Finite Spinodal Instability Regulated Growth in Cosmological Perturbation Theory

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Abstract

We present a fully variational construction of a conservative relativistic scalar-fluid model in which the free-energy density of a conserved number field includes a positive gradient regulator. The model is formulated on a Lorentzian manifold with explicit projection into the fluid rest frame, and the stress–energy tensor is derived from the action by functional variation with respect to the metric. We demonstrate covariant conservation of the stress–energy tensor under particle-number conservation and establish consistency of the Euler–Lagrange equations.

Embedded in a spatially flat Friedmann–Robertson–Walker spacetime, the theory preserves standard homogeneous cosmological evolution while modifying the linear perturbation spectrum. We derive the full set of linearized fluid and Einstein equations in longitudinal gauge and obtain, under clearly stated subhorizon and quasi-static approximations, a modified growth equation for density perturbations containing both k^2 and k^4 terms. The gradient regulator guarantees ultraviolet stability. When the effective compressibility becomes negative over a finite interval in scale factor, the dispersion relation admits a bounded instability band with a calculable fastest-growing mode.

The resulting scale-dependent growth factor defines a controlled and parameterizable deviation from pressureless matter evolution. Mathematical well-posedness, dimensional consistency, and stability structure are examined in detail. The model remains conservative, local, and ultraviolet stable while introducing a single additional length scale associated with the gradient regulator.

1 Introduction

The standard theory of cosmological structure formation treats nonrelativistic matter either as a collisionless component governed by the Vlasov equation or, in an effective limit, as a pressureless perfect fluid. In the matter-dominated regime and at linear order in scalar perturbations, the density contrast evolves according to a second-order differential equation in which gravitational attraction is balanced only by Hubble friction. In this limit, the growth of perturbations is scale independent on subhorizon scales.

Effective descriptions of matter often arise through coarse-graining procedures in which microscopic degrees of freedom are integrated out. Such procedures may generate additional stress contributions in the effective stress–energy tensor, including gradient terms associated with finite correlation lengths. In continuum mechanics, gradient-regularized free-energy functionals appear in the theory of capillarity and phase transitions, where they give rise to Korteweg-type stresses. In field-theoretic language, these terms correspond to higher-derivative operators suppressed by a characteristic length scale.

The purpose of the present work is to construct and analyze a minimal relativistic scalar-fluid model incorporating a positive gradient regulator in the free-energy density of a conserved number field. The construction is explicitly variational and conservative. No dissipative transport coefficients are introduced, and no modifications to the Einstein–Hilbert action are assumed. The homogeneous cosmological background remains unaltered, and all new effects arise exclusively in the perturbative sector.

The analysis proceeds in several stages. First, the geometric and variational formulation of the scalar fluid is developed in detail. The stress–energy tensor is obtained by functional differentiation of the action with respect to the metric, and its covariant conservation is established under the continuity equation. Second, the theory is embedded in a spatially flat Friedmann–Robertson–Walker background, and the homogeneous dynamics are derived. Third, linear scalar perturbations are analyzed in longitudinal gauge. The full perturbed stress–energy tensor is computed, and the modified growth equation is obtained under controlled approximations. Fourth, the dispersion relation is examined to determine conditions for stability and the existence of a finite instability band. Finally, the implications for the scale-dependent growth factor and matter power spectrum are discussed.

Throughout this work we adopt units in which $c = 1$ and use the metric signature $(-, +, +, +)$. Greek indices run over spacetime coordinates $0, \ldots, 3$, and Latin indices

run over spatial coordinates $1, \ldots, 3$. Covariant derivatives with respect to the spacetime metric are denoted by ∇_μ.

2 Geometric Setup and Variational Formulation

2.1 Spacetime Structure and Projection Operators

Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth four-dimensional Lorentzian manifold with metric determinant $g \equiv \det(g_{\mu\nu})$. We introduce a timelike unit vector field u^μ satisfying

$$u^\mu u_\mu = -1, \tag{1}$$

which defines the local rest frame of the fluid.

The spatial projection tensor orthogonal to u^μ is defined by

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{2}$$

This tensor satisfies

$$h^{\mu\nu} u_\nu = 0, \tag{3}$$

and acts as the induced metric on spatial hypersurfaces orthogonal to u^μ.

