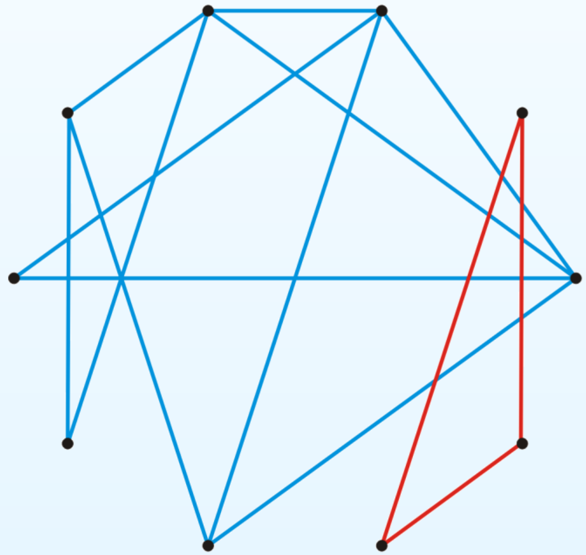
Connectivity

Two vertices v_i and v_j are said to be connected if there exists a path between them

A graph is connected if there exists a path between any two vertices



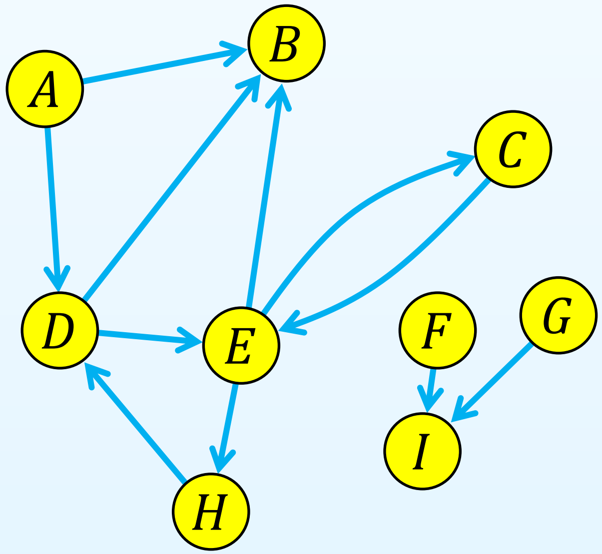
a connected graph



an unconnected graph

Strongly and Weakly Connectivity

- A graph is strongly connected if every pair of vertices (v_i and v_j) contains a path between each other
- A graph is weakly connected if there does not exist any path between any two pairs of vertices
  - Replacing all directed edges with undirected edges produces a connected (undirected) graph



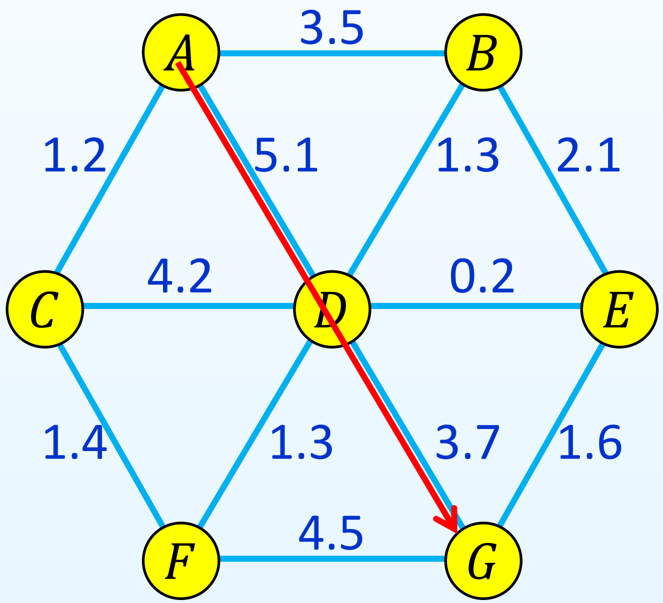
Strongly connected subgraph:
$\{C, D, E, H\}$

Weakly connected subgraph:
$\{A, B, C, D, E, H\}$

Weighted Graphs

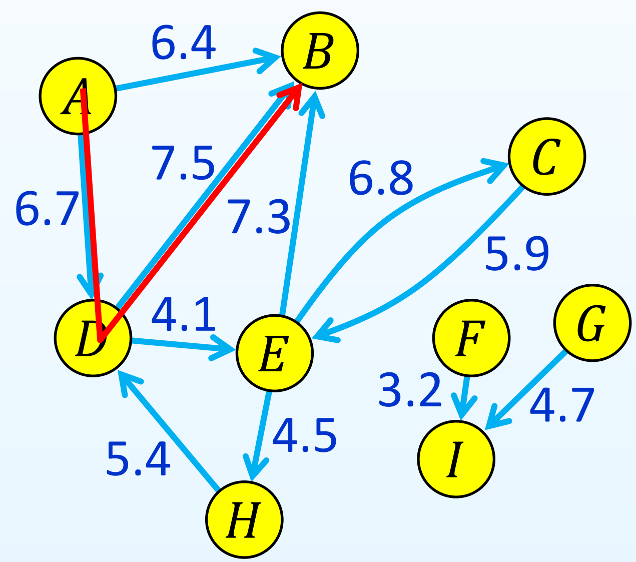
A weighted graph is a graph $G = (V, E, W)$ in which a weight $w_{ij} \in W$ is associated with each edge $(v_i, v_j) \in E$

- If G is an undirected graph



$\langle A, D, G \rangle$ is a path of length 8.8

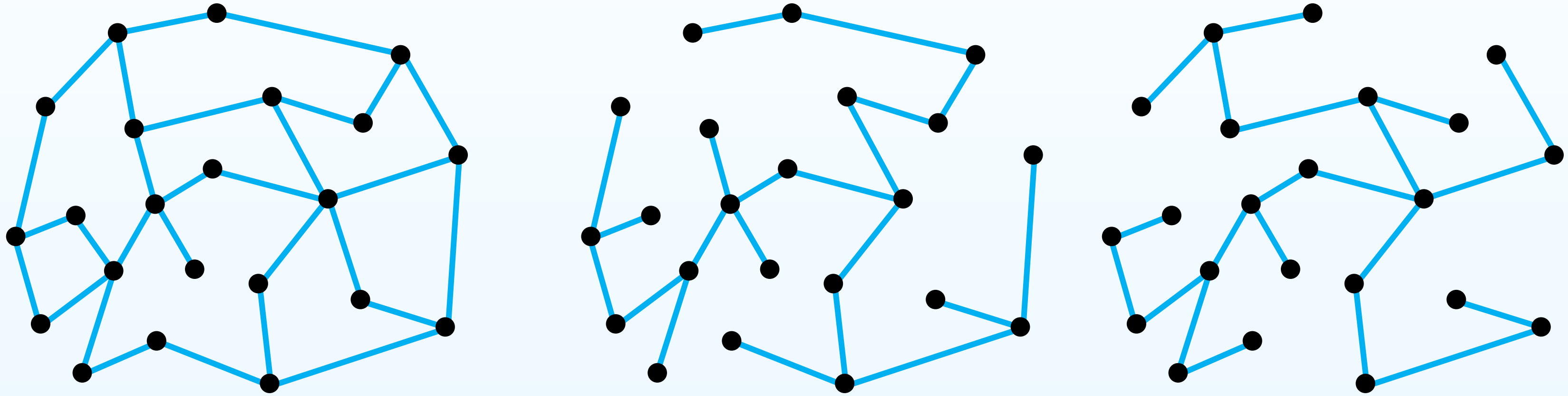
- If G is a directed graph



$\langle A, D, B \rangle$ is a path of length 14.2

Trees

- Trees are special cases of a graph
  - Each is connected, and
  - There is a unique path between any two vertices

