

CME 307 / MS&E 311: Optimization

LP modeling and solution techniques

Professor Udell

Management Science and Engineering  
Stanford

January 24, 2024

## Course survey

You're interested in

- ▶ duality
- ▶ modeling real-world problems
- ▶ hyperparameter and blackbox optimization
- ▶ fairness and ethics in optimization
- ▶ ...

# Outline

LP standard form

Modeling

LP inequality form

Solving LPs

Duality

Using duality

Large-scale linear programming

Integer programming

## Linear programming: standard form

standard form linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array}$$

optimal value  $p^*$ , solution  $x^*$  (if it exists)

- ▶ any  $x$  with  $Ax = b$  and  $x \geq 0$  is called a **feasible point**
- ▶ if problem is infeasible, we say  $p^* = \infty$
- ▶  $p^*$  can be finite or  $-\infty$

## Linear programming: standard form

standard form linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array}$$

optimal value  $p^*$ , solution  $x^*$  (if it exists)

- ▶ any  $x$  with  $Ax = b$  and  $x \geq 0$  is called a **feasible point**
- ▶ if problem is infeasible, we say  $p^* = \infty$
- ▶  $p^*$  can be finite or  $-\infty$

**Q:** if  $p^* = -\infty$ , does a solution exist? is it unique?  
what about  $p^* = \infty$ ?

## Linear programming: standard form

standard form linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array}$$

optimal value  $p^*$ , solution  $x^*$  (if it exists)

- ▶ any  $x$  with  $Ax = b$  and  $x \geq 0$  is called a **feasible point**
- ▶ if problem is infeasible, we say  $p^* = \infty$
- ▶  $p^*$  can be finite or  $-\infty$

**Q:** if  $p^* = -\infty$ , does a solution exist? is it unique?

what about  $p^* = \infty$ ?

henceforth assume  $A \in \mathbf{R}^{m \times n}$  has full row rank  $m$

**Q:** why? how to check?

## Linear programming: standard form

standard form linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array}$$

optimal value  $p^*$ , solution  $x^*$  (if it exists)

- ▶ any  $x$  with  $Ax = b$  and  $x \geq 0$  is called a **feasible point**
- ▶ if problem is infeasible, we say  $p^* = \infty$
- ▶  $p^*$  can be finite or  $-\infty$

**Q:** if  $p^* = -\infty$ , does a solution exist? is it unique?

what about  $p^* = \infty$ ?

henceforth assume  $A \in \mathbf{R}^{m \times n}$  has full row rank  $m$

**Q:** why? how to check?

**A:** otherwise infeasible or redundant rows; use gaussian elimination to check and remove

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

► span of  $A$ :



## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ?

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ? factor-solve with QR or SVD; form normal equations  $A^T Ax = A^T b$  and use CG; other Krylov methods like LSQR (positive definite), MINRES (indefinite), GMRES (general)

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ? factor-solve with QR or SVD; form normal equations  $A^T Ax = A^T b$  and use CG; other Krylov methods like LSQR (positive definite), MINRES (indefinite), GMRES (general)
- ▶ solution to  $Ax = b$  is unique if



## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ? factor-solve with QR or SVD; form normal equations  $A^T Ax = A^T b$  and use CG; other Krylov methods like LSQR (positive definite), MINRES (indefinite), GMRES (general)
- ▶ solution to  $Ax = b$  is unique if  $m = n$  and  $A$  is full rank

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ? factor-solve with QR or SVD; form normal equations  $A^T Ax = A^T b$  and use CG; other Krylov methods like LSQR (positive definite), MINRES (indefinite), GMRES (general)
- ▶ solution to  $Ax = b$  is unique if  $m = n$  and  $A$  is full rank
- ▶ if  $m < n$  and  $A$  is full rank
  - ▶ solution set is a hyperplane of dimension

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ? factor-solve with QR or SVD; form normal equations  $A^T Ax = A^T b$  and use CG; other Krylov methods like LSQR (positive definite), MINRES (indefinite), GMRES (general)
- ▶ solution to  $Ax = b$  is unique if  $m = n$  and  $A$  is full rank
- ▶ if  $m < n$  and  $A$  is full rank
  - ▶ solution set is a hyperplane of dimension  $n - m$
  - ▶ null space of  $A$ ,  $\text{nullspace}(A)$ , is a hyperplane of dimension

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ? factor-solve with QR or SVD; form normal equations  $A^T Ax = A^T b$  and use CG; other Krylov methods like LSQR (positive definite), MINRES (indefinite), GMRES (general)
- ▶ solution to  $Ax = b$  is unique if  $m = n$  and  $A$  is full rank
- ▶ if  $m < n$  and  $A$  is full rank
  - ▶ solution set is a hyperplane of dimension  $n - m$
  - ▶ null space of  $A$ ,  $\text{nullspace}(A)$ , is a hyperplane of dimension  $n - m$
  - ▶ solution set is  $\{x : Ax = b\} = \{x_0 + Vz\}$  where columns of  $V \in \mathbf{R}^{n \times n-m}$  span  $\text{nullspace}(A)$

## Linear algebra review

matrix  $A \in \mathbf{R}^{m \times n}$

- ▶ span of  $A$ :  $\text{span}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$
- ▶ nullspace of  $A$ :  $\text{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\} \subseteq \mathbf{R}^n$
- ▶ how to compute basis for span and nullspace of  $A$ ?  
can use QR factorization or SVD
- ▶ how to solve  $Ax = b$ ? factor-solve with QR or SVD; form normal equations  $A^T Ax = A^T b$  and use CG; other Krylov methods like LSQR (positive definite), MINRES (indefinite), GMRES (general)
- ▶ solution to  $Ax = b$  is unique if  $m = n$  and  $A$  is full rank
- ▶ if  $m < n$  and  $A$  is full rank
  - ▶ solution set is a hyperplane of dimension  $n - m$
  - ▶ null space of  $A$ ,  $\text{nullspace}(A)$ , is a hyperplane of dimension  $n - m$
  - ▶ solution set is  $\{x : Ax = b\} = \{x_0 + Vz\}$  where columns of  $V \in \mathbf{R}^{n \times n-m}$  span  $\text{nullspace}(A)$

if these are confusing: review linear algebra and prove them all!

## LP example: diet problem

- ▶  $x_i$  servings of food  $i$
- ▶  $c_i$  cost per serving
- ▶  $a_{ij}$  amount of nutrient  $j$  in food  $i$
- ▶  $b_j$  required amount of nutrient  $j$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

## LP example: diet problem

- ▶  $x_i$  servings of food  $i$
- ▶  $c_i$  cost per serving
- ▶  $a_{ij}$  amount of nutrient  $j$  in food  $i$
- ▶  $b_j$  required amount of nutrient  $j$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

extensions:

- ▶ foods come from recipes?  $x = By$

## LP example: diet problem

- ▶  $x_i$  servings of food  $i$
- ▶  $c_i$  cost per serving
- ▶  $a_{ij}$  amount of nutrient  $j$  in food  $i$
- ▶  $b_j$  required amount of nutrient  $j$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

extensions:

- ▶ foods come from recipes?
- ▶ ensure diversity in diet?



