The Volatility-Confined LPPL Model: A Consistent Model of 'Explosive' Financial Bubbles With Mean-Reverting Residuals

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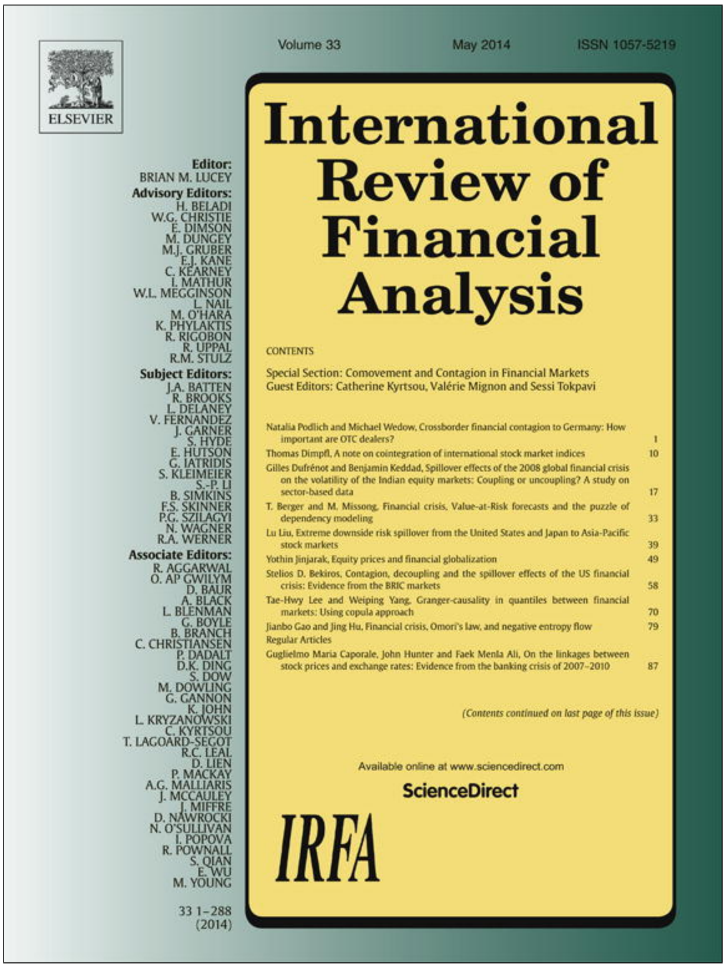
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CONTENTS
Special Section: Comovement and Contagion in Financial Markets
Guest Editors: Catherine Kyrtsou, Valérie Mignon and Sessi Tokpavi
Natalia Podlich and Michael Wedow, Crossborder financial contagion to Germany: How important are OTC dealers? 1
Thomas Dimpfl, A note on cointegration of international stock market indices 10
Gilles Dufrénot and Benjamin Keddad, Spillover effects of the 2008 global financial crisis on the volatility of the Indian equity markets: Coupling or uncoupling? A study on sector-based data 17
T. Berger and M. Missong, Financial crisis, Value-at-Risk forecasts and the puzzle of dependency modeling 33
Lu Liu, Extreme downside risk spillover from the United States and Japan to Asia-Pacific stock markets 39
Yothin Jinjarak, Equity prices and financial globalization 49
Stelios D. Bekiros, Contagion, decoupling and the spillover effects of the US financial crisis: Evidence from the BRIC markets 58
Tae-Hwy Lee and Weiping Yang, Granger-causality in quantiles between financial markets: Using copula approach 70
Jianbo Gao and Jing Hu, Financial crisis, Omori's law, and negative entropy flow 79
Regular Articles
Guglielmo Maria Caporale, John Hunter and Faek Menla Ali, On the linkages between stock prices and exchange rates: Evidence from the banking crisis of 2007–2010 87
(Contents continued on last page of this issue)
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International Review of Financial Analysis 33 (2014) 210–225

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The volatility-confined LPPL model: A consistent model of 'explosive' financial bubbles with mean-reverting residuals

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ABSTRACT

Using the concept of the stochastic discount factor with critical behavior, we present a self-consistent model for explosive financial bubbles, which combines a mean-reverting volatility process and a stochastic conditional return which reflects nonlinear positive feedbacks and continuous updates of the investors' beliefs and sentiments. The conditional expected returns exhibit faster-than-exponential acceleration decorated by accelerating oscillations, called "log-periodic power law" (LPPL). Tests on residuals show a remarkable, low rate (0.2%) of false positives when applied to a GARCH benchmark. When tested on the S&P500 US index from Jan. 3, 1950 to Nov. 21, 2008, the model correctly identifies the bubbles ending in Oct. 1987, in Oct. 1997, and in Aug. 1998 and the ITC bubble ending on the first quarter of 2000. Different unit-root tests confirm the high relevance of the model specification. Our model also provides a diagnostic for the duration of bubbles: applied to the period before the Oct. 1987 crash, there is clear evidence that the bubble started at least 4 years earlier. We confirm the validity and universality of the volatility-confined LPPL model on seven other major bubbles that have occurred in the World in the last two decades. Using Bayesian inference, we find a very strong statistical preference for our model compared with a standard benchmark, in contradiction with Chang and Feigenbaum (2006) which used a unit-root model for residuals.

1. Introduction

Given the stalemate in defining and characterizing the anomalous market behaviors called "bubbles", we propose to define them by three characteristics: (i) faster-than-exponential growth of the price (singular "power law" behavior), (ii) accelerated succession of transient increases followed by corrections captured by a so-called "log-periodic" component and (iii) mean-reverting behavior of the residuals developing around the two first components (the "self-consistent" condition). The combination of ingredients (i) and (ii) constitutes the log-periodic power law (LPPL) model. We formalize these concepts within a self-consistent model for explosive financial bubbles, with nonlinear positive feedbacks with mean-reversal residuals, that we refer to as the "volatility-confined LPPL model". The conditional expected returns exhibit faster-than-exponential acceleration decorated by accelerating oscillations. An essential advance of our model compared with previous specifications such as that of Johansen, Sornette, and Ledoit (1999) is to allow for stochastic conditional expectations of returns that describe continuous updates of the investors' beliefs and sentiments.

We develop two complementary economic interpretations of the volatility-confined LPPL model: (i) a rational-expectation (RE) model of rational bubbles with combined Wiener and Ornstein–Uhlenbeck innovations describing the dynamics of rational traders coexisting with noise traders driving the crash hazard rate; and (ii) a behavioral specification of the dynamics of the stochastic discount factor describing the overall combined decisions of both rational and noise traders.

Tests on residuals show a remarkable, low rate (0.2%) of false positives when applied to a GARCH benchmark. When tested on the S&P500 index from Jan. 3, 1950 to Nov. 21, 2008, the model correctly identifies the bubbles ending in Oct. 1987, in Oct. 1997 and in the summer of 1998 and the ITC bubble ending on the first quarter of 2000. Different unit-root tests confirm the high relevance of the model specification. Our model also provides a diagnostic for the duration of bubbles: applied to the period before the Oct. 1987 crash, there is clear evidence that the bubble started at least 4 years earlier. Using

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Bayesian inference, we find a very strong statistical preference for our model compared with a standard benchmark, in contradiction with the result of Chang and Feigenbaum (2006). Our positive result stems from the mean-reverting structure of the residuals of the conditional returns modeling the bubbles, which is shown to be essential in order to obtain a consistent model. Absent in previous specifications, this feature constitutes the main advance of this work, leading to the novel positive results. The same tests performed on seven major bubbles (Hong Kong 1997, ITC 2000 bubble, Oil bubble ending July 2008, the Chinese bubble ending in October 2007 and others) suggest that our proposed volatility-confined LPPL model provides a consistent universal description of financial bubbles, namely a super-exponential acceleration of price decorated with log-periodic oscillations with mean-reverting residuals.

The present work offers an innovative way to break the stalemate in the ex-ante detection of bubbles, which has been much discussed in the literature. For instance, Gurkaynak (2008) summarizes econometric approaches applied to the detection of financial bubbles, stating that the "econometric detection of asset price bubbles cannot be achieved with a satisfactory degree of certainty. For each paper that finds evidence of bubbles, there is another one that fits the data equally well without allowing for a bubble. We are still unable to distinguish bubbles from time-varying or regime-switching fundamentals, while many small sample econometrics problems of bubble tests remain unresolved."

Bubbles are often defined as exponentially explosive prices, which are followed by a sudden collapse. As summarized for instance by Gurkaynak (2008), the problem with this definition is that, any exponentially growing price regime that one would call a bubble, can be also rationalized by a fundamental valuation model. This is related to the problem that the fundamental price is not directly observable, giving no indisputable anchor to understand how observed prices may deviate from fundamental values, which are also growing exponentially due to the power of compounding and of proportional growth. This is reflected in the standard financial models that essentially almost always start by assuming a proportional growth rate, hence an exponential growth of the price. This was exemplified during the last Internet bubble culminating in 2000 by fundamental pricing models, which incorporated real options in the fundamental valuation, basically justifying any price. Mauboussin and Hiler (1999) were among the most vocal proponents of the proposition offered close to the peak of the Internet bubble, that better business models, the network effect, first-to-scale advantages, and real options effect could account rationally for the high prices of dot-com and other New Economy companies. These interesting views, expounded in early 1999, were in synchrony with the general positive sentiments of the bull market of 1999 and preceding years. They participated in the general optimistic view and added to the strength of the herd. Later, after the collapse of the bubble, these explanations seemed less attractive.

Our model addresses in an innovative way this problem of defining and identifying bubbles. It extends in a novel direction a class of processes that have been proposed to incorporate the positive feedback mechanisms that can push prices upward faster-than-exponentially. This faster-than-exponential characteristic is one of the main diagnostics that we consider for a bubble. Many financial economists recognize that positive feedbacks and in particular herding is a key factor for the growth of bubbles. Herding can result from a variety of mechanisms, such as anticipation by rational investors of noise trader strategies (Long, Shleifer, Summers, & Waldmann, 1990), agency costs and monetary incentives given to competing fund managers (Dass, Massa, & Patgiri, 2008) sometimes leading to extreme Ponzi schemes (Dimitriadi, 2004), rational imitation in the presence of uncertainty (Roehner & Sornette, 2000), and social imitation. The relevance of social imitation or "word-of-mouth" effects has a long history (see for instance Hong, Kubik, & Stein, 2005; Shiller, 2000, for recent evidence). Recently, by analyzing controlled laboratory experiments, the existence of positive feedback mechanisms in the price formation process in which higher prices drive upward future returns was unambiguously identified and shown to lead to transient super-exponential bubbles in which the growth rate grows itself[1] (Hüsler et al., 2013). Our approach is to build on previous specifications that describe such faster-than-exponential growths of price (coined hereafter "super-exponential") (Lin & Sornette, 2013; Sornette & Andersen, 2002; Sornette, Takayasu, & Zhou, 2003).

The Johansen–Ledoit–Sornette (JLS) model (Johansen, Ledoit, & Sornette, 2000; Johansen et al., 1999) constitutes a first attempt to formulate these ingredients into a traditional asset pricing model. Starting from the rational expectation model of bubbles and crashes developed by Blanchard (1979) and by Blanchard and Watson (1982), the JLS model considers the critical properties inherent in the self-organization of complex systems. In the JLS model, the financial market is composed of two types of investors: perfectly rational investors who have rational expectations and irrational traders who are prone to exhibit herding behavior. The dynamics of the price is described by the usual geometric Brownian motion plus a jump process controlled by its crash hazard rate. The noise traders drive the crash hazard rate according to their collective herding behavior, leading its critical behavior. Due to the no-arbitrage condition, this is translated into a price dynamics exhibiting super-exponential acceleration, with possible additional so-called "log-periodic" oscillations associated with a hierarchical organization and dynamics of noise traders. Using the stochastic discount factor (SDF), Sornette and Zhou (2006) extended the JLS model to include inter-temporal parameters and fundamental economic factors.