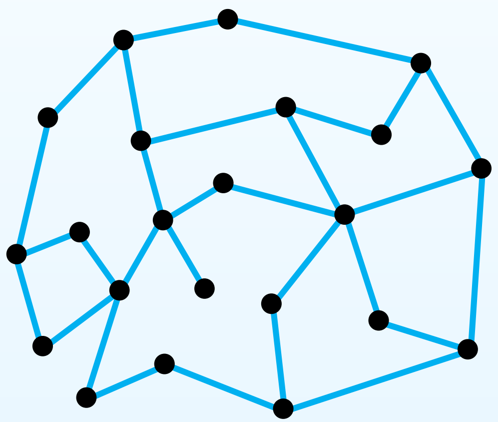


the original graph　　　example of trees t_1 and t_2

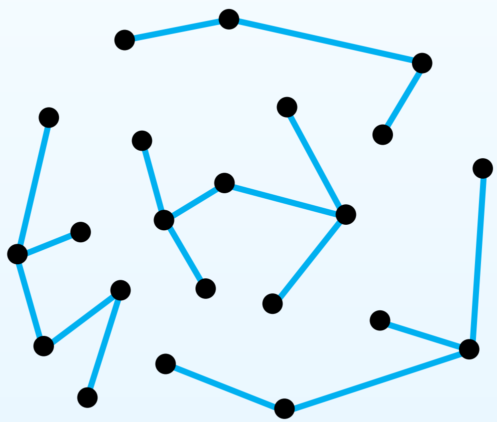
- **Consequences**
  - The number of edges is $|E| = |V| - 1$
  - The graph (tree) is acyclic
  - Adding one edge creates a cycle
  - Removing any one edge creates two disjoint non-empty subgraphs

Forests

- A forest is any graph that has no cycles
  - The number of edges is $|E| < |V|$
  - The number of trees is $|V| - |E|$
  - Removing any one edge adds one more tree to the forest



the original graph

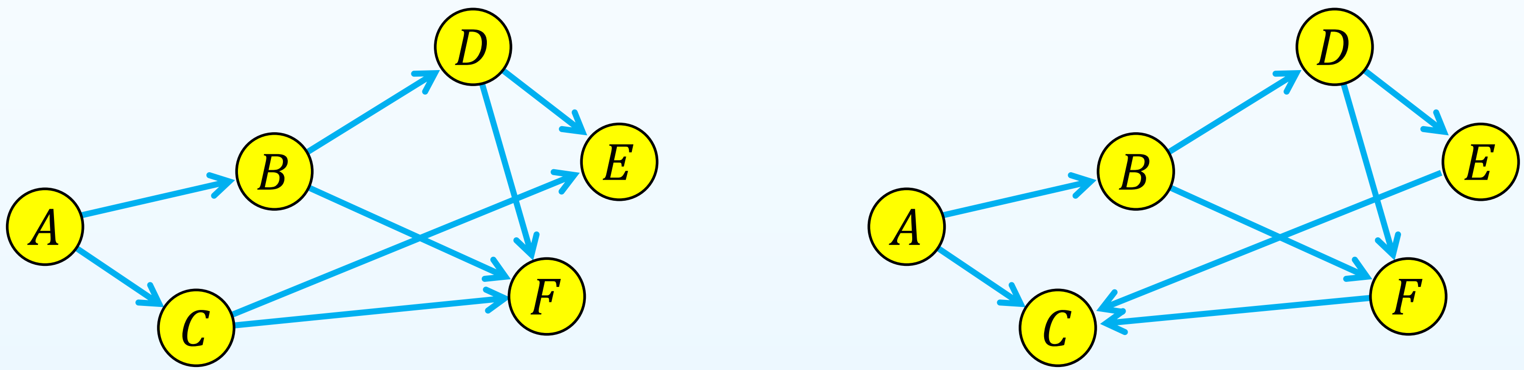


example of a forest with 4 trees

Directed Acyclic Graphs

A directed acyclic graph (DAG) is a directed graph which has no cycles

- Graphical representation of partial order on a finite number of elements



example of two DAGs

Note that adding the edge (C, A) will make them not DAG

Graph Representations

- Space and time are analyzed in terms of
  - Number of vertices ($|V|$) and
  - Number of edges ($|E|$)

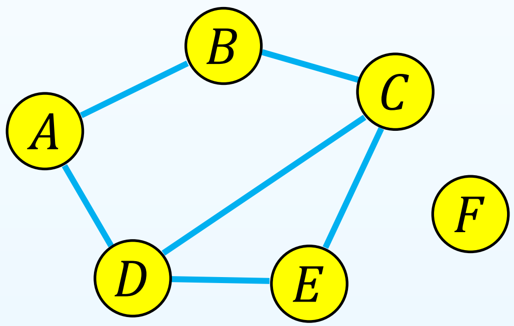
- There are two ways of representing graphs:
  - The adjacency matrix representation
  - The adjacency list representation

Adjacency Matrix

Require more memory but faster

- If G is an undirected graph

$$m_{ij} = \begin{cases} 1 & if\ (v_i, v_j) \in E \\ 0 & otherwise \end{cases}$$



$$M = \begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

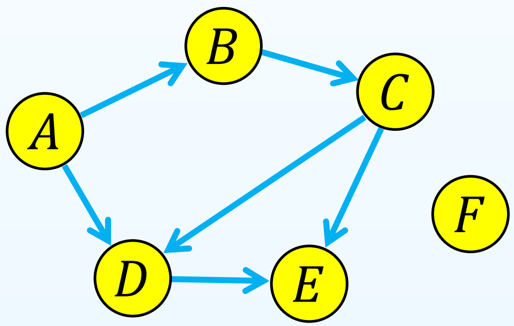
- Memory requirement is $\Theta(|V|^2)$
- Determining if v_i is adjacent to v_j is $\Theta(1)$
- Finding all neighbors of v_i is $\Theta(|V|)$

Adjacency Matrix

Require more memory but faster

- If G is a directed graph

$$m_{ij} = \begin{cases} 1 & if\ (v_i, v_j) \in E \\ 0 & otherwise \end{cases}$$



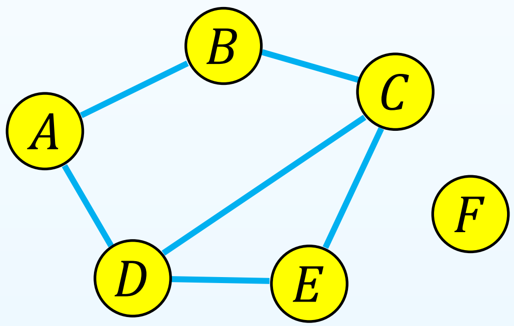
$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

