Lecture 18 : Robust Optimization

December 1, 2025

Quick Announcements

- Will standardize midterm scores
- Preferences for midterm weight - due on Wednesday
- Homework 5 due on Friday (Dec 5)
- My office hours this week - extended schedule (check Google calendar link)
- Any questions?

Outline for Today and Wednesday

1. Introduction
   - Some Motivating Examples
   - A History Detour
   - Pros and Cons of Probabilistic Models

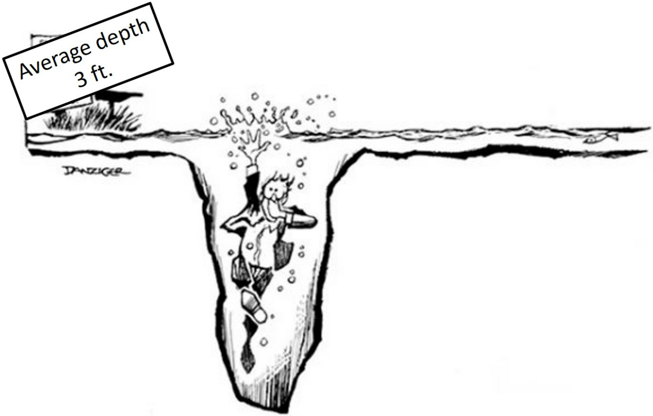
2. Robust Optimization
   - Basic Premises
   - Modeling with Basic Uncertainty Sets
   - Reformulating and Solving Robust Models
   - Extensions
   - Some Applications
   - Distributionally Robust Optimization
   - Calibrating Uncertainty Sets
   - Connections with Other Areas

3. Dynamic Robust Optimization
   - Properly Writing a Robust DP
   - An Inventory Example
   - Tractable Approximations with Decision Rules
   - Some Practical Issues on Bellman Optimality
   - An Application in Monitoring

Introduction

The Flaw of Averages

Optimization based on nominal values can lead to severe pitfalls...



Taken from *"Flaw of averages"* Sam Savage (2009, 2012)

How Robust Are Optimal Solutions?

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- Aharon Ben-Tal and Arkadi Nemirovski: Consider a **real-world scheduling problem** problem (PILOT4) in NETLIB Library
  - One of the constraints is the following linear constraint $\bar{a}^{\mathsf{T}} x \geq b$:

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&-15.79081 \cdot x_{826} - \mathit{8.598819} \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geq 23.387405
\end{aligned}
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- What if these coefficients are just 0.1% inaccurate?
  - i.e., suppose the true a is not $\bar{a}$, but $|a_i - \bar{a}_i| \leq 0.001|\bar{a}_i|$?

- Will the optimal solution to the problem still be feasible?

- How can we test?

How Robust Are Optimal Solutions?

- Original constraint: $\bar{a}^{\mathsf{T}} x \geq b$, optimal solution $x^\star$
- Suppose true $a \in \mathbb{R}^n$ satisfies $|a_i - \bar{a}_i| \leq 0.001|\bar{a}_i|, \forall\, i$
- How to determine if $a^{\mathsf{T}} x^\star \geq b$ holds for true a?

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$$\begin{aligned} \min_{a} \;& a^{\mathsf{T}} x^\star - b \\ \text{s.t.} \;& |a_i - \bar{a}_i| \leq 0.001|\bar{a}_i|, \forall\, i \end{aligned}$$

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- OK, but perhaps we're too conservative?
  - Suppose $a_i = \bar{a}_i + \epsilon_i|\bar{a}_i|$, where $\epsilon_i \sim \text{Uniform}[-0.001, 0.001]$
  - Using Monte-Carlo simulation with 1,000 samples:
    - $\mathbb{P}(\text{infeasible}) = 50\%$, $\mathbb{P}(\text{violation} > 150\%) = 18\%$, $\mathbb{E}[\text{violation}] = 125\%$

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- Disturbing that nominal solutions are likely highly infeasible
- Turns out to be the case for many **NETLIB** problems
- We should **capture uncertainty more explicitly** apriori!

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- Decision Maker (DM) must chose x, without knowing z

- DM incurs a **cost** $C(x, z)$

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- Decision Maker (DM) must chose x, without knowing z
- DM incurs a **cost** $C(x, z)$
- How to model z? How to properly formalize the decision problem?
- "Standard" probabilistic model:
  - There is a unique probability distribution $\mathbb{P}$ for z
  - DM considers an objective: $\min_x \ \mathbb{E}_{z \sim \mathbb{P}}\big[C(x, z)\big]$

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_x \ \mathbb{E}_{z\sim\mathbb{P}}\big[C(x, z)\big]$

- What if there are constraints?

$$f_i\,(x, z) \geq 0, \ \forall\, i \in I$$

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- Need to be a bit more precise in which **sense** we want to satisfy them!
  - expectation constraint: $\mathbb{E}_{\mathbb{P}}[\,f_i\,(x, z)] \geq 0,\; \forall\, i$
  - chance constraint:
    individual: $\mathbb{P}[\,f_i\left(x, z\right) \geq 0\,] \geq 1 - \epsilon,\; \forall\, i$
    joint: $\mathbb{P}[\,f_i\left(x, z\right) \geq 0,\; \forall\, i] \geq 1 - \epsilon$
  - robust (a.s.) constraint: $F\left(x, z\right) \geq 0, \forall\, z$

- Which of these are "easy" to check / enforce?

