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Fitting the Log Periodic Power Law to financial crashes: a critical analysis*

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February 4, 2010

Abstract

A number of papers claim that a Log Periodic Power Law (LPPL) fitted to financial market bubbles that precede large market falls or ‘crashes’, contain parameters that are confined within certain ranges. The mechanism that has been claimed as underlying the LPPL, is based on influence percolation and a martingale condition. This paper examines these claims and the robustness of the LPPL for capturing large falls in the Hang Seng stock market index, over a 30-year period, including the current global downturn. We identify 11 crashes on the Hang Seng market over the period 1970 to 2008. The fitted LPPLs have parameter values within the ranges specified *post hoc* by Johansen and Sornette [12] for only seven of these crashes. Interestingly, the LPPL fit could have predicted the substantial fall in the Hang Seng index during the recent global downturn. We also find that influence percolation combined with a martingale condition holds for only half of the pre-crash bubbles previously reported. Overall, the mechanism posited as underlying the LPPL does not do so, and the data used to support the fit of the LPPL to bubbles does so only partially.

1 Introduction

Two broad finance theories make predictions about stock prices. They are the efficient market hypothesis (EMH) and the rational bubbles view (RBV).

*We acknowledge the great help provided by Anders Johansen particularly in answering the many queries that we had about his data analysis and in providing us with the program that he used to estimate the parameters of the LPPL.

Both theories begin from the standpoint that an asset has a fundamental value, defined as the market's expected discounted present value of the future cash flows associated with the asset. Empirical tests of both the EMH and the RBV often fail to explain large market price falls or 'crashes', as such crashes are not usually associated with any specific news item.¹ For example, Cutler et al. [6] find that of the 50 largest daily price falls in aggregate stock prices for the period 1946-1987, the majority are not accompanied by external news of specific importance.²

Empirical tests of the RBV have also had limited success in identifying price bubbles prior to large price falls (see, e.g., [2] and [27]). In particular, Donaldson and Kamstra [7] estimate a non-linear ARMA-ARCH artificial neural network model which allows them to reject the claim that the 1929 stock market crash was the outcome of a bubble. One reason for the failure of tests of the RBV is the difficulty of explicitly isolating an asset's fundamental value from the component of the bubble tied to the asset's market price.

An alternative approach to modeling stock market crashes and their bubbles is to fit a Log-Periodic Power-Law (LPPL) to asset prices.³ The notion that statistical description of financial crashes as manifestations of power law accelerations essentially suggests that stock market crashes obey a particular power law with log-periodic fluctuations. Sornette et al. [25], Sornette and Johansen [23] and Lillo and Mantegna [17] show that this model is able to capture a shift over time in the log-periodic oscillations of financial prices that are associated with market crashes.

The analogy of financial crashes as being similar in their statistical signatures to critical points as depicted in natural phenomena has, however, been argued to be unrealistic. Laloux et al. [15] express doubts about the validity of a seven-parameter model fitted to highly noisy data. They argue that such a model would suffer from severe over-fitting. Also, some log-periodic precursors do not always lead to crashes but to a smooth draw-down or to an even greater draw-up. This suggests that there is no universality in the manner financial bubbles are manifested. Indeed, some evidence (see, e.g., [8]) shows that the predicted time of a crash is sensitive to the size of the event-window used to predict the crash.

Whilst the LPPL model is not perfect, it is empirically appealing as it provides a forecast of the date by which a financial crash will occur. This is an important attribute relative to other methods of financial risk assessment. For example, Novak and Beirlant [20, p. 461] argue Extreme Value Theory provides

¹ Our definition of a stock market crash is similar to that of Hong and Stein [9], in that they represent unusual large market falls that are not followed by large public news events and such falls are market wide in nature.

² Recently, several specific theoretical models of stock market crashes have been put forward. In Romer's [21] symmetric rational asset-price model, neither rational behavior nor external news plays an important role in stock market crashes. Also both the Hong and Stein [9] and Barley and Veronesi [1] models assume that economically significant differences in the views of investors can lead to stock market crashes when they are revealed.

³ Kumar et al. [13] also apply logit models to microeconomic and financial data and show that currency crashes can be predicted.

a means of predicting “...the magnitude of a market crash but not the day of the event.” As such, the LPPL model appears to contain important statistical attributes that require serious empirical consideration. For example, the LPPL model contains a component that captures the market’s excessive volatility prior to a crash. This feature is consistent with several theoretical models of financial crashes as well as with empirical results [16, 5].

There are several critical considerations associated with fitting an LPPL to financial data and we take issue with some of them as follows: first, studies that support the LPPL (see e.g., [11]) show that the parameter estimates for this model are confined within certain ranges and that it is these that are the indicators of market crashes. This view considerably restricts the number of classes of permissible models that fit bubbles preceding stock market crashes to just those LPPLs whose parameters fall within the specified ranges rather than to any 7 parameter fitted LPPL. Second, the mechanism underlying the LPPL is such that prices must increase throughout the bubble, which is largely in line with the rational bubbles literature, but which is not what has been found in empirical fits of the LPPL (see Section 3.5). Finally, we do not feel that there has been sufficient critical analysis of the LPPL and its goodness-of-fit to market bubbles. In particular, a goodness-of-fit test is rarely applied and the sensitivity of the parameters of the fitted LPPL is usually not reported (see Section 5.5).⁴

The remaining main sections of this paper are as follows: Section 2 introduces the LPPL; Section 3 describes the mechanism underlying the LPPL and tests it using already published results; Section 4 gives some details of the procedure used for finding the parameters of an LPPL so that it best fits the data; Section 5 fits the LPPL to pre-crash bubbles on the Hang Seng index, checks on the fits already published and tests whether the parameters of the fitted LPPLs do in fact predict crashes.

2 The LPPL

The simplest form of the LPPL can be written as:

$$y_t = A + B(t_c - t)^\beta \{1 + C \cos(\omega \log(t_c - t) + \phi)\}, \quad (1)$$

where:

⁴ Laloux et al. [15, p. 4] report two instances when financial crashes were predicted ex ante. The prediction was correct in one case but not in the other despite both predictions being published prior to the expected crash date. Indeed, they conclude that “...recent claims on the predictability of crashes are at this point not trust worthy”.

$y_t > 0$	is the price (index), or the log of the price, at time t ;
$A > 0$	is the value that y_t would have if the bubble were to last until the critical time t_c ;
$B < 0$	is the increase in y_t over the time unit before the crash, if C were to be close to zero;
C	is the magnitude of the fluctuations around the exponential growth, as a proportion;
$t_c > 0$	is the critical time;
$t < t_c$	is any time into the bubble, preceding t_c ;
$\beta = 0.33 \pm 0.18$	is the exponent of the power law growth;
$\omega = 6.36 \pm 1.56$	is the frequency of the fluctuations during the bubble;
$0 \leq \phi \leq 2\pi$	is a shift parameter.

The ranges of values given for both β and ω are based on the observed parameters of crashes for many stock markets [10]. These ranges for β and ω , rather than any goodness-of-fit test, are used to identify the bubbles that precede crashes.

