Integer Optimization

Lecture 8

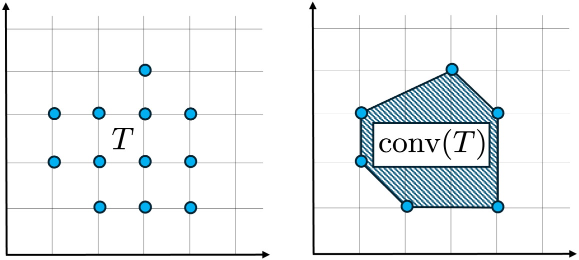
October 16, 2024

Strength of IP Formulation

- Different formulations of the same IP can result in **different LP relaxations**

- *What is an "ideal" formulation?*

Strength of IP Formulation



(a) (b)

- T : all feasible points to an IP and $\mathrm{conv}(T)$ is their convex hull
  - T finite because we assumed bounded feasible set
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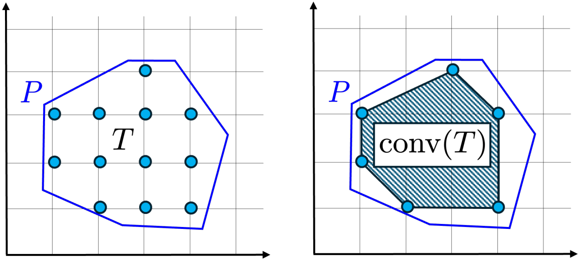

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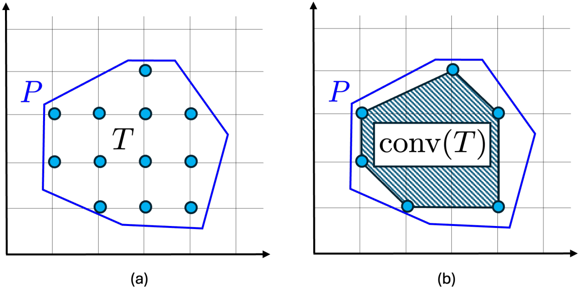

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- If P is the feasible region of the LP relaxation, then

$$T \subseteq \text{conv}\,(T) \subseteq P.$$

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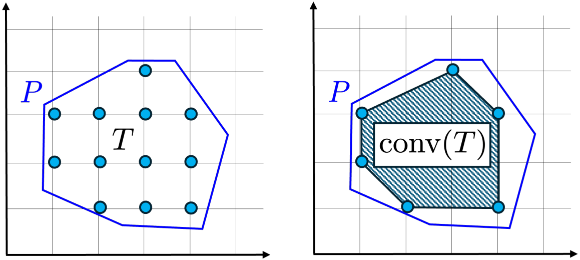

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- If P is the feasible region of the LP relaxation, then

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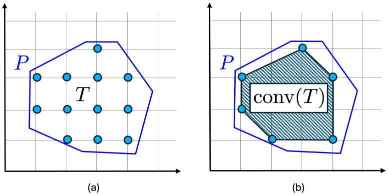
- The "closer" P hugs $\mathrm{conv}\,(T)$, the better!

Key Take-Aways and Next Steps



- **Quality of IP formulation** : how closely its LP relaxation approximates $\mathrm{conv}(T)$
- Formulations A, B equivalent for an IP. A is **stronger than** B if $P_A \subset P_B$
- **Constraints** play a more subtle role in IPs than in LPs
  - Adding valid constraints for T that cut off fractional points from P is very useful!
  - More constraints not necessarily worse in IP!

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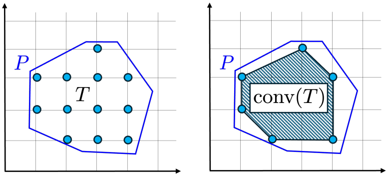

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1. Discuss a few **ideal formulations** : $P = \mathrm{conv}\,(T)$
2. Discuss how to **improve** formulations by adding **cuts**
3. Discuss **algorithms/solution approaches**

Ideal Formulations

Setup:

- $P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ polyhedral set, with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$
- **Goal:** conditions on A so that P **is integral**, i.e., $P = \mathrm{conv}\left(x \in P : x \in \mathbb{Z}^n\right)$

Can anyone recall Cramer's rule?

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Proposition (Cramer's Rule)

Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. For $b \in \mathbb{R}^n$,

$$Ax = b \quad \Longrightarrow \quad x = A^{-1}b \quad \Longrightarrow \quad x_i = \frac{\det(A^i)}{\det(A)}, \ \forall i,$$

where A^i is the matrix with columns $A^i_j = A_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$ and $A^i_i = b$.

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If $\det(A) \in \{1, -1\}$, that would be nice!

(Total) Unimodularity

Definition

1. $A \in \mathbb{Z}^{m \times n}$ of full row rank is **unimodular** if the $\det(A_B) \in \{1, -1\}$ for every basis B.
2. $A \in \mathbb{Z}^{m \times n}$ is **totally unimodular** if the determinant of each square submatrix of A is 0, 1, or -1.

