Lecture 8 : Duality in Convex Optimization

October 15, 2025

Today's Agenda: Convex Duality

Primal Problem

$$
\begin{aligned}
(\mathcal{P}) \ \text{minimize}_x \quad & f_0(x) \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& x \in X.
\end{aligned} \tag{1}
$$

- Convex domain $X \subseteq \mathbb{R}^n$
- Every function $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$ (real-valued), **convex**
- Equality constraints $Ax = b$ can be included in X

Today's Agenda: Convex Duality

Primal Problem

$$(\mathcal{P})\ \text{minimize}_x \quad \begin{aligned} & f_0(x) \\ & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & x \in X. \end{aligned} \tag{1}$$

- Convex domain $X \subseteq \mathbb{R}^n$
- Every function $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$ (real-valued), **convex**
- Equality constraints $Ax = b$ can be included in X

- Many developments deal with the "interior" of X

Definition : Interior

The **interior** of a set X is the set of all points $x \in X$ so that:

$$\exists\, r > 0 \ : \ B(x, r) := \{y : \|y - x\| \le r\} \subseteq X$$

Today's Agenda: Convex Duality

Primal Problem

$$
\begin{aligned}
(\mathcal{P})\ \text{minimize}_x \quad & f_0(x) \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& x \in X.
\end{aligned}
\tag{1}
$$

- Convex domain $X \subseteq \mathbb{R}^n$
- Every function $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$ (real-valued), **convex**
- Equality constraints $Ax = b$ can be included in X

- Many developments deal with the "interior" of X

Definition : Interior

The **interior** of a set X is the set of all points $x \in X$ so that:

$$\exists\, r > 0 \ :\ B(x, r) := \{y : \|y - x\| \leq r\} \subseteq X$$

What is the interior of a set X that is not full-dimensional?

Relative Interior

- **Recall**: **Affine hull** of X is $\mathrm{aff}(X) := \left\{\theta_1 x_1 + \cdots + \theta_k x_k \;:\; x_i \in X,\, \sum_{i=1}^k \theta_i = 1\right\}$

Relative Interior

- **Recall**: **Affine hull** of X is $\mathrm{aff}(X) := \{\theta_1 x_1 + \cdots + \theta_k x_k \ : \ x_i \in X, \sum_{i=1}^k \theta_i = 1\}$

Definition Relative Interior

The **relative interior** of a set X is:

$$\operatorname{rel\,int}(X) := \left\{x \in X \ : \ \exists\, r > 0 \text{ so that } B(x, r) \cap \mathrm{aff}(X) \subseteq X\right\}. \tag{2}$$

What is the relative interior of the following sets?

- $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \in [0, 1]^2\}$
- $\{(x, y) \in \mathbb{R}^2 \mid x + y = 1, x \geq 0, y \geq 0\}$
- $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Convex Duality

Primal Problem

$$
\begin{aligned}
(\mathcal{P}) \text{ minimize}_x \quad & f_0(x) \\
& f_i(x) \leq 0, \quad i = 1, \dots, m \\
& x \in X.
\end{aligned}
$$

- Convex domain $X \subseteq \mathbb{R}^n$
- Every function $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$ (real-valued), **convex**
- Equality constraints $Ax = b$ can be included in X

- Assume $\mathrm{rel\,int}(X) \neq \emptyset$
- Assume that $(\mathcal{P})$ has an optimal solution $x^\star$, optimal value $p^\star = f_0(x^\star)$

- **Core questions:**
  1. For x feasible for $(\mathcal{P})$, how to **quantify the optimality gap** $f_0(x) - p^\star$?
  2. How to certify that $x^\star$ is **optimal** in $(\mathcal{P})$?

Convex Duality

Primal Problem

$$
\begin{aligned}
(\mathcal{P}) \ \text{minimize}_x \quad & f_0(x) \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& x \in X.
\end{aligned}
$$

- To construct **lower bounds** for $(\mathcal{P})$, define the **Lagrangian function**: for $\lambda \geq 0$,

$$
\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x)
$$

Convex Duality

Primal Problem

$$(\mathcal{P}) \text{ minimize}_x \quad \begin{array}{ll} f_0(x) & \\ f_i(x) \leq 0, & i = 1, \ldots, m \\ x \in X. & \end{array}$$

- To construct **lower bounds** for $(\mathcal{P})$, define the **Lagrangian function**: for $\lambda \geq 0$,

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x)$$

- By construction, $\mathcal{L}(x, \lambda) \leq f_0(x)$ for any x feasible in $(\mathcal{P})$

Convex Duality

Primal Problem

$$(\mathcal{P})\ \text{minimize}_x \quad \begin{aligned} & f_0(x) \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & x \in X. \end{aligned}$$

- To construct **lower bounds** for $(\mathcal{P})$, define the **Lagrangian function**: for $\lambda \geq 0$,

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x)$$

- By construction, $\mathcal{L}(x, \lambda) \leq f_0(x)$ for any x feasible in $(\mathcal{P})$
- For a lower bound on $p^\star$, minimize $\mathcal{L}(x, \lambda)$ over $x \in X$ to get:

$$g(\lambda) := \inf_{x \in X} \mathcal{L}(x, \lambda).$$

Dual Problem

$$(\mathcal{D}) \quad \sup_{\lambda \geq 0} g(\lambda).$$

Q: Is the dual $(\mathcal{D})$ a convex optimization problem?

