Graph Theory

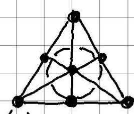
An attempt to explain pure mathematics Ø5.17.2Ø19

- Mathematics that is developed with an eye to practical applications is "Applied math." While "pure math" which is a "charming pastime." Pure math is real math.

To understand math you must understand pure math. Most have never seen pure math. No real feel for it. Consequently most people don't understand mathematics. Pure mathematics is the foundation of applied mathematics, you will see pure math under applications.

Euclidean Geometry - developed in Greece between 6ØØ & 3ØØ BCE & codified at the end of that period by Euclid in the Elements. The Elements is the "Archetype" of pure math & a paradigm that mathematicians have emulated ever since its appearance. It begins abruptly with a list of definitions, followed by a list of basic assumptions or "axioms". Euclid states ten axioms, there other he did not write down. The work consists of a single deductive chain of 465 theorems.

To Euclid a theorem was significant, 05.17.2019 or not, in & of itself; it did not become more signifi'cant if applications were discovered, or less so if not discovered. { String Theory } { octonions }

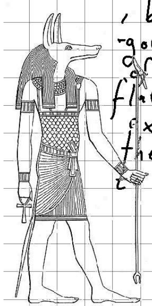


Euclid saw applications as external factors having no bearing on a theorem's inherent quality. The theorems are included for their own sake.
Attitude of self-sufficiency is the hallmark of pure mathematics.
The Elements is the most successful textbook ever written.
Use Euclidean Geometry as an example of pure mathematics -> a branch of pure math most are familiar with.

Games -> pure math is a box of games. One of these games is Euclidean Geometry.
Games have 4 components: objects to play with, an opening arrangement, rules, & a goal. e.g. chess.

The rules of chess tell how the pieces 05.17.2019 move; that is, they specify how arrangements can be created from the opening arrangement.
The goal is checkmate. Which can be described as an arrangement having certain desirable properties a "nice" arrangement. The opening arrangement is the list of axioms, which are accepted without proof.
The analogy with opening arrangement of chessmen may not be apparent, but it is quite strong. First, to begin -> the opening arrangement must start with "that" arrangement & no other. Same -way as Euclidean's axioms.' Second, the opening arrangement of chess-figures are related at the outset? -> The rules of Euclidean Geometry are the rules of formal logic, which is nothing but an "etherealized" version of "common sense," we may absorb from culture as we age. The goal of Euclidean Geometry is to produce as many "nice," arrangements as possible, that is to prove "profound & surprising theorems.' Checkmate terminated a chessmate -> Geometry is open-ended?
Games have a feature in common with pure math, It is subtle but important. It is that the objects which a game is played - having no mean -

-ing outside of the context of the game. Chess figures, for example, are significant only in reference to chess. 05.17.2019
They have no correspondence with anything external to the game. Some chess-masters can play without a board or pieces of any-kind.
The "essence" of chess is its abstract structure.
→ What is relevant is the number & geometric arrangement of the 'squares,' the 'number of types of pieces & the number of pieces of each type, the 'quantitative-geometric power of each piece, etc. So it is with pure math. Euclid's words "plane," "point," & line suggest that geometry deals with flat surface, tiny dots, stretched lines. , but this is only implied interpretation. It is analy-gous to the interpretation of chess as some type of "battle." Geometry is no more a study of flat surfaces & dots than chess is to military exercise. 'Planes, points, or lines → Say that they are "objects" related to each other — in accordance to axioms. These are but convenient names.

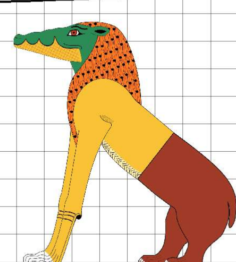


i.e convenient names for the 3 types of objects geometers play with. Any other names will do as well. You can replace these names with symbols such as #, $, ? ... etc Even if you erase all the diagrams in the elements & replace them, it would not make a difference. 'Geometric diagrams are to geometers what board & pieces are to chess players : i.e visual aids, helpful but not indispensable. 05.18.2019

- Why study pure Mathematics?

Plato's academy → "Let no one ignorant of geometry enter here!" → Pure math is held at high regards. Pure math is also applicable

See "lost in Math: how Beauty Leads Physics Astray"
also see → Boston MOS
Amir D. Aczel, Brian Greene, video on Youtube
March 22, 2011 - published



Because pure mathematics has no 05.18.2019
inherent correspondence with the outside world,
we are free to make it correspond, to "interpret
it, in any way choose. * See "fish Bowl" - Stephen
Hawking ->
" A few years ago the city council of
Monza, Italy, barred pet owners from keeping
Goldfish in curved bowls" ... "saying that it
is cruel to keep a fish in a bowl with curved sides
because, gazing out, the fish would have a distorted
reality."
" Goldfish could still formulate scientific
laws (analogy to humans) governing the motion of the
objects they observe outside their bowl. e.g. due
to distortion, a freely moving object would be
observed by the goldfish to move along a <u>curved path.</u>
... the Goldfish could formulate laws from their
distorted frame of reference that would <u>always hold
true.</u>
See Ptolemy (150 ACE) ~ described the
motion of celestail bodies. The Earth stood still
everything else circled around it. You can even use
math to describe this model. See * Carl Sagan

Pure math contin... most branches 05.18.2019
of pure math can be interpreted in such a way
that the axioms & theorems because approximately
true statements about the external world.

- A branch of pure math utterly lacking in significant
interpretations would be boring to the community
of pure mathematicians & the would soon die out
from lack of interest -> * see String Theory.

