

# Quantifying Paradox and Coherence in Structured Systems

## Abstract

Paradoxes are formalized as quantifiable properties of symbolic systems, distinct from physical reality, as articulated by Rutt (2025) [?]. Through functorial defects, cohomological obstructions, and phase-coherence leakage, we develop entropy-based diagnostics for detecting inconsistencies in structured domains. These mechanisms are integrated with RSVP theory's semantic fields and Yuan et al.s (2024) causal emergence framework [?], providing a unified approach to analyzing symbolic and informational coherence. Extended theoretical derivations, illustrative examples, and analytical expansions illustrate applications to artificial intelligence (AI) reasoning and complex systems analysis, with causal emergence offering insights into resolving micro-level paradoxes at macro-level abstractions.

## 1 Introduction

### 1.1 Motivation: Paradoxes as Symbolic Artifacts

Paradoxes present critical challenges for formal reasoning and cognitive systems. In AI, unresolved paradoxes can generate undecidable states or conflicting inferences. In cognitive science, they reveal the limitations of symbolic processing, and in network theory, paradoxes resemble desynchronization phenomena in complex dynamical systems. Formal quantification provides a robust framework to monitor, diagnose, and potentially resolve paradoxes.

### 1.2 Symbolic Systems vs. Physical Reality

Rutt (2025) emphasizes that paradoxes emerge from symbolic representations rather than physical reality [?]. For example, the Cretan liar paradox (Im a Cretan and all Cretans always lie) becomes contradictory only after semantic interpretation. Gödel's incompleteness theorems demonstrate that sufficiently expressive formal systems contain undecidable propositions, highlighting intrinsic limitations without invoking the physical world. This perspective frames paradoxes as diagnostic artifacts, prompting model refinement.

### 1.3 Objectives

Quantify paradox using category-theoretic (functorial defects), cohomological (obstruction classes), and oscillatory (phase-coherence leakage) metrics.

Link these metrics to entropy production for systemic diagnostics.

Integrate RSVP theory's semantic fields with causal emergence to analyze multi-scale coherence transitions.

## 1.4 Paper Structure

Section 2: Foundational concepts.

Section 3: Formal mathematical framework.

Section 4: Entropy-based diagnostics.

Section 5: Alignment with RSVP and causal emergence.

Section 6: Extended case studies.

Section 7: Discussion and implications.

Section 8: Conclusion.

## 2 Background and Prerequisites

### 2.1 Symbolic Systems and Paradox

Symbolic systems assign meaning to sequences of tokens through rules and evaluation functions. Paradoxes arise from internal inconsistencies, exemplified by:

Cretan Liar: Self-reference yields logical contradiction.

Gödel Sentences: Undecidable propositions emerge in sufficiently expressive formal systems.

Currys Paradox: Exploits self-referential implication to generate contradiction.

Berry Paradox: Semantic constructions (the smallest positive integer not definable in under twenty words) highlight limitations in syntactic representation.

Paradoxes can be mitigated or resolved under alternative logical frameworks (e.g., paraconsistent or relevance logics), emphasizing their representational, rather than ontological, nature.

### 2.2 RSVP Theory Essentials

RSVP theory formalizes structured systems using three interrelated fields:

Scalar Field ( $\mathcal{S}$ ): Represents semantic or informational potential.

Vector Field ( $\mathcal{V}$ ): Captures directional flow of information or causality.

Entropy Field ( $\mathcal{E}$ ): Measures uncertainty, misalignment, or incoherence.

Semantic projections map plenum states to symbolic states. Misalignment in  $\mathcal{S}$ ,  $\mathcal{V}$ , or  $\mathcal{E}$  indicates paradoxes or coherence failures.

### 2.3 Causal Emergence

Causal emergence identifies macro-level causal structures that reduce or resolve micro-level inconsistencies [?]:

Effective Information (EI): Quantifies specificity of macro-level causation.

Transfer Entropy (TE): Measures directed information flow between system components.

Time-Density Metrics: Assess temporal structure in emergent causal interactions.

Mapping micro-level paradoxes (e.g.,  $\mathcal{D}_F$ , Obs) to macro-level emergent structures provides a mechanism to integrate RSVP projections with coherent system behavior.

## 2.4 Mathematical Prerequisites

Categories and Functors: Objects (states) and morphisms (transformations) with structure-preserving mappings.

Sheaves and Cohomology: Assign consistent local data to open sets; detect global inconsistencies via cocycles.

Entropy Measures: Shannon entropy and KullbackLeibler divergence.

Oscillatory Dynamics: Kuramoto-type models with order parameter capture phase coherence.

## 3 Functorial and Cohomological Formalism

### 3.1 Functorial Defects

For a plenum category  $\mathcal{P}$  and symbolic category  $\mathcal{S}$ , projection  $F$  defines:

$$\Delta_F(f, g) = F(g \circ f) - F(g) \circ F(f), \quad (1)$$

with aggregated defect:

$$\mathcal{D}_F(x, t) = \sum_{\gamma \in \Gamma_x} w_\gamma(x) \|\Delta_F(\gamma)\|. \quad (2)$$

Residual entropy current:

$$\nabla \cdot \mathbf{J}_{\text{res}} = \kappa \mathcal{D}_F, \quad \partial_t S_{\mathcal{S}} + \nabla \cdot (\mathbf{J}_{\mathcal{S}} + \mathbf{J}_{\text{res}}) = \sigma_{\mathcal{S}} + \kappa \mathcal{D}_F. \quad (3)$$

Higher-order defects for triples of morphisms can capture compounded inconsistencies in more complex feedback loops.

### 3.2 Cohomological Obstructions

For sheaf  $\mathcal{F}$  over  $U$ :

$$\text{Obs}(F, \gamma) = [\{h_{ijk}^\gamma\}] \in H^2(U, \mathcal{A}_\gamma), \quad (4)$$

with norm:

$$\|\text{Obs}\| = \inf_{[c]=\text{Obs}} \left( \sum_{i,j,k} \int_{U_{ijk}} \|c_{ijk}(x)\|^2 S_{\mathcal{P}}(x) dx \right)^{1/2}. \quad (5)$$

Nonzero Obs signals that no global homotopy-coherent lift exists.

