CME 307: Optimization	CME 307 / MS&E 311 / OIT 676

Lecture 2: LP Geometry

Fall 2025 — *Prof. Udell*

1 LP geometry: standard form

We study the geometry of feasibility for the standard-form linear program

$$
\begin{array}{ll}
\text{minimize} & c^\top x \\
\text{subject to} & Ax = b, \\
& x \geq 0, \\
\text{variable} & x \in \mathbb{R}^n,
\end{array}
$$

focusing on two equivalent geometric views and basic convexity facts that will underlie extreme points and basic feasible solutions (BFS) later in the unit. Writing the LP in standard form makes it easy to understand the geometry of the problem that motivates the simplex algorithm.

1.1 View 1: conic hull of the columns

Let $A = [A_1, \ldots, A_n] \in \mathbb{R}^{m\times n}$ with columns $A_i \in \mathbb{R}^m$.

Definition 1.1 (Cone)**.** A set $K \subseteq \mathbb{R}^m$ is a *cone* if it contains the ray from the origin to any vector in the cone:

$$v \in K, \alpha \geq 0 \implies \alpha v \in K.$$

We will most often be interested in a cone generated from a collection of vectors:

Definition 1.2 (Conic hull of vectors)**.** The conic hull of the vectors $A_1, \ldots, A_n \in \mathbb{R}^m$ is

$$\operatorname{cone}(A_1, \ldots, A_n) := \left\{ \sum_{i=1}^n A_i x_i \mid x \geq 0 \right\}.$$

Exercise 1.3. To check this definition makes sense, prove that $\operatorname{cone}(A_1, \ldots, A_n)$ is a cone.

Proposition 1.4. *The image of the nonnegative orthant is a finitely generated cone.*

$$\{Ax \mid x \geq 0\} = \operatorname{cone}(A_1, \ldots, A_n).$$

Proof. If $x \geq 0$, then $Ax = \sum_{i=1}^n A_i x_i \in \operatorname{cone}(a_1, \ldots, a_n)$. Conversely, any $y = \sum_{i=1}^n A_i \alpha_i$ with $\alpha_i \geq 0$ equals Ax for $x = (\alpha_1, \ldots, \alpha_n) \geq 0$. $\square$

Proposition 1.5 (Feasibility criterion (conic view)). *The LP is feasible if and only if* $b \in \mathrm{cone}(a_1, \ldots, a_n)$.

Proof. By definition, feasibility requires an $x \geq 0$ with $Ax = b$. By Proposition 1.4, this holds if and only if b lies in the cone generated by the columns. $\square$

1.2 Carathéodory for cones*

Theorem 1.6 (Carathéodory for cones). *If* $b \in \mathrm{cone}(A_1, \ldots, A_n) \subset \mathbb{R}^m$, *then* b *can be expressed using at most* m *generators: there exists an index set* $S \subseteq 1, \ldots, n$ *with* $|S| \leq m$ *and coefficients* $\{x_i\}_{i \in S}$, $x_i \geq 0$, *such that* $b = \sum_{i \in S} \alpha_i A_i$.

Proof sketch. Start from any conic representation $b = \sum_{i=1}^{n} x_i a_i$ with $x_i \geq 0$. Let $S = \{i : \alpha_i > 0\}$. If $|S| \leq m$, we are done. Otherwise the set $A_{i i \in S}$ is linearly dependent in $\mathbb{R}^m$, so there exists a nonzero $\delta \in \mathbb{R}^{|S|}$ with $\sum_{i \in S} \delta_i A_i = 0$. We can assume δ has at least one entry that is negative. (Otherwise, use $-\delta$ instead.) Move along the ray $x \mapsto x + t\delta$ for $t > 0$, increasing t until at least one active coefficient hits zero, strictly reducing $|S|$. Iterate until $|S| \leq m$. $\square$

Corollary 1.7 (Feasible solution with few positives). *If the LP is feasible, then there exists a feasible* x *with at most* m *positive entries.*

Proof. Apply Theorem 1.6 to $b \in \mathrm{cone}(A_1, \ldots, A_n)$ and read the nonzero x_i as the positive components of x. $\square$

1.3 View 2: intersection of an affine space with halfspaces

Definition 1.8 (Hyperplane, halfspace, orthant). A *hyperplane* is an affine set $\{x \in \mathbb{R}^n \mid Ax = b\}$. A *halfspace* is a set of the form $\{x \in \mathbb{R}^n \mid a^\top x \leq \beta\}$. The *nonnegative orthant* is $\mathbb{R}^n_+ = \{x \mid x \geq 0\}$; it is the intersection of the coordinate halfspaces $\{x \mid e_i^\top x \geq 0\}$.