Empirical studies that fit the LPPL to financial data make a number of claims:

1. The mechanism that characterizes traders on financial markets is one in which they mutually influence each other within local neighborhoods. This leads, in turn to co-ordinated behavior through a martingale condition, which in the extreme can lead to a bubble and then a crash (see e.g., [11]).
2. Financial crashes are preceded by bubbles with fluctuations. Both the bubble and the crash can be captured by the LPPL when specific bounds are imposed on the critical parameters β and ω (see e.g., [10, 12]).
3. The established parameters are sufficient to distinguish between LPPL fits that precede a crash from those that do not (see e.g., [24]).⁵

In this paper, we examine the first two of the above claims and suggest a new approach for testing them. The third claim is more controversial; it only makes sense to evaluate it once we have a positive evaluation of the second claim.

3 Is the Underlying Mechanism Correct?

3.1 The underlying mechanism

The mechanism driving the change in price during a bubble as posited in Johansen et al. [11] is based on rational expectations, namely, that the expected price rise must compensate for the expected risk. The mechanism is a stochastic process such that the conditional expected value of the asset at time $t + 1$,

⁵ Recently Lin et al. [18] carried out such an evaluation on a variant of the LPPL model.

given all previous data before and up to t , is equal to its price at time t . The martingale condition is formulated by Johansen et al. [11] as:

$$dp \leftarrow \kappa p(t) h(t) dt, \quad (2)$$

where: dp is the expected change in price, conditional on no crash occurring over the next time interval dt , at equilibrium;
 $p(t)$ is the price at time t ;
 κ is the proportion by which the price is expected to drop during a crash, if it were to occur;
 $h(t)$ is the hazard rate at time t , i.e. the chance the crash will occur in the next unit of time, given that it has not occurred already.

Under this martingale condition, investors will buy shares at time t if their expectation is that the price at time $t+1$ will exceed the price at t by more than the associated risk; that is: $E(p(t+1)) > p(t) + dp$. This buying would drive up today's price. So the expected rise in price between today and tomorrow will be less (assuming that the expected price tomorrow remains constant); this buying will continue until the expected rise is in line with the perceived risk according to Eq. 2. And, vice versa, if investors believe that the expected rise in price tomorrow will be insufficient to compensate for the risk, i.e. $E(p(t+1)) < p(t) + dp$, then they will try to sell today, going short if necessary, thus driving today's price down.

Note that all the terms on the right side of Eq. 2 are positive, so $dp > 0$, i.e., the price must always be expected to be increasing during a bubble. This condition was not treated as a constraint in early work (see, e.g., [11]) and as such gives us the opportunity of treating this requirement as a testable prediction.⁶

We now follow the consequences of Eq. 2 for the behavior of prices. Rearranging Eq. 2 gives us:

$$\begin{aligned} \frac{1}{p(t)} dp &= \kappa h(t) dt, \\ \log p(t) &= \kappa \int_{t_0}^t h(t') dt'. \end{aligned} \quad (3)$$

To capture the behavior of the price, the hazard rate, $h(t)$, needs to be specified. Here, Johansen et al. [11] posit a model in which each trader i is in one of two states, either bull (+1) or bear (-1). At the next time step, the position of trader i is given by:

$$\text{sign} \left(K \sum_{j \in N(i)} s_j + \sigma \epsilon_i \right), \quad (4)$$

⁶ Sornette and Zhou [26], a later work, does treat this condition as a constraint on the permissible parameter values.

where: K is the coupling strength between traders;
 $N(i)$ is the set of traders who influence trader i ;
 s_j is the current state of trader j ;
 σ is the tendency towards idiosyncratic behavior for all traders;
 ϵ_i is a random draw from a zero mean unit variance normal distribution.

The relevant parameter determining the behavior of a collection of such traders is the ratio K/σ , which determines a critical value of K , say K_c . If $K \ll K_c$ then the collection is in a disordered state. However, as K approaches K_c order begins to appear in the collection, with a majority of traders having the same state. As the value of K approaches K_c from below, the system becomes more sensitive to small initial perturbations. At the critical value, K_c , all the traders will have the same state, either +1 or -1. Johansen et al. [11] further assume that: i) the coupling strength K increases smoothly over time up to K_c ; and ii) the hazard rate is proportional to K . They do not justify these assumptions but the first one might be based on assuming that, as the frequency of fluctuations increases, traders become less sure of their own judgment and rely more on the judgment of their neighbors. In the next sections, we consider the evolution of K over time.

3.2 Simple power law hazard rate

In the simplest scenario K evolves linearly with time. Assuming that each trader has four neighbors arranged in a regular two dimensional grid, then the *susceptibility* of the system near the critical value, K_c , can be shown to be given by the approximation:

$$\chi \approx B''(K_c - K)^{-\gamma}, \quad (5)$$

where $B'' > 0$ and $0 < \gamma < 1$ (see [11]). The three assumptions taken together give:

$$h(t) \approx B'(t_c - t)^{-\alpha}, \quad (6)$$

where $0 < \alpha < 1$. Substituting in Eq. 3 for h as given by Eq. 6 and integrating gives:

$$\begin{aligned} \log p(t) &= \kappa \int_{t_0}^t B'(t_c - t')^{-\alpha} dt' = \frac{-\kappa B'}{1 - \alpha} [(t_c - t)^{1-\alpha}]_{t_0}^t \\ &= \frac{-\kappa B'}{1 - \alpha} ((t_c - t)^{1-\alpha} - (t_c - t_0)^{1-\alpha}). \\ \text{At } t = t_c, \log p(t_c) &= \frac{-\kappa B'}{1 - \alpha} (0 - (t_c - t_0)^{1-\alpha}). \\ \text{So } \log p(t) &= \log p(t_c) + \frac{\kappa B'}{1 - \alpha} (t_c - t)^{1-\alpha} \\ &= A + B(t_c - t)^\beta, \end{aligned} \quad (7)$$

where: $A = \log p(t_c)$, $B = \kappa B'/(1 - \alpha)$ and $\beta = 1 - \alpha$. This is a simple exponential growth.

