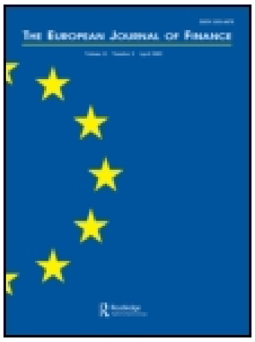

The European Journal of Finance

ISSN: 1351-847X (Print) 1466-4364 (Online) Journal homepage: https://www.tandfonline.com/loi/rejf20

Diagnostics of rational expectation financial bubbles with stochastic mean-reverting termination times

L. Lin & D. Sornette

To cite this article: L. Lin & D. Sornette (2013) Diagnostics of rational expectation financial bubbles with stochastic mean-reverting termination times, The European Journal of Finance, 19:5, 344-365, DOI: 10.1080/1351847X.2011.607004

To link to this article: https://doi.org/10.1080/1351847X.2011.607004

Published online: 19 Sep 2011.

The European Journal of Finance, 2013
Vol. 19, No. 5, 344–365, http://dx.doi.org/10.1080/1351847X.2011.607004

Diagnostics of rational expectation financial bubbles with stochastic mean-reverting termination times

L. Lin[a]* and D. Sornette[b,c]

[a]*School of Economics and Management, Beihang University, 100191 Beijing, China;* [b]*Chair of Entrepreneurial Risks, Department of Management, Technology and Economics, ETH Zurich, Kreuplatz 5, CH-8032 Zurich, Switzerland;* [c]*Swiss Finance Institute, c/o University of Geneva, 40 blvd. Du Pont d'Arve, CH 1211 Geneva 4, Switzerland*

We propose two rational expectation models of transient financial bubbles with heterogeneous arbitrageurs and positive feedbacks leading to self-reinforcing transient stochastic faster-than-exponential price dynamics. As a result of the nonlinear feedbacks, the termination of a bubble is found to be characterized by a finite-time singularity in the bubble price formation process ending at some potential critical time $\tilde{t}_c$, which follows a mean-reverting stationary dynamics. Because of the heterogeneity of the rational agents' expectations, there is a synchronization problem for the optimal exit times determined by these arbitrageurs, which leads to the survival of the bubble almost all the way to its theoretical end time. The explicit exact analytical solutions of the two models provide nonlinear transformations which allow us to develop novel tests for the presence of bubbles in financial time series. Avoiding the difficult problem of parameter estimation of the stochastic differential equation describing the price dynamics, the derived operational procedures allow us to diagnose bubbles that are in the making and to forecast their termination time. The tests have been performed on four financial markets, the US S&P500 index from 1 February 1980 to 31 October 2008, the US NASDAQ Composite index from 1 January 1980 to 31 July 2008, the Hong Kong Hang Seng index from 1 December 1986 to 30 November 2008 and the US Dow Jones Industrial Average Index from 3 January 1920 to 31 December 1931. Our results suggest the feasibility of advance bubble warning using stochastic models that embody the mechanism of positive feedback.

Keywords: bubble; super-exponential regime; rational expectation; critical time; finite-time-singularity

1. Introduction

Bubbles and crashes in financial markets are of global significance because of their effects on the lives and livelihoods of a majority of the world's population. While pundits and experts alike line up after the fact to claim that a particular bubble was obvious in hindsight, the real-time development of the bubble is often characterized by either a deafening silence or a cacophony of contradictory opinions. Here, we propose two models of financial bubbles, from which we develop the corresponding operational procedures to diagnose bubbles that are in the making and to forecast their termination time. The tests performed on four financial markets, the US S&P500 index from 1 February 1980 to 31 October 2008, the US NASDAQ Composite index from 1 January 1980 to 31 July 2008, the Hong Kong Hang Seng index from 1 December 1986

*Corresponding author. Email: elliot@sem.buaa.edu.cn

to 30 November 2008 and the US Dow Jones Industrial Average Index from 3 January 1920 to 31 December 1931, demonstrate the feasibility of advance bubble warning on the major market regimes that were followed by crashes or extended market downturns. The empirical results support the hypothesis that financial bubbles result from positive feedbacks operating on the price and/or on its momentum, leading to faster-than-exponential transients.

These results should be appreciated from the perspective of the present state of art on modeling and detecting bubbles. There is no really satisfactory theory of bubbles, which both encompasses its different possible mechanisms and adheres to reasonable economic principles (no or limited arbitrage, equilibrium or quasi-equilibrium with only transient deviations, bounded rationality). Part of the reason that the literature is still uncertain on even how to define a bubble is that an exponentially growing price can always be argued to result from some fundamental economic factor (Lux and Sornette 2002; Gurkaynak 2008). This is related to the problem that the fundamental price is not directly observable, giving no strong anchor to understand observed prices. Another fundamental difficulty is to go beyond equilibrium to out-of-equilibrium set-ups (Brock 1993; Brock and Hommes 1999; Chiarella, Dieci, and He 2008; Hommes and Wagener 2008).

Two conditions are in general invoked as being necessary for prices to deviate from the fundamental value. First, there must be some degree of irrationality in the market. That is, investors' demand for stocks must be driven by something other than fundamentals, like overconfidence in the future. Second, even if a market has such investors, the general argument is that rational investors will drive prices back to fundamental value. For this *not* to happen, there needs to be some limits on arbitrage. Shleifer and Vishny (1997) provide a description for various limits of arbitrage. With respect to the equity market, clearly the most important impediment to arbitrage are short sales restrictions. Roughly 70% of mutual funds explicitly state (in Securities and Exchange Commission Form N-SAR) that they are not permitted to sell short (Almazan et al. 2004). Seventy-nine percent of equity mutual funds make no use of derivatives whatsoever (either futures or options), suggesting further that funds do not take synthetically short positions (Koski and Pontiff 1999). These figures indicate that the vast majority of funds never take short positions. Then, the argument goes that bubbles can develop because prices reflect mainly the remaining optimistic opinions and not the negative views of pessimistic traders who are already out of the market, and who would take short positions, if given the opportunity.

One important class of theories shows that there can be large movements in asset prices due to the combined effects of heterogeneous beliefs and short-sales constraints. The basic idea finds its root in the original capital asset pricing model theories, in particular, the model of Lintner (1969) of asset prices with investors having heterogeneous beliefs. Lintner and many others after him, show that widely inflated prices can occur (Miller 1977; Harrison and Kreps 1978; Jarrow 1980; Chen, Hong, and Stein 2002; Duffie, Garleanu, and Pedersen 2002; Scheinkman and Xiong 2003). In these models that assume heterogeneous beliefs and short sales restrictions, the asset prices are determined at equilibrium to the extent that they reflect the heterogeneous beliefs about payoffs. But short sales restrictions force the pessimistic investors out of the market, leaving only optimistic investors and thus inflated asset price levels. However, when short sales restrictions no longer bind investors, then prices fall back down. This provides a possible account of the bursting of the Internet bubble that developed in 1998–2000. Many of these models take into account explicitly the relationship between the number of publicly tradable shares of an asset and the propensity for speculative bubbles to form. So far, the theoretical models based on agents with heterogeneous beliefs facing short sales restrictions are considered among the most convincing models to explain the burst of the Internet bubbles.

The role of 'noise traders' in fostering positive feedback trading has been emphasized by a number of models. For instance, DeLong et al. (1990) introduced a model of market bubbles and crashes which exploits this idea of the role of noise traders in the development of bubbles, as a possible mechanism for why asset prices may deviate from the fundamentals over rather long time periods. Their work was followed by a number of behavioral models based on the idea that trend chasing by one class of agents produces momentum in stock prices (Barberis, Shleifer, and Vishny 1998; Daniel, Hirshleifer, and Subrahmanyam 1998; Hong, Kubik, and Stein 2005). An influential empirical evidence on momentum strategies came from the work of Jegadeesh and Titman (1993, 2001), which established that stock returns exhibit momentum behavior at intermediate horizons. Strategies which buy stocks that have performed well in the past and sell stocks that have performed poorly in the past generate significant positive returns over 3- to 12-month holding periods. De Bondt and Thaler (1985) documented long-term reversals in stock returns. Stocks that perform poorly in the past perform better over the next 3– 5 years than stocks that perform well in the past. These findings present a serious challenge to the view that markets are semi-strong-form efficient.

It is important to understand what mechanisms prevent arbitrageurs from removing a bubble as soon as they see one. Abreu and Brunnermeier (2003) have proposed that bubbles continue to grow due to a failure of synchronization of rational traders, so that the later choose to ride rather than arbitrage bubbles. Abreu and Brunnermeier (2003) consider a market where arbitrageurs face synchronization risk and, as a consequence, delay usage of arbitrage opportunities. Rational arbitrageurs are supposed to know that the market will eventually collapse. They know that the bubble will burst as soon as a sufficient number of (rational) traders will sell out. However, the dispersion of rational arbitrageurs' opinions on market timing and the consequent uncertainty on the synchronization of their sell-off are delaying this collapse, allowing the bubble to grow. In this framework, bubbles persist in the short and intermediate terms because short sellers face synchronization risk, that is, uncertainty regarding the timing of the correction. As a result, arbitrageurs who conclude that the arbitrageurs are yet unlikely to trade against the bubble find it optimal to ride the still-growing bubble for a while.

Bhattacharya and Yu (2008) provide a summary of recent efforts to expand on the above concepts, in particular to address the two main questions of (i) the cause(s) of bubbles and crashes and (ii) the possibility to diagnose them *ex ante*. Many financial economists recognize that positive feedbacks and in particular herding is a key factor for the growth of bubbles. Herding can result from a variety of mechanisms, such as anticipation by rational investors of noise traders' strategies (DeLong et al. 1990), agency costs and monetary incentives given to competing fund managers (Dass, Massa, and Patgiri 2008) sometimes leading to the extreme Ponzi schemes, rational imitation in the presence of uncertainty (Roehner and Sornette 2000) and social imitation. The bubble models developed here build strongly on this accepted notion of herding. We refer to Kaizoji and Sornette (2008) for an extensive review complementing this brief survey.