In the Johansen et al. (1999, 2000) model, the logarithmic return is drawn from a normal distribution with a time-varying drift,

$$r_i = \ln p_{t_{i+1}} - \ln p_{t_i} \sim N\left(\Delta H_{t_{i+1},t_i}, \sigma^2 (t_{i+1} - t_i)\right), \quad \Delta H_{t_{i+1},t_i} = H_{t_{i+1}} - H_{t_i}, \tag{1}$$

where

$$H_{t_i} = A - B(t_c - t_i)^{\beta} \left[1 + \frac{C}{\sqrt{1 + \left(\frac{\omega}{\beta}\right)^2}} \cos(\omega \ln(t_c - t_i)) + \phi) \right]. \tag{2}$$

This so-called log-periodic power law (LPPL) dynamics given by Eq. (2) has been previously proposed in different forms in various papers (see for instance Drozdz, Grummer, Ruf, & Speth, 2003; Feigenbaum, 2001; Feigenbaum & Freund, 1996; Johansen & Sornette, 1999, 2001; Sornette, 2004b; Sornette, Johansen, & Bouchaud, 1996; Zhou & Sornette, 2003a). The power law $A - B(t_c - t_i)^{\beta}$ expresses the super-exponential acceleration of prices due to positive feedback mechanisms, alluded to above. Indeed, for $B > 0$ and $0 < \beta < 1$, the rate of change of H_{t_i} diverges as $t_i \to t_c^{-}$. The term proportional to $\cos(\omega \ln(t_c - t_i) + \phi)$ describes a correction to this super-exponential behavior, which has the symmetry of discrete scale invariance (DSI) (Sornette, 1998). This formulation (2) results from analogies with critical phase transitions (or bifurcations) occurring in complex adaptive systems with many interacting agents. The key insight is that spontaneous patterns of organization between investors emerge from repetitive interactions at the micro-level, possibly catalyzed by top-down feedbacks provided for instance by the media and macro-economic readings, which are translated into observable bubble regimes and crashes. A common mathematical signature of such critical behavior is found in the power law singularities that accompany the faster-than-exponential growth. The additional acceleration oscillations may result from the existence of a discrete hierarchy of the organization of traders (Sornette & Johansen, 1998), or from the

[1] Recall that an exponentially growing price is such that its average return (or growth rate) is constant.

interplay between the inertia of transforming information into decision together with nonlinear momentum and price-reversal trading styles (Ide & Sornette, 2002).

Previous tests of the LPPL model (1) with Eq. (2) and its variants belong to the following three main types:

1. Non-parametric tests of the super-exponential behavior and especially of the log-periodic oscillatory structure applied to residuals of prices time series (Zhou & Sornette, 2002, 2003a,b);
2. Nonlinear least-square fits of price and log-price time series (Jiang et al., 2010; Johansen & Sornette, 2001; Sornette & Johansen, 2001; Sornette, Woodard, & Zhou, 2009; Yan, Woodard, & Sornette, 2012; Zhou & Sornette, 2008);
3. Bayesian methods applied to the time series of returns (Chang & Feigenbaum, 2006).

Each type has limitations.

- Non-parametric approaches to the LPPL models have focused essentially on testing the statistical significance of the log-periodic component of price residuals in bubble regimes ending with crashes. In themselves, they do not provide complete tests of the LPPL model (1) with Eq. (2) and its variants.
- Calibrating directly price or log-price time series may produce spurious high measures of goodness of fits (Granger & Newbold, 1974; Phillips, 1986). As a consequence of their non-stationarity, the goodness of fit may not reflect the properties of the underlying data generating process. Indeed, prices or log-prices are to a good approximation generated by non-stationary unit-root processes, obtained from the integration of approximately stationary returns. Such integration mechanically reddens the spectrum, damping the high-frequency component and augmenting the low-frequency component of the time series, which may lead to the illusion that the generating process is deterministic.
- This problem has led Feigenbaum (2001) and Chang and Feigenbaum (2006) to propose tests of the LPPL model applied to the return time series. Indeed, the LPPL model (1) with Eq. (2) also predicts a LPPL structure for the returns. The difficulty with this approach is that, until now, direct filters of the LPPL patterns from daily returns have been unable to detect a signal that is predicted to be one to two-orders-of magnitude smaller than the background noise (Feigenbaum (2001); see however Sornette and Johansen (2001) for a more positive reinterpretation of Feigenbaum's results). The standard financial econometric response to this problem is to work with monthly or quarterly time scales, so that the volatility is reduced in relative value compared to the drift, approximately by the square root of the number of days in a month or in a quarter. Unfortunately, this is hardly applicable to the problem of detecting and calibrating financial bubbles since the signal we are looking for is transient, by definition of a bubble. Therefore, the luxury of long time series spanning many months or quarters is not available. If a bubble expands over 4 years, this provides only 48 months and 16 quarters, not sufficient to calibrate econometric models. Chang and Feigenbaum (2006) later made the first attempt to employ a Bayesian method that is better suited for the analysis of complicated time-series models like the JLS model expressed in terms of returns. Through the comparison of marginal likelihoods, they discovered that, if they did not consider crash probabilities, a null hypothesis model without log-periodical structure outperforms the JLS model. And if the JLS model was true, they found that parameter estimates obtained by curve fitting have small posterior probability. Even though the LPPL hypothesis might be correct, they concluded that researchers should abandon the class of models in which the LPPL structure is revealed through the expected return trajectory.

These problems can be fundamentally traced back to the fact that the JLS model describes a deterministic time-varying drift decorated by a non-stationary stochastic random walk component. In accordance with rational expectation, this predetermined deterministic price path is the unbiased expectation of a representative rational agent in the market, while the stochastic component describes the estimation errors as well as genuine stochastic components contributing to the price formation process, such as in random utility and sunspot models. The problem is that the stochastic random walk component is a variance-increasing process, so that the deterministic trajectory strays farther and farther away from the observable price path. This is the reason why direct calibration of prices is inconsistent with the estimation of the unbiased expectation of prices. And, as we shall demonstrate below, this is also the reason for the lack of power of the Bayesian approaches applied to the return time series.

In this context, the innovation of our approach is to modify the JLS model by a new specification of the residuals, that makes the process consistent with direct price calibration, thus addressing the issues raised by Granger and Newbold (1974) and Phillips (1986). In a nutshell, the realized observable price path during bubbles is attributed to a deterministic LPPL component, while the estimation errors by rational investors are modeled by a mean-reversal Ornstein–Uhlenbeck (O–U) process.[2] While keeping the structure of the model based on time-varying expectations of future returns, the daily logarithmic returns are no longer described by a deterministic drift decorated by a Gaussian-distributed white noise. Instead, specifying a mean-reversal noise component, the no-arbitrage condition predicts that the conditional expected returns become stochastic, which represents the on-going reassessment by investors of the future returns.

Section 2 presents the new model, which we call the "volatility-confined LPPL model", from two different perspectives, a first derivation based on rational expectation and an equivalent demonstration using the stochastic discount factor. Section 3 presents a first battery of empirical statistical tests. Applying direct calibrations of the new LPPL specification to prices generated by GARCH(p, q) processes show that the rate of false positives in terms of the diagnostic of bubble regimes when there are none is smaller than 0.2%. Using tests on residuals of the price calibration method applied to shrinking windows converging on the crash of October 1987, we are able to identify a clear bubble regime starting about 4 years before the crash occurred. Section 4 implements the Bayesian analysis, extending the approach of Chang and Feigenbaum (2006) to our LPPL specification with O − U residuals. The results show a very strong statistical significance of the LPPL model versus a standard benchmark, as the marginal likelihood calculated from the data within bubbles prior to the Oct. 1987 crash is about 150 times larger than that of models in which daily returns have no LPPL structure. Section 5 presents the results of the tests of Section 3 to seven other major bubbles (Hong Kong 1997, ITC 2000 bubble, Oil bubble ending July 2008, the Chinese bubble ending in October 2007 and others). These results confirm that our proposed volatility-confined LPPL model provides a consistent universal description of financial bubbles, namely a super-exponential acceleration of price decorated with log-periodic oscillations with mean-reverting residuals. Section 6 concludes.

2. Volatility-confined LPPL model

Our volatility-confined LPPL model can be obtained in two ways: (i) using the traditional economic framework of rational expectational bubbles and (ii) on the basis of the behavioral stochastic discount factor (BSDF). Although both derivations lead to the same specification, they provide different and complementary economic interpretations. In the following two subsections, we present in turn these two derivations.

2.1. The model based on the rational expectation (RE) condition

Let us consider a financial market in which a regime shift occurs, changing from a standard GARCH process into a bubble phase. The

[2] In discrete times, it becomes a stationary AR(1) process.

price dynamics in the bubble regime is assumed to be given by the following process.

$$\frac{dI}{I} = \mu(t)dt + \sigma_Y dY + \sigma_W dW - \kappa dj, \tag{3a}$$

$$dY = -\alpha Y dt + dW. \tag{3b}$$

The symbol I denotes the stock index or price of the asset and W denotes the standard Wiener process. The time-varying drift leading to the price acceleration which is characteristic of a bubble regime is represented by $\mu(t)$ and the jump process j takes the value zero before the crash and one afterwards. The constant κ denotes the percentage price drop during a crash. The stochastic process Y plays an important role in the model. For $0 < \alpha < 1$, Y is an Ornstein–Uhlenbeck process, so that dY and Y are both stationary. As we shall see, this property ensures that the calibration of the LPPL model to the price time series is consistent, which was not the case for the standard JLS model in the absence of Y. Eq. (3b) describes a self-stabilization mechanism occurring in the market that confines the volatility to remain bounded during the bubble gestation all the way until the downward jump (or crash) occurs. For $\alpha = 0$ or in the absence of Y, the model recovers the original form of the price dynamics in the JLS model. The JLS model is therefore nothing but a special case of our model (3a) with (3b). The corresponding version in discrete time of Eq. (3a) with (3b) reads

$$lnI_{t+1} - lnI_t = \mu_t + \sigma_Y (Y_{t+1} - Y_t) + \sigma_W \varepsilon_t - \kappa \Delta j_t, \tag{4a}$$

$$Y_{t+1} = (1-\alpha)Y_t + \varepsilon_t, \tag{4b}$$

where $\varepsilon_t \sim N(0, 1)$.

As is standard, we require that the price process defined by Eq. (4a) should obey the no-arbitrage condition. Hence, according to the fundamental theorem of asset pricing, in this bubble regime there exists a risk neutral probability measure $\mathcal{Q}$ different from the original objective probability measure $\mathcal{P}$. The market as a whole exhibits a RE behavior in the sense of this risk neutral measure, i.e. the current observable price of any asset is equal to the value of its expected discount future price in terms of $\mathcal{Q}$. Mathematically, this reads

$$I_t = e^{-r(t^{'}-t)} \mathbb{E}_t^{\mathcal{Q}}[I_{t^{'}}] \quad \forall t^{'} > t, \tag{5}$$

where the expectation operator $\mathbb{E}_t^{\mathcal{Q}}\left[\ddot{A}\atop n\right]$ represents the conditional expectation under the probability measure $\mathcal{Q}$ at time t. The factor r quantifies the difference between the risk-free interest rate r_f and the dividend growth rate $\delta(r = r_f - \delta)$. The risk neutral probability $\mathcal{Q}$ essentially indicates the density of state-price securities[3] in terms of their prices across all possible states during a bubble, which can be translated as usual in terms of the stochastic discount factor (SDF) Λ. The SDF is a very special portfolio in the market and can be used as the pricing kernel to obtain the current price for arbitrary asset. In details, the product of the SDF with the value process $I(t)$ of any admissible self-financing trading strategy implemented by trading on a financial asset must satisfy

$$\Lambda(t)I(t) = \mathbb{E}_t^{\mathcal{P}}\left[\Lambda\left(t^{'}\right)I\left(t^{'}\right)\right] \quad \forall t^{'} > t, \tag{6}$$

where the expectation operator $E_t^{\mathcal{P}}\left[\ddot{A}\atop n\right]$ represents the expectation conditional on all current disclosed information under the original objective probability measure $\mathcal{P}$.

[3] A state-price security is a contract binding with a state, which promises a unit-valued payoff if and only if the specific state occurs in the future.

We assume a standard form for the dynamics of the stochastic discount factor (SDF):

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \rho_Y dY - \rho_W dW. \tag{7}$$

The terms $\rho_Y dY$ and $\rho_W dW$ allow transforming the objective drift of the return process into its corresponding risk-neutral version, via the no-arbitrage condition (6). The SDF can be interpreted as the excess return over the spot interest that an asset must earn per unit of risk variance associated respectively with the two processes Y and W. Only these two stochastic processes need to be considered in the dynamics of Λ_t since any others that are uncorrelated with Y and W do not contribute to the pricing of the assets considered here.

To be in line with the above assumptions and conditions, the simplest specification for the drift term $\mu(t)$ of the price process (3a) reads (see Appendix A for detailed derivation)

$$\mu(t) = (r + \rho\Sigma) + \kappa h(t) + \alpha(\sigma_Y - \rho_Y)Y_t, \tag{8}$$

where $\rho\Sigma$ is the short symbol for the term $\sum\sum_{i,j=Y,W} \rho_i \sigma_j$, which is the required excess return remunerating all risks at the exception of the crash risk associated with the jump of amplitude κ. The term $h(t)$, defined by $\mathbb{E}_t^{\mathcal{P}}[dj] = h(t)dt$, denotes the crash hazard rate. The dynamics of $h(t)$ plays a very important role here, as it does in the JLS model. The expression (8) includes the risk premium that needs to remunerate rational investors for being exposed to the risk of a crash, whose frequency is determined by the crash hazard rate $h(t)$. Here, as in the JLS model, we assume that the crash hazard rate $h(t)$ is driven by the behavior of "noise traders", who herd into successive phases of euphoria and panics. Assuming a dynamics of local imitations and herding on a hierarchical network of social influences as in the JLS model, this leads to the crash hazard rate following a LPPL (log-periodic power law) process of the type (2).