Dimension. The hyperplane $\{x \in \mathbb{R}^n \mid Ax = b\}$ is an affine subspace of dimension $n - \mathrm{Rank}(A)$ (equivalently, codimension $\mathrm{Rank}(A)$). In the common case of a single linear equation $a^T x = \beta$, the hyperplane has dimension $n - 1$.

Proposition 1.9 (Feasible set as an intersection). *The feasible set*

$$F = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$$

is the intersection of the hyperplane(s) $Ax = b$ *with the nonnegative orthant.*

Proof. Direct from the definition. $\square$

Definition 1.10 (Polyhedron)**.** A *polyhedron* is an intersection of finitely many halfspaces and hyperplanes (e.g., $x \mid Ax \leq b,\ Cx = d$).

In standard form, F is a (possibly unbounded) polyhedron.

2 Convexity facts and consequences

Definition 2.1 (Convex combination; convex set; convex hull)**.** For $x, y \in \mathbb{R}^n$ and $\theta \in [0, 1]$, the point $\theta x + (1-\theta)y$ is a *convex combination.* A set $C \subseteq \mathbb{R}^n$ is *convex* if it contains every convex combination of its points. The *convex hull* of $S \subseteq \mathbb{R}^n$ is

$$\operatorname{conv}(S) = \Big\{ \sum_{i=1}^{k} \theta_i x_i \;\Big|\; x_i \in S,\ \theta_i \geq 0,\ \sum_{i=1}^{k} \theta_i = 1 \Big\}.$$

Proposition 2.2 (Basic convexity facts)**.** *Hyperplanes and halfspaces are convex, and intersections of convex sets are convex.*

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Consequently, the feasible set $F = \{x \mid Ax = b,\ x \geq 0\}$ is convex.

Proof. Hyperplanes are convex. Let $H = \{x \in \mathbb{R}^n \mid a^T x = b\}$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. If $x, y \in H$ and $\theta \in [0, 1]$, then

$$a^T\big(\theta x + (1-\theta)y\big) = \theta\, a^T x + (1-\theta)\, a^T y = \theta b + (1-\theta)b = b,$$

so $\theta x + (1-\theta)y \in H$.

Halfspaces are convex. Let $S = \{x \in \mathbb{R}^n \mid a^T x \leq b\}$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. If $x, y \in S$ and $\theta \in [0, 1]$, then

$$a^T\big(\theta x + (1-\theta)y\big) = \theta\, a^T x + (1-\theta)\, a^T y \leq \theta b + (1-\theta)b = b,$$

so $\theta x + (1-\theta)y \in S$.

Intersections of convex sets are convex. Let $\{C_i\}_{i \in I}$ be a family of convex sets and $C := \bigcap_{i \in I} C_i$. If $x, y \in C$, then $x, y \in C_i$ for every $i \in I$. Since each C_i is convex, $\theta x + (1-\theta)y \in C_i$ for all $i \in I$ and all $\theta \in [0, 1]$. Hence $\theta x + (1-\theta)y \in \bigcap_{i \in I} C_i = C$. □

Exercise 2.4. Show that the conic hull of a set of vectors $\operatorname{cone}(A_1, \ldots, A_n)$ is convex.

Definition 2.5 (Polytope)**.** A *polytope* is the convex hull of finitely many points.

Any bounded polyhedron is a polytope.

2.1 Worked examples

Example 2.6 (Checking feasibility via the column cone)**.** Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since $b = 1 \cdot a_1 + 1 \cdot a_3$ with $a_1 = (1, 0)$, $a_3 = (1, 1)$, we have $b \in \operatorname{cone}(a_1, a_2, a_3)$; the certificate is $x = (1, 0, 1) \geq 0$ with $Ax = b$. If instead $b = (-1, 1)$, then $b \notin \operatorname{cone}(a_1, a_2, a_3)$, so the LP is infeasible by Proposition 1.5.