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- Even if f is "well-behaved," may need more assumptions on $\mathbb{P}$
  - e.g., f convex in x, concave in z
  - log-concave density for chance constraints
  - convex support

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- Where is $\mathbb{P}$ coming from?
- When is it reasonable to assume $\mathbb{P}$ known?
- What if $\mathbb{P}$ is not the actual distribution?
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- Perhaps we have historical samples $z_1, \ldots, z_N$
- Use empirical distribution $\mathbb{P} = \sum_{i=1}^{N} \frac{1}{N}\delta(z_i)$?
- Future like the past...
- ...

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- **Very** popular modeling framework, but...
- Theory challenging when analyzing **complex, real-world** phenomena
  - poor data, changing environments (future $\neq$ past), many agents, ...
- Framework not geared towards **computing decisions**
  - Limited computational tractability, particularly in higher dimensions
- With $C = -u(\cdot)$ (u utility function), unclear if this is a good behavioral model

An Alternative Model of Uncertainty

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- Let's admit **explicitly** that our model of reality is **incorrect**
- From **classical view**: *"we know distribution* $\mathbb{P}$ *for* z*, and solve:* $\min_x \mathbb{E}_{\mathbb{P}}\big[C(x,z)\big]$*"*

  to **robust view**: *"we only know that* $\mathbb{P} \in \mathcal{P}$*, and solve:* $\min_x \max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[C(x,z)\big]$*"*

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Long history of **robust decision-making** and **model misspecification**:

- **Economics:**
  - Knight (1921) - risk vs. Knightian uncertainty, Wald (1939), von Neumann (1944)
  - Savage (1951): minimax regret, Scarf (1958): robust Newsvendor model
  - Schmeidler, Gilboa (1980s): axiomatic frameworks; Ben-Haim (1980s)
  - Hansen & Sargent (2008): "*Robustness*" - robust control in macroeconomics
  - Bergemann & Morris (2012): "*Robust mechanism design*" book, Carroll (2015), ...
- **Engineering and robust control:** Bertsekas (1970s), Doyle (1980s), etc.
- **Computer science:** complexity analysis
- **Statistics:** M-estimators Huber (1981)
- **Operations Research:**
  - Early work by Soyster (1973), Libura (1980), Bard (1984), Kouvelis (1997)
  - **Robust Optimization**: Ben-Tal, Nemirovski, El-Ghaoui ('90s), Bertsimas, Sim ('00s)
  - Two books: Ben-Tal, El-Ghaoui, Nemirovski (2009), Bertsimas, den Hertog (2020)
  - Many tutorials!

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Why robust optimization? (in my view)

1. Very sensible
2. Modest modeling requirements
3. Modest in its premise: *"always under-promises, and over-delivers"*
4. Tractable: quickly becoming "technology"
5. Very sensible results: can rationalize simple rules in complex problems

“Classical” Robust Optimization

"Classical" Robust Optimization (RO)

- Only information about z: values belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$\boxed{(P) \qquad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} \; C(x, z) \\ &\text{s.t. } f_i(x, z) \leq 0, \forall z \in \mathcal{U}, \; \forall i \in I \end{aligned}}$$

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1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x,z)$
   - Other options possible, based on notions of **regret**

- Conservative?

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- Is there a probabilistic interpretation?
  - Objective $= \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{z \sim \mathbb{P}}[C(x, z)]$ where $\mathcal{P}$ is the set of all measures with support $\mathcal{U}$
  - So we are assuming that the only information about $\mathbb{P}$ is the support $\mathcal{U}$

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Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z

What is the optimal value of the following robust LP?

$$\begin{aligned} \min_{x} \max_{a \in \mathcal{U}} \quad & -(x_1 + x_2) \\ \text{such that} \quad & x_1 \leq a_1, && \forall a \in \mathcal{U} \\ & x_2 \leq a_2, && \forall a \in \mathcal{U} \qquad \text{where } \mathcal{U} = \left\{(a_1, a_2) \in [0,1]^2 \; : \; a_1 + a_2 \leq 1\right\} \\ & x_1 + x_2 \leq 1, && \forall a \in \mathcal{U}. \end{aligned}$$

