Linear Algebra Review

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Matrices

- Matrix: 2 dimensional array of data arranged into **rows** and **columns** e.g.

$$A = \begin{bmatrix} 460 & 4 & 12 \\ 70 & 1 & 5 \\ 155 & 3 & 8 \\ 429 & 6 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 321 & 5 & 21 & 10 & 4 \\ 704 & 2 & 43 & 67 & 1 \end{bmatrix}$$

- By convention: we use capital letter to specify the name of a matrix e.g. A, B, X etc.
- Dimensions of a matrix are (**rows** × **columns**)
  - Dimensions of A are (4 × 3)
  - Dimensions of B are (2 × 5)

Matrices – Indexing

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{34} \end{bmatrix} \qquad A = \begin{bmatrix} 460 & 4 & 12 \\ 70 & 1 & 5 \\ 155 & 3 & 8 \\ 429 & 6 & 10 \end{bmatrix}$$

- Indexing into a matrix M: is done by specifying an ordered index M_{ij} where i is the row and j is the column e.g.
  - $\mathrm{A}_{11} = 460$
  - $\mathrm{A}_{42} = 6$
  - $\mathrm{A}_{23} = 5$
  - $\mathrm{A}_{44} = \text{undefined}$ (the matrix doesn't have a 4th column)

Vectors

- Vector: A matrix with only one column i.e. an $(\mathrm{n} \times 1)$ matrix (by convention).
- By convention: we use a small (possibly bold) letter to specify the name of a vector e.g. $\boldsymbol{x}$, $\boldsymbol{a}$ etc.

$$\boldsymbol{y} = \begin{bmatrix} 21 \\ 88 \\ 76 \\ 53 \end{bmatrix} \qquad \boldsymbol{a} = \begin{bmatrix} 7 \\ 8 \\ 6 \end{bmatrix}$$

- Dimensions of a matrix are (**rows × 1**)
  - Dimensions of $\boldsymbol{y}$ are (4 × 1)
  - Dimensions of $\boldsymbol{a}$ are (3 × 1)

Vectors – Indexing

$$\boldsymbol{y} = \begin{bmatrix} 21 \\ 88 \\ 76 \\ 53 \end{bmatrix} \qquad \boldsymbol{a} = \begin{bmatrix} 7 \\ 8 \\ 6 \end{bmatrix}$$

- Indexing into a vector $\boldsymbol{m}$: is done by specifying the item index $\boldsymbol{m}_i$ where i is the row e.g.
  - $\boldsymbol{y}_1 = 21$
  - $\boldsymbol{y}_4 = 53$
  - $\boldsymbol{a}_2 = 8$
  - $\boldsymbol{a}_4 = \text{undefined}$ (the vector doesn't have a 4^{th} row)

Addition and Subtraction

- 2 Matrices can be added or subtracted if and only if their dimensions match
- If their dimensions match: corresponding indices in the matrices are added up or subtracted e.g.

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 8 & 0 \\ 7 & -3 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 12 & 1 \\ 9 & 0 \\ 9 & -1 \end{bmatrix}$$

- If their dimensions don't match: you get an error:

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 321 & 5 & 21 & 10 & 4 \\ 704 & 2 & 43 & 67 & 1 \end{bmatrix} = \text{undefined}$$ (their dimensions don't match)

Scalar Multiplication

- A value (a.k.a scalar) can be multiplied into a matrix
- Every item in the matrix is then multiplied by that number e.g.

$$2 \times \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 8 & 2 \\ 4 & 6 \\ 6 & 0 \end{bmatrix}$$

- A value (a.k.a scalar) can also be divided into a matrix; this is the same as scalar multiplying the matrix by the inverse of that number
- Every item in the matrix is then divided by that number e.g.

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} / 2 = \frac{1}{2} \times \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} {}^{1}/_{2} & 2 \\ 2 & {}^{1}/_{2} \\ 1 & {}^{3}/_{2} \\ {}^{3}/_{2} & 0 \end{bmatrix}$$

Matrix Operations –Combination of Operations

- Operations can be combined into one expression e.g.
- Every item in the matrix is then multiplied by that number e.g.

