CME 307 / MS&E 311: Optimization

LP modeling and solution techniques

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Course survey

You're interested in

- duality
- modeling real-world problems
- hyperparameter and blackbox optimization
- fairness and ethics in optimization
- ...

Outline

definitions

geometry

modeling

Duality

Linear programming: standard form

standard form linear program (LP)

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & Ax = b : & \quad \text{dual } y \\ & x \geq 0 & \end{array}$$

optimal value $p^\star$, solution $x^\star$ (if it exists)

- any x with $Ax = b$ and $x \geq 0$ is called a **feasible point**
- if problem is infeasible, we say $p^\star = \infty$
- $p^\star$ can be finite or $-\infty$

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what about $p^\star = \infty$?

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Q: why? how to check?

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henceforth assume $A \in \mathbf{R}^{m \times n}$ has full row rank m
Q: why? how to check?
A: otherwise infeasible or redundant rows; use gaussian elimination to check and remove

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matrix $A \in \mathbf{R}^{m \times n}$

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  - solution set is a hyperplane of dimension $n - m$
  - null space of A, $\mathbf{nullspace}(A)$, is a hyperplane of dimension

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  - null space of A, $\mathbf{nullspace}(A)$, is a hyperplane of dimension $n - m$
  - solution set is $\{x : Ax = b\} = \{x_0 + Vz\}$ where columns of $V \in \mathbf{R}^{n \times n-m}$ span $\mathbf{nullspace}(A)$

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if these are confusing: review linear algebra and prove them all!

LP example: diet problem

- x_i servings of food i
- c_i cost per serving
- a_{ij} amount of nutrient j in food i
- b_j required amount of nutrient j

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extensions:

- foods come from recipes? $x = By$

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- ranges of nutrients?

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extensions:

- foods come from recipes?
- ensure diversity in diet? $y \leq u$
- ranges of nutrients? $l \leq y \leq u$

Outline

definitions

geometry

modeling

Duality

Geometry of LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

the **feasible set** is the set of points x that satisfy all constraints

- interpretation: add up columns of A so they match b
- $Ax = b$ defines a **hyperplane**
- $x_i \geq 0$ is a **halfspace**
- $x \geq 0$ is the **positive orthant**

Geometry of LP: convexity

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- define the **feasible set** $\{x : Ax = b,\ x \geq 0\}$
- define **convex set**: C is convex if for any $x, y \in C$,

$$\theta x + (1-\theta) y \in C, \qquad \theta \in [0,1]$$

- prove: the feasible set is convex
- define **extreme point**: x is an extreme point of C if it cannot be written as a linear combination of other points in C:

$$x \in C \quad \text{and} \quad x = \theta y + (1-\theta) z \quad \Longrightarrow \quad x = y = z$$

Geometry of LP: polytopes

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- define **polytope** P: convex hull of finite set of points $v_1, \ldots, v_k \in \mathbf{R}^n$:

$$P = \{x \in \mathbf{R}^n \mid x = \sum_{i=1}^{k} \theta_i v_i,\ \theta_i \geq 0,\ \sum_{i=1}^{k} \theta_i = 1\}$$

- if feasible set is bounded, it is a polytope
- prove: if a solution exists, then some extreme point of the feasible set is optimal

Solving LPs

algorithms:

- enumerate all vertices and check
- fourier-motzkin elimination
- simplex method
- ellipsoid method
- interior point methods
- first-order methods
- ...

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remarks:

- enumeration and elimination are simple but not practical
- simplex was the first practical algorithm; still used today
- ellipsoid method is the first polynomial-time algorithm; not practical
- interior point methods are polynomial-time and practical
- first-order methods are practical and scale to large problems

Discuss: how to solve LPs?

write down a method to solve LPs; discuss in groups

Enumerate vertices of LP

can generate all extreme points of LP: for each $S \subseteq \{1, \ldots, n\}$ with $|S| = m$,

- $A_S \in \mathbf{R}^{m \times m}$, submatrix of A with columns in S, is invertible
- solve $A_S x_S = b$ for x_S and set $x_{\bar{S}} = 0$
- if $x_S \geq 0$, then x is a feasible extreme point (a **basic feasible solution** BFS)
- evaluate objective $c^T x$

the best BFS is optimal!

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the best BFS is optimal!

problem: how many BFSs are there?
n choose m is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ ("exponentially many")

Simplex algorithm

basic idea: local search on the vertices of the feasible set

- start at BFS x and evaluate objective $c^T x$
- move to a neighboring BFS x' with better objective $c^T x'$
- repeat until no improvement possible

later:

- how to find an initial BFS?
- how to find a neighboring BFS with better objective?
- how to prove optimality?

LP inequality form

another common form for LP is **inequality form**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

how to transform to standard form?

- inequality constraints $Ax \leq b$?

LP inequality form

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- free variable $x \in \mathbf{R}^n$?

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how to transform to standard form?

- inequality constraints $Ax \leq b$? slack variables $s \geq 0$
- free variable $x \in \mathbf{R}^n$? split into positive and negative parts

we will see later that these forms are also related by **duality**

LP example: production planning

- x_i units of product i
- c_i cost per unit
- a_{ij} amount of resource j used by product i
- b_j amount of resource j available
- d_i demand for product i
- u_i upper bound on production of product i
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extensions:

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extensions:

- fixed cost for product i? $c^T x + f^T z$, $z_i \in \{0, 1\}$, $x_i \leq M z_i$ for M large

Geometry of LP: inequality form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

- $Ax \leq b$ defines a **polyhedron**
- (a polytope is a bounded polyhedron)
- the set of constraints that hold with equality is the **active set**

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Let's do some modeling!

