# Linear Algebra Notes

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## Contents

## 1   Vectors and Spaces

### 1.1   Parametric representations of lines

**Example 1.1.** Suppose that $L_1$ and $L_2$ are lines in the plane, that the x-intercepts of $L_1$ and $L_2$ are 5 and -1, respectively, and that the respective y-intercepts are 5 and 1. Find the point of intersection of $L_1$ and $L_2$.

**Solution 1.1.** Pick two points on $L_1$, i.e. $(5, 0)$ and $(0, 5)$. Let $\mathbf{a} = \langle 5, \ 0 \rangle$ and $\mathbf{b} = \langle 0, \ 5 \rangle$. Now, $L_1$ can be represented by the following:

$$L_1 = \{(\mathbf{a} - \mathbf{b})t + \mathbf{b} \mid t \in \mathbb{R}\}$$

$$= \left\{ \begin{bmatrix} 5t \\ -5t + 5 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

The $(\mathbf{a} - \mathbf{b})t$ represents $L_1$ parameterized through the origin, so we apply a translation of $\mathbf{a}$ or $\mathbf{b}$. Similarly, let $\mathbf{c} = \langle -1, 0 \rangle$ and $\mathbf{d} = \langle 0, 1 \rangle$.

$$L_2 = \{(\mathbf{c} - \mathbf{d})s + \mathbf{d} \mid s \in \mathbb{R}\}$$
$$= \left\{ \begin{bmatrix} -s \\ -s+1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$$

The point of intersection is where $L_{1_x} = L_{2_x}$ and $L_{1_y} = L_{2_y}$, allowing us to define a system of equations.

$$5t = -s \qquad\qquad\qquad -5t + 5 = -s + 1$$

$$s = -2$$
$$x = -s = 2$$
$$y = -s + 1 = 3$$

The point of intersection is $(2, 3)$. ∎

## 1.2 Linear dependence

**Definition 1.1** (Linear combination). Let $V = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ where $\mathbf{v_i} \in \mathbb{R}^n$. A *linear combination* of $V$ is defined to be $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k}$ where $c_i \in \mathbb{R}$.

**Definition 1.2** (Linearly dependent). Let $V = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ where $\mathbf{v_i} \in \mathbb{R}^n$. $V$ is *linearly dependent* if and only if $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$ where $c_i \in \mathbb{R}$ and there exists at least one $c_i$ such that $c_i \neq 0$. In other words, $V$ is *linearly independent* if and only if $c_1, c_2, \ldots, c_k = 0$ is the only solution.

**Definition 1.3** (Span). Given a set $S$ of vectors, the *span* of S, denoted span$(S)$, is the set of all linear combinations of $S$.

**Example 1.2.** Let $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Is $S$ linearly dependent?

**Solution 1.2.**

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} c_2 + \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} c_3 = 0$$

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ -c_1 + c_2 &= 0 \\ 2c_1 + 3c_2 + 2c_3 &= 0 \end{aligned}$$

Solving this system, we find that the only solution is $c_1 = c_2 = c_3 = 0$. Thus, $S$ is linearly independent. ∎

We can go further by saying that the span of $S$ is $\mathbb{R}^3$. This can be proven by setting the linear combination of $S$ to an arbitrary 3-dimensional vector and isolating for the scalars $c_1$, $c_2$, and $c_3$, which tells us that any given vector in $\mathbb{R}^3$ can be represented in a specific linear combination of $S$. This can be thought of intuitively as well: if $S$ is linearly independent, each vector of $S$ introduces new directionality; there is no vector in $S$ that can be defined as a linear combination of the other vectors.

## 1.3   Linear subspaces

**Definition 1.4** (Linear subspace). A set of vectors $V \subseteq \mathbb{R}^n$ is defined to be a linear/vector *subspace* of $\mathbb{R}^n$ if and only if it contains $\mathbf{0}$, it is closed under scalar multiplication, and it is closed under addition:

$$\mathbf{0} \in V$$
$$\forall\, c \in \mathbb{R}, \; c\mathbf{v} \in V$$
$$\forall\, \mathbf{v_1}, \mathbf{v_2} \in V, \mathbf{v_1} + \mathbf{v_2} \in V$$

**Theorem 1.1.** *Let* $V = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$. *It holds true that* $\mathrm{span}(V)$ *is a subspace of* $\mathbb{R}^n$.