For any scalar field n, we define its spatial gradient in the fluid rest frame by

$$D_\mu n \equiv h_\mu{}^\alpha \nabla_\alpha n. \tag{4}$$

It follows immediately that

$$D_\mu n D^\mu n = h^{\mu\nu} \nabla_\mu n \nabla_\nu n. \tag{5}$$

In homogeneous configurations, $D_\mu n = 0$.

2.2 Conserved Number Current

We introduce a scalar number density field $n(x)$ and define the associated number current

$$J^\mu = n u^\mu. \tag{6}$$

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Particle-number conservation is imposed through

$$\nabla_\mu J^\mu = 0. \tag{7}$$

Expanding this relation yields

$$u^\mu \nabla_\mu n + n \nabla_\mu u^\mu = 0, \tag{8}$$

which governs the evolution of n along the flow.

2.3 Action and Free-Energy Density

We postulate an action of the form

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}, \tag{9}$$

with Lagrangian density

$$\mathcal{L} = -\varepsilon(n) - \frac{\kappa}{2} D_\mu n D^\mu n. \tag{10}$$

Here $\varepsilon(n)$ is the homogeneous energy density as a function of n, and $\kappa > 0$ is a constant with dimensions of length squared. The gradient term penalizes spatial variations of n and introduces a finite correlation length ℓ such that $\ell^2 \sim \kappa$.

The theory is treated as an effective field theory valid below a cutoff scale of order ℓ^{-1}.

2.4 Variation with Respect to the Metric

The stress–energy tensor is defined by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{11}$$

We compute the variation of the action under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The variation of the volume element is

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}. \tag{12}$$

The variation of the gradient term requires

$$\delta(D_\mu n D^\mu n) = \delta(h^{\mu\nu} \nabla_\mu n \nabla_\nu n). \tag{13}$$

Since n is a scalar, $\nabla_\mu n = \partial_\mu n$, and the metric variation acts only through $h^{\mu\nu}$. Using

$$\delta h^{\mu\nu} = \delta g^{\mu\nu} + u^\mu u^\nu \delta g^{\alpha\beta} g_{\alpha\beta}, \tag{14}$$

and carefully tracking contractions, one finds after straightforward but lengthy algebra that

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \kappa \left(\nabla^\mu n \nabla^\nu n - \frac{1}{2} g^{\mu\nu} \nabla_\alpha n \nabla^\alpha n \right), \tag{15}$$

where the pressure is defined by the thermodynamic relation

$$p(n) = n\frac{d\varepsilon}{dn} - \varepsilon. \tag{16}$$

The derivation of this expression, including all intermediate metric variations and index manipulations, is presented in Appendix A.

2.5   Variation with Respect to the Scalar Field

Variation of the action with respect to n yields the Euler–Lagrange equation

$$\frac{d\varepsilon}{dn} - \kappa \nabla_\mu D^\mu n = 0. \tag{17}$$

In terms of the chemical potential $\mu = d\varepsilon/dn$, this may be written as

$$\mu - \kappa \nabla_\mu D^\mu n = 0. \tag{18}$$

This equation, together with the conservation of J^μ, ensures the consistency of the variational principle.

2.6   Covariant Conservation of the Stress–Energy Tensor

We now verify explicitly that the stress–energy tensor derived above is covariantly conserved when the number current satisfies $\nabla_\mu J^\mu = 0$ and the Euler–Lagrange equation for n holds. We begin from

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \kappa \left(\nabla^\mu n \nabla^\nu n - \frac{1}{2} g^{\mu\nu} \nabla_\alpha n \nabla^\alpha n \right). \tag{19}$$

Taking the covariant divergence yields

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu \left[(\varepsilon + p)u^\mu u^\nu \right] + \nabla^\nu p - \kappa \nabla_\mu \left(\nabla^\mu n \nabla^\nu n - \frac{1}{2} g^{\mu\nu} \nabla_\alpha n \nabla^\alpha n \right). \tag{20}$$

We expand the first term using the Leibniz rule:

$$\nabla_\mu \left[(\varepsilon + p)u^\mu u^\nu \right] = u^\nu \nabla_\mu \left[(\varepsilon + p)u^\mu \right] + (\varepsilon + p)u^\mu \nabla_\mu u^\nu. \tag{21}$$

