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Stat 430 - Karin Dorman
HW 2: Sep 9, 2010

1.

(a) The marginal distributions for X and Y are given below.

$$f_X(x) = \int_0^\infty xe^{-x(y+1)}dy$$

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)}dx$$

(b) The conditional distributions are given below.

$$p_{XY}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dx}$$

$$p_{XY}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dy}$$

2.

(a) The probability $P(T_1 > T_2)$ is given below.

$$\begin{aligned}
P(T_1 > T_2) &= \int_0^\infty P(T_1 > t_2, T_2 = t_2)dt_2 \\
&= \int_0^\infty P(T_1 > t_2 | T_2 = t_2) \cdot P(T_2 = t_2)dt_2 \\
&= \int_0^\infty P(T_1 > t_2 | T_2 = t_2) \cdot (\beta e^{-\beta t_2})dt_2 \\
&= \int_0^\infty \left(\int_{t_2}^\infty \alpha e^{-\alpha t_1} dt_1 \right) \cdot (\beta e^{-\beta t_2})dt_2 \\
&= \int_0^\infty (e^{-\alpha t_2})(\beta e^{-\beta t_2})dt_2 \\
&= \beta \int_0^\infty e^{-t_2(\alpha+\beta)}dt_2 \\
&= \beta \left[e^{-t_2(\alpha+\beta)} \cdot -\frac{1}{\alpha+\beta} \right]_0^\infty \\
&= -\frac{\beta}{\alpha+\beta} e^{-t_2(\alpha+\beta)} \Big|_0^\infty \\
&= 0 + \frac{\beta}{\alpha+\beta} \cdot 1 \\
&= \frac{\beta}{\alpha+\beta}
\end{aligned}$$

(b) The probability $P(T_1 > 2T_2)$ is given below.

$$\begin{aligned}
P(T_1 > 2T_2) &= \int_0^\infty P(T_1 > 2t_2, T_2 = t_2)dt_2 \\
&= \int_0^\infty P(T_1 > 2t_2 | T_2 = t_2) \cdot P(T_2 = t_2)dt_2 \\
&= \int_0^\infty P(T_1 > 2t_2 | T_2 = t_2) \cdot (\beta e^{-\beta t_2})dt_2 \\
&= \int_0^\infty \left(\int_{2t_2}^\infty \alpha e^{-\alpha t_1} dt_1 \right) \cdot (\beta e^{-\beta t_2})dt_2 \\
&= \int_0^\infty \left(e^{-\alpha 2 t_2} \right) (\beta e^{-\beta t_2})dt_2 \\
&= \beta \int_0^\infty e^{-t_2(2\alpha+\beta)} dt_2 \\
&= \beta \left[e^{-t_2(2\alpha+\beta)} \cdot -\frac{1}{2\alpha+\beta} \right]_0^\infty \\
&= -\frac{\beta}{2\alpha+\beta} e^{-t_2(2\alpha+\beta)} \Big|_0^\infty \\
&= 0 + \frac{\beta}{2\alpha+\beta} \cdot 1 \\
&= \frac{\beta}{2\alpha+\beta}
\end{aligned}$$

3.

To find the joint distribution, we can use the formula $f_{X_1X_2}(x_1, x_2) = f(x_2|x_1)f_{X_1}(x_1)$. From the problem statement, we have

$$f_{X_1}(x_1) = 1$$

$$f(x_2|x_1) = \frac{1}{x_1}$$

so the joint distribution $f_{X_1X_2}(x_1, x_2)$ is defined as follows.

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{x_1} \cdot 1 = \frac{1}{x_1}$$

With this joint distribution, we can easily obtain the marginal distributions for X_1 and X_2, as shown below.

$$f_{X_1} = \int_0^{x_1} \frac{1}{x_1} dx_2 = \frac{x_2}{x_1}\Big|_{x_2=0}^{x_2=x_1} = \frac{x_1}{x_1} - 0 = 1$$

$$f_{X_2} = \int_{x_2}^1 \frac{1}{x_1} dx_1 = ln(x_1)\Big|_{x_2}^1 = 0 - ln(x_2) = -ln(x_2)$$

4.

Since X and Y are iid, the joint distribution $f_{XY}(x, y)$ can be defined as follows.

$$f_{XY}(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

We can then determine the distribution of $Z = X + Y$ as follows.

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{XY}(x, y = z - x)dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+(z-x)^2\right)} \right) dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} e^{-\frac{x^2}{2}} e^{-\frac{(z-x)^2}{2}} \right) dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} e^{-\frac{x^2}{2}} e^{-\frac{(z^2-2xz-x^2)}{2}} \right) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{-\frac{z^2}{4}} e^{\left(-x-\frac{z}{x}\right)^2} \right) dx \\
&= \frac{1}{2\pi} \cdot e^{-\frac{z^2}{4}} \cdot \sqrt{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\pi}} \cdot e^{\left(-x-\frac{z}{x}\right)^2} \right) dx \\
&= \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}}
\end{aligned}
$$

Thus we can see that $Z \sim$NORM$(0, 2)$.

5.

If X and Y follow standard uniform densities, the joint distribution of X and Y is simply $f_{XY}(x, y) = 1$.

We can then find the density of $Z = \frac{X}{Y}$ as follows. If $0 \leq z \leq 1$, then we have the following.

$$
\begin{aligned}
f_Z(z) &= \int_x^1 f_{XY}(zy, y)|J| \\
&= \int_x^1 2ydy
\end{aligned}
$$