## LP example: diet problem

- ▶  $x_i$  servings of food  $i$
- ▶  $c_i$  cost per serving
- ▶  $a_{ij}$  amount of nutrient  $j$  in food  $i$
- ▶  $b_j$  required amount of nutrient  $j$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

extensions:

- ▶ foods come from recipes?
- ▶ ensure diversity in diet?  $y \leq u$

## LP example: diet problem

- ▶  $x_i$  servings of food  $i$
- ▶  $c_i$  cost per serving
- ▶  $a_{ij}$  amount of nutrient  $j$  in food  $i$
- ▶  $b_j$  required amount of nutrient  $j$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

extensions:

- ▶ foods come from recipes?
- ▶ ensure diversity in diet?  $y \leq u$
- ▶ ranges of nutrients?

## LP example: diet problem

- ▶  $x_i$  servings of food  $i$
- ▶  $c_i$  cost per serving
- ▶  $a_{ij}$  amount of nutrient  $j$  in food  $i$
- ▶  $b_j$  required amount of nutrient  $j$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

extensions:

- ▶ foods come from recipes?
- ▶ ensure diversity in diet?  $y \leq u$
- ▶ ranges of nutrients?  $l \leq y \leq u$

## Geometry of LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

the **feasible set** is the set of points  $x$  that satisfy all constraints

- ▶ interpretation: add up columns of  $A$  so they match  $b$
- ▶  $Ax = b$  defines a **hyperplane**
- ▶  $x_i \geq 0$  is a **halfspace**
- ▶  $x \geq 0$  is the **positive orthant**

## Geometry of LP: convexity

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

- ▶ define the **feasible set**  $\{x : Ax = b, x \geq 0\}$

## Geometry of LP: convexity

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

- ▶ define the **feasible set**  $\{x : Ax = b, x \geq 0\}$
- ▶ define **convex set**:  $C$  is convex if for any  $x, y \in C$ ,

$$\theta x + (1 - \theta)y \in C, \quad \theta \in [0, 1]$$

## Geometry of LP: convexity

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

- ▶ define the **feasible set**  $\{x : Ax = b, x \geq 0\}$
- ▶ define **convex set**:  $C$  is convex if for any  $x, y \in C$ ,

$$\theta x + (1 - \theta)y \in C, \quad \theta \in [0, 1]$$

- ▶ fact: the feasible set is convex

## Geometry of LP: convexity

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

- ▶ define the **feasible set**  $\{x : Ax = b, x \geq 0\}$
- ▶ define **convex set**:  $C$  is convex if for any  $x, y \in C$ ,

$$\theta x + (1 - \theta)y \in C, \quad \theta \in [0, 1]$$

- ▶ fact: the feasible set is convex
- ▶ define **extreme point**:  $x$  is extreme in  $C$  if it cannot be written as a linear combination of other points in  $C$ :

$$x \in C \quad \text{and} \quad x = \theta y + (1 - \theta)z \quad \implies \quad x = y = z$$



## Geometry of LP: convexity

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

- ▶ define the **feasible set**  $\{x : Ax = b, x \geq 0\}$
- ▶ define **convex set**:  $C$  is convex if for any  $x, y \in C$ ,

$$\theta x + (1 - \theta)y \in C, \quad \theta \in [0, 1]$$

- ▶ fact: the feasible set is convex
- ▶ define **extreme point**:  $x$  is extreme in  $C$  if it cannot be written as a linear combination of other points in  $C$ :

$$x \in C \quad \text{and} \quad x = \theta y + (1 - \theta)z \quad \implies \quad x = y = z$$

- ▶ fact: if a solution exists, then some extreme point of the feasible set is optimal

## Geometry of LP: polytopes

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

- ▶ define **polytope**  $P$ : convex hull of its extreme points  $v_1, \dots, v_k \in \mathbf{R}^n$ :

$$P = \{x \in \mathbf{R}^n \mid x = \sum_{i=1}^k \theta_i v_i, \theta_i \geq 0, \sum_{i=1}^k \theta_i = 1\}$$

- ▶ if feasible set is bounded, it is a polytope
- ▶ prove: if a solution exists, then some extreme point of the feasible set is optimal

# Outline

LP standard form

## Modeling

LP inequality form

Solving LPs

Duality

Using duality

Large-scale linear programming

Integer programming

## Let's do some modeling!

- ▶ OptiMUS: <https://optimus-solver.vercel.app/>
- ▶ power systems: [https://jump.dev/JuMP.jl/stable/tutorials/applications/power\\_systems/](https://jump.dev/JuMP.jl/stable/tutorials/applications/power_systems/)
- ▶ multicast routing:  
<https://colab.research.google.com/drive/1iOn1T1Muh51KaA7mf7UIQOdhSFZhZyry?usp=sharing>

## Let's do some modeling!

- ▶ OptiMUS: <https://optimus-solver.vercel.app/>
- ▶ power systems: [https://jump.dev/JuMP.jl/stable/tutorials/applications/power\\_systems/](https://jump.dev/JuMP.jl/stable/tutorials/applications/power_systems/)
- ▶ multicast routing:  
<https://colab.research.google.com/drive/1iOn1T1Muh51KaA7mf7UIQOdhSFZhZyry?usp=sharing>

practical solvers for MILP:

- ▶ Gurobi and COPT (cardinal optimizer) are the state-of-the-art commercial solvers
- ▶ GLPK is a free solver that is not as fast
- ▶ JuliaOpt/JuMP is a modeling language in Julia that calls solvers like Gurobi and is specialized for MILP applications
- ▶ CVX\* (including CVXPY in python) are modeling languages that call solvers like Gurobi with good support for convex problems
- ▶ OptiMUS is a LLM-based modeling tool for MILP

## Modeling challenges

model the following as standard form LPs:

1. **inequality constraints.**  $Ax \leq b$
2. **free variable.**  $x \in \mathbf{R}$
3. **absolute value.** constraint  $|x| \leq 10$
4. **piecewise linear.** objective  $\max(x_1, x_2)$
5. **assignment.** e.g., every class is assigned exactly one classroom
6. **logic.** e.g., class enrollment  $\leq$  capacity of assigned room
7. **flow.** e.g., the least cost way to ship an item from  $s$  to  $t$

## Modeling challenges

model the following as standard form LPs:

1. **inequality constraints.**  $Ax \leq b$
2. **free variable.**  $x \in \mathbf{R}$
3. **absolute value.** constraint  $|x| \leq 10$
4. **piecewise linear.** objective  $\max(x_1, x_2)$
5. **assignment.** e.g., every class is assigned exactly one classroom
6. **logic.** e.g., class enrollment  $\leq$  capacity of assigned room
7. **flow.** e.g., the least cost way to ship an item from  $s$  to  $t$

(see chapter 1 of Bertsimas and Tsitsiklis for more details on 1–6. see <https://colab.research.google.com/drive/1iOn1T1Muh51KaA7mf7UIQOdhSFZhZyry?usp=sharing> for a detailed treatment of a flow problem.)