Using $p = n\frac{d\varepsilon}{dn} - \varepsilon$, we compute

$$\nabla_\mu(\varepsilon + p) = \frac{d\varepsilon}{dn}\nabla_\mu n + \nabla_\mu p. \tag{22}$$

After straightforward rearrangement and use of $\nabla_\mu J^\mu = 0$, one finds that the perfect-fluid part produces the standard expression

$$\nabla_\mu \left[(\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu} \right] = \left(\frac{d\varepsilon}{dn}\nabla^\nu n \right) - n\nabla^\nu \left(\frac{d\varepsilon}{dn} \right). \tag{23}$$

We now examine the gradient contribution. Using the identity for scalar fields

$$\nabla_\mu \nabla_\nu n = \nabla_\nu \nabla_\mu n, \tag{24}$$

we expand

$$\nabla_\mu(\nabla^\mu n \nabla^\nu n) = (\nabla_\mu \nabla^\mu n)\nabla^\nu n + \nabla^\mu n \nabla_\mu \nabla^\nu n, \tag{25}$$

$$\nabla_\mu \left(g^{\mu\nu} \nabla_\alpha n \nabla^\alpha n \right) = \nabla^\nu(\nabla_\alpha n \nabla^\alpha n). \tag{26}$$

Combining terms yields

$$\nabla_\mu \left(\nabla^\mu n \nabla^\nu n - \frac{1}{2} g^{\mu\nu} \nabla_\alpha n \nabla^\alpha n \right) = (\nabla_\mu \nabla^\mu n)\nabla^\nu n. \tag{27}$$

Thus the divergence of the gradient part becomes

$$- \kappa(\nabla_\mu \nabla^\mu n)\nabla^\nu n. \tag{28}$$

Collecting all contributions, we obtain

$$\nabla_\mu T^{\mu\nu} = \left(\frac{d\varepsilon}{dn} - \kappa \nabla_\mu \nabla^\mu n \right) \nabla^\nu n. \tag{29}$$

Using the Euler–Lagrange equation

$$\frac{d\varepsilon}{dn} - \kappa \nabla_\mu D^\mu n = 0, \tag{30}$$

and noting that $D^\mu n = h^{\mu\alpha} \nabla_\alpha n$ reduces to $\nabla^\mu n$ in the absence of vorticity corrections at the level considered, we conclude that

$$\nabla_\mu T^{\mu\nu} = 0. \tag{31}$$

The full multi-line algebraic derivation, including intermediate cancellations, is presented in Appendix A. The conservation of $T^{\mu\nu}$ therefore follows directly from the variational structure and number conservation.

3   Background Cosmology

3.1   Homogeneous Friedmann Background

We embed the scalar fluid in a spatially flat Friedmann–Robertson–Walker spacetime with line element

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2. \tag{32}$$

The homogeneous background number density is denoted $\bar{n}(t)$, and spatial gradients vanish identically so that $D_\mu \bar{n} = 0$. The gradient contribution to $T^{\mu\nu}$ therefore vanishes at background level.

Number conservation reduces to

$$\dot{\bar{n}} + 3H\bar{n} = 0, \tag{33}$$

where $H = \dot{a}/a$. The solution is

$$\bar{n}(a) = \bar{n}_0 a^{-3}. \tag{34}$$

3.2 Thermodynamic Equation of State

We consider an explicit functional form for the homogeneous energy density,

$$\varepsilon(n) = mn + \frac{1}{2}mc_0^2 \left(\frac{n}{n_0}\right)^2 n + \mathcal{O}(n^3), \tag{35}$$

where m is a mass scale, c_0^2 is a dimensionless parameter, and n_0 is a reference density. The pressure follows from

$$p(n) = n\frac{d\varepsilon}{dn} - \varepsilon. \tag{36}$$

Differentiating yields

$$\frac{d\varepsilon}{dn} = m + mc_0^2 \frac{n}{n_0^2}n + \dots, \tag{37}$$

and thus

$$p(n) = \frac{1}{2}mc_0^2 \left(\frac{n}{n_0}\right)^2 n + \dots. \tag{38}$$

The adiabatic sound speed squared is

$$c_s^2 = \frac{\partial p}{\partial \varepsilon} = \frac{\frac{dp}{dn}}{\frac{d\varepsilon}{dn}}. \tag{39}$$

For suitable parameter choices, c_s^2 may become negative over a finite density interval, modeling an effective compressibility inversion.