Example 2.7 (Intersection view in $\mathbb{R}^3$)**.** For $A = \mathbf{1}^\top$ and $b = 1$, the feasible set is $\{x \geq 0 \mid x_1 + x_2 + x_3 = 1\}$, the standard 2-simplex. This is a convex shape (in fact, a triangle), illustrating Proposition 1.9 and Proposition 2.3.

Summary Feasibility for standard-form LPs admits two complementary geometric characterizations: (i) the right-hand side b must lie in the cone spanned by the columns of A; and (ii) the feasible x live at the intersection of an affine space with the positive orthant. These yield immediate convexity of the feasible region and, via Carathéodory, sparse feasible representations.

3 LP geometry: inequality form

Definition 3.1 (Inequality form of an LP)**.** An *inequality-form linear program* is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ \text{variable} & x \in \mathbb{R}^n, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

The inequality form is particularly common in the context of convex optimization, as it generalizes well to other convex cones. Geometrically, each constraint $a_i^T x \leq b_i$ describes a *halfspace*, and the feasible set is the intersection of finitely many halfspaces, i.e., a polyhedron.

Proposition 3.2 (Equivalence of forms)**.** *The standard-form LP and inequality-form LP have the same expressive power. In particular:*

(i) Any inequality $Ax \leq b$ can be written as an equality with a slack variable*:*

$$Ax \leq b \iff Ax + s = b, \quad s \geq 0.$$

(ii) Any free variable $x_j \in \mathbb{R}$ *can be expressed as the difference of two nonnegative variables:*

$$x_j = x_j^+ - x_j^-, \quad x_j^+, x_j^- \geq 0.$$

Hence feasible regions of either form are polyhedra.

Proof. (i) If $Ax \le b$, define $s := b - Ax \ge 0$, so $Ax + s = b$. Conversely, if $Ax + s = b$ with $s \ge 0$, then $Ax \le b$.

(ii) For any $x_j \in \mathbb{R}$, set $x_j^+ = \max(x_j, 0)$, $x_j^- = \max(-x_j, 0)$, so $x_j = x_j^+ - x_j^-$ and $x_j^+, x_j^- \ge 0$. □

Example 3.3 (Production planning)**.** Suppose a factory produces n products. Let x_i denote the quantity of product i.

- Each unit of product i requires a_{ji} units of resource j.
- Resource j has availability b_j.
- The production cost per unit is c_i.

The problem is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \\ & 0 \le x \le d, \\ \text{variable} & x \in \mathbb{R}^n, \end{array}$$

where $d \in \mathbb{R}^n$ encodes demand limits.
As an extension, we can include a fixed charge f_i if product i is produced at all. To model this, introduce binary $z_i \in \{0, 1\}$ with $x_i \le M z_i$, giving the mixed-integer linear program

$$\begin{array}{ll} \text{minimize} & c^T x + f^T z \\ \text{subject to} & Ax \le b \\ & 0 \le x \le d, \\ & x_i \le M z_i, \\ \text{variables} & x \in \mathbb{R}^n, z \in \{0, 1\}^n \end{array}$$

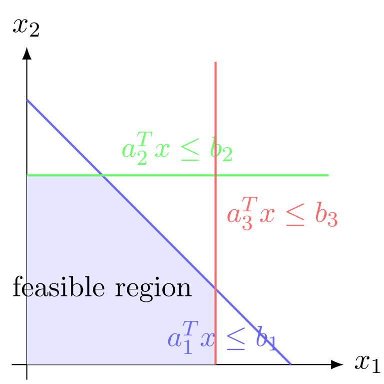

Figure 1: Intersection of halfspaces in $\mathbb{R}^2$ yields a polyhedron.

4 Solutions of a linear program

We will now introduce three ways to characterize the solutions of a linear program, which will help us understand the properties of potential solution, and algorithms to find a solution.

4.1 Active constraints and slacks

Definition 4.1 (Active constraints)**.** Consider the inequality-form feasible region

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid Ax \le b\}.$$

- Constraint i is *active at* x if $a_i^T x = b_i$.
- The *active set* at x is
$$\mathcal{A}(x) := \{\, i \mid a_i^T x = b_i \,\}.$$
- For nonnegativity constraints $x \ge 0$, the bound on coordinate j is active at x if $x_j = 0$.