- **Unimodularity** allows handling standard form $\{x \in \mathbb{Z}_+^n \mid Ax = b\}$
- **Total Unimodularity (TU)** allows handling inequality form $\{x \in \mathbb{Z}_+^n \mid Ax \leq b\}$

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$$\text{e.g.} \quad A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

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- Will provide easier ways to test for U and TU, but first let's see why we care...

(Total) Unimodularity Yields Integral LP Relaxations

Theorem

1. *The matrix $A \in \mathbb{Z}^{m \times n}$ of full row rank* **is unimodular if and only if** *the polyhedron $P(b) = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$* **is integral** *for all $b \in \mathbb{Z}^m$ with $P(b) \neq \emptyset$.*
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Proof. (a) "$\Rightarrow$" Because A unimodular, for any $b \in \mathbb{Z}^m$ with $P(b) \neq \emptyset$, any basic feasible solution $x = (x_B, x_N) \in P(b)$ must satisfy $x_B = A_B^{-1} b \in \mathbb{Z}^{|B|}$.

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- Then $A_B^{-1} \cdot b = z + A_B^{-1} e_i$
- By choosing z large so $z + A_B^{-1} e_i \geq 0$, we obtain a b.f.s. for $P(b)$
- Because $P(b)$ integral, $A_B^{-1} e_i$ must be integral
- Repeat argument for all e_i to proves that A_B^{-1} is integral.

(b) Similar logic, omitted (see notes)

Checking for Total Unimodularity

Proposition

Consider a matrix $A \in \{0, 1, -1\}^{m \times n}$. The following are equivalent:

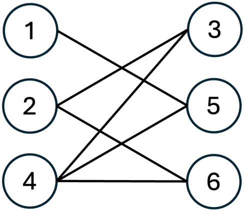
1. *A is totally unimodular.*
2. *$A^\intercal$ is totally unimodular.*
3. *$[A^\intercal \; -A^\intercal \; I \; -I]$ is totally unimodular.*
4. *$\{x \in \mathbb{R}^n_+ \mid Ax = b, 0 \le x \le u\}$ is integral for all integral b, u.*
5. *$\{x \mid a \le Ax \le b, \ell \le x \le u\}$ is integral for all integral a, b, ℓ, u.*
6. *Each collection of columns of A can be partitioned into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries $0, +1$, and -1. (By part 2, a similar result also holds for the rows of A.)*
7. *Each nonsingular submatrix of A has a row with an odd number of non-zero components.*
8. *The sum of entries in any square submatrix with even row and column sums is divisible by four.*
9. *No square submatrix of A has determinant $+2$ or -2.*

#6 perhaps most useful in practice...

Examples of TU Matrices #1

- $G = (\mathcal{N}, \mathcal{E})$ **undirected graph**

- $A \in \{0,1\}^{|\mathcal{N}| \times |\mathcal{E}|}$ is the node-edge incidence matrix of G
  $A_{i,e} = 1$ *if and only if* $i \in e$



	$\{1,5\}$	$\{2,3\}$	$\{2,6\}$	$\{4,3\}$	$\{4,5\}$	$\{4,6\}$
1	1	0	0	0	0	0
2	0	1	1	0	0	0
3	0	1	0	1	0	0
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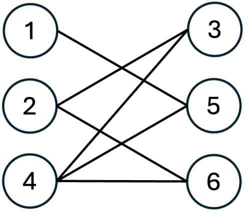

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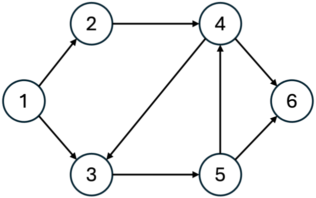
- A **is TU if and only if** G **is bipartite**
- Bipartite matching problems have integral LP relaxations...

Examples of TU Matrices #2

- $D = (V, A)$ is a **directed graph**
- M is the $V \times A$ incidence matrix of D

$$M_{v,a} = \begin{cases} 1 & \text{if and only if } a = (\cdot, v) \text{ (arc } a \text{ enters node } v\text{)} \\ -1 & \text{if and only if } a = (v, \cdot) \text{ (arc } a \text{ leaves node } v\text{)} \\ 0 & \text{otherwise.} \end{cases}$$



	(1,2)	(1,3)	(2,4)	(4,3)	(3,5)	(5,4)	(4,6)	(5,6)
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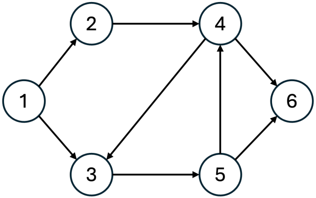

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- Then M **is TU**
- **Network flow problems** (e.g., Prosche Motors) with integral arc capacities and integral supply/demand have integral LP relaxations

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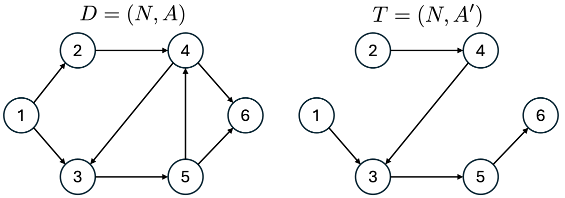