3.3 Log periodic hazard rate

To introduce log periodic fluctuations into the exponential growth function we need a different form of interconnected structure. Such a structure is assumed to be equivalent to one created by: i) starting with a pair of linked traders; ii) replacing each link in the current network by a diamond with four links and two new nodes diagonally opposite each other. This process continues until some stopping criterion is met. Then (see [11]):

$$\begin{aligned}\chi &\approx B''(K_c - K)^{-\gamma} + C''(K_c - K)^{-\gamma} \cos(\omega \log(K_c - K) + \phi') + \dots \\ \text{So } h(t) &\approx B'(t_c - t)^{-\alpha} [1 + C' \cos(\omega \log(t_c - t) + \phi')], \text{ from Eq. 6.}\end{aligned}\quad (8)$$

Substituting for h in Eq. 3 from Eq. 8 gives:

$$\log p(t) = \kappa \int_{t_0}^t B'(t_c - t')^{-\alpha} \{1 + C' \cos(\omega \log(t_c - t') + \phi')\} dt'. \quad (9)$$

Substituting $\beta = 1 - \alpha$ and $\psi(t') = \omega \log(t_c - t') + \phi'$ in the integral

$$\begin{aligned}\int_{t_0}^t (t_c - t')^{-\alpha} \cos(\omega \log(t_c - t') + \phi') dt' &= \int_{t_0}^t (t_c - t')^{\beta-1} \cos \psi(t') dt' \\ &= \left[\frac{-(t_c - t')^\beta}{\omega^2 + \beta^2} \{ \omega \sin \psi(t') + \beta \cos \psi(t') \} \right]_{t_0}^t.\end{aligned}\quad (10)$$

Integrating Eq. 9 using Eq. 10 gives:

$$\begin{aligned}\log p(t) &= \kappa \left[\frac{-B'}{1-\alpha} (t_c - t')^{1-\alpha} - \frac{B'C'(t_c - t')^\beta}{\omega^2 + \beta^2} (\omega \sin \psi(t') + \beta \cos \psi(t')) \right]_{t_0}^t \\ &= \kappa \left[\frac{-B'}{\beta} \{ (t_c - t)^\beta - (t_c - t_0)^\beta \} \right. \\ &\quad \left. - \frac{B'C'}{\omega^2 + \beta^2} \{ (t_c - t)^\beta (\omega \sin \psi(t) + \beta \cos \psi(t)) \right. \\ &\quad \left. - (t_c - t_0)^\beta (\omega \sin \psi(t_0) + \beta \cos \psi(t_0)) \} \right], \\ \log p(t_c) &= \kappa \left[\frac{B'(t_c - t_0)^\beta}{\beta} + \frac{B'C'(t_c - t_0)^\beta}{\omega^2 + \beta^2} (\omega \sin \psi(t_0) + \beta \cos \psi(t_0)) \right], \\ \log p(t) &= \log p(t_c) - \frac{\kappa B'(t_c - t)^\beta}{\beta} - \frac{\kappa B'C'}{\omega^2 + \beta^2} \{ (t_c - t)^\beta (\omega \sin \psi(t) + \beta \cos \psi(t)) \} \\ &= A + B(t_c - t)^\beta [1 + C''(\omega \sin \psi(t) + \beta \cos \psi(t))] \\ &\approx A + B(t_c - t)^\beta [1 + C \cos(\psi(t) + \phi')] \\ &\approx A + B(t_c - t)^\beta [1 + C \cos(\omega \log(t_c - t) + \phi)],\end{aligned}\quad (11)$$

where $A = \log p(t_c)$, $B = -\kappa B'/\beta$, and $C'' = \beta C'/(\omega^2 + \beta^2)$, which is the LPPL of Eq. 1 with $y_t = \log(p_t)$.

3.4 Index: raw versus log

From Eq. (11), it is the log of the price index that needs to be fitted to the LPPL although the LPPL is often fitted to the raw index data. Johansen and Sornette [12] justify the use of the raw data by assuming that the price drop in the crash is proportional to the price over and above the fundamental value rather than being proportional simply to the price. That is, they replace the condition 2 by:

$$dp \leftarrow \kappa(p(t) - p_1)h(t)dt, \quad (12)$$

where p_1 is the fundamental value, which they do not further define.

Johansen and Sornette [12] introduce the assumption that the rise in price since the beginning of the bubble is much less than the amount by which the price at the beginning of the bubble is above the fundamental value. Thus

$$p(t) - p(t_0) \ll p(t_0) - p_1, \quad (13)$$

where t_0 is the time of the beginning of the bubble. Even if the asset's fundamental value is not estimated in the model, the above assumption is weakly testable. If the price rise during the bubble is greater than the price at the beginning of the bubble, i.e. $p(t) > 2p(t_0)$, then the condition of Eq. 13 cannot be fulfilled unless the fundamental price is negative. We assume that this is not what is intended. So we can test whether or not this assumption is met.

Integrating Eq. 12 from the moment when the bubble starts, t_0 , and using Eq. 13 gives:

$$\begin{aligned} p(t) &= p(t_0) + \int_{t_0}^t dp \\ &= p(t_0) + \kappa \int_{t_0}^t (p(t') - p_1)h(t')dt' \\ &\approx p(t_0) + \kappa(p(t_0) - p_1) \int_{t_0}^t h(t')dt'. \end{aligned} \quad (14)$$

Provided the assumption in Eq. 13 is met, Eq. 14 can be used to fit the LPPL to raw price (as done, e.g., in [12]) rather than the log price data.

3.5 Tests of the underlying mechanism

Chang and Feigenbaum [4] tested the mechanism underlying the LPPL using S&P index data for the bubble preceding the 1987 crash. They tested whether the observed price changes could be fitted to a LPPL and whether price changes could be predicted by a random walk model. To do so they first extended the LPPL as given in Eq. 1 by adding:

- a random term with zero mean and variance estimated from the data. This noise term is necessary to compute a likelihood for the observed data deviations from the predicted LPPL.

- a positive upward drift term estimated from the data. This addition to the LPPL, while frequently made in financial time series, is unnecessary here as exponential growth is posited in the LPPL.

Then they estimated the likelihood of the observed change in price since the previous day $t - 1$, and selected parameters that maximized the sum of these likelihoods over the entire bubble.

With a time series there is a choice of which next point to take as being the most likely: either the predicted value or the predicted change since $t - 1$. Using the model's prediction of the value at t ignores the value at $t - 1$; this is what Johansen et al [11] implicitly assume when they minimize the RMSE for the fitted LPPL against the data. On the other hand, using the predicted change since $t - 1$ ignores any deviation that the price at $t - 1$ already has from the model's prediction for $t - 1$. This is what Chang and Feigenbaum [4] explicitly do. So they have established simply that the mechanism underlying their adaptation of the LPPL, when judged for each time point separately, is not to be preferred to the random walk model, which is hardly surprising given what is known about the random nature of stock market prices.

While most of the assumptions underlying the mechanism from which the LPPL is derived are untestable (or even questionable), there is one that is testable: the hazard rate h must be positive. This implies that the price must always rise. If the fitted LPPL does not have this property, then the assumption that $h(t)$ in Eq. 2 is a probability, must be rejected. Figure 1 shows the LPPL fitted to the raw Hang Seng index data for the bubble preceding the 1989 crash. The fit is similar to Figure 8 of Sornette and Johansen [24]. The LPPL in Figure 1 has a negative slope some of the time. The same is true in 16 of the 30 cases reported by Sornette and Johansen [12, 24].⁷ That is, the fitted LPPL predicts that the price decreases at some time points. Unless we are mistaken, this empirical fact is sufficient to reject the martingale condition as being the mechanism underlying the LPPL fit to pre-crash bubbles. However, this does not affect the ability or otherwise of the LPPL to fit the data.