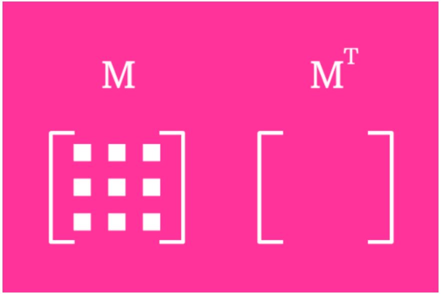
$$2 \times \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 12 & 24 \\ 18 & 21 \\ 15 & 18 \\ 9 & 10 \end{bmatrix} / 3$$

$$= \begin{bmatrix} 2 & 8 \\ 8 & 2 \\ 4 & 6 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 6 & 7 \\ 5 & 6 \\ 3 & {}^{10}/_{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 2 & -5 \\ -1 & 1 \\ 4 & -{}^{10}/_{3} \end{bmatrix}$$

Matrix Transpose

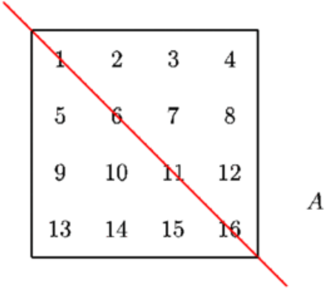
- Given a matrix A, the transpose of the matrix is represented as A^T
- The rows of A become the columns of A^T as in the graphic below:



Source: https://physics.blogberth.com/2017/08/11/fuckyeahphysicaon-the-transpose-of-a-matrix-in-this-post-i/

Matrix Transpose

- Given a matrix A, the transpose of the matrix is represented as A^T
- Another way to see it is: It's a version of the matrix that has been flipped on the diagonal axis as in the graphic below:



$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

Source: https://physics.blogberth.com/2017/08/11/fuckyeahphysicaon-the-transpose-of-a-matrix-in-this-post-i/

Matrix Transpose

- Given a matrix A, the transpose of the matrix is represented as A^T
- Another way to see it is: It's a version of the matrix that has been flipped on the diagonal axis as in the graphic below:

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Source:
https://physics.blogberth.com/2017/08/11/fuckyeahphysicaon-the-transpose-of-a-matrix-in-this-post-i/

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication can only be done if the columns of the first matrix match the rows of the second matrix
- Given the following matrices to be multiplied:

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix}$$

$$(4 \times 2) \qquad\qquad (2 \times 3)$$

These Match:
We can multiply them

This will be the dimension
of the result i.e.
(4×3)

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$\begin{array}{c c} \text{Row 1} & \\ & \end{array}\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \qquad \begin{array}{c} \text{Col 1} \\ \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \text{Col 1} \\ \end{array}$$

$$= \begin{array}{c} \text{Row 1} \\ \\ \\ \end{array}\begin{bmatrix} {\color{blue}1} \times {\color{red}3} + {\color{blue}4} \times {\color{red}2} & & \\ & & \\ & & \end{bmatrix}$$

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$\begin{array}{r} \text{Row 1} \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \quad \begin{array}{c} \begin{array}{ccc} & \text{Col 2} & \end{array} \\ \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{r} \\ \text{Row 1} \\ \\ \\ \end{array} = \begin{array}{c} \begin{array}{ccc} & \text{Col 2} & \end{array} \\ \begin{bmatrix} 1 \times 3 + 4 \times 2 & 1 \times 5 + 4 \times 6 & \\ & & \\ & & \end{bmatrix} \end{array}$$

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$\begin{array}{c} \text{Row 1} \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \qquad \begin{array}{ccc} & & \text{Col 3} \\ \end{array} \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix}$$

$$= \begin{array}{c} \text{Row 1} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} & & \text{Col 3} \end{array} \\ \begin{bmatrix} 1 \times 3 + 4 \times 2 & 1 \times 5 + 4 \times 6 & 1 \times 1 + 4 \times 0 \\ & & \\ & & \end{bmatrix} \end{array}$$

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$\begin{array}{r} \\ \text{Row 2} \\ \\ \\ \end{array} \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \qquad \begin{array}{ccc} \text{Col 1} & & \\ \end{array} \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{Col 1} \end{array}$$

$$\begin{array}{r} \\ \text{Row 2} \\ \\ \end{array} = \begin{bmatrix} 1 \times 3 + 4 \times 2 & 1 \times 5 + 4 \times 6 & 1 \times 1 + 4 \times 0 \\ 4 \times 3 + 1 \times 2 & & \\ & & \end{bmatrix}$$

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$
\begin{array}{c c}
 & \\
\text{Row 2} & \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccc} & \text{Col 2} & \end{array} \\
\begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix}
\end{array}
$$

$$
\begin{array}{c c}
 & \begin{array}{ccc} & \text{Col 2} & \end{array} \\
\text{Row 2} = & \begin{bmatrix} 1 \times 3 + 4 \times 2 & 1 \times 5 + 4 \times 6 & 1 \times 1 + 4 \times 0 \\ 4 \times 3 + 1 \times 2 & {\color{blue}4} \times {\color{red}5} + {\color{blue}1} \times {\color{red}6} & \\ & & \end{bmatrix}
\end{array}
$$

