Duality - Continued

Lecture 5

October 7, 2024

Recap From Last Time

We obtained the following primal-dual pair of problems:

Primal $(\mathscr{P})$			**Dual** $(\mathscr{D})$		
minimize$_x$	$c^\mathsf{T} x$		maximize$_p$	$p^\mathsf{T} b$	
$(p_i \rightarrow)$	$a_i^\mathsf{T} x \geq b_i,$	$i \in M_1,$		$p_i \geq 0,$	$i \in M_1,$
$(p_i \rightarrow)$	$a_i^\mathsf{T} x \leq b_i,$	$i \in M_2,$		$p_i \leq 0,$	$i \in M_2,$
$(p_i \rightarrow)$	$a_i^\mathsf{T} x = b_i,$	$i \in M_3,$		p_i free,	$i \in M_3,$
	$x_j \geq 0,$	$j \in N_1,$	$(x_j \rightarrow)$	$p^\mathsf{T} A_j \leq c_j,$	$j \in N_1,$
	$x_j \leq 0,$	$j \in N_2,$	$(x_j \rightarrow)$	$p^\mathsf{T} A_j \geq c_j,$	$j \in N_2,$
	x_j free,	$j \in N_3.$	$(x_j \rightarrow)$	$p^\mathsf{T} A_j = c_j,$	$j \in N_3.$

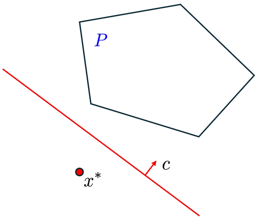
Simple rules to help you derive duals quickly:

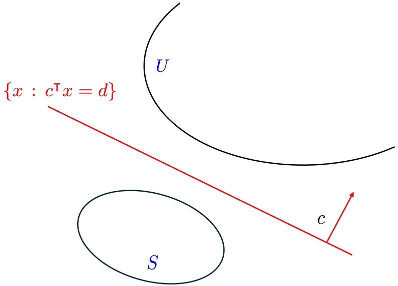
- a dual decision variable for every primal constraint (except variables signs)
  - if "$=$" constraint, dual variable is free
  - if ("$\geq$", minimize) or ("$\leq$", maximize), dual variable ≥ 0
  - if ("$\geq$", maximize) or ("$\leq$", minimize), dual variable ≤ 0
- for every decision variable in the primal, there is a constraint in the dual
  - signs for the constraint derived by reversing the above

Separating Hyperplane Theorem

Theorem (Separating Hyperplane Theorem for Convex Sets)

Let S and U be two nonempty, closed, convex subsets of $\mathbb{R}^n$ such that $S \cap U = \emptyset$ and S is bounded. Then, there exists a vector $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ such that $S \subset \{x \in \mathbb{R}^n : c^\intercal x < d\}$ and $U \subset \{x \in \mathbb{R}^n : c^\intercal x > d\}$.





Separating Hyperplane Theorem - Caveats!

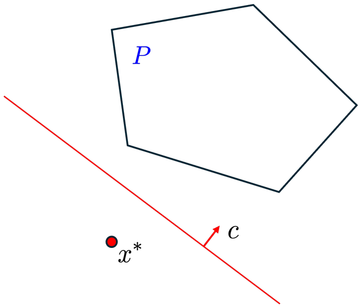
Both conditions in the theorem needed: closed and at least one bounded

Needed For Our Purposes

We proved the first **fundamental result in optimization**!

Corollary (Needed for our purposes...)

If P is a polyhedron and x^* satisfies $x \notin P$, there exists a hyperplane that strictly separates x from P, i.e., $\exists c \neq 0$ such that $c^\intercal x^* < c^\intercal x \, \forall x \in P$.



Farkas Lemma

Time for the **second fundamental result in optimization**!

Farkas Lemma

Theorem (Farkas' Lemma)

For $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$*, exactly one of the following two alternatives holds:*

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For $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, exactly one of the following two alternatives holds:
(a) There exists some $x \geq 0$ such that $Ax = b$.
(b) There exists some vector p such that $p^T A \geq 0$ and $p^T b < 0$.