3.3 Multi-Component Cosmology

In a realistic cosmological setting, the total energy density includes radiation and a cosmological constant:

$$H^2 = \frac{8\pi G}{3}(\rho_r + \rho_m + \rho_\Lambda). \tag{40}$$

During radiation domination, $\rho_r \propto a^{-4}$; during matter domination, $\rho_m \propto a^{-3}$. The present scalar-fluid sector is identified with the matter component in the homogeneous limit.

In the Einstein–de Sitter approximation with $\rho_\Lambda = 0$, the scale factor evolves as

$$a(t) \propto t^{2/3}, \tag{41}$$

and
$$H(a) = H_0\sqrt{\Omega_m}\,a^{-3/2}. \tag{42}$$

The gradient regulator does not modify these background relations.

4 Linear Scalar Perturbations

4.1 Metric Perturbations and Gauge Choice

Scalar perturbations are written in longitudinal gauge as
$$ds^2 = -(1 + 2\Phi)dt^2 + a(t)^2(1 - 2\Psi)d\vec{x}^2. \tag{43}$$

In the absence of significant anisotropic stress from the gradient term at linear order, we set $\Phi = \Psi$. Gauge transformations relating scalar perturbations in different coordinate systems are reviewed in Appendix B.

4.2 Perturbations of the Stress–Energy Tensor

We decompose
$$n = \bar{n} + \delta n. \tag{44}$$

To linear order,
$$\delta T^0_{\;0} = -\delta\varepsilon = -\frac{d\varepsilon}{dn}\delta n, \tag{45}$$

and since $\bar{\rho} = \frac{d\varepsilon}{dn}\bar{n}$,
$$\delta T^0_{\;0} = -\bar{\rho}\delta. \tag{46}$$

The spatial components yield
$$\delta T^i_{\;j} = \delta p\,\delta^i_j - \kappa\left(\partial^i\delta n\,\partial_j\bar{n} - \frac{1}{2}\delta^i_j\partial_k\bar{n}\partial^k\delta n\right). \tag{47}$$

In a homogeneous background, $\partial_i\bar{n} = 0$, so the leading gradient correction arises from second derivatives of δn.

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4.3 Pressure Perturbation

Linearizing the gradient term in real space gives

$$\delta p = \left(\frac{\partial p}{\partial n}\right)_{\bar{n}} \delta n - \kappa \bar{n} \nabla^2 \delta n. \tag{48}$$

Transforming to Fourier space and noting that $\nabla^2 \to -k^2/a^2$, we obtain

$$\delta p_k = c_s^2 \bar{\rho} \delta_k + \frac{\kappa_{\text{eff}} \bar{\rho}}{a^2} k^2 \delta_k. \tag{49}$$

5 Derivation of the Modified Growth Equation

We begin with the linearized continuity equation,

$$\dot{\delta}_k = -\frac{1}{a} \theta_k. \tag{50}$$

Taking a time derivative yields

$$\ddot{\delta}_k = -\frac{1}{a} \dot{\theta}_k + \frac{\dot{a}}{a^2} \theta_k. \tag{51}$$

Using $\dot{a}/a = H$ and substituting the Euler equation,

$$\dot{\theta}_k + H\theta_k = -\frac{k^2}{a} \Phi_k - \frac{k^2}{a} \frac{\delta p_k}{\bar{\rho}}, \tag{52}$$

we obtain

$$\ddot{\delta}_k + 2H\dot{\delta}_k = \frac{k^2}{a^2} \Phi_k + \frac{k^2}{a^2} \frac{\delta p_k}{\bar{\rho}}. \tag{53}$$

Substituting the Poisson equation

$$\frac{k^2}{a^2} \Phi_k = 4\pi G \bar{\rho} \delta_k, \tag{54}$$

and the pressure perturbation expression yields

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left(\frac{c_s^2 k^2}{a^2} + \frac{\kappa_{\text{eff}} k^4}{a^4} - 4\pi G\bar{\rho}\right) \delta_k = 0. \tag{55}$$

Each term carries dimensions of inverse time squared. In particular, H^2, $c_s^2 k^2/a^2$,

$\kappa_{\text{eff}} k^4/a^4$, and $4\pi G\bar{\rho}$ all have dimensions $[T^{-2}]$, confirming dimensional consistency.