## Use slack variables to represent inequality constraints

to represent the following problem in standard form,

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0\end{array}$$



## Use slack variables to represent inequality constraints

to represent the following problem in standard form,

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0\end{array}$$

introduce slack variable  $s \in \mathbf{R}^m$ :  $Ax + s = b$ ,  
 $s \geq 0 \iff Ax \leq b$

$$\begin{array}{ll}\text{minimize} & c^T x + 0^T s \\ \text{subject to} & Ax + s = b \\ & x, s \geq 0\end{array}$$

## Split variable into parts to represent free variables

to represent the following problem in standard form,

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b\end{array}$$

## Split variable into parts to represent free variables

to represent the following problem in standard form,

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b\end{array}$$

introduce positive variables  $x_+, x_-$  so  $x = x_+ - x_-$ :

$$\begin{array}{ll}\text{minimize} & c^T x_+ - c^T x_- \\ \text{subject to} & Ax_+ - Ax_- = b \\ & x_+, x_- \geq 0\end{array}$$

## Use epigraph variables to handle absolute value

to represent the following problem in standard form,

$$\begin{array}{ll}\text{minimize} & \|x\|_1 = \sum_i |x_i| \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

## Use epigraph variables to handle absolute value

to represent the following problem in standard form,

$$\begin{array}{ll}\text{minimize} & \|x\|_1 = \sum_i 1^n |x_i| \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

introduce epigraph variable  $t \in \mathbf{R}^n$  so  $|x_i| \leq t_i$ :

$$\begin{array}{ll}\text{minimize} & 1^T t = \sum_{i=1}^n t_i \geq \|x\|_1 \\ \text{subject to} & Ax = b \\ & -t \leq x \leq t \\ & x, t \geq 0\end{array}$$

## Use epigraph variables to handle absolute value

to represent the following problem in standard form,

$$\begin{array}{ll}\text{minimize} & \|x\|_1 = \sum_i |x_i| \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

introduce epigraph variable  $t \in \mathbf{R}^n$  so  $|x_i| \leq t_i$ :

$$\begin{array}{ll}\text{minimize} & \mathbf{1}^T t = \sum_{i=1}^n t_i \geq \|x\|_1 \\ \text{subject to} & Ax = b \\ & -t \leq x \leq t \\ & x, t \geq 0\end{array}$$

**Q:** Why does this work? For what kinds of functions can we use this trick?

## Use binary variables to handle assignment

every class is assigned exactly one classroom:

define variable  $X_{ij} \in \{0, 1\}$  for each class  $i = 1, \dots, n$  and room  $j = 1, \dots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

## Use binary variables to handle assignment

every class is assigned exactly one classroom:

define variable  $X_{ij} \in \{0, 1\}$  for each class  $i = 1, \dots, n$  and room  $j = 1, \dots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

now solve the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \sum_{j=1}^m C_{ij} X_{ij} \\ \text{subject to} & \sum_{i=1}^n X_{ij} = 1, \forall j \quad (\text{every class assigned one room}) \\ & \sum_{j=1}^m X_{ij} = 1, \forall i \quad (\text{no more than one class per room}) \\ & X_{ij} \in \{0, 1\} \quad (\text{binary variables}) \end{array}$$

where  $C_{ij}$  is the cost of assigning class  $i$  to room  $j$ .



## Use binary variables to handle logic

model class enrollment  $n_i \leq$  capacity  $c_j$  of assigned room:

define variable  $X_{ij} \in \{0, 1\}$  for each class  $i = 1, \dots, n$  and room  $j = 1, \dots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

## Use binary variables to handle logic

model class enrollment  $n_i \leq$  capacity  $c_j$  of assigned room:

define variable  $X_{ij} \in \{0, 1\}$  for each class  $i = 1, \dots, n$  and room  $j = 1, \dots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

solve the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \sum_{j=1}^m C_{ij} X_{ij} \\ \text{subject to} & \sum_{i=1}^n X_{ij} = 1, \quad \forall j \quad (\text{every class assigned one room}) \\ & \sum_{j=1}^m X_{ij} = 1, \quad \forall i \quad (\text{no more than one class per room}) \\ & \sum_{i=1}^n p_i X_{ij} \leq c_j, \quad \forall j \quad (\text{capacity constraint}) \\ & X_{ij} \in \{0, 1\} \quad (\text{binary variables}) \end{array}$$

where  $C_{ij}$  is the cost of assigning class  $i$  to room  $j$ .

## Use binary variables to handle logic

model class enrollment  $n_i \leq$  capacity  $c_j$  of assigned room:

define variable  $X_{ij} \in \{0, 1\}$  for each class  $i = 1, \dots, n$  and room  $j = 1, \dots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

solve the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \sum_{j=1}^m C_{ij} X_{ij} \\ \text{subject to} & \sum_{i=1}^n X_{ij} = 1, \forall j \quad (\text{every class assigned one room}) \\ & \sum_{j=1}^m X_{ij} = 1, \forall i \quad (\text{no more than one class per room}) \\ & \sum_{i=1}^n p_i X_{ij} \leq c_j, \forall j \quad (\text{capacity constraint}) \\ & X_{ij} \in \{0, 1\} \quad (\text{binary variables}) \end{array}$$

where  $C_{ij}$  is the cost of assigning class  $i$  to room  $j$ .

what if we want  $p$  to be a variable, too?

## ...or use a big-M relaxation!

model class enrollment  $n_i \leq$  capacity  $c_j$  of assigned room:  
define variable  $X_{ij} \in \{0, 1\}$  for each class  $i = 1, \dots, n$  and room  $j = 1, \dots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

suppose  $M$  is a very large number.

## ...or use a big-M relaxation!

model class enrollment  $n_i \leq$  capacity  $c_j$  of assigned room:  
define variable  $X_{ij} \in \{0, 1\}$  for each class  $i = 1, \dots, n$  and room  $j = 1, \dots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

suppose  $M$  is a very large number. solve the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \sum_{j=1}^m C_{ij} X_{ij} \\ \text{subject to} & \sum_{i=1}^n X_{ij} = 1, \forall j \quad (\text{every class assigned one room}) \\ & \sum_{j=1}^m X_{ij} = 1, \forall i \quad (\text{no more than one class per room}) \\ & p_i \leq c_j + (1 - X_{ij})M, \forall i, j \quad (\text{capacity constraint}) \\ & X_{ij} \in \{0, 1\} \quad (\text{binary variables}) \end{array}$$

where  $C_{ij}$  is the cost of assigning class  $i$  to room  $j$ .

# Outline

LP standard form

Modeling

**LP inequality form**

Solving LPs

Duality

Using duality

Large-scale linear programming

Integer programming

## LP inequality form

another common form for LP is **inequality form**

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

how to transform to standard form?

- ▶ inequality constraints  $Ax \leq b$ ?

## LP inequality form

another common form for LP is **inequality form**

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

how to transform to standard form?

- ▶ inequality constraints  $Ax \leq b$ ? slack variables  $s \geq 0$
- ▶ free variable  $x \in \mathbf{R}^n$ ?



## LP inequality form

another common form for LP is **inequality form**

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

how to transform to standard form?

- ▶ inequality constraints  $Ax \leq b$ ? slack variables  $s \geq 0$
- ▶ free variable  $x \in \mathbf{R}^n$ ? split into positive and negative parts

we will see later that these forms are also related by **duality**

## LP example: production planning

- ▶  $x_i$  units of product  $i$
- ▶  $c_i$  cost per unit
- ▶  $a_{ij}$  amount of resource  $j$  used by product  $i$
- ▶  $b_j$  amount of resource  $j$  available
- ▶  $d_i$  demand for product  $i$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & 0 \leq x \leq d\end{array}$$

## LP example: production planning

- ▶  $x_i$  units of product  $i$
- ▶  $c_i$  cost per unit
- ▶  $a_{ij}$  amount of resource  $j$  used by product  $i$
- ▶  $b_j$  amount of resource  $j$  available
- ▶  $d_i$  demand for product  $i$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & 0 \leq x \leq d\end{array}$$

extensions:

- ▶ fixed cost for producing product  $i$  at all?