### 3.3 Phase-Coherence Leakage

For ensemble phases  $\phi_j$ :

$$Re^{i\Phi} = \frac{1}{N} \sum_{j=1}^N e^{i\phi_j}, \quad \mathcal{E}_\phi = 1 - R, \quad \sigma_\phi = \eta \mathcal{E}_\phi + \zeta \text{Var}(\dot{\phi}_j). \quad (6)$$

Temporal desynchronization serves as an oscillatory measure of paradox or incoherence.

## 4 Combined Entropy-Based Diagnostics

Total entropy:

$$\sigma_{\text{total}} = \sigma_{\mathcal{P}} + \kappa \mathcal{D}_F + \gamma_1 \| \text{Obs} \| + \sigma_{\phi}. \quad (7)$$

Threshold-based paradox detection:

$$\int_U \mathcal{D}_F dx > \epsilon_D \quad \vee \quad P(U, t) > \Theta \quad \vee \quad \max_{\gamma \subset U} \Delta_{\phi}(\gamma) > \eta_{\phi}. \quad (8)$$

Sensitivity analyses can refine thresholds using analytic expansions.

## 5 Alignment with RSVP and Causal Emergence

Functorial Defects: Mismatch in  $\mathcal{S}$ ; correspond to micro-causal inefficiencies.

Cohomological Obstructions: Gluing failure in  $\mathcal{E}$ ; parallel macro-level causal barriers.

Phase Leakage: Temporal desynchronization; mirrors emergent temporal incoherence.

Macro-level causal patterns can resolve paradoxes by reorganizing micro-level inconsistencies, a direct link to Yuan et al.s causal emergence metrics.

## 6 Extended Case Studies

### 6.1 Classical Symbolic Paradoxes

Cretan Liar.

Gödel Sentences: Undecidable propositions yield nonzero obstruction classes  $\text{Obs}$ , indicating structural limitations in the symbolic system.

Currys Paradox: Self-referential implications amplify functorial defects, increasing  $\mathcal{D}_F$  and associated entropy, which flags potential contradictions at multiple levels of abstraction.

Berry Paradox: Semantic ambiguity and encoding constraints manifest as high  $\sigma_{\phi}$  in oscillatory representations, highlighting conflicts between syntactic compression and semantic expressivity.

### 6.2 Analytical Examples in RSVP Framework

Semantic Projection Misalignment: Consider two plenum states  $p_1, p_2$  with a projection  $F$ . If local interactions imply  $F(p_1 \rightarrow p_2)$  but  $F(p_2 \rightarrow p_1)$  contradicts, functorial defect  $\mathcal{D}_F$  arises. Aggregating over all paths in a closed feedback loop yields  $\sigma_{\text{total}}$ , quantifying paradox intensity.

Cohomological Cohesion Test: For a cover  $\{U_i\}$  of plenum domain  $U$ , assign local semantic lifts  $\tilde{F}_i$ . Triple overlaps  $U_{ijk}$  produce cocycles  $h_{ijk}$ . Nontrivial classes indicate unresolved global inconsistencies, quantifying the paradox amplitude  $P$  analytically.

### 6.3 Macro-Level Emergence from Micro-Level Defects

Functorial Resolution via Causal Emergence: Micro-level defects can partially cancel when aggregated into macro-level states. Using effective information EI and transfer

entropy  $T$ , one can identify macro-causal structures where  $\mathcal{D}_F$  and  $\|\text{Obs}\|$  are minimized, effectively resolving local paradoxes.

**Phase-Coherence Integration:** Temporal desynchronization measures  $\sigma_\phi$  can be reduced by enforcing synchrony conditions across interacting oscillatory sub-networks, reflecting emergent coherence.

**Analytical Insight:** Let  $\mathcal{M}$  denote macro-level aggregation of microstates. Then for macro-causal mapping  $G$ , the effective paradox metric satisfies:

$$\mathcal{D}_G \leq \sum_{i=1}^N \mathcal{D}_F(p_i), \quad (9)$$

## 6.4 Taxonomy of Symbolic Paradoxes

Paradoxes can be classified according to their structural origin and the nature of their coherence failure. Let  $\mathcal{T}$  denote the category of paradox types:

1. Self-Referential Paradoxes ( $\mathcal{T}_{\text{self}}$ )

Examples: Liar paradox, Curry's paradox.

Characterized by feedback loops in symbolic morphisms:  $\gamma : s \rightarrow s$ .

Functorial defect:  $\Delta_F(\gamma)$  systematically, leading to persistent entropy production

$\sigma_{\text{total}}$ .

2. Semantic Ambiguity Paradoxes ( $\mathcal{T}_{\text{amb}}$ )

Examples: Berry paradox, the smallest positive integer not nameable in under 10 words.

Arise from underspecified or contradictory semantic assignments.

Cohomological obstruction Obs due to gluing failure across local semantic charts.

3. Rule-Conflict Paradoxes ( $\mathcal{T}_{\text{rule}}$ )

Examples: Russell's paradox in set theory.

Generated by incompatible axioms; functorial mapping fails at the level of morphism composition.

Phase-coherence leakage  $\sigma_\phi$  occurs when iterative applications of rules desynchronize symbolic state trajectories.

4. Combinatorial Explosion Paradoxes ( $\mathcal{T}_{\text{comb}}$ )

High-dimensional paradoxes arising from exhaustive enumeration, e.g., certain decision-theoretic constructions.

Metric:  $\sigma_{\text{total}}$  grows combinatorially with system size, providing a quantitative handle on computational complexity-induced paradoxes.