Compared with the JLS model, the new ingredient Y in Eq. (3b) is directly imported into the expression (8) by the term $\alpha(\sigma_Y - \rho_Y)Y_t$. Hence, the return $\mathbb{E}_t^{\mathcal{P}}[\mu(t')]$ that is anticipated at time t for the time horizon up to t' is a function of the specific stochastic realization Y_t of the O–U process Y, which is known at t. This property captures the possible belief updates of RE investors. According to rational expectation, a RE investor always makes an unbiased conditional estimation of the actual return, which is adjusted continuously in response to the flow of available information, i.e., $\mu(t_1) = E_{t_1}^{\mathcal{P}}\left(\frac{dI_{t_1}}{I_{t_1}}\right) \neq E_{t_2}^{\mathcal{P}}\left(\frac{dI_{t_2}}{I_{t_2}}\right) = \mu(t_2)$, for $t_1 \neq t_2$.

Substituting Eq. (8) into Eq. (3a), we obtain

$$\frac{dI}{I} = [r + \rho\Sigma + \kappa h(t)]dt - \alpha\rho_Y Y dt + (\sigma_Y + \sigma_W)dW. \tag{9}$$

Similarly, substituting Eq. (8) into Eq. (4a), we obtain the discrete formulation for the dynamics of the logarithmic returns:

$$\begin{aligned} tlnI_{t+1} - lnI_t &= \mu_t + \sigma_Y (Y_{t+1} - Y_t) + \sigma_W \varepsilon_t, \\ &= [r + \rho\Sigma + \kappa h(t)] - \alpha\rho_Y Y_t + (\sigma_Y + \sigma_W)\varepsilon_t, \end{aligned} \tag{10a}$$

$$= [r + \rho\Sigma + \kappa h(t)] + \rho_Y (Y_{t+1} - Y_t) + (\sigma_Y + \sigma_W - \rho_Y)\varepsilon_t. \tag{10b}$$

As explained below Eq. (8), following the JLS model, we assume that the crash hazard rate $h(t)$ follows a deterministic time-dependence that describes the collective behavior of noise traders approaching a critical time at which the probability per unit time for a crash to occur peaks sharply. Using a model of social imitation on a hierarchical network of social influences, JLS obtained a crash hazard rate obeying a LPPL process. Since r, ρ, Σ and κ are assumed constant, the term $r + \rho\Sigma + \kappa h(t)$ is following a LPPL deterministic process $\Delta H(t) = H(t+1) - H(t)$, where $H(t)$ is given by expression (2).

Then, using Eq. (10b), the residual $\nu_t \equiv \ln I_t - H(t)$ of the logarithm of the asset value with respect to the deterministic LPPL process is given by

$$\nu_{t+1} - \nu_t = \rho_Y (Y_{t+1} - Y_t) + (\sigma_Y + \sigma_W - \rho_Y)\varepsilon_t. \tag{11}$$

Operationally, the process ν_t is nothing but the residuals of the nonlinear calibration of the process $H(t)$ to the asset price time series $\ln I_t$.

We make the hypothesis that price regimes where bubbles dominate are characterized by a strong deterministic component $H(t)$ in the log-price dynamics. As a consequence, one can expect that the residuals ν_t remain bounded, so that the log-price remains "guided" by $H(t)$. If $H(t)$ was stochastic, we would say that $\ln I_t$ and $H(t)$ are cointegrated (Granger & Hallman, 1991). Translated in the context of expression (11), this implies that we consider the case where $\sigma_Y + \sigma_W \approx \rho_Y$ with $|\sigma_Y + \sigma_W - \rho_Y| \ll \rho_Y$. This condition means that the excess return over the spot interest rate that an asset must earn per unit of risk variance associated respectively with the two processes Y and W almost exactly compensates these two risks when quantified by their respective volatilities. In this limit, the residuals ν_t are stationary and can be taken proportional to Y_t, i.e., they follow an AR(1) process. Thus, we assume

$$\Delta\nu_t = \nu_{t+1} - \nu_t = -\alpha\nu_t + u_t \tag{12}$$

where u_t is a Gaussian white noise. From Eqs. (10b) and (11) and using the definition of $\Delta H(t)$, we get

$$lnI_{t+1} - lnI_t = \Delta H(t) + \Delta\nu_t. \tag{13}$$

Combining Eqs. (13) and (12), the recursive formula for the logarithmic asset prices reads

$$lnI_{t+1} = lnI_t + \Delta H_t - \alpha(lnI_t - H_t) + u_t. \tag{14}$$

Equivalently, the equation for the logarithmic return is

$$r_{i+1} = lnI_{t_{i+1}} - lnI_{t_i} \sim N\left(\Delta H_{t_{i+1},t_i} - \alpha\left(lnI_{t_i} - H_{t_i}\right), \sigma_u^2 (t_{i+1} - t_i)\right), \Delta H_{t_{i+1},t_i} = H_{t_{i+1}} - H_{t_i}. \tag{15}$$

Compared with the conditional probability distribution given by expression (1) valid for the JLS model, our model introduces a new stochastic term in the drift. This new term $\alpha(lnI_{t_i} - H_{t_i})$ ensures that the log-price fluctuates around while remaining in the neighborhood of the LPPL trajectory H_t. This formulation ensures the consistency of modeling the log-price by the deterministic LPPL component as a global observable emergent macroscopic characteristics. We refer to model (15) as the "volatility-confined LPPL model." Obviously, this modeling strategy leading to the general form (15) holds for arbitrary deterministic models H_t.

2.2. Derivation of the model using the behavioral stochastic discount factor (BSDF) with critical behavior

We now present an alternative derivation of the volatility-confined model (15) with a LPPL drift trajectory (2), using an angle that is completely different from the RE bubble model of the previous section. Our alternative derivation describes the dynamics of the impact of herding investors on asset prices via a novel specification of the stochastic discount factor. This different approach is motivated by several weaknesses of the RE model.

- The RE model segments rather artificially the respective roles of noise traders on the one hand and of RE investors on the other hand. The former are assumed to control the crash hazard rate only via their herding behavior, and their impact on price is indirect through the no-arbitrage condition representing the actions of RE investors that link the conditional expected return to the crash hazard rate.
- Within the logic of the RE model, notwithstanding the deterministic predictability of the crash hazard rate obtained via the corresponding deterministic price component, the RE investors cannot on average make profit: the RE investors are remunerated from taking the risk of being exposed to a crash. Over all possible scenarios, their expected gain is zero. But some RE agents endowed with different preferences could in principle arbitrage the risk-neutral agents. The homogeneity of the RE agent preferences is therefore a limitation of the model.

Rather than using the interplay between the noise traders driving the crash hazard rate and the risk-adverse rational investors acting as market makers, we attribute the characteristics of the price behavior to the internal dynamics of the *market sentiment*. We propose to capture the critical behavior of an asset price resulting from the emergent collective organization of the complex financial system by a specification of the stochastic discount factor (SDF).

The starting point is to recognize the existence of critical dynamics (in the sense of complex systems) occurring in financial markets. The critical dynamics reflects the herding behavior of imitative investors, which leads to increasing correlations between the agents translated into financial bubbles. Such behaviors result from imperfective information, the use of heuristics and possible biases in the judgments of heterogeneous investors. It is therefore natural to combine insights from the field of behavioral finance and the concepts of criticality developed in the theory of complex systems (Sornette, 2004a).

We therefore adopt a behavioral finance perspective, and we refer to Shefrin, who extended the SDF into a so-called behavioral SDF (BSDF). The BSDF is supposed to provide a behaviorally-based synthesis of different theories of asset pricing (Shefrin, 2005). In this approach, the BSDF can be interpreted as a market sentiment factor which, according to Shefrin, is not a scalar but a stochastic process reflecting the deviation of subjective beliefs described by a certain representative agent (the market itself) relative to objective beliefs and of the market equilibrium time discount factor relative to the situation when all investors hold objectively correct beliefs. Expressed with discrete times, the BSDF can be defined as

$$\Lambda^{ST}(x_t) = \frac{\nu(x_t)}{\Pi(x_t)} = \left[\frac{\mathbb{P}_R(x_t)}{\Pi(x_t)} \cdot \frac{\delta_R^t}{\delta_{R,\Pi}^t}\right] \cdot \delta_{R,\Pi}^t [g(x_t)]^{-\gamma_R(x_t)}, \tag{16}$$

where the exponent ST stresses that the BSDF embodies the "sentiment" of the market. The term $\nu(x_t)$ is the state price of the basic security associated with the time-state pair (t, x_t). Π denotes the objective probability density and P_R is the representative investor's subjective belief density distribution, which can be derived by aggregating the heterogeneous investor's subjective beliefs given a set of adequate state prices. γ_R denotes the coefficient of relative risk aversion of the market. g is the interest discount factor used to discount future payoffs. The term $\frac{\mathbb{P}_R(x_t)}{\Pi(x_t)} \cdot \frac{\delta_R^t}{\delta_{R,\Pi}^t}$ is the product of the deviation of the market subjective beliefs to objective beliefs and of the market equilibrium time discount factor relative to the objective discount factor. Therefore, it plays the role of a *market sentiment* factor, which we denote by $\Phi(x_t)$ below. Notice that the remaining terms of Eq. (16) correspond to the traditional SDF, which we still denote by Λ. This leads one to express $\Lambda^{ST}(x_t)$ as the product of $\Phi(x_t)$ and Λ, or in continuous time, as

$$\Lambda(t)^{ST} = \Phi(t)\Lambda(t), \tag{17}$$

with

$$\Phi(x_t) \equiv \frac{\mathbb{P}_R(x_t)}{\Pi(x_t)} \cdot \frac{\delta_R^t}{\delta_{R,\Pi}^t}, \quad \Lambda(t) = \delta_{R,\Pi}^t [g(x_t)]^{-\gamma_R(x_t)}. \tag{18}$$

Armed with this representation (14), we propose to capture the market critical behavior associated with bubbles through the dynamics of the market sentiment factor, which is assumed to be characterized by the following jump process

$$\frac{d\Phi_t}{\Phi_t} = a\ dt\text{-}b\ dj. \tag{19}$$

The coefficient a is assumed to be small, as it describes the amplitude of the deviations of the market's equilibrium discount rate from the objective discount rate in "normal" times. In contrast, the term dj governs the occurrence of a possible catastrophe of the market sentiment resulting from a critical collective amplification of beliefs leading to a run-away. When the market operates close to a critical point, increasing large crowds of herding investors gather in their social imitational network to drive down the market's sentiment which may, as a result, fall sharply with some probability. In such a case, investors underestimate the likelihood of many possible states (as described by expression (18)) and, accordingly, the prices of state securities are subtly moved by the market to prevent arbitrage opportunities. During a bubble regime, the large majority of states are reflecting a bullish market, and an abrupt decrease of sentiment reflects that traders collectively distrust the ability of the market to continue its previous accelerated rise.

Different from the price dynamics represented by Eq. (3a), we assume no inherent jumps of price for all objective states. Thus, we use the same model (4a) as in the previous subsection, except for the jump term. In the present interpretation, a crash does not refer to the performance of the market price but to a collective phenomenon associated with the aggregate subjective agents' beliefs, which is represented by the BSDF. We also assume that the expectation $\mathbb{E}_t^{\mathcal{P}}(dj) = h(t)dt$ of the jump process dj defines the hazard rate $h(t)$. The difference with the RE model of Section 2.1 is that, here, $h(t)$ represents the probability for an overwhelming synchronized loss of trust to occur, conditional on the fact that the loss of trust has not yet happened. As in Section 2.1, we assume that $h(t)$ follows a deterministic time-dependence with LPPL properties that are typical of a critical behavior on a hierarchical network. Using Eqs. (17) and (19), we have

$$\begin{aligned}\frac{d\Lambda_t^{ST}}{\Lambda_t^{ST}} &= \frac{d(\Phi_t\Lambda_t)}{\Phi_t\Lambda_t} = \frac{d\Phi_t}{\Phi_t} + \frac{d\Lambda_t}{\Lambda_t} + \frac{d\Phi_t}{\Phi_t}\frac{d\Lambda_t}{\Lambda_t}\\ &= -[r\text{-}a]dt\text{-}bdj - \rho_Y dY - \rho_W dW,\end{aligned} \tag{20}$$

where we have used the process (7) for the SDF $\Lambda(t)$.