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$
\begin{array}{c}
\\ \text{Row 2} \\ \\ \\
\end{array}
\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix}
\quad
\begin{array}{ccc}
 & & \text{Col 3} \\
\end{array}
\begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix}
$$

$$
\begin{array}{ccc}
 & & \text{Col 3}
\end{array}
$$

$$
\begin{array}{c} \\ \text{Row 2} = \\ \\ \end{array}
\begin{bmatrix}
1 \times 3 + 4 \times 2 & 1 \times 5 + 4 \times 6 & 1 \times 1 + 4 \times 0 \\
4 \times 3 + 1 \times 2 & 4 \times 5 + 1 \times 6 & 4 \times 1 + 1 \times 0 \\
 & &
\end{bmatrix}
$$

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$
\begin{array}{cc}
 & \\
 & \\
\text{Row 3} & \\
\text{Row 4} &
\end{array}
\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix}
\quad
\begin{array}{ccc}
\text{Col 1} & \text{Col 2} & \text{Col 3}
\end{array}
\\
\qquad\qquad\qquad\qquad
\begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix}
1\times 3 + 4\times 2 & 1\times 5 + 4\times 6 & 1\times 1 + 4\times 0 \\
4\times 3 + 1\times 2 & 4\times 5 + 1\times 6 & 4\times 1 + 1\times 0 \\
2\times 3 + 3\times 2 & 2\times 5 + 3\times 6 & 2\times 1 + 3\times 0 \\
3\times 3 + 0\times 2 & 3\times 5 + 0\times 6 & 3\times 1 + 0\times 0
\end{bmatrix}
$$

Matrix – Matrix Multiplication

- Matrix – Matrix multiplication is carried out by carrying out an inner product between each row of the first matrix with each column of the second matrix e.g.
- Given the following matrices to be multiplied:

$$\begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 29 & 1 \\ 14 & 26 & 4 \\ 12 & 28 & 2 \\ 9 & 15 & 3 \end{bmatrix}$$

(4×2) (2×3) (4×3)

These Match:
We can multiply them

This will be the dimension
of the result i.e.
(4×3)

Matrix – Vector Multiplication

- Matrix – Vector is a special case of Matrix – Matrix multiplication, where the second (not first!) matrix is a 1-column matrix
- The operation is carried out in exactly the same way
- Always results in a vector

$$\begin{bmatrix} 3 & 5 & 1 & 4 \\ 2 & 6 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$$

(2×4) (4×1)

These Match:
We can multiply
them

The result:
(2×1)

Matrix – Vector Multiplication

- Matrix – Vector is a special case of Matrix – Matrix multiplication, where the second (not first!) matrix is a 1-column matrix
- The operation is carried out in exactly the same way
- Always results in a vector

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \end{array} \begin{bmatrix} 3 & 5 & 1 & 4 \\ 2 & 6 & 0 & 7 \end{bmatrix} \quad \begin{array}{c} \text{Col 1} \\ \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \end{array}$$

$$\begin{array}{l} \\ \text{Row 1} \\ = \\ \text{Row 2} \end{array} \begin{array}{c} \text{Col 1} \\ \begin{bmatrix} 3 \times 1 + 5 \times 4 + 1 \times 2 + 4 \times 3 \\ 2 \times 1 + 6 \times 4 + 0 \times 2 + 7 \times 3 \end{bmatrix} \end{array}$$

Matrix – Vector Multiplication

- Matrix – Vector is a special case of Matrix – Matrix multiplication, where the second (not first!) matrix is a 1-column matrix
- The operation is carried out in exactly the same way
- Always results in a vector

$$\begin{bmatrix} 3 & 5 & 1 & 4 \\ 2 & 6 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 37 \\ 47 \end{bmatrix}$$

(2×4) (4×1) (2×1)

These Match:
We can multiply
them

The result:
(2×1)

Matrix – Vector Multiplication For Linear Regression

- We can use Matrix-Vector multiplication to compute predictions for many examples quickly e.g. given

$$\begin{array}{cccccc} & & \text{Size} & \text{\#Rooms} & \text{Age} & \text{\#Garages} \end{array}$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 460 & 4 & 12 & 2 \\ 1 & 70 & 1 & 5 & 0 \\ 1 & 155 & 3 & 8 & 2 \\ 1 & 429 & 6 & 10 & 3 \end{bmatrix} \qquad \boldsymbol{\theta} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