This paper takes its roots in the Johansen–Ledoit–Sornette (JLS) model (Johansen, Sornette, and Ledoit 1999; Johansen, Ledoit, and Sornette 2002) formulated in the Blanchard–Watson framework of rational expectation bubbles (Blanchard 1979; Blanchard and Watson 1982). The JLS model combined a representation of the herding behavior of noise traders controlling a crash hazard rate with the arbitrage response of rational traders on the asset price. One implication of the JLS model is the transient faster-than-exponential acceleration of the price due to the positive feedback associated with the herding behavior of noise traders. This faster-than-exponential pattern can theoretically culminate in a finite-time singularity (FTS), which characterizes the end of the bubble and the time at which the crash is the most probable. Other models have explored further the hypothesis that bubbles can be the result of positive feedbacks and that the dynamical signature of bubbles derives from the interplay between fundamental value investment and more technical analysis. The former can be embodied in nonlinear extensions of the standard financial Black–Scholes model of log-price variations (Corcos et al. 2002; Ide and Sornette 2002; Sornette and Andersen 2002; Andersen and Sornette 2004). The latter requires more significant extensions to account for the competition between inertia between analysis and decisions, positive momentum feedbacks and negative value investment feedbacks (Sornette and Ide 2002). Close to our present formulation, Sornette and Andersen (2002) and Andersen and Sornette (2004) develop a nonlinear generalization of the Black–Scholes process which can be solved analytically. The nonlinear feedback is acting as the effect of price on future growth, according to the view that high prices lead to a wealth effect that drives behavioral investors to invest more aggressively.

The present paper adds to the literature by developing two related models of transient bubbles in which their terminations occur at some potential critical time $\tilde{t}_c$, which follows a mean-reverting stationary process with a fixed unconditional mean T_c. These models provide straightforward ways to determine the potential critical time without confronting the difficult problem of parameter estimation of the stochastic differential equation describing the dynamics of price. In our models, rational arbitrageurs can diagnose bubbles but do not know precisely when they end. These investors are assumed to form rational expectations of the potential critical time but not necessarily of the detailed price process itself, which form a much weaker condition for investors' rationality. Furthermore, we assume that rational arbitrageurs hold consistent expectation for the potential critical time with unbiased errors. Although rational arbitrageurs know that the bubble will burst at its critical time, they cannot make a deterministic prediction of this time and therefore of when other arbitrageurs will sell out, because they have little knowledge about others' belief about the process governing the stochastic critical time $\tilde{t}_c$. Our rational investors continuously update their beliefs on the probable termination of the bubble, according to their observation of the development of the bubble. They exit the market by maximizing their expected payoff, based on their subjective perception of the market bubble risk and the knowledge of the bubble dynamics. Because of the heterogeneity of these rational agents' expectations, there is a synchronization problem between these arbitrageurs, which leads to the survival of the bubble almost all the way to its theoretical end time. In this respect, our model is reminiscent to that of Abreu and Brunnermeier (2003), since the resilience of the bubble results from the lack of synchronization between arbitrageurs on the decision to exit the bubble due to the heterogeneity of their optimal exit times. Our models have the advantage of being quantitatively testable and their concrete implementation provides diagnostics of bubbles in real time series, as we demonstrate below.

The first model, which leads to an FTS in the price dynamics with stochastic critical time is presented in Section 2. This model generalizes Sornette and Andersen's model to allow for a mean reversal dynamics of the bubble end. Section 3 presents a second model leading to a FTS in the momentum price dynamics with stochastic critical time. Both models exemplify the importance of a positive feedback, which is quantified by a unique exponent m. A value of m larger than 1 (respectively, 2) characterize a bubble regime in the first (respectively, second) model. The two models can be solved exactly in explicit analytical forms. These solutions provide nonlinear transformations which allow us to develop novel tests for the presence of bubbles in financial time series. These two classes of tests, one for each bubble model, are developed, respectively, in Sections 4.1 and 4.2 and applied on four financial markets, the US S&P500 index from 1 February 1980 to 31 October 2008, the US NASDAQ Composite index from 1 January 1980 to 31 July 2008, the Hong Kong Hang Seng index from 1 December 1986 to 30 November 2008 and the

US Dow Jones Industrial Average Index from 3 January 1920 to 31 December 1931. Section 5 concludes.

2. First bubble model: FTS in the price dynamics with stochastic critical time

Financial bubbles are often viewed as being characterized by anomalously high growth rates resulting from temporary over-optimistic beliefs in a 'new economy' or in a 'paradigm shift' of the fundamental structure of productivity gains. However, this definition is unsatisfactory because a high growth rate associated with an exponentially growing price can always be justified by some other fundamental valuation models which use higher discount factors and larger dividend growth expectations, or introduce new accounting rules incorporating, for instance, the benefits of real options. This definition also leaves a large ambiguity as to when the bubble is supposed to end, or when a crash might occur.

In contrast, we define a bubble as a transient faster-than-exponential growth of the price, which would end in a FTS in the absence of a crash or change of regime. Such 'super-exponential regime' results from the existence of positive nonlinear feedback mechanisms amplifying past price increases into even faster growth rates. These positive nonlinear feedback mechanisms may be due to a variety of causes, including derivative hedging strategies, portfolio insurance methods or to imitative behaviors of bounded rational arbitrageurs and of noise traders. While herding has largely been documented to be a trait of noise traders, it is actually rational for bounded rational agents to also enter into social imitation, as the collective behavior may reveal information otherwise hidden to the agents. As a result of the nonlinear positive feedbacks, the bubble price becomes less and less coupled to the market fundamentals, and the super-exponential growth of the price makes the market more and more unstable. In this scenario, the end of the bubble conditional in the absence of a crash occurs at a critical time at which the market becomes maximally unstable. The end of the bubble is therefore the time when the crash is the most probable. With or without a crash, the end of the bubble signals the end of the transient super-exponential growth, and the transition to a different regime, with unspecified characteristics.

Here, we assume that sophisticated market participants are indeed aware of the current bubble state, and that they know the price is growing towards its final singularity that will occur at some future random critical time, before which the market may collapse with a finite probability. It should be stressed that a bubble can never persist until the underlying critical time since the infinite profit that would result within a finite time is not compatible with the assumption of a finite money supply. Our bounded rational sophisticated arbitrageurs are able to form unbiased rational expectations of the critical time corresponding to the end of the bubble at which the crash is the most probable. We assume that these sophisticated arbitrageurs enter sequentially into the market, attracted by the potential large gains, given their anticipation of the crash risk quantified by their estimation of the critical end time of the bubble which is formed when they enter the market. Because their anticipations of the bubble demise are heterogeneous, they solve an optimal timing problem with distinct inputs, which leads to different exit strategies. The heterogeneity in their exit strategies is common knowledge among these arbitrageurs, and results in a lack of coordination, ensuring the persistence of the bubble. This *synchronization problem* is analogous to that identified by Abreu and Brunnermeier (2003). However, for Abreu and Brunnermeier, the lack of synchronization stems from the existence of heterogeneous beliefs on the start of the bubble, i.e. arbitrageurs have 'sequential awareness' and do not know whether they have learnt the information on the mispricing early or late relative to other rational arbitrageurs. In contrast, our model emphasizes that the lack of synchronization results from the heterogeneous beliefs concerning future price variation either before or after the end of the bubble. Many reports both in the academic and professional literature state that sophisticated participants like hedge-funds correctly diagnosed the presence of a bubble and actually 'surfed' the bubbles, attracted by the potential large gains. Many reported that the largest uncertainty was how long it would continue its course (Gurkaynak 2008; Sullivan 2009)

Less sophisticated traders investing in the market have little knowledge on the bubble duration and their action add noise which is assumed to have an influence only on the critical time characterizing the end of the bubble, while the super-exponential growth of the price remains robust. In the bubble regime, a well-defined nonlinear exponent characterizes the positive feedbacks at the origin of the bubble. The noisy character of the critical time t_c of the end of the bubble is modeled by an Ornstein–Uhlenbeck process. Intuitively, the price trail resembles the trace of a bug climbing erratically along a hanging curved rope attached to a vibrating support. Such asset price dynamics during the development of a bubble has an explicit and self-consistent economic implication. This robust super-exponential growth of the price that terminates with the FTS allows the sophisticated agents among the population of investors to form a consensus on the termination time of the bubble, which is nothing but the underlying singular point. On the other hand, it is precisely the formation of the consensus that sustains the super-exponential growth regime during the bubble. Whenever a sophisticated arbitrageur does market timing according to the consensus, he will involve himself into the synchronization problem, which supports the bubble and its persistent dynamics.

Specifically, the price dynamics in the bubble regime is assumed to be described by the following stochastic differential equation:

$$\mathrm{d}p = \mu p^m (1 + \theta(p,t))\,\mathrm{d}t + \sigma p^m\,\mathrm{d}W, \tag{1}$$

where the exponent $m > 1$ embodies the positive feedback mechanism, in which a high price p pushes even further the demand so that the return and its volatility tend to be a nonlinear accelerating function of p. When $m > 1$, we will show that the price diverges in finite time. The time at which this divergence occurs is referred to as the critical time $\tilde{t}_c$. As we will see later, the term $\theta(x,t)$ is a time-varying regulator term that governs the behavior of $\tilde{t}_c$, μ is the instantaneous return rate, σ is the volatility of the returns and W is the standard Wiener process. This model recovers the standard Black–Scholes model of the geometric random walk with drift μ and standard deviation σ for $m = 1$ and $\theta = 0$.

Let us consider first the case where $\theta(p,t) = \sigma = 0$, so that expression (1) reduces to $\mathrm{d}p = \mu p^m\,\mathrm{d}t$, whose solution is

$$p = K(t_c - t)^{-\beta}, \tag{2}$$

where $\beta = 1/(m-1)$, $K = (\beta/\mu)^\beta$, $t_c = p_0^{-(m-1)}/(m-1)\mu$ and p_0 denotes the price at the start time of the bubble taken to be $t = 0$. Since $\beta > 0$ for $m > 1$, expression (2) exemplifies the existence of a finite-time (or 'movable', Bender and Orszag 1999) singularity of the price that goes to infinity in finite time as $t \to t_c^-$. This pathological behavior is the direct consequence of the positive feedback embodied in the condition $m > 1$.

Motivated by this simple analytical solution, we now consider the more general process (1) and carefully specify $\theta(p,t)$ in order to obtain a price process with stochastic finite-time singularities and an explicit analytical expression. We postulate the following specific form for the process governing $\theta(p,t)$,

$$\theta(p,t) = \alpha \tilde{t}_c(t) + \tfrac{1}{2} m\mu\sigma^2 [p(t)]^{m-1}, \tag{3}$$

where

$$\mathrm{d}\tilde{t}_{\mathrm{c}} = -\alpha \tilde{t}_{\mathrm{c}}\,\mathrm{d}t + \left(\frac{\sigma}{\mu}\right)\mathrm{d}W \tag{4}$$

follows an Ornstein–Uhlenbeck process with zero unconditional mean. The Wiener process in Equation (4) is the same as the one in Equation (1). We obtain the following result.