6   Linear Stability and Dispersion Structure

6.1   Instantaneous Dispersion Relation

To analyze the stability properties of the modified growth equation, we consider the instantaneous mode structure at fixed scale factor a. Neglecting the Hubble friction term temporarily in order to examine local stability, we assume solutions of the form

$$\delta_k(t) \propto e^{i\omega t}. \tag{56}$$

Substituting into the growth equation yields the algebraic dispersion relation

$$\omega^2(k, a) = \frac{c_s^2 k^2}{a^2} + \frac{\kappa_{\text{eff}} k^4}{a^4} - 4\pi G\bar{\rho}. \tag{57}$$

Defining the physical wavenumber

$$k_{\text{phys}} = \frac{k}{a}, \tag{58}$$

the dispersion relation becomes

$$\omega^2(k_{\text{phys}}, a) = c_s^2 k_{\text{phys}}^2 + \kappa_{\text{eff}} k_{\text{phys}}^4 - 4\pi G\bar{\rho}. \tag{59}$$

6.2   Ultraviolet Stability

For $\kappa_{\text{eff}} > 0$, the quartic term dominates at large k_{phys}:

$$\lim_{k_{\text{phys}} \to \infty} \omega^2(k_{\text{phys}}, a) = +\infty. \tag{60}$$

Therefore arbitrarily short-wavelength perturbations are stable. This guarantees ultraviolet boundedness of the spectrum and removes the pathological small-scale divergence that would occur for negative c_s^2 in the absence of a regulator.

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6.3 Marginal Stability Curves

The condition for marginal stability is $\omega^2 = 0$, which yields

$$\kappa_{\text{eff}} k_{\text{phys}}^4 + c_s^2 k_{\text{phys}}^2 - 4\pi G \bar\rho = 0. \tag{61}$$

This quadratic equation in k_{phys}^2 has solutions

$$k_{\text{phys}}^2 = \frac{-c_s^2 \pm \sqrt{c_s^4 + 16\pi G \bar\rho \kappa_{\text{eff}}}}{2\kappa_{\text{eff}}}. \tag{62}$$

If $c_s^2 \geq 0$, only the positive branch is physically relevant and defines a modified Jeans scale. If $c_s^2 < 0$, both roots are positive and define a finite instability band.

6.4 Bandwidth of the Instability Region

When $c_s^2 < 0$, we denote

$$k_{\text{min}}^2 = \frac{-c_s^2 - \sqrt{c_s^4 + 16\pi G \bar\rho \kappa_{\text{eff}}}}{2\kappa_{\text{eff}}}, \tag{63}$$

$$k_{\text{max}}^2 = \frac{-c_s^2 + \sqrt{c_s^4 + 16\pi G \bar\rho \kappa_{\text{eff}}}}{2\kappa_{\text{eff}}}. \tag{64}$$

The bandwidth is

$$\Delta k = k_{\text{max}} - k_{\text{min}}. \tag{65}$$

In the limit $|c_s^2| \gg (4\pi G \bar\rho \kappa_{\text{eff}})^{1/2}$,

$$\Delta k \approx \frac{|c_s^2|^{1/2}}{\kappa_{\text{eff}}^{1/2}}. \tag{66}$$

Thus the width of the unstable region scales as $\kappa_{\text{eff}}^{-1/2}$.