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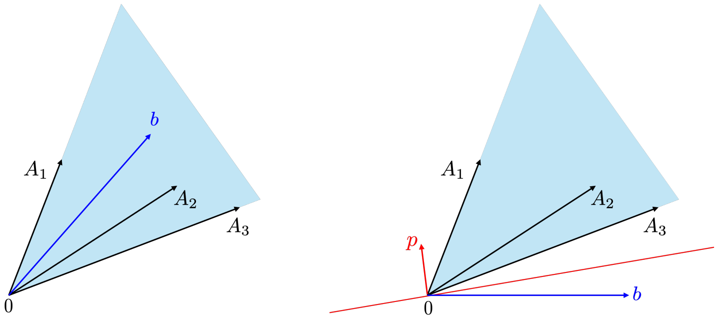

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Proof. "(a) $\Rightarrow$ not (b)."

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Proof. "(a) $\Rightarrow$ not (b)."

(a) implies $\exists x \geq 0 : Ax = b$.

(b) implies $\exists p : p^T A \geq 0$.

But then $p^T b = p^T Ax \geq 0$, so (b) cannot hold.

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"not (a) $\Rightarrow$ (b)." Want to use the separating hyperplane theorem.

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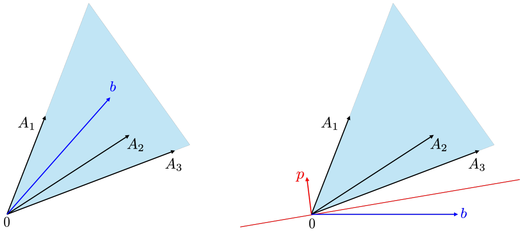
- Assume $\nexists x \geq 0 : Ax = b$. This implies that $b \notin S$ where:

$$S := \{Ax \ : \ x \geq 0\} = \{y \ : \ \exists\, x \geq 0 \text{ such that } y = Ax\}.$$



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- S is convex.
- To apply separating hyperplane theorem, need S **closed**!

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  - S is the projection of $\bar{S} := \{(x, y) \,:\, x \geq 0,\, y = Ax\}$ on the y variables.

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$\Rightarrow S$ is closed.

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- $S := \{Ax : x \geq 0\} = \{y : \exists\, x \geq 0 \text{ such that } y = Ax\}$ is convex and closed.
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- Every column A_i of A satisfies $\lambda A_i \in S$ for every $\lambda > 0$, so

$$\frac{p^\intercal b}{\lambda} < p^\intercal A_i,\ \forall \lambda > 0$$

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- Limit $\lambda \to \infty$ implies $p^\intercal A_i \geq 0$. ■

Farkas Lemma Implications

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We proved the **second fundamental result in optimization**!

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This has some important implications:

- Suppose your primal problem $(\mathscr{P})$ was the standard-form LP:

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- Farkas Lemma states that either $(\mathscr{P})$ is feasible or ...
  ... there exists p (satisfying $p^\intercal A \leq c^\intercal$) that is a **certificate of infeasibility**!

Strong Duality

(W.L.O.G.) Consider the following primal-dual pair:

$$(\mathscr{P}) \quad \text{minimize } c^\mathsf{T} x \qquad\qquad (\mathscr{D}) \quad \text{maximize } p^\mathsf{T} b$$
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Theorem (Strong Duality)

If $(\mathscr{P})$ has an optimal solution, so does $(\mathscr{D})$, and their optimal values are equal.

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Proof.

- Assume $(\mathscr{P})$ has optimal solution x^*
- Will prove that $(\mathscr{D})$ admits feasible solution p such that $p^\intercal b = c^\intercal x^*$

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- Let $\mathcal{F} = \{i \mid a_i^\mathsf{T} x^* = b_i\}$ be indices of active constraints at x^*
- Show that c can be written as conic combination of constraints $\{a_i : i \in \mathcal{F}\}$

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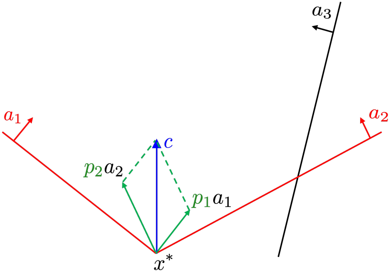

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- First, we show that for any vector d, the following implication holds:

$$a_i^\mathsf{T} d \geq 0, \forall\, i \in \mathcal{F} \quad \Rightarrow \quad c^\mathsf{T} d \geq 0.$$

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$(\mathscr{P})$ minimize $c^\intercal x$ subject to $Ax \geq b$