Convex Duality

Primal Problem

$$(\mathcal{P}) \quad \text{minimize}_x \quad \begin{array}{ll} f_0(x) & \\ f_i(x) \leq 0, & i = 1, \ldots, m \\ x \in X. & \end{array}$$

- To construct **lower bounds** for $(\mathcal{P})$, define the **Lagrangian function**: for $\lambda \geq 0$,

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x)$$

- By construction, $\mathcal{L}(x, \lambda) \leq f_0(x)$ for any x feasible in $(\mathcal{P})$
- For a lower bound on $p^\star$, minimize $\mathcal{L}(x, \lambda)$ over $x \in X$ to get:

$$g(\lambda) := \inf_{x \in X} \mathcal{L}(x, \lambda).$$

Dual Problem

$$(\mathcal{D}) \quad \sup_{\lambda \geq 0} g(\lambda).$$

Q: Is the dual $(\mathcal{D})$ a convex optimization problem?

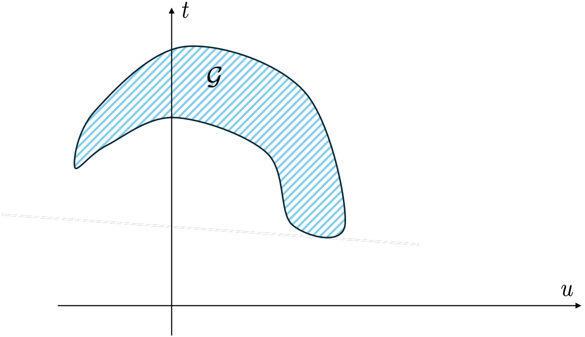
Geometric Interpretation

Primal-Dual Pair

$$(\mathcal{P})\ p^\star := \inf_{x\in X}\ f_0(x) \qquad\qquad (\mathcal{D})\quad d^\star := \sup_{\lambda\geq 0}\ g(\lambda)$$

$$f_i(x) \leq 0,\ \ i = 1,\ldots,m$$

- Suppose $X = \mathbb{R}^n$ and $(\mathcal{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u,t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



What do feasible points in $(\mathcal{P})$ correspond to? Where is $p^\star$?

How to express the Lagrangian $\mathcal{L}(x,\lambda)$ using the t, u variables?

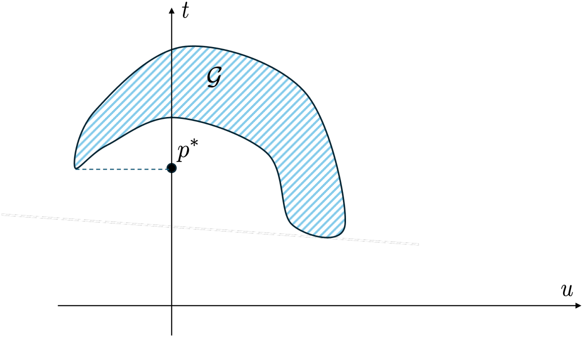
Geometric Interpretation

Primal-Dual Pair

$$(\mathcal{P})\ p^\star := \inf_{x\in X} f_0(x) \qquad\qquad (\mathcal{D})\quad d^\star := \sup_{\lambda\geq 0} g(\lambda)$$

$$f_i(x) \leq 0,\ i = 1,\ldots,m$$

- Suppose $X = \mathbb{R}^n$ and $(\mathcal{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u,t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



$\mathcal{L}(x, \lambda)$ is the same as $t + \lambda \cdot u$.

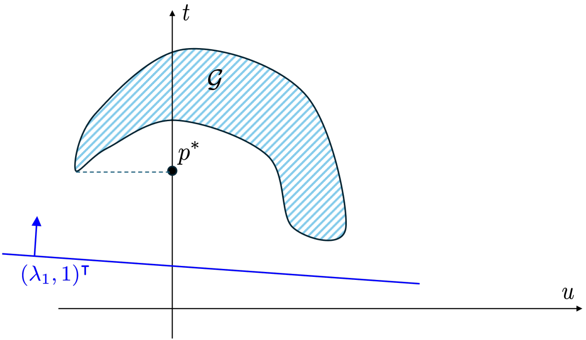
Geometric Interpretation

Primal-Dual Pair

$$(\mathcal{P})\ p^\star := \inf_{x\in X} f_0(x) \qquad\qquad (\mathcal{D})\quad d^\star := \sup_{\lambda\geq 0} g(\lambda)$$

$$f_i(x) \leq 0,\ \ i = 1, \ldots, m$$

- Suppose $X = \mathbb{R}^n$ and $(\mathcal{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u,t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



For $\lambda \geq 0$, we have $g(\lambda) = \inf_{x\in X}\big(f_0(x) + \lambda f_1(x)\big) = \inf_{(u,t)\in\mathcal{G}}(t + \lambda \cdot u)$

What is the value of $g(\lambda_1)$ in this figure?