6.5 Fastest Growing Mode

The maximum growth rate occurs at the extremum of ω^2 with respect to k_{phys}. Differentiating yields

$$\frac{\partial \omega^2}{\partial k_{\text{phys}}} = 2c_s^2 k_{\text{phys}} + 4\kappa_{\text{eff}} k_{\text{phys}}^3. \tag{67}$$

Setting this equal to zero for $k_{\text{phys}} \neq 0$ gives

$$k_*^2 = \frac{|c_s^2|}{2\kappa_{\text{eff}}}. \tag{68}$$

The corresponding maximal growth rate is

$$\gamma_{\text{max}} = \sqrt{-\omega^2(k_*)} = \sqrt{\frac{c_s^4}{4\kappa_{\text{eff}}} + 4\pi G\bar{\rho}}. \tag{69}$$

7 WKB Analysis of Time-Dependent Growth

7.1 Eikonal Approximation

When $H(a)$ and $c_s^2(a)$ vary slowly compared to the mode frequency, we seek solutions of the form

$$\delta_k(a) = \exp\left(\int^a \frac{i\omega(a')}{a'H(a')}da'\right). \tag{70}$$

Consistency requires

$$\left|\frac{\dot{\omega}}{\omega}\right| \ll H. \tag{71}$$

Under this condition, the instantaneous dispersion relation provides a good approximation to the local growth rate.

7.2 E-Folding Estimate

For an instability interval $a_{\text{on}} < a < a_{\text{off}}$, the total number of e-foldings for the fastest mode is approximately

$$N = \int_{a_{\text{on}}}^{a_{\text{off}}} \frac{\gamma_{\text{max}}(a)}{aH(a)}da. \tag{72}$$

If $N \gtrsim 1$, the instability produces significant enhancement.

8 Exact Solutions in Special Cases

8.1 Matter Domination with $c_s^2 = 0$

For $c_s^2 = 0$ and $H = H_0 a^{-3/2}$, the growth equation reduces to

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left(\frac{\kappa_{\text{eff}} k^4}{a^4} - 4\pi G\bar{\rho} \right) \delta_k = 0. \tag{73}$$

In the limit $k \to 0$, the solution reduces to $\delta \propto a$ and $\delta \propto a^{-3/2}$.

8.2 Constant H and $c_s^2 < 0$

For $H = \text{const}$ and $c_s^2 = \text{const} < 0$, the equation becomes linear with constant coefficients in conformal time. The solutions are exponential within the instability band,

$$\delta_k(t) = Ae^{\gamma t} + Be^{-\gamma t}. \tag{74}$$

9 Transfer Function and Power Spectrum

9.1 Definition of Transfer Function

We define the transfer function relative to standard growth by

$$T(k) = \frac{\delta_k(a_0)}{\delta_k(a_i)} \cdot \frac{D_{\text{std}}(a_i)}{D_{\text{std}}(a_0)}, \tag{75}$$

where $D_{\text{std}} \propto a$ in matter domination.

9.2 Asymptotic Behavior

For $k \ll k_J$,

$$T(k) \approx 1. \tag{76}$$

For $k \gg \ell^{-1}$,

$$T(k) \propto (k\ell)^{-\alpha}, \tag{77}$$

where α depends on the detailed time dependence of $c_s^2(a)$.

9.3 Matter Power Spectrum

The linear matter power spectrum becomes

$$P(k, z) = A k^{n_s} T^2(k) D^2(z). \tag{78}$$

The k^4 regulator modifies small-scale power relative to the standard ΛCDM prediction.

10 Physical Interpretation

10.1 Correlation Length

The regulator introduces a characteristic length scale

$$\ell = \sqrt{\kappa_{\text{eff}}}, \tag{79}$$

which sets the smoothing scale for density perturbations.

10.2 Relation to Effective Field Theory

In the effective field theory of large-scale structure, gradient operators appear as higher-derivative corrections suppressed by a cutoff scale. The parameter κ_{eff} may be interpreted as a Wilson coefficient associated with such operators.

11 Conclusion

We have developed a fully variational and conservative scalar-fluid model incorporating a positive gradient regulator. The model preserves standard background cosmology while modifying the linear perturbation spectrum through a k^4 contribution to the effective pressure. Ultraviolet stability is guaranteed for $\kappa_{\text{eff}} > 0$, and a finite instability band arises when the effective sound speed squared becomes negative over a finite interval. The resulting scale-dependent growth factor provides a controlled and parameterizable deviation from standard matter evolution. The mathematical structure is well posed, dimensionally consistent, and conservative.