PROPOSITION 1 *Provided that $\theta(p,t)$ follows the process* (3) *with* (4), *the solution of Equation* (1) *can be written under a form similar to Equation* (2) *as follows*:

$$p(t) = K(\tilde{T}_{\mathrm{c}} - t)^{-\beta}, \tag{5}$$

with

$$\beta = \frac{1}{m-1}, \quad K = \left(\frac{\beta}{\mu}\right)^{\beta}, \quad T_{\mathrm{c}} = \frac{\beta}{\mu} p_0^{-1/\beta}, \quad \tilde{T}_{\mathrm{c}} = T_{\mathrm{c}} + \tilde{t}_{\mathrm{c}}. \tag{6}$$

The proof of Proposition (1) is given in Appendix 1.

Note that

$$\mathbb{E}(\tilde{T}_{\mathrm{c}}) = \mathbb{E}(T_{\mathrm{c}} + \tilde{t}_{\mathrm{c}}) = T_{\mathrm{c}}. \tag{7}$$

Thus, T_{c} given by the third term in Equation (6) is the expected end time of the bubble. This expected end time T_{c} is the rational expectation consensus on the bubble termination formed by sophisticated arbitrageurs who are observing the stochastic critical time $\tilde{T}_{\mathrm{c}}$ of the bubble demise. The stochastic behavior of $\widetilde{T_{\mathrm{c}}}$ expresses that the end of the bubble cannot be known with certainty but is instead a stochastic variable. Each trading day t reveals a different 'actual' critical time $T_{\mathrm{c}}(t) = T_{\mathrm{c}} + t_{\mathrm{c}}(t)$, which is vibrating around its consensus expected value T_{c}. Actually, Equation (7) precisely reflects the rational expectation financial bubble concept that the financial market as a whole can determine an unbiased estimation of the bubble termination time of bubble. The presence of rational expectation requires that the dynamics of the stochastic $\widetilde{T_{\mathrm{c}}}$ should be a mean-reverting stationary process, which is indeed the case as seen from the last term in Equation (6) in which $\tilde{t}_{\mathrm{c}}(t)$ is an Ornstein–Uhlenbeck process (4). Thus, our model is consistent in the sense of rational expectation of the bubble termination time. This is different from the rational expectation of the price formed by each rational arbitrageur in standard rational expectation bubble models.

Our model assumes that there is no coordination mechanism that would ensure the exchange of information among the sophisticated arbitrageurs concerning their expectation of the end of the bubble. The arbitrageurs reveal their private information only upon entering the market. We assume that they do so sequentially, based on their heterogeneous beliefs on the process $\tilde{t}_{\mathrm{c}}$. And the process $\tilde{t}_{\mathrm{c}}$ is an emergent property that results from the aggregation of market beliefs rather than from the action of single arbitrageur. Being aware of the escalating level of the bubble that has not yet burst, each arbitrageur will ride the bubble for a while and identify the best exit strategy according to the maximization of her risk-adjusted return based on her belief.

Let us denote $t_i > 0$ the time at which the i's arbitrageur has entered the market. Being aware of the form (5) of the price dynamics, at each instant t, the rational arbitrageur is aware of her exposition to the risk that a crash might occur in the next instant, conditional on the fact that it has not yet happened, which is quantified by the crash hazard rate $h(t)$. Suppose $1 - \Pi(t)$ denote the probability that the crash will not happen until time t. Given the explosive form (5) of the price dynamics, we assume that the arbitrageur is guessing that the crash hazard rate should be of the same form, that is,

$$h(t) = \frac{\pi(t)}{1 - \Pi(t)} \propto (T_c - t)^{-\beta}, \tag{8}$$

where $\Pi(t)$ is the conditional cumulative distribution function of the bursting date and $\pi(t)$ represents the associated conditional density. T_c denotes the rational expectation consensus on the end of the bubble for each arbitrageur who has entered the market. The exponent β reflects the integrated degree of positive feedback contributed by all traders in the market. The second relationship for Equation (8) embodies that all sophisticated arbitrageurs are fully aware that a bubble is present and that the market may undergo a critical transition at the end of the bubble, embodied by the accelerated power law divergence of the market susceptibility at the critical time, which is then translated into the perception of the singular behavior of the crash hazard rate as formed by the arbitrageurs.

The occurrence of the market collapse is posited to be triggered when a sufficiently large number η of arbitrageurs have exited the market, leading to a large price movement, amplified by the herding of noise traders. Their cumulative effect is accounted for by a postulated percent loss κ associated with the crash, which itself is a random variable. Given such an environment, the date to exit the market for a given rational arbitrageur i determines her best trade strategy according to a risk benefit–cost analysis: the optimal exit time for arbitrager i is the time when her marginal expected gain becomes zero. When this occurs, perceived crash loss risks become larger than expected gains. This condition reads

$$\mathbb{E}^i[(1 - \Pi(t)) \cdot \mathrm{d}p - \pi(t)\,\mathrm{d}t \cdot \kappa p] = 0. \tag{9}$$

The first term $(1 - \Pi(t)) \cdot \mathrm{d}p$ represents each arbitrageur's instantaneous marginal benefit from t to $t + \mathrm{d}t$ provided that the burst of the bubble has not yet happened. The second term $\pi(t)\mathrm{d}t \cdot \kappa p$ is the instantaneous marginal cost each arbitrageur will have to bear when the bubble bursts. After eliminating the term $\mathrm{d}t$, the condition (9) reads

$$(1 - \Pi(t))\mathbb{E}^i(\mu p^m(1 + \theta(p, t))) = \pi(t)\mathbb{E}^i(\kappa p). \tag{10}$$

We can thus state the following:

PROPOSITION 2 *Given a population of heterogeneous arbitrageurs,* (i) *who have different beliefs on the future price variation but* (ii) *who form a consensus on the anticipated critical time T_c and* (iii) *who perceive the crash hazard rate to obey a singular behavior given by expression* (8), *a given arbitrageur i decides to exit the market at the date t_i^{ex} which is the solution of*

$$\frac{\mathbb{E}^i[\mathrm{d}p(\tilde{t}_c, t_i^{\mathrm{ex}})]}{\mathbb{E}^i[\kappa p(\tilde{t}_c, t_i^{\mathrm{ex}})]} = h(t_i^{\mathrm{ex}}) \propto (T_c - t_i^{\mathrm{ex}})^{-\beta}. \tag{11}$$

Although for arbitrary i and j, $\mathbb{E}^i(\tilde{t}_c) = \mathbb{E}^j(\tilde{t}_c) \equiv 0$, *since* $\mathbb{E}^i(\cdot) \neq \mathbb{E}^j(\cdot)$, *we have* $\mathbb{E}^i[dp(\tilde{t}_c, t)] \neq \mathbb{E}^j[\mathrm{d}p(\tilde{t}_c, t)]$ *and* $\mathbb{E}^i[\kappa p(\tilde{t}_c, t)] \neq \mathbb{E}^j[\kappa p(\tilde{t}_c, t)]$. *This naturally implies* $t_i^{\mathrm{ex}} \neq t_j^{\mathrm{ex}}$. *Notwithstanding the fact that the presence of the bubble is common knowledge among all rational arbitrageurs, the absence of synchronization of their market exit allows the bubble to persist and run its course up to a time close to its expected value* (7).

For the price process (1) with (3), Equation (11) yields

$$\frac{\mathbb{E}^i(\mu p^m(1+\alpha\tilde{t}_c(t_i^{ex}) + (1/2\mu)m\sigma^2[p(t_i^{ex})]^{m-1}))}{\mathbb{E}^i(\kappa p)} = h(t_i^{ex}) \propto (T_c^* - t_i^{ex})^{-\beta}. \tag{12}$$

This *synchronization problem* is analogous to that identified by Abreu and Brunnermeier (2003), with the important difference that we emphasize that the lack of synchronization results from the heterogeneous beliefs concerning the future price variation.

The corresponding observable logarithmical return for the asset price reads

$$\begin{aligned} r_\tau(t) = \ln p(t+\tau) - \ln p(t) &= -\beta \ln\left(\frac{T_c + \tilde{t}_c(t+\tau) - (t+\tau)}{T_c + \tilde{t}_c(t) - t}\right) \\ &= -\beta \ln\left(1 + \frac{\Delta\tilde{t}_c(t) - \tau}{T_c + \tilde{t}_c(t) - t}\right), \end{aligned} \tag{13}$$

where τ is the time interval between two observations of the price. In the case where T_c is large enough such that $T_c + \tilde{t}_c(t) - t \gg \Delta\tilde{t}_c(t) - \tau$, expression (13) can be approximated by its first-order Taylor expansion:

$$r_\tau(t) = \frac{\beta}{T_c + \tilde{t}_c(t) - t}(\tau - \Delta\tilde{t}_c(t)) = \frac{1}{(m-1)(T_c + \tilde{t}_c(t) - t)}(\tau - \Delta\tilde{t}_c(t)). \tag{14}$$

Therefore, under the condition $m \to 1$ and $T_c + \tilde{t}_c(t) - t \to \infty$, the logarithmical return $r_\tau(t)$ is driven by the change $\Delta\tilde{t}_c(t)$ of the critical time on each trading day. Although $\tilde{t}_c$ follows an Ornstein–Uhlenbeck process, i.e. $\Delta\tilde{t}_c(t) = -\alpha\tilde{t}_c(t) + \varepsilon_t$, the existence of correlation between successive returns will be hardly detectable if α is small enough. Conversely, if $\tilde{t}_c$ follows a unit root process, the logarithmical return $r_\tau(t)$ is only dependent on the realization of the gaussian noise term ε. In this sense, only if T_c is not too far in the future, m is sufficiently larger than 1 and $\tilde{t}_c$ is stationary, can we diagnose the existence of a bubble, characterized by the emergence of a transient super-exponential price growth of the form (5).

3. Second bubble model: FTS in the momentum price dynamics with stochastic critical time

The price process (5) of the first bubble model might appear extreme, in the sense that the price diverges on the approach of the critical time $\tilde{T}_c$ of the end of the bubble. However, such a divergence cannot run its full course in our model due to the divergence of the crash hazard rate which ensures that a crash will occur before. The critical time is thus a ghost-like time, which is out-of-reach, and the price process (5) describes a transient run-up that would diverge only in the hypothetical absence of any arbitrageur. Here, we consider an alternative model in which the price remains always finite but the faster-than-exponential growth associated with the bubble is embodied into the price momentum, i.e. the derivative of the logarithm of the price or logarithmic return.