- We can compute the h_θ for ALL of the examples in one go by computing $h_\theta(\boldsymbol{X}) = \boldsymbol{X\theta}$

$$h_\theta(\boldsymbol{X}) = \boldsymbol{X\theta} = \begin{bmatrix} \theta_0 + 460 \cdot \theta_1 + 4 \cdot \theta_2 + 12 \cdot \theta_3 + 2 \cdot \theta_4 \\ \theta_0 + 70 \cdot \theta_1 + 1 \cdot \theta_2 + 5 \cdot \theta_3 + 0 \cdot \theta_4 \\ \theta_0 + 155 \cdot \theta_1 + 3 \cdot \theta_2 + 8 \cdot \theta_3 + 2 \cdot \theta_4 \\ \theta_0 + 429 \cdot \theta_1 + 6 \cdot \theta_2 + 10 \cdot \theta_3 + 3 \cdot \theta_4 \end{bmatrix}$$

Matrix Transpose – Applied to Vector Products

- Given two vectors $\boldsymbol{x}$ and $\boldsymbol{\theta}$ of the same dimensions

$$\boldsymbol{x} = \begin{array}{c l} \begin{bmatrix} 1 \\ 460 \\ 4 \\ 12 \\ 2 \end{bmatrix} & \begin{array}{l} \\ \text{Size} \\ \text{\#Rooms} \\ \text{Age} \\ \text{\#Garages} \end{array} \end{array} \qquad \boldsymbol{\theta} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

- The inner (dot) product of the two vectors $\boldsymbol{x} \cdot \boldsymbol{\theta}$ can be expressed in terms of a matrix multiplication operation:
  - $\boldsymbol{h_\theta}(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{\theta} = \boldsymbol{x}^{\boldsymbol{T}} \boldsymbol{\theta} = \boldsymbol{\theta}^{\boldsymbol{T}} \boldsymbol{x}$

$$[1 \;\; 460 \;\; 4 \;\; 12 \;\; 2] \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = [\theta_0 \theta_1 \theta_2 \theta_3 \theta_4] \begin{bmatrix} 1 \\ 460 \\ 4 \\ 12 \\ 2 \end{bmatrix} = \boldsymbol{\theta_0} + 460 \cdot \boldsymbol{\theta_1} + 4 \cdot \boldsymbol{\theta_2} + 12 \cdot \boldsymbol{\theta_3} + 2 \cdot \boldsymbol{\theta_4}$$

Matrix Multiplication Characteristics

- Matrix multiplication is not commutative i.e. given two matrices A and B:
  - $AB \neq BA$

- E.g. given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$

- Then:

$$AB = \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}$$

- But:

$$BA = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Matrix Multiplication Characteristics

- Matrix multiplication is not commutative i.e. given two matrices A and B:
  - $AB \neq BA$

- In fact, in some cases, AB may be defined, but BA may not be defined e.g.

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix}$$

(4×2) (2×3)

These Match:

We can multiply them

(2×3) (4×2)

NO MATCH!

We can't multiply them

Identity Matrix

- The identity matrix is a square matrix of varying size (>1) with a 1s on the diagonal and 0s everywhere else e.g.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Represented by the symbol I by convention
- Any matrix multiplied by an appropriate size I gives back the same matrix i.e.
    - $I \cdot \mathrm{A} = \mathrm{A} \cdot I = A$
    - It is the matrix-equivalent of the number 1 for scalar numbers e.g. $1 \cdot 4 = 4 \cdot 1 = 4$
- E.g.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix}$$

Matrix Inverse

- Given a scalar number, multiplying the number by its inverse results in 1 e.g.
    - $3 \cdot (3^{-1}) = 1$
    - $52 \cdot (52^{-1}) = 1$

- Not all numbers have an inverse e.g. $0 \cdot (0^{-1})$ is undefined; 0 doesn't have an inverse

- For matrices, multiplying the matrix by its inverse results in the identity matrix I.
    - $\mathrm{A} \cdot \mathrm{A}^{-1} = I$
- E.g.

$$\underset{\mathrm{A}}{\begin{bmatrix} 4 & 8 \\ 7 & 5 \end{bmatrix}} \underset{\mathrm{A}^{-1}}{\begin{bmatrix} 5/36 & -2/9 \\ -7/36 & 1/9 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Matrix Inverse

- Given a scalar number, multiplying the number by its inverse results in 1 e.g.
    - $3 \cdot (3^{-1}) = 1$
    - $52 \cdot (52^{-1}) = 1$

- Not all numbers have an inverse e.g. $0 \cdot (0^{-1})$ is undefined; 0 doesn't have an inverse

- Not all matrices have inverses:
    - Non-square matrices don't have inverses
    - Some square matrices don't have inverses

- Therefore, for a matrix to have an inverse:
    - it **must** be square
    - but **not** all square matrices have inverses

Matrix Pseudo-Inverse

- Given a matrix A that doesn't have an inverse i.e.:
  - It is non-square OR
  - It is square but doesn't have an inverse

- The pseudo-inverse of the matrix A^+ can be computed such that:
  - $A^+ \cdot A \approx I$

THE END

Of Linear Algebra Review