Linear Algebra Notes

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1 Vectors and Spaces

1.1 Parametric representations of lines

Example 1.1. Suppose that L_1 and L_2 are lines in the plane, that the x-intercepts of L_1 and L_2 are 5 and -1, respectively, and that the respective y-intercepts are 5 and 1. Find the point of intersection of L_1 and L_2.

Solution 1.1. Pick two points on L_1, i.e. $(5, 0)$ and $(0, 5)$. Let $\mathbf{a} = \langle 5,\ 0\rangle$ and $\mathbf{b} = \langle 0,\ 5\rangle$. Now, L_1 can be represented by the following:

$$\begin{aligned} L_1 =& \{(\mathbf{a} - \mathbf{b})t + \mathbf{b} \mid t \in \mathbb{R}\} \\ =& \left\{ \begin{bmatrix} 5t \\ -5t + 5 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\} \end{aligned}$$

The $(\mathbf{a} - \mathbf{b})t$ represents L_1 parameterized through the origin, so we apply a translation of $\mathbf{a}$ or $\mathbf{b}$. Similarly, let $\mathbf{c} = \langle -1, 0\rangle$ and $\mathbf{d} = \langle 0, 1\rangle$.

$$\begin{aligned} L_2 =& \{(\mathbf{c} - \mathbf{d})s + \mathbf{d} \mid s \in \mathbb{R}\} \\ =& \left\{ \begin{bmatrix} -s \\ -s + 1 \end{bmatrix} \middle| \, s \in \mathbb{R} \right\} \end{aligned}$$

The point of intersection is where $L_{1_x} = L_{2_x}$ and $L_{1_y} = L_{2_y}$, allowing us to define a system of equations.

$$5t = -s \qquad\qquad -5t + 5 = -s + 1$$

$$\begin{aligned} s =& -2 \\ x =& -s = 2 \\ y =& -s + 1 = 3 \end{aligned}$$

The point of intersection is (2, 3). □

1.2 Linear dependence

Definition 1.1 (Linear combination)**.** Let $V = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ where $\mathbf{v_i} \in \mathbb{R}^n$. A *linear combination* of V is defined to be $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k}$ where $c_i \in \mathbb{R}$.

Definition 1.2 (Linearly dependent)**.** Let $V = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ where $\mathbf{v_i} \in \mathbb{R}^n$. V is *linearly dependent* if and only if $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$ where $c_i \in \mathbb{R}$ and there exists at least one c_i such that $c_i \neq 0$. In other words, V is *linearly independent* if and only if $c_1, c_2, \dots, c_k = 0$ is the only solution.

Definition 1.3 (Span)**.** Given a set S of vectors, the *span* of S, denoted $\text{span}(S)$, is the set of all linear combinations of S.

Example 1.2. Let $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Is S linearly dependent?

Solution 1.2.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} c_2 + \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} c_3 = 0$$

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ -c_1 + c_2 &= 0 \\ 2c_1 + 3c_2 + 2c_3 &= 0 \end{aligned}$$

Solving this system, we find that the only solution is $c_1 = c_2 = c_3 = 0$. Thus, S is linearly independent. □

We can go further by saying that the span of S is $\mathbb{R}^3$. This can be proven by setting the linear combination of S to an arbitrary 3-dimensional vector and isolating for the scalars c_1, c_2, and c_3, which tells us that any given vector in $\mathbb{R}^3$ can be represented in a specific linear combination of S. This can be thought of intuitively as well: if S is linearly independent, each vector of S introduces new directionality; there is no vector in S that can be defined as a linear combination of the other vectors.

1.3 Linear subspaces

Definition 1.4 (Linear subspace)**.** A set of vectors $V \subseteq \mathbb{R}^n$ is defined to be a linear/vector *subspace* of $\mathbb{R}^n$ if and only if it contains $\mathbf{0}$, it is closed under scalar multiplication, and it is closed under addition:

$$\begin{aligned} \mathbf{0} &\in V \\ \forall\ c \in \mathbb{R},\ c\mathbf{v} &\in V \\ \forall\ \mathbf{v_1}, \mathbf{v_2} \in V, \mathbf{v_1} + \mathbf{v_2} &\in V \end{aligned}$$

Theorem 1.1. *Let* $V = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$*. It holds true that* $\text{span}(V)$ *is a subspace of* $\mathbb{R}^n$*.*

Proof. If $\text{span}(V)$ is a valid linear subspace, it must contain the zero vector, it must be closed under scalar multiplication, and it must be closed under addition.

By definition, the span of V is the set of all linear combinations of V:

$$\text{span}(V) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} \; \forall \; c_i \in \mathbb{R}\} \tag{1}$$

1. Inclusion of zero vector:

$$\text{Let } c_1, c_2, \ldots, c_n = 0.$$

$$\implies c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$$

$$\implies \mathbf{0} \in \text{span}(V)$$

2. Closure under scalar multiplication:

$$\text{Let } \mathbf{a} \in \text{span}(V).$$

$$\iff c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{a}$$

$$d(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}) = d\mathbf{a} \text{ where } d \in \mathbb{R}$$

$$\implies dc_1\mathbf{v_1} + dc_2\mathbf{v_2} + \cdots + dc_n\mathbf{v_n} = d\mathbf{a}$$

dc_i is just a scalar, meaning $d\mathbf{a}$ is another linear combination of V:

$$d\mathbf{a} \in \text{span}(V)$$

3. Closure under addition:

$$\text{Let } \mathbf{a}, \mathbf{b} \in \text{span}(V).$$

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{a}$$

$$d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_n\mathbf{v_n} = \mathbf{b}$$

$$(c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots + (c_n + d_n)\mathbf{v_n} = \mathbf{a} + \mathbf{b}$$

Again, $(c_i + d_i)$ is just a scalar, meaning $\mathbf{a} + \mathbf{b}$ is another linear combination of V:

$$\mathbf{a} + \mathbf{b} \in \text{span}(V)$$

□

Definition 1.5 (Basis)**.** Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be linearly independent. It follows that S is a *basis* for the subspace $V = \text{span}(S)$.