Defining $y(t) = \ln p(t)$, we assume the following process for $y(t)$:

$$\mathrm{d}y = x(1+\gamma(x,t))\,\mathrm{d}t + \left(\frac{\sigma}{\mu}\right) x\,\mathrm{d}W, \tag{15}$$

$$\mathrm{d}x = \mu x^m(1+\theta(x,t))\,\mathrm{d}t + \sigma x^m\,\mathrm{d}W, \tag{16}$$

where the same Wiener process $W(t)$ acts on both dy and dx. The process $x(t)$ plays the role of an effective price momentum. To see this, consider the special case $\gamma(x,t) = 0$. Then, expression (15) reduces to $\mathrm{d}y = x\mathrm{d}t + (\sigma/\mu)x\mathrm{d}W$, which shows that $x(t)\,\mathrm{d}t = \mathbb{E}[\mathrm{d}y]$ and thus $x(t)$ is the average momentum of the price, defined as the instantaneous time derivative of the expected logarithm of the price. The dynamics of the log-price described by Equation (15) with Equation (16) is similar to previous models (Bouchaud and Cont 1998; Farmer 2002; Ide and Sornette 2002; Sornette and Ide 2002), which argued for the presence of some inertia in the price formation process. This inertia is related to the momentum effect (Jegadeesh and Titman 1993, 2001; Carhart 1997; Xue 2003; Cooper, Gutierrez, and Hameed 2004). Intuitively, a price process involving both dy and dx holds if the price variation from today to tomorrow is based in part on decisions using analyses of the price change between yesterday (and possibly earlier times) and today.

In the (unrealistic) deterministic limit $\gamma(x,t) = \theta(x,t) = \sigma = 0$, the two equations (15) and (16) reduce to the deterministic equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \mu \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^m, \tag{17}$$

whose solution reads, for $m > 2$,

$$y(t) = A - B(T_\mathrm{c} - t)^{1-\beta}, \tag{18}$$

where $\beta = 1/(m-1)$, $T_\mathrm{c} = (\beta/\mu)(\mathrm{d}p/\mathrm{d}t|_{t=t_0})^{-1/\beta}$, $B = (1/(1-\beta))(\mu/\beta)^{-\beta}$ and $A = p(T_\mathrm{c})$. The condition $m > 2$ ensures that $0 < 1-\beta < 1$. Therefore, the log-price $y(t)$ exhibits a FTS at T_c. But this FTS is of a different type than in the model of the previous section: here, $y(t)$ remains finite at $t = T_\mathrm{c}$ and equal to some value $A = p(T_\mathrm{c})$. The singularity is expressed via the divergence of the momentum $x(t) = \mathrm{d}y/\mathrm{d}t$ which diverges at $t = T_\mathrm{c}$. As in the previous model, this FTS embodies the positive feedback mechanism, in which a high price momentum x pushes even further the demand so that the return and its volatility tend to be nonlinear accelerating functions of x. In the previous model, it is the price that provides a feedback on further price moves, rather than the price momentum used here.

Motivated by this simple analytical solution (18), we complement the general process (15) and (16) by specifying $\gamma(x,t)$ and $\theta(p,t)$ in order to obtain analytical solutions with stochastic finite-time singularities in the momentum with finite prices. We postulate the following specific processes:

$$\gamma(x,t) = \alpha\tilde{t}_\mathrm{c}(t) + \frac{\sigma^2}{2\mu}[x(t)]^{m-1}, \tag{19}$$

$$\theta(x,t) = \alpha\tilde{t}_\mathrm{c}(t) + \tfrac{1}{2}m\mu\sigma^2[x(t)]^{m-1}, \tag{20}$$

where

$$\mathrm{d}\tilde{t}_\mathrm{c} = -\alpha\tilde{t}_\mathrm{c}\mathrm{d}t + \left(\frac{\sigma}{\mu}\right)\mathrm{d}W \tag{21}$$

follows an Ornstein–Uhlenbeck process with zero unconditional mean. The Wiener process in Equation (21) is the same as the one in Equations (15) and (16), which reflects that the same series of news or shocks move log-price, momentum and anticipated critical time. We obtain the following result.

Proposition 3 *Provided that $\gamma(x,t)$ and $\theta(p,t)$ follow the processes given, respectively, by Equations* (19) *and* (20), *then the solution of Equations* (15) *and* (16) *for the log-price $y(t) =$*

$\ln p(t)$ *can be written under a form similar to expression* (18) *as follows*:

$$y(t) = A - B(T_c + \tilde{t}_c(t) - t)^{1-\beta}, \tag{22}$$

where

$$\beta = \frac{1}{m-1}, \quad T_c = \frac{\beta}{\mu} x_0^{1/\beta}, \quad x_0 := x(t=0), \quad B = \frac{1}{1-\beta}\left(\frac{\beta}{\mu}\right)^{\beta}, \tag{23}$$

and A is a constant.

The proof of Proposition 3 is given in Appendix 2.

Expression (22) describes a log-price trajectory exhibiting a FTS occurring at an unknown future critical time $T_c + \tilde{t}_c(t)$ which itself follows an Ornstein–Uhlenbeck walk. In the same manner as in the previous model of the preceding section, we assume that there is no coordination mechanism that would ensure the exchange of information among the sophisticated arbitrageurs concerning their expectation of the time $T_c + \tilde{t}_c(t)$ of the end of the bubble. Following step by step the same reasoning as in the previous section, we conclude that Proposition 2 also holds for the present model. Being aware of the escalating level of the bubble that has not yet burst, each arbitrageur will ride the bubble for a while and identify the best exit strategy according to the maximization of her risk-adjusted return based on her belief. Proposition 2 determines that the exit time t_i^{ex} for arbitrageur i is the solution of

$$\frac{\mathbb{E}^i[p(t_i^{\mathrm{ex}})x(t_i^{\mathrm{ex}})(1+\gamma(x(t_i^{\mathrm{ex}}), t_i^{\mathrm{ex}})) + (\sigma^2/2\mu^2)p(t_i^{\mathrm{ex}})[x(t_i^{\mathrm{ex}})]^2]}{\mathbb{E}^i[\kappa p(t_i^{\mathrm{ex}})]} = h(t_i^{\mathrm{ex}}) \propto (T_c - t_i^{\mathrm{ex}})^{-\beta}. \tag{24}$$

The observable logarithmic return for the asset price corresponding to Equation (22) reads

$$r_\tau(t) = y(t+\tau) - y(t) = -B[(T_c + \tilde{t}_c(t+\tau) - (t+\tau))^{1-\beta} - (T_c + \tilde{t}_c(t) - t)^{1-\beta}] \tag{25}$$

$$= -B(T_c + \tilde{t}_c(t) - t)^{1-\beta}\left[\left(1 + \frac{\Delta\tilde{t}_c(t) - \tau}{T_c + \tilde{t}_c(t) - t}\right)^{1-\beta} - 1\right]. \tag{26}$$

For future potential critical times T_c sufficiently far away from the present time t such that $T_c + \tilde{t}_c(t) - t \gg \Delta\tilde{t}_c(t) - \tau$, expression (26) can be simplified into

$$r_\tau(t) = -\frac{(1-\beta)B}{(T_c + \tilde{t}_c(t) - t)^{\beta}}(\Delta\tilde{t}_c(t) - \tau) = \frac{\mu^{-\beta}}{[(m-1)(T_c + \tilde{t}_c(t) - t)]^{\beta}} \cdot (\tau - \Delta\tilde{t}_c(t)). \tag{27}$$

Equation (27) has a structure similar to that of Equation (14). For a weak positive feedback of the momentum on itself ($m \to 2^+$) and when T_c is large enough so that $\mu(T_c + \tilde{t}_c(t) - t)$ is slowly varying, then the logarithmical return r_s is essentially driven by $\Delta\tilde{t}_c(t)$, i.e. the change of critical time disclosed by every trading day is the main stochastic process. The Geometric Brownian Motion (GBM) is then recovered as an approximation in this limit when the correlation of the Orstein–Uhlenbeck process driving the critical time goes to zero.

4. Empirical test of the tow bubble models

We have proposed two models in which a financial bubble is characterized by a transient faster-than-exponential growth culminating into a FTS at some potential critical time. Because both models reduce to a standard GBM in appropriate limits, the diagnostic of a bubble lies in the conjunction of three pieces of evidence that characterize specific deviations from the GBM regime: (i) the reconstructed time series of the critical time $\tilde{t}_c$ should be stationary and thus reject a standard unit-root test; (ii) the critical exponent m should be significant larger than 1, a condition for the existence of the super-exponential regime proposed to characterize bubbles; (iii) the calibrated potential critical date T_c should be close to the end of the time window in which the calibration of the models are made.

4.1 *Construction of alarms from the first model*

Given a financial time series of close prices at the daily scale, our purpose is to develop a procedure using the model of Section 2 to diagnose the presence of bubbles. Suppose we use time windows of N trading days that we slide with a time step of Δ days from the beginning to the end of the available financial time series. The number of such windows is therefore equal to the total number of trading days in the financial time series minus N and divided by Δ. For each window, the purpose is to decide if the model of Section 2 diagnoses an on-going bubble or not and then to compare with the actual subsequent realization of bubble burst that we consider as the validation step.

For each observed window $[t_i - N, t_i]$ ending at t_i, we transform the price time series in that window into a critical time series by inverting expression (5) for $\tilde{T}_{c,i}(t)$:

$$\tilde{T}_{c,i}(t) = \exp\left[\frac{\ln K - \ln p(t)}{\beta}\right] + t, \quad t = t_i - N + 1, \ldots, t_i. \tag{28}$$

The critical time series $\tilde{T}_{c,i}(t)$ is defined over the window i ending at t_i. If the model was exact and no stochastic component was present, and in absence of estimation errors, $\tilde{T}_{c,i}(t)$ would be a constant equal to T_c defined in (6). In the presence of an expected strong stochastic component, we estimate $T_{c,i}$ according to (7) as the arithmetical average of $\tilde{T}_{c,i}(t)$

$$T_{c,i} = \frac{1}{N}\sum_{t=t_i-N+1}^{t_i} \tilde{T}_{c,i}(t). \tag{29}$$

We can then construct $\tilde{t}_{c,i}(t)$ as

$$\tilde{t}_{c,i}(t) = \tilde{T}_{c,i}(t) - T_{c,i}. \tag{30}$$

The transformation (28) from a non-stationary possibly explosive price process $p(t)$ into what should be a stationary time series $\tilde{T}_{c,i}(t)$ in absence of misspecification is a key element of our methodology for bubble detection that avoids the problems documented by Granger and Newbold (1974) and Phillips (1986) resulting from direct calibration of price or log-price time series.

It will not have escaped the attention of the reader that transformation (28) requires the knowledge of the two unknown parameters K and β that specify the bubble process (5). We propose to calibrate these two parameters by applying an optimization procedure as follows, which will be implemented through adopting Taboo search (Cvijomic and Klinowski 1995). Recall that a crucial ingredient of the bubble model is the mean reversal nature of the underlying critical time $\tilde{t}_{c,i}$. This suggest us to apply a unit root test on the reconstructed time series $t_{c,i}(t)$, which has no linear trend by construction, and determine the optimal values K^* and β^* as those that minimize the false positive rate to reject non-stationarity of the time series $\tilde{t}_c$ mistakenly.