4 Fitting the LPPL Parameters

The seven parameters of the LPPL in Eq. 1 have to be estimated from the window of data points in the bubble. The chosen values of these parameters should be the ones that minimize the root mean squared error (RMSE) between the data and the LPPL prediction for each day of the bubble. The squared error between the prediction from the fitted curve from Eq. 1 and the data is:

⁷These 16 pre-crash bubbles are: the Dow Jones (1929, '62), S & P ('37, '87), Hang Seng ('89, '94), Argentina ('91, '92, '97) and various stock market crashes of 1994 (Indonesia, Korea, Malaysia, Philippines) and 1997 (Indonesia, Mexico, Peru).

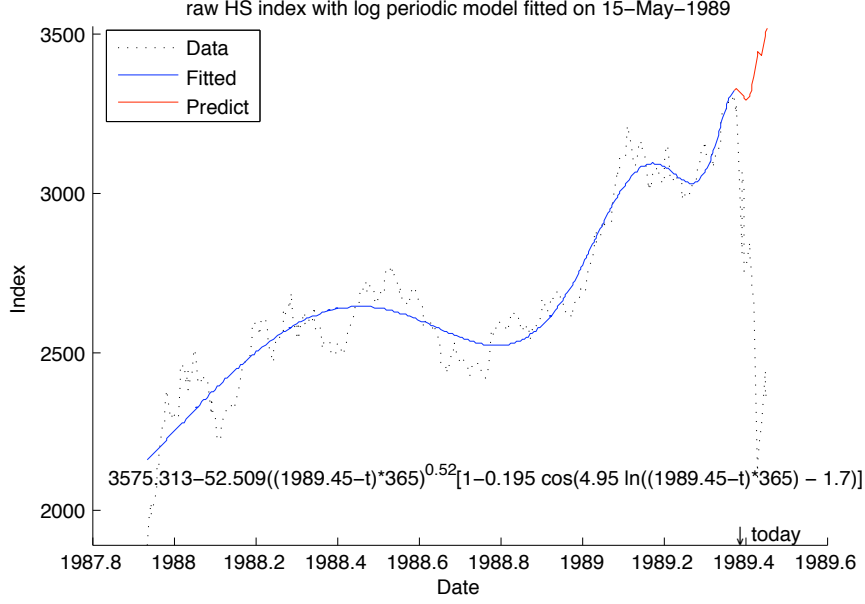


Figure 1: LPPL fit to the bubble preceding the 1989 crash on Hang Seng.

$$SE = \sum_{t=t_1}^{t_n} (y_t - \hat{y}_t)^2 = \sum_{t=t_1}^{t_n} \{y_t - A - B(t_c - t)^\beta (1 + C \cos(\omega \log(t_c - t) + \phi))\}^2, \quad (15)$$

where: y_t is the data point, either the price index or its log;
 \hat{y}_t is the data point as predicted by the model;
 n is the number of weekdays in the bubble;
 t_i is the calendar day date of the i^{th} weekday
from the beginning of the bubble.

Partially differentiating Eq. 15 with respect to the parameters A, B and C gives us three linear equations from which the values of A, B and C that minimize the RMSE are derived, given the other four parameters: β, ω, t_c and ϕ . To find suitable values for these four parameters a search method is required. The method used in [12] and [24], hereafter collectively called the *JS studies*, was:

- First to make a grid of points for the parameters ω and t_c , from each of which a Taboo search was conducted to find the best value of β and ϕ , i.e. the ones for which, with A, B and C chosen to minimize the RMSE, gave the lowest RMSE.
- To select from these points those for which $0 < \beta < 1$.

- From these points to conduct a Nelder-Mead Simplex search [19], with all the four search parameters free (and A, B and C chosen to minimize the RMSE).

We presume that the reason that any fit with $\beta \geq 1$ was rejected is because then the increase in the index is exponentially *declining* whereas the underlying mechanism requires it to be increasing. An alternative technique would have been to place no restriction on the value of β , and if a value of $\beta \geq 1$ is found, to reject the model, as we have done for the requirement that the fitted LPPL never decreases (see Section 3.5).

Similar to the JS studies, we use a preliminary search procedure based on a grid to provide seeds for the Nelder-Mead Simplex method, as implemented in Matlab [14]. It is based on choosing different values for the two parameters ω and β , as these are the critical parameters for determining whether the fitted LPPL is a crash precursor or not (see Eq. 1). The algorithm and the parameter values used are shown in the Appendix. Note that instead of the crash date, t_c , we use t_{2c} , the number of days between the day on which the estimate is being made and the predicted critical date.

5 Empirical Results

5.1 Data and descriptive statistics

Table 1: Descriptive statistics for changes in the log of the Hang Seng stock index.

	N	Mean	Variance	Skewness	Kurtosis	J-B
HS Index returns	10152	0.00045 ^b	0.00035	-1.25934 ^a	31.58011 ^a	424542.78.9 ^a

The mean and variance are multiplied by 100.

J-B is the Jarque-Bera statistic for testing the null hypothesis of normality.

^a and ^b denote statistical significance at a 1- and 5- percent level, respectively.

To test the LPPL, daily observations for the Hang Seng were downloaded from Data-stream. The data spans the period 1st January 1970 to 31st December 2008. We analyze the Hang Seng index since it is commonly believed that this market has had several crashes, thus giving us ample opportunity to test the LPPL.⁸ Descriptive statistics, shown in Table 1, reveal that the mean log changes of the Hang Seng index series are significantly different from zero. Both skewness and (excess) kurtosis are significant such that the Jarque-Bera test rejects the null of normality at a 1 percent level. Notice that skewness is highly significant and negative. This finding suggests that the Hang Seng stock

⁸ This suggests that stock market crashes can be common. Indeed, using a statistical method to identify outliers, Schluter and Trede [22] show that the 1987 stock market crash of the Dow Jones Industrial index was not a structurally unusual event.

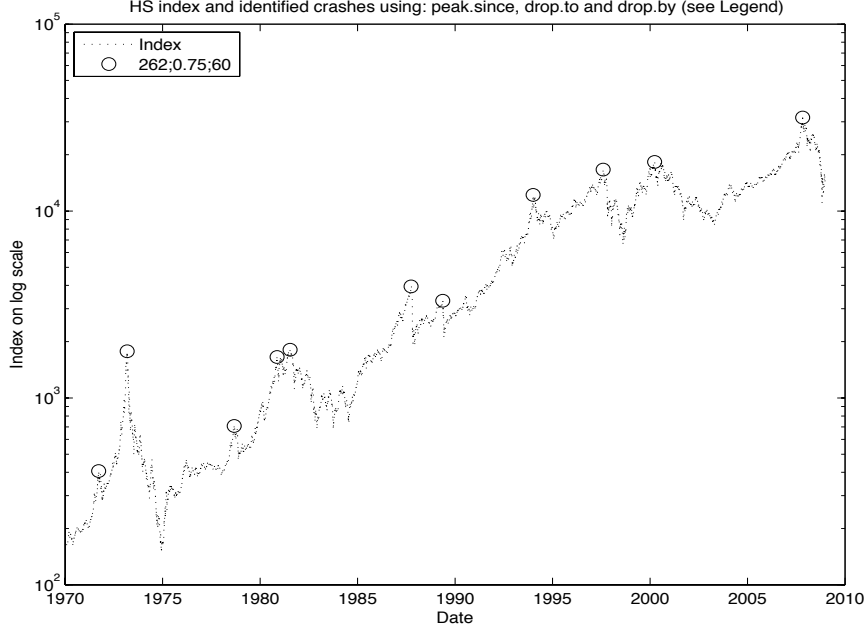


Figure 2: The Hang Seng index 1970 to 2008, showing those peaks that are initiators of crashes.

market can be very sensitive to stock market crashes. That is, volatility feedback can increase the probability of large negative returns and in turn, increase the potential for crashes [3].

