Appendix: Mathematical Formalization of the Unified Active Inference Architecture

A. Aspect Relegation Theory (ART)

A.1 Hierarchical Generative Model

Let $\ell \in \{1, \dots, L\}$ denote levels in a hierarchical predictive coding model.

- Observations: $y^{(\ell)}$
- Predictions: $\hat{y}^{(\ell)} = g^{(\ell)}(\theta^{(\ell)})$, where $p(\theta^{(\ell)}|\theta^{(\ell+1)})$ encodes hierarchical priors
- Prediction errors:
$$\epsilon^{(\ell)} = y^{(\ell)} - \hat{y}^{(\ell)}$$

A.2 Precision Estimation

Precision is the inverse variance of the prediction error, estimated over a temporal window:

$$\pi^{(\ell)} = \frac{1}{\frac{1}{N}\sum_{t=1}^{N}(\epsilon_t^{(\ell)})^2}$$

A.3 Reflex Arc Gating Function

Reflex Arcs select between System 1 and System 2 probabilistically:

$$P(\Gamma^{(\ell)} = \mathrm{S1}) = \sigma\left(\beta(\pi^{(\ell)} - \pi_{\mathrm{thresh}}) + \gamma(\mathcal{C}_{\mathrm{thresh}} - \mathcal{C}(T))\right)$$

where σ is the sigmoid function, and β, γ are scaling parameters.

A.4 Task Complexity Estimation

Task complexity is domain-specific, e.g., for semantic graphs:

$$\mathcal{C}(T) = H(G) = -\sum_{i \in V} p(i) \log p(i)$$

where $p(i)$ is the probability of visiting node i.

A.5 Free Energy Objective

The variational free energy and ART loss are:

$$\mathcal{L} = \sum_{\ell=1}^{L}\left[\pi^{(\ell)}(\epsilon^{(\ell)})^2 + \mathrm{KL}(q(\theta^{(\ell)}) \parallel p(\theta^{(\ell)}|\theta^{(\ell+1)}))\right] + \lambda \cdot \mathbb{E}[E(\Gamma^{(\ell)})]$$

where $E(\Gamma^{(\ell)})$ is the cognitive cost.

A.6 Adaptive Threshold Updates

Thresholds are adapted via gradient descent:

$$\pi_{\mathrm{thresh}} \leftarrow \pi_{\mathrm{thresh}} - \eta \frac{\partial \mathcal{L}}{\partial \pi_{\mathrm{thresh}}}, \quad \mathcal{C}_{\mathrm{thresh}} \leftarrow \mathcal{C}_{\mathrm{thresh}} - \eta \frac{\partial \mathcal{L}}{\partial \mathcal{C}_{\mathrm{thresh}}}$$

B. Domain-Specific Systems

B.1 Haplopraxis: Sensorimotor Precision and Task Entropy

- Prediction error: $\epsilon_S(t) = y_t - \hat{y}_t$, where $\hat{y}_t = f(a_t, \theta^{(\ell)})$
- Precision: $\pi_S = \frac{1}{\frac{1}{N}\sum_{t=1}^{N} \epsilon_S(t)^2}$
- Task complexity: $\mathcal{C}_{\text{Hap}} = H(\tau) = -\sum_{n\in\tau} p(n)\log p(n)$, where $p(n) = \frac{\text{count}(n)}{\sum_{m\in\tau}\text{count}(m)}$
- Free energy:

$$F_S = \pi_S \cdot \mathbb{E}[\epsilon_S^2] + \text{KL}(q(\theta^{(\ell)}) \parallel p(\theta^{(\ell)}|\theta^{(\ell+1)})) + \lambda\mathcal{C}_{\text{Hap}}$$

B.2 Yarncrawler: Mythic Schema Update Dynamics

- Belief transition:

$$b(v_j, t+1) = \sum_{v_i\in V} b(v_i, t)\cdot P(v_j \mid v_i), \quad P(v_j \mid v_i) = \frac{\exp(\pi_C \cdot w_{ij})}{\sum_{k\in V}\exp(\pi_C \cdot w_{ik})}$$

- Prediction error: $\epsilon_C = \text{KL}(b(v) \parallel \hat{b}(v)) = \sum_{v_i\in V} b(v_i)\log\frac{b(v_i)}{\hat{b}(v_i)}$, where $\hat{b}(v) = p(v|\theta^{(\ell+1)})$
- Complexity: $\mathcal{C}_{\text{Yarn}} = H(G) = -\sum_{v_i\in V} b(v_i)\log b(v_i)$
- Free energy:

$$F_C = \pi_C \cdot \epsilon_C + \text{KL}(q(\theta^{(\ell)}) \parallel p(\theta^{(\ell)}|\theta^{(\ell+1)})) + \lambda H(G)$$