*Proof.* If $\mathrm{span}(V)$ is a valid linear subspace, it must contain the zero vector, it must be closed under scalar multiplication, and it must be closed under addition.

By definition, the span of $V$ is the set of all linear combinations of $V$:

$$\mathrm{span}(V) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} \; \forall\, c_i \in \mathbb{R}\}$$

1. Inclusion of zero vector:

$$\text{Let } c_1, c_2, \ldots, c_n = 0.$$

$$\implies c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$$

$$\implies \mathbf{0} \in \mathrm{span}(V)$$

2. Closure under scalar multiplication:

$$\text{Let } \mathbf{a} \in \mathrm{span}(V).$$

$$\iff c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{a}$$

$$d(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = d\mathbf{a} \text{ where } d \in \mathbb{R}$$

$$\implies dc_1\mathbf{v_1} + dc_2\mathbf{v_2} + \cdots + dc_n\mathbf{v_n} = d\mathbf{a}$$

$dc_i$ is just a scalar, meaning $d\mathbf{a}$ is another linear combination of $V$:

$$d\mathbf{a} \in \text{span}(V)$$

3. Closure under addition:

Let $\mathbf{a}, \mathbf{b} \in \text{span}(V)$.

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{a}$$
$$d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_n\mathbf{v_n} = \mathbf{b}$$
$$(c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots + (c_n + d_n)\mathbf{v_n} = \mathbf{a} + \mathbf{b}$$

Again, $(c_i + d_i)$ is just a scalar, meaning $\mathbf{a} + \mathbf{b}$ is another linear combination of $V$:

$$\mathbf{a} + \mathbf{b} \in \text{span}(V)$$

∎

**Definition 1.5** (Basis). Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be linearly independent. It follows that S is a *basis* for the subspace $V = \text{span}(S)$.

**Lemma 1.1.** *Any vector in a subspace $V$ is the result of a unique linear combination of some basis for $V$.*

*Proof.* Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis for the subspace $V$. Suppose $\mathbf{a} \in V$.

$$\mathbf{a} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} \text{ where } c_i \in \mathbb{R}$$

Assume that $\mathbf{a}$ can be represented by another linear combination of $S$.

$$\mathbf{a} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_n\mathbf{v_n} \text{ for some } d_j \neq c_j$$
$$\mathbf{0} = (c_1 - d_1)\mathbf{v_1} + (c_2 - d_2)\mathbf{v_2} + \cdots + (c_n - d_n)\mathbf{v_n}$$

S is linearly independent so the scalar $(c_i - d_i)$ must be zero for $1 \leq i \leq n$. This implies that $c_i = d_i$ which contradicts the statement that some $d_j \neq c_j$. This further implies that $\mathbf{a}$ cannot be represented by more than one linear combination of $S$. ∎

## 1.4 Dot product

**Definition 1.6.** The *dot product* of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, denoted $\mathbf{a} \cdot \mathbf{b}$, is defined to be $\mathbf{a} \cdot \mathbf{b} = \mathbf{a_1}\mathbf{b_1} + \mathbf{a_2}\mathbf{b_2} + \cdots + \mathbf{a_n}\mathbf{b_n}$.

**Lemma 1.2.** $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

*Proof.* Let $\mathbf{a} \in \mathbb{R}^n$.

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$
$$\|\mathbf{a}\|^2 = |a_1^2 + a_2^2 + \cdots + a_n^2|$$
$$= a_1^2 + a_2^2 + \cdots + a_n^2$$
$$= \mathbf{a} \cdot \mathbf{a}$$

∎

### 1.4.1 Properties

Dot products are commutative $(\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a})$, distributive $(\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c})$, and associative with scalars $(c(\mathbf{a} \cdot \mathbf{b}) = (c\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{b}))$. These properties can be easily proven with the definition of the dot product.