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   Optimal value 0. In RO, **each constraint must be satisfied separately, robustly.**

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$$\boxed{f_i(x, z) \leq 0, \forall\, z \in \mathcal{U}} \quad \Leftrightarrow \quad \boxed{\sup_{z \in \mathcal{U}} f_i(x, z) \leq 0}$$

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   Many RO models are in this *epigraph reformulation*, and focus on constraints

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4. Without loss, we can consider a problem where z only appears in constraints
5. DM only responsible for objective and constraints when $z \in \mathcal{U}$
   - If $z \notin \mathcal{U}$ actually occurs, all bets are off
   - Can extend framework to ensure **gradual** degradation of performance:

     Globalized robust counterparts (Ben-Tal & Nemirovski)

"Classical" Robust Optimization (RO)

$$
(P) \qquad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} \; C(x, z) \\ &\text{s.t. } f_i(x, z) \leq 0, \forall z \in \mathcal{U}, \; \forall i \in I \end{aligned}
$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem
4. Without loss, we can consider a problem where z only appears in constraints
5. DM only responsible for objective and constraints when $z \in \mathcal{U}$
6. Robust model seems to lead to a **difficult** optimization problem
   - For any given x, checking constraints/solving the "adversary" problem may be tough
   - We must also solve our original problem of finding x!

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1. How to model $\mathcal{U}$
2. How to formulate and solve the **robust counterpart**
3. Why is this useful, in theory and in practice

Intuition for Some Basic Uncertainty Sets

- Recall PILOT4; how to build some "safety buffers" for constraint like #372:

$$-15.79081 \cdot x_{826} - \mathit{8.598819} \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} - \ldots$$
$$-0.946049 \cdot x_{916} \geq 23.387405$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^{\mathsf{T}} x \leq b, \ \forall\, z \in \mathcal{U}}$$

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- P is a known matrix; z is primitive uncertainty

- **Q:** Why this more general form?

  A: For modeling flexibility:

  - Suppose the same physical quantity (i.e., coefficient) appears in multiple constraints
  - Can capture "correlations", e.g., with a factor model

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$$\mathcal{U}_{\text{box}} := \{z : -\rho \leq z_i \leq \rho\} = \{z : \|z\|_{\infty} \leq \rho\}$$

"Too conservative?"

- In PILOT4, **robust** solution has objective value within 1% of that of $x^{\star}$
- Recall that $x^{\star}$ would violate this constraint by 450%
- Sometimes we don't sacrifice too much for robustness!

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- Or what if we gave "nature" a **budget** on how many coefficients it could change:

$$\mathcal{U}_{\text{budget}} := \{z \,:\, \|z\|_\infty \leq \rho,\ \|z\|_1 \leq \Gamma\rho\}$$

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- How to formulate the robust counterpart? How to set ρ, Γ? How to use in practice?

Robust Counterpart (RC) for Box Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^{\mathsf{T}} x \leq b, \ \forall\, z \in \mathcal{U}}$$

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Remarks.

- To formulate the RC for (2), we must introduce a set of **auxiliary decision variables** y
  - these are **decision variables**, chosen together with x
- *How many* **auxiliary variables** *are needed to derive the RC for* (2)*?*
- *How many* **constraints** *are needed to derive the RC for* (2)*?*
- *Suppose we were solving* $\min_x \{c^{\mathsf{T}} x \,:\, Ax \le b\}$*, with* $A \in \mathcal{U}_{\text{polyhedral}} \subset \mathbb{R}^{m \times n}$ *and* $D \in \mathbb{R}^{p \times q}$*.*

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  - the RC is still an LP, with $n + m \cdot p$ variables, $m \cdot (1 + p + q)$ constraints

Robust Counterpart (RC) for Ellipsoidal Uncertainty Set

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Intermezzo: $\max \{q^{\mathsf{T}} z : \|z\|_2 \leq \rho\}$ *or* $\max \{q^{\mathsf{T}} z : z^{\mathsf{T}} z \leq \rho^2\}$

Lagrange: $z = q/\lambda$, *and* $\lambda = \|q\|_2/\rho$.

Optimal objective value: $\frac{q^{\mathsf{T}} q}{\lambda} = \rho \|q\|_2$.

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Optimal objective value: $\frac{q^{\mathsf{T}}q}{\lambda} = \rho\|q\|_2$.