Assuming that the financial market is complete and in absence of risk-free arbitrage, the product of the asset price and of the BSDF should conform to the no arbitrage pricing condition $\mathbb{E}_t^{\mathcal{P}}\left[\Lambda_{t'}^{ST} I_{t'}\right)] = \Lambda_t^{ST} I_t$. With Eqs. (20) and (4a) (with no jump) and using a derivation similar to that presented in Appendix A, the no-arbitrage condition leads to

$$\frac{dI}{I} = [r + \rho\Sigma - a + bh(t)]dt - \alpha\rho_Y Y dt + (\sigma_Y + \sigma_W)dW. \tag{21}$$

This equation has the same structure as expression (9) obtained with the RE model, with just a redefinition of the constants $r + \rho\Sigma \to r + \rho\Sigma - a$ and $\kappa \to b$. With the same price dynamics, the conditional probability distributions of returns for the RE model and for the BSDF formulation are identical. It is this model (21) or equivalently Eq. (9) that we will calibrate and test in the next Sections.

But, before doing so, let us further elaborate on the economic meaning of the above derivation based on the concept of the BSDF with critical behavior. In contrast with the RE model, there is no need here for a representative rational investor playing the role of a market maker fixing the price on the basis of his rational expectations. The existence of the BSDF captures the intuition that the market as a whole is developing biased beliefs during the bubble gestation. Since the BSDF embodies the beliefs of both rational and noise traders to provide the pricing kernel associated with the dynamics of the market, the segmentation of the roles of the rational and of noise traders assumed in the RE model disappears. Besides, the drift term in Eq. (21) acting during the bubble implies that, even without the requirement of compensation for the risk of future price drops, the fear of a global loss of trust will naturally be pushing the price up according to the increasing hazard rate. Essentially, the market requires a risk premium associated with a sentiment risk. Provided that the market is complete, sufficiently many state securities are available at any time to all investors that allow a perfect replication of the asset value. Therefore, at any time t, the state securities for all state $\left\{x_{t'}^{\downarrow} | t' > t\right\}$ can be reduced to derivatives via adequate stripping techniques. Here, the subscript ↓ is used to differentiate the state in which the subjective belief jump occurs and the normal state $x_{t'}^{\star}$ with no jump in the BSDF process. Note that the two belief states $x_{t'}^{\downarrow}$ and $x_{t'}^{\star}$ are not differentiated in the price dynamics (and we can group them under the joint symbol $x_{t'}$) because there are no intrinsic jump states for the price. Recall that the state security ensures a unit payoff if and only if state $x_{t'}$ occurs. Compared with a market that does not exhibit any sentiment collapse risk and with a return $1/\nu(x_{t'})$ for the state $x_{t'}$, the market structure that we model requires a return equal to $1/\nu\left(x_{t'}^{\downarrow}\right)$ in the presence of a sentiment drop risk. According to Eq. (16), $\mathbb{P}_R\left(x_{t'}^{\downarrow}\right) < \mathbb{P}_R = \left(x_{t'}^{\star}\right)$, so for the state price $\upsilon\left(x_{t'}^{\downarrow}\right) < \upsilon(x_{t'})$. Hence, our market model requires a larger expected return during the bubble phase. Moreover, the mechanism for the crash in this interpretation is quite different from that of the RE model. In the RE model, the price downward jump is treated as the crash and is explicitly introduced into the price dynamics. In contrast, within the BSDF framework, the crash is interpreted as a critical event of the market sentiment, which does not cause a synchronous drop of price but may act as a trigger to the subsequent burst of the bubble. The downward jump of the market sentiment reflects the fact that traders collectively distrust the market. As a result, a substantial fraction of traders begin to sell-off and withdraw money from the financial market. Hence, the collapse of the unsustainable bubble is triggered, unfolding through a process that we do not need to specify here, since we focus on modeling the bubble process, i.e. the pre-crash regime.

3. Tests based on the Ornstein–Uhlenbeck structure of residuals of the LPPL model

We now describe a first series of empirical tests performed using model (21) (or equivalently (9)), supplemented by the LPPL specification (2). One key feature is the Ornstein–Uhlenbeck (O–U) structure of the residuals. This suggests that evaluations of our model of a bubble regime should test both for the presence of significant LPPL signatures as well as for the O–U property of residuals. According to Eq. (12), this translates into an AR(1) test for the residuals obtained by fitting the asset price trajectory using a LPPL process (2). We will therefore use two strategies. The first one developed in this section calibrates the asset price and then tests for the O–U properties for the residuals. The second one, which is implemented in Section 4, uses the equivalent specification (15) on the asset returns to develop a Bayesian inference test.

3.1. Evaluation of GARCH processes to test for errors of type I (false positive)

Recall that the purpose of this paper is to test the claim that financial bubbles can be diagnosed from their super-exponential price dynamics, possibly decorated by log-periodic accelerating oscillations. A first approach is to test whether standard financial processes exhibit such signatures. As an illustration, let us consider the GARCH (1,1) model

$$\begin{aligned}lnI_t - lnI_{t-1} &= \mu_0 + \sigma_t z_t\\ \sigma_t^2 &= \sigma_0^2 + a_G(lnI_{t-1} - lnI_{t-2} - \mu_0)^2 + b_G\sigma_{t-1}^2,\end{aligned} \tag{22}$$

Table 1
Test of the LPPL specifications (2) to synthetic time series generated with the GARCH model (22), with the LPPL conditions (23), and the unit-root tests on the residuals. For each type of samples, 1000 time series have been generated.

Type of samples	Percentage of LPPL condition satisfied	Signif. level	Percentage of not rejecting H_0		False positive rate
			Phillips–Perron	Dickey–Fuller	
Random length	0.2%	$\alpha = 0.01$	94.1%	94.1%	0.2%
		$\alpha = 0.001$	72.8%	72.8%	0.2%
Fixed length	0.1%	$\alpha = 0.01$	93.8%	93.8%	0.1%
		$\alpha = 0.001$	72.7%	72.7%	0.0%

where the innovation z is distributed according to the Student-n distribution (with n degrees of freedom). Estimating this GARCH(1,1) model on the S&P500 index for the US market from Jan. 3, 1950 to Nov. 21, 2008 at the daily time scale (such that one unit time increment in Eq. (22) corresponds to one day) yields the following parameters: conditional mean of return $\mu_0 = 5.4 \times 10^{-4}$, conditional variance $\sigma_0^2 = 5.1 \times 10^{-7}$, ARCH coefficient $a_G = 0.07$, GARCH coefficient $b_G = 0.926$ and number of degrees of freedom of the student distribution $n = 7$.

Calibrating the LPPL specification (2) to a given price trajectory will always provide some output for the parameters and the residuals. In order to qualify the LPPL calibration, we impose the following restrictions on the parameters

$$\begin{aligned} B &> 0 \\ 0.1 \le \beta &\le 0.9 \\ |C| &< 1 \\ 6 \le \omega &\le 13. \end{aligned} \tag{23}$$

These conditions (23) can be regarded as the “stylized features of LPPL”, which were documented in many previous investigations (see Johansen, 2004; Johansen & Sornette, 2006, for reviews documenting these stylized facts). The two first conditions $B > 0$ and $0.1 \le \beta \le 0.9$ ensure a faster-than-exponential acceleration of the log-price with a vertical slope at the critical time t_c. The restriction $|C| < 1$ in Eq. (23) was introduced by Bothmer and Meister (2003) to ensure that the hazard rate $h(t)$ remains always positive. For the sake of brevity, we refer to conditions (23) as the LPPL conditions. The last condition $6 \le \omega \le 13$ constrains the log-periodic oscillations to be neither too fast (otherwise they would fit the random component of the data), nor too slow (otherwise they would provide a contribution to the trend, see Huang, Johansen, Lee, Saleur, and Sornette (2000) for the conditions on the statistical significance of log-periodicity). Recently, Brée and Joseph (in press) revisited the validity of the bounds imposed on ω by calibrating the LPPL model to the Hang Seng stock market index over the period 1970 to 2008. Considering eleven different market periods identified as eleven bubbles, they found that only one third of them had their parameters obeying the bounds (23).[4] Sornette, Woodard, Yan, and Zhou (2013) cautioned that the different algorithms (unbound Nelder–Mead Simplex search, bounded Taboo search and others) could explore only imperfectly the likelihood landscape, which is in general characterized by many competing maxima for neighboring values of ω. This can be partially alleviated by the improved algorithm proposed by Filimonov and Sornette (2013). In order to avoid tinkering with the parameters, we rely on the bounds proposed earlier (Johansen & Sornette, 2006) and apply the restriction $6 \le \omega \le 13$ in our procedure developed below. Moreover, we also impose that the critical time t_c should be no further than one year beyond the last data point used in the fit.

Table 1 shows the results obtained by calibrating the LPPL specification (2) to synthetic time series generated with the GARCH model (22), with the LPPL conditions (23), and the unit-root tests on the residuals. We have performed these tests on two sets of 1000 synthetic GARCH time series: (i) samples of random lengths, with lengths uniformly distributed from 750 days to 1500 days and (ii) samples of fixed length of 1500 days. The unit-root tests are the Phillips–Perron test and the Dickey–Fuller test, which are such that a rejection of the null hypothesis H_0 implies that the residuals are stationary (and therefore are compatible with the Ornstein–Uhlenbeck process posited in our model presented in the previous Section 2). Table 1 shows first a very small rate of false positives, i.e., less than 0.2% of the 2000 GARCH-generated time series are found to obey the LPPL conditions, and would thus be diagnosed as being in a bubble regime. Secondly, the unit-root tests show that, for most residual time series obtained as the difference between the synthetic GARCH time series and their LPPL calibration, one cannot reject the null, i.e. the residuals are non-stationary. This confirms that our model is not a good fit to synthetic GARCH time series.

3.2. Tests on the S&P500 US index from Jan. 3, 1950 to Nov. 21, 2008

We now apply the same procedure as in the previous Section to the S&P500 index in the US from Jan. 3, 1950 to Nov. 21, 2008. But we do not have of course the luxury of a large sample of different realizations, as for the synthetically generated GARCH time series. Instead, we generate two sets of time windows of 750 successive trading days. The first (respectively second) set is obtained by sliding windows of 750 days over the whole duration of our data sets with time increments of 25 days (respectively 50 days), referred to as windows of type I and II respectively. The first (second) set has 563 (262) windows.

In Table 2, we can see that, for set I (respectively II), a fraction $P_{LPPL} = 2.49\%$ (respectively 2.84%) of the windows obey the LPPL conditions (23). This is more than a factor of ten larger than the corresponding fraction for the synthetic GARCH time series. For this fraction of windows that obey the LPPL conditions, all of them reject the two unit-root tests for non-stationarity, showing that the time windows, which qualify as being in a bubble regime according to our model, also give residuals that are stationary, as required from the Ornstein–Uhlenbeck specification of the residues of our model. In contrast, Table 2 shows that, as for Table 1, the large majority of windows give residuals for which the null unit-root hypothesis of non-stationarity cannot be rejected. This means that, for most windows that do not obey the LPPL conditions, their residuals are non-stationary, providing two reasons for diagnosing these windows as being in a non-bubble regime. This result, together with the 100% rate of rejection of the null hypothesis for non-stationarity for the subset of windows that obey the LPPL conditions, provides strong support for our model. In contrast, for windows that are diagnosed to be in a bubble regime, their residues are automatically stationary, in accordance with our model. A crucial additional evidence is provided by Table 3, which lists the windows that obey the LPPL conditions. We find that all of them correspond to periods preceding well-known crashes. This confirms that our method for identifying bubbles exhibits a very low rate of errors of type I (false positives).

Summarizing our results obtained so far, we can state that about 97–97.5% of the time intervals of 750 trading days within the period from Jan. 3, 1950 to Nov. 21, 2008 correspond to non-bubble regimes, rather well described by a GARCH process. We have been able to characterize LPPL signatures of bubbles that occupy about 2.5–3% of the whole time interval. These percentages suggest a highly selective and

[4] Most of the estimated ω were found inside the interval [4.3; 7.5].

Table 2
Test of the LPPL specifications (2) and the unit-root tests on the residuals, for time series of 750 consecutive trading days of the S&P500 US index in the interval from Jan. 3, 1950 to Nov. 21, 2008. The first (respectively second) set of windows is obtained by sliding windows of 750 days over the whole duration of our data set with time increments of 25 days (respectively 50 days). P_{LPPL} denotes the fraction of windows that satisfy the LPPL condition. $P_{\mathrm{StationaryResi.|LPPL}}$ is the conditional probability that, out of the fraction P_{LPPL} of windows that satisfy the LPPL condition, the null unit-root test for non-stationarity is rejected for the residuals.