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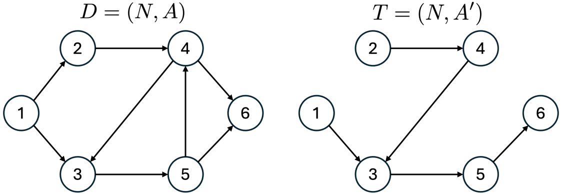

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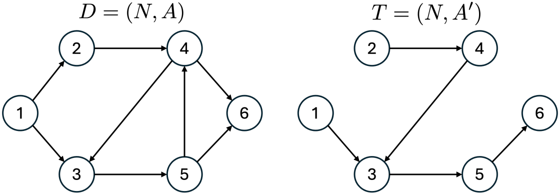

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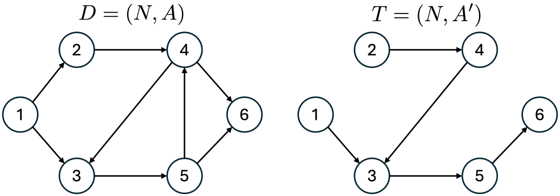

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- Then M **is TU**

- All previous examples were **special cases** of this

- Paul Seymour: **all TU matrices** generated from network matrices and **two** other matrices

Dual Integrality and Submodular Functions

- Alternative way to show integrality of polyhedra based on **LP** duality
- Simple observation: to show that LP relaxation is integral, it suffices to check that the optimal value of any LP with integer cost vector c is an integer

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Proposition

P polyhedron with at least one extreme point. Then P is integral **if and only if** *the optimal value $Z_{LP} := \min\{c^\intercal x \mid x \in P\}$ is an integer for all $c \in \mathbb{Z}^n$.*

Proof. Straightforward; omitted.

- To show integrality of P, we **construct an integral dual-optimal solution** (for any $c \in \mathbb{Z}^n$)

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- Simple observation: to show that LP relaxation is integral, it suffices to check that the optimal value of any LP with integer cost vector c is an integer

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P polyhedron with at least one extreme point. Then P is integral **if and only if** *the optimal value $Z_{LP} := \min\{c^\intercal x \mid x \in P\}$ is an integer for all $c \in \mathbb{Z}^n$.*

Proof. Straightforward; omitted.

- To show integrality of P, we **construct an integral dual-optimal solution** (for any $c \in \mathbb{Z}^n$)
- Our discussion here is quite specific
  - broader concepts possible related to Totally Dual Integrality
  - if interested, see notes for references

Submodular Functions

Definition

A function $f(S)$ defined on subsets S of a finite set $N = \{1, \ldots, n\}$ is **submodular** if

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T), \quad \forall S, T \subset N \tag{1}$$

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  - Submodular functions exhibit **"diminishing returns"** or **"decreasing differences"**
  - Might resemble concavity in economic intuition, but **not** computationally! (submodular functions are more like **convex** functions!)

Submodular Functions - Equivalent Definitions

Proposition

A set function $f : 2^N \to \mathbb{R}$ *is* **submodular if and only if***:*

(a) *For any* $S, T \subseteq N$ *such that* $S \subseteq T$ *and* $k \notin T$*:*

$$f(S \cup \{k\}) - f(S) \geq f(T \cup \{k\}) - f(T).$$

(b) *For any* $S \subseteq N$ *and any* j, k *with* $j, k \notin S$ *and* $j \neq k$*:*

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- Subsequently, interested in non-negative and **increasing** submodular functions

$$f(S) \leq f(T), \quad \forall S \subset T \subseteq N.$$

Submodular Functions - Equivalent Definitions

- **Linear functions.** For $w \in \mathbb{R}^n$, $f(A) = \sum_{i \in A} w_i$ is both sub- and super-modular.

- **Composition 2.** If $w \geq 0$ and g concave, then $f(S) = g\left(\sum_{i \in S} w_i\right)$ is submodular.

- **Optimal TSP cost on tree graphs.** Consider **undirected tree graph** $G = (N, E)$ with a positive cost for traversing the edges ($c_e \geq 0$ for every edge $e \in E$). For every $S \subseteq N$, define $f(S)$ as the optimal (i.e., smallest) cost for a TSP that goes through all the nodes in S. Then, $f(S)$ is submodular.

- **Network optimization:** consider directed graph with capacities on edges that constrain how much flow can be transported; if $f(S)$ is the maximum flow that can be received at a set of sink nodes S, $f(S)$ is submodular.

- **Inventory and supply chain management:** perishable inventory systems, dual sourcing, and inventory control problems with trans-shipment.