Hence robust counterpart (RC) is:

$$\boxed{\bar{a}^{\mathsf{T}}x + \rho\|P^{\mathsf{T}}x\|_2 \le b.}$$

RC for Linear Optimization Problems with Classical Sets

The robust counterpart for $\boxed{(\bar{a} + Pz)^{\mathsf{T}} x \leq b, \ \forall\, z \in \mathcal{U}}$ is:

U-set	$\mathcal{U}$	Robust Counterpart	Tractability
Box	$\|z\|_\infty \leq \rho$	$\bar{a}^{\mathsf{T}} x + \rho \|P^{\mathsf{T}} x\|_1 \leq b$	LO
Ellipsoidal	$\|z\|_2 \leq \rho$	$\bar{a}^{\mathsf{T}} x + \rho \|P^{\mathsf{T}} x\|_2 \leq b$	CQO
Polyhedral	$Dz \leq d$	$\exists y \ : \ \begin{cases} \bar{a}^{\mathsf{T}} x + d^{\mathsf{T}} y \leq b \\ D^{\mathsf{T}} y = P^{\mathsf{T}} x \\ y \geq 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leq \rho \\ \|z\|_1 \leq \Gamma \end{cases}$	$\exists y \ : \ \bar{a}^{\mathsf{T}} x + \rho \|y\|_1 + \Gamma \|P^{\mathsf{T}} x - y\|_\infty \leq b$	LO

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Polyhedral	$Dz \leq d$	$\exists y \; : \; \begin{cases} \bar{a}^\top x + d^\top y \leq b \\ D^\top y = P^\top x \\ y \geq 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leq \rho \\ \|z\|_1 \leq \Gamma \end{cases}$	$\exists y \; : \; \bar{a}^\top x + \rho \|y\|_1 + \Gamma \|P^\top x - y\|_\infty \leq b$	LO

- Problems above can be handled by large-scale modern solvers, e.g., Gurobi
- Some software now also handling automatic problem re-formulation
- If some of the decisions x are integer, problems above become MI-LPs/CQPs
- Several important extensions

Extensions

1. **Uncertainty in the right-hand side:** $(\bar{a} + P\boldsymbol{z})^\mathsf{T} x \leq b + p^\mathsf{T} z, \; \forall\, \boldsymbol{z} \in \mathcal{U}$

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2. **General convex uncertainty set:** $\mathcal{U} = \{z \,:\, h_k(z) \leq 0,\ \forall k \in K\}$, $h_k(\cdot)$ convex?

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2. **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leq 0, \forall k \in K\}$, $h_k(\cdot)$ convex?

   $\Leftrightarrow \quad \exists \{w_k, u_k\}_{k \in K} : \bar{a}^\mathsf{T} x + \sum_k u_k h_k^*(w_k / u_k) \leq b, \sum_k w^k = P^\mathsf{T} x, u \geq 0.$

   h_k^* is Fenchel conjugate of h_k. Works if we have a tractable representation of h_k^*.

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Extensions

1. **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^{\mathsf{T}}x \leq b + p^{\mathsf{T}}z,\ \forall\, z \in \mathcal{U}$

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   Constraint is now **linear** in (z, d) and $\mathcal{U}^+$ is a convex uncertainty set - apply #2.

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   Tractable if f has "easy" piece-wise description: $f(x, z) = \max_{k \in K} f_k(x, z)$, where f_k corresponds to one of cases above

Used in many applications

- supply chain management [Ben-Tal et al., 2005, Bertsimas and Thiele, 2006, ...]
- logistics and transportation [Baron et al., 2011, ...]
- scheduling [Lin et al., 2004, Yamashita et al., 2007, Mittal et al., 2014, ...]
- revenue management [Perakis and Roels, 2010, Adida and Perakis, 2006, ...]
- project management [Wiesemann et al., 2012, Ben-Tal et al., 2009, ...]
- energy generation and distribution [Zhao et al., 2013, Lorca and Sun, 2015, ...]
- portfolio optimization [Goldfarb and Iyengar, 2003, Tütüncü and Koenig, 2004, Ceria and Stubbs, 2006, Pinar and Tütüncü, 2005, Bertsimas and Pachamanova, 2008, ...]
- healthcare [Borfeld et al., 2008, Hanne et al., 2009, Chen et al., 2011, I., Trichakis, Yoon (2018), ...]
- humanitarian [Uichano 2017, den Hertog et al., 2019, ...]

Two Important Caveats for Robust Models

Example: Facility Location Problem (Baron et al. 2011)

Need to decide where to open facilities, how much capacity to install, and how to assign customer demands over a future planning horizon, in order to maximize profits.

