Examples

1.

Poisson distribution

Using the pmf of the Poisson distribution, we obtain the following probability generating function.

$$\begin{aligned} A(s) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} s^k \\ &= e^{-\lambda}\left(1 + \frac{\lambda^1 s^1}{1!} + \frac{\lambda^2 s^2}{2!} + \ldots\right) \\ &= e^{-\lambda}\left(1 + \frac{(\lambda s)^1}{1!} + \frac{(\lambda s)^2}{2!} + \ldots\right) \\ &= e^{-\lambda}\left(e^{\lambda s}\right) = e^{\lambda s - \lambda} \\ &= e^{\lambda(s-1)} \end{aligned} \tag{1}$$

Taking the first and second derivatives of this pgf we obtain the following functions.

$$A'(s) = \lambda s e^{\lambda(s-1)} \tag{2}$$

$$\begin{aligned} A''(s) &= (\lambda s)' e^{\lambda(s-)} + \lambda s (e^{\lambda(s-1)})' \\ &= \lambda e^{\lambda(s-1)} + \lambda^2 s^2 e^{\lambda(s-1)} \\ &= e^{\lambda(s-1)}\left(\lambda^2 s^2 + \lambda\right) \end{aligned} \tag{3}$$

Using functions (2) and (3) we can calculate the mean and variance for the Poisson distribution.

$$E = A'(1) = \lambda \cdot 1 \cdot 1 = \lambda \tag{4}$$

$$\begin{aligned} Var &= A''(1) + A'(1) - A'(1)^2 \\ &= (\lambda^2 + \lambda) + (\lambda) - (\lambda)^2 \\ &= 2\lambda \end{aligned} \tag{5}$$

Binomial distribution

Using the pmf of the binomial distribution, we obtain the following probability generating function.

$$\begin{aligned} A(s) &= \sum_{k=0}^{\infty} \binom{n}{k} p^k q^{n-k} s^k \\ &= \sum_{k=0}^{\infty} \binom{n}{k} (ps)^k q^{n-k} \\ &= q^n + \binom{n}{1}(ps)^1 q^{n-1} + \binom{n}{2}(ps)^2 q^{n-2} + \ldots \\ &= (ps+q)^n \end{aligned} \tag{6}$$

Taking the first and second derivatives of this pgf we obtain the following functions.

$$A'(s) = np(ps+q)^{n-1} \tag{7}$$

$$A''(s) = np^2(n-1)(ps+q)^{n-2} \tag{8}$$

Using functions 7 and 8 we can calculate the mean and variance for the binomial distribution.

$$\begin{aligned} E &= A'(1) = pn(p+q)^{n-1} \\ &= np(1)^{n-1} \\ &= np \end{aligned} \tag{9}$$

$$\begin{aligned} Var &= A''(1) + A'(1) - A'(1)^2 \\ &= np^2(n-1) + np - n^2p^2 \\ &= np(p(n-1) + 1 - np) \\ &= np(1-p) \end{aligned} \tag{10}$$

2.

$$\begin{aligned} \sum_{k=0}^{\infty} kp_k &= 1p_1 + 2p_2 + 3p_3 + ... \\ &= p_1 + p_2 + p_3 + ... + p_2 + p_3 + p_4 + ... + p_3 + p_4 + p_5 + ... \\ &= \sum_{k=0}^{\infty}(p_{k+1} + p_{k+2} + p_{k+3} + ...) \\ &= \sum_{k=0}^{\infty} t_k = E[X] = T(1) = P'(1) \end{aligned} \tag{11}$$

3.

Each X_i is an independent Bernoulli random variable. If we let $a_k = (a_0, a_1)$, then we can obtain the pgf of the Bernoulli distribution.

$$\begin{aligned} A(s) &= \sum_{k=0}^{\infty} a_k s^k \\ &= a_0 + a_1 s \\ &= q + ps \end{aligned} \tag{12}$$

If S is the sum of n such random variables, then we can get the pgf of S using equation 1.7.

$$\begin{aligned} A_{X_1+...+X_n}(s) &= \prod_{i=1}^{n} A_{X_i}(s) \\ &= \prod_{i=1}^{n}(q+ps) \\ &= (q+ps)^n \end{aligned} \tag{13}$$

So the pgf for S is the pgf for the binomial distribution.

4.

asdf

Problems

1.

Because $D(s) = (s - s_1)D^*(s)$, we can write $D'(s)$ as follows.

$$D'(s) = (s - s_1)'D^*(s) + (s - s_1)D^{*'}(s)$$

If we assume that s_1 is the shared root, then we have the following

$$\rho_1 = \frac{(s_1 - s_1)N^*(s_1)}{D^*(s_1) + (s_1 - s_1)D^{*\prime}(s_1)} = \frac{0}{D^*(s_1)} = 0$$

The remainder of the ρ_i values are unchanged. Consider, without loss of generality, ρ_2.

$$\rho_2 = \frac{N(s_2)}{D'(s_2)} = \frac{(s_2 - s_1)N^*(s_2)}{D^*(s_2) + (s_2 - s_1)D^{*\prime}(s_2)} = \frac{(s_2 - s_1)N^*(s_2)}{(s_2 - s_1)D^{*\prime}(s_2)} = \frac{N^*(s_2)}{D^{*\prime}(s_2)}$$

2.

Let us assume $m = 1$. Therefore $D(s)$ is a 0^{th}-order polynomial (constant) with 1 root. If $D(s)$ is a constant and has a root, then that constant must be 0, so $D(s)$ is the null polynomial.

Now let us assume that conditions hold for $m = k$ (that is, a $(k-1)^{\text{th}}$-order polynomial with k distinct roots must be the null polynomial). Let us then consider when $m = k + 1$ (that is, a k^{th}-order polynomial with $k+1$ distinct roots). Let r be one of these roots. We can write $D(s)$ like so.

$$D(r) - D(s) = (r - s)\left[a_1 + a_2(r + s) + ... + a_k(r^{k-1} + r^{k-2}s + ... + s^{k-1})\right]$$

The bracketed component is a $(k-1)^{\text{th}}$-order polynomial with k distinct roots, so by the induction hypothesis it is the null polynomial. Therefore $D(s)$ is also the null polynomial.

3.

Asdf.

4.