Main Result

- For a submodular function f, consider the problem:

$$\begin{aligned} \text{maximize } & \sum_{j=1}^{n} r_j \cdot x_j \\ & \sum_{j \in S} x_j \leq f(S), \ \forall S \subseteq N \\ & x \in \mathbb{Z}_+^n. \end{aligned}$$

- T: set of feasible integer solutions
- $P(f)$ the feasible set of the LP relaxation:

$$P(f) = \left\{ x \in \mathbb{R}_+^n \;\middle|\; \sum_{j \in S} x_j \leq f(S), \ \forall S \subset N \right\}$$

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Theorem

If f is submodular, increasing, integer valued, and $f(\emptyset) = 0$, then

$$P(f) = \mathrm{conv}(T).$$

Main Result - Proof

To show: f is submodular, increasing, integer-valued, $f(\emptyset) = 0$, then $P(f) = \text{conv}(T)$.

Proof. Consider the linear relaxation and its dual:

$$\begin{aligned} \text{maximize} \ & \sum_{j=1}^{n} r_j x_j \\ & \sum_{j \in S} x_j \leq f(S), \quad S \subset N, \\ & x_j \geq 0, \ j \in N \end{aligned}$$

- Key idea: construct feasible solutions for both, with equal value
- Key intuition: use a **greedy** construction in the primal!

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- Suppose $r_1 \geq r_2 \geq \ldots \geq r_k > 0 \geq r_{k+1} \geq \ldots \geq r_n$.
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- The primal objective: $\displaystyle\sum_{j=1}^{k} r_j \left(f(S^j) - f(S^{j-1}) \right)$

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Proof. Consider the linear relaxation and its dual:

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- We prove that the following x and y are optimal for the primal and dual, respectively.

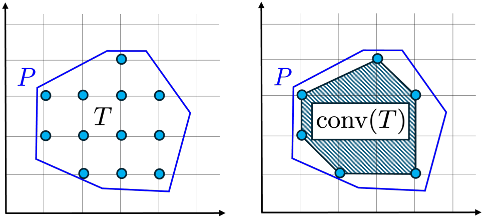
$$x_j = \begin{cases} f(S^j) - f(S^{j-1}), & 1 \leq j \leq k, \\ 0, & j > k. \end{cases} \qquad y_S = \begin{cases} r_j - r_{j+1}, & S = S^j, \quad 1 \leq j < k, \\ r_k, & S = S^k, \\ 0, & \text{otherwise.} \end{cases}$$

- To show y is dual feasible, note that $y_S \geq 0$ and:

$$\sum_{S: j \in S} y_S = y_{S^j} + \ldots + y_{S^k} = r_j, \text{ if } j \leq k \quad \text{and} \quad \sum_{S: j \in S} y_S = 0 \geq r_j, \text{ if } j > k.$$

- The primal objective: $\sum_{j=1}^{k} r_j \left(f(S^j) - f(S^{j-1}) \right)$
- The dual objective $\sum_{j=1}^{k-1} (r_j - r_{j+1}) f(S^j) + r_k f(S^k) = \sum_{j=1}^{k} r_j \left(f(S^j) - f(S^{j-1}) \right)$.

Improving LP Relaxations With Cuts



- **Recall:** T are feasible points to an IP, $\mathrm{conv}\,(T)$ is their convex hull
- P is the feasible region of an LP relaxation to the IP
- Typically, the formulation is **not ideal**:

$$\mathrm{conv}\,(T) \subset P$$

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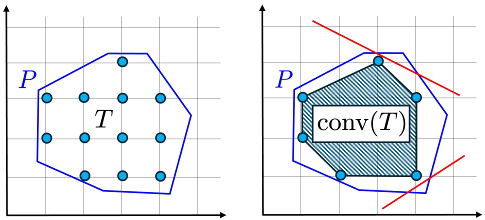

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- Typically, the formulation is **not ideal**:

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- How to **improve it by generating valid cuts**?
  - Linear inequalities **satisfied by** T **and** $\operatorname{conv}(T)$**, but not by** P?

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- **Setup:** A, b, c with rational entries and the IP:

$$\text{minimize}\left\{ c^\mathsf{T} x \ : \ Ax = b, \ x \geq 0, \ x \in \mathbb{Z}^n \right\}$$

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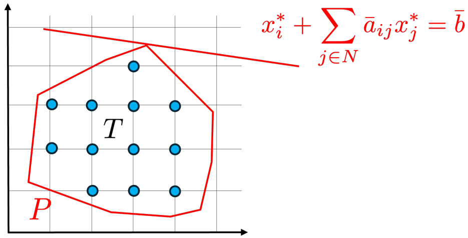

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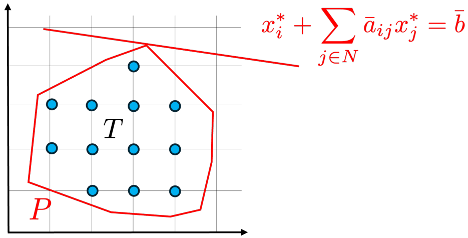

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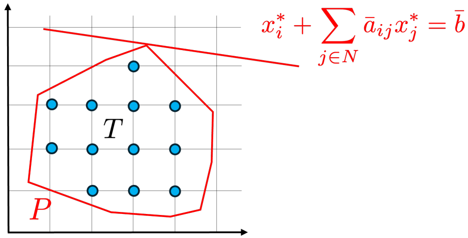