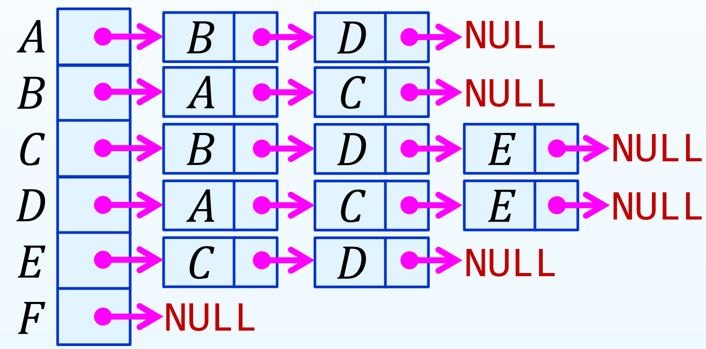
- Memory requirement is $\Theta(|V|^2)$
- Determining if v_i is adjacent to v_j is $\Theta(1)$
- Finding all neighbors of v_i is $\Theta(|V|)$

Adjacency List

Each vertex is associated with a list of its neighbors

- If G is an undirected graph



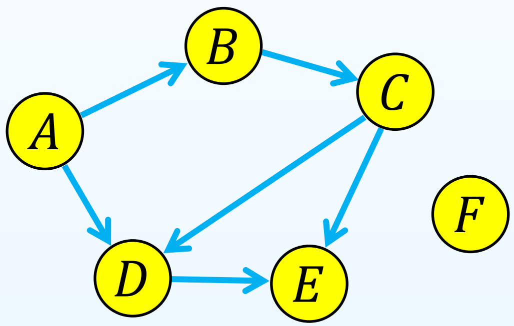


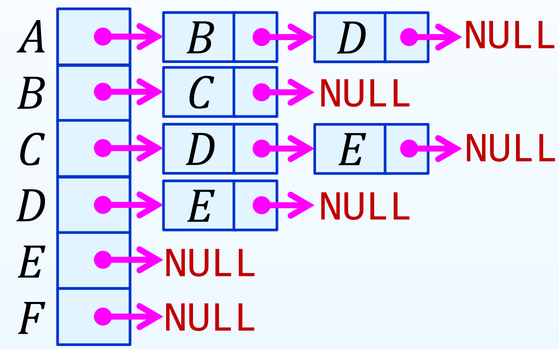
- Memory requirement is $\Theta(|V| + 2|E|) = \Theta(|V| + |E|)$
- On average, determining if v_i is adjacent to v_j is $\Theta(|E|/|V|)$
- On average, finding all neighbors of v_i is $\Theta(|E|/|V|)$

Adjacency List

Each vertex is associated with a list of its neighbors

- If G is a directed graph





- Memory requirement is $\Theta(|V| + |E|)$
- On average, determining if v_i is adjacent to v_j is $\Theta(|E|/|V|)$
- On average, finding all neighbors of v_i is $\Theta(|E|/|V|)$

Outline

- Graph ADT and Representations
- Graph Traversals
- Connectivity
- Bipartite Graphs
- Topological Sort

Strategies

- Traversals of graphs are also called searches

- We can use either
  - Breadth-first traversal requiring a queue, or
  - Depth-first traversal requiring a stack

- We will have to track which vertices have been visited, requiring $\Theta(|V|)$ memory

Breadth-First Traversal

Procedures:

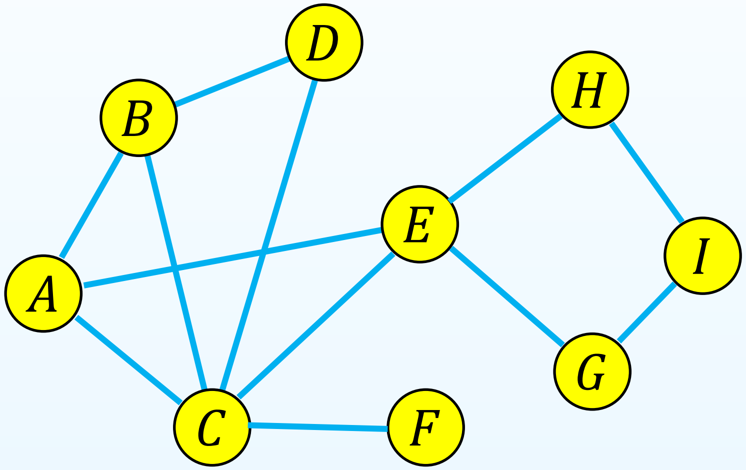
- Choose a vertex, mark it as visited and insert into queue
- While the queue is not empty
  - Remove the front vertex v from the queue
  - For each vertex adjacent to v that has not been visited
    - Mark it visited, and
    - Insert it into the queue

Note that if the queue is empty and there are no unvisited vertices, the graph is connected

The size of the queue is $O(|V|)$

Example

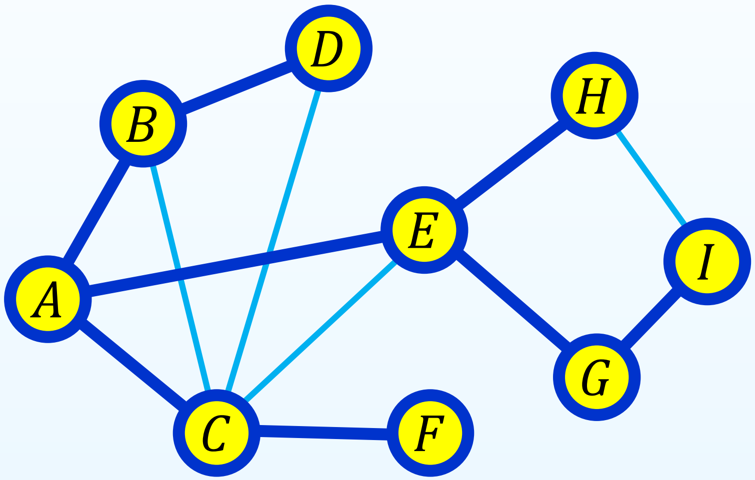
Consider this graph:



Example: Breadth-First Traversal

Consider this graph:

- Start with the vertex A



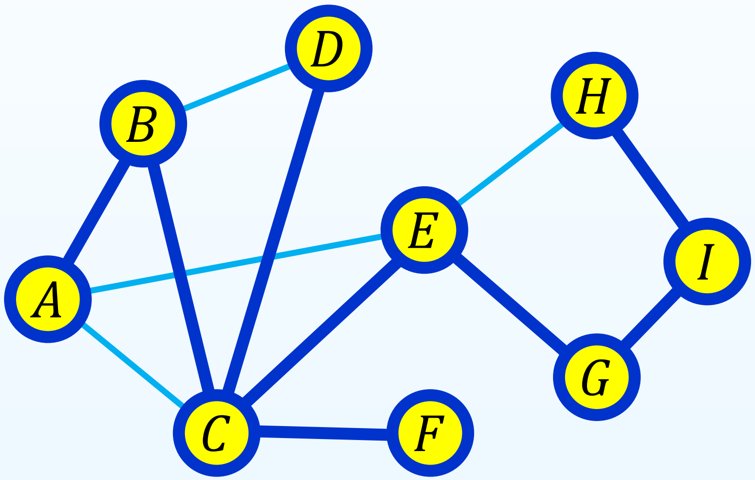
Output: $A \quad B \quad C \quad E \quad D \quad F \quad G \quad H \quad I$

Time complexity: $\Theta(|V| + |E|)$

Example: Depth-First Traversal

Consider this graph:

- Start with the vertex A



Output: $A \quad B \quad C \quad D \quad E \quad G \quad I \quad H \quad F$

Time complexity: $\Theta(|V| + |E|)$

Applications

- Determining connectivity and finding connected subgraphs
- Determining the path length from one vertex to all others
- Testing if a graph is bipartite
- Determining maximum flow
- …