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- So $\nexists d : a_i^\intercal d \geq 0, \forall\, i \in \mathcal{F}, \, c^\intercal d < 0$
- Farkas Lemma : alternative (b) is not true, so alternative (a) must be true:

$$\exists \{p_i\}_{i \in \mathcal{F}} : p_i \geq 0, \; c = \sum_{i \in \mathcal{F}} p_i a_i$$

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- $c^{\mathsf{T}} d \geq 0$ because otherwise $c^{\mathsf{T}}(x^* + \epsilon d) < c^{\mathsf{T}} x^*$ would contradict x^* optimal
- So $\nexists d : a_i^{\mathsf{T}} d \geq 0, \forall\, i \in \mathcal{F}, \, c^{\mathsf{T}} d < 0$
- Farkas Lemma : alternative (b) is not true, so alternative (a) must be true:

$$\exists \{p_i\}_{i \in \mathcal{F}} : p_i \geq 0, \; c = \sum_{i \in \mathcal{F}} p_i a_i$$

- Let $p_i = 0$ for $i \notin \mathcal{F} \quad \Rightarrow \quad \exists p$ feasible for $(\mathscr{D})$
- $p^{\mathsf{T}} b = \sum_{i \in \mathcal{F}} p_i b_i = \sum_{i \in \mathcal{F}} p_i a_i^{\mathsf{T}} x^* = c^{\mathsf{T}} x^*$ $\blacksquare$

Implications

Strong duality leaves only a few possibilities for a primal-dual pair:

		Dual		
		Finite Optimum	Unbounded	Infeasible
Primal	Finite Optimum	?	?	?
	Unbounded	?	?	?
	Infeasible	?	?	?

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Primal	Finite Optimum	Possible	Impossible	Impossible
	Unbounded	Impossible	Impossible	Possible
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Example

Is this primal feasible? What is its dual?

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\begin{aligned}
\text{minimize} \quad & x_1 + 2x_2 \\
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Application in Robust Optimization

- We have LP with constraints $Ax \leq b$. One of the constraints is:

$$a^\intercal x \leq b, \tag{1}$$

  where a satisfies $a \in \mathcal{A}$ and $\mathcal{A}$ is polyhedral

- We seek decisions x that are **robustly feasible**, i.e.,

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  Infinitely many constraints : "semi-infinite" LP. *Any ideas?*

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Polynomially-Sized CVaR Representation

- Recall homework: ensure CVaR of portfolio payoff exceeds a lower limit
- CVaR was defined as the average over the k-smallest values (for suitable integer k)
- If payoffs in the scenarios are $v_1, v_2, \ldots, v_n$, the key constraint is:

$$\sum_{i=1}^{k} v_{[i]} \geq b, \tag{3}$$

where $v_{[1]} \leq v_{[2]} \leq \cdots \leq v_{[n]}$ is the sorted vector of payoffs.

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  where $v_{[1]} \leq v_{[2]} \leq \cdots \leq v_{[n]}$ is the sorted vector of payoffs.
- Can write one constraint for each vector in $\{0,1\}^n$ with exactly k values of 1.
- *How to formulate with a polynomial number of variables and constraints?*

Optimality for Standard-Form LPs

$$(\mathscr{P}) \ \min c^\mathsf{T} x \qquad Ax = b, \quad x \geq 0$$

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Primal optimality $\Leftrightarrow$ Dual feasibility

Simplex terminates when finding a dual-feasible solution!

Solve $(\mathscr{P})$ or $(\mathscr{D})$?

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- Suppose we have x^*, p^* and must solve a **larger** problem. *Any ideas?*
- Modern solvers include primal and dual simplex and allow concurrent runs

Dual Variables As Marginal Costs

$$(\mathscr{P})\ \min c^\intercal x \qquad\qquad (\mathscr{D})\ \max p^\intercal b$$
$$Ax = b, \quad x \geq 0 \qquad\qquad p^\intercal A \leq c^\intercal$$

- Solved the LP and obtained x^* and p^*
- Want to show that p^* is **gradient of the optimal cost with respect to** b ("almost everywhere")
- Related to **sensitivity analysis**
  How do the optimal value and solution depend on problem data A, b, c?