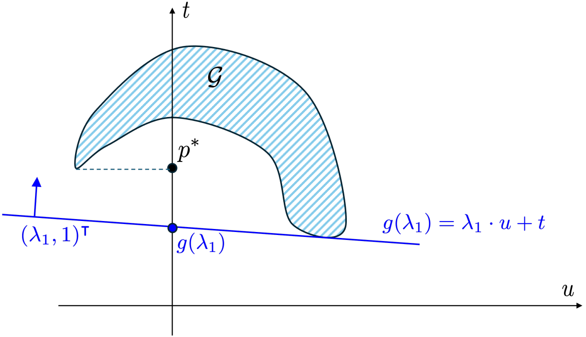
Geometric Interpretation

Primal-Dual Pair

$$(\mathcal{P})\ p^\star := \inf_{x\in X} f_0(x) \qquad\qquad (\mathcal{D})\quad d^\star := \sup_{\lambda\geq 0} g(\lambda)$$

$$f_i(x) \leq 0,\ i = 1,\dots,m$$

- Suppose $X = \mathbb{R}^n$ and $(\mathcal{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u,t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



The optimal pairs (u,t) yield a **supporting hyperplane** for $\mathcal{G}$

Intersection with $t = 0$ is value of $g(\lambda_1)$

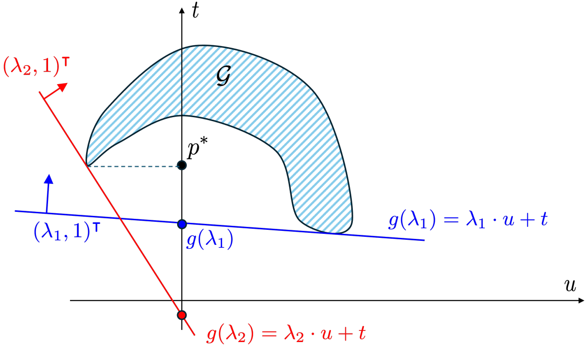
Geometric Interpretation

Primal-Dual Pair

$$(\mathcal{P})\ p^\star := \inf_{x\in X} f_0(x) \qquad\qquad (\mathcal{D})\quad d^\star := \sup_{\lambda\geq 0} g(\lambda)$$

$$f_i(x) \leq 0,\ i = 1, \ldots, m$$

- Suppose $X = \mathbb{R}^n$ and $(\mathcal{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u, t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



What is the value of $\max_{\lambda\geq 0} g(\lambda)$**?**

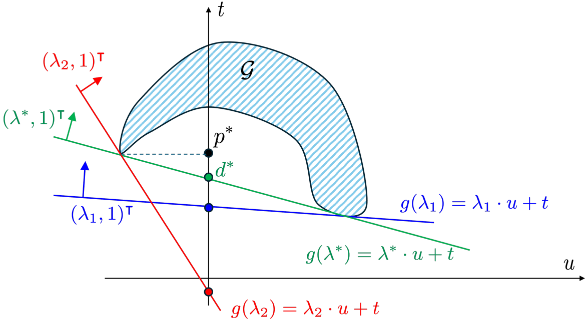
Geometric Interpretation

Primal-Dual Pair

$$(\mathcal{P})\ p^\star := \inf_{x \in X} f_0(x) \qquad\qquad (\mathcal{D}) \quad d^\star := \sup_{\lambda \geq 0} g(\lambda)$$

$$f_i(x) \leq 0,\ i = 1, \ldots, m$$

- Suppose $X = \mathbb{R}^n$ and $(\mathcal{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u, t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



Here, strong duality does not hold: $d^\star < p^\star$. But the set $\mathcal{G}$ is not convex!

Strong Duality in Convex Optimization?

Strong Duality in Convex Optimization?

Non-zero duality gap

Let $X = \{(x, y) \mid y \geq 1\}$ and consider the problem:

$$\underset{(x,y)\in X}{\text{minimize}} \; e^{-x}$$

$$x^2/y \leq 0.$$

- **Is this a convex optimization problem?**
- **What are $p^\star$, $\mathcal{L}$, g, $d^\star$?**
- **Does $p^\star = d^\star$ hold for any primal convex optimization problem if $p^\star$ finite?**

Conditions Leading to Strong Duality

Primal Problem

$$
\begin{aligned}
(\mathcal{P})\ \text{minimize}_x \quad & f_0(x) \\
& f_i(x) \le 0, \quad i = 1, \dots, m \\
& x \in X.
\end{aligned}
$$

Conditions Leading to Strong Duality

Primal Problem

$$
\begin{aligned}
(\mathcal{P})\ \text{minimize}_x \quad & f_0(x) \\
& f_i(x) \le 0, \quad i = 1, \dots, m \\
& x \in X.
\end{aligned}
$$