5.2 Identifying a crash

As indicated earlier, we use a definition of stock market crashes similar to that of Hong and Stein [9]. To test whether or not the LPPL can predict crashes we first need to identify the crash itself. Usually a stock market crash is taken to mean a very large and unusual price fall. In our application, a crash can span more than one day. This is consistent with the October 1987 stock market crash.

There are two situations when we might falsely claim that a crash has occurred. One is when the index is on the way up in a bubble and then there is a large drop, but it turns out that the drop is temporary and the bubble continues. The other is when, on the way down during a crash, the index experiences a recovery and so we identify the beginning of a new bubble but the recovery is temporary and the anti bubble is still in effect. To avoid those situations, we identify a peak as one initiating a crash as follows:

- A period of 262 weekdays prior to the peak for which there is no value higher

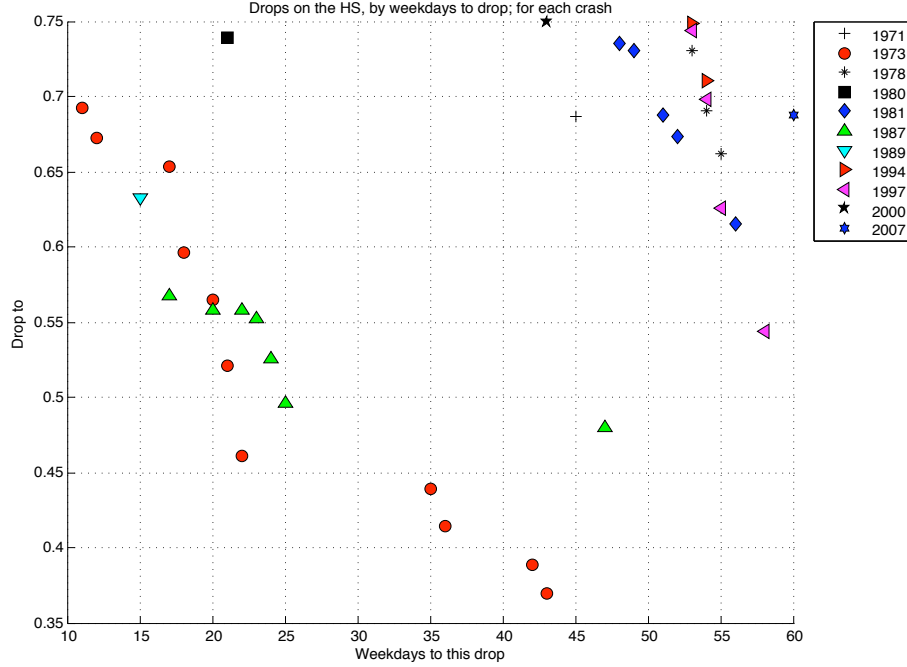


Figure 3: Drops from peaks on Hang Seng index 1970 to 2008.

than the peak's.

- A drop in price of 25%, i.e. down to 0.75 of the peak price, which is in line with the 1987 crash.
- A period of 60 weekdays within which the drop in price needs to occur.

We first want to determine whether the application of these criteria enables us to capture the eight crashes on the Hang Seng index, as identified in the JS studies. Indeed, we identify crashes at the same time points as in the JS studies, except for one additional crash in 1981 (see Figure 2). To exclude the price fall in 1981 from being classified as a crash, we would have to increase the drop-to criterion or reduce the drop-by criterion. Doing either would also exclude some of the other peaks as initiating crashes, viz. those peaks that immediately preceded the crashes of 1978, 1994, 1997, all of which are identified as crash initiators in the JS studies (see Figure 3). Thus the rule they apply seems somewhat imprecise. It is true that the 1981 crash occurs shortly after the 1980 crash, so we might exclude the 1980 peak as initiating a crash, but rather being a part of the bubble preceding the 1981 crash, but this is not what was done in Sornette and Johansen [24]. It would also be possible to exclude fitting an LPPL to the bubble preceding the 1981 crash on the grounds that this bubble is too short – just 7 months long. However, another bubble (the one

preceding the crash 1971) was fitted even though it lasted only 6 months. As such, the bubble preceding the 1981 crash should have been included in the JS studies, unless one insists on having more than say 7 months of data preceding a crash. On balance, we believe that it is appropriate to include the 1981 crash we have identified, giving us nine crashes for the period of the JS studies. Overall, the criteria for identifying a crash does not appear to be consistently applied in the JS studies.

In the period after the JS studies, i.e. between 2000 and 2008, our criteria identify two additional peaks as initiating crashes; these are in 2000 and in 2007. The two bubbles preceding these crashes provide a post-hoc test of the hypothesis underlying the LPPL Eq. 1.

5.3 Troughs and bubble beginnings

Having decided that a peak is the initiator of a crash, the data window to be used for fitting the LPPL to the preceding bubble needs to be carefully selected. In the JS studies the start of the data window is taken to be the day on which the index reaches its lowest value “prior to the change in trend” [12]. In real time matters are not so simple, since one does not know if the index will drop still further in the future. So for real time analysis we would need to take as the end of the previous crash the lowest point since the last crash, up until now.

Moreover, Johansen and Sornette [12] sometimes move the beginning of the bubble from the lowest point since the previous crash to a later time as in their Asian and Latin-American study. This was done if “at the trough the next bubble had not yet begun” (Johansen, personal communication). From the JS studies, we deduce that this was done for four of the eight crashes they identified on the Hang Seng:

- 1971 crash: forward 2 months, from 5/1/1971 to 10/3/1971;
- 1978 crash: forward 3 years and 1 month, from 10/12/1974 to 13/1/1978;
- 1987 crash: forward 1 year and 8 months, from 2/12/1982 to 23/7/1984;
- 1994 crash: forward 2 years and 2 months, from 5/6/1989 to 19/8/1991.

These are indicated by squares in Figure 4.

It is clear why Johansen and Sornette [12] moved the beginning of the bubbles for the 1978 and 1987 crashes to times later than the trough proceeding the crash. For 1978 there was a long period of stable prices which is clearly not part of a bubble. For 1987 the year and 8 months following the trough are characterized by two mini bubbles and two peaks (which with other crash criteria would themselves be considered initiators of crashes). It is not so clear why they moved the start points of the other two bubbles (preceding the 1971 and 1994 crashes) forward.