Pure math originates from abstractions of the "real"
world ~ from the physical world, just as geometry
begins with "idealized" dot and lines.
Once the abstractions have been made the mathemati-
-cal game comes into play. { independent - existance -
& evolves under its own laws }

" number rules the universe " - Pythagoras

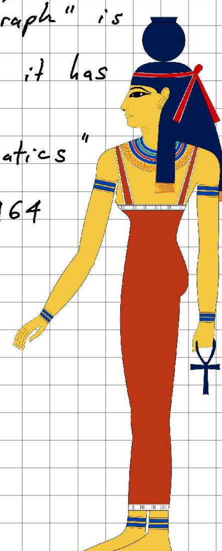
" Math is the only true metaphysics " - Kelvin

See "Ship of Theseus" -> Time Travel, <u>One electron</u>

Our common sense, or world view, is ∅5.18.2∅19
not "common" to all people. It is shaped by
the culture we inhabit. It is like a pair of
glasses, "few of us ever manage to take off
* see → Joseph Campell, power of Myth" So confirmat-
-ion is everywhere we look. "Of this embedded material
perhaps the most fundamental is logic, the standard
by which judge reasoning to be "correct", recorded by
Aristotle - Organon (35∅ BCE)." "I use
logic all the time in math, & it seems to yield
"correct" results, but in mathematics "correct" means
"logical", see (catch - 22)" "I can't defend logic
because I can't remove my eye-glasses." Not
suprisingly, since logic is the study of deduction.
→ See Bertrand Russell & pure mathematics is
the only completely deductive study, logic is inex-
-tricably interwined with pure mathematics.
"Which came first, logic or mathematics?
"chicken or the egg, dinosaur came first" Aristotle
abstracted the laws of Logic in some part from
pure math he studied at Plato's academy.
Pure math can be fun. - A few might be
Philosophers taking Bertrand Russell's advice -
to create a healthy Philosophy you should →

→ renounce metaphysics but be ∅5.18.2∅19
a good mathematician." You can also use it to
sharpen your mind.

The objects of this game are called "graphs", should have
been called → network Theory "Graph" is
a network of dots & lines.
Graph Theory is new → the bulk of it has
been developed since 189∅"
See → "Fantasia Mathematica"
→ Nature of growth of Modern Mathematics"
Edna Kramer → 197∅
Philosophy of Mathematics → S. Barker 1964
Laws of Form Spencer Brown, 1973



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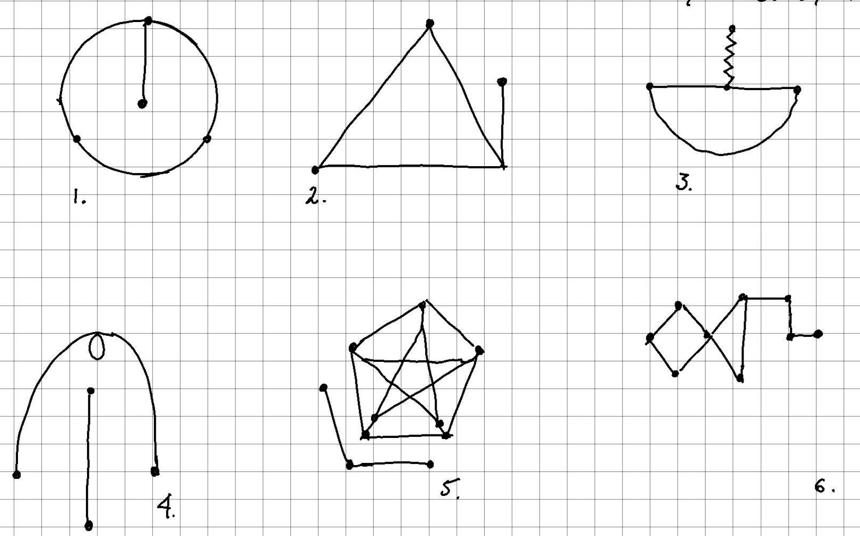


Graphs 1, 2, 3 are isomorphic Ø5. 21. 2Ø19

Sets: A set is a collection of distinct objects, none of which is the set itself

A collection → e.g $A = \{1, 2, 3, A\}$ for now will be excluded

The words "object" & collection are left undefined
There is an interpretation of "set" as being a bunch of things, an aggregate of entities, etc.
Words such as "object" & "collection" are used in a slightly unconventional fashion – "collections" of "infinitely" many things, or just one thing, or even "no things" NULL.
An object in mathematics could be "anything."
e.g. a wave in physics is a mathematical object with certain properties → like a JSON.

However, in mathematics these objects 05. 21. 2019
must be "mathematically" true, e.g. like a Soccer
ball, a Dinosaur, even unicorns and/or Peter Pan are
all true. A figure such as a shape/form which
= simultaneously has all the geometric properties of
both a triangle & a circle cannot be "mathemat-
ically" true. ' There is no proof for such a
figure/form as of now.

A set is usually denoted by listing the
names of the objects, seperated by commas,
within a pair of braces.

{ 12, z, \$, Alamo, 🤔 6, Quetzalcoatl }

is a set.

The elements of the sets {objects}, could be key-value
pairs, properties - such as some variable - need not
have anything in common. You can do this →

$$\underbrace{\text{" Let } A = \{ 12, z, \$, \text{Alamo}, 🤔, 6, \text{Quetzalcoatl} \}\text{"}}_{\text{Mathematical Notation}}$$

Definition: A set containing no elem- 05. 21. 2019
ents is called a null set or empty set.

→ An empty set plays the same role in
set theory as 0 plays in Arithmetic - but do not
confuse the two. While an empty set is still a set,
0 is a number. 0 can tell you how many objects are
in an empty set.

Definition: 3: A set A is said to be a subset of
a set B, denoted " $A \subset B$ ", if every of A
is also an Element of B.

e.g. If $A = \{1, 2, 3\}$ and $B = \{\&, 3, +, 1, 2\}$,
then $A \subset B$, $A \subset A$, and $B \subset B$.

* every set is a subset of itself.

Convention: We agree to consider an empty set to
be a subset of every set.