Slater Condition

The functions $f_1, \dots, f_m : X \subseteq \mathbb{R}^n \to \mathbb{R}$ satisfy **the Slater condition on** X if there exists $x \in \operatorname{rel\,int}(X)$ such that

$$f_j(x) < 0, \quad j = 1, \dots, m.$$

Conditions Leading to Strong Duality

Primal Problem

$$
\begin{aligned}
(\mathcal{P})\ \text{minimize}_x \quad & f_0(x) \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& x \in X.
\end{aligned}
$$

Slater Condition

The functions $f_1, \ldots, f_m : X \subseteq \mathbb{R}^n \to \mathbb{R}$ satisfy **the Slater condition on X** if there exists $x \in \operatorname{rel\,int}(X)$ such that

$$f_j(x) < 0, \quad j = 1, \ldots, m.$$

- A point x that is **strictly feasible**
- If **all** $f_i(x)$ are **affine**, we do not need this (i.e., feasibility is enough)
- If **some** f_i are affine, we only require $f_i(x) < 0$ for the **non-linear** f_i

Strong Duality in Convex Optimization

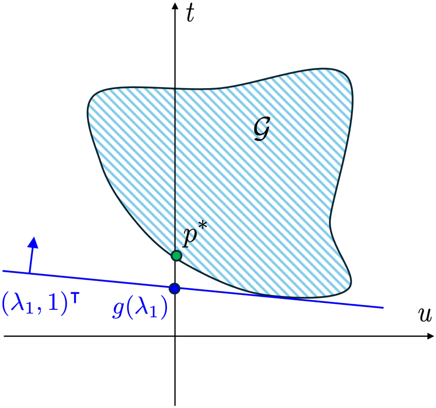
Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.



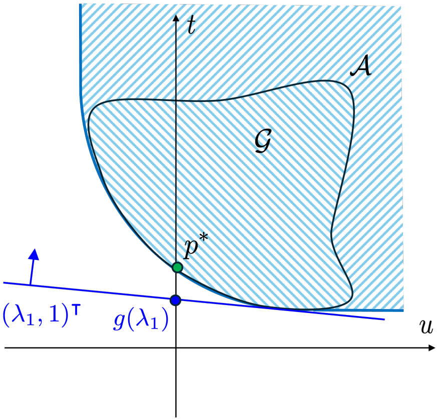
Geometric intuition for proof:

- Recall $\mathcal{G} := \{(u, t) \in \mathbb{R}^{m+1} : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$ (above, $m = 1$)

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

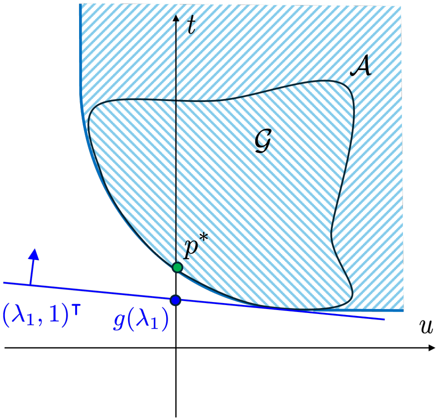


- Recall $\mathcal{G} := \{(u, t) \in \mathbb{R}^{m+1} : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$ (above, $m = 1$)
- Same $p^\star$ if we replace $\mathcal{G}$ with $\mathcal{A} = \{(u, t) \in \mathbb{R}^{m+1} : \exists x \in \mathbb{R}^n,\ t \geq f_0(x),\ u \geq f_1(x)\}$

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

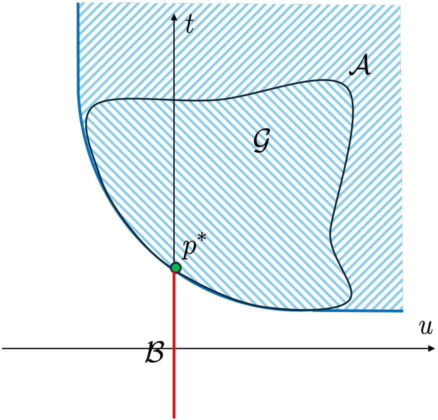


- Same $p^\star$ if we replace $\mathcal{G}$ with $\mathcal{A} = \{(u, t) \in \mathbb{R}^{m+1} : \exists x \in \mathbb{R}^n,\ t \geq f_0(x),\ u \geq f_1(x)\}$
- **Is $\mathcal{A}$ a convex set?**

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

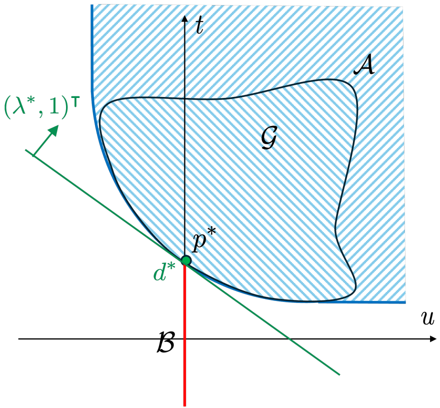


- Define $\mathcal{B} := \left\{(0, t) \in \mathbb{R}^m \times \mathbb{R} \mid t < p^\star\right\}$
- **Claim.** $\mathcal{A} \cap \mathcal{B} = \emptyset$