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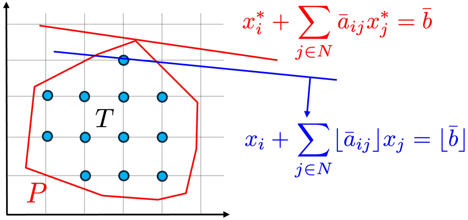

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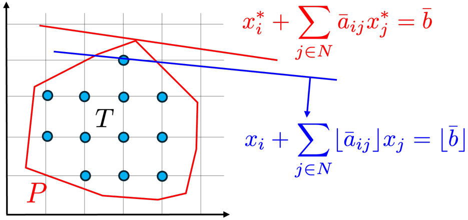
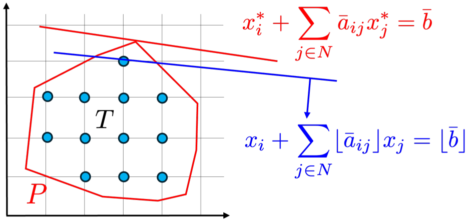


- This inequality is **satisfied by all integer solutions** $x \in T$
- It is **not satisfied by** x^* because $x_i^* = \bar{b}$ is fractional
- **Gomory cut**

Improving LP Relaxations With Cuts

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b} \rfloor, \ \forall x \in T$$



- **Gomory cut**
- Systematically adding all the Gomory cuts lead to first **cutting algorithm** for IP
  1. Solve the linear relaxation and get an optimal solution x^*
  2. If x^* is integer stop
  3. If not, add a cut (i.e., linear inequality that all integer solutions satisfy but that x^* does not satisfy) and go to step 1 again.
- Can show that this is guaranteed to terminate
- *Which simplex algorithm would you use in Step 1?*
- *If you're wondering how this works for $Ax \leq b$ or why it terminates, see notes!*

Lift-and-Project

- Balas, Céria and Cornuéjols introduced a new approach
- **Binary** IP, feasible set $x \in P \cap \{0,1\}^n$ where $P := \{x \in \mathbb{R}^n \ : \ Ax \geq b, \, x \geq 0\}$
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$$x_j(Ax - b) \geq 0, \quad (1 - x_j)(Ax - b) \geq 0.$$

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- **Claims.** (i) Every **binary** $x \in P$ satisfies $x \in P_j$. (ii) $P_j \subseteq P$.
- $\bigcap_{j=1}^n P_j$ is called the **lift-and-project closure**. Clearly, $\bigcap_{j=1}^n P_j \subseteq P$
- Bonami and Minoux : 35 Mixed 0-1 IPs from MIPLIB library, lift-and-project closure reduces integrality gap by **37% on average**

Other Cuts

- **Mixed-Integer Rounding (MIR) Cuts**: designed for general integer variables
- **Knapsack Cover Cuts**: applied for knapsack constraint

$$w \geq 0, w^\intercal x \leq K \;\; \Rightarrow$$

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$$w \geq 0, w^{\mathsf{T}} x \leq K \;\Rightarrow\; \sum_{i} x_i \leq |C| - 1 \text{ for any } C \;:\; \sum_{i \in C} w_i > K \quad \text{(Cover)}$$

- **Clique Cuts**: used to strengthen $\sum_{i=1}^{n} x_i \leq 1$ when some of the x_i form a **clique**
- **Flow Cover** and **Flow Path Cuts**: specialized cuts for network flow problems
- **Lattice-Free Cuts, Multi-Branch Split Cuts**
- **Comb Inequalities** for TSP
- Solvers like Gurobi have many of these built-in and allow adding custom cuts
- Adding “good” cuts is problem-dependent; requires good understanding of combinatorial structure

Solving IPs

IPs “hard,” but many methods devised

- **Exact algorithms**: guaranteed to find optimal solution, but may take exponential number of iterations
  - cutting planes
  - branch and bound
  - branch and cut
  - lift-and-project methods
  - dynamic programming methods
- **Approximation algorithms**: suboptimal solution with a bound on the degree of its suboptimality, in polynomial time
- **Heuristic algorithms**: suboptimal solution, typically no guarantees on its quality; typically run fast
  - local search methods
  - simulated annealing
  - ...

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Suppose we have **binary** variables **x, y, z** and **minimize an objective**
Maintain upper bound **U** and lower bound **L** on optimal value

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- At optimality, get: $x_F=0$, $y_F=0.3$, $z_F=1$
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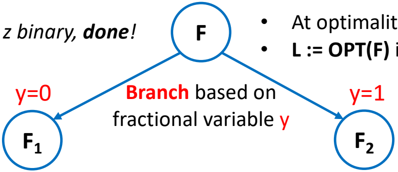

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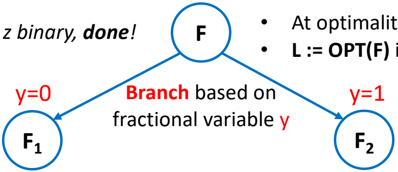

Branch based on fractional variable y

F₁: Solve with y=0, $0 \leq x,z \leq 1$
- Optimal: $x_{F1}=0.5$, $y_{F1}=0$, $z_{F1}=1$
- Optimal value **OPT(F_1)**