This "convention" can in fact be proved, but only
by a logical contortion that seems inappropriate
at the present. Thus this is a convention rather than
a theorem.

e.g. if J is an empty set A and B are the sets of the previous example, then $J \subset A$, $J \subset B$, and $J \subset J$. ϕ5.22.2ϕ19

Def 4.: A set A is said to be equal to set B, denoted "A = B", if $A \subset B$ and $B \subset A$. Thus A = B if A and B consists of exactly the same objects. It follows that the order in which the elements of a set appear is irrelevant.
e.g. $\{1, 2, 3\} = \{2, 1, 3\}$ since $\{1, 2, 3\} \subset \{2, 1, 3\}$

```
{
  "Username" : "Bigfoot",
  "password" : "Blink183"
}
{
  "password" : "Blink183",
  "username" : "Bigfoot"
}
```

both json $\subset$ of each other

Theorem 1: There is only one empty set. ϕ5.22.2ϕ19

Proof: Let J & K be empty sets. Then by convention $J \subset K$ & $K \subset J$, so $J = K$ ~ i.e by def 4. Thus all empty sets are equal; i.e there is only one empty set.

Notation: The empty set shall be denoted by ϕ or containing one element, that element being the symbol ϕ. In a view of the word "distinct" in def 1. collections like "{1, 9, 9, 9 &, 137}", in which an object is listed more than once, once, ARE NOT sets. The definition was designed to exclude this sort of thing because "{1, 9, 9, 9, & 137}" is a redundancy. You are being told 3x that the collection contains a number 9.
Pure mathematics is excruciatingly careful about definitions & proofs.

Paradox: In the 6^{th} century B.C.E, Crotona, Italy → Pythagoreans living there

The Pythagorean paradox → Let AB and CD be two finite straight lines. 05.22.2019

The finite straight XY is a "common measure" of AB and CD if there are whole numbers m & n so that XY laid down m times is the same length as CD.

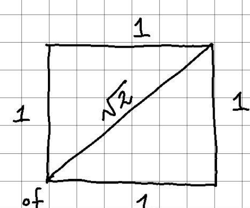
e.g. if AB were a yard long and CD 10 inches, a line segment XY of 2 inches would be a common measure with $m = 18$ & $n = 5$; laying down XY eighteen times would produce a length of 10 inches, the same as CD. XY = 1/4 inch would be another common measure, this time with $m = 144$ & $n = 40$. It became intuitively evident to the Pythagoreans ~ that a common measure can be found for any pair of finite straight lines -- though of course it may be necessary to take XY quite small in order to measure both AB & CD exactly.

Since $AB/CD = \dfrac{m\,(XY)}{n\,(XY)} = \dfrac{m}{n}$, i.e. a rational number.

aka a quotient of whole numbers [i.e. the quotient of whole numbers is always a rational number.

But wait...! 05.23.2019

Now take a square with a side equal to 1 & draw a diagonal is $\sqrt{2}$, & so the quotient of the length of the diagonal & length of one of the sides is $\sqrt{2}$. If the Pythagorean community's intuition is correct → then



inserting that the quotient of the two lengths is always a rational number, $\sqrt{2}$ must be a rational number.

Hippasus 5th century BCE (probably, Hippasus) discovered that $\sqrt{2}$ cannot be a rational number → Hold on to your tunics → The proof → Any rational number can be reduced to "lowest terms", i.e. express-ed as a quotient of whole numbers that have no factors beside 1 in common. e.g. $\dfrac{360}{75} = \dfrac{24}{5}$, & 24 & 5 have no common factor.

Therefore, if $\sqrt{2}$ were a rational number, it would be possible to express $\sqrt{2}$ as $\sqrt{2} = \frac{p}{q}$, where p & q are whole numbers without a common factor → Squaring both sides gives $\frac{p^2}{q^2} = 2$, multiplying both sides by q^2 gives → $2q^2 = p^2$. This means that p^2 must be even, because it is twice another whole number, q^2. 05.23.2019.

Pythagoreans had shown/proved that the square of an odd number is odd. Here we see that p^2 is even → implying p is even also (if p were odd, p^2 would be odd) p in this cas is even; i.e, p is the double of some other whole number → $p = 2r$, where r = some whole number. Substituting $p = 2r$ into this equation → $2q^2 = p^2$ gives $2q^2 = (2r)^2$ or $2q^2 = 4r^2$. Divide both sides by 2 gives $q^2 = 2r^2$ so q^2, being twice the whole number r^2 is even, and so is q, which is also even.

However since p & q are both even, as we have shown → then they both have a common factor of 2, which contradicts the earlier statement where p and q have no common factor.

"Hence the assumption that $\sqrt{2}$ is rational leads to a contradiction & so is logically untenable." 05.23.2019

This why you should trust logic over intuition. But the axioms of which this logic is derived from is intuitive i.e ~ unsusceptible to proof. intuition plays a big role in the discovery of theorems, or otherwise mathematicians would be spending most of their time trying to prove false theorems. Intuitive evidence is not accepted as conclusive.

Definitions & axioms are carefully framed with an eye to hidden assumptions.

"Theorems are stated as precisely as possible; & though they are discovered by intuition, they are demonstrated by logic alone. This habit of mind is called by mathematicians "rigor." It is the characteristics of pure math."

Russell's Paradox: "During the last half of the 19^{th} century several branches of pure math were being simultaneously rebuilt because of paradoxes that had arisen a few decades prior. This rebuilding

was based on the notion of "set," 05.23.2019
which was defined as →

Old Definition: "A set is a collection of distinct objects."
The set concept seemed like a good foundation because it was simple & obviously free of logical difficulty. (a good starting point) Mathematicians were confident that with it they could build a logical edifice so solid as to never again be shaken by a paradox.
But, then, in 1902, Bertrand Russell presented his infamous paradox." He observed that if we define a set to be merely a collection of distinct objects, we have to include a self-referent collection's such as $A = \{1, 2, 3, A\}$, in which the set is one of its own elements. → This is known as an extraordinary set. In which a set is a set that is an element of itself, & ordinary set is a set i.e not an element of itself!
Remember: "Being an element of" of and "being a subset of" ⟷ are not the same.
Every set is a subset of itself, but only a "strange" set = like $A = \{1, 2, 3, A\}$ is an element of itself!