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\text{subject to} \quad & \sum_{i\in\mathcal{F}} X_{ij\tau} \le {\color{red}D_{j\tau}}, \qquad j\in\mathcal{N},\ \tau\in\mathcal{T}, \\
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& X \ge 0,\ \ I\in\{0,1\}^{|\mathcal{F}|}
\end{aligned}$$

Parameters

$\mathcal{T}$: discrete planning horizon, indexed by τ
$\mathcal{F}$: potential facility locations, indexed by i
$\mathcal{N}$: demand node locations, indexed by j
p: unit price of goods
c_i: cost per unit of production at facility i
C_i: cost per unit of capacity for facility i
K_i: cost of opening a facility at location i
c^s_{ij}: cost of shipping units from i to j
$D_{j\tau}$: demand in period τ at location j

Decision variables

$X_{ij\tau}$: quantity of demand j in period τ satisfied by i
$P_{i\tau}$: quantity produced at facility i in period τ
I_i: whether facility i is open (0/1)
Z_i: capacity of facility i if open

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Step 2. Identify all uncertain parameters and model the uncertainty set $\mathcal{U}$.
Baron et al. 2011 captured uncertain demands with an ellipsoidal uncertainty set:

$$
\mathcal{U} = \left\{ D\in\mathbb{R}^{|\mathcal{N}|\cdot|\mathcal{T}|} \;\middle|\; \sum_{j\in\mathcal{N}}\sum_{t\in\mathcal{T}}\left(\frac{D_{jt}-\bar{D}_{jt}}{\epsilon_t \bar{D}_{jt}}\right)^2 \le \rho^2 \right\},
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$\{\bar{D}_{jt}\}_{j\in\mathcal{N};t\in\mathcal{T}}$ are “nominal” demands, ϵ_t is allowed deviation (%), ρ is the size of the ellipsoid

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Equivalently, can write $D_{jt} = \bar{D}_{jt}(1 + \epsilon_t \cdot z_{jt})$, where $z \in \mathcal{U} = \{z \in \mathbb{R}^{|\mathcal{N}|\cdot|\mathcal{T}|} : \|z\|_2 \le \rho\}$

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Step 3. Derive robust counterpart for the problem. Here, a Conic Quadratic program.

Compare Two Models

Our initial model, with **decisions for quantities** X:

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Another model, with **decisions for fractions of demands** Y:

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- For fixed D, are these **deterministic/nominal** models **equivalent**?

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$$
\begin{aligned}
\max_{Y,I,Z,P} \quad & \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{N}}(p-c^s_{ij})Y_{ij\tau}D_{j\tau} - \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}} c_i P_{i\tau} - \sum_{i\in\mathcal{F}}(C_i Z_i - K_i I_i) \\
\text{subject to} \quad & \sum_{i\in\mathcal{F}} Y_{ij\tau} \le 1, \qquad j\in\mathcal{N},\ \tau\in\mathcal{T}, \\
& \sum_{j\in\mathcal{N}} Y_{ij\tau}D_{j\tau} \le P_{i\tau}, \qquad i\in\mathcal{F},\ \tau\in\mathcal{T}, \\
& P_{i\tau} \le Z_i,\ \ Z_i \le M\cdot I_i, \qquad i\in\mathcal{F},\ \tau\in\mathcal{T} \\
& Y \ge 0,\ \ I\in\{0,1\}^{|\mathcal{F}|}
\end{aligned} \tag{3}
$$

- For fixed D, are these **deterministic/nominal** models **equivalent**? **Yes!**
- Are their **robust counterparts equivalent**?

Compare Two Models

Our initial model, with **decisions for quantities** X:

$$
\begin{aligned}
\max_{X,I,Z,P} \quad & \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{N}}(p-c^s_{ij})X_{ij\tau} - \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}} c_i P_{i\tau} - \sum_{i\in\mathcal{F}}(C_i Z_i + K_i I_i) \\
\text{subject to} \quad & \sum_{i\in\mathcal{F}} X_{ij\tau} \le D_{j\tau}, \qquad j\in\mathcal{N},\ \tau\in\mathcal{T}, \\
& \sum_{j\in\mathcal{N}} X_{ij\tau} \le P_{i\tau}, \qquad i\in\mathcal{F},\ \tau\in\mathcal{T}, \\
& P_{i\tau} \le Z_i,\ \ Z_i \le M\cdot I_i, \qquad i\in\mathcal{F},\ \tau\in\mathcal{T} \quad (M \text{ is a large constant}) \\
& X \ge 0,\ \ I\in\{0,1\}^{|\mathcal{F}|}
\end{aligned}
$$

Another model, with **decisions for fractions of demands** Y:

$$
\begin{aligned}
\max_{Y,I,Z,P} \quad & \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{N}}(p-c^s_{ij})Y_{ij\tau}D_{j\tau} - \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}} c_i P_{i\tau} - \sum_{i\in\mathcal{F}}(C_i Z_i - K_i I_i) \\
\text{subject to} \quad & \sum_{i\in\mathcal{F}} Y_{ij\tau} \le 1, \qquad j\in\mathcal{N},\ \tau\in\mathcal{T}, \\
& \sum_{j\in\mathcal{N}} Y_{ij\tau}D_{j\tau} \le P_{i\tau}, \qquad i\in\mathcal{F},\ \tau\in\mathcal{T}, \\
& P_{i\tau} \le Z_i,\ \ Z_i \le M\cdot I_i, \qquad i\in\mathcal{F},\ \tau\in\mathcal{T} \\
& Y \ge 0,\ \ I\in\{0,1\}^{|\mathcal{F}|}
\end{aligned}
\tag{3}
$$

- For fixed D, are these **deterministic/nominal** models **equivalent**? **Yes!**
- Are their **robust counterparts equivalent**? **No!**
  - The feasible set in the second formulation is **larger**
  - Second formulation implements ordering quantities that **depend on demand**!

