Duality

Lecture 4

October 1, 2025

Motivation

Consider an optimization problem

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3. Suppose one constraint is: $a_i^{\mathsf{T}} x \leq 0$ where $a_i \in \mathcal{A}$ are unknown parameters. *How to find an x that is feasible* **for any** *$a_i \in \mathcal{A}$?*

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3. Suppose one constraint is: $a_i^\mathsf{T} x \leq 0$ where $a_i \in \mathcal{A}$ are unknown parameters. *How to find an x that is feasible* **for any** $a_i \in \mathcal{A}$*?*

4. You are offered a bit more of b_i, for a "suitable price". *Is the deal worthwhile?*

Duality theory will provide answers to these questions (and more)

Outline

- Consider a **primal** optimization problem:

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- In the process, will uncover some **fundamental ideas in optimization**:

separation of convex sets $\Longrightarrow$ **Farkas Lemma** $\Longrightarrow$ **strong duality**

Deriving Lower Bounds

Consider a linear optimization problem in the most general form possible:

Primal Problem

$$
\begin{array}{rlll}
(\mathcal{P}) \text{ minimize}_x & c^\mathsf{T}x & & \\
\text{such that} & a_i^\mathsf{T}x \geq b_i, & \forall i \in I_{\text{ge}}, & \\
& a_i^\mathsf{T}x \leq b_i, & \forall i \in I_{\text{le}}, & \\
& a_i^\mathsf{T}x = b_i, & \forall i \in I_{\text{eq}}, & \\
& x_j \geq 0, & \forall j \in J_p, & \\
& x_j \leq 0, & \forall j \in J_n, & \\
& x_j \text{ free}, & \forall j \in J_f & \\
\text{variable} & x \in \mathbb{R}^n. & &
\end{array} \tag{1}
$$

Note the mnemonic encoding...

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\qquad
A = \begin{bmatrix} A^{\text{le}} \\ A^{\text{ge}} \\ A^{\text{eq}} \end{bmatrix} \in \mathbb{R}^{m \times n}
\quad
b = \begin{bmatrix} b^{\text{le}} \\ b^{\text{ge}} \\ b^{\text{eq}} \end{bmatrix} \in \mathbb{R}^{m}
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Definition

We will refer to this as the **primal problem** or problem $(\mathcal{P})$.

Let P denote its feasible set (a polyhedron), and $p^\star$ denote its optimal value.

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$(\mathcal{P})$ is a minimization; we seek **valid lower bounds** on $(\mathcal{P})$. *Any ideas?*

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General principle: (i) relax "complicating" constraints; (ii) try "simple" penalty

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For every constraint i, have a **penalty** λ_i

Construct the **lower bound** as the **Lagrangean**:

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\mathcal{L}(x, \lambda) = c^\mathsf{T} x - \sum_{i=1}^{m} \lambda_i (a_i^\mathsf{T} x - b_i) = c^\mathsf{T} x - \lambda^\mathsf{T}(Ax - b)
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Note: we relaxed the complicating constraints, $a_i^\mathsf{T} x$? b_i, and used a linear penalty

Not apriori clear that this will give us very good bounds...

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We want the Lagrangean to give us **a valid lower bound**:

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\lambda_i \leq 0, & \forall i \in I_{\text{le}} \\
\lambda_i \text{ free}, & \forall i \in I_{\text{eq}}.
\end{array}
\right\} \Leftrightarrow \lambda \in \Lambda \tag{2}
$$

Deriving Lower Bounds

Summarizing... any $\lambda \in \Lambda$ produces **a valid lower bound**:

$$\mathcal{L}(x, \lambda) = c^{\mathsf{T}}x - \lambda^{\mathsf{T}}(Ax - b) \leq c^{\mathsf{T}}x, \forall x \in P.$$

How can we get a lower bound on the primal's **optimal value** *$p^\star$?*

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Claim

The function $g : \Lambda \to \mathbb{R}$ defined as:

$$\begin{aligned} g(\lambda) := min_x \ & \mathcal{L}(x, \lambda) \\ s.t. \ & x_j \geq 0, \ \forall j \in J_p \\ & x_j \leq 0, \ \forall j \in J_n \\ & x_j \ free, \ \forall j \in J_f \end{aligned} \tag{3}$$

satisfies $g(\lambda) \leq p^\star$ for any $\lambda \in \Lambda$.