We proceed in two steps. We first search in the space of the two parameters $\ln K$ and β and select an elite list of the 10 best candidate pairs $(\ln K, \beta)$ (when they exist) that maximize the probability for an absence of linear trend in the corresponding $\tilde{t}_{c,i}$ time series. This procedure is implemented with the t-test statistic for the coefficient γ_1 in the following regression:

$$t_{c,i} = \gamma_0 + \gamma_1 t, \quad t = t_i - N + 1, \ldots, t_i. \tag{31}$$

The smaller the absolute value of the t-test statistic for γ_1, the larger the probability to accept the existence of a regression towards the mean. Then, for those selected pairs obtained from the first step, we choose the pair $(\ln K^*, \beta^*)$ with the smallest t-test statistic for the Dickey–Fuller unit root test (without intercept) for the corresponding time series $\tilde{t}_{c,i}$. We supplement this procedure with a variance stability test performed on this selected time series $\tilde{t}_{c,i}$ to verify its stationary in the structural sense. This is realized by the F-test at the 90% significance level for the homogeneity of variance between the second half of the $\tilde{t}_{c,i}$ time series and the whole $\tilde{t}_{c,i}$ time series. If we cannot reject the homogeneity of variance, the obtained optimal pair $(\ln K^*, \beta^*)$ from step 2 should be regarded as the calibrated parameter for the model that best 'fit' the data, in the sense of providing the closest approximation to a stationary time series for $\tilde{t}_{c,i}$ given by expression (30), which characterizes the underlying termination of bubbles. From $(\ln K^*, \beta^*)$, the exponent m in Equation (1) can be determined from the two first formula in Equation (6). Recall that the larger m is above 1, the faster is the final growth of the price towards the critical point.

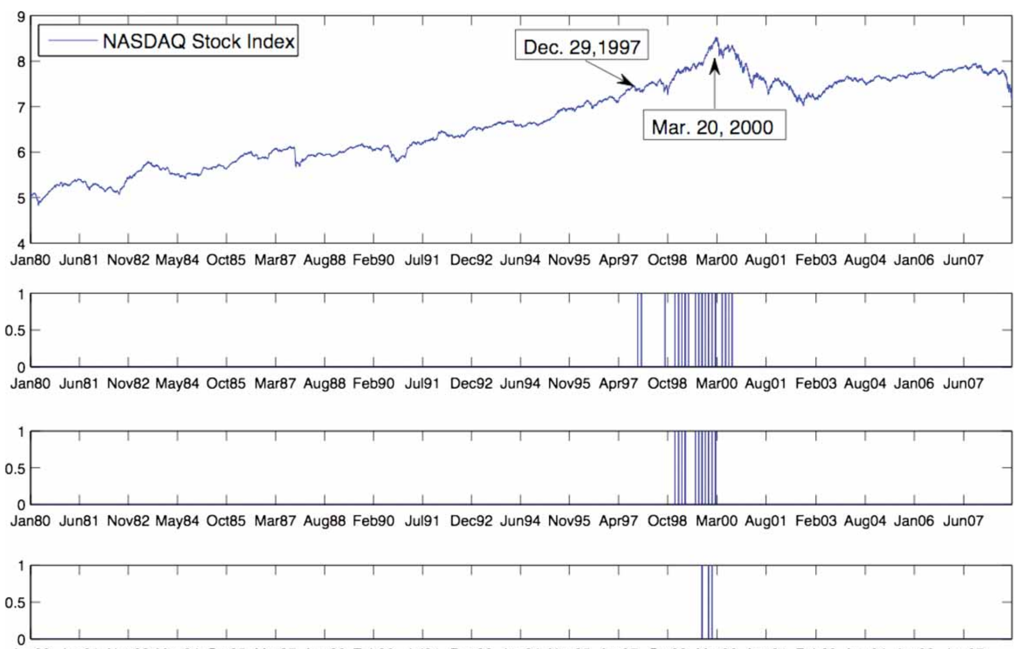

Figure 1. Logarithm of the historical NASDAQ Composite index and corresponding alarms shown in the three lower panels as vertical lines indicating the ends of the windows of 750 trading days in which our procedure using the first bubble model of Section 2 flags a diagnostic for the presence of bubble. The three lower panels correspond to alarms for which $T_{c,i} - t_i < 750$, $T_{c,i} - t_i < 500$ and $T_{c,i} - t_i < 250$, from top to bottom. By definition, the set of alarms of the lowest panel is included in the set of alarms of the middle panel which is itself included in the set of alarms of the upper panel. The exponents m found for the upper panel corresponding to $T_{c,i} - t_i < 750$ have a mean of 2.30 with a standard deviation of 0.46.

For a given window i, a diagnostic for the presence of bubble is flagged and an alarm is declared when

(i) The time series $\tilde{t}_{c,i}$ associated with the optimal pairs $(\ln K^*, \beta^*)$ rejects the unit-root test at the 95% significance level,
(ii) $\beta^* > 0$ such that $m > 1$ (signature of a positive feedback on price) and
(iii) the distance $S_{c,i} \equiv T_{c,i} - t_i$ to the critical point is smaller than 750 days, i.e. the estimated critical time of bubble is not too distant in the future. When condition (i) is not satisfied, $S_{c,i}$ is actually a spurious critical time distance, even if it is small.

Figures 1–4 depict all the bubble alarms obtained by applying this procedure to four major stock indices, the US S&P500 index from 1 February 1980 to 31 October 2008, the US NASDAQ Composite index from 1 January 1980 to 31 July 2008, the Hong Kong Hang Seng index from 1 December 1986 to 30 November 2008 and the US Dow Jones Industrial Average Index from 3 January 1920 to 31 December 1931.[1]. An alarm is indicated by a vertical line positioned on the last day t_i of the corresponding window that passes the three criteria (i)–(iii). We refine the diagnostic by presenting three alarm levels, corresponding, respectively, to $S_{c,i} < 750$, $S_{c,i} < 500$ and $S_{c,i} < 250$: the closer the estimated termination of the bubble, the stronger should be the evidence for the bubble as a faster-than-exponential growth. Another indication is the existence of clustering of the alarms. If indeed a bubble is developing, it should be diagnosed repeatedly by several successive windows.

Our bubble diagnostic procedure is novel. It is thus instructive to apply it to time series generated by a known benchmark process without bubbles but that possesses the other main stylized facts

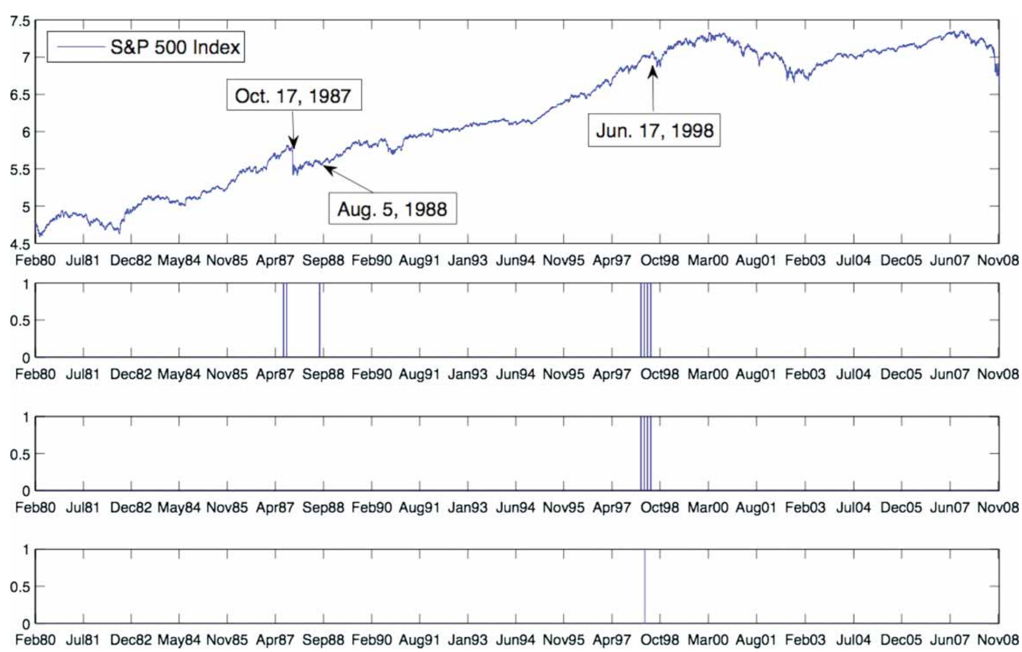

Figure 2. Same as Figure 1 for the S&P500 stock index. The exponents m found for the upper panel corresponding to $T_{c,i} - t_i < 750$ have a mean of 2.38 with a standard deviation of 0.32.

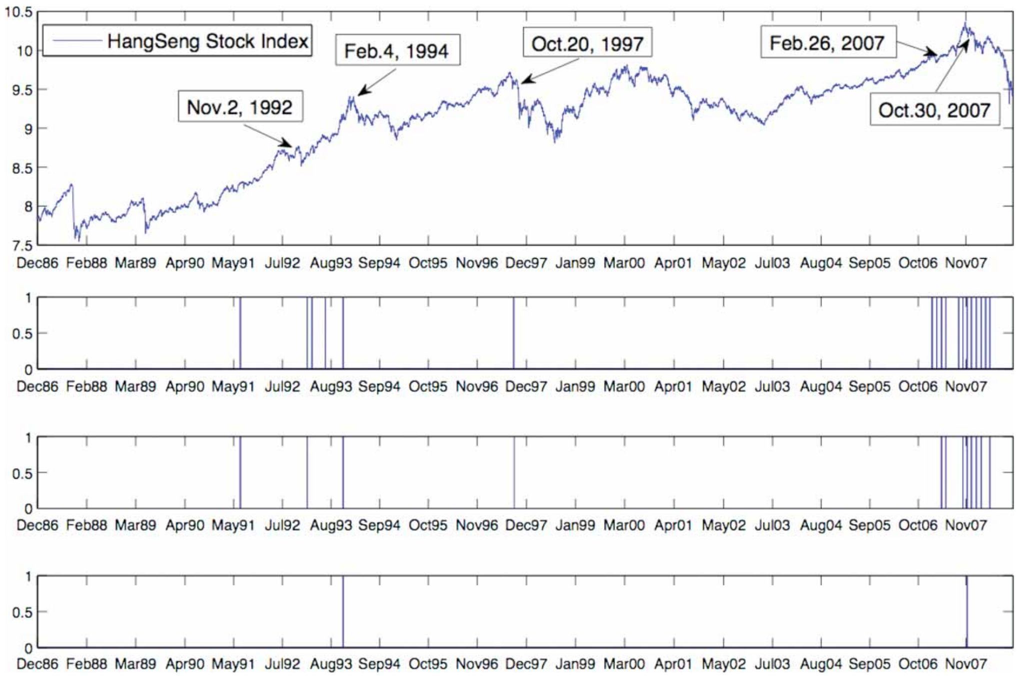

Figure 3. Same as Figure 1 for the Heng Seng index of Hong Kong and corresponding alarms shown in the three lower panels as vertical lines indicating the ends of the windows of 500 trading days. The exponents m found for the upper panel corresponding to $T_{c,i} - t_i < 750$ have a mean of 2.54 with a standard deviation of 0.37.