## LP example: production planning

- ▶  $x_i$  units of product  $i$
- ▶  $c_i$  cost per unit
- ▶  $a_{ij}$  amount of resource  $j$  used by product  $i$
- ▶  $b_j$  amount of resource  $j$  available
- ▶  $d_i$  demand for product  $i$

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & 0 \leq x \leq d\end{array}$$

extensions:

- ▶ fixed cost for producing product  $i$  at all?  
 $c^T x + f^T z$ ,  $z_i \in \{0, 1\}$ ,  $x_i \leq Mz_i$  for  $M$  large

## Geometry of LP: inequality form

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

- ▶  $Ax \leq b$  defines a **polyhedron**

## Geometry of LP: inequality form

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

- ▶  $Ax \leq b$  defines a **polyhedron**
- ▶  $\implies$  feasible set  $P = \{x : Ax \leq b\}$  is a polyhedron

## Geometry of LP: inequality form

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

- ▶  $Ax \leq b$  defines a **polyhedron**
- ▶  $\implies$  feasible set  $P = \{x : Ax \leq b\}$  is a polyhedron
- ▶  $x$  is a **vertex** of polyhedron  $P$  if there is some  $v$  so that

$$v^T x < v^T y, \quad \forall y \in P \setminus \{x\}$$

## Geometry of LP: inequality form

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

- ▶  $Ax \leq b$  defines a **polyhedron**
- ▶  $\implies$  feasible set  $P = \{x : Ax \leq b\}$  is a polyhedron
- ▶  $x$  is a **vertex** of polyhedron  $P$  if there is some  $v$  so that

$$v^T x < v^T y, \quad \forall y \in P \setminus \{x\}$$

**fact:** vertex  $\iff$  extreme point



## Solution of LP is extreme point

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

**fact:** if a solution exists and the feasible set has an extreme point, then some extreme point of the feasible set is optimal

## Solution of LP is extreme point

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

**fact:** if a solution exists and the feasible set has an extreme point, then some extreme point of the feasible set is optimal

**cases:** solution  $x^*$  is unique / not unique

## Solution of LP is extreme point

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

**fact:** if a solution exists and the feasible set has an extreme point, then some extreme point of the feasible set is optimal

**cases:** solution  $x^*$  is unique / not unique

- ▶ unique: so  $c^T x < c^T y$  for all  $y \in P \setminus \{x\}$

## Solution of LP is extreme point

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

**fact:** if a solution exists and the feasible set has an extreme point, then some extreme point of the feasible set is optimal

**cases:** solution  $x^*$  is unique / not unique

- ▶ unique: so  $c^T x < c^T y$  for all  $y \in P \setminus \{x\}$
- ▶ not unique:  $\{X^* : c^T x = c^T x^*, x \in P\}$  is a polyhedron. It is not empty (a solution exists) and its complement is not empty (optimal value is bounded). So, it has at least one vertex. That vertex is also a vertex of  $P$ .

## Basic feasible solution

define:  $x \in \mathbf{R}^n$  is a **basic feasible solution** (BFS) if there is a set  $S \subset \{1, \dots, n\}$  of  $m$  columns so that  $A_S$  is invertible and

$$x_S = A_S^{-1}b, \quad x_{\bar{S}} = 0, \quad x \geq 0.$$

- ▶  $A_S \in \mathbf{R}^{m \times m}$ , submatrix of  $A$  with columns in  $S$ , is invertible
- ▶ BFS  $\iff$  extreme point
- ▶ two BFS with  $S, S'$  are neighbors if they share  $m - 1$  columns:  $|S \cap S'| = m - 1$

## Basic feasible solution

define:  $x \in \mathbf{R}^n$  is a **basic feasible solution** (BFS) if there is a set  $S \subset \{1, \dots, n\}$  of  $m$  columns so that  $A_S$  is invertible and

$$x_S = A_S^{-1}b, \quad x_{\bar{S}} = 0, \quad x \geq 0.$$

- ▶  $A_S \in \mathbf{R}^{m \times m}$ , submatrix of  $A$  with columns in  $S$ , is invertible
- ▶ BFS  $\iff$  extreme point
- ▶ two BFS with  $S, S'$  are neighbors if they share  $m - 1$  columns:  $|S \cap S'| = m - 1$

define: **active set** is set of nonzero variables in  $x$

## Basic feasible solution

define:  $x \in \mathbf{R}^n$  is a **basic feasible solution** (BFS) if there is a set  $S \subset \{1, \dots, n\}$  of  $m$  columns so that  $A_S$  is invertible and

$$x_S = A_S^{-1}b, \quad x_{\bar{S}} = 0, \quad x \geq 0.$$

- ▶  $A_S \in \mathbf{R}^{m \times m}$ , submatrix of  $A$  with columns in  $S$ , is invertible
- ▶ BFS  $\iff$  extreme point
- ▶ two BFS with  $S, S'$  are neighbors if they share  $m - 1$  columns:  $|S \cap S'| = m - 1$

define: **active set** is set of nonzero variables in  $x$

**Q:** how to find a BFS?

## Basic feasible solution

define:  $x \in \mathbf{R}^n$  is a **basic feasible solution** (BFS) if there is a set  $S \subset \{1, \dots, n\}$  of  $m$  columns so that  $A_S$  is invertible and

$$x_S = A_S^{-1}b, \quad x_{\bar{S}} = 0, \quad x \geq 0.$$

- ▶  $A_S \in \mathbf{R}^{m \times m}$ , submatrix of  $A$  with columns in  $S$ , is invertible
- ▶  $\text{BFS} \iff \text{extreme point}$
- ▶ two BFS with  $S, S'$  are neighbors if they share  $m - 1$  columns:  $|S \cap S'| = m - 1$

define: **active set** is set of nonzero variables in  $x$

**Q:** how to find a BFS?

**A:** start at a feasible point; move in a **feasible direction** until you hit another constraint; continue until you reach a BFS



# Outline

LP standard form

Modeling

LP inequality form

**Solving LPs**

Duality

Using duality

Large-scale linear programming

Integer programming

## Solving LPs

algorithms:

- ▶ enumerate all vertices and check
- ▶ fourier-motzkin elimination
- ▶ simplex method
- ▶ ellipsoid method
- ▶ interior point methods
- ▶ first-order methods
- ▶ ...

# Solving LPs

algorithms:

- ▶ enumerate all vertices and check
- ▶ fourier-motzkin elimination
- ▶ simplex method
- ▶ ellipsoid method
- ▶ interior point methods
- ▶ first-order methods
- ▶ ...

remarks:

- ▶ enumeration and elimination are simple but not practical
- ▶ simplex was the first practical algorithm; still used today
- ▶ ellipsoid method is the first polynomial-time algorithm; not practical
- ▶ interior point methods are polynomial-time and practical
- ▶ first-order methods are practical and scale to large problems

## Discuss: how to solve LPs?

write down a method to solve LPs; discuss in groups.

- ▶ idea
- ▶ math
- ▶ pseudocode

complete <https://forms.gle/JbP2fLd6cRVbNUoW9> when you're ready (and before Friday noon)  
(link also available from course schedule)

## Enumerate vertices of LP

can generate all extreme points of LP: for each  $S \subseteq \{1, \dots, n\}$  with  $|S| = m$ ,

- ▶  $A_S \in \mathbf{R}^{m \times m}$ , submatrix of  $A$  with columns in  $S$ , is invertible
- ▶ solve  $A_S x_S = b$  for  $x_S$  and set  $x_{\bar{S}} = 0$
- ▶ if  $x_S \geq 0$ , then  $x$  is a BFS
- ▶ evaluate objective  $c^T x$

the best BFS is optimal!