Days of one step	Number of windows	P_{LPPL}	Signif. level	Percentage of not rejecting H_0		$P_{\mathrm{StationaryResi.|LPPL}}$
				Phillips–Perron	Dickey–Fuller	
25	563	2.49%	$\alpha = 0.01$	96.45%	96.45%	100%
			$\alpha = 0.001$	69.27%	69.27%	100%
50	282	2.84%	$\alpha = 0.01$	96.81%	96.81%	100%
			$\alpha = 0.001$	70.92%	70.92%	100%

efficient detection filter. We test further this selectivity by focusing on the classic crash of October 1987, to test how well we can diagnose a bubble regime preceding it. We consider shrinking windows with increasing starting dates and fixed last date of September 30, 1987. We scan the starting dates with a resolution of 5 days and stop with the shortest window of size equal to 750 trading days. We expect that the LPPL conditions and the rejection of the null unit-test hypothesis for the residuals should be observed increasingly as the starting date of the windows moves upward towards the crash date. Table 4 shows the results for different starting dates, which confirm remarkably well our expectations. The closer the starting date is to the crash date, the larger is the fraction P_{LPPL} of windows that obey the LPPL conditions. Of these, a fraction of $P_{\mathrm{StationaryResi.|LPPL}} = 100\%$ rejects the null unit-root tests of non-stationarity. Compared with the overall fraction of 2.5–3% of windows that pass the LPPL conditions over the whole time interval from Jan. 3, 1950 to Nov. 21, 2008, this fraction rises drastically from about 20% to 100% for the time windows most influenced by the latest part of the time series closest to the crash. This suggests the existence of a regime shift from a GARCH-like process to a LPPL bubble regime as time approaches the Oct. 1987 crash. Note also that all 43 windows that pass the LPPL conditions have starting dates around the end of 1983, suggesting that the bubble that led to the great Oct. 1987 crash started around the beginning of 1984. This result is very interesting in so far that it strengthens the interpretation of crashes as the outcome of a long maturation process, and not due to proximal causes of the previous few days or weeks (Johansen & Sornette, 2006). This confirmed bubble onset time could be rationalized by looking at the economic context in detail. In fact, since about 1984, the share of private consumption as percentage of GDP has outstripped the share of wage as percentage of GDP in U.S. and the gap has grown even larger thereafter (see Fig. 3 of Sornette & Cauwels, 2012). Concomitantly, the discrepancy between the rate of financial profit and growth rate of net capital started to deviate. Moreover, the evolution of the U.S. debt (households + firms + government) in percentage of GDP exhibited an explosive growth around 1984. All of the above evidence suggests that the increase in consumption was mainly paid by debt and by extracting financial profits either from stock market investments or so-called mortgage wealth. This is in line with the process of a self-fulfilling bubble developing in the financial sphere from at least 1984.

The left panel of Fig. 1 shows the fit of the logarithm of the S&P500 US index with expression (2) over the time interval from Jan. 3, 1984 (the first trading day in 1984) to Sep. 30, 1987. The time series of the residuals of this fit is shown in the upper right panel and its partial autoregression correlation function (PACF) is depicted in the lower right panel for lags from 0 to 20 days. All values of the PACF with lags larger than 1 fall within two standard deviations, indicating the absence of linear dependence. Combined with the result of the Phillips–Perron test on this series of residuals shown in Table 5, this suggests that these residuals are both stationary (they reject the unit root test of non-stationarity) and furthermore they can be closely approximated by an AR(1) process with a mean-reverting coefficient $-\alpha \approx -0.03$. This supports our proposal to model the residuals $v(t)$ of the LPPL as generated by a Ornstein–Uhlenbeck process.

Table 3
Windows of the S&P500 US index in the interval from Jan. 3, 1950 to Nov. 21, 2008 that obey the LPPL conditions. Windows of type I (respectively type II) are obtained by sliding a time interval of 750 days over the whole duration of our data sets with time increments of 25 days (respectively 50 days).

Start of window	End of window	Reject H_0 for residuals	Type of sliding step
May. 7, 1984	Apr. 24, 1987	Yes	I
Jun. 12, 1984	Jun. 1, 1987	Yes	I & II
Jun. 18, 1984	Jul. 7, 1987	Yes	I
Mar. 15, 1991	Feb. 16, 1994	Yes	I & II
Mar. 25, 1994	Mar. 13, 1997	Yes	I
May. 3, 1994	Apr. 18, 1997	Yes	I & II
Jun. 8, 1994	May. 23, 1997	Yes	I
Jul. 14, 1994	Jun. 30, 1997	Yes	I & II
Sep. 23, 1994	Sep. 10, 1997	Yes	I & II
Oct. 28, 1994	Oct. 15, 1997	Yes	I
Apr. 28, 1995	Apr. 11, 1998	Yes	I & II
Jun. 5, 1995	May. 15, 1998	Yes	I
Jun. 11, 1995	Jun. 21, 1998	Yes	I & II
Sep. 16, 1996	Sep. 30, 1999	Yes	I & II

4. Bayesian inference for our modified LPPL model with Ornstein–Uhlenbeck residuals

Before proceeding, a caveat should be stressed. The Bayesian inference test presented in this section should not be over-interpreted. While some authors present the method of comparing "marginal likelihoods" (see below) as a kind of panacea (Chang & Feigenbaum, 2006), we refrain from such overblown enthusiasm. As explained below, the Bayesian approach and its marginal likelihoods suffer from severe problems. Our goal in presenting this test is to show that, playing the same "game" as Chang and Feigenbaum (2006) with the same tools, our improved volatility-confined LPPL model significantly reverses their conclusion and outperforms their simple benchmark. This does not mean that we believe that this Bayesian test is the right test, nor does it constitute the ultimate gauge to compare models. Our philosophy is that there is no absolute metric to judge in an absolute way the relative merits of two competing models. One should rather develop more and more tests, each of them revealing different and complementary facets of the competing models with respect to the explanatory power of empirical data. By piling up evidence, one can eventually develop a trust in the value of the proposed model, following a systematic approach to model validation (Sornette et al., 2007). In our opinion, the Bayesian inference test is just one relatively weak test, whose results should not be over-interpreted. It does not seem to be the most appropriate to test the volatility-confined LPPL model, as compared with the other tests presented in the previous and the following Sections. Our positive results presented below in this section stemming from Bayesian

Table 4
Test for the validity of the LPPL conditions and unit-root tests on residuals in windows all ending on Sep. 30, 1987 with different starting dates for the S&P500 US index. The smallest window size is 750 days. P_{LPPL} is the percentage of windows that obey the LPPL conditions in all the test windows. $P_{StationaryResi.|LPPL}$ is the probability that the null unit-root tests for non-stationarity are rejected for the residuals, conditional on the fact that the LPPL conditions are met. The unit-root tests are also the Phillips–Perron and Dickey–Fuller tests (both produce the same results) with a significance level of 0.001. Double stars (**) denotes 1% significance level to reject the null H0 that the residual process has a unit root.

Start of window	Number of samples	Number of series satisfy LPPL condition	P_{LPPL}	$P_{StationaryResi.\|LPPL}$
Jan. 2, 1980	242	43	17.78%	100%**
Jan. 3, 1983	90	43	47.48%	100%**
Sep. 1, 1983	57	42	73.68%	100%**
Dec. 1, 1983	44	43	97.73%	100%**
Mar. 1, 1984	32	32	100%	100%

Inference can merely provide a support for the volatility-confined LPPL model on one facet, but does not constitute an absolute criterion of goodness.

4.1. Likelihood versus Bayesian marginal likelihood approach

We now describe the second series of empirical tests performed using model (21) (or equivalently Eq. (9)), supplemented by the LPPL specification (2). While the previous Section 3 has used the asset price to test for the presence of LPPL conditions and has then tested for the Orstein–Uhlenbeck (O–U) properties for the residuals, here we use the other equivalent specification (15) on the asset returns to develop a Bayesian inference test.

Our approach parallels that of Chang and Feigenbaum (2006) for the implementation of the Bayesian inference. But a fundamental difference is that, while their implementation used the specification (1), our model (15) contains the additional term $\alpha(\ln p_t - H_t)$, stemming from the intrinsic guiding mechanism associated with the O–U model of the residuals decorating the deterministic LPPL bubble trajectory. We show below that this new term makes all the difference in establishing the statistical significance of LPPL properties of asset returns.

The use of Bayesian inference, compared with maximum likelihood in the context of models specifying the dynamics of financial results, may be motivated as follows. Eq. (1) suggests that one might detect directly the LPPL signature in returns by removing the effects caused by the intrinsic guiding mechanism associated with the O–U model of the residuals. Defining the random variable $\Psi_{t_i} = -\alpha(\ln - p_t - H_t)$, we define the *adjusted return* as

$$r_{t_i}^{Ad} = r_t - \Psi_t = \Delta H_t + u_t. \tag{24}$$

Recall that ΔH_t results directly from the hazard rate and contains the LPPL signal. The residual u_t should then be a white noise process. The adjusted returns r_t^{Ad} defined by Eq. (24) for the S&P500 US index from Jan. 3, 1984 to Sept. 30, 1987 are shown in Fig. 2. The continuous curve shows ΔH_t, where the parameters for the process H_t are obtained by a nonlinear least square fit as in the previous Section. Unsurprisingly, one can see that the deterministic component is very small compared with the typical amplitude of the adjusted returns. Note that the same relative smallness of the LPPL signal viewed in the return time series has been noted earlier (Chang & Feigenbaum, 2006; Feigenbaum, 2001). It is not clear how to develop a test that directly tests for the existence of a significant LPPL component in the time series of adjusted returns shown in Fig. 2.

The general weakness of the likelihood analysis of log-periodicity on returns is not a surprise when viewed from the perspective offered by the analysis of Huang et al. (2000). Using numerically intensive Monte-Carlo simulations, Huang et al. (2000) showed that, for regularly sampled time series as is the case for financial time series, the log-periodic signal is much more significant in the cumulative signal than in its first difference (and that using the cumulative signal does not create spurious log-periodicity), due to the well-known fact that integration corresponds to low-pass filtering. This

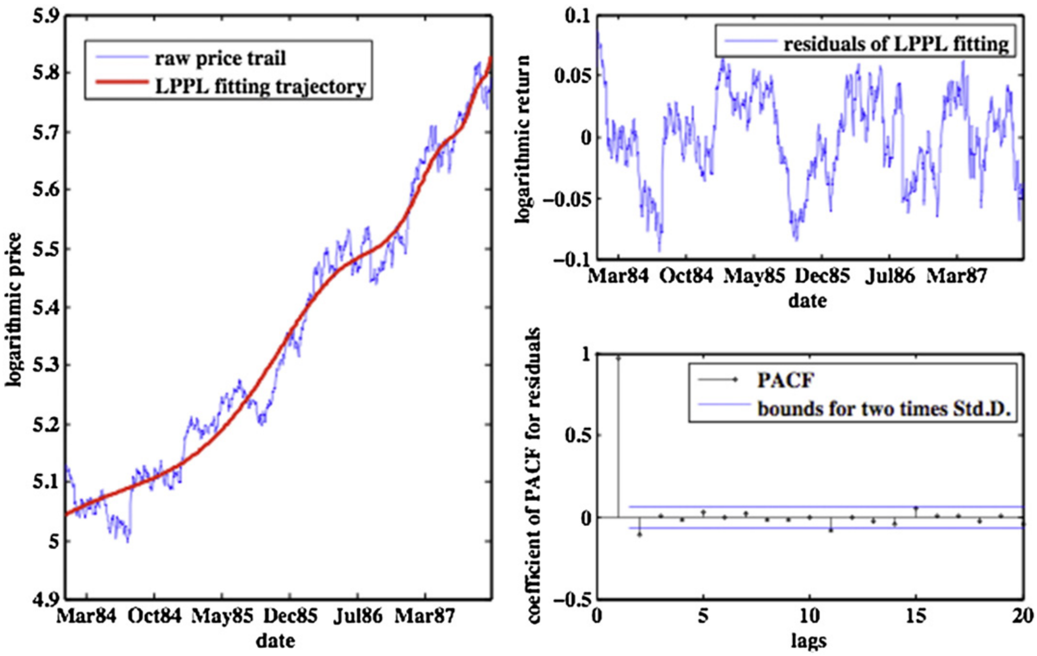

Fig. 1. Left panel: fit of the logarithm of the S&P500 US index with expression (2) over the time interval from Jan. 3, 1984 (the first trading day in 1984) to Sep. 30, 1987. Upper right panel: time series of the residuals of the fit shown in the left panel. Lower right panel: partial autoregression correlation function (PACF) of the residuals. The value of the PACF at lag 1 is equal to 0.9709. For lags larger than 1, the PACF is bounded between ± two standard deviations.

