CME 307 / MS&E 311: Optimization

LP modeling and solution techniques

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Course survey

You're interested in

- duality
- modeling real-world problems
- hyperparameter and blackbox optimization
- fairness and ethics in optimization
- ...

Outline

Linear programming: standard form

standard form linear program (LP)

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\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b : \qquad \text{dual } y \\
& x \geq 0
\end{array}
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optimal value $p^\star$, solution $x^\star$ (if it exists)

- any x with $Ax = b$ and $x \geq 0$ is called a **feasible point**
- if problem is infeasible, we say $p^\star = \infty$
- $p^\star$ can be finite or $-\infty$

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what about $p^\star = \infty$?

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henceforth assume $A \in \mathbf{R}^{m \times n}$ has full row rank m
Q: why? how to check?

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Q: why? how to check?
A: otherwise infeasible or redundant rows; use gaussian elimination to check and remove

Linear algebra review

matrix $A \in \mathbf{R}^{m \times n}$

- span of A:

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  - null space of A, $\mathbf{nullspace}(A)$, is a hyperplane of dimension $n - m$
  - solution set is $\{x : Ax = b\} = \{x_0 + Vz\}$ where columns of $V \in \mathbf{R}^{n \times n-m}$ span $\mathbf{nullspace}(A)$

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if these are confusing: review linear algebra and prove them all!

LP example: diet problem

- x_i servings of food i
- c_i cost per serving
- a_{ij} amount of nutrient j in food i
- b_j required amount of nutrient j

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- foods come from recipes? $x = By$

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Geometry of LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

the **feasible set** is the set of points x that satisfy all constraints

- interpretation: add up columns of A so they match b
- $Ax = b$ defines a **hyperplane**
- $x_i \geq 0$ is a **halfspace**
- $x \geq 0$ is the **positive orthant**

Geometry of LP: convexity

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- define the **feasible set** $\{x : Ax = b,\ x \geq 0\}$

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$$\theta x + (1-\theta) y \in C, \qquad \theta \in [0,1]$$

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$$x \in C \quad \text{and} \quad x = \theta y + (1-\theta) z \quad \Longrightarrow \quad x = y = z$$

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- fact: if a solution exists, then some extreme point of the feasible set is optimal

Geometry of LP: polytopes

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- define **polytope** P: convex hull of its extreme points $v_1, \ldots, v_k \in \mathbf{R}^n$:

$$P = \{x \in \mathbf{R}^n \mid x = \sum_{i=1}^{k} \theta_i v_i,\ \theta_i \geq 0,\ \sum_{i=1}^{k} \theta_i = 1\}$$

- if feasible set is bounded, it is a polytope
- prove: if a solution exists, then some extreme point of the feasible set is optimal

Outline

Let's do some modeling!

- OptiMUS: https://optimus-solver.vercel.app/
- power systems: https://jump.dev/JuMP.jl/stable/tutorials/applications/power_systems/
- multicast routing: https://colab.research.google.com/drive/1iOn1T1Muh51KaA7mf7UIQOdhSFZhZyry?usp=sharing

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practical solvers for MILP:

- Gurobi and COPT (cardinal optimizer) are the state-of-the-art commercial solvers
- GLPK is a free solver that is not as fast
- JuliaOpt/JuMP is a modeling language in Julia that calls solvers like Gurobi and is specialized for MILP applications
- CVX* (including CVXPY in python) are modeling languages that call solvers like Gurobi with good support for convex problems
- OptiMUS is a LLM-based modeling tool for MILP

Modeling challenges

model the following as standard form LPs:

- **inequality constraints.** $Ax \leq b$
- **free variable.** $x \in \mathbf{R}$
- **absolute value.** constraint $|x| \leq 10$
- **piecewise linear.** objective $\max(x_1, x_2)$
- **assignment.** *e.g.*, every class is assigned exactly one classroom
- **logic.** *e.g.*, class enrollment $\leq$ capacity of assigned room
- **flow.** *e.g.*, the least cost way to ship an item from s to t

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LP standard form

Modeling

LP inequality form

Solving LPs

Duality

LP inequality form

another common form for LP is **inequality form**

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how to transform to standard form?

- inequality constraints $Ax \leq b$?

LP inequality form

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- free variable $x \in \mathbf{R}^n$?

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how to transform to standard form?