## Enumerate vertices of LP

can generate all extreme points of LP: for each  $S \subseteq \{1, \dots, n\}$  with  $|S| = m$ ,

- ▶  $A_S \in \mathbf{R}^{m \times m}$ , submatrix of  $A$  with columns in  $S$ , is invertible
- ▶ solve  $A_S x_S = b$  for  $x_S$  and set  $x_{\bar{S}} = 0$
- ▶ if  $x_S \geq 0$ , then  $x$  is a BFS
- ▶ evaluate objective  $c^T x$

the best BFS is optimal!

**problem:** how many BFSs are there?

## Enumerate vertices of LP

can generate all extreme points of LP: for each  $S \subseteq \{1, \dots, n\}$  with  $|S| = m$ ,

- ▶  $A_S \in \mathbf{R}^{m \times m}$ , submatrix of  $A$  with columns in  $S$ , is invertible
- ▶ solve  $A_S x_S = b$  for  $x_S$  and set  $x_{\bar{S}} = 0$
- ▶ if  $x_S \geq 0$ , then  $x$  is a BFS
- ▶ evaluate objective  $c^T x$

the best BFS is optimal!

**problem:** how many BFSs are there?

$n$  choose  $m$  is  $\binom{n}{m} = \frac{n!}{m!(n-m)!}$  (“exponentially many”)

## Simplex algorithm

basic idea: local search on the vertices of the feasible set

- ▶ start at BFS  $x$  and evaluate objective  $c^T x$
- ▶ move to a neighboring BFS  $x'$  with better objective  $c^T x'$
- ▶ repeat until no improvement possible



## Simplex algorithm

basic idea: local search on the vertices of the feasible set

- ▶ start at BFS  $x$  and evaluate objective  $c^T x$
- ▶ move to a neighboring BFS  $x'$  with better objective  $c^T x'$
- ▶ repeat until no improvement possible

discuss in groups:

- ▶ how to find an initial BFS?
- ▶ how to find a neighboring BFS with better objective?
- ▶ how to prove optimality?

## Finding an initial BFS

solve an auxiliary problem for which a BFS is known:

$$\begin{array}{ll}\text{minimize} & \sum_{i=1}^m z_i \\ \text{subject to} & Ax + Dz = b \\ & x, z \geq 0\end{array}$$

where  $D \in \mathbf{R}^{m \times m}$  is a diagonal matrix with  $D_{ii} = \mathbf{sign}(b_i)$  for  $i = 1, \dots, m$ .

- ▶  $x = 0, z = |b|$  is a BFS of this problem
- ▶  $(x, z) = (x, 0)$  is a BFS of this problem  $\iff x$  is a BFS of the original problem

## Find a better neighboring BFS

start with BFS  $x$  with active set  $S$  and turn on variable  $j \notin S$

$$x^+ \leftarrow x + \theta d, \quad \theta > 0$$

where  $d_j = 1$  and  $d_i = 0$  for  $i \notin S \cup \{j\}$ . need to solve for  $d_S$ .

## Find a better neighboring BFS

start with BFS  $x$  with active set  $S$  and turn on variable  $j \notin S$

$$x^+ \leftarrow x + \theta d, \quad \theta > 0$$

where  $d_j = 1$  and  $d_i = 0$  for  $i \notin S \cup \{j\}$ . need to solve for  $d_S$ .

- need to stay feasible wrt equality constraints, so

$$Ax = b, \quad A(x + \theta d) = b, \implies Ad = 0$$

## Find a better neighboring BFS

start with BFS  $x$  with active set  $S$  and turn on variable  $j \notin S$

$$x^+ \leftarrow x + \theta d, \quad \theta > 0$$

where  $d_j = 1$  and  $d_i = 0$  for  $i \notin S \cup \{j\}$ . need to solve for  $d_S$ .

- ▶ need to stay feasible wrt equality constraints, so

$$Ax = b, \quad A(x + \theta d) = b, \implies Ad = 0$$

- ▶ construct the  **$j$ th basic direction**

$$Ad = A_S d_S + A_j = 0 \implies d_S = -A_S^{-1} A_j$$

## Find a better neighboring BFS

start with BFS  $x$  with active set  $S$  and turn on variable  $j \notin S$

$$x^+ \leftarrow x + \theta d, \quad \theta > 0$$

where  $d_j = 1$  and  $d_i = 0$  for  $i \notin S \cup \{j\}$ . need to solve for  $d_S$ .

- ▶ need to stay feasible wrt equality constraints, so

$$Ax = b, \quad A(x + \theta d) = b, \implies Ad = 0$$

- ▶ construct the  **$j$ th basic direction**

$$Ad = A_S d_S + A_j = 0 \implies d_S = -A_S^{-1} A_j$$

- ▶ if  $x_S > 0$  is **non-degenerate**, then  $\exists \theta > 0$  st  $x^+ \geq 0$

## Find a better neighboring BFS

start with BFS  $x$  with active set  $S$  and turn on variable  $j \notin S$

$$x^+ \leftarrow x + \theta d, \quad \theta > 0$$

where  $d_j = 1$  and  $d_i = 0$  for  $i \notin S \cup \{j\}$ . need to solve for  $d_S$ .

- ▶ need to stay feasible wrt equality constraints, so

$$Ax = b, \quad A(x + \theta d) = b, \implies Ad = 0$$

- ▶ construct the  **$j$ th basic direction**

$$Ad = A_S d_S + A_j = 0 \implies d_S = -A_S^{-1} A_j$$

- ▶ if  $x_S > 0$  is **non-degenerate**, then  $\exists \theta > 0$  st  $x^+ \geq 0$
- ▶ how does objective change?

$$c^T x^+ = c^T x + \theta c^T d = c^T x + \theta c_j - \theta c_S^T A_S^{-1} A_j$$

## Reduced cost

define **reduced cost**  $\bar{c}_j = c_j - c_S^T A_S^{-1} A_j, j \notin S$



## Reduced cost

define **reduced cost**  $\bar{c}_j = c_j - c_S^T A_S^{-1} A_j, j \notin S$

fact:

- ▶ if  $\bar{c} \geq 0$ ,  $x$  is optimal
- ▶ if  $x$  is optimal and nondegenerate ( $x_S > 0$ ), then  $\bar{c} \geq 0$

# Outline

LP standard form

Modeling

LP inequality form

Solving LPs

**Duality**

Using duality

Large-scale linear programming

Integer programming

## Why duality?

- ▶ certify optimality
  - ▶ turn  $\forall$  into  $\exists$
  - ▶ use dual lower bound to derive stopping conditions
- ▶ new algorithms based on the dual
  - ▶ solve dual, then recover primal solution

## Duality notation

- ▶ inner product

$$y^T x = \langle y, x \rangle = y \cdot x = \sum_{i=1}^n y_i x_i$$

- ▶ conjugate

$$\langle y, Ax \rangle = \langle A^T y, x \rangle$$

## Warmup: Farkas lemma

### Theorem (Farkas lemma)

*Given  $A \in \mathbf{R}^{m \times n}$  and  $b \in \mathbf{R}^m$ , exactly one of the following is true:*

