Exercice 3

$\exists X \in \mathbb{R}^3$ tel que $X \neq 0$ et $AX = X$ (-1 vp de A).

$\chi_A = \chi_{A^T} = \chi_{A^{-1}}$ et $\chi_A(\lambda) = \chi_{A^{-1}}\left(\frac{1}{\lambda}\right)(-1)^3 \lambda^3 \det(A)$

$= -\chi_{A^{-1}}\left(\frac{1}{\lambda}\right) \lambda^3 \det(A)$

$\Rightarrow \begin{cases} \det(A) = \det(A^{-1}) \\ \operatorname{tr}(A) = \operatorname{tr}(A^{-1}) \end{cases}$

$\lambda^3 - \operatorname{tr}(A)\lambda^2 + b\lambda - \det(A) = \left(\left(\frac{1}{\lambda}\right)^3 - \frac{\operatorname{tr}(A^{-1})}{\lambda^2} + \frac{b}{\lambda} - \det(A^{-1})\right)\lambda^3 \det A$

$\lambda^3 - \operatorname{tr}(A)\lambda^2 + b\lambda - \det(A) = -\det(A) + \operatorname{tr}(A^{-1})\det(A)\lambda - b\lambda^2\det(A) + \lambda^3$

En identifiant, $b = \operatorname{tr}(A^{-1})\det(A) = \operatorname{tr}(A)\det(A)$

$\det(A) = \det(A^{-1}) = \frac{1}{\det(A)} \Rightarrow \det(A)^2 = 1 \Rightarrow \boxed{\det(A) = 1}$

car $\det(A) \geq 0$

donc $\chi_A = \lambda^3 - \operatorname{tr}(A)\lambda^2 + \operatorname{tr}(A) - 1$

$\chi_A(1) = 1^3 - \operatorname{tr}(A) + \operatorname{tr}(A) - 1 = 0$

$\boxed{1 \text{ vp de } A}$

Exercice 4.5

On considère $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^k - x - 1 \end{cases}$

f dérivable sur $\mathbb{R}$ et $\forall x \in \mathbb{R}$, $f'(x) = kx^{k-1} - 1$

$f'(x) \geq 0 \iff x^{k-1} \geq \frac{1}{k}$

Or $k-1$ est pair donc :

$f'(x) \geq 0 \iff x \geq \left(\frac{1}{k}\right)^{\frac{1}{k-1}}$ ou $x \leq -\left(\frac{1}{k}\right)^{\frac{1}{k-1}}$

x	$-\infty$		$-\left(\frac{1}{k}\right)^{\frac{1}{k-1}}$		$\left(\frac{1}{k}\right)^{\frac{1}{k-1}}$		$+\infty$
Signe de f'		$+$	0	$-$	0	$+$	
Var de f	$-\infty$	$\nearrow$		$\searrow$		$\nearrow$	$+\infty$

$$f\left(-\left(\frac{1}{k}\right)^{\frac{1}{k-1}}\right) = (-1)^k \left(\frac{1}{k}\right)^{\frac{k}{k-1}} + \left(\frac{1}{k}\right)^{\frac{1}{k-1}} - 1$$

$$= -\left(\frac{1}{k}\right)^{\frac{k}{k-1}} + \underbrace{\underbrace{\left(\frac{1}{k}\right)^{\frac{1}{k-1}}}_{\leq 1} - 1}_{\leq 0} \quad \boxed{<0}$$

et donc par le thm de la bijection, l'équation $f(x) = 0$ admet une unique solution qu'on note α et $\alpha > 0$ car $\alpha \geq \left(\frac{1}{k}\right)^{\frac{1}{k-1}}$

Ainsi $Sp_{\mathbb{R}}(A) \subset \{\alpha\}$

Dans $\mathbb{C}$: $\chi_A = (X - \alpha) \prod_{i=1}^{p} (X - \omega_i)^{m_i} (X - \overline{\omega_i})^{m_i}$

et on $\boxed{\det(A)} = \prod_{i=1}^{p} \omega_i^{m_i} \, \overline{\omega_i}^{m_i} \times \alpha$

$$= \prod_{i=1}^{p} |\omega_i|^{2m_i} \alpha \quad \boxed{>0}$$