05.23.2019
The paradox is → by letting S be the collection of all ordinary sets & nothing else. "I" claim that S is a set. S is a collection of objects, i.e. an object being any thing concievable. So S is a set. Every Set is either ordinary or extra-ordinary, So this means S must be ordinary or extraordinary. S is going to be neither. Recall → S: is a set, it contains every ordinary set, and it contains nothing else. If S itself were an ordinary set, it would be an element of itself & thus extra-ordinary! The contrary is also true: If S were an extraordinary set, it would not be an element of itself, so S cannot be either ordinary or extraordinary!!! But being that S is a set it must be either one or the other.
Mathematicians got around this "law of excluded Middle," i.e a rule of logic that says "any meaningful statement is either true or false (binary) If there were "a middle ground," then the argument that S is neither extraordinary or ordinary would lose its effect. Rejecting the law of excluding Middle seems fundamental to reason!

There is a minority who do not reject the "Law of Excluded Middle" aka the intuitionists. 05.24.2019

1910 - Russell resolved his own paradox with the theory of types - says we can avoid the paradox if we tack an "extra" clause onto the definition of "set" excluding collections that are elements of themselves.

Def 1:

Under this new definition → there are no such things as "extraordinary", all sets are ordinary. Meaning that S becomes the collection of all sets. Furthermore, S itself is no longer a set, for if S were a set it would be an element of itself. S in some sense → would be "too big" to be a set.

For collections like S we can invent a new name, e.g. "class" to indicate that they are qualitatively different from sets. Hence, the Paradox is gone! Leaving some chaff → ∴ S is not an extraordinary set, because extraordinary sets do not exist; neither is S an ordinary set because S is not a set at all! S is a class

"The subtlety of Russell's paradox can be seen in the fact that mathematicians all over the world had worked with the theory of sets 20 years before the discovery of this loophole 05.28.2019

Standards of Rigor are now expected. Statements are now scrutinized more than ever.

Graphs → Definition 5: A graph is an object consisting of two sets called its vertex set & its edge set. The vertex set is a finite non-empty set.

The edge set may be empty, but otherwise its elements are two-element subsets of the vertex set.

e.g. The set $\{P, Q, R, S, T, U\}$

& $\{\{P, Q\}, \{P, R\}, \{Q, R\}, \{S, U\}\}$

constitute a graph → call this graph "G"

2. $\{1, 2, 3, 4, 5\}$ and $\emptyset$ constitute a 2nd graph, "H."

3. "J" is a graph having vertex set {1, 2, 3, 4, 5} & edge set {{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {4, 5}} 05.28.2019

Definition 6: The elements of the vertex set of a graph are called vertices [vertex] and the elements of the edge set are called edges. Denote the number of vertices by 'v' and the number of edges by 'e.'

e.g. > The vertices of the graph G are P, Q, R, S, T, U.
The edges of the graph G are {P, Q}, {P, R} {Q, R}, & {S, U}. G has v = 6 and e = 4

The vertices of H are 1, 2, 3, 4, 5. H has no edges. Where H, v = 5 and e = 0

J has v = 5 and e = 10

Definition 7. If {X, Y} is an edge of a graph, we say that {X, Y} joins or connects the vertices X and Y, and that X and Y are adjacent to one another 05.28.2019
Meaning that the edge {X, Y} is an incident to each of X & Y, and each of X and Y is incident to {X, Y}.

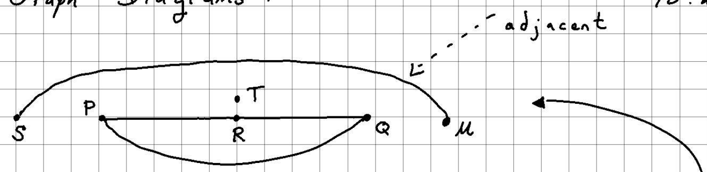
Two edges incident to the same vertex are called adjacent edges. A vertex incident to no edges at all is isolated

e.g. In the Graph G, {P, Q} joins P and Q; P & Q are adjacent; P and S are not adjacent; {P, Q} is incident to Q;

{P, Q} is not incident to R; P is incident to {P, R}; P is not incident to {Q, R}; {P, R} and {Q, R} are adjacent

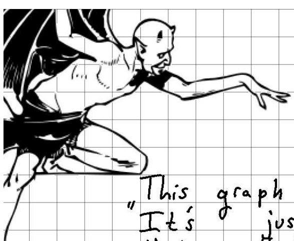
{P, R} and {S, U} are not adjacent; & T is isolated.

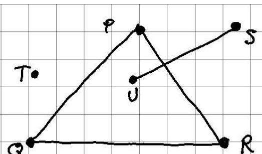
Graph Diagrams : ϕ5. 28. 2ϕ19



Customary to draw diagrams of graphs.
e.g this is the Graph Diagram of the Graph G
The vertices have been represented by "heavy" dots, and vertex adjacency has been represented by connecting the corresponding dots.

Diagrams can be help grasp the graph, but do not rely too much on them.
"Diagrams have properties over & above those consequent to their being graph representations.
The diagram should not have unit value, angles, positions relative to each other (i.e dots)
They only have the abstract attributes of the definition of the Graph → which in this case is
"G is the graph having vertex set {P, Q, R, S, T, u} & edge set of {{P,Q}, {P, R}, {Q,R}, {S, U}}





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"This graph representation is the same as the previous
"It's just another picture of Graph G"
Notice that it has the same vertices & the same connections. ~ Graph diagrams may look different but could be essentially the same.

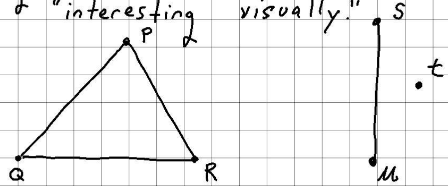
Definition 8 : Say that two graphs are equal if they have equal vertex sets and equal edge sets.
We will see that two graphs are equal if they represent equal vertex sets and equal edge sets.