- ▶ *there exists  $x \in \mathbf{R}^n$  so that  $Ax = b$  and  $x \geq 0$*
- ▶ *there exists  $y \in \mathbf{R}^m$  so that  $A^T y \geq 0$  and  $\langle b, y \rangle < 0$*

$\implies$  can efficiently certify infeasibility of a linear program

## Warmup: Farkas lemma

### Theorem (Farkas lemma)

Given  $A \in \mathbf{R}^{m \times n}$  and  $b \in \mathbf{R}^m$ , exactly one of the following is true:

- ▶ there exists  $x \in \mathbf{R}^n$  so that  $Ax = b$  and  $x \geq 0$
- ▶ there exists  $y \in \mathbf{R}^m$  so that  $A^T y \geq 0$  and  $\langle b, y \rangle < 0$

$\implies$  can efficiently certify infeasibility of a linear program

**proof:** suppose we have  $x \in \mathbf{R}^n$  so that  $Ax = b$  and  $x \geq 0$ .  
then for any  $y \in \mathbf{R}^m$ ,

$$\begin{aligned} 0 &= \langle y, b - Ax \rangle = \langle y, b \rangle - \langle A^T y, x \rangle \\ \langle y, b \rangle &= \langle A^T y, x \rangle \end{aligned}$$

so if  $A^T y \geq 0$ , then use  $x \geq 0$  to conclude  $\langle y, b \rangle \geq 0$ .

## Warmup: Farkas lemma

### Theorem (Farkas lemma)

Given  $A \in \mathbf{R}^{m \times n}$  and  $b \in \mathbf{R}^m$ , exactly one of the following is true:

- ▶ there exists  $x \in \mathbf{R}^n$  so that  $Ax = b$  and  $x \geq 0$
- ▶ there exists  $y \in \mathbf{R}^m$  so that  $A^T y \geq 0$  and  $\langle b, y \rangle < 0$

$\implies$  can efficiently certify infeasibility of a linear program

**proof:** suppose we have  $x \in \mathbf{R}^n$  so that  $Ax = b$  and  $x \geq 0$ .  
then for any  $y \in \mathbf{R}^m$ ,

$$\begin{aligned} 0 &= \langle y, b - Ax \rangle = \langle y, b \rangle - \langle A^T y, x \rangle \\ \langle y, b \rangle &= \langle A^T y, x \rangle \end{aligned}$$

so if  $A^T y \geq 0$ , then use  $x \geq 0$  to conclude  $\langle y, b \rangle \geq 0$ .

(opposite direction is similar)

## Lagrange duality

primal problem with solution  $x^* \in \mathbf{R}^n$ , optimal value  $p^*$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array} \quad (\mathcal{P})$$

if  $x$  is feasible, then  $Ax = b$ , so  $\langle y, Ax - b \rangle = 0$  for  $y \in \mathbf{R}^m$ .



## Lagrange duality

primal problem with solution  $x^* \in \mathbf{R}^n$ , optimal value  $p^*$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array} \quad (\mathcal{P})$$

if  $x$  is feasible, then  $Ax = b$ , so  $\langle y, Ax - b \rangle = 0$  for  $y \in \mathbf{R}^m$ .

define the **Lagrangian**

$$\mathcal{L}(x, y) := c^T x - \langle y, Ax - b \rangle$$

## Lagrange duality

primal problem with solution  $x^* \in \mathbf{R}^n$ , optimal value  $p^*$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array} \quad (\mathcal{P})$$

if  $x$  is feasible, then  $Ax = b$ , so  $\langle y, Ax - b \rangle = 0$  for  $y \in \mathbf{R}^m$ .

define the **Lagrangian**

$$\begin{aligned} \mathcal{L}(x, y) &:= c^T x - \langle y, Ax - b \rangle \\ p^* &= \inf_{x: Ax=b, x \geq 0} \mathcal{L}(x, y) \geq \inf_{x \geq 0} \mathcal{L}(x, y) \end{aligned}$$

## Lagrange duality

primal problem with solution  $x^* \in \mathbf{R}^n$ , optimal value  $p^*$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b : \quad \text{dual } y \\ & x \geq 0 \end{array} \quad (\mathcal{P})$$

if  $x$  is feasible, then  $Ax = b$ , so  $\langle y, Ax - b \rangle = 0$  for  $y \in \mathbf{R}^m$ .

define the **Lagrangian**

$$\begin{aligned} \mathcal{L}(x, y) &:= c^T x - \langle y, Ax - b \rangle \\ p^* &= \inf_{x: Ax=b, x \geq 0} \mathcal{L}(x, y) \geq \inf_{x \geq 0} \mathcal{L}(x, y) \\ &= \inf_{x \geq 0} c^T x + \langle y, b - Ax \rangle \\ &= \langle y, b \rangle + \inf_{x \geq 0} \left( c^T x - \langle A^T y, x \rangle \right) \\ &= \langle y, b \rangle + \inf_{x \geq 0} \left( \langle c - A^T y, x \rangle \right) \end{aligned}$$

unbounded below unless  $c - A^T y \geq 0$ .

## Lagrange duality, ctd

we have a lower bound on  $p^*$  for any  $y$ , and a useful one whenever  $c + A^T y = 0$ . maximize bound:

$$p^* \geq \begin{array}{ll} \text{maximize} & \langle y, b \rangle \\ \text{subject to} & A^T y \leq c \\ \text{variable} & y \in \mathbf{R}^m \end{array}$$

define the **dual function**

$$g(y) = \begin{cases} \langle y, b \rangle & A^T y \leq c \\ -\infty & \text{otherwise} \end{cases}$$

## Lagrange duality

**weak duality** asserts that  $p^* \geq g(y)$  for all  $y \in \mathbf{R}^m$ .

$$\begin{aligned} p^* &\geq g(y) \quad \forall y \in \mathbf{R}^m \\ &\geq \underbrace{\sup_y g(y)}_{\mathcal{D}} =: d^* \end{aligned}$$

$p^* \geq d^*$  dual optimal value

## Strong duality

### Definition (Duality gap)

The **duality gap** for a primal-dual pair  $(x, y)$  is  $c^T x - b^T y \geq 0$

by weak duality, duality gap is always nonnegative

## Strong duality

### Definition (Duality gap)

The **duality gap** for a primal-dual pair  $(x, y)$  is  $c^T x - b^T y \geq 0$

by weak duality, duality gap is always nonnegative

### Definition (Strong duality)

A primal-dual pair  $(x^*, y^*)$  satisfies **strong duality** if

$$p^* = d^* \iff c^T x - b^T y = 0$$

## Strong duality

### Definition (Duality gap)

The **duality gap** for a primal-dual pair  $(x, y)$  is  $c^T x - b^T y \geq 0$

by weak duality, duality gap is always nonnegative

### Definition (Strong duality)

A primal-dual pair  $(x^*, y^*)$  satisfies **strong duality** if

$$p^* = d^* \iff c^T x - b^T y = 0$$

strong duality holds

- ▶ for feasible LPs
- ▶ (later) for convex problems under **constraint qualification** aka **Slater's condition**. feasible region has an **interior point**  $x$  so that all inequality constraints hold strictly

strong duality fails if either primal or dual problem is infeasible or unbounded



## Strong duality for LPs

primal and dual LP in standard form:

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

$$\begin{array}{ll}\text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c\end{array}$$