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

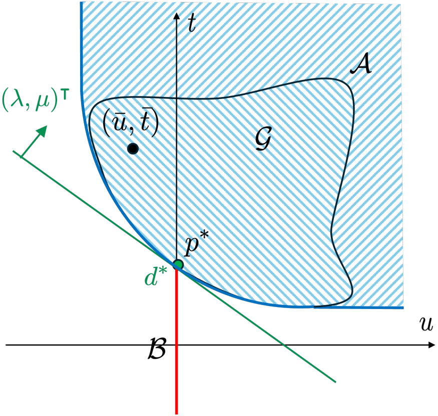


- The Separating Hyperplane Theorem will give us the optimal $\lambda^\star$ and $p^\star = d^\star$

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.



- The Slater point will guarantee that the hyperplane is not vertical

Strong Duality in Convex Optimization

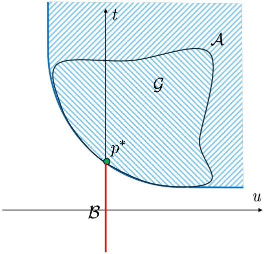
Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- Define the **convex** set

$$\mathcal{A} = \big\{(u,t) \in \mathbb{R}^m \times \mathbb{R} : \exists\, x \in X, \\ t \geq f_0(x), u_i \geq f_i(x),\ i = 1, \ldots, m\big\}.$$

- Define the **convex** set $\mathcal{B} = \{(0,t) \in \mathbb{R}^m \times \mathbb{R} \mid t < p^\star\}$.
- $\mathcal{A} \cap \mathcal{B} = \emptyset$.



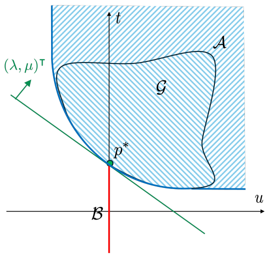
Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- Define the **convex** set

$$\mathcal{A} = \{(u,t) \in \mathbb{R}^m \times \mathbb{R} : \exists\, x \in X,\\ t \geq f_0(x), u_i \geq f_i(x),\ i = 1, \ldots, m\}.$$



- Define the **convex** set $\mathcal{B} = \{(0,t) \in \mathbb{R}^m \times \mathbb{R} \mid t < p^\star\}$.
- $\mathcal{A} \cap \mathcal{B} = \emptyset$.
- (Non-strict) Separating Hyperplane Theorem:

$$\exists\,(\lambda, \mu) \in \mathbb{R}^{m+1},\ \ b \in \mathbb{R} : \begin{cases} (1) & (\lambda, \mu) \neq 0, \\ (2) & \lambda^\intercal u + \mu t \geq b,\ \forall (u,t) \in A \\ (3) & \lambda^\intercal u + \mu t \leq b,\ \forall (u,t) \in B. \end{cases}$$

Strong Duality in Convex Optimization

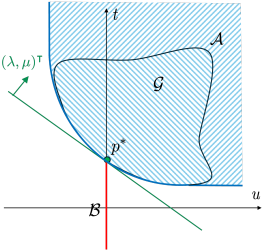
Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \dots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- Separating Hyperplane Theorem:

$$\exists\,(\lambda, \mu) \in \mathbb{R}^{m+1}, \;\; b \in \mathbb{R} : \begin{cases} (1) & (\lambda, \mu) \neq 0, \\ (2) & \lambda^\mathsf{T} u + \mu t \geq b, \forall\,(u, t) \in A \\ (3) & \lambda^\mathsf{T} u + \mu t \leq b, \forall\,(u, t) \in B. \end{cases}$$

- (2) implies $\lambda \geq 0$ and $\mu \geq 0$.
  Otherwise, $\inf_{(u,t)\in\mathcal{A}}(\lambda^\mathsf{T} u + \mu t) = -\infty$ so $\not\geq b$.



Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

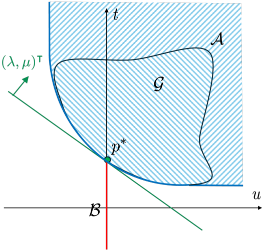
Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- Separating Hyperplane Theorem:

$$\exists\, (\lambda, \mu) \in \mathbb{R}^{m+1},\;\; b \in \mathbb{R} : \begin{cases} (1) & (\lambda, \mu) \neq 0, \\ (2) & \lambda^{\mathsf{T}} u + \mu t \geq b, \forall\, (u, t) \in A \\ (3) & \lambda^{\mathsf{T}} u + \mu t \leq b, \forall\, (u, t) \in B. \end{cases}$$

- (2) implies $\lambda \geq 0$ and $\mu \geq 0$.
  Otherwise, $\inf_{(u,t)\in\mathcal{A}}(\lambda^{\mathsf{T}} u + \mu t) = -\infty$ so $\not\geq b$.