The above definition is NOT circular → as it defines "equality" in terms of equality, it may seem that it is, (circular)
→ In definition 4 → using to define the new concepts of equality of graphs and equality of graph diagrams

05. 29. 2019

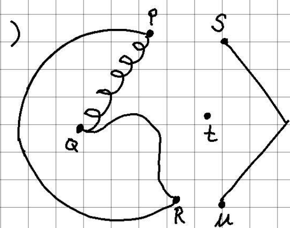
e.g. The graph G having vertex set $\{P, Q, R, S, T, U\}$ and edge set $\{\{P, Q\}, \{P, R\}, \{Q\ R\}, \{S, U\}\}$ is equal to the graph L having vertex set ↳ $\{T, P, R, Q, U, S\}$ & edge set $\{Q, R\}$, $\{R, P\}$, $\{U, S\}$, $\{P, Q\}\}$; & G is equal to any other graph differing from G only by a permutation of set elements

2. The 2nd part of the definition isn't surprising either * at the intellectual level, it is & "interesting visually."



a.1

(a.2)



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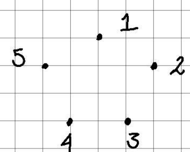
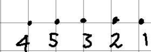
a.3



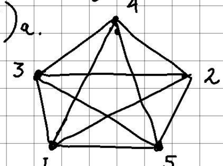
a.4



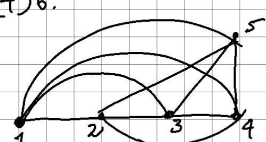
The two diagrams in a.1, a.2 are equal so are the figures before a.1, & a.2 → are all equal i.e they all represent G; i.e they have vertices → P, Q, R, S, T, U & no others & edges, $\{P, Q\}$, $\{P, R\}$, $\{Q, R\}$, $\{S, U\}$ & no others — apparent differences are irrelevant.

(3.) Graphs a.3, a.4 are equal. They represent the graph H mentioned in the last section

(4)a.



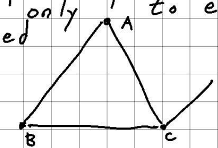
(4)b.



4. The figures are 4a. & 4b. are equal. They represent J from the previous example. 06.14.2019

To verify that the two graphs are equal {diagrams}, you don't need to write down their vertex & edge sets.
All you need to do → is check that they have the same number of vertices, i.e the vertices have the same names → and that the graphs have the same adjacency structure

e.g → without writing anything down you can tell {verify} that the diagrams in a.1, & a.2 are equal, by → "both graphs have six vertices labeled → P, Q, R, S, T & U. In both graphs P is adjacent to Q & R & nothing else., Q is adjacent to P & R & to nothing else, R is adjacent to P & Q & nothing else, S & U are adjacent only to each other. An T is isolated

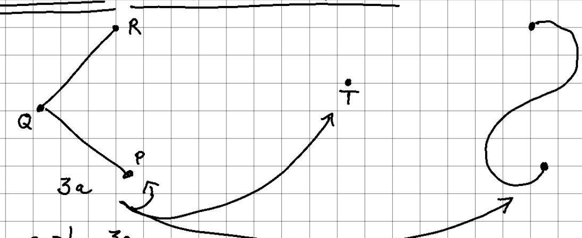


← this is not a graph

The reason as to why the object is not a graph. Is because an edge by definition is a set of two vertices, hence cannot exist without a vertex at each end. A graph can have a vertex without incident edges. 06.14.2019

→ i.e {an isolated vertex}; note → now that edges, on the other hand, must always have two incident vertices.

3.)



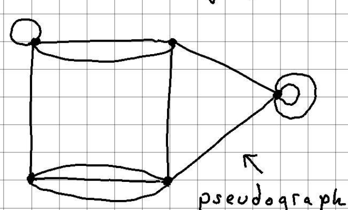
The graph 3a is not equal to the graph at (a.1), notice there is no edge connecting P to R
Vertex adjacency is not like electric current: It is not transmitted through the intermediary vertex Q.

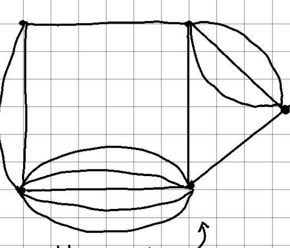
The graphs in 4a, 4b, ∅6. 14. 2∅19
have a number of edge-crossings.
An edge-crossing is not a vertex,
but are just incidental features of a diagram
they do not exist in 3-D renderings of these
diagrams. Heavy dots should be used for vertices,
as to avoid any of this 2-D ambiguities

5. Sharp corners on edges are not vertices either
in figure a.2 → the vertices P, Q, R, S,
T, & U, → none of the sharp corners
on edges are vertices.

6. The definition of graph precludes "loops"
(vertices joined to themselves)
& "skeins" ↳ several edges joining the same pair of
vertices. This is because a loop would translate
abstractly as an "edge" of the form {A, A}
which cannot exist →, as {A, A} is not
a set ~ definition #5 ; & a skein would imply
the multiple inclusion of an element {A, B}
in the edge collection, which would prevent
the collection of edges from being a set as →

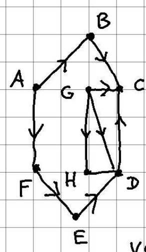
→ required. By allowing skeins but not loops ∅6.14.2∅19
you will end up with a multigraph. If you
allow both loops & skeins you will have what is
called a "pseudograph"





7. The edges of a graph are unidirected, i.e, any
impression that the notation {A, B}
may give of an edge "going from A toward B"
is unintentional. ↳ {A, B} = {B, A}. Any
"graph like" "thing" in which the edges have direction
is called a "diagraph."

Common graphs → Some graphs have acquired more
or less standard names →



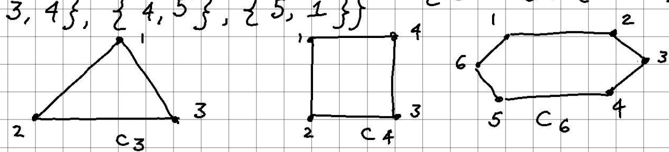
digraph

→ contin... ∅6. 14. 2∅19

less standard names because they turn up so frequently

Def 9) If v is an integer greater than or equal to 3, the cyclic graph on v vertices, denoted "C_v" is the graph having vertex set ↳ $\{1, 2, 3 \ldots v,\}$ & an edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{(v-1), v\}, \{v, 1\}\}$ • This is a cyclic graph, make sure to recognize them • In this case there is one cyclic graph for each integer, beginning with 3.