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- Let $P(b) := \{x : Ax = b, x \geq 0\}$ and $F(b)$ denote the optimal cost
- Assume that dual is feasible: $\{p : p^\intercal A \leq c^\intercal\} \neq \emptyset$, so $F(b) > -\infty$
- Want to show that $F(b)$ is **piecewise linear and convex**

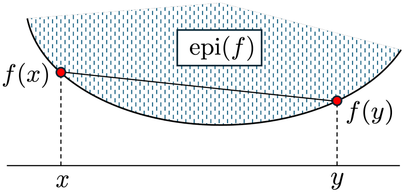
Convex and Concave Functions

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **convex** if X is a convex set and

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall x, y \in X \text{ and } \lambda \in [0,1]. \tag{6}$$

A function is **concave** if $-f$ is convex.



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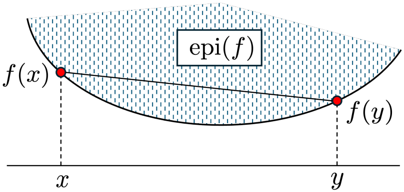

Equivalent definition in terms of **epigraph**:

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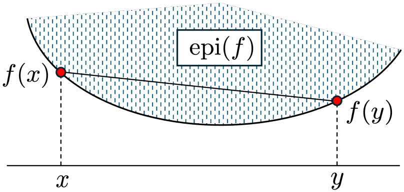

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f is convex if and only if epi(f) is a convex set.

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$$F(b) := \min\left\{c^\intercal x : Ax = b,\ x \geq 0\right\} \equiv \max\left\{p^\intercal b \,:\, p^\intercal A \leq c^\intercal\right\}$$

Theorem

$F(b)$ *is a convex and piece-wise linear function of* b *on* $S := \{b : P(b) \neq \emptyset\}$.

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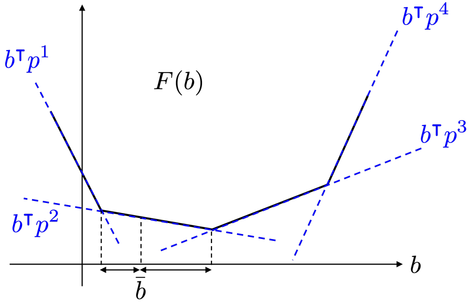

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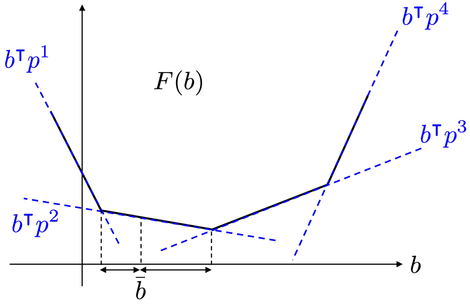

How to complete proof that $F(b)$ is convex?

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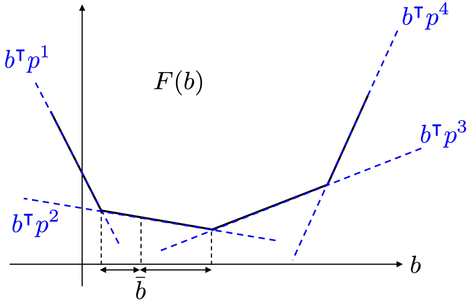

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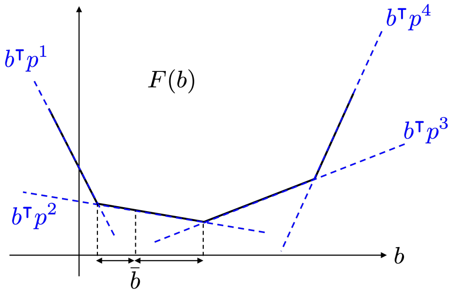

- At any $b = \bar{b}$ where $F(b)$ is differentiable, p^* **is the gradient of** $F(b)$
- p_i^* acts as a **marginal cost** or **shadow price** for the i-th constraint r.h.s. b_i
- p_i allows estimating **exact change in** $F(b)$ **in a range around** $\bar{b}$
- Modern solvers give direct access to p_i^* and the range

  Gurobipy: for constraint **c**, the attribute **c.Pi** is p_i^* and the range is from **c.SARHSLow** to **c.SARHSUp**