- OptiMUS: https://optimus-solver.vercel.app/
- power systems: https://jump.dev/JuMP.jl/stable/tutorials/applications/power_systems/
- multicast routing: https://colab.research.google.com/drive/1iOn1T1Muh51KaA7mf7UIQOdhSFZhZyry?usp=sharing

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practical solvers for MILP:

- Gurobi and COPT (cardinal optimizer) are the state-of-the-art commercial solvers
- GLPK is a free solver that is not as fast
- JuliaOpt/JuMP is a modeling language in Julia that calls solvers like Gurobi and is specialized for MILP applications
- CVX* (including CVXPY in python) are modeling languages that call solvers like Gurobi with good support for convex problems
- OptiMUS is a LLM-based modeling tool for MILP

Modeling challenges

- $|x|$
- $\max(x, y)$
- assignment constraints: *e.g.*,
  every class is assigned exactly one classroom
- flow constraints: *e.g.*,
  find the least cost way to ship an item from s to t
- logical constraints: *e.g.*,
  class enrollment must be less than the capacity of its assigned room

Closing announcements

- Fill out exit survey by Friday (linked from schedule)

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Why duality?

- certify optimality
  - turn $\forall$ into $\exists$
  - use dual lower bound to derive stopping conditions
- new algorithms based on the dual
  - solve dual, then recover primal solution

Warmup: Farkas lemma

Theorem (Farkas lemma)

Given $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$, exactly one of the following is true:

- *there exists $x \in \mathbf{R}^n$ so that $Ax = b$ and $x \geq 0$*
- *there exists $y \in \mathbf{R}^m$ so that $A^T y \geq 0$ and $\langle b, y \rangle < 0$*

$\Longrightarrow$ can efficiently certify infeasibility of a linear program

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proof: suppose we have $x \in \mathbf{R}^n$ so that $Ax = b$ and $x \geq 0$. then for any $y \in \mathbf{R}^m$,

$$
\begin{aligned}
0 &= \langle y, b - Ax \rangle = \langle y, b \rangle - \langle A^T y, x \rangle \\
\langle y, b \rangle &= \langle A^T y, x \rangle
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so if $A^T y \geq 0$, then use $x \geq 0$ to conclude $\langle y, b \rangle \geq 0$.

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(opposite direction is similar)

Lagrange duality

primal problem with solution $x^\star \in \mathbf{R}^n$, optimal value $p^\star$:

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & Ax = b : & \quad \text{dual } y \\ & x \geq 0 & \end{array} \qquad (\mathcal{P})$$

if x is feasible, then $Ax = b$, so $\langle y, Ax - b \rangle = 0$ for $y \in \mathbf{R}^m$.

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define the **Lagrangian**

$$\mathcal{L}(x, y) \quad := \quad c^T x - \langle y, b - Ax \rangle$$

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$$\begin{aligned} \mathcal{L}(x, y) &:= c^T x - \langle y, b - Ax \rangle \\ p^\star &= \inf_{x : Ax = b} \mathcal{L}(x, y) \geq \inf_x \mathcal{L}(x, y) \end{aligned}$$

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if x is feasible, then $Ax = b$, so $\langle y, Ax - b\rangle = 0$ for $y \in \mathbf{R}^m$.
define the **Lagrangian**

$$\begin{aligned} \mathcal{L}(x,y) &:= c^T x - \langle y, b - Ax\rangle \\ p^\star &= \inf_{x:Ax=b} \mathcal{L}(x,y) \geq \inf_x \mathcal{L}(x,y) \\ &= \inf_x c^T x + \langle y, -b + Ax\rangle \\ &= \langle y, -b\rangle + \inf_x \left(c^T x + \langle A^T y, x\rangle\right) \\ &= \langle y, -b\rangle + \inf_x \left(\langle c + A^T y, x\rangle\right) \end{aligned}$$

unbounded below unless $c + A^T y = 0$. true for any y, so
maximize bound:

Lagrange duality

inequality holds for any $y \in \mathbf{R}^m$, so we have proved **weak duality**

$$\begin{aligned} p^\star &\geq g(y) \quad \forall y \in \mathbf{R}^m \\ &\geq \underbrace{\sup_y g(y)}_{\mathcal{D}} =: d^\star \end{aligned} \tag{1}$$

dual optimal value $d^\star \leq p^\star$

Strong duality

Definition (Duality gap)

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strong duality holds

- for feasible LPs (pf later)
- for convex problems under **constraint qualification** aka **Slater's condition**. feasible region has an **interior point** x so that all inequality constraints hold strictly

strong duality fails if either primal or dual problem is infeasible or unbounded

Strong duality for LPs

primal and dual LP in standard form: (derive!)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

claim: if primal LP has a bounded feasible solution $x^\star$, then strong duality holds
i.e., dual LP has a bounded feasible solution $y^\star$ and $p^\star = d^\star$

Proof of strong duality for LPs

consider the following system with variables $x' \in \mathbf{R}^n$, $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^\star \tau - 1, \quad (x', \tau) \geq 0$$

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$$Ax + b = 0, \ x \geq 0 \qquad \text{or} \qquad A^T y \geq 0, \quad b^T y < 0$$

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use second system to show y/σ is dual feasible and optimal

Strong duality and complementary slackness

Definition (complementary slackness)

The primal-dual pair x and y are **complementary** if

$$\langle y, b - Ax \rangle = 0$$

They satisfy **strict complementary slackness** if $y_i(b_i - a_i^T x) = 0$ for $i = 1, \dots, n$.

for conic problem, strong duality $\iff$ complementary slackness

$$\begin{aligned}
\langle y, s \rangle &= \langle y, b - Ax \rangle \\
&= \langle y, b \rangle - \langle A^* y, x \rangle \\
&= \langle y, b \rangle - \langle c, x \rangle
\end{aligned}$$