The **robust counterparts** of **equivalent** deterministic models **may be different**!

You should always try to allow your formulation to be as flexible as possible!

Another Caveat...

Are Robust Solutions "Efficient"?

$$\max_{x \in \mathcal{X}} \quad \min_{u \in \mathcal{U}} \; u^{\mathsf{T}} x$$

- Feasible set of solutions $\mathcal{X} = \left\{x \in \mathbb{R}^n : Ax \leq b\right\}$
- Uncertainty set of objective coefficients $\mathcal{U} = \{u \in \mathbb{R}^n : Du \geq d\}$

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- Uncertainty set of objective coefficients $\mathcal{U} = \{u \in \mathbb{R}^n : Du \geq d\}$

- Classical RO framework results in
  - Optimal value $J^\star_{\text{RO}}$
  - Set of robustly optimal solutions

$$X^{\text{RO}} = \left\{x \in \mathcal{X} : \exists\, y \geq 0 \text{ such that } D^\mathsf{T} y = x, \quad y^\mathsf{T} d \geq J^\star_{\text{RO}}\right\}$$

Set of Robustly Optimal Solutions

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- What if an uncertainty scenario materializes that does not correspond to the worst-case?
- Are there any guarantees that no other solution $\bar{x}$ exists that, apart from protecting us from worst-case scenarios, also performs better overall, under all possible uncertainty realizations?

Pareto Robustly Optimal solutions (I. & Trichakis 2014)

$$\max_{x \in \mathcal{X}} \quad \min_{u \in \mathcal{U}} u^\mathsf{T} x \tag{4}$$

Definition

A solution x is called a **Pareto Robustly Optimal (PRO) solution** for Problem (4) if

(a) it is robustly optimal, i.e., $x \in X^{\mathrm{RO}}$, and

(b) there is no $\bar{x} \in \mathcal{X}$ such that

$$u^\mathsf{T}\bar{x} \geq u^\mathsf{T}x, \quad \forall u \in \mathcal{U}, \quad \text{and}$$
$$\bar{u}^\mathsf{T}\bar{x} > \bar{u}^\mathsf{T}x, \quad \text{for some } \bar{u} \in \mathcal{U}.$$

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$$\bar{u}^{\mathsf{T}} \bar{x} > \bar{u}^{\mathsf{T}} x, \quad \text{for some } \bar{u} \in \mathcal{U}.$$

- $X^{\mathrm{PRO}} \subseteq X^{\mathrm{RO}}$: set of all PRO solutions

Some questions

- Given a RO solution, is it also PRO?
- How can one find a PRO solution?
- Can we optimize over X^{PRO}?
- Can we characterize X^{PRO}?
  - Is it non-empty?
  - Is it convex?
  - When is $X^{\mathrm{PRO}} = X^{\mathrm{RO}}$?
- How does the notion generalize in other RO formulations?

Finding PRO solutions

Theorem

Given a solution $x \in X^{\mathrm{RO}}$ and an arbitrary point $\bar{p} \in \mathrm{ri}(\mathcal{U})$, consider the following linear optimization problem:

$$\begin{array}{ll} \text{maximize} & \bar{p}^\mathsf{T} y \\ \text{subject to} & y \in \mathcal{U}^g \\ & x + y \in \mathcal{X}. \end{array}$$

Then, either

- $\mathcal{U}^* \{y \in \mathbb{R}^n \,:\, y^\mathsf{T} u \geq 0, \forall\, u \in \mathcal{U}\}$ is the dual of $\mathcal{U}$

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Then, either

- *the optimal value is zero and $x \in X^{\mathrm{PRO}}$, or*

- $\mathcal{U}^* \{y \in \mathbb{R}^n : y^{\mathsf{T}} u \geq 0, \forall\, u \in \mathcal{U}\}$ is the dual of $\mathcal{U}$

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Then, either

- *the optimal value is zero and $x \in X^{\mathrm{PRO}}$, or*
- *the optimal value is strictly positive and $\bar{x} = x + y^\star \in X^{\mathrm{PRO}}$, for any optimal $y^\star$.*

- $\mathcal{U}^* \{y \in \mathbb{R}^n : y^{\mathsf{T}} u \geq 0, \forall\, u \in \mathcal{U}\}$ is the dual of $\mathcal{U}$

