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September 7, 2010

Stat 430 - Karin Dorman
HW 1: Sep 2, 2010

1.

(a) The sample space is defined as

$$\Omega = \{000, 010, 001, 011, 100, 110, 101, 111\}$$

where each element $\omega \in \Omega$ corresponds to the 3 bits received.

(b) The event that 1 was transmitted and received correctly is $C = \{011, 110, 101, 111\}$.

(c) The set $A = \{A_1 = \{111\}, A_2 = \{101, 110, 011\}, A_3 = \{010, 001, 100\}, A_4 = \{000\}\}$ partitions the sample space Ω based on the number of mistransmitted bits.

(d) The number of mistransmitted bits follows a binomial distribution.

2.

(a) $P = \frac{7}{18} \cdot \frac{6}{17} + \frac{8}{18} \cdot \frac{7}{17} + \frac{9}{18} \cdot \frac{8}{17} = \frac{(7 \cdot 6)+(8 \cdot 7)+(9 \cdot 8)}{18 \cdot 17} = \frac{42+56+72}{306} = \frac{170}{306} = 0.5555556$.

(b) There field is $120 \cdot 53.3 = 6396$ square yards in size. Each 20x20 yard square contains 400 square yards, so there are $6396/400 = 15.99$ squares in the field. Since there are 18 boxes, we would expect to see $18/15.99 = 1.125704$ boxes in each 20x20 square of the field.

(c) It is reasonable to expect that the number X of boxes in a 20x20 square follows a Poisson distribution with $\lambda = 1.126$. The following simulation in R supports this choice.

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> rpois(16, 1.126)
[1] 2 0 0 0 2 1 1 3 1 2 1 1 2 1 1 1
```

3.

Define the following events.

- C: The probability that a claim is submitted
- H: The probability that a client is high risk
- M: The probability that a client is medium risk
- L: The probability that a client is low risk

We are given the following probabilities.

- $P(H) = 0.1$
- $P(M) = 0.2$
- $P(L) = 0.7$
- $P(C|H) = 0.02$
- $P(C|M) = 0.01$
- $P(C|L) = 0.00125$

We are asked to find $P(H|C)$. Using Bayes' rule, we have

$$\begin{aligned} P(H|C) &= \frac{P(C|H)P(H)}{P(C|H)P(H) + P(C|M)P(M) + P(C|L)P(L)} \\ &= \frac{(.02)(.1)}{(.02)(.1) + (.01)(.2) + (.00125)(.7)} \\ &= \frac{0.002}{0.002 + 0.002 + 0.00875} \\ &= 0.4102564 \end{aligned}$$

4.

(a) If events A and B are disjoint when $P(A) > 0$ and $P(B) > 0$, then A and B are independent.

(b) The events A and B are not independent when $A \subset B$.

(c) If $P(A \cup B) = P(A)P(B^C) + P(B)$, then events A and B are independent.

5.

Let the random variable X be defined as the probability that team A will win a "Best of n" tournament. X follows a binomial distribution. When $n = 5$, X is computed as follows.

$$X = \binom{n}{k} p^k (1-p)^{n-k} = \binom{5}{3} (.4)^3 (.6)^2 = 0.2304$$

When $n = 7$, X is computed as follows.

$$X = \binom{n}{k} p^k (1-p)^{n-k} = \binom{7}{4} (.4)^4 (.6)^3 = 0.193536$$

Therefore, team A would do well to elect a "Best of 5" tournament.

6.

To show that $F(x)$ is a CDF, we must demonstrate the following properties.

- The left limit of F is 0, or in other words, $\lim_{x \to 0} F(x) = 0$. Observe the following.

$$\lim_{x \to 0} 1 - exp\left\{-\alpha x^\beta\right\} = 1 - e^0 = 1 - 1 = 0$$

- The right limit of F is 1, or in other words, $\lim_{x \to \infty} F(x) = 1$. Observe the following.

$$\lim_{x \to \infty} 1 - exp\left\{-\alpha x^\beta\right\} = 1 - \frac{1}{e^\infty} = 1 - 0 = 1$$

- The function is increasing monotonically. Observe the following.

$$\frac{d}{dx}\left[1 - exp\left\{-\alpha x^\beta\right\}\right] = 0 - \left[exp\left\{-\alpha x^\beta\right\} \cdot \left(-\alpha \beta x^{\beta - 1}\right)\right] = \alpha \beta x^{\beta-1} exp\left\{-\alpha x^\beta\right\}$$

  This function, representing the slope of the F, is always positive. Therefore F is strictly monotonically increasing.

- The function is right continuous, or in other words, $\lim_{x->x_o^+} F(x) = F(x_o)$. Because F is defined within the specified domain and is (infinitely) differentiable within the specified domain, it is also continuous.

The density, as derived above, is as follows.

$$\frac{d}{dx}F(x) = f(x) = \alpha\beta x^{\beta-1}exp\left\{-\alpha x^{\beta}\right\}$$

7.

The exponential distribution has a PDF defined as follows.

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

The corresponding CDF is defined as follows.

$$F(x) = \int_0^x \lambda e^{-\lambda u} du = -e^{-\lambda x}$$

The upper quartile is the value $q_{.75}$ such that $F(q_{.75}) = .75$. We can determine this value by finding the inverse $F^{-1}(\cdot)$ of $F(\cdot)$ and plugging in .75. The inverse of the CDF is

$$F^{-1}(p) = \frac{-ln(1-p)}{\lambda}$$

so the upper quartile is

$$q_{.75} = F^{-1}(.75) = \frac{-ln(.25)}{\lambda} = \frac{1}{\lambda} \cdot 1.386294$$

8.

Observe the following.

$$\begin{aligned} P(|X-\mu| \leq 0.675\sigma) = 0.5 \quad &\Leftrightarrow \quad P(-0.675\sigma \leq X-\mu \leq 0.675\sigma) = 0.5 \\ &\Leftrightarrow \quad P\left(-0.675 \leq \frac{X-\mu}{\sigma} \leq 0.675\right) = 0.5 \end{aligned}$$

The central term in the last equation follows a standard normal distribution, so we can calculate the probability as follows.

$$\begin{aligned} P\left(-0.675 \leq \frac{X-\mu}{\sigma} \leq 0.675\right) &= P\left(\frac{X-\mu}{\sigma} \leq 0.675\right) - P\left(\frac{X-\mu}{\sigma} \leq -0.675\right) \\ &= (0.75016) - (0.24984) \\ &= 0.50032 \end{aligned}$$

9.

Recall that the normal distribution is defined as follows.

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$$

Now consider that $y(z) = e^z \Rightarrow z = y^{-1}(y) = lny$. Applying this change of variable we have

$$
\begin{aligned}
f_Y(y) &= f_Z(y^{-1}(y)) \cdot \frac{d}{dy} y^{-1}(y) \\
&= \frac{1}{\sqrt{2\pi}\sigma} exp\left\{ -\frac{(lny-\mu)^2}{2\sigma^2} \right\} \cdot \frac{d}{dy} lny \\
&= \frac{1}{y\sqrt{2\pi}\sigma} exp\left\{ -\frac{(lny-\mu)^2}{2\sigma^2} \right\}
\end{aligned}
$$

which is the log-normal density.