CME 307 / MS&E 311 / OIT 676: Optimization

LP geometry, modeling and solution techniques

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Management Science and Engineering
Stanford

September 29, 2025

Course survey

you're interested in:

- modeling real-world problems, from finance and economics to energy systems and trajectory planning
- robustness and modeling under uncertainty
- understanding core optimization concepts like duality
- ...

questions:

- what readings are required?
- what projects are allowed?
- Friday section?
- programming requirements?

Quiz Wednesday

- Take paper when you arrive
- Write your SUNetID[1] and your name.
- We will project the question.
- You will have 5 minutes to write your answer on the paper.
- When time is up, put your pen or pencil down and stop writing.

[1] Your SUNetID is your Stanford University Network ID, and consists of letters, **not** numbers. Your email is `yourSUNetID@stanford.edu`.

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since we are calibrating what is possible with a 5 minute quiz, we will grade on a **check** basis.

- **no check:** no serious attempt
- **check**: a serious attempt
- **check+**: correct or nearly correct

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Exit ticket

Please fill out the exit ticket (linked from course schedule spreadsheet) before you leave:

- Helps us improve the course

We will not grade this, but we will read it!

Course overview

the practice of optimization consists of:

- modeling (real-world problems $\rightarrow$ optimization problems)
- analysis (properties of optimization problems)
- algorithms (how to solve optimization problems)

all three are necessary for success in optimization!

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- last Monday: mostly modeling
- today: mostly analysis
- next Monday: algorithms and some more modeling

Outline

LP standard form

LP inequality form

What kinds of points can be optimal?

Solving LPs

Modeling

Linear programming: standard form

standard form linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

optimal value $p^\star$, solution $x^\star$ (if it exists)

- any x with $Ax = b$ and $x \geq 0$ is called a **feasible point**
- if problem is infeasible, we say $p^\star = \infty$
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Q: why? how to check?

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Q: why? how to check?
A: otherwise infeasible or redundant rows; use gaussian elimination to check and remove

LP example: diet problem

We an planning a backpacking trip, and want to minimize the total weight of the food packed subject to nutritional requirements. We have a list of essential nutrients and how much an active person needs of each. We also know the weight of each food, and how much of each nutrient is in each food.

- x_j servings of food j, $j = 1, \ldots, n$
- c_j weight per serving
- a_{ij} amount of nutrient i in food j
- b_i required amount of nutrient i, $i = 1, \ldots, m$

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- define the **cone** generated by $A = [A_1, \ldots A_n]$:

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- LP is feasible if hyperplane $\{x \mid Ax = b\}$ intersects the positive orthant

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some useful convex sets:

- a hyperplane is convex
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- the intersection of convex sets is convex
- the feasible set $\{x : Ax = b,\ x \geq 0\}$ is convex

Outline

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LP inequality form

What kinds of points can be optimal?

Solving LPs

Modeling

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another useful form for LP is **inequality form**

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interpretation: halfspaces

- $a_i^T x \leq b_i$ defines a **halfspace**
- $Ax \leq b$ defines a **polyhedron**: intersection of halfspaces
- LP is feasible if polyhedron $\{x \mid Ax \leq b\}$ is nonempty

LP example: production planning

- x_i units of product i
- c_i cost per unit
- a_{ij} amount of resource j used by product i
- b_j amount of resource j available
- d_i demand for product i

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & 0 \leq x \leq d \end{array}$$

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  $c^T x + f^T z$, $z_i \in \{0, 1\}$, $x_i \leq M z_i$ for M large

LP inequality form to standard form

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$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array} \qquad \rightarrow \qquad \begin{array}{ll} \text{minimize} & c^T (x_+ - x_-) \\ \text{subject to} & A(x_+ - x_-) + s = b \\ & s, x_+, x_- \geq 0 \end{array}$$

so both forms have the same expressive power, and feasible sets are polyhedra

Active constraints and variables

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active variables. for nonnegative variable $x \geq 0$, variable i is **active** if $x_i > 0$

example: active slack variables are dual to active constraints

$$\begin{aligned} Ax \leq b \quad &\Longleftrightarrow \quad Ax + s = b,\ s \geq 0 \\ a_i^T x = b_i \quad &\Longleftrightarrow \quad s_i = 0 \\ \text{constraint } i \text{ is active} \quad &\Longleftrightarrow \quad \text{slack variable } s_i \text{ is inactive} \end{aligned}$$

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Q: Does there always exist an extreme solution?

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we say this hyperplane **supports** S at x

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fact: x is a vertex of $S \implies x$ is an extreme point of S
proof: x is a vertex of S. suppose its defining vector is c and consider the optimization problem

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interpretation: $\{z : c^T z = c^T x\}$ is a hyperplane that intersects S only at x.
we say this hyperplane **supports** S at x

fact: x is a vertex of $S \implies x$ is an extreme point of S
proof: x is a vertex of S. suppose its defining vector is c and consider the optimization problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in S \end{array}$$

x is the unique optimum of this problem, so the proof of this statement follows from the previous proof.