- (3) simplifies to $\mu t \leq b$ for all $t < p^\star$, so $\mu p^\star \leq b$.



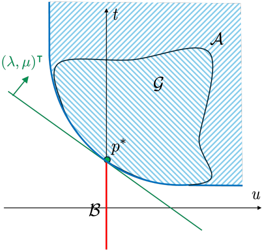
Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- Separating Hyperplane Theorem:

$$\exists\,(\lambda,\mu) \in \mathbb{R}^{m+1}, \quad b \in \mathbb{R} : \begin{cases} (1) & (\lambda,\mu) \neq 0, \\ (2) & \lambda^\intercal u + \mu t \geq b, \forall\,(u,t) \in A \\ (3) & \lambda^\intercal u + \mu t \leq b, \forall\,(u,t) \in B. \end{cases}$$



- (2) implies $\lambda \geq 0$ and $\mu \geq 0$.
  Otherwise, $\inf_{(u,t)\in\mathcal{A}}(\lambda^\intercal u + \mu t) = -\infty$ so $\not\geq b$.

- (3) simplifies to $\mu t \leq b$ for all $t < p^\star$, so $\mu p^\star \leq b$.

- Recap: We found $\lambda \geq 0, \mu \geq 0$:

$$\mathcal{L}(x,\lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star, \,\forall\, x \in X$$

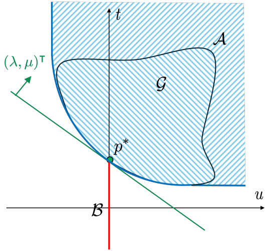
Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- We found $\lambda \geq 0, \mu \geq 0$:

$$(4)\ \mathcal{L}(x, \lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star,\ \forall\, x \in X$$



- **Case 1.** $\mu > 0$ (non-vertical hyper-plane)

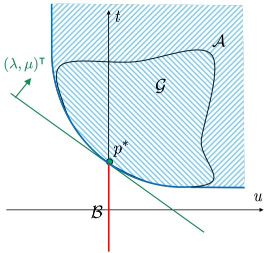
Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- We found $\lambda \geq 0, \mu \geq 0$:

$$(4)\ \mathcal{L}(x, \lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star,\ \forall\, x \in X$$



- **Case 1.** $\mu > 0$ (non-vertical hyper-plane)
- Divide (4) by μ to get: $\mathcal{L}(x, \lambda/\mu) \geq p^\star,\ \forall\, x \in X$.
- This implies $g(\lambda/\mu) := \inf_{x \in X} \mathcal{L}(x, \lambda/\mu) \geq p^\star$.

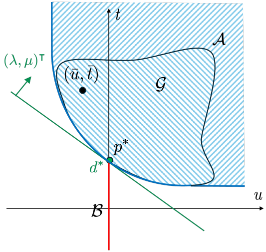
Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- We found $\lambda \geq 0, \mu \geq 0$:

$$(4)\ \mathcal{L}(x, \lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star,\ \forall\, x \in X$$



- **Case 1.** $\mu > 0$ (non-vertical hyper-plane)
- Divide (4) by μ to get: $\mathcal{L}(x, \lambda/\mu) \geq p^\star,\ \forall\, x \in X$.
- This implies $g(\lambda/\mu) := \inf_{x \in X} \mathcal{L}(x, \lambda/\mu) \geq p^\star$.

- Weak duality: $g(\lambda/\mu) \leq p^\star$, so $g(\lambda/\mu) = p^\star$.
- Strong duality holds: $p^\star = d^\star$.

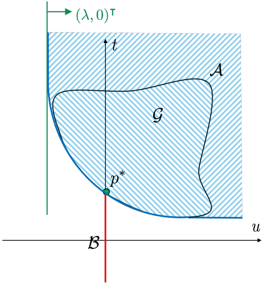
Strong Duality in Convex Optimization

Strong Duality in Convex Optimization

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- We found $\lambda \geq 0, \mu \geq 0$:

$$(4)\ \mathcal{L}(x, \lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star,\ \forall\, x \in X$$



- **Case 2.** $\mu = 0$ (vertical hyperplane)
- $\mu = 0$ so (4) implies $\sum_{i=1}^{m} \lambda_i f_i(x) \geq 0,\ \forall\, x \in X$

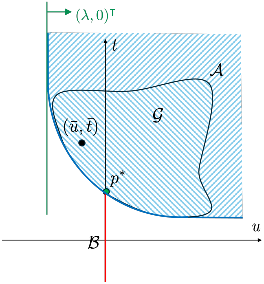
Strong Duality in Convex Optimization

Strong Duality in Convex Optimization

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- We found $\lambda \geq 0, \mu \geq 0$:

$$(4)\ \mathcal{L}(x, \lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star,\ \forall\, x \in X$$



- **Case 2.** $\mu = 0$ (vertical hyperplane)
- $\mu = 0$ so (4) implies $\sum_{i=1}^{m} \lambda_i f_i(x) \geq 0,\ \forall\, x \in X$
- $\bar{x}$ satisfies Slater condition $\Rightarrow f_i(\bar{x}) < 0$ for $i = 1, \ldots, m$
- This together with $\lambda \geq 0$ implies that $\lambda = 0$
- Contradicts that $(\lambda, \mu) \neq 0$.