In the JS studies, a model fit is only made if there are at least 131 weekdays of data between the trough and the crash. Changing the number of days could lead to different bubbles being considered as crash precursors. To illustrate this

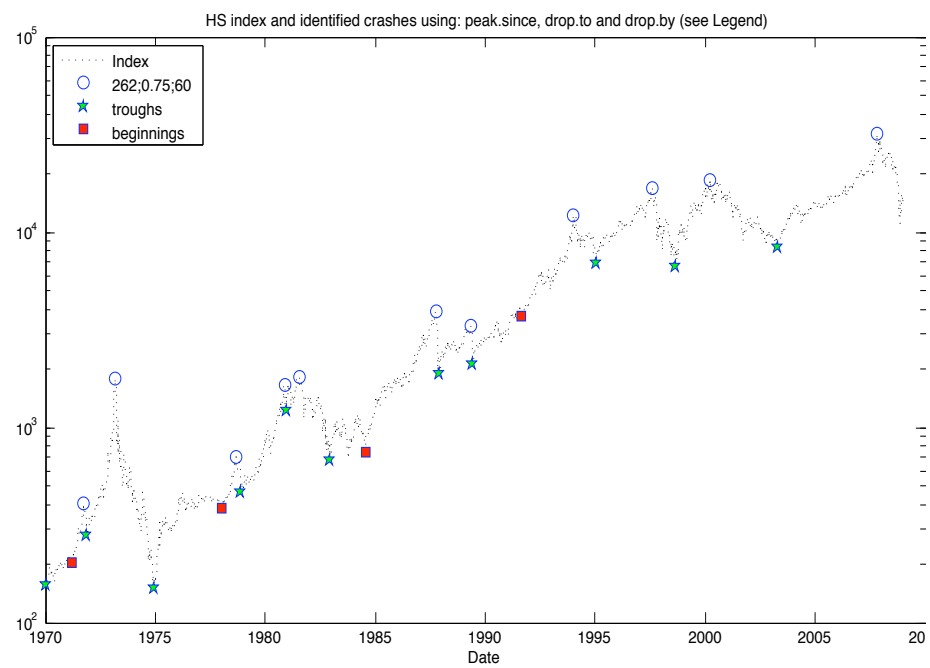


Figure 4: Troughs and other beginnings of bubbles on Hang Seng 1970 to 2008.

for the Hang Seng data, there are only 155 weekdays between the end of the 1980 crash and the peak in 1981 when it appears that another crash occurred. To require (say) 262 weekdays would result in insufficient data, and thus exclude the bubbles before both the 1981 and the 1971 crashes, thus affecting the results. This means that one needs to be very careful in implementing the rule, given the data under consideration.

5.4 Fitting to the raw index

Table 2: Ratio of raw Hang Seng index on the last day to index at the beginning of the bubble.

Bubble:		Raw Hang Seng:		Ratio:
beginning at t_0	ending on t_e	$p(t_0)$	$p(t_e)$	$p(t_e)/p(t_0)$
*10-Mar-1971	20-Sep-1971	201	406	2.02!
22-Nov-1971	09-Mar-1973	279	1775	6.36!
*13-Jan-1978	04-Sep-1978	383	707	1.85
20-Nov-1978	13-Nov-1980	468	1655	3.54!
12-Dec-1980	17-Jul-1981	1222	1810	1.48
*23-Jul-1984	01-Oct-1987	747	3950	5.29!
07-Dec-1987	15-May-1989	1895	3310	1.75
*19-Aug-1991	04-Jan-1994	3723	12201	3.28!
23-Jan-1995	07-Aug-1997	6968	16673	2.39!
13-Aug-1998	28-Mar-2000	6660	18302	2.75!
23-Apr-2003	30-Oct-2007	8520	31638	3.71!

t_0 the day the bubble began.

t_e the last day of the bubble.

* Bubble beginning moved to a time later than the trough between peaks.

! The ratio $p(t_e)/p(t_0) > 2$, so the raw index should not be used (see Eq. 13).

In the JS studies, for all but the 1973 crash, the LPPL has been fitted to the bubble in the raw index rather than to the log of the index. For this to be justified, the inequality in Eq. 13 must hold. That is, the price rise during the bubble must be considerably less than the difference between the price at the beginning of the bubble and the fundamental price. If we make the reasonable assumption that the fundamental price cannot be negative, then at any time during the bubble the price must at the very least not be more than double that at the beginning of the bubble. If we compare the price at the time of fitting, prior to the crash, to that at the beginning of the bubble, then we see in Table 2 that this condition is met for only two of the eight bubbles found in the JS studies. For the remaining six bubbles this condition does not hold, i.e. the price more than doubled during the bubble, so the inequality in Eq. 13, which is the assumption upon which the raw rather than the log of the index can be

chosen, was violated. Despite this, in the JS studies five of these six fits of the LPPL are made to the raw index rather than to its log; they should not have been.

5.5 Sensitivity to search parameter values

Identifying an LPPL fit to a bubble as one that precedes a crash depends on the two critical parameters β and ω ; so it is important to examine how sensitive the RMSE of the fit is to variations in these parameters. We use the bubble preceding the 1989 crash on the Hang Seng to examine the sensitivity of the RMSE of the LPPL fit to variations in each of the four search parameters (β, ω, ϕ and t_c); the other three parameters (A, B and C) are always set using these four (see Section 4). The results are shown in Figure 5. The circle indicates the chosen parameter value. While the chosen values of the search parameters are at global minima, the RMSE is highly sensitive to small fluctuations in the value chosen for ω . The sensitivity diagrams for the other Hang Seng bubbles listed in Table 2 are similar to those shown in Figure 5. In general, the search procedure can easily get trapped in a local minimum for the ω parameter. Consequently the value found by the search procedure for ω may not be the one that leads to the minimum RMSE. As the value of ω is used in predicting whether or not the bubble will be followed by a crash, this is a serious problem.