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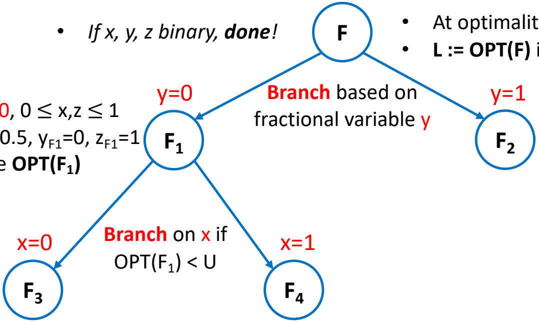

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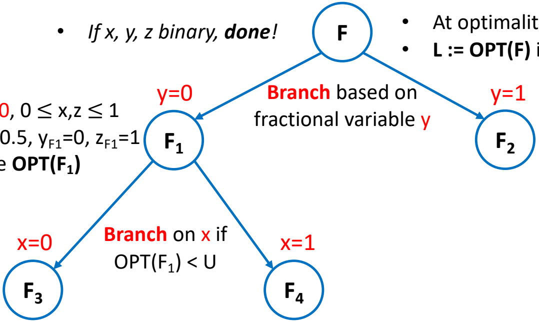

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- At optimality: get $z_{F3}=1$
- A **feasible** solution!
- Update **upper bound U := OPT(F_3)**
- If **U** – **L** $\leq$ tolerance, **stop**

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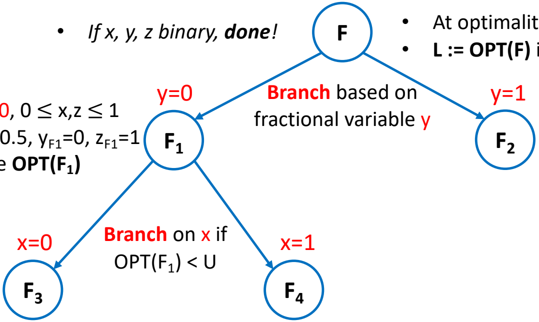

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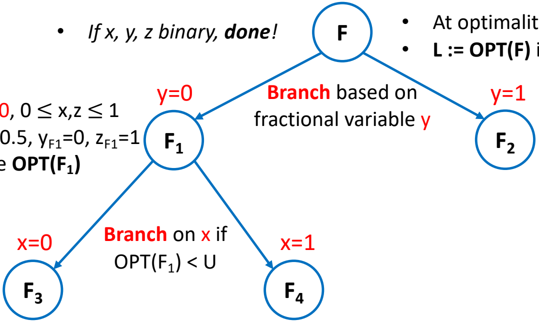

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F_2: Solve with y=1, 0 <= x,z <=1
- Optimal: $x_{F2}=0$, $y_{F2}=1$, $z_{F2}=0.2$
- Optimal value **OPT(F_2)**

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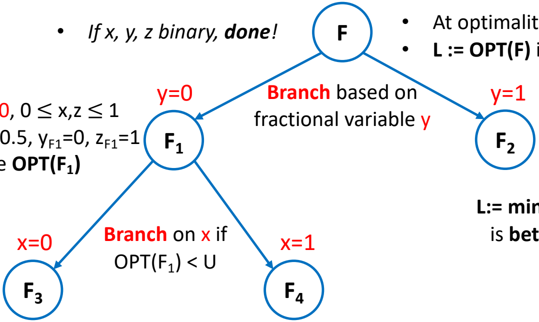

F_1: Solve with $y=0$, $0 \leq x,z \leq 1$
- Optimal: $x_{F1}=0.5$, $y_{F1}=0$, $z_{F1}=1$
- Optimal value **$OPT(F_1)$**

F_2: Solve with $y=1$, $0 <= x,z <=1$
- Optimal: $x_{F2}=0$, $y_{F2}=1$, $z_{F2}=0.2$
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$L := \min(OPT(F_1), OPT(F_2))$
is **better** lower bound!

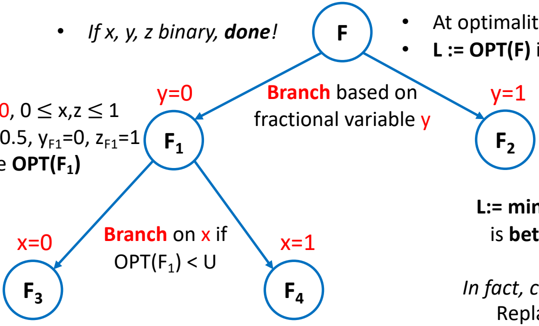
F_3: Solve with $x=y=0$, $0 \leq z \leq 1$
- At optimality: get $z_{F3}=1$
- A **feasible** solution!
- Update **upper bound U := $OPT(F_3)$**
- If **U – L** $\leq$ tolerance, **stop**

F_4: Infeasible!