- inequality constraints $Ax \leq b$? slack variables $s \geq 0$
- free variable $x \in \mathbf{R}^n$? split into positive and negative parts

we will see later that these forms are also related by **duality**

LP example: production planning

- x_i units of product i
- c_i cost per unit
- a_{ij} amount of resource j used by product i
- b_j amount of resource j available
- d_i demand for product i
- u_i max production of product i
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extensions:

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  $c^T x + f^T z$, $z_i \in \{0, 1\}$, $x_i \leq M z_i$ for M large

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fact: vertex $\iff$ extreme point

Solution of LP is extreme point

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fact: if a solution exists and the feasible set has an extreme point, then some extreme point of the feasible set is optimal

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- unique: so $c^T x < c^T y$ for all $y \in P \setminus \{x\}$
- not unique: $\{X^\star : c^T x = c^T x^\star, x \in P\}$ is a polyhedron. It is not empty (a solution exists) and its complement is not empty (optimal value is bounded). So, it has at least one vertex. That vertex is also a vertex of P.

Basic feasible solution

define: $x \in \mathbf{R}^n$ is a **basic feasible solution** (BFS) if there is a set S of m linearly independent active constraints so that

$$x_S = A_S^{-1} b, \qquad x_{\bar{S}} = 0.$$

- $A_S \in \mathbf{R}^{m \times m}$, submatrix of A with columns in S, is invertible
- BFS $\iff$ extreme point
- two BFS with S, S' are neighbors if they share $m = 1$ constraints: $|S \cap S'| = m = 1$

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Q: how to find a BFS?
A: start at a feasible point; move in a **feasible direction** until you hit another constraint; continue until you reach a BFS

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Duality

Solving LPs

algorithms:

- enumerate all vertices and check
- fourier-motzkin elimination
- simplex method
- ellipsoid method
- interior point methods
- first-order methods
- ...

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remarks:

- enumeration and elimination are simple but not practical
- simplex was the first practical algorithm; still used today
- ellipsoid method is the first polynomial-time algorithm; not practical
- interior point methods are polynomial-time and practical
- first-order methods are practical and scale to large problems

Discuss: how to solve LPs?

write down a method to solve LPs; discuss in groups.

- idea
- math
- pseudocode

complete https://forms.gle/JbP2fLd6cRVbNUoW9 when you're ready (and before Friday noon)
(link also available from course schedule)

Enumerate vertices of LP

can generate all extreme points of LP: for each $S \subseteq \{1, \dots, n\}$ with $|S| = m$,

- $A_S \in \mathbf{R}^{m \times m}$, submatrix of A with columns in S, is invertible
- solve $A_S x_S = b$ for x_S and set $x_{\bar{S}} = 0$
- if $x_S \geq 0$, then x is a BFS
- evaluate objective $c^T x$

the best BFS is optimal!

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problem: how many BFSs are there?
n choose m is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ ("exponentially many")

Simplex algorithm

basic idea: local search on the vertices of the feasible set

- start at BFS x and evaluate objective $c^T x$
- move to a neighboring BFS x' with better objective $c^T x'$
- repeat until no improvement possible

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discuss in groups:

- how to find an initial BFS?
- how to find a neighboring BFS with better objective?
- how to prove optimality?

Finding an initial BFS

solve an auxiliary problem for which a BFS is known:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} z_i \\ \text{subject to} & Ax + Dz = b \\ & x, z \geq 0 \end{array}$$

where $D \in \mathbf{R}^{m \times m}$ is a diagonal matrix with $D_{ii} = \mathbf{sign}(b_i)$ for $i = 1, \ldots, m$.

- $x = 0$, $z = b$ is a BFS of this problem
- $(x, z) = (x, 0)$ is a BFS of this problem $\iff$ x is a BFS of the original problem

Find a better neighboring BFS

start with BFS x with active set S and turn on variable $j \notin S$

$$x^+ \leftarrow x + \theta d, \qquad \theta > 0$$

where $d_j = 1$ and $d_i = 0$ for $i \notin S \cup \{j\}$. need to solve for d_S.

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- construct the j**th basic direction**

$$Ad = A_S d_S + A_j = 0 \implies d_S = -A_S^{-1} A_j$$

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- if $x_S > 0$ is **non-degenerate**, then $\exists \theta > 0$ st $x^+ \geq 0$
- how does objective change?

$$c^T x^+ = c^T x + \theta c_j^T d = c^T x + c_j - \theta c_S^T A_S^{-1} A_j$$

Reduced cost

define **reduced cost** $\bar{c}_j = c_j - c_S^T A_S^{-1} A_j$, $j \notin S$

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fact:

- if $\bar{c} \geq 0$, x is optimal
- if x is optimal and nondegenerate ($x_S > 0$), then $\bar{c} \geq 0$

Outline

LP standard form

Modeling

LP inequality form

Solving LPs

Duality

Why duality?