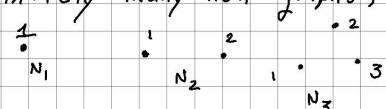
To find out what particular member of the family looks like, e.g. at C_5, you replace all occurrences of "v" with 5, in the definition.
↳ $\{1, 2, 3, 4, 5\}$ edge at $\{\{1, 2\}, \{2, 3,\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$



All are cyclic

def 1∅ : If v is a positive integer, ∅6. 14. 2∅19
the null graph on v vertices, denoted "N_v", is the graph having vertex set → $\{1, 2, 3, \ldots, v\}$ & no edges

There are infinitely many null graphs, one for each integer. ↳

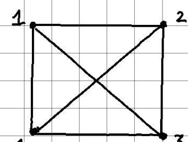


The previous graph H has been changed to N_5.
As defined in Defs → All graphs have at least one vertex → a vertex set be nonempty, as per defs.
Thus N_1 is the smallest possible graph & the only one (essentially) for which $v = 1$

Here are graphs that are "opposite" to the null graphs.

Def 11. If v is a positive integer, the complete graph on v vertices, denoted "K_v", is the graph having vertex set → $\{1, 2, 3, \ldots, v\}$ & all possible edges. ∅6. 14. 2∅19

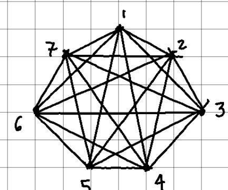
K_2



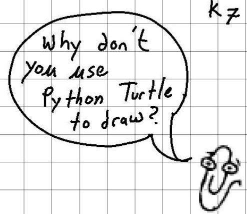
K_4

These are complete graphs. The graph J from before, here, has been renamed to $\underline{K}_5$. K_1 & N_1 are equal.

As are K_3 & C_3



K_7



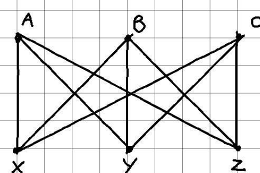
∘ Definition 12 ∘

The utility graph, denoted "UG," is the graph having vertex set ↳

↳ $\{A, B, C, X, Y, Z\}$ & edge set $\{\{A, X\}, \{\{A, Y\}, \{A, Z\},$
$\{B, X\}, \{B, Y\}, \{B, Z\}$ →

$\{C, X\}, \{C, Y\}, \{C, Z\}$ ∅6. 14. 2∅19

would look like this



There is only one "utility" graph!

i.e unlike "cyclic graphs" → "null graphs", or "complete graphs"

"The utility graph gets its name from a puzzle in which three houses → named (A, B, & C) & three utility companies (X, Y, Z) are represented by dots on paper. The task is to connect each house to each utility without crossing any lines.

It cannot be done!

Create a Python turtle graph with 16 edges → see if you can make one with 1∅∅∅

K_{1000}

Develop a formula to count edges 06.19.2019
& see how theorems are derived from graph theory.

1. Draw the first complete "few" graphs.
2. Next count their edges.

v	e
1	0
2	1
3	3
4	6
5	10
6	15
7	21
8	?
.	

e = edges → notice any patterns? notice that the value of e's go up by consecutive integers. From 0 to 1 → the jump is 1 from 1 to 3 it is 2, from 3 to 6 it's 3, from 6 → 10 it's 4 etc each e is the sum of the two numbers in the row above it. The 1 in the e column is 1 + 0, the sum of the row above; 3 is 2+1, the 6 is 3 + 3, 10 is 4 + 6, etc

Conjecture →

For any v,
(Number of edges of K_{v+1}) = (number of vertices of K_v) + (number of edges K_v)

```
# Insert Turtle Graphic here, for K = 16, i.e. 16 edges
```

The number of vertices of K_v is v, 06.19.2019
the conjecture may be rewritten as follows →

Conjecture: For any v,

(number edges K_{v+1}) = v + (no. edges K_v)

For testing the conjecture → try K_8 → the next row on the table

According to this conjecture, (number of edges K_8) = 7 + (number of edges K_7) = 7 + 21 = 28

Now draw K_8 & count edges.

K_8 does have 28 edges.

The conjecture may "look" good → but all it says about K_{1000} is that (number of edges of (K_{1000}) = 999 + (no. of edges K_{999})

edges ↳ don't know the number of edges of K_{999} either → (K_{999}) = 998 + (no edges K_{998})

You could keep going down to the beginning, i.e by counting → 06.28.2019

what is needed is a pattern that doesn't involve any previous e's

Look for such a pattern →

Each e is half the product of its v & the previous v. e.g.

$$1 = (1/2)(2 \cdot 1), \quad 3 = (1/2)(3 \cdot 2)$$

$$6 = (1/2)(4 \cdot 3), \quad 10 = (1/2)(5 \cdot 4)$$

giving a new conjecture : For any v

$$(\text{no. Edges } K_{v+1}) = (1/2)(v+1)(v)$$

check the squares with the first conjecture. The first conjecture said that

$$(\text{no. edges } K_{v+1}) = v + (\text{no. edges } K_v)$$

by the second conjecture $(\text{no. edges } K_{v+1}) = (1/2)(v+1)(v)$ & no. edges (K_v) →

→ $= (1/2)(v) \cdot (v-1)$; 06.28.2019

substitute these values into the conjecture

$$(1/2)(v+1)(v) = v + (1/2)(v)(v-1)$$

$$(1/2)(v+1)(v) = (1/2)(2v) + (1/2)(v)(v-1)$$

$$(1/2)(v+1)(v) = (1/2)[2v + v(v-1)]$$

$$(1/2)(v+1)(v) = (1/2)v[2 + v - 1]$$

$$(1/2)(v+1)(v) = (1/2)v(v+1)$$

The last eq is true → so conclude → that the two conjectures are both true & false → but are they compatible? → we don't know for a fact that either one is actually true

°Notice → that the 2nd conjecture → if true, is the sort of thing you would want.