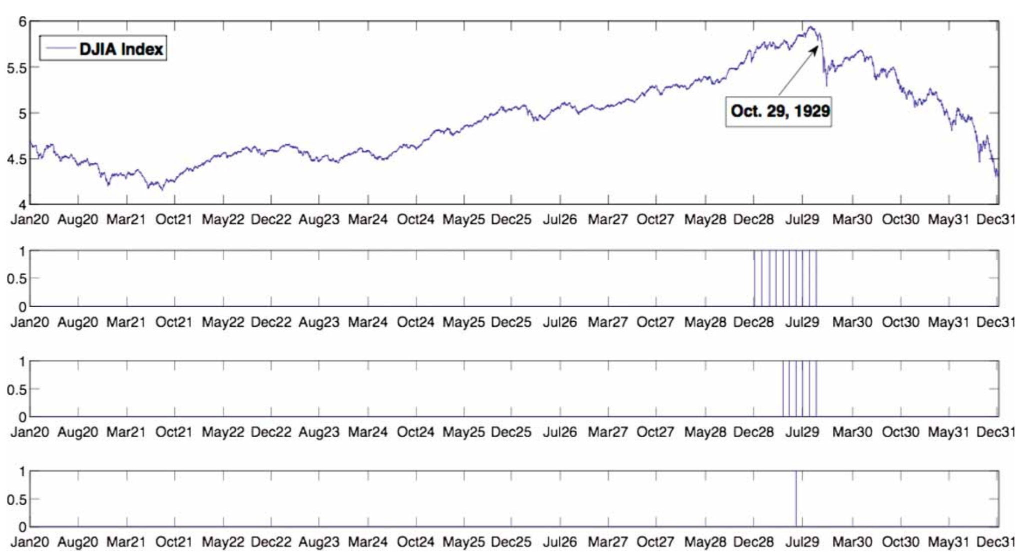

Figure 4. Same as Figure 1 for the Dow Jones Industrial Average index and corresponding alarms shown in the three lower panels as vertical lines indicating the ends of the windows of 750 trading days. The exponents m found for the upper panel corresponding to $T_{c,i} - t_i < 750$ have a mean of 2.35 with a standard deviation of 0.23.

characterizing financial time series. This provides a useful information of the false alarm rate (for the diagnostic of bubbles) for such a benchmark. We thus generated 1000 GARCH(1,1) time series with parameters estimated from the S&P500 index from 3 January 1985 to 30 September 1987 at the daily time scale. Out of these 1000 benchmark time series, we find only 43 among them that fulfill our criteria for the triggering of a bubble alarm with $S_c < 750$. This implies a probability of errors of type I error smaller than 5% for the bubble ending with 'Black Monday' shown in Figure 2. Out of 1000 GARCH-generated time series which parameters fitting the S&P500 index from 28 April 1995 to 16 April 1998, we find that only 22 of them meet our criteria for $S_c < 250$, which again indicates that the alarms flagged in 1998 have also a rate of false positives smaller that 5%.

4.2 *Construction of alarms from the second model*

Similarly to the procedure described in Section 4.1, we transform a given price time series in a given window i of N successive trading days into what should be a stationary time series of underlying critical times, supposing that the bubble price series is indeed described by the bubble model of Section 3 in which positive feedback is implemented through the momentum instead of

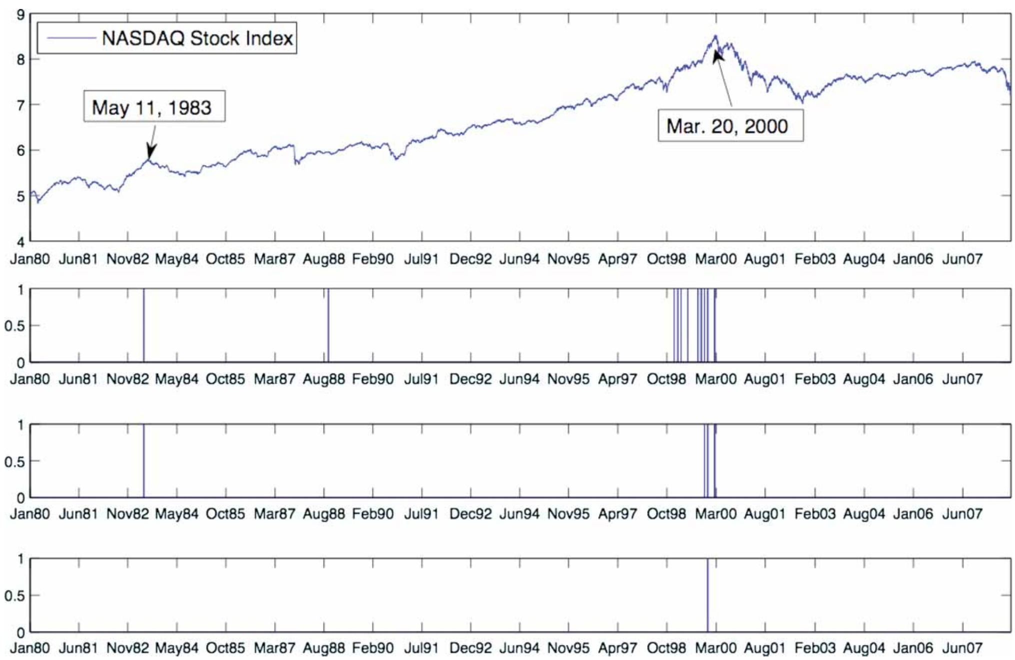

Figure 5. Logarithm of the historical NASDAQ Composite index and corresponding alarms shown in the three lower panels as vertical lines indicating the ends of the windows of 750 trading days in which our procedure using the second bubble model of Section 3 flags a diagnostic for the presence of bubble. The three lower panels correspond to alarms for which $T_{c,i} - t_i < 250$, $T_{c,i} - t_i < 90$ and $T_{c,i} - t_i < 50$, from top to bottom. from top to bottom. By definition, the set of alarms of the lowest panel is included in the set of alarms of the middle panel which is itself included in the set of alarms of the upper panel.The exponent m corresponding to the alarm in the lowest panel is 2.30.

the price. Inverting expression (22), we get, similarly to expression (28),

$$\tilde{T}_{c,i}(t) = \left(\frac{A - \ln p(t)}{B}\right)^{1/(1-\beta)} + t, \quad t = t_i - N + 1, \ldots, t_N. \tag{32}$$

The critical time series $\tilde{T}_{c,i}(t)$ is defined within window i ending at t_i. If the model was exact and no stochastic component was present, and in absence of estimation errors, $\tilde{T}_{c,i}(t)$ would be a constant equal to T_c defined in Equation (23). The expected potential critical time $T_{c,i}$ of a bubble, if any, is then estimated by expression (29) and the fluctuations around $T_{c,i}$ are described by $\tilde{t}_{c,i}(t)$ defined by Equation (30).

As for the first bubble model, the transformation (32) requires the determination of parameters in advance, here the triplet (A, B, β). For this, we proceed exactly as in the previous subsection, with t-test for the regression coefficient γ_1 in Equation (31) applied to the time series $\tilde{t}_{c,i}(t)$, followed by the selection of the best triplet (A^*, B^*, β^*) that minimize the t-test statistic of Dickey–Fuller unit-root test to $\tilde{t}_{c,i}(t)$ to make it as stationary as possible. The search of the additional parameter A is performed in an interval bound from above by $2 * \max_{t_1 \le t \le t_N} \ln p(t)$. Then, for a given window i, a diagnostic for the presence of bubble is flagged and an alarm is declared when

(i) The time series $\tilde{t}_c$ resulting from the optimal triplet (A^*, B^*, β^*) can reject the unit-root test at the 95% significance level;
(ii) $0 < \beta^* < 1$ such that $m > 2$ (the signature of a positive feedback in the momentum price dynamics model) and
(iii) $S_{c,i} = T_{c,i} - t_i < 250$, i.e. the estimated distance to the critical time ought to be small enough.

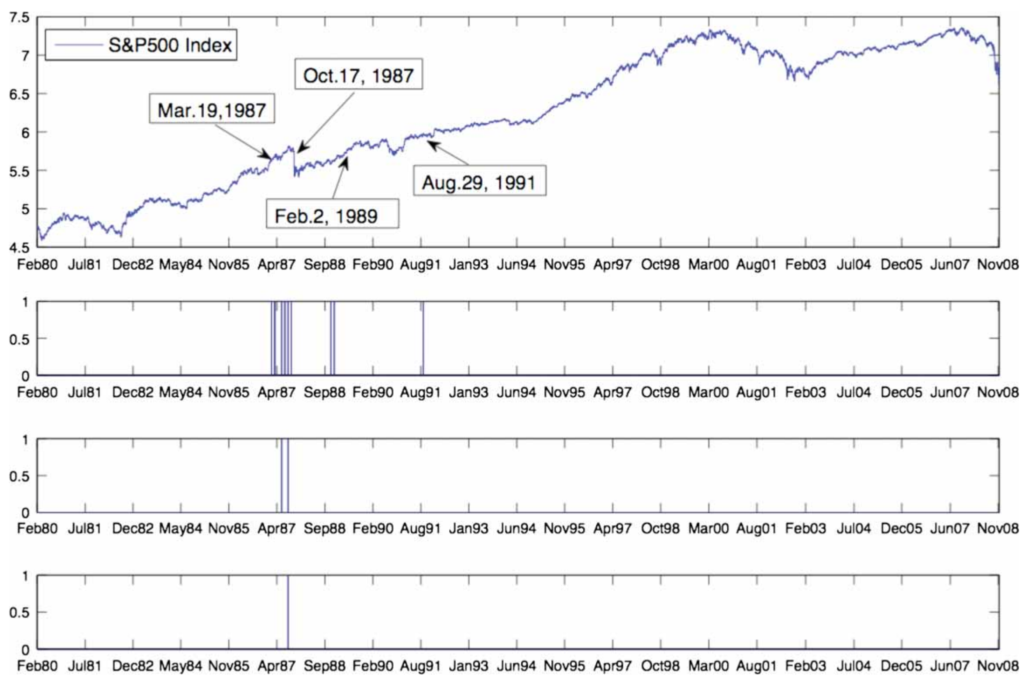

Figure 6. Same as Figure 5 for the S&P500 stock index. The exponent m corresponding to the alarm in the lowest panel is 2.59.