Explicit Equality Constraints

- In applications, useful to make the **equality constraints explicit**:

$$\begin{aligned} \text{minimize}_{x\in X} \ & f_0(x) \\ \text{subject to } & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & Ax = b. \end{aligned}$$

  where $f_i, i = 0, \ldots, m$ are convex and $A \in \mathbb{R}^{p\times n}$ has rank p.

Explicit Equality Constraints

- In applications, useful to make the **equality constraints explicit**:

$$\begin{aligned}\text{minimize}_{x \in X} \ & f_0(x) \\ \text{subject to } & f_i(x) \leq 0, \quad i = 1, \dots, m, \\ & Ax = b.\end{aligned}$$

  where $f_i, i = 0, \dots, m$ are convex and $A \in \mathbb{R}^{p \times n}$ has rank p.

- With $\nu \in \mathbb{R}^p$ denoting Lagrange multipliers for $Ax = b$, Lagrangian is:

$$\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \nu^\mathsf{T}(Ax - b),$$

Explicit Equality Constraints

- In applications, useful to make the **equality constraints explicit**:

$$\begin{aligned} \text{minimize}_{x \in X} \; & f_0(x) \\ \text{subject to } & f_i(x) \le 0, \quad i = 1, \ldots, m, \\ & Ax = b. \end{aligned}$$

  where $f_i, i = 0, \ldots, m$ are convex and $A \in \mathbb{R}^{p \times n}$ has rank p.

- With $\nu \in \mathbb{R}^p$ denoting Lagrange multipliers for $Ax = b$, Lagrangian is:

$$\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \nu^{\mathsf{T}}(Ax - b),$$

- With $g(\lambda, \nu) := \inf_{x \in X} \mathcal{L}(x, \lambda, \nu)$, the dual problem becomes:

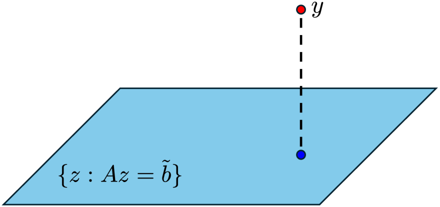
$$\begin{aligned} \text{maximize}_{\lambda, \nu} \; & g(\lambda, \nu) \\ \text{subject to } & \lambda \ge 0. \end{aligned}$$

  No sign constraints on ν!

Minimum Euclidean Distance Problem

- Given $y \in \mathbb{R}^n$ and affine set $\{z : Az = \tilde{b}\}$
- $A \in \mathbb{R}^{p \times n}$, $\tilde{b} \in \mathbb{R}^p$ has rank p

$$\min_z \left\{ \|z - y\|_2^2 \ : \ Az = \tilde{b} \right\}$$



- Change of variables $x := z - y$ and with $b := \tilde{b} - Ay$,

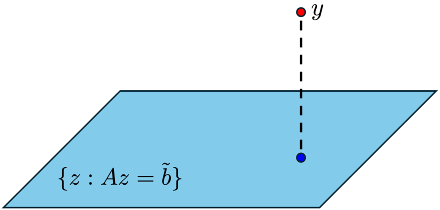
$$\min_x \left\{ \|x\|_2^2 \ : \ Ax = b \right\}$$

- **What is the optimal value $p^\star$?**

Minimum Euclidean Distance Problem

- Given $y \in \mathbb{R}^n$ and affine set $\{z : Az = \tilde{b}\}$
- $A \in \mathbb{R}^{p \times n}$ is full rank $p < n$. $\tilde{b} \in \mathbb{R}^p$.

$$\min_z \left\{ \|z - y\|_2^2 \ : \ Az = \tilde{b} \right\}$$



Quadratic Programs - Preliminaries

Unconstrained Quadratic Program

For $Q = Q^{\mathsf{T}}$, consider the following unconstrained problem:

$$\min\ f(x) := \frac{1}{2}x^{\mathsf{T}}Qx + q^{\mathsf{T}}x$$

- **What is the optimal value $p^{\star}$?**

Quadratic Programs - Preliminaries

Unconstrained Quadratic Program

For $Q = Q^\mathsf{T}$, consider the following unconstrained problem:

$$\min\ f(x) := \frac{1}{2}x^\mathsf{T}Qx + q^\mathsf{T}x$$

- **What is the optimal value $p^\star$?**

$$\nabla_x f(x) = 0 \quad \Leftrightarrow \quad Qx = -q$$

$$p^\star = \begin{cases} -\frac{1}{2}q^\mathsf{T}Q^\dagger q & \text{if } Q \succeq 0 \text{ and } q \in \mathcal{R}(Q) \\ -\infty & \text{otherwise.} \end{cases}$$