Branch and Bound

Suppose we have **binary** variables **x, y, z** and **minimize an objective**
Maintain upper bound **U** and lower bound **L** on optimal value

Root node: solve LP relaxation
$0 \leq x, y, z \leq 1$



- *If x, y, z binary, **done**!*

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L:= min(OPT(F_1), OPT(F_2))
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In fact, could do even better!
Replace **OPT(F_1)** with
min(OPT(F_3), OPT(F_4)) = OPT(F_3)

F_3: Solve with x=y=0, $0 \leq z \leq 1$
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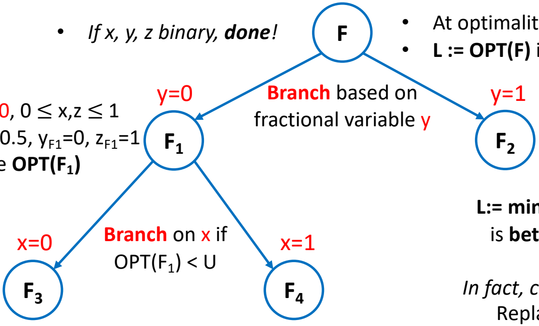

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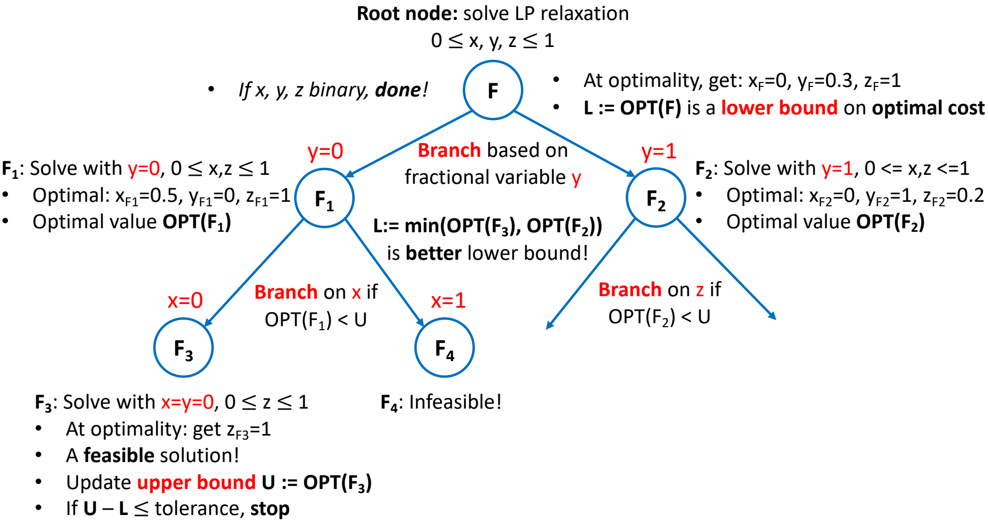
F_4: Infeasible!

Trade-off between exploring depth-first vs. breadth-first
- **depth-first** gets to a binary solution “faster”
- **breadth-first** allow improving lower bounds

Branch and Bound

Suppose we have **binary** variables **x, y, z** and **minimize an objective**
Maintain upper bound **U** and lower bound **L** on optimal value



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F

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Branch based on fractional variable y

y=0

F1

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- Optimal: $x_{F1}=0.5$, $y_{F1}=0$, $z_{F1}=1$
- Optimal value **OPT(F_1)**

y=1

F2

F_2: Solve with y=1, 0 <= x,z <=1

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L:= min(OPT(F_3), OPT(F_2)) is **better** lower bound!

Branch on x if OPT(F_1) < U

x=0

F3

x=1

F4

Branch on z if OPT(F_2) < U

...

...

F_3: Solve with x=y=0, $0 \leq z \leq 1$

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- A **feasible** solution!
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F_4: Infeasible!

Keep branching & bounding until you achieve desired tolerance (or get proof of optimality)

Branch and Bound

- More general formulation: let F be set of feasible solutions to an IP
  1. Maintain upper bound U, lower bound L on problem's objective
  2. Partition F into finite collection of subsets F_i
  3. Choose an unsolved subproblem and solve it; only need a **lower bound** $\ell(F_i)$ on cost:

$$\ell(F_i) \leq \min_{x \in F_i} c^\mathsf{T} x.$$

  4. If $\ell(F_i) \geq U$, no need to explore subproblem F_i further!
  5. Otherwise, partition F_i further and update collection of subproblems/nodes to explore
  6. If we get a feasible solution, update the upper bound U
  7. If $U - L \leq \epsilon$, stop
  8. When solving all children of a given node, can update lower bound at the node

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- Many **choices**:
  1. How to **explore subproblems**: "breadth-first search" vs "depth-first search" vs...
  2. How to **derive lower bound** $\ell(F_i)$: LP relaxation vs. Lagrangean duality
  3. Improve LP relaxations by **adding cuts**: **branch-and-cut** approaches
  4. How to **partition a problem** into subproblems? We used $x_i \leq \lfloor x_i^* \rfloor$ and $x_i \geq \lceil x_i^* \rceil$