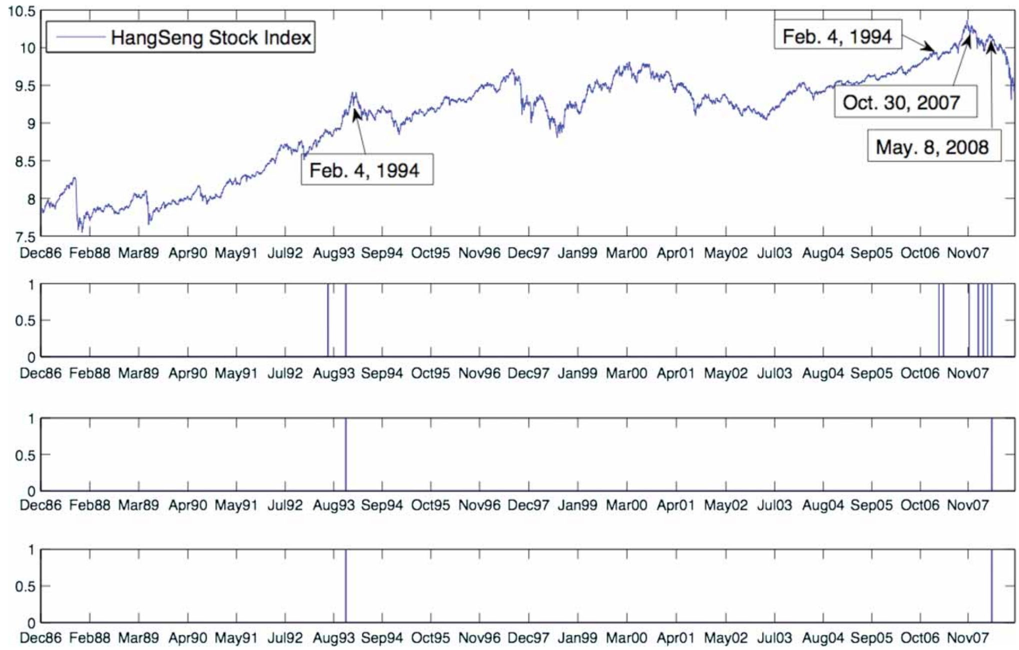

Figure 7. Same as Figure 5 for the Heng Seng index of Hong Kong and corresponding alarms shown in the three lower panels as vertical lines indicating the ends of the windows of 500 trading days. The exponents *m* corresponding to the two alarms in the lowest panel are, respectively, 2.23 and 2.13.

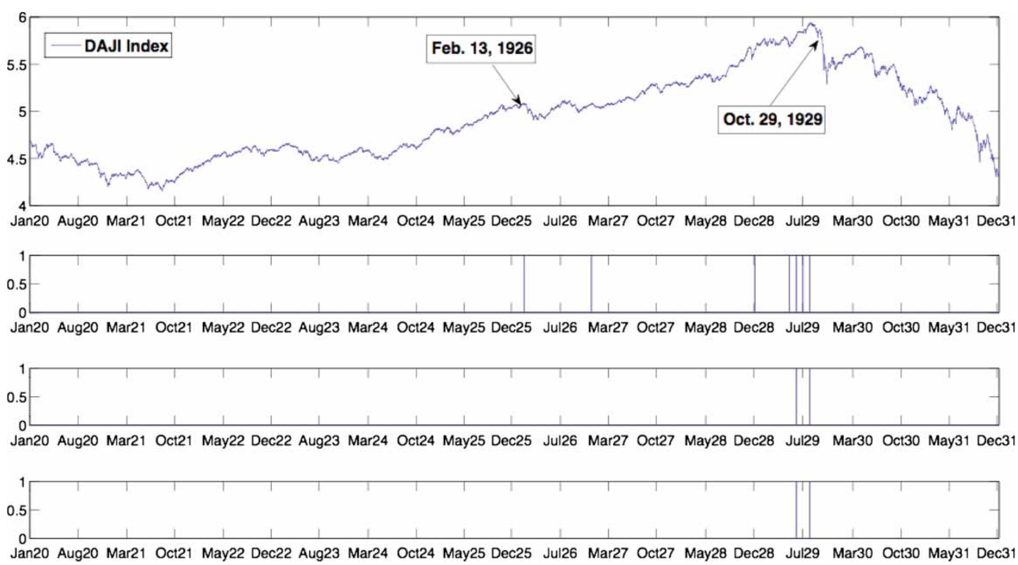

Figure 8. Same as Figure 5 for the S&P500 stock index. The exponents *m* corresponding to the two alarms in the lowest panel are, respectively, 3.74 and 3.06.

Figures 5–8, respectively, depict all the bubble alarms obtained by applying the above procedure to the same four major stock indices studied in the previous subsection. The lengths of the sliding windows are the same as for the first model, that is, $N = 750$ for US stock index and $N = 500$ for Hong Kong stock index. Similarly, we consider three refinement levels for the bubble diagnostic, with $S_{c,i} < 250$, $S_{c,i} < 90$ and $S_{c,i} < 50$. Note that the condition $S_{c,i} < 50$ is much more stringent than its counterpart, i.e. for the shorter level $S_{c,i} < 250$ in the first bubble model. This is due to the fact that a FTS occurring in the price momentum is a weaker singularity that a FTS occurring in the price itself. The signal for a bubble is expected to be weaker, and only detected quite close to the critical time.

We also perform tests with GARCH-generated synthetic time series in order to estimate the false positive rate for alarms with the second model. Out of 1000 synthetic time series generated with the GARCH(1,1) model fitted to the NASDAQ index from 17 March 1997 to 6 March 2000, only 35 series (respectively, 29 series) generated alarms with $S_{c,i} < 90$ ($S_{c,i} < 50$) (and passing conditions (i) and (ii). We thus conclude that the rate of false positives is smaller that 5%.

5. Concluding remarks

We have developed two rational expectation models of financial bubbles with heterogeneous rational arbitrageurs. Two key ingredients characterize these models: (i) the existence of a positive feedback quantified by a nonlinear power-law dependence of price growth as a function of either price or momentum; (ii) the stochastic mean-reversion dynamics of the termination time of the bubble. The first model characterizes a bubble as a faster-than-exponential accelerating stochastic price ending in a FTS at a stochastic critical time. The second model views a bubble as a regime characterized by an accelerating momentum ending at a FTS, also with at a stochastic critical time. This second model has the additional feature of taking into account the existence of some inertia in the price formation process, which is related to the momentum effect.

In these two models, the heterogeneous arbitrageurs exhibit distinct perception for the rising risk of a crash as the bubble develops. Each arbitrageur is assumed to know the price formation process and to determine her exit time so as to maximize her expected gain. The resulting distribution of exit times lead to a synchronization problem, preventing arbitraging of the bubble and allowing it to continue its course up to close to its potential critical time.

The explicit analytical solutions of the two models allow us to propose nonlinear transformations of the price time series into stochastic critical time series. The qualification of a bubble regime then boils down to characterize the nature of the transformed stochastic critical time series, thereby avoiding the difficult problem of parameter estimation of the stochastic differential equation describing the price dynamics. We develop an operational procedure that qualifies the existence of a running bubble (i) if the critical time series is found to reject a standard unit-root test at a high confidence level, (ii) if the exponent m of the nonlinear power law characterizing the positive feedback is sufficient large and (iii) if the expected critical time is not too distant from the time of the analysis.

The two procedures derived from the two bubble models have been applied to four financial markets, the US S&P500 index from 1 February 1980 to 31 October 2008, the US NASDAQ Composite index from 1 January 1980 to 31 July 2008, the Hong Kong Hang Seng index from 1 December 1986 to 30 November 2008 and the US Dow Jones Industrial Average Index from 3 January 1920 to 31 December 1931. Specificall, we have developed criteria to flag an alarm for the presence of a bubble, that we validate by determining if the diagnosed bubble is followed by a crash in short order. Remarkably, we find that the major known crashes over these periods are correctly identified with few false alarms. The method using the second bubble model in terms of an FTS of the price momentum seems to be more reliable with fewer false alarms and a better detection of the two principal bubbles phases characterizing the last 30 years or so. This is further confirmed by the application of our method to the bubble developing in the 1920s that ended in the well-known crash on 'Black Tuesday' in 1929. In this later case, we find again that model 2 provides a refinement and more precise diagnostic of the impending bubble end. Except for one false alarm in 1927, the alarms generated by model 2 give two strongest alarms ($t_c - t_{end} < 50$ days) for the bubble end. Compared with model 1, model 2 improves the precision of warning since it generates the strongest alarm about 2 months before the big crash (generated in 26 August 1929) while the most important alarms ($t_c - t_{end} < 250$ days) of model 1 are generated about 3 months (in 27 July 1929) before the big crash.

These results suggest the feasibility of advance bubble warning using stochastic models that embody the mechanism of positive feedback.

Note

1. We choose sliding windows of length $N = 750$ for the US stock market and $N = 500$ for the Hong Kong stock market. The reduced size for the Hong Kong market is required due to the larger frequency of bubbles found in that market, so as to avoid overlaps of bubbles in a same window that would confuse our procedure.

References

Abreu, D., and M. K. Brunnermeier. 2003. Bubbles and crashes. *Econometrica* 71, no. 1: 173–204.

Almazan, A., K.C. Brown, M. Carlson, and D. A. Chapman. 2004. Why constrain your mutual fund manager? *Journal of Financial Economics* 73, 289–321.

Andersen, J. V., and D. Sornette. 2004. Fearless versus fearful speculative financial bubbles. *Physica A* 337, nos. 3–4: 565–85.

Barberis, N., A. Shleifer, and R. Vishny. 1998. A model of investor sentiment. *Journal of Financial Economics* 49, no. 3: 307–43.

Bender, C., and S. Orszag. 1999. *Advanced mathematical methods for scientists and engineers: Asymptotic methods and perturbation theory*. Heidelberg: Springer.

Bhattacharya, U., and X. Yu. 2008. The causes and consequences of recent financial market bubbles: An introduction. *Review of Financial Studies* 21, no. 1: 3–10.

Blanchard, O. J. 1979. Speculative bubbles, crashes and rational expectations. *Economics Letters* 3, 387–389.