### 1.4.2 Geometric representation

The dot product of two vectors can also be represented in relation to the angle between them, $\theta$, by the following: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta)$.
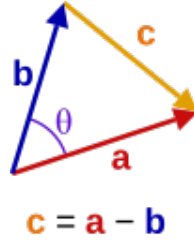


Figure 1: The angle $\theta$ in the triangle constructed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - \mathbf{b}$

*Proof.* Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{a}, \mathbf{b} \neq 0$. Using the Law of Cosines,

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos(\theta)$$
$$\|\mathbf{a}\|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\| \cos(\theta)$$
$$-2(\mathbf{a} \cdot \mathbf{b}) = -2\|\mathbf{a}\|\|\mathbf{b}\| \cos(\theta)$$
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos(\theta)$$

■

### 1.4.3 Interpretation

The geometric representation makes it easy to recognize that $\mathbf{a} \cdot \mathbf{b}$ is maximized when $\theta = 0°$, which is when $\mathbf{a}$ and $\mathbf{b}$ are collinear in the same direction. On the other hand, $\mathbf{a} \cdot \mathbf{b}$ is minimized when $\theta = 180°$, which is when $\mathbf{a}$ and $\mathbf{b}$ are collinear in opposite directions. When $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other (i.e. $\theta = 90°$), $\mathbf{a} \cdot \mathbf{b} = 0$. This means that we can interpret the dot product as a measure of collinearity.

**Definition 1.7** (Orthogonal). Vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are *orthogonal* to each other if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

### 1.4.4 Cauchy-Schwarz Inequality

**Lemma 1.3.** *For some scalar $c \geq 0$ and some vector $\mathbf{a}$, $c\|\mathbf{a}\| = \|c\mathbf{a}\|$.*

*Proof.* Let $c \geq 0, c \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{R}^n$.

$$
\begin{aligned}
c\|\mathbf{a}\| &= c\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\
&= \sqrt{c^2(a_1^2 + a_2^2 + \cdots + a_n^2)} \\
&= \sqrt{(ca_1)^2 + (ca_2)^2 + \cdots + (ca_n)^2} \\
&= \|c\mathbf{a}\|
\end{aligned}
$$

■

**Theorem 1.2** (Cauchy-Schwarz inequality). *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $\|\mathbf{x}\|\|\mathbf{y}\| \geq |\mathbf{x} \cdot \mathbf{y}|$. *Furthermore,* $\|\mathbf{x}\|\|\mathbf{y}\| = |\mathbf{x} \cdot \mathbf{y}|$ *if and only if* $\mathbf{y} = c\mathbf{x}$ *for some* $c \in \mathbb{R}$.

*Proof.* Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be nonzero. Let $p(t) = \|t\mathbf{y} - \mathbf{x}\|^2$. Note that $p(t) \geq 0$ for all $t \in \mathbb{R}$.

$$
\begin{aligned}
p(t) &= (t\mathbf{y} - \mathbf{x}) \cdot (t\mathbf{y} - \mathbf{x}) \\
&= t^2\|\mathbf{y}\|^2 - 2t(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2
\end{aligned}
$$

Let $a = \|\mathbf{y}\|^2$. Let $b = 2(\mathbf{x} \cdot \mathbf{y})$. Let $c = \|\mathbf{x}\|^2$. Note that $a \neq 0$ because $\mathbf{y} \neq \mathbf{0}$.

$$
\begin{aligned}
p(t) &= at^2 - bt + c \geq 0 \\
p\left(\frac{b}{2a}\right) &= a\left(\frac{b}{2a}\right)^2 - b\left(\frac{b}{2a}\right) + c \\
&= \frac{b^2}{4a} - \frac{b^2}{2a} + c \\
&= -\frac{b^2}{4a} + c \geq 0 \\
\implies c &\geq \frac{b^2}{4a}
\end{aligned}
$$