Introducing *slack variables* gives a dual view of this geometry:

$$Ax \le b \quad \iff \quad Ax + s = b, \quad s \ge 0.$$

Then constraint i is active at x precisely when the corresponding slack variable is zero:

$$a_i^T x = b_i \quad \iff \quad s_i = 0.$$

Example 4.2 (Active vs inactive constraints in $\mathbb{R}^2$)**.** Consider the feasible set

$$\mathcal{F} = \{(x_1, x_2) \mid x_1 \ge 0,\ x_2 \ge 0,\ x_1 + x_2 \le 3\}.$$

- At $x = (0, 3)$, two constraints are active: $x_1 \ge 0$ and $x_1 + x_2 \le 3$.
- At $x = (1, 2)$, only one constraint is active: $x_1 + x_2 \le 3$.

4.2 Extreme points of a convex set

Definition 4.3 (Extreme point)**.** Let $C \subseteq \mathbb{R}^n$ be convex. A point $x \in C$ is an *extreme point* of C if it cannot be written as a nontrivial convex combination of other points in C. That is,

$$x = \theta y + (1 - \theta) z, \quad y, z \in C,\ \theta \in (0, 1) \quad \Longrightarrow \quad y = z = x.$$

Theorem 4.4 (Uniqueness $\implies$ extremality)**.** *If $x^\star$ is the unique minimizer of the linear program*

$$\min_{x \in S} c^T x,$$

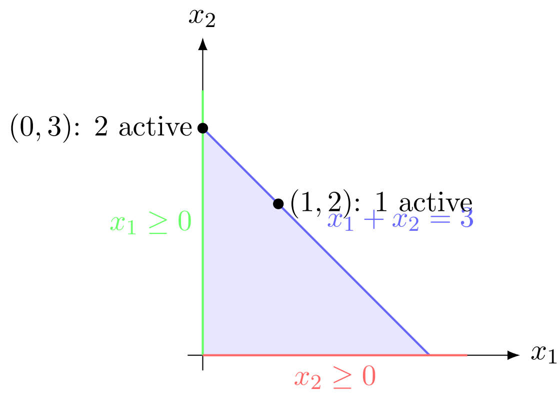

Figure 2: Active vs inactive constraints. At $(0, 3)$ two facets meet (both active); at $(1, 2)$ only one constraint is active.

then $x^\star$ is an extreme point of S.

Proof sketch. Suppose $x^\star$ were not extreme. Then there exist $y, z \in S$, $y \neq z$, and $\theta \in (0, 1)$ such that $x^\star = \theta y + (1 - \theta)z$. By linearity of the objective,

$$c^T x^\star = c^T\big(\theta y + (1 - \theta)z\big) = \theta c^T y + (1 - \theta)c^T z.$$

Since $x^\star$ is optimal, we must have $c^T y = c^T z = c^T x^\star$. Thus y and z are distinct minimizers, contradicting uniqueness. Therefore $x^\star$ must be extreme. $\square$

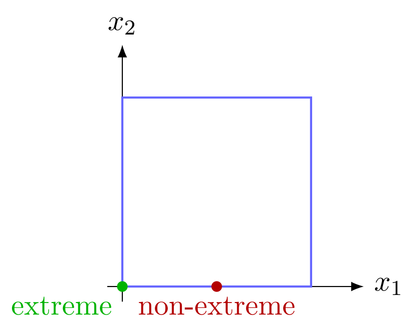

Figure 3: Extreme vs. non-extreme points. Corners of a polytope are extreme; interior edge points are not.

4.3 Vertices and supporting hyperplanes

Definition 4.5 (Vertex)**.** Let $S \subseteq \mathbb{R}^n$. A point $x \in S$ is a *vertex* of S if there exists $c \in \mathbb{R}^n$ such that

$$c^T x < c^T y \quad \text{for all } y \in S \setminus \{x\}.$$

Equivalently, the hyperplane

$$H = \{z \in \mathbb{R}^n \mid c^T z = c^T x\}$$

supports S only at the single point x.