B.3 Womb Body Bioforge: Ecological Inference

- Prediction error: $\epsilon_E = \text{KL}(p(S|y) \parallel p(S|\hat{y}))$
- Precision: $\pi_E = \frac{1}{\mathbb{E}[\epsilon_E^2]}$
- Complexity: $\mathcal{C}_{\text{Bio}} = H(S) = -\sum_{s\in S} p(s)\log p(s)$
- Free energy:

$$F_E = \pi_E \cdot \epsilon_E + \text{KL}(q(\theta^{(\ell)}) \parallel p(\theta^{(\ell)}|\theta^{(\ell+1)})) + \lambda H(S)$$

B.4 Zettelkasten Academizer: Semantic Foraging

- Prediction error: $\epsilon_Z = \sum_{i,j} w_{ij}\cdot d(s_i, s_j)$, where $d(s_i, s_j)$ is the semantic distance between nodes
- Precision: $\pi_Z = \frac{1}{\frac{1}{N}\sum_{t=1}^{N} \epsilon_Z(t)^2}$
- Complexity: $\mathcal{C}_{\text{Zet}} = H(G) = -\sum_{i\in V} p(i)\log p(i)$
- Free energy:

$$F_Z = \pi_Z \cdot \epsilon_Z + \text{KL}(q(\theta^{(\ell)}) \parallel p(\theta^{(\ell)}|\theta^{(\ell+1)})) + \lambda H(G)$$

B.5 Inforganic Codex: Hybrid Cognitive Architecture

- Prediction error: $\epsilon_I(t) = y_t - \hat{y}_t$, where $\hat{y}_t = u(t)$ is the PID control signal:

$$u(t) = K_p\epsilon_I(t) + K_i\int_0^t \epsilon_I(\tau)d\tau + K_d\frac{d\epsilon_I(t)}{dt}$$

- Precision: $\pi_I = \frac{1}{\frac{1}{N}\sum_{t=1}^{N} \epsilon_I(t)^2}$
- Complexity: $\mathcal{C}_{\text{Inf}} = H(M) = -\sum_{m\in M} p(m)\log p(m)$, where M is the trail-based memory state space
- Free energy:

$$F_I = \pi_I \cdot \mathbb{E}[\epsilon_I^2] + \text{KL}(q(\theta^{(\ell)}) \parallel p(\theta^{(\ell)}|\theta^{(\ell+1)})) + \lambda H(M)$$

C. Summary Table

System	Prediction Error	Precision	Task Complexity	Free Energy
Haplopraxis	$\epsilon_S = y - \hat{y}$	$\pi_S = \frac{1}{\mathrm{Var}[\epsilon_S]}$	$H(\tau)$	$\pi_S \mathbb{E}[\epsilon_S^2] + \mathrm{KL}(q \parallel p) + \lambda H(\tau)$
Bioforge	$\epsilon_E = \mathrm{KL}(p \parallel \hat{p})$	$\pi_E = \frac{1}{\mathbb{E}[\epsilon_E^2]}$	$H(S)$	$\pi_E \epsilon_E + \mathrm{KL}(q \parallel p) + \lambda H(S)$
Zettelkasten	$\epsilon_Z = \sum w_{ij} d(s_i, s_j)$	$\pi_Z = \frac{1}{\mathrm{Var}[\epsilon_Z]}$	$H(G)$	$\pi_Z \epsilon_Z + \mathrm{KL}(q \parallel p) + \lambda H(G)$
Yarncrawler	$\epsilon_C = \mathrm{KL}(b \parallel \hat{b})$	$\pi_C = \frac{1}{\mathbb{E}[\epsilon_C^2]}$	$H(G)$	$\pi_C \epsilon_C + \mathrm{KL}(q \parallel p) + \lambda H(G)$
Inforganic Codex	$\epsilon_I = y - \hat{y}$	$\pi_I = \frac{1}{\mathrm{Var}[\epsilon_I]}$	$H(M)$	$\pi_I \mathbb{E}[\epsilon_I^2] + \mathrm{KL}(q \parallel p) + \lambda H(M)$

Table 1: Summary of domain-specific terms across systems. The KL-divergence term is $\mathrm{KL}(q(\theta^{(\ell)}) \parallel p(\theta^{(\ell)} | \theta^{(\ell+1)}))$.