These categories map naturally onto RSVPs semantic fields:

$$\begin{aligned} \mathcal{S} &\longleftrightarrow \text{functorial defects,} \\ \mathcal{V} &\longleftrightarrow \text{directional information flow,} \\ \mathcal{E} &\longleftrightarrow \text{global coherence/entropy balance.} \end{aligned}$$

## 6.5 Textual Entropy-Flow Representation

Although graphical diagrams are omitted, we formalize entropy flow in textual terms:

Consider a symbolic system  $\mathcal{S}$  over plenum domain  $\mathcal{P}$ .

Define entropy flux chains as sequences  $\mathcal{C} = (c_1, c_2, \dots, c_n)$ , where each element represents contributions from functorial, cohomological, or oscillatory mechanisms.

The total chain entropy is:

$$\Sigma_{\mathcal{C}} = \sum_i \|\mathbf{J}_{\mathcal{S}}^{(i)}\| + \sum_i \|\mathbf{J}_{\text{res}}^{(i)}\| + \sum_i \sigma_{\phi}^{(i)}. \quad (10)$$

Positive contributions indicate unresolved paradoxes.

Cancellation between micro-level chains reduces macro-level  $\Sigma_{\mathcal{C}}$ , representing emergent coherence.

Example (Textual Chain Representation):

Node s1:  $\Delta_F = 0.3, \text{Obs} = 0.2, \text{sigma}_p \text{hi} = 0.1$

Node s2:  $\Delta_F = 0.1, \text{Obs} = 0, \text{sigma}_p \text{hi} = 0.05$

Node s3:  $\Delta_F = 0.2, \text{Obs} = 0.1, \text{sigma}_p \text{hi} = 0.2$

Macro-chain sum:  $\Sigma_{\mathcal{C}} = 0.3 + 0.2 + 0.1 + 0.1 + 0 + 0.05 + 0.2 + 0.1 + 0.2 = 1.25$

This textual framework permits formal reasoning about entropy flow without graphical depictions and can be generalized to arbitrary hierarchical structures.

## 6.6 Multi-Step Macro-Level Coherence

Let  $\mathcal{M}_k$  denote the macro-level system after  $k$  aggregation steps of microstates  $p_i$ .

### 6.6.1 Iterative Functorial Aggregation

Define macro-projection  $G_k$  recursively:

$$G_0 = F, \quad G_{k+1} = \text{Aggregate}(G_k(\mathcal{M}_k)), \quad k = 0, 1, \dots, K. \quad (11)$$

Aggregation rules reduce  $\mathcal{D}_F$ .

Effective paradox metric at step  $k$ :

$$P_k = \alpha \|\text{Obs}_{G_k}\| + \beta \mathcal{L}_{G_k}. \quad (12)$$

### 6.6.2 Hierarchical Phase-Coherence Adjustment

For oscillatory microstates  $\phi_j^{(k)}$ , define coarse-grained phases  $\bar{\phi}^{(k)}$ .

Phase leakage at macro-level:

$$\sigma_{\phi}^{(k)} = 1 - \frac{1}{|\mathcal{M}_k|} \left| \sum_{j \in \mathcal{M}_k} e^{i\phi_j} \right|. \quad (13)$$

### 6.6.3 Entropy Flow and Emergence

Total entropy at step  $k$ :

$$\sigma_{\text{total}}^{(k)} = \sigma_{\mathcal{P}}^{(k)} + \kappa \mathcal{D}_{G_k} + \gamma_1 \|\text{Obs}_{G_k}\| + \sigma_{\phi}^{(k)}. \quad (14)$$

## 7 Discussion

Paradoxes, when formalized in this framework, become measurable aspects of symbolic and informational coherence. Key implications:

**AI Reasoning:** Entropy-based diagnostics provide a systematic method for detecting logical inconsistencies in automated reasoning systems, supporting robust inference pipelines.

**Cognitive Modeling:** Phase-coherence and functorial defects model the cognitive processing of contradictory information, offering quantitative tools for studying human reasoning under uncertainty and paradoxical stimuli.

**Complex Systems Analysis:** Cohomological obstructions capture global network-level inconsistencies, linking local misalignments to systemic failures or emergent patterns.

**Causal Emergence Integration:** Macro-level emergent structures can absorb micro-level inconsistencies, demonstrating that systemic coherence is a higher-order property rather than a mere sum of parts.

**Limitations and Open Questions:**

**Computational scalability:** High-dimensional symbolic and plenum spaces pose numerical challenges.

**Multi-scale interactions:** Optimal aggregation strategies for micro-to-macro causal mappings remain an open problem.

**Stochastic perturbations:** Real-world systems introduce noise, requiring refined measures for paradox detection and mitigation.

## 8 Extended Discussion and Future Directions

### 8.1 Theoretical Implications

The formalism developed in this work establishes a multi-layered approach to understanding and mitigating paradoxes in structured systems. By combining functorial defects, cohomological obstructions, and phase-coherence leakage, we have constructed a hierarchy-aware framework that quantifies symbolic inconsistencies and entropy production. Key theoretical insights include:

1. **Micro-to-Macro Coherence:** The integration of RSVP fields with causal emergence demonstrates that local paradoxes and inconsistencies do not necessarily propagate irreversibly; rather, they can be absorbed, redistributed, or resolved at higher abstraction levels, aligning with Yuan et al.s (2024) principles of emergent causality.

2. **Entropy as a Diagnostic and Control Metric:** Total entropy production,  $\sigma_{\text{total}}$ , functions not only as a descriptive measure of inconsistency but also as a control target for optimization. Hierarchical entropy minimization enables structured systems to maintain coherence across layers while suppressing paradox amplitudes.

3. **Interplay of Structure and Dynamics:** Functorial defects capture structural misalignments, cohomological obstructions identify global inconsistencies, and phase-coherence leakage reflects dynamic temporal misalignments. These mechanisms together reveal how structural, topological, and temporal aspects of a system interact to produce symbolic paradoxes.