Basic feasible solution

recall the standard form LP

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$$x_S = A_S^{-1} b, \qquad x_{\bar{S}} = 0, \qquad x \geq 0.$$

- $A_S = \{A_{S_1}, \ldots, A_{S_m}\} \in \mathbb{R}^{m \times m}$ is submatrix of A with columns in S
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Q: how to find a BFS?

A: choose m linearly independent columns of A and set $x = A_S^{-1} b$; check $x \geq 0$.

Extreme point $\iff$ vertex $\iff$ BFS

fact. consider the feasible set $F = \{x \mid Ax = b,\ x \geq 0\}$ in $\mathbb{R}^n$. the following are equivalent:

- x is an extreme point of F
- x is a vertex of F
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implications: since any polyhedron $Ax \leq b$ can be written as $Ax = b,\ x \geq 0$,

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we have already shown that vertex $\Longrightarrow$ extreme point. need to show

- extreme point $\Longrightarrow$ BFS
- BFS $\Longrightarrow$ vertex

Extreme point $\implies$ BFS

we will show the contrapositive: x is not a BFS $\implies$ x is not an extreme point

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there is no $S \subseteq [n]$ so that A_S is invertible, $\hat{x}_S = A_S^{-1}b$, and $\hat{x}_{\bar{S}} = 0$.

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consider $\hat{S} = \{i : x_i^\star > 0\}$, the active set of variables in $\hat{x}$.

- if $A_{\hat{S}}$ were full rank $|\hat{S}|$, either it would be invertible or we could complete $A_{\hat{S}}$ to an invertible A_S with $\hat{S} \subseteq S$, contradicting that $\hat{x}$ is not a BFS,
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extend this vector to $d \in \mathbb{R}^n$ by appending zeros, so $Ad = A_{\hat{S}} d_{\hat{S}} = 0$.
now for $\epsilon \leq \min_{i \in \hat{S}} \hat{x}_i / \max_{i \in \hat{S}} |d_i|$, define $x^+, x^- \in \mathbb{R}^n$ as

$$x^+ = \hat{x} + \epsilon d, \qquad x^- = \hat{x} - \epsilon d.$$

these are feasible:

- $x^+, x^- \geq 0$ by our choice of ϵ,
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so $\hat{x} = \frac{1}{2}x^+ + \frac{1}{2}x^-$ is not extreme in F.

BFS $\implies$ vertex

suppose $x^\star$ is a BFS of F with active set S and A_S invertible. define $c \in \mathbb{R}^n$ as

$$c_i = \begin{cases} 0 & \text{if } i \in S \\ 1 & \text{otherwise} \end{cases}$$

so $c^T x^\star = 0$.

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so $c^T x^\star = 0$.

- $x^\star$ is the only point in F supported on S, as $\mathrm{nullspace}(A_S) = 0$,
- so any other feasible point $x \in F$ has a positive objective value $c^T x > 0$.

hence $x^\star$ is a vertex of F with defining vector c.

Outline

Solving LPs

algorithms:

- enumerate all vertices and check
- fourier-motzkin elimination
- simplex method
- ellipsoid method
- interior point methods
- first-order methods
- ...

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remarks:

- enumeration and elimination are simple but not practical
- simplex was the first practical algorithm; still used today
- ellipsoid method is the first polynomial-time algorithm; not practical
- interior point methods are polynomial-time and practical
- first-order methods are practical and scale to large problems

Example of Fourier-Motzkin elimination

consider the system of inequalities

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ -x_1 + x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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elimination method also shows projection of a polyhedron is a (closed) polyhedron

Enumerate vertices of LP

can generate all extreme points of LP: for each $S \subseteq \{1, \ldots, n\}$ with $|S| = m$,

- $A_S \in \mathbb{R}^{m \times m}$, submatrix of A with columns in S, is invertible
- solve $A_S x_S = b$ for x_S and set $x_{\bar{S}} = 0$
- if $x_S \geq 0$, then x is a BFS
- evaluate objective $c^T x$

the best BFS is optimal!

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n choose m is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ ("exponentially many")

Simplex algorithm

basic idea: local search on the vertices of the feasible set

- start at BFS x and evaluate objective $c^T x$
- move to a neighboring BFS x' with better objective $c^T x'$
- repeat until no improvement possible

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discuss in groups:

- how to find an initial BFS?
- how to find a neighboring BFS with better objective?
- how to prove optimality?

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where $D \in \mathbb{R}^{m \times m}$ is a diagonal matrix with $D_{ii} = \textbf{sign}(b_i)$ for $i = 1, \ldots, m$.