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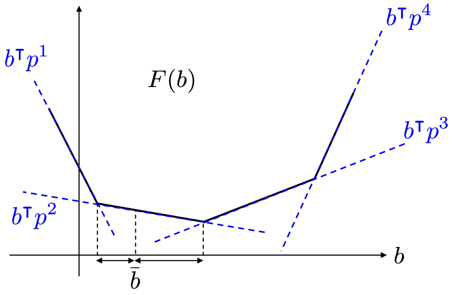

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- All such p^i are valid **subgradients** of $F(b)$

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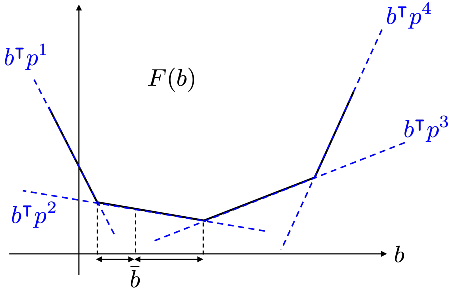

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Definition (Subgradient.)

F convex, defined on (convex) set S. A vector p is a **subgradient** of F at $\bar{b} \in S$ if

$$F(\bar{b}) + p^\intercal (b - \bar{b}) \leq F(b), \quad \forall b \in S.$$

Optimal Duals As Subgradients

Theorem

Suppose $F(b) := \min\{c^\mathsf{T} x : Ax = b,\ x \geq 0\} \equiv \max\{p^\mathsf{T} b \,:\, p^\mathsf{T} A \leq c^\mathsf{T}\} > -\infty$*.*
Then p *is optimal for the dual* **if and only if** *it is a subgradient of* F *at* $\bar{b}$*.*

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- This implies $p^\intercal b \leq F(b)$
- But then, $p^\intercal b - p^\intercal \bar{b} \leq F(b) - F(\bar{b})$

We conclude that p is a subgradient of F at $\bar{b}$

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Proof. For the reverse direction, let p be a subgradient of F at $\bar{b}$, that is,

$$F(\bar{b}) + p^\intercal(b - \bar{b}) \leq F(b), \quad \forall b \in S. \tag{8}$$

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- Because this is true for any $x \geq 0$, we must have $p^\intercal A \leq c^\intercal$. *Why?*

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- This implies that p is dual-feasible
- With $x = 0$, we obtain $F(\bar{b}) \leq p^\intercal \bar{b}$
- Using weak duality, every dual-feasible q satisfies $q^\intercal \bar{b} \leq F(\bar{b}) \leq p^\intercal \bar{b}$

We conclude that p is optimal.

Global Dependency On c

Let $G(c) := \min\{c^\intercal x : Ax = b,\ x \geq 0\} \equiv \max\{p^\intercal b \,:\, p^\intercal A \leq c^\intercal\}$

Theorem

For an LP in standard form,

1. *The set $T := \{c : G(c) > -\infty\}$ is convex.*
2. *$G(c)$ is a* **concave** *function of c on the set T.*
3. *If for some c the LP has a* **unique** *optimal solution x^*, then G is linear in the vicinity of c and its gradient is x^*.*

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- The optimal primal solution x^* is a **shadow price for the dual constraints**
- x^* remains optimal for a range of change in each objective coefficient c_j
- Modern solvers also allow obtaining the range directly

  Gurobipy: attributes **SAObjLow** and **SAObjUp** for each decision variable

Signs of Dual Variables Revisited

These ideas carry over directly to **primals in general form**:

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\begin{array}{llll}
F(b,c) := \min_x & c^\intercal x & & \\
& a_i^\intercal x \geq b_i, & i \in M_1, & \\
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- Given two of these, can figure out the third one!
- *What is the sign of the shadow price for a ...*

  $\leq$ constraint in a **minimization** problem ?

  $\geq$ constraint in a **minimization** problem ?