- certify optimality
  - turn $\forall$ into $\exists$
  - use dual lower bound to derive stopping conditions
- new algorithms based on the dual
  - solve dual, then recover primal solution

Warmup: Farkas lemma

Theorem (Farkas lemma)

Given $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, exactly one of the following is true:

- *there exists $x \in \mathbf{R}^n$ so that $Ax = b$ and $x \geq 0$*
- *there exists $y \in \mathbf{R}^m$ so that $A^T y \geq 0$ and $\langle b, y\rangle < 0$*

$\Longrightarrow$ can efficiently certify infeasibility of a linear program

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$\implies$ can efficiently certify infeasibility of a linear program

proof: suppose we have $x \in \mathbf{R}^n$ so that $Ax = b$ and $x \geq 0$. then for any $y \in \mathbf{R}^m$,

$$\begin{aligned} 0 &= \langle y, b - Ax \rangle = \langle y, b \rangle - \langle A^T y, x \rangle \\ \langle y, b \rangle &= \langle A^T y, x \rangle \end{aligned}$$

so if $A^T y \geq 0$, then use $x \geq 0$ to conclude $\langle y, b \rangle \geq 0$.

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(opposite direction is similar)

Lagrange duality

primal problem with solution $x^\star \in \mathbf{R}^n$, optimal value $p^\star$:

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & Ax = b : & \qquad \text{dual } y \\ & x \geq 0 & \end{array} \qquad (\mathcal{P})$$

if x is feasible, then $Ax = b$, so $\langle y, Ax - b \rangle = 0$ for $y \in \mathbf{R}^m$.

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$$\mathcal{L}(x, y) \;\; := \;\; c^T x - \langle y, Ax - b \rangle$$

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unbounded below unless $c - A^T y \geq 0$.

Lagrange duality, ctd

we have a lower bound on $p^\star$ for any y, and a useful one whenever $c + A^T y = 0$. maximize bound:

$$p^\star \geq \begin{array}{ll} \text{maximize} & \langle y, b \rangle \\ \text{subject to} & A^T y \leq c \\ \text{variable} & y \in \mathbf{R}^m \end{array}$$

define the **dual function**

$$g(y) = \begin{cases} \langle y, b \rangle & A^T y \leq c \\ -\infty & \textit{otherwise} \end{cases}$$

Lagrange duality

weak duality asserts that $p^\star \geq g(y)$ for all $y \in \mathbf{R}^m$.

$$\begin{aligned} p^\star &\geq g(y) \quad \forall y \in \mathbf{R}^m \\ &\geq \underbrace{\sup_y g(y)}_{\mathcal{D}} =: d^\star \end{aligned}$$

$p^\star \geq d^\star$ dual optimal value

Strong duality

Definition (Duality gap)

The **duality gap** for a primal-dual pair (x, y) is $c^T x - b^T y \geq 0$

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strong duality holds

- for feasible LPs
- (later) for convex problems under **constraint qualification** aka **Slater's condition**. feasible region has an **interior point** x so that all inequality constraints hold strictly

strong duality fails if either primal or dual problem is infeasible or unbounded

Strong duality for LPs

primal and dual LP in standard form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

claim: if primal LP has a bounded feasible solution $x^\star$, then strong duality holds
i.e., dual LP has a bounded feasible solution $y^\star$ and $p^\star = d^\star$

Proof of strong duality for LPs

consider the following system with variables $x' \in \mathbf{R}^n$, $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^\star \tau - 1, \quad (x', \tau) \geq 0$$

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claim: this system has no solution. pf by contradiction:

- if $\tau > 0$, then x'/τ is feasible for LP and $c^T x'/\tau < p^\star$
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use second system to show y/σ is dual feasible and optimal

Strong duality and complementary slackness

Definition (complementary slackness)

The primal-dual pair x and y are **complementary** if

$$\langle y, b - Ax \rangle = 0$$

They satisfy **strict complementary slackness** if $y_i(b_i - a_i^T x) = 0$ for $i = 1, \ldots, n$.

for conic problem, strong duality $\iff$ complementary slackness

$$\begin{aligned}
\langle y, s \rangle &= \langle y, b - Ax \rangle \\
&= \langle y, b \rangle - \langle A^* y, x \rangle \\
&= \langle y, b \rangle - \langle c, x \rangle
\end{aligned}$$