5.6 The ‘best’ fits of the LPPL

We now fit the LPPL to the raw data for each of the bubbles preceding the 11 crashes identified for the Heng Seng index over the period 1970 to 2008 (as selected by the criteria in Section 5.2), using the minimum RMSE as the criterion for best fit. For each crash:

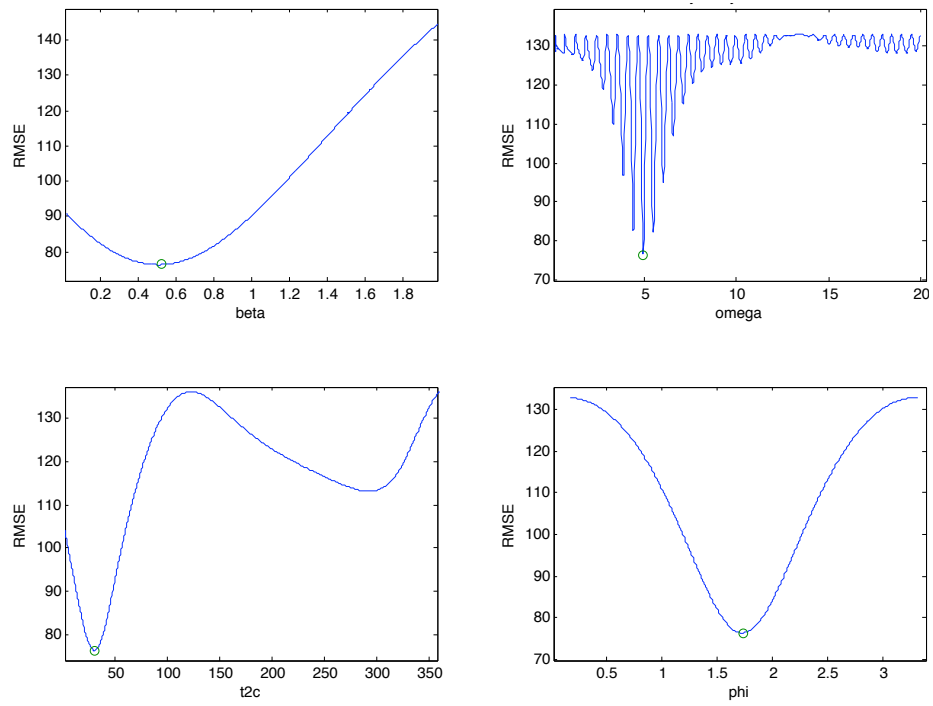


Figure 5: Sensitivity of the RMSE to the parameters of the LPPL for 1989 Hang Seng crash.

Table 3: The bubbles and crashes of the Hang Seng index and LPPL fits to the raw bubble data.

Bubble:	Ref:	A HSI	B HSI	C	β	ω rads	$t2c$ days	ϕ rads	RMSE HSI
from/to		low: high:	0		0.15 0.51	4.80 7.92	1 ?	0 π	
*10-Mar-1971	SJ	594	-132	-0.033	0.20	4.30	7	0.50	7.58
20-Sep- 1971		539	-101	-0.047	0.22	4.30	3	0.25	6.11
22-Nov-1971	SJ	11	-3	0.003	0.11	8.70	2	0.05	0.0722
09-Mar- 1973	<i>log</i>	65	-56	-0.001	0.01	11.1	20	1.32	0.0538
	<i>log</i>	8	-0	-0.177	0.57	1.47	2	3.14	0.0549
	<i>raw</i>	2443	-485	-0.114	0.26	1.45	2	3.14	40.91
*13-Jan-1978	SJ	816	-50	-0.053	0.40	5.90	6	0.17	10.09
04-Sep- 1978		741	-23	0.072	0.51	5.30	1	0.00	10.12
20-Nov-1978	SJ	1998	-231	-0.044	0.29	7.24	3	1.80	46.72
13-Nov- 1980		41164	-38080	0.001	0.01	7.51	52	3.06	35.02
		7929	-5352	0.008	0.05	6.79	26	1.55	35.55
		1998	-231	-0.044	0.29	7.24	3	2.63	37.00
12-Dec-1980									
17-Jul- 1981		1753	-0	-0.890	2.41	3.02	1	3.14	40.46
		1817	-3	-0.567	1	4.75	12	0.35	49.24
		1946	-11	-0.399	0.76	5.89	36	0.00	54.95
*23-Jul-1984	JS	5262	-542	-0.007	0.29	5.60	22	1.60	133.86
01-Oct- 1987		5779	-711	0.048	0.27	5.68	34	2.63	68.47
07-Dec-1987	SJ	3403	-32	-0.023	0.57	4.90	34	0.50	133.21
15-May- 1989		3575	-53	-0.195	0.52	4.95	31	1.74	76.33
*19-Aug-1991	JS	21421	-7614	0.024	0.12	6.30	4	0.60	322.80
04-Jan- 1994		212635	-194575	-0.002	0.27	5.95	1	3.13	272.82
		14038	-1717	-0.028	0.26	6.43	4	3.14	281.36
23-Jan-1995	JS	20359	-1149	-0.019	0.34	7.50	51	0.80	531.79
07-Aug- 1997		20255	-1201	-0.048	0.33	7.47	51	2.29	438.79
13-Aug-1998									
28-Mar- 2000		21918	-16	0.073	1.00	18.35	290	0.00	710.99
		24095	-97	-0.057	0.76	17.51	264	3.14	720.17
		19503	-372	0.111	0.52	5.7	9	2.07	744.15
23-Apr-2003									
30-Oct- 2007		38940	-6408	0.019	0.20	5.41	1	3.14	693.61

HSI is the units of the Hang Seng Index.

Ref : JS denotes Johansen and Sornette [12]; SJ denotes Sornette and Johansen [24].

* Bubble beginning moved to a time later than the trough between peaks.

β was constrained to be ≥ 0.01 , so a $\beta = 0.01$ indicates that the optimal value of $\beta \leq 0.01$.

$t2c$ number of days from date of the fit until predicted crash date, i.e. $t2c = t_c - \text{today}$.

Bold highlights those values of β and ω that fall well outside the range specified in Eq. 1, shown in the top two rows of the table.

- The first line of Table 3 shows the parameters of the LPPL fit as given in the JS studies, but with the linear parameters A, B and C recalculated for time expressed in days rather than years. As the RMSE was not reported for the JS studies (except for the LPPL fitted to the bubble preceding the 1997 crash) this too has been recalculated by us.
- The second line shows the parameters for our best fit to the raw data. The results are based on the raw data, despite our reservations about its appropriateness (Section 5.4), because we want to compare our results with those of the JS studies.⁹
- If this is not within the bounds for a crash prediction, then subsequent lines show the next best fit that is (or might be).

Variation in the values of the critical parameters β and ω sufficiently large to take them across their acceptable boundaries lead to only quite small fluctuations in the RMSE. This can be seen, for example, for the crashes of 1973 and 1980 (see Table 3).

We were interested in comparing our LPPL fits to those found in the JS studies. However, given the high sensitivity of the RMSE to small changes in the value of ω (see Section 5.5) and as the values for β and ω were reported to only one decimal place in the JS studies, our re-calculated RMSEs will be different from those that were obtained in these studies. We can see this in the bubble ending in the crash of 1997, where we have not only our recalculated RMSE using the parameters rounded to one decimal place, but also the RMSE using the unrounded parameter values as found by Johansen et al. [11]; the latter fit is considerably better than our recalculation (RMSE=436 rather than 532 Hang Seng Index units). This improvement is almost certainly due to using the exact rather than the rounded value of ω . So caution needs to be taken when comparing the RMSEs for the fits reported in the JS studies and our fits.