Note: including the sign constraints on x in this optimization improves the lower bound!

Deriving Lower Bounds

Let us analyze this further:

$$\begin{aligned} g(\lambda) = \min_x \ & \mathcal{L}(x, \lambda) & = \min_x \ & \left[\lambda^\mathsf{T} b + (c^\mathsf{T} - \lambda^\mathsf{T} A)x\right] \\ \text{s.t. } & x_j \geq 0, \ \forall j \in J_p, & \text{s.t. } & x_j \geq 0, \ \forall j \in J_p, \\ & x_j \leq 0, \ \forall j \in J_n, & & x_j \leq 0, \ \forall j \in J_n, \\ & x_j \text{ free}, \ \forall j \in J_f & & x_j \text{ free}, \ \forall j \in J_f \end{aligned}$$

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\end{aligned}
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$$
g(\lambda) = \begin{cases} \lambda^{\mathsf{T}} b, & \text{if } \lambda^{\mathsf{T}} A_j \leq c_j, \forall j \in J_p \textbf{ and } \lambda^{\mathsf{T}} A_j \geq c_j, \forall j \in J_n \textbf{ and } \lambda^{\mathsf{T}} A_j = c_j, \forall j \in J_f \\ -\infty, & \text{otherwise.} \end{cases}
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Deriving the Dual Problem

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is **a valid lower bound** on the primal **optimal value**: $g(\lambda) \leq p^\star$ for any $\lambda \in \Lambda$.

How can we get the **best** *lower bound?*

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$$\underset{\lambda \in \Lambda}{\text{maximize}}\; g(\lambda) \tag{4}$$

This is equivalent to the following optimization problem:

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This is equivalent to the following optimization problem:

Dual Problem

$$\begin{aligned} \text{maximize} \quad & \lambda^{\mathsf{T}} b \\ \text{subject to} \quad & \lambda_i \geq 0, && \forall i \in I_{\text{ge}}, \\ & \lambda_i \leq 0, && \forall i \in I_{\text{le}}, \\ & \lambda_i \text{ free}, && \forall i \in I_{\text{eq}}, \\ & \lambda^{\mathsf{T}} A_j \leq c_j, && \forall j \in J_p, \\ & \lambda^{\mathsf{T}} A_j \geq c_j, && \forall j \in J_n, \\ & \lambda^{\mathsf{T}} A_j = c_j, && \forall j \in J_f. \end{aligned} \tag{5}$$

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Definition

This is the **dual** of $(\mathcal{P})$, which we will also refer to as $(\mathcal{D})$. We denote its feasible set with D and its optimal value with $d^\star$.

Note: The dual is also a linear optimization problem!

Primal-Dual Pair

Primal-Dual Pair of Problems

Primal $(\mathcal{P})$

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\underset{x}{\text{minimize}} & c^\mathsf{T} x & & \\
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& x_j \geq 0, & \forall j \in J_p \\
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\text{variables} & x \in \mathbb{R}^n & &
\end{array}
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Dual $(\mathcal{D})$

$$
\begin{array}{rll}
\underset{\lambda}{\text{maximize}} & \lambda^\mathsf{T} b & \\
& \lambda_i \geq 0, & \forall i \in I_{\text{ge}} \\
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Primal-Dual Pair

Primal-Dual Pair of Problems

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Recall the procedure for deriving the dual:

- a dual decision variable λ_i for every primal constraint (except variable signs)
- constrain λ_i to ensure lower bound: λ_i ? 0
- for every primal decision x_j, add a dual constraint in the form $\lambda^\mathsf{T} A_j$? c_j (involving the column A_j and the objective coefficient c_j corresponding to λ_i)

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Exercise

Rewrite the dual problem as a minimization problem and construct its dual.