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- *What is the dependency of the optimal objective on the r.h.s. of a ...*

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Theorem (Complementary Slackness)

Let x and p be feasible solutions for $(\mathscr{P})$ and $(\mathscr{D})$, respectively. Then x and p **are optimal solutions** *for $(\mathscr{P})$ and $(\mathscr{D})$* **if and only if***:*

$$p_i(a_i^\intercal x - b_i) = 0, \; \forall i$$

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Theorem (**Strict** C.S. **Standard-Form LPs**)

Consider the following primal-dual pair of LPs:

$$(\mathscr{P})\ \min c^\intercal x \qquad\qquad (\mathscr{D})\ \max p^\intercal b$$
$$Ax = b, x \geq 0 \qquad\qquad p^\intercal A \leq c^\intercal$$

If $(\mathscr{P})$ and $(\mathscr{D})$ are feasible, they admit optimal solutions x^* and p^* satisfying **strict complementarity***: $x_j^* > 0 \ \Leftrightarrow\ p^\intercal A_j = c_j$.*

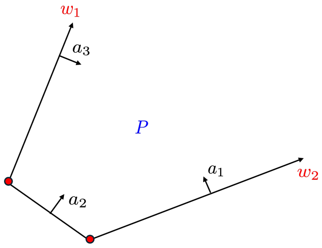
Representation of Polyhedra

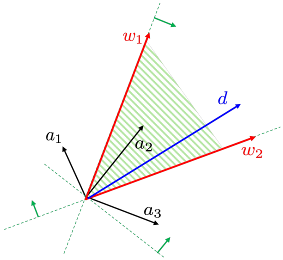
Important consequence of duality: alternative representation of all polyhedra

Definition (Extreme rays of a polyhedron)

Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n \,:\, Ax \geq b\}$. Then:

1. $\mathcal{C} := \{d \in \mathbb{R}^n \,:\, Ad \geq 0\}$ is called the **recession cone** of P.
2. Any $d \in \mathcal{C}$ with $d \neq 0$ is called a **ray** of P.
3. Any ray d that satisfies $a_i^\intercal d = 0$ for $n-1$ linearly independent a_i is called an **extreme ray** of P.





Representation of Polyhedra

Theorem (Resolution Theorem)

Let $P = \{x \in \mathbb{R}^n \,:\, Ax \geq b\}$ be a non-empty polyhedron, $x^1, x^2, \ldots, x^k$ be its extreme points, and $w^1, w^2, \ldots, w^r$ be its extreme rays. Then $P = Q$, where

$$Q := \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \;:\; \lambda \geq 0,\ \theta \geq 0,\ e^\mathsf{T} \lambda = 1 \right\}.$$

Proof. Proving $Q \subseteq P$ is immediate. To prove $P \subseteq Q$, assume $\exists z \in P$ with $z \notin Q$. Consider the following primal-dual pair:

$$(\mathscr{P}) \quad \max_{\lambda \geq 0, \theta \geq 0} \; \sum_{i=1}^{k} 0\lambda_i + \sum_{j=1}^{r} 0\theta_j$$

$$\sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j = z$$

$$\sum_{i=1}^{k} \lambda_i = 1$$

$$(\mathscr{D}) \quad \min_{p,q} \; p^\mathsf{T} z + q$$

$$p^\mathsf{T} x_i + q \geq 0, \quad i = 1, \ldots, k,$$

$$p^\mathsf{T} w_j \geq 0, \qquad j = 1, \ldots, r,$$

Is $(\mathscr{P})$ feasible? Is $(\mathscr{D})$ feasible? What are the optimal values?

Representation of Polyhedra - cntd

$$P := \{x \in \mathbb{R}^n : Ax \geq b\} = Q := \left\{ \sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^r \theta_j w^j : \lambda \geq 0, \theta \geq 0, e^\mathsf{T} \lambda = 1 \right\}.$$

Proof - cont'd. Assume $\exists z \in P$ with $z \notin Q$. Consider the following primal-dual pair:

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$$\sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^r \theta_j w^j = z \qquad\qquad p^\mathsf{T} x_i + q \geq 0, \quad i = 1, \ldots, k,$$

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- $(\mathscr{P})$ is infeasible because $z \notin Q$
- $(\mathscr{D})$ is feasible with $p = q = 0$, so its optimal value is $-\infty \Rightarrow \exists (p,q) : p^\mathsf{T} z + q < 0$

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$$(\mathscr{D}) \quad \min_{p,q} p^\intercal z + q \qquad p^\intercal x_i + q \geq 0, \quad i = 1, \ldots, k, \qquad p^\intercal w_j \geq 0, \quad j = 1, \ldots, r,$$

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- If optimal cost finite, $\exists x^i$ optimal. But $z \in P$ and $p^\intercal z < p^\intercal x_i$ lead to $\lightning$
- If cost is $-\infty$, $\exists w^j : p^\intercal w^j < 0$, which is also a $\lightning$