With the 2nd conjecture → there's no problem to finding the number of edges → of

→ K_{1000} : so number of edges → 06.28.2019

$$K_{1000} = \left(\tfrac{1}{2}\right)(1000)(999) = \left(\tfrac{1}{2}\right)(999{,}000)$$

↳ = 499,500 → So the conjecture would have to be true for any complete graph. You can see that the number of edges in a graph is $\left(\frac{1}{2}\right)$ of the number of edge-ends

- 2nd Conjecture → The number of edges in a complete graph K_v is given by the formula

$$e = \left(\tfrac{1}{2}\right) v(v-1)$$

- Proof: We will count the number of edge-ends in K_v. K_v has v vertices.

Each vertex is joined to the other $(v-1)$ vertices, so that at each vertex there are

↳ $(v-1)$ edge-ends. Therefore the total number of edge-ends in K_v is $v(v-1)$. → This means that every edge has two ends,

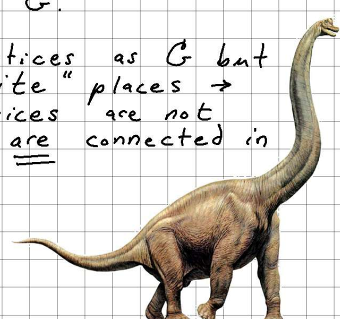
so the number of edges in K_v is $\left(\frac{1}{2}\right)$ the number of edge-ends, or 07.01.2019

$$\left(\tfrac{1}{2}\right) v(v-1)$$

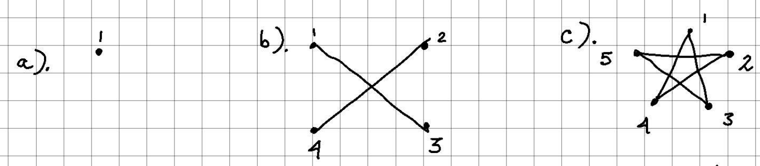
Complements & subgraphs

Definition 13. I G is a graph then the complement of G, denoted a "$\bar{G}$", is a graph having the same vertex set; the edge set of $\bar{G}$ consists of all two-element subsets of the vertex set which have not already been included in the edge set of G.

Therefore $\bar{G}$ has the same vertices as G but the edge are in the "opposite" places → This means if two vertices are not connected in G, then they are connected in $\bar{G}$ and vice versa ↔



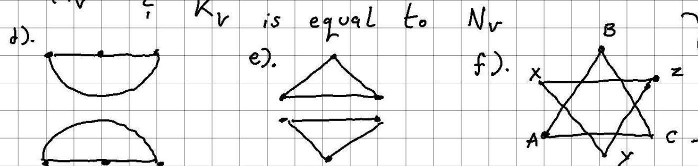
e.g. ⊳ 07. 01. 2019



cyclic graphs

The complements of these cyclic graphs

Null graphs & complete graphs are complementary
i.e ⊳ for every positive integer v $\bar{N}_v$ is equal to K_v & $\bar{K}_v$ is equal to N_v

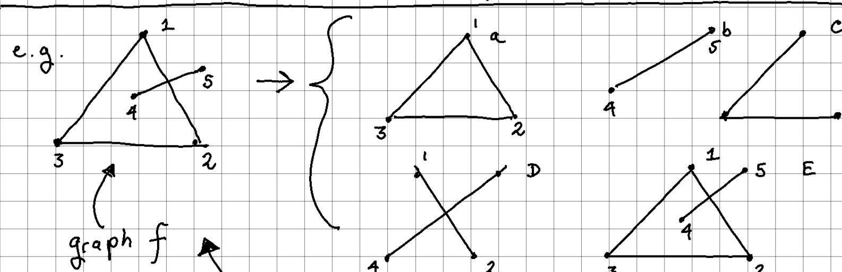


$\overline{UG}$

~ so that for any graph G, the complement $\bar{G}$ is also a graph → so we may take its complement $\bar{\bar{G}}$; would result in going back to the original G graph 07. 01. 2019

• Def 14 → A graph H is a subgraph of a graph G if the vertex set of H is a subset of the vertex set of G & the edge set of H is a subset of the edge set of G.

e.g.



graph f

These are all the subset of f

53 png page left blank, where 52 == 53

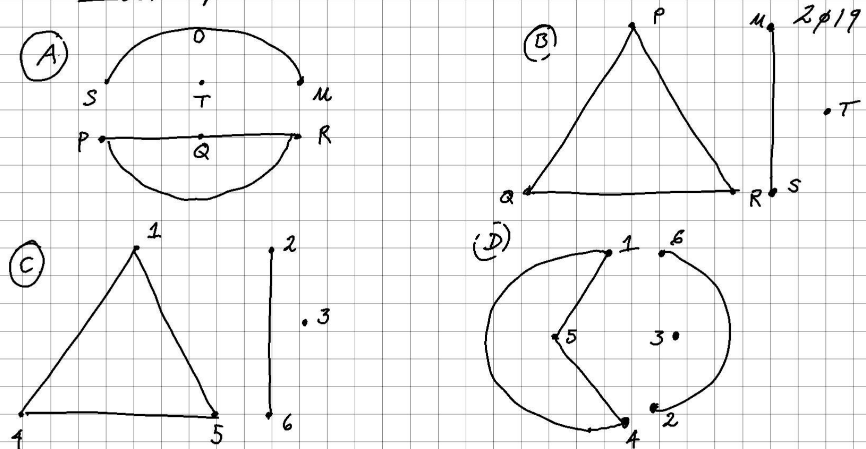
observe that all first 5 graphs are all subgraphs of the 6th. 12.Ø4. 2Ø19

> A subgraph is a graph contained in another graph. Note that since every set is a subset of itself, every graph is a subgraph of itself.