**claim:** if primal LP has a bounded feasible solution  $x^*$ , then strong duality holds

*i.e.*, dual LP has a bounded feasible solution  $y^*$  and  $p^* = d^*$

## Proof of strong duality for LPs

consider the following system with variables  $x' \in \mathbf{R}^n$ ,  $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^* \tau - 1, \quad (x', \tau) \geq 0$$

## Proof of strong duality for LPs

consider the following system with variables  $x' \in \mathbf{R}^n$ ,  $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^* \tau - 1, \quad (x', \tau) \geq 0$$

**claim:** this system has no solution. pf by contradiction:

- ▶ if  $\tau > 0$ , then  $x'/\tau$  is feasible for LP and  $c^T x'/\tau < p^*$
- ▶ if  $\tau = 0$ , then  $x^* + x'$  is feasible for LP and  $c^T(x^* + x') < p^*$

## Proof of strong duality for LPs

consider the following system with variables  $x' \in \mathbf{R}^n$ ,  $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^* \tau - 1, \quad (x', \tau) \geq 0$$

**claim:** this system has no solution. pf by contradiction:

- ▶ if  $\tau > 0$ , then  $x'/\tau$  is feasible for LP and  $c^T x'/\tau < p^*$
- ▶ if  $\tau = 0$ , then  $x^* + x'$  is feasible for LP and  $c^T(x^* + x') < p^*$

**so use Farkas' lemma:**

$$Ax + b = 0, \quad x \geq 0 \quad \text{or} \quad A^T y \geq 0, \quad b^T y < 0$$

## Proof of strong duality for LPs

consider the following system with variables  $x' \in \mathbf{R}^n$ ,  $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^* \tau - 1, \quad (x', \tau) \geq 0$$

**claim:** this system has no solution. pf by contradiction:

- ▶ if  $\tau > 0$ , then  $x'/\tau$  is feasible for LP and  $c^T x'/\tau < p^*$
- ▶ if  $\tau = 0$ , then  $x^* + x'$  is feasible for LP and  $c^T(x^* + x') < p^*$

**so use Farkas' lemma:**

$$\begin{array}{ll} Ax + b = 0, \quad x \geq 0 & \text{or} \quad A^T y \geq 0, \quad b^T y < 0 \\ \begin{bmatrix} A & -b \\ c^T & -p^* \end{bmatrix} \begin{bmatrix} x \\ \tau \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \text{or} \quad \begin{bmatrix} A^T & c \\ -b^T & -p^* \end{bmatrix} \begin{bmatrix} y \\ \sigma \end{bmatrix} \geq 0, \quad \sigma > 0 \end{array}$$

## Proof of strong duality for LPs

consider the following system with variables  $x' \in \mathbf{R}^n$ ,  $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^* \tau - 1, \quad (x', \tau) \geq 0$$

**claim:** this system has no solution. pf by contradiction:

- ▶ if  $\tau > 0$ , then  $x'/\tau$  is feasible for LP and  $c^T x'/\tau < p^*$
- ▶ if  $\tau = 0$ , then  $x^* + x'$  is feasible for LP and  $c^T(x^* + x') < p^*$

**so use Farkas' lemma:**

$$\begin{array}{ll} Ax + b = 0, \quad x \geq 0 & \text{or} \quad A^T y \geq 0, \quad b^T y < 0 \\ \begin{bmatrix} A & -b \\ c^T & -p^* \end{bmatrix} \begin{bmatrix} x \\ \tau \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \text{or} \quad \begin{bmatrix} A^T & c \\ -b^T & -p^* \end{bmatrix} \begin{bmatrix} y \\ \sigma \end{bmatrix} \geq 0, \quad \sigma > 0 \end{array}$$

use second system to show  $y/\sigma$  is dual feasible and optimal

## Strong duality and complementary slackness

### Definition (complementary slackness)

The primal-dual pair  $x$  and  $y$  are **complementary** if

$$\langle y, b - Ax \rangle = 0$$

They satisfy **strict complementary slackness** if  $y_i(b_i - a_i^T x) = 0$  for  $i = 1, \dots, n$ .

strong duality  $\iff$  complementary slackness

$$\begin{aligned}\langle y, s \rangle &= \langle y, b - Ax \rangle \\ &= \langle y, b \rangle - \langle A^T y, x \rangle \\ &= \langle y, b \rangle - \langle c, x \rangle\end{aligned}$$

# Outline

LP standard form

Modeling

LP inequality form

Solving LPs

Duality

**Using duality**

Large-scale linear programming

Integer programming



## Duality as stopping condition

want to optimize until **primal suboptimality**  $p^* - c^T x \geq 0$  or **dual suboptimality**  $d^* - b^T y \geq 0$  are small enough. how?

## Duality as stopping condition

want to optimize until **primal suboptimality**  $p^* - c^T x \geq 0$  or **dual suboptimality**  $d^* - b^T y \geq 0$  are small enough. how?

duality gap  $c^T x - b^T y \geq 0$  bounds both!

for  $x$  feasible,  $y$  dual feasible,

$$c^T x \geq c^T x^* \geq b^T y^* \geq b^T y$$

## How to use duality to estimate sensitivity?

primal and dual LP in standard form:

$$\begin{array}{ll} p^* = \min & c^T x \\ & \text{subject to } Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} d^* = \max & b^T y \\ & \text{subject to } A^T y \leq c \end{array}$$

optimal primal and dual solution  $x^*, y^*$

perturbed problem: primal and dual LP in standard form:

$$\begin{array}{ll} \tilde{p}^* = \min & c^T x \\ & \text{subject to } Ax = b + \epsilon \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \tilde{d}^* = \max & (b + \epsilon)^T y \\ & \text{subject to } A^T y \leq c \end{array}$$

## How to use duality to estimate sensitivity?

primal and dual LP in standard form:

$$p^* = \begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad d^* = \begin{array}{ll} \max & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

optimal primal and dual solution  $x^*, y^*$

perturbed problem: primal and dual LP in standard form:

$$\tilde{p}^* = \begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b + \epsilon \\ & x \geq 0 \end{array} \quad \tilde{d}^* = \begin{array}{ll} \max & (b + \epsilon)^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

$y^*$  is feasible for perturbed problem, so

$$\tilde{p}^* = \tilde{d}^* \geq (b + \epsilon)^T y^* = d^* + \epsilon^T y^*$$

# Outline

LP standard form

Modeling

LP inequality form

Solving LPs

Duality

Using duality

**Large-scale linear programming**

Integer programming

## Column / constraint generation

primal and dual LP,  $A \in \mathbf{R}^{m \times n}$ ,  $n \gg m$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

## Column / constraint generation

primal and dual LP,  $A \in \mathbf{R}^{m \times n}$ ,  $n \gg m$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

approximate by using  $S \subset \{1, \dots, n\}$ : fewer variables (primal) or constraints (dual)

$$\begin{array}{ll} \text{minimize} & c_S^T x_S \\ \text{subject to} & A_S x_S = b \\ & x_S \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A_S^T y \leq c_S \end{array}$$

## Column / constraint generation

primal and dual LP,  $A \in \mathbf{R}^{m \times n}$ ,  $n \gg m$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

approximate by using  $S \subset \{1, \dots, n\}$ : fewer variables (primal) or constraints (dual)

$$\begin{array}{ll} \text{minimize} & c_S^T x_S \\ \text{subject to} & A_S x_S = b \\ & x_S \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A_S^T y \leq c_S \end{array}$$

if  $x_S$  is optimal for  $\mathcal{P}_S$  and reduced cost  $\bar{c} \geq 0$ , then  $x_S$  is optimal for  $\mathcal{P}$

if  $y$  is optimal for  $\mathcal{D}_S$  and feasible for  $\mathcal{D}$ , then  $y$  is optimal for  $\mathcal{D}$