Asset Pricing and No-Arbitrage

- Investment world with $n+1$ securities indexed by $i = 0, \ldots, n$
- $i = 0$ denotes cash; the other securities can be anything (stocks, derivatives, ...)
- We have two periods: current period **c**, future period **f**

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- Future period: prices are uncertain; there are m possible states of the world $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, each occurring with positive probability, and prices are:
  - cash is riskless: $S_0^f = R = 1 + r$, where r is the risk-free rate of return
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Asset Pricing and No-Arbitrage

- Investment world with $n + 1$ securities indexed by $i = 0, \ldots, n$
- $i = 0$ denotes cash; the other securities can be anything (stocks, derivatives, ...)
- We have two periods: current period **c**, future period **f**
- Current period: prices of securities are S_i^c for $i = 1, \ldots, n$; cash: $S_0^c = 1$
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  - cash is riskless: $S_0^f = R = 1 + r$, where r is the risk-free rate of return
  - security $i > 1$ will have price $S_i^f(\omega_j)$ in state of world ω_j
- If we purchase x_i of each security i:
  - we incur immediate cost $\sum_{i=0}^n S_i^c x_i$
  - we have future cashflow $\sum_{i=0}^n S_i^f(\omega) \cdot x_i$ if state of world is $\omega \in \Omega$

Asset Pricing and No-Arbitrage

Definition (Arbitrage)

An **arbitrage** is a trading strategy that either has a positive initial cashflow and has no risk of a loss later (type A) or that requires no initial cash input, has no risk of loss, and has a positive probability of making profits in the future (type B).

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$$\begin{aligned} \sum_{i=0}^{n} S_i^c \cdot x_i &< 0 && \text{(positive initial cashflow)} \\ \sum_{i=0}^{n} S_i^f(\omega) \cdot x_i &\geq 0, \, \forall \, \omega \in \Omega && \text{(no risk of loss)} \end{aligned} \tag{9}$$

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Asset Pricing and No-Arbitrage

Definition (R.N.P.M.)

A **risk-neutral probability measure** on the set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ is a vector $p \in \mathbb{R}^m$ so that $p > 0$ and $\sum_{j=1}^m p_j = 1$ and for every security $S_i, i = 0, \ldots, n$,

$$S_i^c = \frac{1}{R}\left(\sum_{j=1}^{m} p_j S_i^f(\omega_j)\right) = \frac{1}{R}\mathbb{E}_p[S_i^f].$$

- Above, $\mathbb{E}_p[S]$ is the expected value of the random variable S under the probability distribution $p := (p_1, p_2, \ldots, p_m)$
- The definition states that the current price/value of every asset, S_i^c, exactly equals **the discounted expected price/value in the future**
- The expectation is taken with respect to the R.N.P.M.
- Discounting is done at the risk-free interest rate R

Asset Pricing and No-Arbitrage

Theorem (Asset Pricing Theorem)

A risk-neutral probability measure exists **if and only if** *there is no arbitrage.*

Proof. Consider the following linear program with variables x_i, for $i = 0, \ldots, n$:

$$\begin{aligned} \min_x \; & \sum_{i=0}^{n} S_i^c \cdot x_i \\ \text{s.t.} \; & \sum_{i=0}^{n} S_i^f(\omega_j) \cdot x_i \geq 0, j = 1, \ldots, m. \end{aligned} \tag{11}$$

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- Suppose no type-A arbitrage. Then, no type-B arbitrage if and only if all constraints are tight for all optimal solutions of (11): $\sum_{i=0}^{n} S_i^f(\omega_j) \cdot x_i^* = 0$, for $j = 1, \ldots, m$

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&\max_p 0 \\
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- Dual constraint for $i = 0$ implies $\sum_{j=1}^{m} p_j^* = \frac{1}{R}$, so taking $p^* \cdot R$ yields a RNPM.

The converse direction is proved in an identical manner. ■

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  - each itinerary refers to an origin-destination-fare class combination
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- Requirements: $A \in \{0,1\}^{F \cdot I}$ with $A_{f,i} = 1 \Leftrightarrow$ itinerary i needs seat on flight leg f

Resource matrix A :

	Itinerary 1	Itinerary 2	...	Itinerary $\|I\|$
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- Goal: decide how many itineraries of each type to sell to maximize revenue

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- Broader principle of how to price "products" through resource usage/cost