- $Q^\dagger$ is the (Moore-Penrose) pseudo-inverse of Q
- For A with singular value decomposition $A = U\Sigma V^\mathsf{T}$, $A^\dagger := V\Sigma^{-1}U^\mathsf{T}$
- Equals $(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ if $\text{rank}(A) = n$ and $A^\mathsf{T}(AA^\mathsf{T})^{-1}$ if $\text{rank}(A) = m$

QPs and QCQPs

Quadratic Programs

A **Quadratic Program (QP)** is an optimization problem of the form:

$$
\begin{aligned}
\min \ & \frac{1}{2}x^{\mathsf{T}}Qx + c^{\mathsf{T}}x \\
& A_1x = b_1 \\
& A_2x \leq b_2
\end{aligned}
$$

where $Q = Q^{\mathsf{T}}$.

QPs and QCQPs

Quadratic Programs

A **Quadratic Program (QP)** is an optimization problem of the form:

$$\begin{aligned} \min\ & \frac{1}{2}x^T Q x + c^T x \\ & A_1 x = b_1 \\ & A_2 x \leq b_2 \end{aligned}$$

where $Q = Q^T$.

Quadratically Constrained Quadratic Programs

A **Quadratically Constrained Quadratic Program (QCQP)** is a problem:

$$\begin{aligned} \min\ & \frac{1}{2}x^T Q_0 x + c^T x \\ & x^T Q_i x + q_i^T x + b_i \leq 0, i = 1, \dots, m \\ & Ax = b \end{aligned}$$

where $Q_i, i = 0, \dots, m$ are **symmetric** matrices.

Convex if $Q_0 \succeq 0, Q_i \succeq 0$. Gurobi can now handle **non-convex** QCQPs!

Two Problems to Warm Up

QP with Inequality Constraint

$$\underset{x}{\text{minimize}}\ \frac{1}{2}x^{\mathsf{T}}Qx + c^{\mathsf{T}}x$$
$$Ax \leq b$$

where $Q \succ 0$ is a **positive definite** matrix.

QCQP

$$\begin{aligned} \text{minimize}\ & \frac{1}{2}x^{\mathsf{T}}Q_0x + q_0^{\mathsf{T}}x + r_0 \\ \text{subject to}\ & \frac{1}{2}x^{\mathsf{T}}Q_ix + q_i^{\mathsf{T}}x + r_i \leq 0, \quad i = 1, \ldots, m, \end{aligned}$$

where $Q_0 \succ 0$ and $Q_i \succeq 0$

- **What is the Lagrangian? What is the dual? Does Slater Condition hold?**

A Non-Convex QCQP

A Special Non-Convex QCQP

For $A = A^\mathsf{T}$ and $A \not\succeq 0$, consider:

$$\begin{aligned} \text{minimize } & x^\mathsf{T} A x + 2b^\mathsf{T} x \\ & x^\mathsf{T} x \leq 1 \end{aligned}$$

- Lagrangian is:

$$\mathcal{L}(x, \lambda) = x^\mathsf{T} A x + 2b^\mathsf{T} x + \lambda(x^\mathsf{T} x - 1) = x^\mathsf{T}(A + \lambda I)x + 2b^\mathsf{T} x - \lambda,$$

$$g(\lambda) = \begin{cases} -b^\mathsf{T}(A + \lambda I)^\dagger b - \lambda & A + \lambda I \succeq 0,\ b \in \mathcal{R}(A + \lambda I), \\ -\infty & \text{otherwise,} \end{cases}$$

  where $M^\dagger$ is the (Moore-Penrose) pseudo-inverse of M

- The dual problem is

$$\begin{aligned} \text{maximize}_{\lambda \geq 0} \quad & -b^\mathsf{T}(A + \lambda I)^\dagger b - \lambda \\ \text{subject to} \quad & A + \lambda I \succeq 0,\ b \in \mathcal{R}(A + \lambda I) \end{aligned}$$

- Readily solved because it can be expressed as

$$\text{maximize}\left\{ -\sum_{i=1}^{n} \frac{(q_i^\mathsf{T} b)^2}{\lambda_i + \lambda} - \lambda \ :\ \lambda \geq -\lambda_{\min}(A) \right\}$$

  where λ_i, q_i are eigen-decomposition of A and $(q_i^\mathsf{T} b)^2/0 := 0$ if $q_i^\mathsf{T} b = 0$ and ∞ otherwise.

A **Non-Convex** QCQP

A Special Non-Convex QCQP

For $A = A^\mathsf{T}$ and $A \not\succeq 0$, consider:

$$\begin{aligned} \text{minimize } & x^\mathsf{T} A x + 2b^\mathsf{T} x \\ & x^\mathsf{T} x \leq 1 \end{aligned}$$

- Slater condition trivially satisfied!
- We actually have **zero duality gap**, $p^\star = d^\star$!
- A more general result: strong duality for any quadratic optimization problem with two constraints $\ell \leq x^\mathsf{T} Q x \leq u$ if Q and A are simultaneously diagonalizable