Substituting back $a$, $b$, and $c$:

$$
\begin{aligned}
\|\mathbf{x}\|^2 &\geq \frac{4(\mathbf{x} \cdot \mathbf{y})^2}{4\|\mathbf{y}\|^2} \\
\|\mathbf{x}\|^2\|\mathbf{y}\|^2 &\geq (\mathbf{x} \cdot \mathbf{y})^2 \\
\|\mathbf{x}\|\|\mathbf{y}\| &\geq |\mathbf{x} \cdot \mathbf{y}|
\end{aligned}
$$

Consider the case of the equality:

$$
\begin{aligned}
\|\mathbf{x}\|\|\mathbf{y}\| &= |\mathbf{x} \cdot \mathbf{y}| \\
\|\mathbf{x}\|\|\mathbf{y}\| &= |\|\mathbf{x}\|\|\mathbf{y}\| \cos(\theta)|
\end{aligned}
$$

This can be true if and only if $\theta = 0°$ or $\theta = 180°$, meaning $\mathbf{x}$ and $\mathbf{y}$ must be collinear with each other.

While we initially assumed that $\mathbf{x}$ and $\mathbf{y}$ were nonzero, it is easy to see that the inequality still holds when $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$. ■

**Theorem 1.3** (Triangle inequality). *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. *This inequality becomes an equality if and only if* $\mathbf{y} = c\mathbf{x}$ *for some* $c \in \mathbb{R}$ *such that* $c \geq 0$. *In other words, the sum of the lengths of two sides of a triangle is always greater than the length of its third side.*
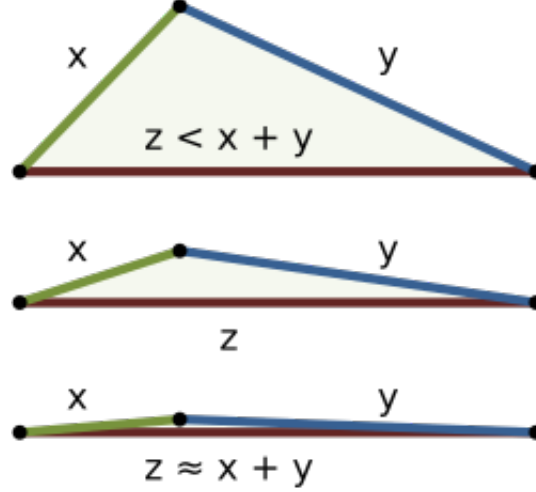


Figure 2: As two sides of a triangle get closer to being collinear, the sum of their lengths gets closer to the length of the third side.

*Proof.* Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. By the Cauchy-Schwarz inequality (Theorem 1.2),

$$\|\mathbf{x}\|\|\mathbf{y}\| \geq |\mathbf{x} \cdot \mathbf{y}| \geq \mathbf{x} \cdot \mathbf{y}$$

Note that $|\mathbf{x} \cdot \mathbf{y}| = \mathbf{x} \cdot \mathbf{y}$ if and only if $\mathbf{x} \cdot \mathbf{y} \geq 0$. Else, $|\mathbf{x} \cdot \mathbf{y}| > \mathbf{x} \cdot \mathbf{y}$. Additionally, $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ if and only if $\mathbf{y} = c\mathbf{x}$ for some $c \in \mathbb{R}$, as stated in the Cauchy-Schwarz inequality. So, if $c \geq 0$ and $\mathbf{y} = c\mathbf{x}$, then $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$.

Now, consider the following:

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2$$

Since $\|\mathbf{x}\|\|\mathbf{y}\| \geq \mathbf{x} \cdot \mathbf{y}$, we can create the inequality $\|\mathbf{x} + \mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2$. Note that this inequality becomes an equality if and only if $\mathbf{y} = c\mathbf{x}$.

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2$$
$$\|\mathbf{x} + \mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$$
$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

∎

## 1.5  Cross product

**Definition 1.8.** The *cross product* of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, denoted $\mathbf{a} \times \mathbf{b}$, is defined to be

$$
\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}
$$

*Remark.* A way to easily remember how to calculate the cross product is by writing it as a determinant (covered in Section ??):

$$
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
$$