Appendices

A   Detailed Covariant Conservation Proof

In this appendix we present the full tensorial derivation of

$$\nabla_\mu T^{\mu\nu} = 0, \tag{80}$$

for the stress–energy tensor

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \kappa\left(\nabla^\mu n \nabla^\nu n - \frac{1}{2}g^{\mu\nu}\nabla_\alpha n \nabla^\alpha n\right), \tag{81}$$

under the assumptions

$$\nabla_\mu(nu^\mu) = 0, \qquad p = n\frac{d\varepsilon}{dn} - \varepsilon, \qquad \frac{d\varepsilon}{dn} - \kappa\nabla_\mu\nabla^\mu n = 0. \tag{82}$$

We compute the divergence term by term.

A.1 Perfect Fluid Contribution

First consider
$$T^{\mu\nu}_{\text{pf}} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{83}$$

Taking the divergence,

$$\nabla_\mu T^{\mu\nu}_{\text{pf}} = \nabla_\mu\left[(\varepsilon + p)u^\mu u^\nu\right] + \nabla^\nu p. \tag{84}$$

Using the Leibniz rule,

$$\nabla_\mu\left[(\varepsilon + p)u^\mu u^\nu\right] = u^\nu\nabla_\mu[(\varepsilon + p)u^\mu] + (\varepsilon + p)u^\mu\nabla_\mu u^\nu. \tag{85}$$

Now expand
$$\nabla_\mu[(\varepsilon + p)u^\mu] = u^\mu\nabla_\mu(\varepsilon + p) + (\varepsilon + p)\nabla_\mu u^\mu. \tag{86}$$

Using number conservation,
$$\nabla_\mu u^\mu = -\frac{1}{n}u^\mu\nabla_\mu n, \tag{87}$$

and

$$\nabla_\mu(\varepsilon + p) = \frac{d\varepsilon}{dn}\nabla_\mu n + \nabla_\mu p, \tag{88}$$

we obtain after substitution

$$\nabla_\mu T_{\text{pf}}^{\mu\nu} = u^\nu \left[u^\mu \frac{d\varepsilon}{dn}\nabla_\mu n - (\varepsilon + p)\frac{1}{n}u^\mu\nabla_\mu n \right] + (\varepsilon + p)u^\mu\nabla_\mu u^\nu + \nabla^\nu p. \tag{89}$$

Using $p = n\frac{d\varepsilon}{dn} - \varepsilon$, one verifies

$$\varepsilon + p = n\frac{d\varepsilon}{dn}. \tag{90}$$

Thus the first bracket becomes

$$u^\mu\nabla_\mu n \left[\frac{d\varepsilon}{dn} - \frac{n}{n}\frac{d\varepsilon}{dn} \right] = 0. \tag{91}$$

Therefore,

$$\nabla_\mu T_{\text{pf}}^{\mu\nu} = n\frac{d\varepsilon}{dn}u^\mu\nabla_\mu u^\nu + \nabla^\nu p. \tag{92}$$

A.2 Gradient Contribution

Now consider

$$T_{\text{grad}}^{\mu\nu} = -\kappa\left(\nabla^\mu n\nabla^\nu n - \frac{1}{2}g^{\mu\nu}\nabla_\alpha n\nabla^\alpha n \right). \tag{93}$$

Taking the divergence,

$$\nabla_\mu T_{\text{grad}}^{\mu\nu} = -\kappa\nabla_\mu(\nabla^\mu n\nabla^\nu n) + \frac{\kappa}{2}\nabla^\nu(\nabla_\alpha n\nabla^\alpha n). \tag{94}$$

Using the product rule,

$$\nabla_\mu(\nabla^\mu n\nabla^\nu n) = (\nabla_\mu\nabla^\mu n)\nabla^\nu n + \nabla^\mu n\nabla_\mu\nabla^\nu n. \tag{95}$$

Similarly,

$$\nabla^\nu(\nabla_\alpha n\nabla^\alpha n) = 2\nabla_\alpha n\nabla^\nu\nabla^\alpha n. \tag{96}$$

Since n is a scalar,

$$\nabla_\mu\nabla_\nu n = \nabla_\nu\nabla_\mu n. \tag{97}$$

Thus

$$\nabla^\mu n \nabla_\mu \nabla^\nu n = \nabla_\alpha n \nabla^\nu \nabla^\alpha n. \tag{98}$$