Remarks

- Finding a point $\bar{u} \in \text{ri}(\mathcal{U})$ can be done efficiently using LP techniques
- Testing whether $x \in X^{\mathrm{RO}}$ is no harder than solving the classical RO problem in this setting
- Finding a PRO solution $x \in X^{\mathrm{PRO}}$ is no harder than solving the classical RO problem in this setting

Corollaries

- If $\bar{u} \in \mathrm{ri}(\mathcal{U})$, all optimal solutions to the problem below are PRO:

$$\begin{array}{ll} \text{maximize} & \bar{u}^\mathsf{T} x \\ \text{subject to} & x \in X^{\mathrm{RO}} \end{array}$$

- If $0 \in \mathrm{ri}(\mathcal{U})$, then $X^{\mathrm{PRO}} = X^{\mathrm{RO}}$

- If $\bar{u} \in \mathrm{ri}(\mathcal{U})$, then $X^{\mathrm{PRO}} = X^{\mathrm{RO}}$ if and only if the optimal value of this LP is zero:

$$\begin{array}{ll} \text{maximize} & \bar{u}^\mathsf{T} y \\ \text{subject to} & x \in X^{\mathrm{RO}} \\ & y \in \mathcal{U}^g \\ & x + y \in \mathcal{X} \end{array}$$

Optimizing over / Understanding X^{PRO}

- Secondary objective r: can we solve

$$\begin{array}{ll} \text{maximize} & r^{\mathsf{T}}x \\ \text{subject to} & x \in X^{\mathrm{PRO}}? \end{array}$$

- Interesting case: $X^{\mathrm{RO}} \neq X^{\mathrm{PRO}}$

Optimizing over / Understanding X^{PRO}

- Secondary objective r: can we solve

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Proposition

X^{PRO} is not necessarily convex.

- $\mathcal{X} = \{x \in \mathbb{R}^4_+ \ : \ x_1 \leq 1,\, x_2 + x_3 \leq 6,\, x_3 + x_4 \leq 5,\, x_2 + x_4 \leq 5\}$
- $\mathcal{U} = \mathrm{conv}\Big(\big\{e_i,\, i \in \{1, \ldots, 4\}\big\}\Big)$
- $J^\star_{\mathrm{RO}} = 1$, and $X^{\mathrm{RO}} = \{x \in X \ : \ x \geq \mathbf{1}\}$
- $x^1 = \begin{bmatrix}1 & 2 & 4 & 1\end{bmatrix}^{\mathsf{T}}$, $x^2 = \begin{bmatrix}1 & 4 & 2 & 1\end{bmatrix}^{\mathsf{T}} \in X^{\mathrm{PRO}}$
- $0.5\, x^1 + 0.5\, x^2$ is Pareto dominated by $\begin{bmatrix}1 & 3 & 3 & 2\end{bmatrix}^{\mathsf{T}} \in X^{\mathrm{RO}}$.

Optimizing over / Understanding X^{PRO}

- Secondary objective r: can we solve

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Proposition

If $X^{\mathrm{RO}} \neq X^{\mathrm{PRO}}$, then $X^{\mathrm{PRO}} \cap \mathrm{ri}(X^{\mathrm{RO}}) = \emptyset$.

- Whether solution to nominal RO is PRO depends on algorithm used for solving LP

- Simplex **better for RO problems** than interior point methods

What Are The Gains?

Example (Portfolio)

- $n + 1$ assets, with returns r_i
- $r_i = \mu_i + \sigma_i \zeta_i$, $i = 1, \ldots, n$, $r_{n+1} = \mu_{n+1}$
- ζ unknown, $U = \{\zeta \in \mathbb{R}^n : -\mathbf{1} \leq \zeta \leq \mathbf{1}, \mathbf{1}^\mathsf{T}\zeta = 0\}$
- Objective: select weights x to maximize worst-case portfolio return

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- Objective: select weights x to maximize worst-case portfolio return

Example (Inventory)

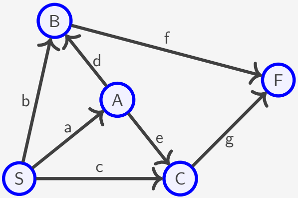
- One warehouse, N retailers where uncertain demand is realized
- Transportation, holding costs and profit margins differ for each retailer
- Demand driven by market factors $d_i = d_i^0 + q_i^\mathsf{T} z, \quad i = 1, \ldots, N$
- Market factors z are uncertain

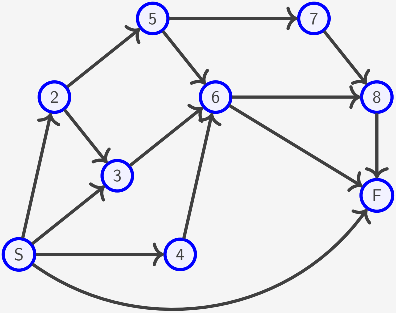
$$z \in \mathcal{U} = \{z \in \mathbb{R}^N : -b \cdot \mathbf{1} \leq z \leq b \cdot \mathbf{1},\ -B \leq \mathbf{1}^\mathsf{T} z \leq B\}$$