Table 5
Phillips–Perron unit root test on residuals of the calibration of the S&P 500 index by the LPPL model (1) with (2) over the interval from Jan. 3 1984 to Sep. 30 1987.

Phillips–Perron test statistic		Adj. t-stat −4.008		Prob.* <0.01
Test critical values	1%	−2.567		
	5%	−1.941		
Model	Coefficient α	Std. error	t-Statistic	Prob.
$v_{t+1} = -\alpha v_t + u_t$	0.029	0.0077	−3.789	0.0002
R-squared	0.015	Mean dependent var		−9.95E−05
Adjusted R-squared	0.015	S.D. dependent var		0.0084
S.E. of regression	0.0084	AIC		−6.7286
Sum squared resid	0.0662	SC		−6.7234
Log likelihood	3186.97	Durbin–Watson stat		1.7928

suggests that working on returns, while being the standard of econometric studies, may actually be sub-optimal in this case. Sornette and Johansen (2001) summarized in their Section 9 the Monte-Carlo tests which have been performed by various groups to address specifically this problem, including (Feigenbaum & Freund, 1996, 1998), both on synthetically generated price levels and on randomly chosen time intervals of real financial time series: these tests show the high statistical significance of log-periodicity in the log-price trajectory before the crash of October 1987 and on several other bubbles.

4.2. Set-up of the Bayesian inference test

We thereupon turn to the method of Bayesian inference to investigate the statistical significance of LPPL features in the return time series. Following the philosophy attached to Bayesian analysis, two models can be compared by estimating the ratio of the posterior probability for each model given the data, this ratio being called the *Bayesian factor*. Let M_0 denote the benchmark model and Ξ_0 its corresponding set of parameters. Similarly, let M_1 denote an alternative model with its set of parameters Ξ_1. Then, the Bayesian factor of model M_1 compared with model M_0 is defined as

$$B_{M_1,M_0} = \frac{p(\Xi_1|Q;M_1)}{p(\Xi_0|Q;M_0)} = \frac{\dfrac{\int p\left(\theta_{M_1};M_1\right)p\left(Q|\theta_{M_1};M_1\right)d\theta_{M_1}}{p(Q)}}{\dfrac{\int p\left(\theta_{M_0};M_0\right)p\left(Q|\theta_{M_0};M_0\right)d\theta_{M_0}}{p(Q)}} = \frac{\int p\left(\theta_{M_1};M_1\right)p\left(Q|\theta_{M_1};M_1\right)d\theta_{M_1}}{\int p\left(\theta_{M_0};M_0\right)p\left(Q|\theta_{M_0};M_0\right)d\theta_{M_0}}. \tag{25}$$

In this expression, θ_M denotes the vector of parameters for model M. The term $p(\Xi|Q;\ M)$ represents the posterior probability for the set of parameters in model M, given the observed data Q. The term $p(\theta_M;\ M)$ is the prior probability chosen for the parameters θ in model M. Within the framework of Bayesian hypothesis testing, if B_{M_1,M_0} is larger than 1, one should accept the alternative model because the posterior probability for its parameters has enjoyed a larger increase from its initial prior basis level, which implies that the alternative model can explain the data better than the reference model. If the prior probabilities are not too restrictive, and for a large sufficient data set, Bayesian inference amounts to comparing the likelihood function of each model and the Bayesian factor test tends asymptotically for large data sets to the likelihood ratio test.

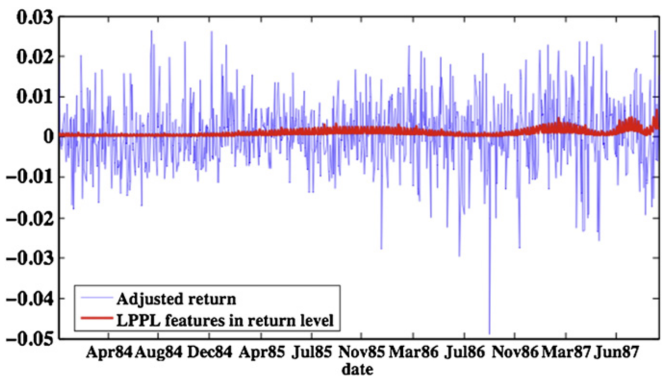

Fig. 2. Time series of adjusted returns defined by expression (24) for the S&P500 US index from Jan. 3, 1984 to Sept. 30, 1987. The smooth continuous line shows the LPPL term ΔH_t, where H_t is defined by Eq. (2).

Let us consider the time series of returns $\{q_i\}$ sampled at the time instants $t \in \{t_0, t_1, t_2, \ldots, t_N\}$. For the reference model, as in Chang and Feigenbaum (2006), we choose the Black–Scholes model whose logarithmic returns are given by

$$r_i \sim N\left(\mu(t_i - t_{i\text{-}1}), \sigma^2(t_i - t_{i-1})\right). \tag{26}$$

The drift μ is drawn from the prior distribution $N(\mu_r, \sigma_r)$. The variance σ^2 of daily returns is specified in terms of its inverse $\tau = \frac{1}{\sigma^2}$, known as the "precision" in the language of Bayesian analysis. The precision describes how precisely the random variable will be known and thus the higher it is, the better. The precision is supposed to be drawn as $\tau \sim \Gamma(\alpha_\tau, \beta_\tau)$.

The alternate hypothesis model is our volatility-confined LPPL model. Recalling expression (15) with our present notations, the returns of the alternative model are described by

$$r_i \sim N\left(\Delta H_{i,i-1} - \alpha(q_{i-1} - H_{i-1})\sigma_u^2(t_i - t_{i-1})\right) \tag{27}$$

where

$$\Delta H_{i,i-1} = B(t_c\text{-}t_{i-1})^\beta\left[1 + \frac{C}{\sqrt{1+\left(\frac{\omega}{\beta}\right)^2}}\cos(\omega \ln(t_c\text{-}t_{i-1})) + \phi)\right] - B(t_c\text{-}t_i)^\beta\left[1 + \frac{C}{\sqrt{1+\left(\frac{\omega}{\beta}\right)^2}}\cos(\omega\ \ln(t_c\text{-}t_i)) + \phi)\right].$$

The LPPL characteristics of the model for $\Delta H_{i,i-1}$ are encoded in the vector of parameters $\boldsymbol{\xi} = (A, B, C, \beta, \omega, \phi, t_c)$. For simplicity, we assume that these parameters are drawn independently from the following prior distributions[5]:

$$\begin{aligned} A &\sim N(\mu_A, \sigma_A) \\ B &\sim \Gamma(\alpha_B, \beta_B) \\ C &\sim U(0,1) \\ \beta &\sim B\left(\alpha_\beta, \beta_\beta\right) \\ \omega &\sim \Gamma(\alpha_\omega, \beta_\omega) \\ \phi &\sim U(0, 2\pi) \\ t_c - t_N &\sim \Gamma\left(\alpha_{t_c}, \beta_{t_c}\right) \end{aligned} \tag{28}$$

where Γ, B and U denote the Γ-distribution, B-distribution and uniform distribution respectively. In practice of Bayesian inference, the Γ-distribution and B-distribution are often adopted as prior probability distribution.[6] The Γ-distribution is usually used to describe non-

[5] To avoid confusion when comparing with Chang and Feigenbaum (2006), it should be noted that the notation for the gamma distributions used here is related to that of Chang and Feigenbaum (2006) under the transformation $\beta \to 1/\beta$.

[6] Γ-distribution and B-distribution are also called *conjugate prior family* because, by adopting a prior density of Beta (Gamma) form, one also obtains a posterior density of Beta (Gamma) form, but with different parameters. Although there is no obligation to adopt conjugate priors, the conjugate prior property is very convenient because it avoids having to integrate numerically to find the normalizing constant in the posterior density (Young & Smith, 2005).

negative variables, and has the density function $f(x;\alpha,\beta)=\beta^{-\alpha}\Gamma^{-1}(\alpha)x^{a-1}exp\left(\frac{x}{\beta}\right)$, with $E(X)=\alpha\beta$ and $Var(X)=\alpha\beta^2$. $\Gamma(z)$ is the gamma function defined as $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$. The random variable realized between 0 and 1 is usually assigned with a beta prior density, which is $f(x;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-\beta)^{\beta-1}$, where $B(z)$ is the beta function satisfying $B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$. Accordingly, the mean and variance of the variable with B-distribution are $E(X)=\frac{\alpha}{\alpha+\beta}$ and $Var(X)=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$. Then, the full set of parameters of the volatility-confined LPPL model is $\Xi=(\mu,\tau,\alpha,\boldsymbol{\xi})$. The prior density for our model is given explicitly by the product of all marginal priors for each parameter

$$p(\theta_{LPPL};\mathrm{LPPL})=\frac{1}{\sqrt{2\pi}\sigma_r}exp\left[-\frac{(\mu-\mu_r)^2}{2\sigma_r^2}\right]\times f_\Gamma(\tau;\alpha_\tau,\beta\tau)\times f_\Gamma(\alpha;\alpha_\alpha,\beta_\alpha)$$
$$\times\frac{1}{\sqrt{2\pi}\sigma_A}exp\left[-\frac{(\mu-\mu_A)^2}{2\sigma_A^2}\right]\times f_\Gamma(B;\alpha_B,\beta_B)\times f_B\left(\beta;\alpha_\beta,\beta_\beta\right)$$
$$\times f_\Gamma(\omega;\alpha_\omega,\beta\omega)\times\frac{1}{2\pi}\times f_\Gamma\left(t_c-t_N;\alpha_{t_c-t_N},\beta_{t_c-t_N}\right), \tag{29}$$

for $\theta_{LPPL}\in\Xi=\mathbb{R}^2\times\mathbb{R}_+^3\times[0,1]^3\times[0,2\pi)\times[t_N,\infty)$. According to Eq. (27), given θ_{LPPL} and q_{i-1}, the density for q_i is

$$p(q_i|q_{i-1},\theta_{LPPL};\mathrm{LPPL})$$
$$=\sqrt{\frac{\tau}{2\pi(t_i-t_{i-1})}}exp\left[-\frac{\tau\left(q_i-q_{i-1}+\alpha(q_{i-1}-H_{i-1})-\Delta H_{i,i-1}\right)^2}{2(t_i-t_{i-1})}\right]. \tag{30}$$

Thus, the conditional density of the returns given the prior parameters reads

$$p(Q|\theta_{LPPL};\mathrm{LPPL})=\prod_{i=1}^N p(q_i|q_{i-1},\theta_{LPPL};\mathrm{LPPL}), \tag{31}$$

and the log marginal likelihood needed for the computation of the Bayesian factor is given by

$$\mathscr{L}=ln\left(\int_\Xi p(\theta_{LPPL})p(Q|\theta_{LPPL};\mathrm{LPPL})d\theta_{LPPL}\right). \tag{32}$$

Expression (32) defines nothing but a smoothing of the likelihood function performed with respect to some a priori weight for the input parameters.

4.3. Numerical implementation of the Bayesian inference test

We now proceed to the calculation of expression (32) for $\mathscr{L}_{\mathrm{LPPL}}$ and $\mathscr{L}_{\mathrm{BS}}$ and obtain the Bayesian factor. To implement the Bayesian inference test, we consider the same data set as before, namely the S&P 500 US index, but concentrating on the period from Jan. 3, 1984 to Sep. 30, 1987 to correspond with our previous analysis. The calculations are performed with time corresponding to the market time (i.e., trading days). The constant drift μ, the precision τ, coefficients B and C, exponent β, angular log-periodic frequency ω and phase term ϕ are assigned with the same priors as those in (Chang & Feigenbaum, 2006). The coefficient A, which is the final expected price at the critical time t_c, is taken from a normal distribution with $E[A]=6$ and $Var[A]=0.05$ to roughly agree with the range of price fluctuations near the critical time. Since t_c can be a few days or months after the real crash, but with the most probable value just being the crash day, we choose $E[t_c-t_N]=30$ and standard deviation as $\sqrt{Var[t_c-t_N]}=30$. Additionally, we choose $E[\alpha]=\sqrt{Var[\alpha]}=0.05$, which roughly reflects our estimated results obtained from the test using shrinking windows with a fixed last date of Sep. 30, 1987 and with time increments of 5 days. The chosen priors are given by the following expressions:

$$\mu\sim N\left(0.0003,(0.01)^2\right)$$
$$\tau\sim\Gamma\left(1.0,10^5\right)$$
$$\alpha\sim\Gamma(1.0,0.05)$$
$$A\sim N(6,0.05)$$
$$B\sim\Gamma(1,0.01)$$
$$C\sim U(0,1)$$
$$\beta\sim B(40,30)$$
$$\omega\sim\Gamma(16,0.4)$$
$$\phi\sim U(0,2\pi)$$
$$t_c\text{-}t_N\sim\Gamma(1,30).$$

The integrals in Eq. (32) for the log marginal likelihood have been estimated by the Monte-Carlo method with 10,000 sampling values for each integral component. In order to ascertain the validity of our numerical estimation of $\mathscr{L}_{\mathrm{LPPL}}$ in Eq. (32) and to estimate its confidence interval, we have repeated these calculations 100 times. We also performed the same calculations for $\mathscr{L}_{\mathrm{BS}}$ and finally get

$$\begin{aligned}\mathscr{L}_{\mathrm{LPPL}}(2.5\%-97.5\%)&=3173.546-3176.983\\ \mathscr{L}_{\mathrm{BS}}(2.5\%-97.5\%)&=3169.808-3170.097.\end{aligned} \tag{33}$$

A difference of the average log likelihood $\overline{\mathscr{L}}_{\mathrm{LPPL}}-\overline{\mathscr{L}}_{\mathrm{BS}}$ of about 5 translates into a very large Bayesian factor $exp(\overline{\mathscr{L}}_{\mathrm{LPPL}}-\overline{\mathscr{L}}_{\mathrm{BS}})\approx e^5\simeq150$. The Bayesian inference test therefore suggests that our volatility-confined LPPL model strongly outperforms the Black–Scholes benchmark.