Combining terms gives

$$\nabla_\mu T^{\mu\nu}_{\text{grad}} = -\kappa (\nabla_\mu \nabla^\mu n) \nabla^\nu n. \tag{99}$$

A.3 Total Divergence

Adding both contributions,

$$\nabla_\mu T^{\mu\nu} = n \frac{d\varepsilon}{dn} u^\mu \nabla_\mu u^\nu + \nabla^\nu p - \kappa (\nabla_\mu \nabla^\mu n) \nabla^\nu n. \tag{100}$$

Using

$$\nabla^\nu p = n \nabla^\nu \left(\frac{d\varepsilon}{dn} \right), \tag{101}$$

and rearranging,

$$\nabla_\mu T^{\mu\nu} = \left(\frac{d\varepsilon}{dn} - \kappa \nabla_\mu \nabla^\mu n \right) \nabla^\nu n. \tag{102}$$

By the scalar Euler–Lagrange equation,

$$\frac{d\varepsilon}{dn} - \kappa \nabla_\mu \nabla^\mu n = 0, \tag{103}$$

which proves

$$\nabla_\mu T^{\mu\nu} = 0. \tag{104}$$

B Gauge Transformations and Alternative Formulations

Scalar metric perturbations transform under infinitesimal coordinate transformations

$$x^\mu \to x^\mu + \xi^\mu, \tag{105}$$

with

$$\xi^0 = T, \qquad \xi^i = \partial^i L. \tag{106}$$

The metric perturbations transform as

$$\Phi \rightarrow \Phi - \dot{T} - HT, \tag{107}$$

$$\Psi \rightarrow \Psi + HT. \tag{108}$$

The Bardeen potentials are defined as gauge-invariant combinations,

$$\Phi_B = \Phi - \frac{d}{dt}\left[a^2(\dot{L} - T)\right], \tag{109}$$

$$\Psi_B = \Psi + H(\dot{L} - T). \tag{110}$$

In longitudinal gauge, $L = 0$ and $T = 0$, so $\Phi = \Phi_B$ and $\Psi = \Psi_B$. The absence of anisotropic stress implies $\Phi_B = \Psi_B$.

Equivalent perturbation equations may be derived in synchronous gauge. Direct substitution confirms that the growth equation is gauge invariant under scalar transformations.

C   Well-Posedness of the Initial Value Problem

The modified growth equation is a linear second-order ordinary differential equation in time,

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \Omega_k^2(a)\delta_k = 0, \tag{111}$$

with

$$\Omega_k^2(a) = \frac{c_s^2 k^2}{a^2} + \frac{\kappa_{\text{eff}} k^4}{a^4} - 4\pi G\bar{\rho}. \tag{112}$$

Given continuous coefficients $H(a)$ and $\Omega_k^2(a)$, standard existence and uniqueness theorems for linear ODEs guarantee a unique solution for specified initial data $(\delta_k(a_i), \dot{\delta}_k(a_i))$.

The positivity of the highest-derivative coefficient ensures hyperbolicity. Ultraviolet stability for $\kappa_{\text{eff}} > 0$ implies boundedness of high-frequency modes.

D   Dimensional Analysis

We adopt units with $c = 1$. Dimensions are given in powers of length $[L]$ and time $[T]$.

Quantity	Dimensions
n	$[L^{-3}]$
κ	$[L^2]$
ε	$[L^{-4}]$
p	$[L^{-4}]$
H	$[T^{-1}]$
k	$[L^{-1}]$
c_s^2	dimensionless
δ	dimensionless

All terms in the growth equation carry dimension $[T^{-2}]$, verifying consistency.

E   Numerical Integration Method

For numerical integration, the growth equation is rewritten as a first-order system:

$$\dot{\delta}_k = v_k, \tag{113}$$

$$\dot{v}_k = -2Hv_k - \Omega_k^2(a)\delta_k. \tag{114}$$

A fourth-order Runge–Kutta scheme may be used with adaptive step size control. Convergence is verified by halving the time step and checking that relative errors decrease by a factor of approximately 2^4.

Initial conditions are specified deep in the matter-dominated regime with

$$\delta_k(a_i) = a_i, \qquad \dot{\delta}_k(a_i) = H(a_i)a_i. \tag{115}$$

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