Numerical experiments

Example (Project management)

- A PERT diagram given by directed, acyclic graph $G = (\mathcal{N}, \mathcal{E})$
- $\mathcal{N}$ are project events, $\mathcal{E}$ are project activities / tasks





Numerical experiments

Example (Project management)

- A PERT diagram given by directed, acyclic graph $G = (\mathcal{N}, \mathcal{E})$
- $\mathcal{N}$ are project events, $\mathcal{E}$ are project activities / tasks
- Task $e \in \mathcal{E}$ has uncertain duration $\tau_e = \tau_e^0 + \delta_e$

$$\delta \in \mathcal{U} := \left\{\delta \in \mathbb{R}_+^{|\mathcal{E}|} \; : \; \delta \leq b \cdot \mathbf{1}, \;\; \mathbf{1}^\mathsf{T}\delta_e \leq B\right\}$$

- Task $e \in \mathcal{E}$ can be expedited by allocating a budgeted resource x_e

$$\tau_e = \tau_e^0 + \delta_e - x_e$$
$$\mathbf{1}^\mathsf{T} x \leq C$$

- Goal: find resource allocation x to minimize worst-case completion time

Results – finance and inventory examples (10K instances)

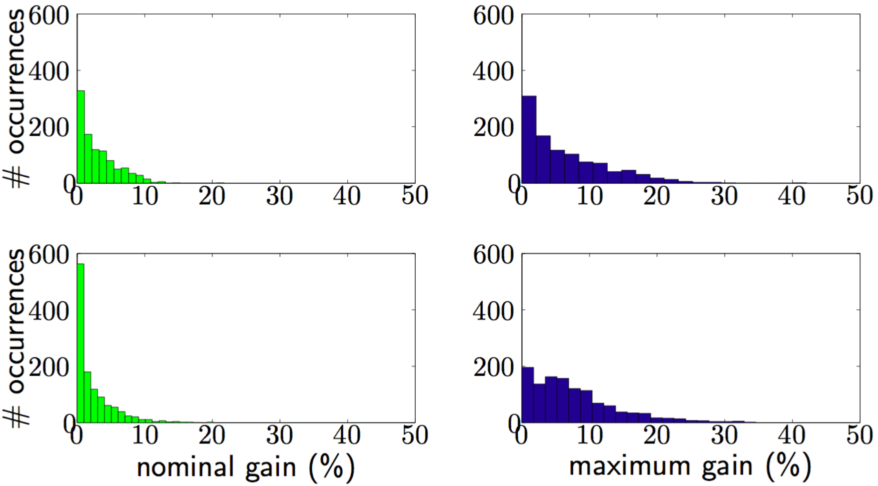
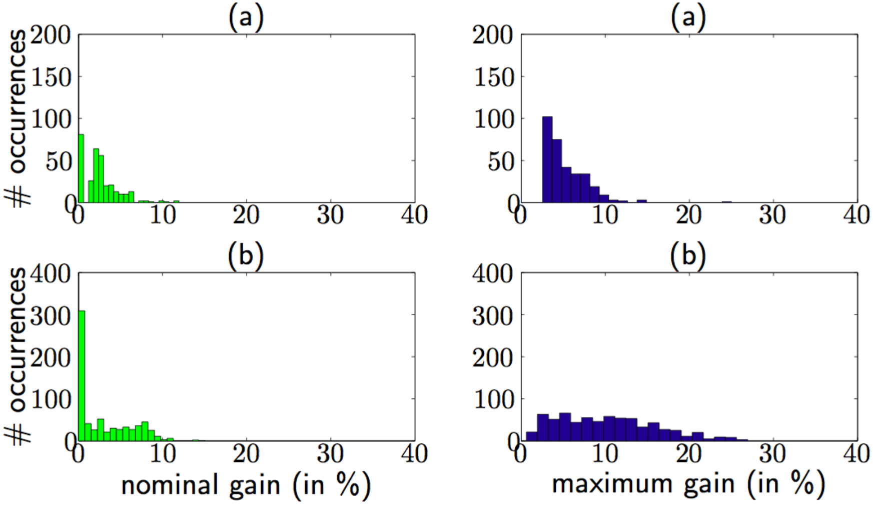


Figure: TOP: portfolio example. BOTTOM: inventory example. LEFT: performance gains in nominal scenario. RIGHT: maximal performance gains.

Results – two project management networks (10K instances)



Careful To Avoid Naïve Inefficiencies In Robust Models!

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