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- $(x, z) = (x, 0)$ is a BFS of this problem $\iff$ x is a BFS of the original problem

Find a better neighboring BFS

start with BFS x with active set S, $x_S > 0$. (called a **non-degenerate** BFS.)
construct the j**th basic direction** d^j by turning on variable $j \notin S$

$$x^+ \leftarrow x + \theta d^j, \qquad \theta > 0$$

where $d_j^j = 1$ and $d_i^j = 0$ for $i \notin S \cup \{j\}$. need to solve for d_S^j.

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- how does objective change if we move to $x^+ = x + \theta d^j$?

$$c^T x^+ - c^T x = \theta c^T d^j = \theta c_j - \theta c_S^T A_S^{-1} A_j$$

Reduced cost

define **reduced cost** $\bar{c}_j = c_j - c_S^T A_S^{-1} A_j$, $j \notin S$

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we will show:

- if $\bar{c} \geq 0$, x is optimal
- if x is optimal and nondegenerate ($x_S > 0$), then $\bar{c} \geq 0$

(handling degenerate x is more complicated, we will skip it)

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three steps to the proof:

- every feasible direction at x is contained in $\mathbf{cone}(\{d_j \mid j \notin S\})$

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  feasible directions d must satisfy, for some $\theta \geq 0$,

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  by convexity (consider segment from x to $x' \in F$)

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$$\begin{aligned} p^\star = \min_{x' \in F} c^T x' \quad &\geq \quad \min_{\alpha \geq 0} c^T (x + \sum_{j \notin S} \alpha_j d_j) \\ &= \quad c^T x + \min \sum \alpha_j \bar{c}_j = c^T x \end{aligned}$$

Outline

LP standard form

LP inequality form

What kinds of points can be optimal?

Solving LPs

Modeling

LP solvers

practical solvers for MILP:

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- gurobipy is a python interface to Gurobi
- CVX* (including CVXPY in python) are modeling languages that call solvers with good support for convex problems

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  - power systems https://jump.dev/JuMP.jl/stable/tutorials/applications/power_systems/

Oro Verde case + tutorial

https://github.com/stanford-cme-307/demos/tree/main/gurobipy

Modeling challenges

model the following as standard form LPs:

1. **inequality constraints.** $Ax \leq b$
2. **free variable.** $x \in \mathbb{R}$
3. **absolute value.** constraint $|x| \leq 10$
4. **piecewise linear.** objective $\max(x_1, x_2)$
5. **assignment.** *e.g.*, every class is assigned exactly one classroom
6. **logic.** *e.g.*, class enrollment $\leq$ capacity of assigned room
7. **(big-M).** $Ax \leq b$ if $x \geq 10$
8. **flow.** *e.g.*, the least cost way to ship an item from s to t

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(see chapter 1 of Bertsimas and Tsitsiklis for more details on 1–6. see https://github.com/stanford-cme-307/demos/blob/main/Mullticast_Routing_Demonstration.ipynb for a detailed treatment of a flow problem.)

Use slack variables to represent inequality constraints

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$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

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introduce slack variable $s \in \mathbb{R}^m$: $Ax + s = b$, $s \geq 0 \iff Ax \leq b$

$$\begin{array}{ll} \text{minimize} & c^T x + 0^T s \\ \text{subject to} & Ax + s = b \\ & x, s \geq 0 \end{array}$$

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introduce positive variables x_+, x_- so $x = x_+ - x_-$:

$$\begin{array}{ll} \text{minimize} & c^T x_+ - c^T x_- \\ \text{subject to} & Ax_+ - Ax_- = b \\ & x_+, x_- \geq 0 \end{array}$$

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$$\begin{array}{ll} \text{minimize} & \|x\|_1 = \sum_{i=1}^{n} |x_i| \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

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Q: Why does this work? For what kinds of functions can we use this trick?

Use binary variables to handle assignment

every class is assigned exactly one classroom:
define variable $X_{ij} \in \{0, 1\}$ for each class $i = 1, \ldots, n$ and room $j = 1, \ldots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

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now solve the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} \\ \text{subject to} & \sum_{j=1}^{m} X_{ij} = 1, \ \forall i \quad \text{(every class assigned one room)} \\ & \sum_{i=1}^{n} X_{ij} \leq 1, \ \forall j \text{(no more than one class per room)} \\ & X_{ij} \in \{0, 1\} \qquad\qquad\qquad\qquad \text{(binary variables)} \end{array}$$

where C_{ij} is the cost of assigning class i to room j.

Use binary variables to handle logic

model class enrollment $p_i \leq$ capacity c_j of assigned room:
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what if we want enrollment p to be a variable, too?

... or use a big-M relaxation!

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