* Intuitively a subgraph is the result of "attacking" a graph with an eraser, i.e with 2 qualifications:

→ you wouldn't need to erase anything, since every graph is a subgraph of itself; you may erase the edges as much as you wish do not erase a vertex without also erasing all edges incedent to it → or you will no longer have a graph

Isomorphism → 12.Ø4. 2Ø19



A & B are equal || however → $B \neq C$

C & D are equal || $C \neq B$

Which does not seem right ? 12.04.2019

The only difference between b & c lies in how the vertices have been labeled.

As you have defined the term "equal"

according to def (8) →

We will say that two graphs are equal if they have equal vertex sets & equal edge sets.
And we will say that two graph diagrams are equal if they represent equal vertex sets & equal edge sets.

→ observe b & c are not equal, for their vertex sets

{ P, Q, R, S, T, U } &

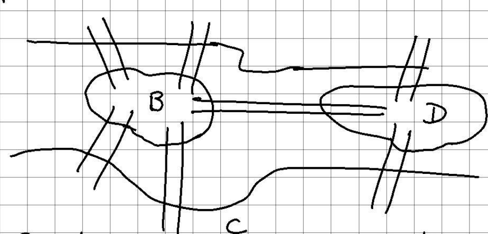
{ 1, 2, 3, 4, 5, 6 } are unequal

sets & their edge sets are unequal also

1.1 Königsberg Bridge Problem

05. 01. 2019

The river Pregel flowed through the city of K in Prussia.
There were 7 bridges over the river P connecting the two islands (B & D) & two opposite banks (A & C)



Starting with any 1 of the 4 place A, B, C, D is it possible to have a walk which passes through each of the 7 bridges once & only once, & return to where you started?

```
#/scr/graph.py
graph = { "a" : ["c"],
          "b" : ["c" "d"],
          "c" : ["a", "b", "d"],
          "d" : ["c"]
          }
# letters as nodes
def generate_edges(graph):
```

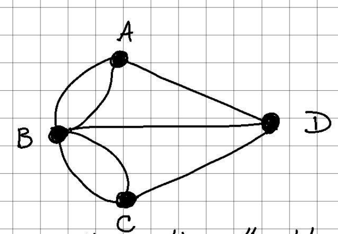
```python
        edges = []
        for node in graph:
            for neighbor in graph[node]:
                edges.append((node, neighbor))

        return edges

    print(generate_edges(graph))
```

Euler - noticed that this 05. 01. 2019
problem was very much different in nature
than the traditional geometry problem.
Euler → 1736, deduced the impossibility
of having such a walk on this bridge.

Euler observed that the Königsberg bridge problem
had nothing to do with traditional geometry -
where the measurements of lengths & angles,
relative locations of vertices count.
The Key ingredients are whether the islands
or banks are connected by a bridge, & by
how many bridges?

Represent the islands or banks by 'dots'
; one for each island or bank, & two dots
are joined by 'K' lines { don't have to be
straight } where $K \geq 0$ when & only when
the respective islands or banks represented
by the dots are connected by $\underline{K}$ bridges
the Königsberg bridge problem is
represented by →

05. 05. 2019



This is called the "multigraph." A multigraph
is a diagram consisting of 'dots' & lines,
where <u>each</u> line joins some '<u>pair</u> of
dots
↳ two dots may be joined by <u>no</u> lines
or any number of lines.

Formally → the 'dot' is called vertex (vertices)
the line is called the "edge."

There are 4 vertices & 7 edges, where each
edge joins some pair of vertices.

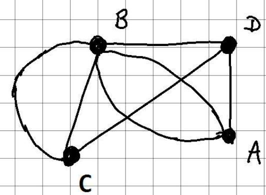
vertices A, & C are <u>not</u> joined by two edges

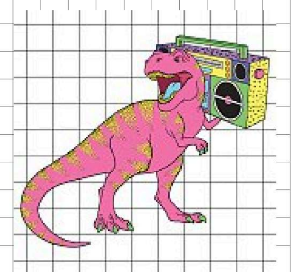
05. 05. 2019

The size & relative locs of the vertices (dots), & the lengths of the edges are immaterial.

Only the 'linking relations' among the vertices & number of edges that join any two vertices count.

Thus the Königsberg bridge problem can be represented like so





05. 05. 2019

There a 6 people: A, B, C, D, E & F at a party

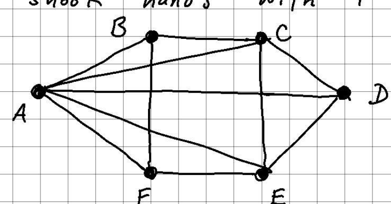
Suppose A shook hands with B, C, D, E & F;

B, in addition, shook hands with C & F

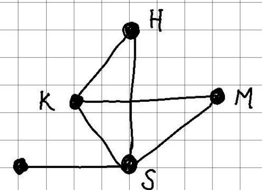
C, → shook hands with D & E

D, ↓ ↓ ↓ E

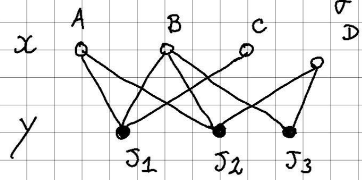
E, shook hands with F



05. 05. 2019



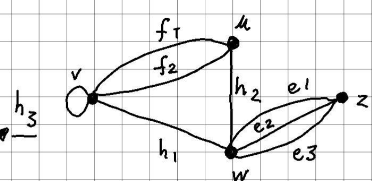
Availability of flights operated by an airline between a number of cities.
The vertices represent cities, two vertices are joined by an edge if there is a flight path available between the two correspond-ing cities



Note every 2 vertices are joined by at most one edge, i.e either no edges or exactly one edge

called simple graphs

05. 05. 2019



- 4 vertices, u, v, w & z
- 8 edges : f_1 & f_2 joining u & v; e_1, e_2 & e_3 joining w & z; h_1 joining v & w; h_2 joining u & w; h_3 joing v to itself. Two or more edges joing the same pair of vertices are parallel edges
  f_1 & f_2 are parallel edges;
  $e1$, e_2 e_3 are parallel edges
  h_3 is a loop : i.e. Any edge joining itself.