Gurobi Output

```
Parameter OutputFlag unchanged
   Value: 1  Min: 0  Max: 1  Default: 1
Gurobi Optimizer version 9.1.2 build v9.1.2rc0 (linux64)
Thread count: 1 physical cores, 2 logical processors, using up to 2 threads
Optimize a model with 55 rows, 105 columns and 310 nonzeros
Model fingerprint: 0x0e3b21e3
Variable types: 5 continuous, 100 integer (100 binary)
Coefficient statistics:
  Matrix range     [5e-02, 1e+00]
  Objective range  [1e+00, 1e+00]
  Bounds range     [1e+00, 1e+00]
  RHS range        [1e+00, 4e+00]
Found heuristic solution: objective -0.0000000
Presolve removed 18 rows and 33 columns
Presolve time: 0.00s
Presolved: 37 rows, 72 columns, 192 nonzeros
Found heuristic solution: objective 1.0190799
Variable types: 0 continuous, 72 integer (68 binary)

Root relaxation: objective 3.139194e+00, 54 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    3.13919    0    7    1.01908    3.13919   208%     -    0s
H    0     0                       2.8417259    3.13919  10.5%     -    0s
H    0     0                       3.0648352    3.13919  2.43%     -    0s
H    0     0                       3.0879121    3.13919  1.66%     -    0s
     0     0    3.10586    0    8    3.08791    3.10586  0.58%     -    0s
     0     0     cutoff    0         3.08791    3.08791  0.00%     -    0s

Cutting planes:
  Gomory: 1
  MIR: 1
  GUB cover: 1
  RLT: 1

Explored 1 nodes (57 simplex iterations) in 0.04 seconds
Thread count was 2 (of 2 available processors)

Solution count 5: 3.08791 3.06484 2.84173 ... -0

Optimal solution found (tolerance 1.00e-04)
Best objective 3.087912087912e+00, best bound 3.087912087912e+00, gap 0.0000%

Solved the optimization problem...
```

Available computational resources

Summary of model
constraints, # variables, sparsity, coefficient values

Can we get close with a heuristic?

Can we simplify the problem?
(presolve)

Branch & Bound
(current node, bound on objective, gap)

Cutting planes applied

Optimal solution found

Lagrangian Duality in IP

- **Good lower bounds critical for MILPs!**

$$Z_{\mathrm{IP}} := \min \left\{ c^\top x \ : \ Ax \geq b, \ Dx \geq d, \ x \in Z^n \right\}$$

- We get a lower bound from LP relaxation:

$$Z_{\mathrm{LP}} := \min \left\{ c^\top x \ : \ Ax \geq b, \ Dx \geq d \right\} \quad \Rightarrow \quad Z_{\mathrm{LP}} \leq Z_{\mathrm{IP}}$$

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- Suppose the **"ugly/hard" constraints** are $Ax \geq b$ ...

  ... and we are able to **minimize efficiently** $c^\intercal x$ **over** $\mathcal{X} := \{x \in \mathbb{Z}^n \mid Dx \geq d\}$

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- Let $p \geq 0$ be dual variables (**Lagrange multipliers**) for $Ax \geq b$; form Lagrangean:

$$\mathcal{L}(x, p) := c^\top x + p^\top (b - Ax)$$

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- Then we can get the following lower bound on Z_{IP}:

$$\forall\, p \geq 0,\; g(p) := \min_{x \in \mathcal{X}} \left[c^\top x + p^\top (b - Ax) \right] \;\Rightarrow\; g(p) \leq Z_{\text{IP}}$$

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- **Important!** We are **not dualizing** all the constraints!
  - We keep the constraints $x \in \mathcal{X}$ because these are "easy"
  - Similar to LP developments: recall how we kept the constraints $x_i \geq 0$ or $x_i \leq 0$
  - What matters is that we can easily compute $g(p)$ for any $p \geq 0$

Lagrangian Duality in IP

- Because $g(p) \leq Z_{\text{IP}}, \forall p \geq 0$, we can look for **the best lower bound**:

$$Z_D := \max_{p \geq 0} g(p) \tag{2}$$

- This is the **Lagrangean dual** of our problem.
  - $g(p)$ **piece-wise linear, concave**; supergradient available
  - Can compute Z_D using first-order-methods
  - **Weak duality holds:** $Z_D \leq Z_{\text{IP}}$
  - Unlike LP, we do **not** have a strong duality result!

Lagrangian Duality in IP

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- Most important result here (recall that $\mathcal{X} := \{x \in \mathbb{Z}^n \mid Dx \geq d\}$)

$$Z_D = \min\left\{c^\top x \ : \ Ax \geq b, \quad x \in \operatorname{conv}(\mathcal{X})\right\}.$$

- Immediate consequence: we get **stronger bounds than from LP relaxation**,

$$Z_{\text{IP}} \leq Z_{\text{D}} \leq Z_{\text{IP}}.$$

- Details, proofs: see notes

Other Methods

- **Dynamic Programming** very powerful
- Can solve in pseudo-polynomial time IPs in **fixed dimension**
- Heuristics can also be powerful
  - Local search
  - Simmulated annealing
  - Genetic algorithms, "ant colony optimization", etc.