Our result contrasts decisively with that of Chang & Feigenbaum (2006). Using our numerical scheme, we were able to reproduce the negative results reported by Chang & Feigenbaum (2006) that the JLS model is not significantly preferred to the benchmark model according to the Bayesian inference test. Thus, our new results cannot be ascribed to a spurious numerical implementation but reveals the importance of the specification of the residuals. The difference can be traced back to the Ornstein–Uhlenbeck model of residuals, which makes the LPPL fit self-consistent. Given the empirical price data, any agnostic economist would have to put more weight on our volatility-confined LPPL model than on the standard benchmark without super-exponential growth and log-periodicity.

4.4. Comparing the power law model (without log-periodicity) and the LPPL model with the Bayesian inference test

It is instructive to calculate the log marginal likelihood for the volatility-confined PL (power law) model. The PL model is the special case of the volatility-confined LPPL model obtained for $C=0$ in expression (2). The PL model keeps the super-exponential component but neglects the log-periodic oscillatory component. The following compares the log-likelihoods of the two models in their 2.5-percentile to 97.5-percentile range obtained over the distribution of their numerical estimations:

$$\begin{aligned}\mathscr{L}_{\mathrm{LPPL}}(2.5\%-97.5\%)&=3173.546-3176.983\\ \mathscr{L}_{\mathrm{PL}}(2.5\%-97.5\%)&=3175.520-3178.425.\end{aligned} \tag{34}$$

This shows that there is no significant gain in the Bayesian factor when going from the PL model to the LPPL model. Actually, the Bayesian factor for the volatility-confined PL model tends to be somewhat larger than that of the volatility-confined LPPL model.

We attribute this result to the stronger impact of the priors of the volatility-confined LPPL model due to its larger number of parameters, compared with the volatility-confined PL model. To show this, let us construct bounds for $\mathscr{L}_{\text{LPPL}} - \mathscr{L}L_{\text{PL}}$:

$$\begin{aligned}\mathscr{L}_{\text{LPPL}}-\mathscr{L}_{\text{PL}} &= ln\left(\frac{\int p(\theta_{LPPL})L(\theta_{LPPL}|Q;\text{LPPL})d\theta_{LPPL}}{\int p(\theta_{PL})L(\theta_{PL}|Q;\text{PL})d\theta_{PL}}\right) \\ &\approx ln\left(\frac{\sum_i p_i(\theta_{PL})L_i(\theta_{LPPL}|Q;\text{LPPL})\cdot p_i(C)\cdot p_i(\omega)\cdot p_i(\phi)}{\sum_i p_i(\theta_{PL})L_i(\theta_{PL}|Q;\text{PL})}\right) \\ &= -ln2\pi + ln\left(\frac{\sum_i p_i(\theta_{PL})L_i(\theta_{LPPL}|Q;\text{LPPL})\cdot p_i(\omega)}{\sum_i p_i(\theta_{PL})L_i(\theta_{PL}|Q;\text{PL})}\right).\end{aligned} \tag{35}$$

Then, under the condition

$$1<\frac{L_i(\theta_{LPPL}|Q;\text{LPPL})}{L_i(\theta_{PL}|Q;\text{PL})}<\frac{1}{p_i(\omega)}, \tag{36}$$

we have

$$\mathscr{L}_{\text{LPPL}}-\mathscr{L}_{\text{PL}}<-ln2\pi\approx-1.84. \tag{37}$$

This bound − 1.84 is close to the mean of the differences of the upper bounds and of the lower bounds (34): [(3176.983 − 3178.425) + (3173.546 − 3175.520)]/2 = − 1.71. This fact, and the large overlaps of the two 95% confidence intervals (38) prevent one to conclude definitively on the relative merits of the volatility-confined PL model versus the volatility-confined LPPL model. What is important for our purpose is that both models outperform the Black–Scholes benchmark very significantly.

This looks good, but as stated in the introduction of this Section, one should not over-interpret the results of the Bayesian inference test. Let us indeed point out that a major difficulty with the Bayesian inference test lies with the fact that the prior distribution represents a somewhat arbitrary preconception, which may severely distort the conclusions. This difficulty cannot really be alleviated by trying different priors and by checking the corresponding posteriors, because all posteriors are false due to always distorting choices of a priori distributions of the parameters (see e.g. Bauwens, Lubrano and Richard, 2000 for discussion). We stress that there is a highly non-trivial assumption underlying the Bayesian inference test, namely that the parameters can be considered as random values: random parameters would need in general an ensemble of different sample realizations (or series of experiments), whereas we are interested here in one particular realization (or sample). In a sense, the Bayes approach to hypothesis testing assumes that some kind of ergodicity on a single sample applies and that the sample is of sufficiently large size. But this needs to be tested and it is not a trivial task.

From a different view point, the Bayes approach amounts to smoothing out the likelihoods corresponding to different parameter values by an a priori density. It is a legitimate question to ask why such smoothing may work. When the sample size n tends to infinity, the maximum-likelihood ML-estimates tend to the true values and the likelihood function under the integral in Eq. (32) "cuts out" only a narrow neighborhood of the true values. Thus, the behavior of the a priori density outside of this neighborhood becomes irrelevant, and the Bayes approach tends to the maximum likelihood approach, of course under the condition that the chosen prior would not ascribe zero weight to the true parameter value. However, when the sample size is moderate or small and the number of parameters is not small, the situation becomes more and more uncertain. The likelihoods can have several (even many) local maxima in the present case of log-periodicity (see Jacobsson, 2009). Proponents of the Bayes approach argue that this multiplicity is overcome by integration (smoothing). But, for finite sample size n, the smoothing in the marginal likelihood may be more harmful (in particular under unfortunate choices of the prior): smoothing and its positive effects (suppression or decreasing multiplicity of local peaks) come at the price of a loss of efficiency. It is indeed easy to construct examples in which a model 1 has a larger likelihood for the true parameters than another model 2, but the Bayes construct of the marginal likelihood incorrectly reverses this ranking, due to the distortion induced by the choice of the prior distribution. We believe that this could explain the somewhat better performance of the LP model compared with the LPPL model within the Bayesian inference tests.

Given this, we have nevertheless pursued, if only for the goal of comparing with the negative results of the same procedure applied to the JLS model by (Chang & Feigenbaum, 2006). In conclusion, we find a decisive preference in favor of the PL and LPPL models against the benchmark model, which supports the claim that the super-exponential property of the price constitutes an important characteristic of financial bubbles.

5. Out-of-sample tests of the volatility-confined LPPL model to diagnose other bubbles

We now apply the above described procedure and tests of significance for the LPPL property to different price time series that contain other historical speculative bubbles. Our goal is to test for the validity and universality of the volatility-confined LPPL model.

We proceed in two steps. For each time series to be analyzed, we first calibrate the nonlinear model (1) with Eq. (2) to the logarithm of the price. If the LPPL parameters determined from the fit for a certain period meet the LPPL conditions (23), a speculative bubble is then diagnosed within this period. The volatility-confined LPPL model is then supposed to be applicable. Second, we test the O–U property as well as the order of autoregression of the residuals obtained from the previous calibration step in the same time interval. This is a test of the stationarity of the residual time series.

We consider some of the most important speculative bubbles that have occurred in the world in the last decades. Specifically, we study the following bubbles listed below

- *The bubble in the USA as well as in other European markets that led to a crash at the end of the summer of 1998* (*the so-called Russian crisis*). From its top in mid-June 1998 to its bottom in the beginning of September 1998, the U.S. S&P 500 stock index lost 19% within 3 months. Before this "slow" crash, the stock market in USA enjoyed a long booming period for almost 7.5 years from its historical lowest points around the January of 1991. This crash event is widely associated with and often attributed to the plunge of the Russian financial market, the devaluation of its currency and the default of the government on its debt obligations. But in our views, the Russian event only acts as the triggering factor for the crash rather than its fundamental cause, and omits the evidence that speculative herding had formed over years preceding it. In fact, the long period boom led to Greenspan's famous comments doubting its sustainability, which was made in December 5, 1996, summarized by the well-known quip "irrational exuberance". To identify this bubble in the U.S. market, we test the S&P 500 stock index and choose the first trading day in 1991 as the start of the first window and the last trading day in April, 1998 as the end of the last window. The strong correction starting in mid-August was not specific to U.S. markets. Some other markets exhibited similar or even stronger losses, such as the UK stock market with the FTSE 100 stock index falling by 18.5% between August and the beginning of October, 1998. Before that, the FTSE exhibited a continuing growth over about 4 years. In order to take our bubble testing procedure to the FTSE, we choose June 1st, 1994 as the start of the spanning windows when the UK financial market just began to recover after a strong correction (about 18%) due to the contagion from the turbulence exhibited by many emerging stock markets in early 1994 and amplified by the sharp raise of interest rates in the U.S. The end of windows is chosen at the last trading day in May, 1998.

- *The booming market in Hong Kong in the mid-1990s ending with the crash of October 1997*. Since capital can flow in and out of the Hong Kong stock market almost unhindered and there are no restrictions on the conversion and remittance of dividends and interest, one may expect the speculative behavior and herding effects to be free to express themselves in their full force. Not surprisingly, there are more bubbles and crashes in the Hong Kong market than any other western developed markets. The Hong Kong market crash of October 1997 also has been presented as a textbook example of speculation taking a course of its own: over investment during the boom resulting in an instability, which left the market vulnerable to so-called speculative attacks. From mid-August 1997, the Hang Seng index ended its rapid expansion of the preceding 2.5 years by a slow and regular decay until October 17, 1997, followed by an abrupt crash, with a drop summing up to 33.4% over two weeks. Here, we choose the last trading day in July, 1997 for the end of windows to test for the presence of a bubble. The start of the windows is chosen at the first trading day in 1995, when the index almost reached its lowest value of the bearish period that began with a "slow crash" around early 1994.
- *The ICT bubble reflecting over-optimistic expectation of a new economy ending in the spring of 2000 with a big crash of the NASDAQ index*. The last few years of 1990s witnessed a growing divergence in the stock market between sectors of dotcom and biotech companies representing the "New Economy" and the sectors of companies belonging to the "Old Economy". From 1998 to 2000, the stock index related to the Internet constructed on S&P 500 index rose an astonishing fourteen-fold, while the non-Internet stock price index remained basically flat. The bull market was in synchrony with some interesting views expounded in early 1999 to justify such a rarely seen prosperous increase, including *economics of scale*, *better business models*, *the network effect*, *first-to-scale advantages*, *and real options effect*, among others. They participated in the general optimistic view and added to the strength of the herd, which finally led to an unsustainable accelerating overvaluation. The Nasdaq composite index that consists mainly of stock related to the "New Economy" dropped precipitously from March 10, 2000, with 37% of market capitalization quickly wiped off until April 17, 2000. To test for the presence of this bubble, we choose the last trading day in February, 2000 as the end of the testing windows. Even though the Nasdaq index tripled from 1990 to 1997, its increase was a factor 4 in the three years preceding the crash. This defines an "inflection point" leading us to choose the first trading day in April of 1997 as the start of the testing windows.
- *The U.S. stock market bubble from 2004 to October 2007*. From mid-2003, the loose monetary policy of the Federal Reserve, together with expansive congressional real estate initiatives, triggered a chain of events, specifically characterized by a cascade of bubbles (see Sornette & Cauwels, 2012; Sornette & Woodard, 2010). What came first is the real-estate bubble, paralleled with a boom of structured credit. The "castles in the air" of bubbling house prices promoted a veritable eruption of investments in structured credit instruments. As a consequence, bubble-like amounts of MBS and CDO were put on the market, associated with deteriorating lending standards. The exuberance, catalyzed by loose monetary policies and what can be called an "illusion in a perpetual money machine" (Sornette & Cauwels, 2012) also spilled over to the stock market, starting in 2004. The stock market run up with characteristic super-exponential growth ended in October, 2007, when several consecutive losses announced a year-long market debacle. We choose December 1, 2004 as the first day and July 15, 2007 as the end of the windows scanning the bubble.
- *The Chinese stock market bubble from 2006 to 2007*. After the effective launch of the *Split Share Structure Reform* in the last quarter of 2005, which unfroze about two-third of the shares owned by the state and other specific legal entities and made them tradable by the public, the Chinese stock market began to recover from 5 years of bearish performance. The Shanghai Stock Exchange Composite (SSEC) index and the Shenzhen Stock Component (SZSC) index rose exuberantly six-fold in just two years. This extreme growth was followed by a dramatic drop of about two-third of their peak values attained in October 2007, occurring over half-a-year. The sharp increase of the stock prices was fuelled by compelling growth stories, confirming from all directions great positive outlooks, based on the rapid fundamental growth of the Chinese economy, e.g., its sky-rocketing export surpluses leading to enormous reserves and liquidity, the strength of its currency and the coming Olympic Games offering another instance of an extraordinary expected prosperity. However, the roller-coaster performance of the Chinese market over 2008 suggests that these two years of 2006 and 2007 were characterized by a bubble (Jiang et al., 2010) with herding, over-optimistic, accompanied by a growing tension of the investors' sentiments. We test the bubble in both Chinese stock indices, with the start of the first window chosen as the first trading day in February 2007 and the end of the last window chosen as the last trading day of 2007.