Proposition 4.6. *Every vertex of a convex set is also an extreme point.*

Proof idea. Suppose $x \in S$ is a vertex. Then x is the unique solution to the optimization problem

$$\min_{y \in S} c^T y.$$

By the previous theorem (uniqueness $\implies$ extremality), x must be an extreme point. $\square$

4.4 Basic feasible solutions (BFS)

Definition 4.7 (Basic feasible solution)**.** Consider the standard-form feasible region

$$F = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\},$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank m. A point $x \in F$ is a *basic feasible solution (BFS)* if there exists an index set $S \subset \{1, \dots, n\}$ with $|S| = m$ such that

$$A_S \in \mathbb{R}^{m \times m} \quad \text{is invertible, and} \quad x_S = A_S^{-1} b, \qquad x_{\bar{S}} = 0, \qquad x \geq 0.$$

Here A_S denotes the submatrix of A with columns indexed by S, and $x_{\bar{S}}$ are the coordinates of x not in S.

Definition 4.8 (Neighboring BFS)**.** Two BFS with bases S, S' are called *neighbors* if they share $m - 1$ basic columns:

$$|S \cap S'| = m - 1.$$

Remark 4.9 (Computation of BFS)**.** To enumerate candidate BFS:

1. Pick a set S of m linearly independent columns of A.
2. Solve $x_S = A_S^{-1} b$.
3. Set $x_{\bar{S}} = 0$.
4. If $x \geq 0$, then x is a BFS.

This procedure generates finitely many BFS, which we will soon show are the only candidates for optimal solutions.

5 Equivalence of extreme points, vertices, and BFS

Theorem 5.1. *For the feasible set*

$$F = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\},$$

the following statements are equivalent:

(i) x *is an* extreme point *of* F*.*

(ii) x *is a* vertex *of* F*.*

(iii) x *is a* basic feasible solution (BFS) *of* F*.*

Hence every polyhedron has finitely many extreme points, and the geometric "corners" are precisely the BFS of any standard-form representation.

Proof sketches. (Vertex $\implies$ Extreme). If x is a vertex, then there exists $c \in \mathbb{R}^n$ such that x is the unique minimizer of $\min_{y \in F} c^T y$. By the uniqueness $\implies$ extremality theorem, this implies x is an extreme point.

(Extreme $\implies$ BFS). Suppose $x^\star$ is an extreme point but not basic. Let $S = \{i \mid x_i^\star > 0\}$. If A_S were rank-deficient, there exists $d \neq 0$ with support in S such that $Ad = 0$. Then for sufficiently small $\varepsilon > 0$, both

$$x^+ = x^\star + \varepsilon d, \qquad x^- = x^\star - \varepsilon d$$

are feasible: they satisfy $Ax^\pm = Ax^\star = b$, and nonnegativity is preserved by choosing ε small enough (via the min-ratio rule). Moreover, $x^\star = \frac{1}{2}(x^+ + x^-)$ with $x^+ \neq x^-$. Thus $x^\star$ is not extreme, contradicting the assumption. Therefore A_S must be nonsingular with $|S| = m$, so $x^\star$ is basic.

(BFS $\implies$ Vertex). Let $x^\star$ be a BFS with basis S. Define $c \in \mathbb{R}^n$ by

$$c_i = 0 \ \ (i \in S), \qquad c_i = 1 \ \ (i \notin S).$$

Then $c^T x^\star = 0$. For any other feasible $x \in F$, at least one coordinate outside S is positive, so $c^T x > 0$. Therefore $x^\star$ is the unique minimizer of $\min_{y \in F} c^T y$, hence a vertex. □

Consequences for optimization

- **Corner optimality.** If an LP has a solution and its feasible set has extreme points, then some extreme point is optimal. This explains why LP solvers can focus on corners of the feasible set.
- **Algorithmic viewpoint.** Enumerating all bases is conceptually simple but requires $\binom{n}{m}$ possibilities. The simplex method instead performs a local search, moving between neighboring BFS, improving the objective until no improving neighbor exists.

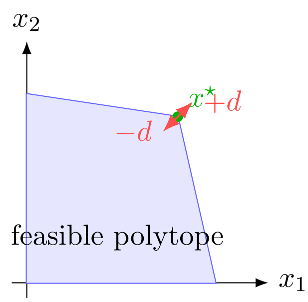

Figure 4: Nullspace perturbation argument for “Extreme $\implies$ BFS”.