4. **Formal Connections to Causal Emergence:** The hierarchical optimization algorithm aligns local entropy suppression with macro-level causal coherence. Micro-level inconsistencies are reconciled through upward and downward projections  $(\Phi, \Psi)$ , demonstrating

a formal mechanism by which causal emergence can be interpreted as paradox-resilient reorganization.

## 8.2 Practical Applications

The framework lends itself to multiple applications in AI, cognitive science, and complex systems:

1. **Artificial Intelligence:** Hierarchical RSVP systems can serve as diagnostic modules for AI reasoning engines, identifying latent paradoxes and optimizing symbolic representations to reduce conflict and enhance decision reliability.
2. **Cognitive Modeling:** Multi-layer semantic hierarchies model cognitive architectures with natural resolution of contradictory information, offering insights into human reasoning under uncertainty and paradoxical stimuli.
3. **Complex Networks:** Social, communication, or biological networks can be analyzed for emergent paradoxes and entropy imbalances. Layered optimization may guide interventions that maintain coherence and stability.
4. **Formal Verification and Logic Design:** Symbolic systems in software and hardware can incorporate entropy-based monitoring to ensure logical consistency, particularly in self-modifying or recursive architectures.

## 8.3 Extensions and Open Problems

While the current work establishes foundational formalism, several extensions remain open:

1. **Stochastic and Nonlinear Dynamics:** Incorporating stochastic perturbations and strongly nonlinear interactions in the hierarchical RSVP system could reveal new regimes of paradox emergence and resolution.
2. **Adaptive Thresholding:** Dynamically adjusting thresholds  $(\epsilon_D, \Theta, \eta_\phi)$  based on real-time feedback could improve responsiveness in AI and cognitive architectures.
3. **Scalability to Large Systems:** Optimizing high-dimensional plenum spaces with many layers presents computational challenges. Approximation schemes and coarse-grained projections could make the framework practical for large-scale simulations.
4. **Empirical Validation:** Implementing hierarchical RSVP in AI architectures or cognitive simulators will provide empirical evidence for the theoretical predictions, enabling refinement of functorial and cohomological parameters.
5. **Integration with Other Semantic Frameworks:** RSVP fields can potentially interface with alternative semantic representation systems, such as vector embeddings or category-theoretic knowledge graphs, allowing comparative studies of paradox dynamics.

## 8.4 Long-Term Vision

The hierarchical RSVP framework, combined with entropy-based diagnostics, represents a step toward robust symbolic reasoning systems capable of detecting, quantifying, and mitigating paradoxes autonomously. This approach bridges the gap between formal logic, information theory, and emergent causal dynamics, providing:

A quantitative theory of paradox applicable across symbolic, cognitive, and computational domains.



A mechanism for emergent coherence, reconciling local inconsistencies with global structural integrity.

A generalizable optimization framework that informs design of resilient AI, networked systems, and multi-layered cognitive architectures.

Future research will explore adaptive, real-time RSVP hierarchies, multi-agent extensions, and integration with causal inference frameworks to produce fully self-consistent, paradox-aware computational environments.

## Appendices

### A. Functorial and Cohomological Derivations

Derivation of  $\mathcal{D}_F$  from categorical morphism norms.

Calculation of  $\|\text{Obs}\|$  using ech cohomology over open covers.

### B. Phase-Coherence Calculations

Oscillatory ensemble order parameters  $R$ , phase entropy  $\sigma_\phi$ .

Analytical derivation of entropy contribution from phase variance.

### C. Causal Emergence Metrics

Effective information EI.

Transfer entropy  $T$  [?].

### D. Notation Table

Symbol Definition

$\mathcal{D}_F$  Functorial defect density  
 $\text{Obs}$  Cohomological obstruction class  
 $\sigma_\phi$  Phase-coherence entropy production  
 $P(U, t)$  Paradox amplitude  
 $R$  Oscillatory order parameter  
 $\Delta_\phi(\gamma)$  Phase misalignment measure

## E: Hierarchical Aggregation and Dynamic Functorial Defect Evolution

### E.1 Dynamic Functorial Defects

Let  $F_t$  denote a time-dependent functor mapping plenum states to symbolic states. For a path  $\gamma$  in  $\mathcal{P}$ , define the instantaneous defect as

$$\Delta_{F_t}(\gamma) := F_t(\gamma) - \prod_{\ell \in \gamma} F_t(\ell), \quad \mathcal{D}_{F_t}(x) := \sum_{\gamma \in \Gamma_x} w_\gamma(x) \|\Delta_{F_t}(\gamma)\|, \quad (15)$$

Evolution Equation: Assume the functor evolves under local feedback from semantic field gradients and vector flows  $\mathcal{V}$ :

$$\frac{dF_t}{dt} = -\eta \frac{\delta \mathcal{D}_{F_t}}{\delta F_t} + \mathcal{H}(\mathcal{S}, \mathcal{V}, \mathcal{E}), \quad (16)$$

The first term drives toward defect minimization.

The second term encodes contextual or macro-level constraints, ensuring that local adjustments respect global coherence.

Remark: This formulation parallels gradient flow in functional spaces, and ensures  $F_t$  evolves continuously under smooth field variations.

## E.2 Hierarchical Aggregation

Let the symbolic domain be organized into a hierarchy of macro-level states  $\mathcal{M}_k$ , where each level aggregates a collection of lower-level microstates:

$$\mathcal{M}_k := \bigcup_{i=1}^{n_k} \mathcal{M}_{k-1}^{(i)}. \quad (17)$$

Aggregation Functor: Define  $G_k$  mapping micro-level symbolic states to macro-level abstractions.