Blanchard, O., and M. Watson. 1982. Bubbles, rational expectations and speculative markets. In *Crisis in economic and financial structure: Bubbles, Bursts and Shocks*, ed. P. Wachtel, 295–316. Lexington: Heath and Company.

Bouchaud, J.-P., and R. Cont. 1998. A Langevin approach to stock market fluctuations and crashes. *European Physical Journal B* 6, 543–50.

Brock, W. 1993. Pathways to randomness in the economy: emergent nonlinearity and chaos in economics and finance. *Estudios Economicos* 8, no. 1: 3–55.

Brock, W., and C. Hommes. 1999. Rational animal spirits. In *The theory of markets*, ed. P.J.J. Herings and G. van der Laan, 109–137. Amsterdam: North-Holland.

Carhart, M. 1997. On persistence in mutual fund performance. *The Journal of Finance* 52, no. 1: 57–82.

Chen, J., H. Hong, and J.C. Stein. 2002. Breadth of ownership and stock returns. *Journal of Financial Economics* 66: 171–205.

Chiarella, C., R. Dieci, and X.-Z. He. 2008. Heterogeneity, market mechanisms, and asset price dynamics Research Paper 231, Quantitative Finance Research Center, University of Technology Sydney, September 2008.

Cooper, M. J., R. Gutierrez, and A. Hameed. 2004. Market states and momentum. *The Journal of Finance* 59, no. 3: 1345–65.

Corcos, A., J.-P. Eckmann, A. Malaspinas, Y. Malevergne, and D. Sornette. 2002. Imitation and contrarian behavior: hyperbolic bubbles, crashes and chaos. *Quantitative Finance* 2: 264–81.

Cvijomic, D., and J. Klinowski. 1995. Taboo search: An approach to the multiple minima problem. *Science* 267: 664–66.

Daniel, K., D. Hirshleifer, and A. Subrahmanyam. 1998. Investor psychology and security market under- and overreactions. *Journal of Finance* 53, 1839–85.

Dass, N., M. Massa, and R. Patgiri. 2008. Mutual funds and bubbles: The surprising role of contractual incentives. *Review of Financial Studies* 21, no. 1: 51–99.

De Bondt, W. M., and R. Thaler. 1985. Does the stock market overreact? *Journal of Finance* 40: 793–805.

DeLong, J. B., A. Shleifer, L.H. Summers, and R.J. Waldmann. 1990. Noise trader risk in financial markets. *Journal of Political Economy* 98, no. 4: 703–38.

Duffie, D., N. Garleanu, and L.H. Pedersen. 2002. Securities lending, shorting, and pricing. *Journal of Financial Economics* 66: 307–39.

Farmer, J. 2002. Market force, ecology and evolution. *Industrial and Corporate Change* 11, no. 5: 895–953.

Granger, C., and P. Newbold. 1974. Spurious regressions in econometrics. *Journal of Econometrics* 2: 111–20.

Gurkaynak, R. S. 2008. Econometric tests of asset price bubbles: taking stock. *Journal of Economic Surveys* 22, no. 1: 166–86.

Harrison, M., and D. Kreps. 1978. Speculative investor behavior in a stock market with heterogeneous expectations. *Quarterly Journal of Economics* 92: 323–36.

Hommes, C., and T. Wagener. 2008. Complex evolutionary systems in behavioral finance. In *Handbook of financial markets: Dynamics and evolution* (Tinbergen Institute Discussion Paper TI 2008-054/1, CeNDEF, School of Economics, University of Amsterdam), eds. T. Hens and K. Schenk-Hoppe. New York: Academic Press.

Hong, H., J.D. Kubik, and J.C. Stein. 2005. Thy neighbor's portfolio: Word-of-mouth effects in the holdings and trades of money managers. *Journal of Finance*, 2801–24.

Ide, K., and D. Sornette. 2002. Oscillatory finite-time singularities in finance, population and rupture. *Physica A* 307, nos. 1–2: 63–106.

Jarrow, R. 1980. Heterogeneous expectations, restrictions on short sales, and equilibrium asset prices. *Journal of Finance* 35: 1105–13.

Jegadeesh, N., and S. Titman. 1993. Returns to buying winners and selling losers: Implications for stock market efficiency. *The Journal of Finance* 48, no. 1: 65–91.

Jegadeesh, N., and S. Titman. 2001. Profitability of momentum strategies: An evaluation of alternative explanations. *Journal of Finance* 54: 699–720.

Johansen, A., O. Ledoit, and D. Sornette. 2000. Crashes as critical points. *International Journal of Teoretical and Applied Finance* 3: 219–55.

Johansen, A., D. Sornette, and O. Ledoit. 1999. Predicting financial crashes using discrete scale invariance. *Journal of Risk* 1, no. 4: 5–32.

Kaizoji, T., and D. Sornette. 2008. Market bubbles and crashes. *The Encyclopedia of Quantitative Finance* (http://arXiv.org/abs/0812.2449). New York: Wiley.

Koski, J., and J. Pontiff. 1999. How are derivatives used? Evidence from the mutual fund industry. *Journal of Finance* 54, no. 2: 791–816.

Lintner, J., 1969. The aggregation of investors' diverse judgments and preferences in purely competitive security markets. *Journal of Financial and Quantitative Analysis* 4: 347–400.

Lux, T., and D. Sornette. 2002. On rational bubbles and fat tails. *Journal of Money, Credit and Banking, Part 1* 34, no. 2: 589–610.

Miller, E. 1977. Risk, uncertainty and divergence of opinion. *Journal of Finance* 32: 1151–68.

Phillips, P. 1986. Understanding spurious regressions in econometrics. *Journal of Econometrics* 31: 311–40.

Roehner, B. M., and D. Sornette. 2000. "Thermometers" of speculative frenzy. *The European Physical Journal B-Condensed Matter* 16, no. 4: 729–39.

Scheinkman, J., and W. Xiong. 2003. Overconfidence and speculative bubbles. *Journal of Political Economy* 111: 1183–219.

Shleifer, A., and R. Vishny. 1997. Limits of arbitrage. *Journal of Finance* 52: 35–55.

Sornette, D., and J.V. Andersen. 2002. A nonlinear super-exponential rational model of speculative financial bubbles. *International Journal of Modern Physics C* 13, no. 2: 171–88.

Sornette, D., and K. Ide. 2002. Theory of self-similar oscillatory finite-time singularities in finance, population and rupture. *International Journal of Modern Physics C* 14, no. 3: 267–75.

Sullivan, R. Summer 2009. Taming global village risk ii: Understanding and mitigating bubbles. *Journal of Portfolio Management* 35, no. 4: 131–41.

Xue, H. 2003. Identifying factors within the apt: A new approach. AFA 2004 San Diego Meetings, Available at SSRN: http://ssrn.com/abstract=462561 or DOI: 10.2139/ssrn.462561.

Appendix 1. Proof of Proposition 1

Proof Let $n = m - 1$ and $z = p^{-n}$. Employing Itö lemma for Equation (1), we have

$$\mathrm{d}z = \frac{\partial z}{\partial p}\mathrm{d}p + \frac{1}{2}\frac{\partial^2 p}{\partial p^2}(\mathrm{d}p)^2 \tag{A1}$$

$$= -np^{-m}(\mu p^m[1+\theta(p,t)]\,\mathrm{d}t + \sigma p^m\,\mathrm{d}W) - \frac{1}{2}n(-n-1)p^{-m-1}\sigma^2 p^{2m}\,\mathrm{d}t \tag{A2}$$

$$= -n\mu(1+\theta(p,t) - \frac{1}{2}(n+1)\sigma^2\mu p^{m-1})\,\mathrm{d}t + n\sigma\,\mathrm{d}W. \tag{A3}$$

Recalling that $\theta(p,t) = \alpha\tilde{t}_c + \frac{1}{2}m\mu\sigma^2 p^{m-1}$, the previous expression can be simplified into

$$\mathrm{d}z = -n\mu\mathrm{d}t + n\mu\left(-\alpha\tilde{t}_c\mathrm{d}t + \frac{\sigma}{\mu}\mathrm{d}W\right) \tag{A4}$$

$$= -n\mu\,\mathrm{d}t + n\mu\,\mathrm{d}\tilde{t}_c. \tag{A5}$$

Integrating both sides of the above equation and with $x(t=0) = p_0^{-n}$, we obtain

$$p = (n\mu)^{-1/n}[T_c + \tilde{t}_c - t]^{-1/n}, \quad T_c = \frac{p_0^{-n}}{n\mu}. \tag{A6}$$

Replacing n by $1/\beta$, this reproduces the solution (5).

Appendix 2. Proof of Proposition 3

Proof We now check that Equation (22) is the solution of the SDEs (15) and (16). For this, we apply Itö lemma on equation (22) by regarding $\ln p_t$ as a function of $\tilde{t}_c$. This leads to

$$\mathrm{d}\ln p = \frac{\partial \ln p}{\partial t}\,\mathrm{d}t + \frac{\partial \ln p}{\partial \tilde{t}_c}\,\mathrm{d}\tilde{t}_c + \frac{1}{2}\frac{\partial^2 \ln p}{\partial \tilde{t}_c^2}(\mathrm{d}\tilde{t}_c)^2 \tag{A7}$$

$$= (1-\beta)B(T_c + \tilde{t}_c - t)^{-\beta}\,\mathrm{d}t - (1-\beta)B(T_c + \tilde{t}_c - t)^{-\beta}\,\mathrm{d}\tilde{t}_c \tag{A8}$$

$$+ \frac{1}{2}\beta(1-\beta)B(T_c + \tilde{t}_c - t)^{-\beta-1}\left(\frac{\sigma}{\mu}\right)^2\,\mathrm{d}t. \tag{A9}$$

Taking into account that $B = (1/(1-\beta))(\beta/\mu)^\beta$ and let Z represent $(\beta/\mu)^\beta[T_c + \tilde{t}_c - t]^{-\beta}$, the above expression can be rewritten as

$$\mathrm{d}\ln p = Z\left[1 + \alpha\tilde{t}_c + \frac{1}{2}\left(\frac{\sigma^2}{\mu}\right)Z^{1/\beta}\right]\mathrm{d}t + Z\left(\frac{\sigma}{\mu}\right)\mathrm{d}W. \tag{A10}$$

On the other hand, it is easy to see that Z is the solution of (16) in the light of Proposition 1, which leads to $Z = x$. Furthermore, we note that $\alpha\tilde{t}_c + (\sigma^2/2\mu)Z^{1/\beta}$ is nothing but $\gamma(Z,t)$. Therefore $\ln p_t$ given by (22) satisfies both Equations (15) and (16).