Of the eight pre-crash bubbles fitted in the JS studies we find virtually the same parameters for the LPPL for six of them; namely, those preceding the crashes of 1971, 1978, 1987, 1989, 1994 and 1997. However, for their other two bubbles we found different parameters as follows:

1973: For this bubble, Sornette and Johansen [24] report the fit to the log of the Hang Seng index, rather than to the raw index. We have used both the log and the raw index. When we fit the log of the index we find a better fit than that reported in [24] with values of both β and ω outside their acceptable ranges. For comparison with other bubbles we also fitted the raw index; we find that the best fitting LPPL has a value for $\beta = 0.26$, which is within the acceptable range of 0.15 – 0.51, but for $\omega = 1.45$, which is well below the lower bound of its critical range of 4.8 – 8.0 (see Equation 1).

⁹ For the crash of 1973 Johansen and Sornette [12] used the log instead of the raw index, so we report both log and raw fits specifically for that year.

1980: We were able to reproduce the fit reported in [24], with a crash predicted 3 days later, but it was not the best fit that we found. Our best fit predicted a crash after 52 days, and had critical parameter values $\omega = 7.51$, which is acceptable, but $\beta = 0.01$, which is outside the acceptable range.

There are three pre-crash bubbles that were not considered in the JS studies; one, in 1981, they did not consider a crash (but see Section 5.2), and two others were later than their period:

1981: We find a best fit for which both $\beta (= 2.41)$ and $\omega (= 3.02)$ are well outside their acceptable ranges. As $\beta > 1$, we surmise that this fit would have been rejected by the criteria used in the JS studies (see Section 4). The first fit that has a $\beta \leq 1$ has $\omega = 4.75$, which is just acceptable, but with a $\beta = 1$, i.e. no power law, so well outside its acceptable range. It might be argued that this peak was too soon (8 months) after the trough following the previous crash of 1980 for an LPPL to be fitted on the grounds of there being insufficient data. But, as we have argued in Section 5.2, we believe it should have been.

2000: Our best fit to the bubble has both critical parameters $\beta (= 1.00)$ and $\omega (= 18.35)$ well outside their respective acceptable ranges. There is a fit that does have these parameters within their acceptable ranges, and predicts a crash after only 9 days; but it is not the best fit.

2007: Our best fit to this bubble has parameters well within the ranges required for a crash and the crash is predicted for the day it actually occurred.

6 Conclusion

The LPPL for pre-crash bubbles on stock markets, as reported in Johansen et al. [11] and the JS studies, has important consequences. Our analysis has led us to the following conclusions.

The mechanism proposed to lead to the LPPL fluctuations as reported in Johansen et al. [11] must be incorrect as it requires the price to be increasing throughout the bubble (a constraint recognized later in [26]), but in half the studies reported the LPPL fitted to the index (or its log) decreases at some point during the bubble. Hence either another explanation is required or the fits have to be redone with a constraint on the parameters that leads to LPPL fits that never decrease. Also, in the JS studies the fits were made to the raw rather than the log of the index for all but one (1973) of the eight bubbles, even though the assumption upon which the use of the raw rather than the log should be used was certainly **not** met in six of these seven bubbles. So, on both counts, these studies should no longer be used to support a conclusion that the proposed mechanism underlies the LPPL.

Identifying crashes and bubble beginnings was not well specified in the JS studies. In particular it is not clear why one peak, that of 1981, was not identified

as a crash initiator. Moreover, moving the trough that marks the beginning of a bubble forward by ‘eye’ in half the data sets is not really satisfactory. While we have taken more care in identifying those peaks that initiated crashes, we have still, for comparison, used the same bubble beginnings as used in the JS studies; in future a clear criterion needs to be established.

In the JS studies, the fits of the LPPL to the data were only accepted if the exponential parameter β was < 1 . That is, the fits showed an exponential *increase*. It would be stronger to reject the LPPL if a $\beta \geq 1$ is found.

In our study the two critical parameters of the fitted LPPLs, β and ω , do fall within acceptable ranges in 7 of the 11 bubbles. Of the remaining four bubbles, an LPPL with critical parameters within their respective acceptable ranges could be found for all but one crash (1973). However, these LPPLs did not have the best fits (minimum RMSE). For one crash (1980) the best fit would be acceptable if the lower end of the acceptable range of β was decreased, i.e. a range of 0.01 – 0.51. For another (1981), a fit with $\beta > 1$ would also have to be ruled out to save the hypothesis. For two crashes (1973 and 2000), there seems to be no saving strategy. That the bubbles leading to the 1981 and 2000 crashes do not satisfy the criteria is particularly negative as these are two of the three crashes for which the ranges on the critical parameters were not set *post hoc* in the JS studies.

Finally, while the objection that with seven parameters a curve can be fitted to any data [15] is not directly relevant as no goodness of fit is measured here, it is indirectly highly relevant. The RMSE of the fit of the LPPL model (Eq. 1) to the data is highly sensitive to small but not to large fluctuations in one of the critical parameters (ω); this makes the search for the LPPL that minimizes the RMSE unreliable. Moreover, substantial fluctuations in both parameters together can result in quite small changes in the RMSE. This suggests that the permissible ranges for these parameters should not be independent of one another.

Despite these criticisms, and because of the partial success of the hypothesis, in particular for predicting the 2007 crash, we believe that it is worth investigating whether fitted LPPLs with critical parameters in acceptable non-independent ranges can be used to give a probabilistic, rather than an all-or-none prediction of an impending crash. This will be the subject of on going work.

Appendix: Search algorithm

0. For each of the four parameters $\beta, \omega, t2c$ and ϕ , fix the lower L and upper U bounds for the seeds. For a subset \mathcal{P} of selected parameters (β and ω), fix the minimum width W to continue searching.
1. Choose as the current seed $S1 \leftarrow (L + U)/2$, the mid point of the current lower and upper bounds.
2. Run the unbounded Nelder-Mead Simplex search from the current seed $S1$, which will return a solution $S2$.
3. Construct a hypercube in the space of \mathcal{P} using $S1$ and $S2$, with their minimum as the bottom corner: $B \leftarrow \min(S1, S2)$; and their maxima as the top corner: $T \leftarrow \max(S1, S2)$.
4. For $p \leftarrow 1 : \text{size}(\mathcal{P})$, i.e. for each of the selected parameters, do:
 - if** $B_p - L_p < W_p$ i.e. if there is too little space under the hypercube on the p^{th} dimension in \mathcal{P} , set $B_p \leftarrow L_p$, i.e. set the bottom of the hypercube on the p^{th} dimension to its lower bound,
 - else** recursively search from step 1, with $L' \leftarrow L$ and $U' \leftarrow U, U'_p \leftarrow B_p$, i.e. search under the hypercube;
 - if** $U_p - T_p < W_p$, i.e. if there is too little space above the hypercube on the p^{th} parameter, set $T_p \leftarrow U_p$, i.e. set the top of the hypercube on the p^{th} parameter to its upper bound,
 - else** recursively search from step 1, with $L' \leftarrow L, L'_p \leftarrow T_p$ and $U' \leftarrow U$, i.e. search above the hypercube.

Initial bounds on the four parameters for selecting seeds

	β	ω	$t2c$	ϕ
		rads	days	rads
lower	0	0	1	0
upper	2	20	260	π
minimum width	0.2	2	-	-

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