Macro-Level Defect: For a macro-path  $\gamma^{(k)}$  at level  $k$ :

$$\mathcal{D}_{F_t}^{(k)} := \sum_{\gamma \in \Gamma^{(k)}} w_{\gamma}^{(k)} \|\Delta_{G_k \circ F_t}(\gamma)\|. \quad (18)$$

Recursive Property: Let  $\lambda_k$  represent the defect dissipation factor at level  $k$ . Then:

$$\mathcal{D}_{F_t}^{(k)} \leq \lambda_k \mathcal{D}_{F_t}^{(k-1)}, \quad (19)$$

## E.3 Time-Dependent Cohomological Obstructions

Let  $\text{Obs}_t(F_t, \gamma)$  denote the dynamic obstruction class over a path  $\gamma$ . Its time evolution is given by

$$\frac{d}{dt} \text{Obs}_t(F_t, \gamma) = \delta_{\text{ech}} \left( \frac{d\eta_{ij,t}}{dt} \right) + \Phi(\mathcal{E}), \quad (20)$$

where  $\tilde{F}_{i,t}$  are time-dependent local lifts.

$\delta_{\text{ech}}$  is the ech coboundary operator.

$\Phi(\mathcal{E})$  encodes the effect of entropy flow on obstruction evolution.

Proposition E.1: If  $\Phi$  is bounded and  $\frac{d\eta_{ij,t}}{dt} \rightarrow 0$  as  $t \rightarrow \infty$ , then  $\text{Obs}_t$  converges to a stationary class  $\text{Obs}_{\infty}$ , representing residual global paradoxes that cannot be resolved by local adjustments.

Proof Sketch:

Assume  $\|\Phi\| \leq M$  as  $t \rightarrow \infty$ .

Local lifts stabilize, implying  $\frac{d\eta_{ij,t}}{dt} \rightarrow 0$ .

Coboundary term vanishes in the limit; only bounded  $\Phi$  contributes.

Convergence of  $\text{Obs}_t$  follows.

## E.4 Hierarchical Aggregation with Dynamic Functors

Macro-Level Obstruction: For macro-level aggregation:

$$\text{Obs}_t^{(k)} := \bigcup_{i=1}^{n_k} \text{Obs}_t^{(k-1),(i)} \cdot G_k(\mathcal{D}_{F_t}^{(k-1),(i)}), \quad (21)$$

Decay Property: Suppose  $\lambda_k$  is uniform across levels; then the macro-obstruction satisfies

$$\|\text{Obs}_t^{(k)}\| \leq \lambda_k \sum_{i=1}^{n_k} \|\text{Obs}_t^{(k-1),(i)}\|, \quad (22)$$

Corollary E.2: Under recursive aggregation, the system exhibits geometric decay of paradox intensity:

$$\|\text{Obs}_t^{(K)}\| \leq \prod_{k=1}^K \lambda_k \|\text{Obs}_t^{(0)}\|, \quad (23)$$

## E.5 Continuous-Time Limit and Field Representations

In the continuum limit, define a functor field  $F(x, t)$  over plenum coordinates  $x$ :

$$\frac{\partial F}{\partial t} = -\eta \frac{\delta \mathcal{D}[F]}{\delta F} + \mathcal{H}[\mathcal{S}, \mathcal{V}, \mathcal{E}]. \quad (24)$$

Macro-Level Fields: Introduce level-dependent projections  $G_k$  mapping micro-fields to macro-fields.

Entropy-Constrained Evolution: Total macro-entropy

$$\sigma_{\text{total}}^{(k)}(t) = \int_{\Omega} \left( \mathcal{D}_F^{(k)} + \|\text{Obs}^{(k)}\| + \sigma_{\phi}^{(k)} \right) dx \quad (25)$$

## E.6 Summary

Appendix E formalizes:

Dynamic Functorial Defects: Time-evolving mappings from plenum to symbolic states.

Hierarchical Aggregation: Macro-level projection operators that reduce residual defects.

Dynamic Cohomological Obstructions: Continuous-time evolution of obstruction classes.

Decay Properties: Geometric attenuation of paradox intensity across hierarchical levels.

Continuum Limit: PDE-style representation of functor fields and entropy evolution.

## F: Analytical Bounds on Defect Dissipation and Optimal Coupling

### F.1 Preliminaries

Let  $\mathcal{D}_{F_t}(x, t)$  denote the dynamic functorial defect density at point  $x$  and time  $t$ , and let  $\sigma_{\phi}(x, t)$  denote the phase-based entropy production. Define the total local defect metric:

$$\Sigma(x, t) := \mathcal{D}_{F_t}(x) + \|\text{Obs}_t(F_t, \gamma_x)\| + \sigma_{\phi}(x, t), \quad (26)$$

We consider a gradient-flow evolution for  $F_t$ :

$$\frac{dF_t}{dt} = -\eta \frac{\delta \mathcal{D}[F_t]}{\delta F_t} + \mathcal{H}[\mathcal{S}, \mathcal{V}, \mathcal{E}], \quad (27)$$

## F.2 Defect Dissipation Rate

Definition F.1 (Defect Dissipation Rate): The global dissipation rate at time  $t$  is

$$R_{\mathcal{D}}(t) := -\frac{d}{dt} \int_{\Omega} \mathcal{D}_{F_t}(x) dx. \quad (28)$$

Proposition F.1 (Upper Bound): Assume  $\mathcal{H}$  is Lipschitz continuous with constant  $L$ . Then

$$0 \leq R_{\mathcal{D}}(t) \leq \eta \int_{\Omega} \left\| \frac{\delta \mathcal{D}[F_t]}{\delta F_t} \right\|^2 dx + L \int_{\Omega} \|\mathcal{H}\| dx. \quad (29)$$

Proof Sketch:

Multiply the evolution equation by  $\frac{\delta \mathcal{D}[F_t]}{\delta F_t}$  and integrate over  $\Omega$ .

Apply the Cauchy-Schwarz inequality and Lipschitz bound for  $\mathcal{H}$ .

Rearrange to yield the stated bound.

Remark: The bound indicates that dissipation scales quadratically with the functional gradient and linearly with higher-order field interactions.

## F.3 Optimal Coupling Parameter

Let  $\eta_{\text{opt}}$  denote the optimal coupling that maximizes defect dissipation while avoiding instability.

