BCB 567 - HW 1
September 23, 2010

1.

A	B	O	Ω	Θ	**Justification**
$lg^k(n)$	n^j	yes	no	no	Evaluated with L'hospital's rule, $\frac{A}{B}$ goes to 0
$n^{\sin n}$	$\sqrt{n}$	no	no	no	Because of the growing, periodic nature of A, B can never bound A
2^n	$2^{\frac{n}{2}}$	no	yes	no	Consider $2^{\frac{n}{2}} = \sqrt{2^n}$. It's clear that $f(n) = \Omega(\sqrt{f(n)})$.
$n^{lg(c)}$	$c^{lg(n)}$	yes	yes	yes	By the algebraic identity $n^{\log c} = c^{\log n}$.
$lg(n!)$	$lg(n^n)$	yes	no	no	By the algebraic identity $n! < n^n$ for $n > 1$.
n^k	c^n	yes	no	no	Polynomial is always upper bound by any exponential.

2.

(i) Let $c = \sum_{i=0}^{d} a_i$ and $n > 0$. Thus we have $\sum_{i=0}^{d} a_i n^i \leq cn^k$. By the definition of $O(f(n))$, we have the following result.

$$p(n) = O(n^k)$$

(ii) Let $0 < c < a_d$ and $n > 0$. Thus we have $\sum_{i=0}^{d} a_i n^i \geq cn^k$. By the definition of $\Omega(f(n))$, we have the following result.

$$p(n) = \Omega(n^k)$$

(iii) Let $0 < c_1 < a_d, c_2 = \sum_{i=0}^{d} a_i$, and $n > 0$. Thus we have $c_1 n^k \leq \sum_{i=0}^{d} a_i n^i \leq c_2 n^k$. By the definition of $\Theta(f(n))$, we have the following result.

$$p(n) = \Theta(n^k)$$

3.

All of the following recurrences are of the form $T(n) = aT\left(\frac{n}{b}\right) + f(n)$.

(i) For the recurrence $T(n) = 4T\left(\frac{n}{2}\right) + n$, we have $a = 4, b = 2$, and $f(n) = n$. In this case, $f(n) = n \in O\left(n^{\log_2 4 - \epsilon}\right) = O\left(n^{2-\epsilon}\right)$ for some $\epsilon > 0$. Thus, by the Master Theorem,

$$T(n) = \Theta\left(n^2\right)$$

(ii) For the recurrence $T(n) = 4T\left(\frac{n}{2}\right) + n^2$, we have $a = 4, b = 2$, and $f(n) = n^2$. In this case, $f(n) = n^2 \in \Theta\left(n^{\log_2 4}\right) = \Theta\left(n^2\right)$. Thus, by the Master Theorem,

$$T(n) = \Theta\left(n^2 \log n\right)$$

(iii) For the recurrence $T(n) = 4T\left(\frac{n}{2}\right) + n^3$, we have $a = 4, b = 2$, and $f(n) = n^3$. In this case, $f(n) = n^3 \in \Omega\left(n^{\log_2 4+\epsilon}\right) = \Omega\left(n^{2+\epsilon}\right)$ for some $\epsilon > 0$. Thus, by the Master Theorem,

$$T(n) = \Theta\left(n^3\right)$$

4.

We must solve the following recurrence.

$$T(n) = 2T\left(\sqrt{n}\right) + 1$$

If we let $m = \log_2 n$, then $n = 2^m$ and $\sqrt{n} = 2^{\frac{m}{2}}$. Therefore our problem now has the following form.

$$T\left(2^m\right) = 2T\left(2^{\frac{m}{2}}\right) + 1$$

To simplify notation, let $S(m) = T\left(2^m\right)$. Therefore the problem now has the following form.

$$S(m) = 2S\left(\frac{m}{2}\right) + 1$$

By the Master Theorem, we have $S(m) = \Theta(m)$, so our recurrence has the following solution.

$$T(n) = \Theta(m) = \Theta(\log n)$$