Outline

- Graph ADT and Representations
- Graph Traversals
- Connectivity
- Bipartite Graphs
- Topological Sort

Connectivity

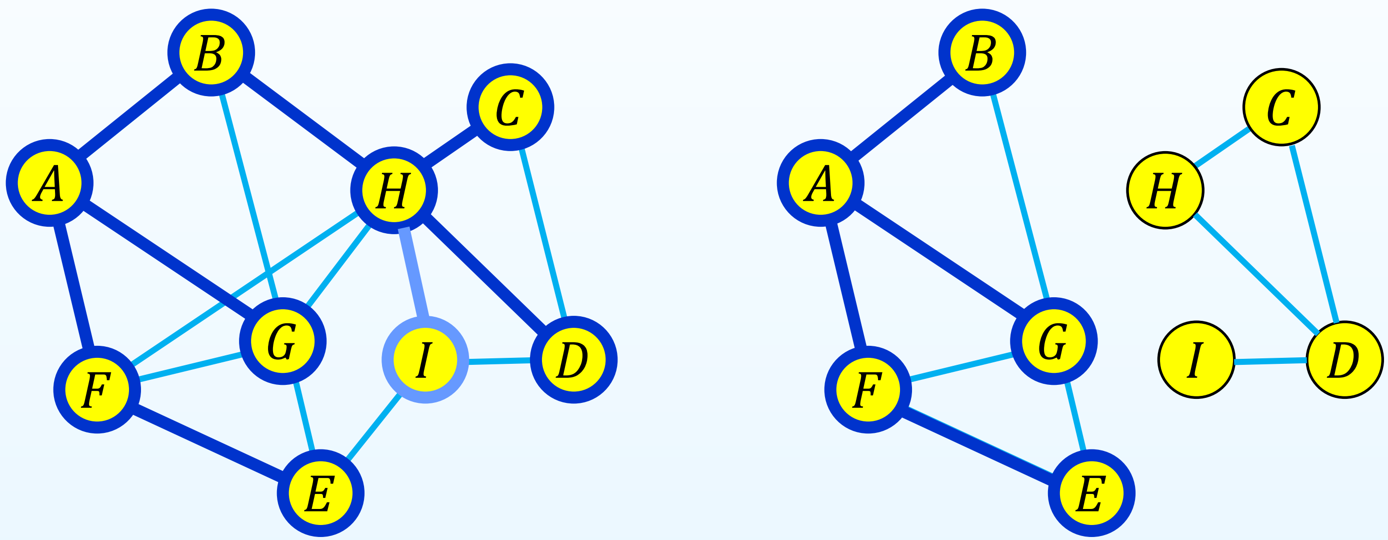
- Determine whether one vertex is connected to another
  - v_i is connected to v_j if there is a path from v_i to v_j

- Strategy:
  - Perform a breadth-first traversal starting at v_i
  - During the traversal,
    - If the vertex v_j ever found to be adjacent to any vertex in front of the queue, return true
    - Otherwise, if the queue is empty, return false

Determining Connectivity

Consider these graphs:

- Is A connected to D?



BFS: $A\ B\ F\ G\ H\ E\ C\ D\ I$

found here!

BFS: $A\ B\ F\ G\ E$

NOT found

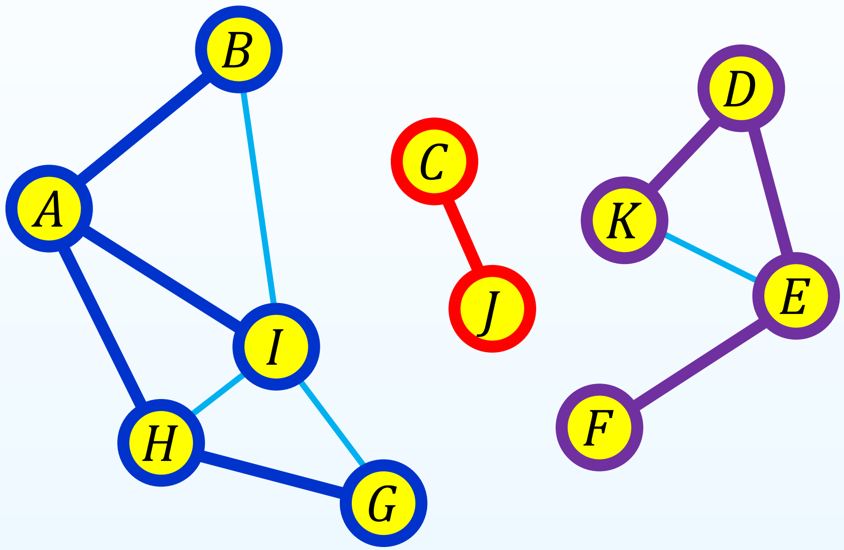
Connected Components

- Previously, if we continued the traversal, we would find all vertices that are connected to A

- Suppose we want to partition the vertices into connected subgraphs
  - While there are unvisited vertices in the graph
    - Select an unvisited vertex and perform a traversal on that vertex
    - Each vertex that is visited in that traversal is added to the set initially containing the initial unvisited vertex
  - Continue until all vertices are visited

Connected Components

Consider this graph:



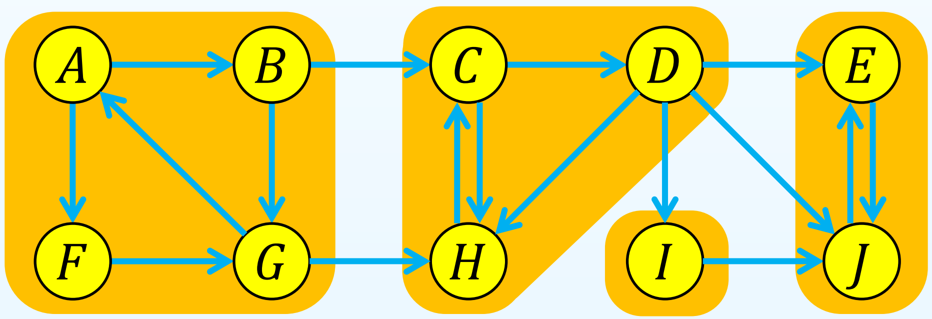
A	B	C	D	E	F	G	H	I	J	K
1	2	6	8	9	11	5	3	4	7	10

There are three connected subgraphs:

$\{A, B, G, H, I\}$, $\{C, J\}$, and $\{D, E, F, K\}$

Strongly Connected Components

- A directed graph is strongly connected if there is a path between all pairs of vertices
- A strongly connected component (SCC) of a directed graph is a maximal strongly connected subgraph



Strongly Connected Components

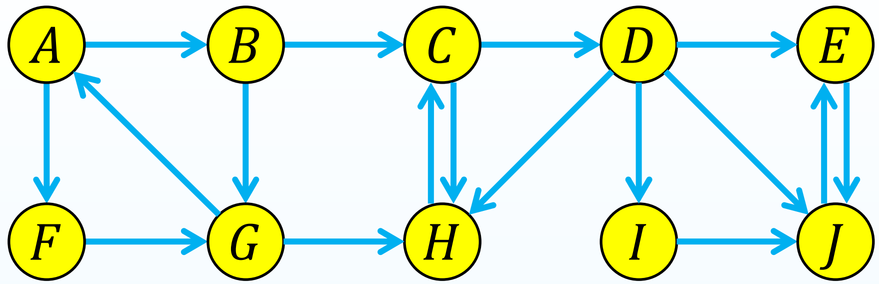
Algorithm:

1. Call DFS(G) to compute exiting times for each vertex
2. Compute G^T
3. Call DFS(G^T), but in the main loop of DFS, consider the vertices in order of decreasing times as computed in (1)
4. Output the vertices of each tree in the depth-first forest of (3) as a separate strongly connected components

The algorithm takes $O(|V| + |E|)$ time

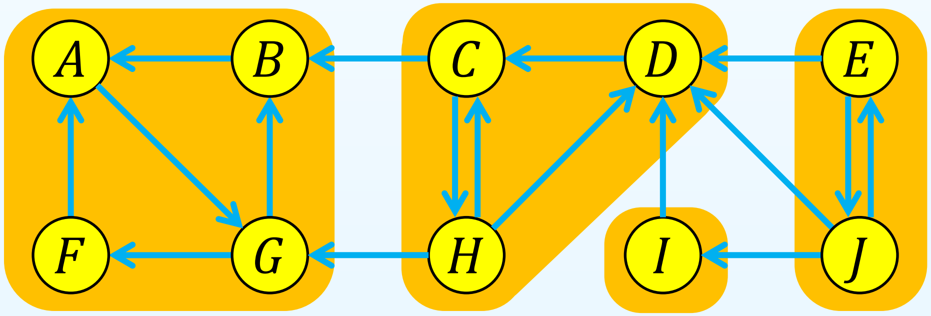
Strongly Connected Components

Step 1:



A		B		C		D		E		F		G		H		I		J	
1	20	2	17	3	14	4	13	5	8	18	19	15	16	9	10	11	12	6	7

Steps 2-4:



A		B		C		D		E		F		G		H		I		J	
1	8	3	4	9	14	11	12	17	20	5	6	2	7	10	13	15	16	18	19

Outline

- Graph ADT and Representations
- Graph Traversals
- Connectivity
- Bipartite Graphs
- Topological Sort

Definition

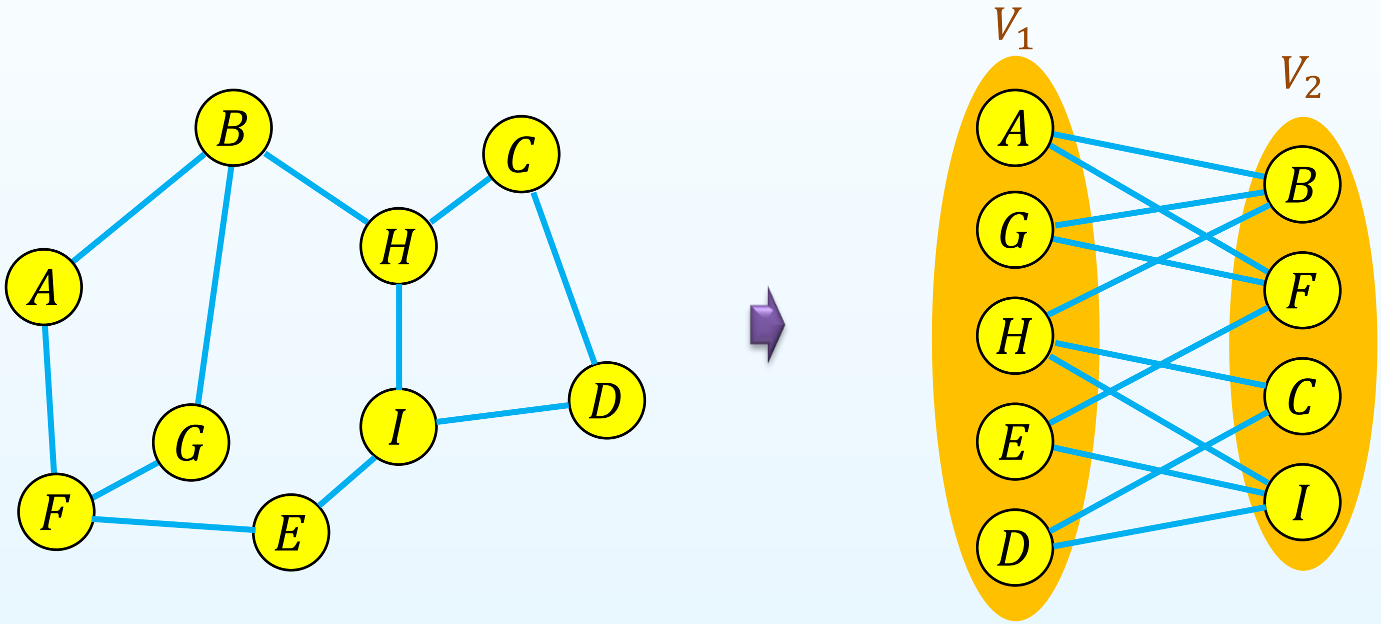
A bipartite graph is a graph where the vertices V can be divided into two disjoint sets V_1 and V_2 such that

- Every edge has one vertex in V_1 and the other in V_2
- In other words, no edges connects two vertices in the same set

Bipartite Graphs

Consider this graph:

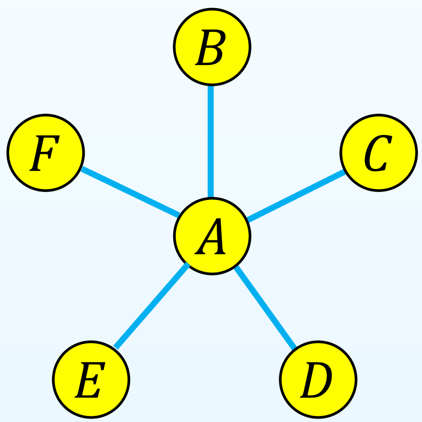
- Is it a bipartite graph?
  - ➡ **YES**, with a little work to decompose the vertices

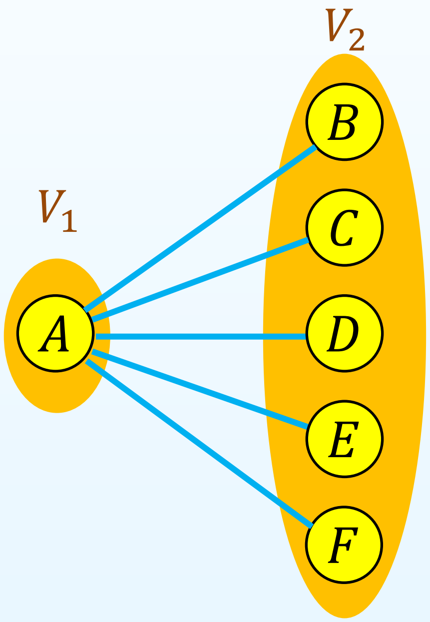


Bipartite Graphs

Consider this graph:

- Is it a bipartite graph?
  - ➡ **YES**, with a little work to decompose the vertices

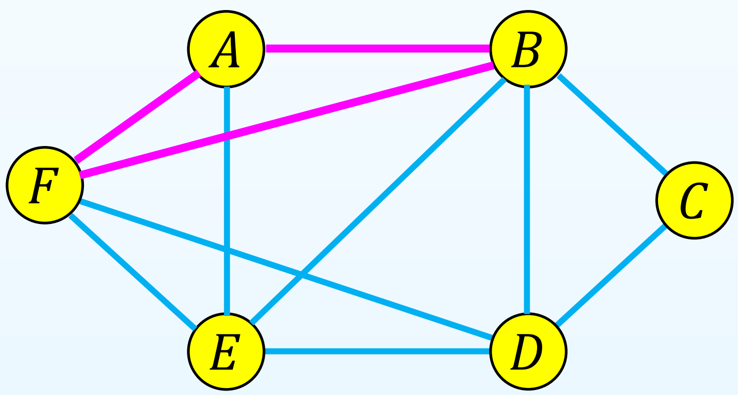




Bipartite Graphs

Consider this graph:

- Is it a bipartite graph?
  - ➡ **NO**, we cannot divide $\{A, B, F\}$, for instance



Bipartite Graphs

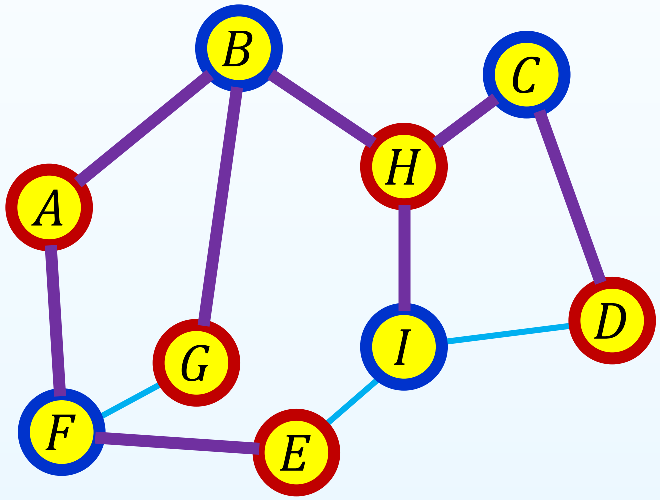
Algorithm: using a breadth-first traversal

- Choose a vertex, mark it belonging to V_1 and insert in into a queue
- While the queue is not empty
  - Dequeue the front vertex v, and
  - Any adjacent vertices that are already marked must belong to the set not containing v, otherwise, the graph is not bipartite (we are done); while
  - Any unmarked adjacent vertices are marked as belonging to the other set and they are inserted into the queue
- If the queue is empty, the graph is bipartite

Bipartite Graphs

Consider the previous graph:

- We will use colors to distinguish the two sets



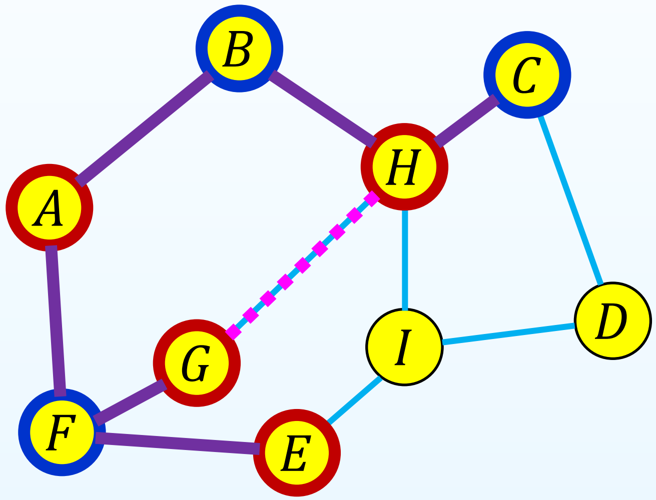
BFS: $A \quad B \quad F \quad G \quad H \quad E \quad C \quad I \quad D$

OK, this graph is bipartite

Bipartite Graphs

Consider the previous graph:

- We will use colors to distinguish the two sets

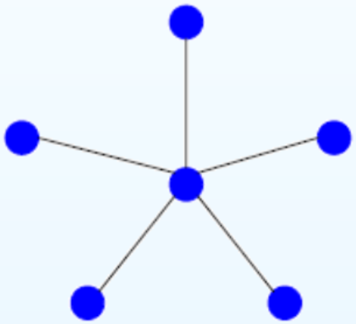


BFS: A B F H E G C

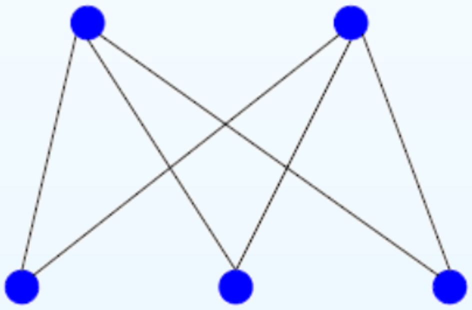
This graph is NOT bipartite

Complete Bipartite Graphs $K_{m,n}$

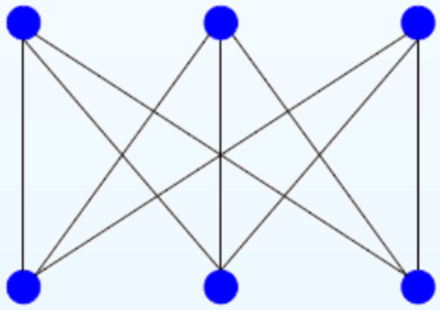
- Vertex set partitioned into two subsets of size m and n
- All vertices in one subset are connected to all vertices in the other subset