Lemma F.2 (Optimal Coupling Condition): Under quadratic approximation  $\mathcal{D} \approx \frac{1}{2} \langle F, \mathcal{L} F \rangle$  with symmetric positive-definite operator  $\mathcal{L}$ :

$$R_{\mathcal{D}}(\eta) = \eta \langle F, \mathcal{L}^2 F \rangle - \eta^2 \langle F, \mathcal{L}^3 F \rangle. \quad (30)$$

Maximization: Solve  $\frac{dR_{\mathcal{D}}}{d\eta} = 0$ :

$$\eta_{\text{opt}} = \frac{\langle F, \mathcal{L}^2 F \rangle}{2 \langle F, \mathcal{L}^3 F \rangle}. \quad (31)$$

This provides a closed-form estimate for the learning rate or coupling parameter that maximizes defect reduction.

## F.4 Bounds for Phase-Coherence Entropy

For oscillatory ensembles with order parameter  $R(t)$ , the phase entropy production satisfies:

$$\sigma_{\phi}(t) = \eta_{\phi} (1 - R(t)) + \zeta \text{Var}(\dot{\phi}_j), \quad (32)$$

Proposition F.3 (Maximum Dissipation Condition): If  $\text{Var}(\dot{\phi}_j) \leq V_{\text{max}}$  and  $R(t) \geq R_{\text{min}}$  for coupling  $K$ :

$$\max \sigma_{\phi} = \eta_{\phi} + \zeta \max \text{Var}(\dot{\phi}_j), \quad (33)$$

Corollary: The optimal oscillator coupling ensures  $R \rightarrow 1$  to balance coherence and dissipation.

## F.5 Hierarchical Aggregation Bounds

Let aggregation levels have dissipation factors  $\lambda_k$ . Define the total macro-level defect:

$$\mathcal{D}_{\text{tot}}^{(K)}(t) := \sum_{k=1}^K \mathcal{D}_{F_t}^{(k)}. \quad (34)$$

Proposition F.4 (Geometric Decay Bound):

$$\mathcal{D}_{\text{tot}}^{(K)}(t) \leq \mathcal{D}_{\text{tot}}^{(0)}(t) \prod_{k=1}^K \lambda_k. \quad (35)$$

Hierarchical aggregation ensures exponential attenuation of micro-level defects in the macro-state.

## F.6 Summary and Practical Implications

Defect dissipation rate is bounded by functional gradients and higher-order interactions.

Optimal coupling can be computed from quadratic approximations of  $\mathcal{D}$ .

Phase entropy is maximized at minimal coherence; coupling must balance coherence and dissipation.

Hierarchical aggregation provides exponential attenuation of residual paradoxes.

These analytical bounds provide actionable guidelines for AI system design, RSVP-inspired symbolic processing, and multi-scale coherence management.

## G: Probabilistic Paradox Suppression and Stochastic Bounds

### G.1 Preliminaries

Let  $\mathcal{S}$  be a symbolic system,  $\mathcal{P}$  its plenum of microstates, and  $F$  the projection functor. Introduce a probability measure  $\mu_{\mathcal{P}}$  over  $\mathcal{P}$  and define the probabilistic paradox metric:

$$\mathbb{P}[P > \Theta] := \mu_{\mathcal{P}}(\{x \in \mathcal{P} \mid P(x) > \Theta\}), \quad (36)$$

### G.2 Stochastic Functorial Defects

Define the random functorial defect as a stochastic variable due to microstate uncertainty:

$$\Delta_F(x) = \Delta_F^0(x) + \xi(x), \quad \mathbb{E}[\xi(x)] = 0, \quad \text{Var}[\xi(x)] = \sigma_{\xi}^2. \quad (37)$$

Proposition G.1 (Expectation and Variance):

$$\mathbb{E}[\mathcal{D}_F] = \sum_{\gamma \in \Gamma} w_{\gamma} \mathbb{E}[\|\Delta_F(\gamma)\|], \quad \text{Var}[\mathcal{D}_F] = \sum_{\gamma} w_{\gamma}^2 \sigma_{\xi}^2. \quad (38)$$

Remark: Random perturbations increase the effective paradox probability, necessitating probabilistic thresholds for robust detection.

### G.3 Probabilistic Cohomological Obstructions

For obstruction class Obs, let  $c_{ijk}^\gamma$  be stochastic 2-cochains with zero-mean fluctuations  $\epsilon_{ijk}$ :

$$c_{ijk}^\gamma = \bar{c}_{ijk}^\gamma + \epsilon_{ijk}, \quad \mathbb{E}[\epsilon_{ijk}] = 0. \quad (39)$$

Proposition G.2 (Bound on Stochastic Obstruction Norm):

$$\mathbb{E}[\|\text{Obs}\|] \leq \|\overline{\text{Obs}}\| + \sqrt{\sum_{i,j,k} \text{Var}[\epsilon_{ijk}]}. \quad (40)$$

The stochastic term inflates the effective obstruction, increasing paradox likelihood in low-coherence regimes.

### G.4 Transfer Entropy and Effective Information Bounds

Let  $X_t, Y_t$  be microstate ensembles at times  $t$ . The transfer entropy measures directed information:

$$T_{X \rightarrow Y} = \sum_{x_t, y_{t+1}, y_t} p(x_t, y_{t+1}, y_t) \log \frac{p(y_{t+1}|y_t, x_t)}{p(y_{t+1}|y_t)}. \quad (41)$$

Proposition G.3 (Probabilistic Upper Bound): For stochastic microstates with variance  $\sigma_y^2$ , the expected transfer entropy satisfies

$$\mathbb{E}[T_{X \rightarrow Y}] \leq \frac{1}{2} \log \left( 1 + \frac{\text{Var}[\mathbb{E}[Y|X]]}{\sigma_y^2} \right), \quad (42)$$

This establishes a quantitative connection between micro-level uncertainty and macro-level causal predictability.

Corollary: Maximizing  $T_{X \rightarrow Y}$  reduces the probabilistic paradox rate, providing an information-theoretic mechanism for paradox suppression.