## Column / constraint generation

primal and dual LP,  $A \in \mathbf{R}^{m \times n}$ ,  $n \gg m$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

approximate by using  $S \subset \{1, \dots, n\}$ : fewer variables (primal) or constraints (dual)

$$\begin{array}{ll} \text{minimize} & c_S^T x_S \\ \text{subject to} & A_S x_S = b \\ & x_S \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A_S^T y \leq c_S \end{array}$$

if  $x_S$  is optimal for  $\mathcal{P}_S$  and reduced cost  $\bar{c} \geq 0$ , then  $x_S$  is optimal for  $\mathcal{P}$

otherwise?

if  $y$  is optimal for  $\mathcal{D}_S$  and feasible for  $\mathcal{D}$ , then  $y$  is optimal for  $\mathcal{D}$

## Column / constraint generation

primal and dual LP,  $A \in \mathbf{R}^{m \times n}$ ,  $n \gg m$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

approximate by using  $S \subset \{1, \dots, n\}$ : fewer variables (primal) or constraints (dual)

$$\begin{array}{ll} \text{minimize} & c_S^T x_S \\ \text{subject to} & A_S x_S = b \\ & x_S \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A_S^T y \leq c_S \end{array}$$

if  $x_S$  is optimal for  $\mathcal{P}_S$  and reduced cost  $\bar{c} \geq 0$ , then  $x_S$  is optimal for  $\mathcal{P}$

if  $y$  is optimal for  $\mathcal{D}_S$  and feasible for  $\mathcal{D}$ , then  $y$  is optimal for  $\mathcal{D}$

otherwise? find  $i$  with  $\bar{c}_i = c_i - c_S^T A_S^{-1} a_i < 0$  (primal) or  $a_i^T y > c_i$  (dual) and add to  $S$

## Column / constraint generation

primal and dual LP,  $A \in \mathbf{R}^{m \times n}$ ,  $n \gg m$ :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

approximate by using  $S \subset \{1, \dots, n\}$ : fewer variables (primal) or constraints (dual)

$$\begin{array}{ll} \text{minimize} & c_S^T x_S \\ \text{subject to} & A_S x_S = b \\ & x_S \geq 0 \end{array} \quad \leftrightarrow^{\text{dual}} \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A_S^T y \leq c_S \end{array}$$

if  $x_S$  is optimal for  $\mathcal{P}_S$  and reduced cost  $\bar{c} \geq 0$ , then  $x_S$  is optimal for  $\mathcal{P}$

if  $y$  is optimal for  $\mathcal{D}_S$  and feasible for  $\mathcal{D}$ , then  $y$  is optimal for  $\mathcal{D}$

otherwise? find  $i$  with  $\bar{c}_i = c_i - c_S^T A_S^{-1} a_i < 0$  (primal) or  $a_i^T y > c_i$  (dual) and add to  $S$

► if dual constraints are all binding,  $A_S^T y = c_S$ , so these

## Presolve

Often many constraints are redundant or can be simplified.

example:

$$\begin{array}{ll}\text{minimize} & x_3 \\ \text{subject to} & x_1 = 1 \\ & x_2 = x_3 - x_1 \\ & x_3 - x_2 \geq 0 \\ & x \geq 0\end{array}$$

a good presolve can often reduce problem from 1000s of variables and constraints down to 10s!

reference: Achterberg, Tobias, et al. "Presolve reductions in mixed integer programming." *INFORMS Journal on Computing* 32.2 (2020): 473-506.

# Outline

LP standard form

Modeling

LP inequality form

Solving LPs

Duality

Using duality

Large-scale linear programming

Integer programming

## MILP solution vs LP solution

mixed-integer linear program (MILP):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + Bz = b \\ & x \geq 0, z \geq 0 \in \mathbb{Z} \end{array} \quad \xrightarrow{\text{relax}} \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + Bz = b \\ & x, z \geq 0 \end{array}$$

## MILP solution vs LP solution

mixed-integer linear program (MILP):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + Bz = b \\ & x \geq 0, z \geq 0 \in \mathbb{Z} \end{array} \quad \xrightarrow{\text{relax}} \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + Bz = b \\ & x, z \geq 0 \end{array}$$

example:

$$\begin{array}{ll} \text{maximize} & x \\ \text{subject to} & x \leq z \\ & x \leq 1 - z \\ & x \geq 0, z \in \{0, 1\} \end{array}$$

## MILP solution vs LP solution

mixed-integer linear program (MILP):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + Bz = b \\ & x \geq 0, z \geq 0 \in \mathbb{Z} \end{array} \quad \xrightarrow{\text{relax}} \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + Bz = b \\ & x, z \geq 0 \end{array}$$

example:

$$\begin{array}{ll} \text{maximize} & x \\ \text{subject to} & x \leq z \\ & x \leq 1 - z \\ & x \geq 0, z \in \{0, 1\} \end{array}$$

draw picture: where is solution of MILP? of LP relaxation?



## Branch and bound

given MILP with integer variable  $z$  in rectangle  $R = (l, u)$ ,  
 $l \leq z \leq u$ , optimal value  $p^*(R)$ , solution  $z^*(R)$

- ▶ solve LP relaxation to produce lower bound  $LB(R) \leq p^*(R)$
- ▶ round  $z$  to nearest feasible integer  $z'$  to produce upper bound  $UB(R) \geq p^*(R)$

## Branch and bound

given MILP with integer variable  $z$  in rectangle  $R = (l, u)$ ,  
 $l \leq z \leq u$ , optimal value  $p^*(R)$ , solution  $z^*(R)$

- ▶ solve LP relaxation to produce lower bound  $LB(R) \leq p^*(R)$
- ▶ round  $z$  to nearest feasible integer  $z'$  to produce upper bound  $UB(R) \geq p^*(R)$

if  $LB(R) = UB(R)$ , then  $p^*(R) = LB(R) = UB(R)$  and we are done.

## Branch and bound

given MILP with integer variable  $z$  in rectangle  $R = (l, u)$ ,  
 $l \leq z \leq u$ , optimal value  $p^*(R)$ , solution  $z^*(R)$

- ▶ solve LP relaxation to produce lower bound  $LB(R) \leq p^*(R)$
- ▶ round  $z$  to nearest feasible integer  $z'$  to produce upper bound  $UB(R) \geq p^*(R)$

if  $LB(R) = UB(R)$ , then  $p^*(R) = LB(R) = UB(R)$  and we are done.

otherwise, branch

- ▶ split  $R$  into two subrectangles  $R_1 = (l_1, u_1)$ ,  $R_2 = (l_2, u_2)$  so that  $\mathbb{Z} \cap R = (\mathbb{Z} \cap R_1) \cup (\mathbb{Z} \cap R_2)$
- ▶ compute bounds  $LB(R_1)$ ,  $UB(R_1)$ ,  $LB(R_2)$ ,  $UB(R_2)$
- ▶  $R \subset R_1 \cup R_2$  so  $LB(R) \leq \min(LB(R_1), LB(R_2))$
- ▶ keep best solution so far  $UB \leftarrow \min(UB, UB(R_1), UB(R_2))$
- ▶ prune: eliminate rectangle from consideration if  $LB(R) > UB$

draw picture in 2D