$K_{1,5}$



$K_{2,3}$



$K_{3,3}$

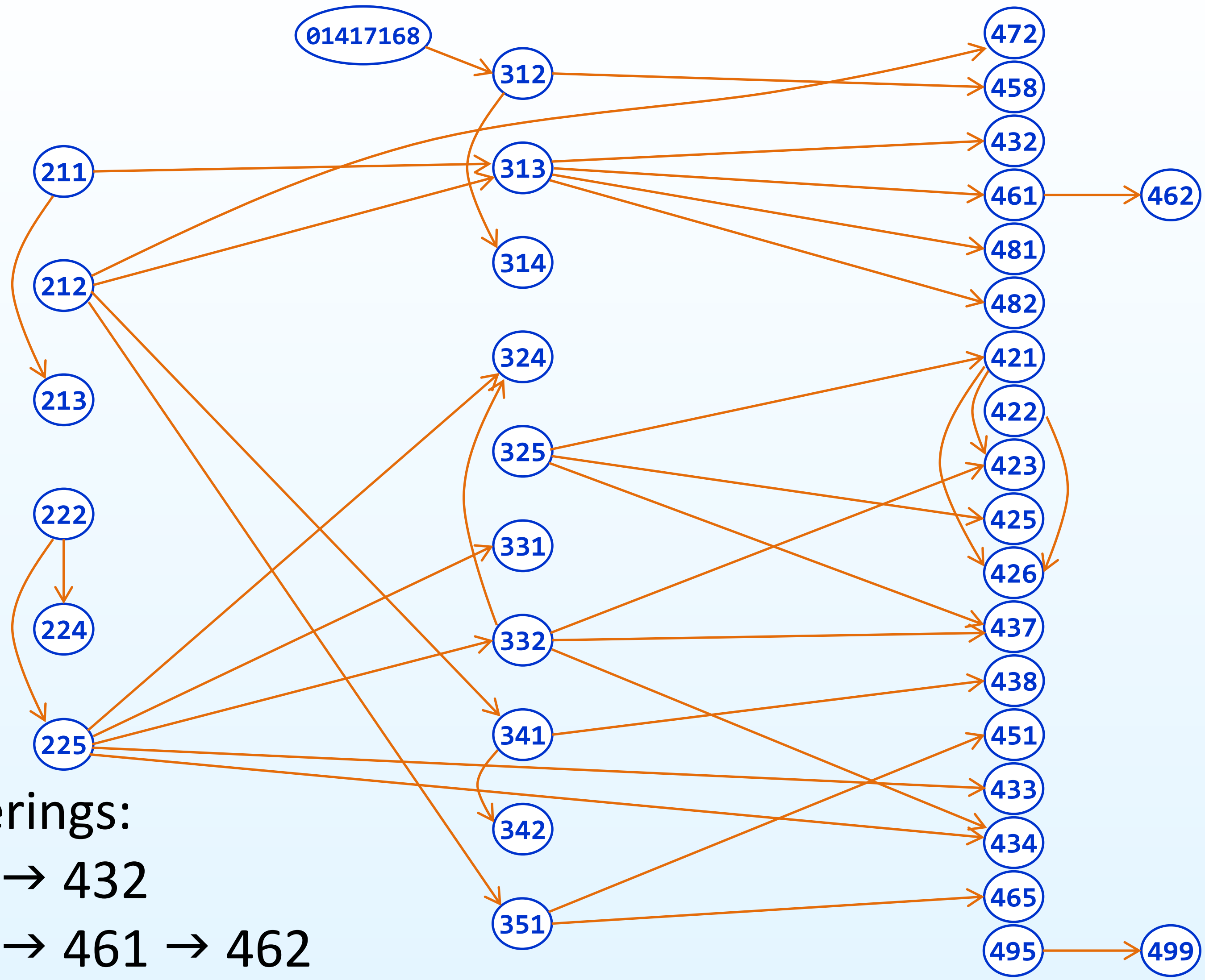
Outline

- Graph ADT and Representations
- Graph Traversals
- Connectivity
- Bipartite Graphs
- Topological Sort

Motivation

- Given a set of tasks with dependencies
  - Is there an order in which we can complete the tasks?
- Dependencies form a partial ordering
  - A partial ordering on a finite number of objects can be represented as a directed acyclic graph (DAG)

Example: CPE Course Prerequisites



Partial orderings:

212 → 313 → 432

212 → 313 → 461 → 462

Definition

Restriction of paths in DAGs

- Given two different vertices v_i and v_j, there cannot both be a path from v_i to v_j and a path from v_j to v_i

Topological sort

- A topological sort of the vertices in a DAG is an ordering

$$v_1, v_2, v_3, \ldots, v_n$$

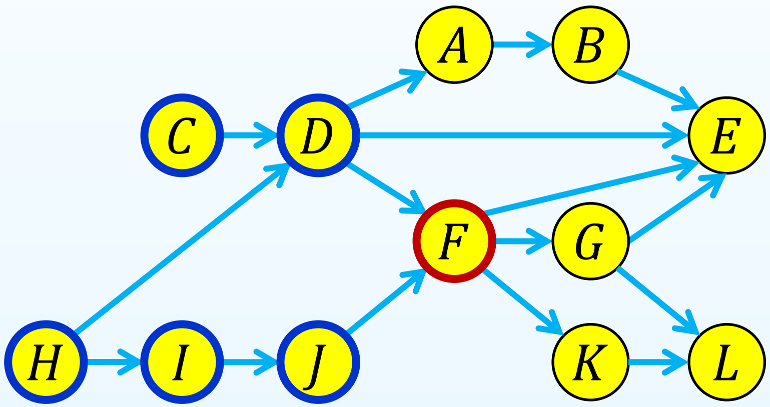
such that if there is a path from v_i to v_j then v_i appears before v_j

Example

Consider this DAG:

- A topological sort is

$$C, H, D, I, A, J, B, F, G, K, E, L$$



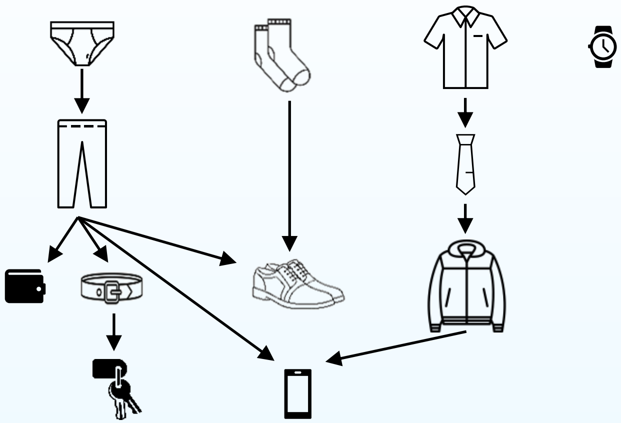
Note that

- There are paths from C, D, H, I, and J to F, so all these must come before F in a topological sort
- Clearly, this sorting need not be unique

Applications

Suppose you want to dress up to dinner

- You must wear the following:
  - jacket, shirt, briefs, socks, tie, etc.
- There are certain constraints:
  - The pants should go on after the briefs
  - Socks are put on before shoes



- One topological sort is

  briefs, pants, wallet, belt, keys, socks, shoes, shirt, tie, jacket, phone, watch
- A more reasonable topological sort is

  briefs, socks, pants, shirt, belt, keys, tie, jacket, wallet, phone, watch, shoes

Topological Sort

Algorithm:

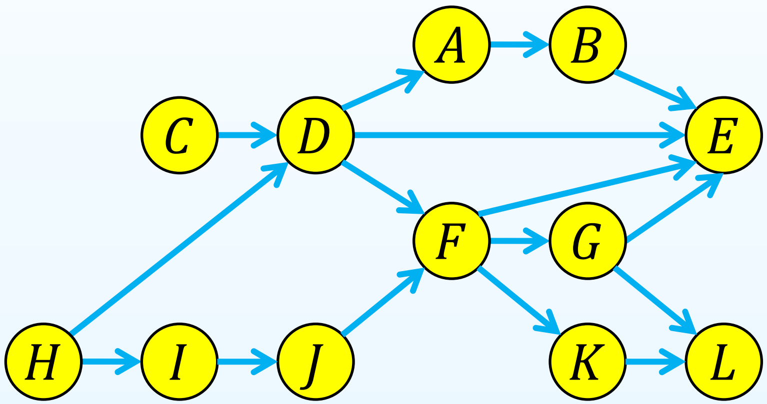
Given a graph, iterate the followings $|V|$ times

- Choose a vertex v that has in-degree zero
- Let v be the next vertex in our topological sort
- Remove v and all edges connected to it

Example

Consider the previous DAG:

- First, create a table of in-degrees of each vertex
- Then, create a queue

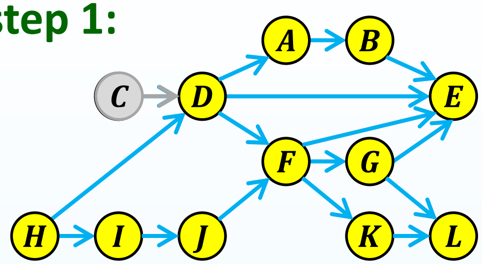


	A	B	C	D	E	F	G	H	I	J	K	L
in-degree	1	1	0	2	4	2	1	0	1	1	1	2

queue	C	H										

Example

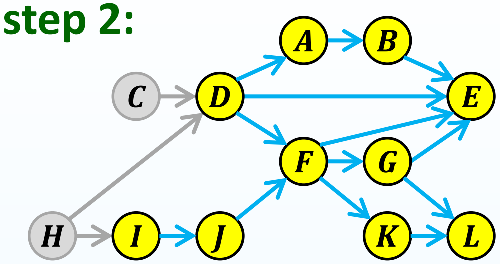
step 1:



	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
in-degree	1	1	0	1	4	2	1	0	1	1	1	2
queue	*H*											

topological sort is
C

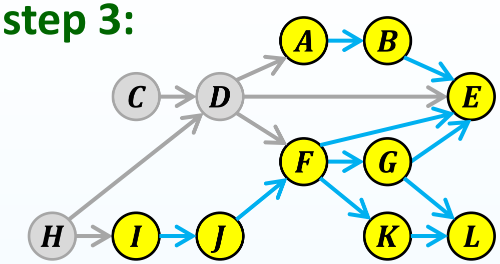
step 2:



	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
in-degree	1	1	0	0	4	2	1	0	0	1	1	2
queue	*D*	*I*										

topological sort is
C H

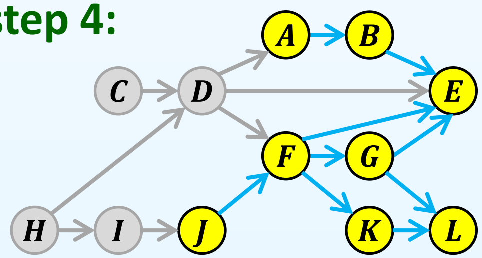
step 3:



	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
in-degree	0	1	0	0	3	1	1	0	0	1	1	2
queue	*I*	*A*										

topological sort is
C H D

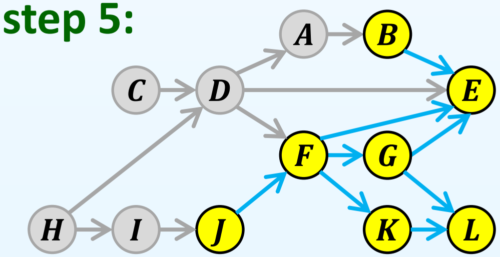
step 4:



	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
in-degree	0	1	0	0	3	1	1	0	0	0	1	2
queue	*A*	*J*										

topological sort is
C H D I

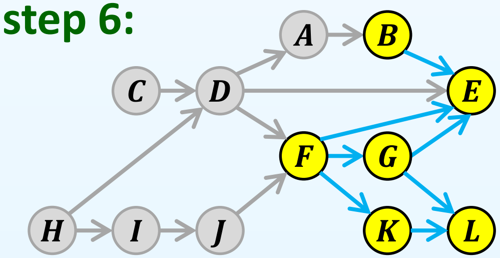
step 5:



	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
in-degree	0	0	0	0	3	1	1	0	0	0	1	2
queue	*J*	*B*										

topological sort is
C H D I A

step 6:

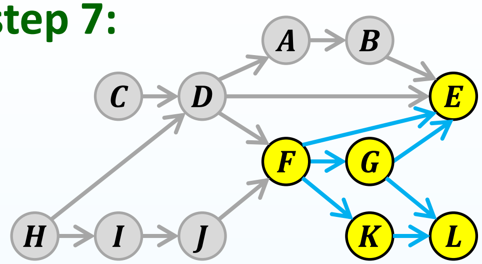


	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
in-degree	0	0	0	0	3	0	1	0	0	0	1	2
queue	*B*	*F*										

topological sort is
C H D I A J

Example

step 7:

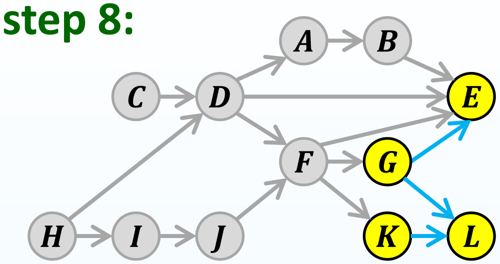


	A	B	C	D	E	F	G	H	I	J	K	L
in-degree	0	0	0	0	2	0	1	0	0	0	1	2
queue	F											

topological sort is

C H D I A J B

step 8:

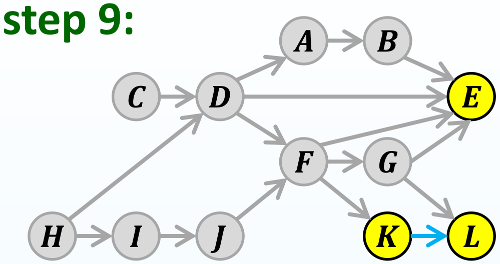


	A	B	C	D	E	F	G	H	I	J	K	L
in-degree	0	0	0	0	1	0	0	0	0	0	0	2
queue	G	K										

topological sort is

C H D I A J B F

step 9:

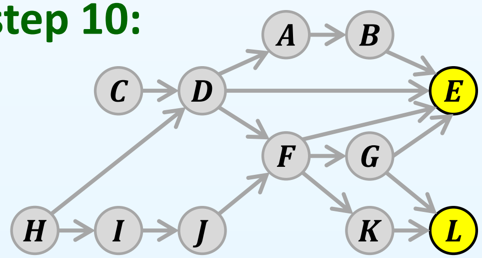


	A	B	C	D	E	F	G	H	I	J	K	L
in-degree	0	0	0	0	0	0	0	0	0	0	0	1
queue	K	E										

topological sort is

C H D I A J B F G

step 10:

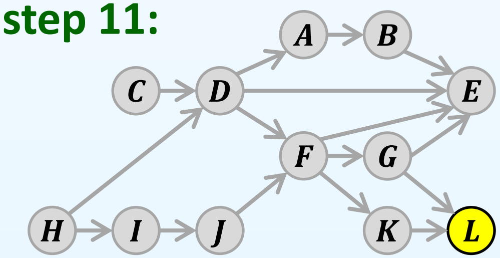


	A	B	C	D	E	F	G	H	I	J	K	L
in-degree	0	0	0	0	0	0	0	0	0	0	0	0
queue	E	L										

topological sort is

C H D I A J B F G K

step 11:

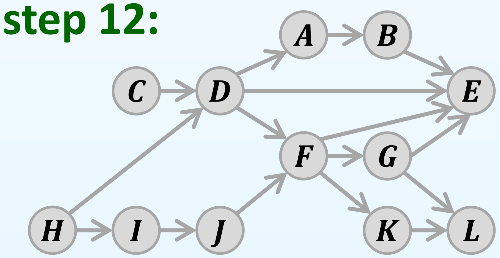


	A	B	C	D	E	F	G	H	I	J	K	L
in-degree	0	0	0	0	0	0	0	0	0	0	0	0
queue	L											

topological sort is

C H D I A J B F G K E

step 12:



	A	B	C	D	E	F	G	H	I	J	K	L
in-degree	0	0	0	0	0	0	0	0	0	0	0	0
queue												

topological sort is

C H D I A J B F G K E L

Complexity Analysis

- Initialize in-degree array
  - ➡ $\Theta(|E|)$
- Initialize queue with zero in-degree vertices
  - ➡ $\Theta(|V|)$
- Dequeue and output vertex
  - All vertices will be enqueued and dequeued once, each take $\Theta(1)$
  - ➡ $\Theta(|V|)$
- Reduce in-degree of all vertices adjacent to a vertex and enqueue any zero in-degree vertices
  - All edges will decrease in-degree of adjacent once, each take $\Theta(1)$
  - ➡ $\Theta(|E|)$

Finally, the topological sort takes $\Theta(|V| + |E|)$ time

Any Question?