### G.5 Probabilistic Paradox Thresholding

Define the stochastic paradox suppression function:

$$\Pi_\Theta(x) := \mathbb{P}[P(x) > \Theta] = \int_\Omega \mathbf{1}_{\{P(x) > \Theta\}} d\mu_{\mathcal{P}}(x), \quad (43)$$

Adjusting  $\Theta$  balances sensitivity and false-positive rates.

In multi-scale RSVP systems, hierarchical aggregation reduces  $\Pi_\Theta$  via macro-level averaging, consistent with causal emergence principles.

### G.6 Hierarchical Stochastic Bounds

Let micro-level defect  $\mathcal{D}_F^{(0)}$  propagate to macro-level defect through aggregation factor  $\lambda_k$ . Stochastic fluctuations yield

$$\text{Var}[\mathcal{D}_F^{(K)}] \leq \sum_{k=1}^K \lambda_k^2 \text{Var}[\mathcal{D}_F^{(k)}]. \quad (44)$$

Ensures variance attenuation at higher abstraction levels.

Implies probabilistic paradoxes are naturally suppressed in well-coupled, multi-scale symbolic systems.

## G.7 Summary

Micro-level stochasticity introduces random functorial defects and cohomological obstructions.

Expected transfer entropy and effective information provide upper bounds for macro-level causal predictability.

Probabilistic paradox thresholds ( $\Pi_\Theta$ ) offer robust detection under uncertainty.

Hierarchical aggregation ensures variance reduction, aligning with causal emergence and RSVP field integration.

These formal bounds guide AI reasoning, symbolic coherence monitoring, and entropy-aware system design.

## H: Entropy Flow in Multi-Layer Semantic Fields

### H.1 Multi-Layer Semantic Field Setup

Let the RSVP plenum consist of layers  $l = 1, \dots, L$  of semantic fields  $\mathcal{S}^{(l)}, \mathcal{V}^{(l)}, \mathcal{E}^{(l)}$ . Each layer represents a level of abstraction:

Scalar Field ( $\mathcal{S}^{(l)}$ ): Semantic potential at layer  $l$ .

Vector Field ( $\mathcal{V}^{(l)}$ ): Directional flow of information or causal influence.

Entropy Field ( $\mathcal{E}^{(l)}$ ): Local uncertainty or symbolic inconsistency.

The projection functor  $F^{(l)}$  maps microstates to layer  $l$  semantic representations.

### H.2 Entropy Continuity Equation

For each layer, define the total entropy density:

$$S^{(l)}(x, t) = S_S^{(l)}(x, t) + S_V^{(l)}(x, t) + S_E^{(l)}(x, t), \quad (45)$$

$$\partial_t S^{(l)} + \nabla \cdot \mathbf{J}^{(l)} = \sigma_{\text{total}}^{(l)}, \quad \mathbf{J}^{(l)} = \mathbf{J}_S^{(l)} + \mathbf{J}_V^{(l)} + \mathbf{J}_E^{(l)}, \quad (46)$$

$$\sigma_{\text{total}}^{(l)} = \kappa \mathcal{D}_F^{(l)} + \gamma \|\text{Obs}^{(l)}\| + \eta \mathcal{E}_\phi^{(l)} + \xi(x, t), \quad (47)$$

### H.3 Inter-Layer Coupling

Layers are coupled via downward and upward semantic projections:

Upward coupling: Micro-to-macro influence

$$\mathcal{S}^{(l+1)} = \Phi^{(l \rightarrow l+1)}(\mathcal{S}^{(l)}, \mathcal{V}^{(l)}, \mathcal{E}^{(l)}), \quad (48)$$

Downward coupling: Macro-to-micro feedback

$$\mathcal{S}^{(l)} = \Psi^{(l+1 \rightarrow l)}(\mathcal{S}^{(l+1)}, \mathcal{V}^{(l+1)}, \mathcal{E}^{(l+1)}). \quad (49)$$

The functions  $\Phi$  and  $\Psi$  enforce semantic coherence across layers, analogous to renormalization flows in physics.

## H.4 Stochastic PDE for Entropy Flow

Define layer-wise stochastic partial differential equations (SPDEs):

$$\partial_t S^{(l)} + \nabla \cdot \mathbf{J}^{(l)} = \kappa \mathcal{D}_F^{(l)} + \gamma \|\text{Obs}^{(l)}\| + \eta \mathcal{E}_\phi^{(l)} + \sigma_\xi^{(l)} \mathcal{W}^{(l)}(x, t), \quad (50)$$

where  $\mathcal{W}^{(l)}$  is layer-specific white noise.

The SPDE formalism captures randomized symbolic inconsistencies, phase desynchronization, and macro-level smoothing due to inter-layer causal emergence.

## H.5 Phase-Coherence Leakage Across Layers

Define layer-wise Kuramoto order parameter  $R^{(l)}$ :

$$R^{(l)}(x, t) e^{i\Phi^{(l)}(x, t)} = \frac{1}{N_l} \sum_{j=1}^{N_l} e^{i\phi_j^{(l)}(x, t)}, \quad \mathcal{E}_\phi^{(l)} = 1 - R^{(l)}. \quad (51)$$

Entropy production from phase leakage propagates across layers via:

$$\mathcal{E}_\phi^{(l+1)} = \Lambda^{(l \rightarrow l+1)}(\mathcal{E}_\phi^{(l)}), \quad \mathcal{E}_\phi^{(l-1)} = \Gamma^{(l \rightarrow l-1)}(\mathcal{E}_\phi^{(l)}), \quad (52)$$

where  $\Lambda, \Gamma$  are projection operators.

## H.6 Total Multi-Layer Entropy

The aggregate entropy across layers:

$$S_{\text{total}}(x, t) = \sum_{l=1}^L S^{(l)}(x, t), \quad \sigma_{\text{total}}(x, t) = \sum_{l=1}^L \sigma_{\text{total}}^{(l)}. \quad (53)$$

Macro-level paradox suppression arises when upward projections dominate stochastic fluctuations, effectively averaging micro-level defects.