Table 6 displays the parameters obtained from the calibration of the LPPL model to these bubbles. One can verify that the LPPL conditions $B > 0$, $0.1 \le \beta \le 0.9$, $6 \le \omega \le 13$, and $|C| < 1$ are met for these bubbles.[7]

Table 7 gives the results of the O–U test for the residuals obtained from calibrating the nonlinear model (1) with Eq. (2) to the logarithm of each time series. Combining the results of the different unit-root tests, we conclude that all indices except one have their residuals qualifying as generating by a stationary process at the 99.9% confidence level. The exception is the Shenzhen stock component index for which the confidence level to reject the null of non-stationarity is 99%. The estimated coefficient α of auto-regression associated with the O–U process is between 0.02 and 0.06. This range of values corresponds approximately to our choice for the prior distribution of the coefficient α in the Bayesian analysis reported in the previous Section. The last columns of Table 7 list the order of the AR model obtained for the residuals. Two criteria of order selection are tested for robustness. In almost all cases, the two different criteria give the same order equal to 1 for the AR model, with only one exception being the Hang Seng index for which the HQ criterion suggests an AR(3).

The above tests performed on these seven bubbles presented in Tables 6 and 7 suggest that our proposed volatility-confined LPPL model, first tested for the bubble and crash of October 1987, is not just fitting a single "story" but provides a consistent universal description of financial bubbles, namely a super-exponential acceleration of price decorated with log-periodic oscillations with mean-reverting residuals.

The overall picture that emerges from these above historical bubbles is supporting and reinforcing the generic scenario documented by Kindleberger (2005) and Sornette (2004b). By studying many historical bubbles, they found that essentially all of them have developed in five acts, corresponding to (i) a first displacement triggered by a recovery from a previous recession or the emergence of a new technology or a novel opportunity for growth, (ii) a take-off where appreciating stock market prices are accompanied by easier access to credit, (iii) exuberance, (iv) critical stage and (v) crash followed by revulsion. The Johansen–Ledoit–Sornette (JLS) model (Johansen et al., 1999, 2000) embodies these five stages, using an adequate mathematical set-up to account for the positive feedbacks due to imitation and herding entangled with objective measures of success during the bubble ascendency. The JLS model specifies the precise

[7] For the SSEC index, the estimated β is found equal to 0.905, which is barely outside the chosen qualifying interval [0.1, 0.9]. Changing slightly by a few days the time window in which the fit is performed puts back the exponent β within the qualifying interval.

Table 6
Parameters obtained from the calibration of the nonlinear model (1) with (2) to the logarithm of the different price indices named in the first column.

Index	t_{start}	t_{end}	t_c	β	ω	ϕ	B	C
S&P500	Jan-03-91	Apr-30-98	Jul-11-98	0.3795	6.3787	4.3364	0.0833	0.7820
FTSE100	Jun-01-94	May-30-98	Aug-26-98	0.4022	12.1644	0.9409	0.0571	0.8076
HangSeng	Jan-03-95	Jul-31-97	Oct-28-97	0.7443	7.4117	4.9672	0.0042	0.7955
NASDAQ	Apr-01-97	Feb-28-00	May-27-00	0.1724	7.3788	3.2314	1.0134	0.9745
S&P500	Dec-01-04	Jul-15-07	Oct-26-07	0.1811	12.9712	1.5361	0.2419	−0.8884
SSEC	Feb-01-06	Oct-31-07	Jan-23-08	0.9050	7.3538	2.3614	0.0054	−0.6277
SZSC	Feb-01-06	Oct-31-07	Dec-14-07	0.8259	6.3039	6.2832	0.0111	0.7344

fingerprints of such endogenous bubbles, in terms of the "log-periodic power law" (LPPL) dynamics. Our present volatility-confined LPPL model extends the JLS model by providing a framework that is consistent with the calibration of the log-price. The stock market dynamics in the western world ending in the Russian crisis, the Hong Kong bubble ending with the crash of October 1997, the ICT bubble ending in the spring of 2000 with a big crash of the NASDAQ index, the U.S. stock market bubble from 2004 to October 2007 and the Chinese stock market bubble from 2006 to 2007 are well described and accounted for by our formalism. Our calibrations presented in Tables 6 and 7 reinforce the hypothesis that all these crises were the result of endogenous processes fuelled by non-sustainable positive feedbacks, leading in finite-time to a change of regime. This might be surprising or even shocking to some, especially when some crashes are associated with political unraveling circumstances. To continue testing this hypothesis, we need to pursue testing historical well-documented cases as well as develop real-time diagnostic processes, as done in the Financial Crisis Observatory at ETH Zurich (www.er.ethz.ch/fco/).

6. Concluding remarks

We have presented a model of bubbles, termed the volatility-confined LPPL model, to describe and diagnose situations when excessive public expectations of future price increases cause prices to be temporarily elevated.

Our contribution to the literature concerning the detection of bubbles consists in focusing on three characteristics: (i) the faster-than-exponential growth of the price of the asset under consideration represented by a singular power law behavior, (ii) an accelerated succession of transient increases followed by corrections captured by a so-called log-periodic component and (iii) a mean-reverting behavior of the residuals developing around the two first components, which by themselves form the log-periodic power law (LPPL) model.

These three properties have been nicely tied together via a rational-expectation (RE) model of bubbles with combined Wiener and Ornstein–Uhlenbeck innovations describing the dynamics of rational traders coexisting with noise traders driving the crash hazard rate. An alternative model has been proposed in terms of a behavioral specification of the dynamics of the stochastic discount factor describing the overall combined decisions of both rational and noise traders.

The test of the volatility-confined LPPL model has proceeded in two steps. First, we calibrated the nonlinear model (1) with expression (2) to the logarithm of the price time series under study and diagnosed a bubble when the LPPL parameters determined from the fit for a certain period meet the LPPL conditions (23). Second, we tested for the stationarity of the residual time series. Applied extensively to GARCH benchmarks and to eight well-known historical bubbles, we found overall that these bubbles obey the conditions for the volatility-confined LPPL model at a very high confidence level (99.9%) and that the rate of false positives is very low, at about 0.2%. These results suggest that we have identified a consistent universal description of financial bubbles, namely a super-exponential acceleration of price decorated with log-periodic oscillations with mean-reverting residuals.

Further validation will come by testing further on other known bubble cases and in real time. These studies are currently underway and will be reported elsewhere.

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Table 7
Stationarity tests on the residuals of the financial indices obtained from the calibration of the nonlinear model (1) with (2) to the logarithm of the different price indices named in the first column. Tripled stars (***) and double stars (**) respectively denote 0.1% and 1% significance levels to reject the null H_0 that the residual process has a unit root. α is the mean-reverting parameter of the Ornstein–Uhnlenbeck generating process of the residuals. The orders of the AR model for the residuals selected using the Schwarz information criterion (SIC) and the Hannan–Quinn criterion are listed in the last two columns.

Index	t_{start}	t_{end}	Unit-root test		Coefficient α	AR order	
			Phillips–Perron	Dickey–Fuller		SIC	HQ
S&P500	Jan-03-91	Apr-30-98	-4.454^{***}	-4.594^{***}	0.022	1	1
FTSE100	Jun-01-94	May-30-98	-4.731^{***}	-4.893^{***}	0.045	1	1
HangSeng	Jan-03-95	Jul-31-97	-3.756^{***}	-3.482^{***}	0.041	1	3
NASDAQ	Apr-01-97	Feb-28-00	-3.849^{***}	-3.759^{***}	0.037	1	1
S&P500	Dec-01-04	Jul-15-07	-4.000^{***}	-4.229^{***}	0.053	1	1
SSEC	Feb-01-06	Oct-31-07	-3.932^{***}	-3.808^{***}	0.064	1	1
SZSC	Feb-01-06	Oct-31-07	-3.111^{**}	-2.960^{**}	0.041	1	1

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Appendix A. Detailed derivation of expression (8)

When the no-arbitrage condition holds, the product of $\Lambda(t)$ with the asset value $I(t)$ must follow a Martingale process (6) or, rewritten in the following differential form,

$$\mathbb{E}_t\left[d\left(\Lambda\left(t'\right)I\left(t'\right)\right)\right]=0 \quad \forall t'>t. \tag{38}$$

Here and below, we shall use the abbreviated symbols $\mathbb{E}_t(\cdot)$ for $\mathbb{E}^{\mathcal{P}}(\cdot|\mathcal{F}_t)$ to represent the expectation conditional on all current disclosed information corresponding to the σ − algebra $\mathcal{F}_t$. From this condition, we obtain

$$\begin{aligned}
0 &= \mathbb{E}_t\left[\frac{d(\Lambda_{t'}I_{t'})}{\Lambda_{t'}I_{t'}}\right]=\mathbb{E}_t\left[\frac{d\Lambda_{t'}}{\Lambda_{t'}}+\frac{dI_{t'}}{I_{t'}}+\frac{d\Lambda_{t'}}{\Lambda_{t'}}\frac{dI_{t'}}{I_{t'}}\right]\\
&=\{-rdt-\rho_Y\mathbb{E}_t(dY)\}+\left\{\mathbb{E}_t\left(\mu\left(t'\right)\right)dt+\sigma_Y\mathbb{E}_t(dY)-\kappa h\left(t'\right)dt\right\}\\
&\quad-\sum_{i,j=Y,W}\sum\rho_i\sigma_j dt=\mathbb{E}_t\left(\mu\left(t'\right)\right)dt\text{-}rdt-\kappa h\left(t'\right)dt-\sum_{i,j=Y,W}\sum\rho_i\sigma_j dt\\
&\quad+(\sigma_Y-\rho_Y)\mathbb{E}_t(dY)=\mathbb{E}_t\left(\mu\left(t'\right)\right)dt\text{-}rdt-\kappa h\left(t'\right)dt-\sum_{i,j=Y,W}\sum\rho_i\sigma_j dt\\
&\quad+(\sigma_Y-\rho_Y)\left(-\alpha e^{-\alpha(t'-t)}Y_t\right)dt.
\end{aligned} \tag{39}$$

The term $\sum_{i,j=Y,W}\sum\rho_i\sigma_j$ is the required excess return remunerating all risks at the exception of the crash risk associated with the jump of amplitude κ. It is denoted below as $\rho\Sigma$ for short. Then, the above equation leads to

$$E_t\left(\mu\left(t'\right)\right)=(r+\rho\Sigma)+\kappa h\left(t'\right)+\alpha(\sigma_Y-\rho_Y)e^{-\alpha(t'\text{-}t)}Y_t. \tag{40}$$

Since $E_t(Y_{t'})=e^{-\alpha(t'-t)}Y_t$ by construction of the O–U process Y_t and the crash hazard rate $h(t)$ is assumed to be deterministic, the simplest specification for the drift term $\mu(t)$ of the price process (3a), which is compatible with Eq. (40), should be

$$\mu(t)=(r+\rho\Sigma)+\kappa h(t)+\alpha(\sigma_Y-\rho_Y)Y_t. \tag{41}$$

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