Layer-resolved diagnostics allow identification of paradox-prone subspaces in  $\mathcal{P}$ .

## H.7 Summary and Implications

Multi-layer SPDEs formalize entropy flow in RSVP semantic fields, incorporating stochastic, functorial, and phase-coherence effects.

Inter-layer coupling captures micro-to-macro coherence emergence, aligning with causal emergence principles.

Phase-coherence leakage provides temporal diagnostic for symbolic desynchronization.

Total entropy serves as a holistic measure of system consistency, guiding AI reasoning and symbolic field design.

# I: Optimization Strategies for Paradox Minimization in Hierarchical RSVP Systems

## I.1 Problem Formulation

Let the RSVP plenum be a hierarchy of semantic layers  $l = 1, \dots, L$ . The objective is to minimize total entropy production across the hierarchy while preserving semantic functionality:



$$\min_{\{F^{(l)}, \Phi^{(l \rightarrow l+1)}, \Psi^{(l+1 \rightarrow l)}\}} \int_{\Omega} \sigma_{\text{total}}(x, t) dx \quad (54)$$

subject to:

$$\begin{cases} \partial_t S^{(l)} + \nabla \cdot \mathbf{J}^{(l)} = \sigma_{\text{total}}^{(l)}, & l = 1, \dots, L \\ S^{(l)}(x, t) \geq 0, & \forall x, t \\ \mathcal{V}^{(l)} \cdot \nabla S^{(l)} \leq \lambda_{\text{max}}^{(l)} \end{cases} \quad (55)$$

where  $\lambda_{\text{max}}^{(l)}$  bounds semantic flux at layer  $l$ .

## I.2 Layer-Wise Functorial Optimization

For each layer, define a layer-specific cost functional penalizing functorial defects and cohomological obstructions:

$$\mathcal{C}^{(l)}[F^{(l)}] = \int_{\Omega} \left[ \kappa \mathcal{D}_F^{(l)}(x) + \gamma \|\text{Obs}^{(l)}(x)\| \right] dx \quad (56)$$

Optimization seeks a functor  $F^{(l)}$  minimizing  $\mathcal{C}^{(l)}$  while respecting boundary conditions of semantic projections. Gradient flow dynamics can be applied:

$$\frac{\partial F^{(l)}}{\partial \tau} = -\frac{\delta \mathcal{C}^{(l)}}{\delta F^{(l)}}, \quad \tau \text{ fictitious optimization time.} \quad (57)$$

## I.3 Inter-Layer Coherence Constraints

Entropy minimization requires coherence across layers. Define a cross-layer penalty:

$$\mathcal{P}^{(l \leftrightarrow l+1)} = \int_{\Omega} \|\mathcal{S}^{(l+1)} - \Phi^{(l \rightarrow l+1)}(\mathcal{S}^{(l)})\|^2 dx \quad (58)$$

Similarly, downward consistency:

$$\mathcal{Q}^{(l \leftrightarrow l-1)} = \int_{\Omega} \|\mathcal{S}^{(l-1)} - \Psi^{(l \rightarrow l-1)}(\mathcal{S}^{(l)})\|^2 dx \quad (59)$$

These terms are incorporated into the total cost functional:

$$\mathcal{J} = \sum_{l=1}^L \mathcal{C}^{(l)} + \lambda_{\text{up}} \sum_{l=1}^{L-1} \mathcal{P}^{(l \leftrightarrow l+1)} + \lambda_{\text{down}} \sum_{l=2}^L \mathcal{Q}^{(l \leftrightarrow l-1)} \quad (60)$$

where  $\lambda_{\text{up}}, \lambda_{\text{down}}$  balance inter-layer coupling.

## I.4 Phase-Coherence Optimization

Let  $\mathcal{V}^{(l)}$  denote the vector field at layer  $l$ . Let  $R^{(l)}$  denote the Kuramoto order parameter. Phase-coherence optimization seeks to maximize  $R^{(l)}$  while minimizing entropy:

$$\max_{\mathcal{V}^{(l)}} \int_{\Omega} R^{(l)}(x, t) dx \quad (61)$$

subject to:

$$\partial_t \mathcal{V}^{(l)} = -\eta \frac{\delta \sigma_{\phi}^{(l)}}{\delta \mathcal{V}^{(l)}} \quad (62)$$

where  $\sigma_{\phi}^{(l)}$  captures phase desynchronization effects.

## I.5 Hierarchical Optimization Algorithm

Initialize layers with default functors  $F^{(l)}$  and projections  $\Phi, \Psi$ .

Layer-wise functorial update: Gradient descent on  $\mathcal{C}^{(l)}$ .

Inter-layer coherence update: Minimize  $\mathcal{P}$  and  $\mathcal{Q}$  with projected gradients.

Phase coherence update: Adjust  $\mathcal{V}^{(l)}$  to maximize  $R^{(l)}$ .

Iterate until  $\sigma_{\text{total}}^{(l)} \leq \epsilon$  for all layers.

This procedure ensures entropy-minimal, paradox-resilient multi-layer semantic fields.

## I.6 Theoretical Guarantees

Convergence: Under convex cost functionals  $\mathcal{C}^{(l)}$  and Lipschitz continuous projections  $\Phi, \Psi$ , the algorithm converges to local minima of  $\mathcal{J}$ .

Paradox Suppression: By minimizing  $\mathcal{D}_F^{(l)}$  and  $\|\text{Obs}^{(l)}\|$ , local paradox amplitudes are reduced.

Causal Emergence Alignment: Macro-level consistency emerges naturally as upward projections average micro-level inconsistencies.

## I.7 Implications and Applications

AI reasoning systems: Enables automated detection and mitigation of paradoxes in symbolic reasoning.

Complex networks: Guides multi-scale coherence maintenance in communication or social networks.

Hierarchical cognitive modeling: Models micro-to-macro information flow in cognition, consistent with causal emergence principles.