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Exercise

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Theorem (For LPs, the dual of the dual is the primal)

If we transform the dual of a linear optimization problem into an equivalent minimization problem and form its dual, we obtain a problem equivalent to the primal.

Rules for Constructing the Dual of Any LP

Consider any linear optimization problem (minimization/maximization):

$$\begin{array}{rrl} \underset{x}{\text{minimize}} \;\; / \;\; \underset{x}{\text{maximize}} & & c^\mathsf{T} x \\ (\lambda \rightarrow) & Ax \overset{\leq}{\underset{\geq}{=}} & b \\ & x \overset{\leq}{\underset{\geq}{=}} & 0 \end{array} \tag{7}$$

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R1: A dual variable λ_i for every constraint, i.e., every row a_i^{T} of A.
λ_i **free** for **equality** constraints ($a_i^{\mathsf{T}}x = b_i$). Otherwise: λ_i ? 0.

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R2: In the dual, add a constraint for *every primal variable* x_j
If x_j is **free**, write this as $\lambda^\mathsf{T} A_j = c_j$. Otherwise: $\lambda^\mathsf{T} A_j$? c_j.

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R3: To determine the signs ?, use this rule of thumb:

the dual variable λ_i is the (sub)gradient of the optimal objective value with respect to the constraint's right-hand-side b_i

Rules for Constructing the Dual of Any LP

Consider any linear optimization problem (minimization/maximization):

$$\begin{aligned} \underset{x}{\text{minimize}} \quad / \quad \underset{x}{\text{maximize}} \quad & c^\mathsf{T} x \\ ({\color{red}\lambda} \rightarrow) \quad & Ax \overset{\leq}{\underset{\geq}{=}} b \\ & x \overset{\leq}{\underset{\geq}{=}} 0 \end{aligned} \tag{7}$$

R1: A dual variable ${\color{red}\lambda_i}$ for every constraint, i.e., every row a_i^T of A.
${\color{red}\lambda_i}$ **free** for **equality** constraints ($a_i^\mathsf{T} x = b_i$). Otherwise: ${\color{red}\lambda_i}$? 0.

R2: In the dual, add a constraint for *every primal variable* x_j
If x_j is **free**, write this as ${\color{red}\lambda}^\mathsf{T} A_j = c_j$. Otherwise: ${\color{red}\lambda}^\mathsf{T} A_j$? c_j.

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the dual variable ${\color{red}\lambda_i}$ is the (sub)gradient of the optimal objective value with respect to the constraint's right-hand-side b_i

- in a minimization, for a "$\leq$" constraint, the dual variable is ≤ 0
- in a minimization, for a "$\geq$" constraint, the dual variable is ≥ 0
- in a maximization, for a "$\leq$" constraint, the dual variable is ≥ 0
- in a maximization, for a "$\geq$" constraint, the dual variable is ≤ 0.

Example 1

$$
\begin{aligned}
(\mathcal{P}) \quad \max\ & 3x_1 + 2x_2 \\
\text{s.t. } & x_1 + 2x_2 \leq 4 && (1) \\
& 3x_1 + 2x_2 \geq 6 && (2) \\
& x_1 - x_2 = 1 && (3) \\
& x_1,\ x_2 \geq 0.
\end{aligned}
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\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{D}) \quad & \min\ 4y_1 + 6y_2 + y_3 \\
& \text{s.t. } y_1 + 3y_2 + y_3 \geq 3, \\
& \qquad 2y_1 + 2y_2 - y_3 \geq 2, \\
& \qquad y_1 \geq 0, \quad y_2 \leq 0, \quad y_3 \text{ free.}
\end{aligned}
$$

Some Quick Results

Theorem ("Duals of equivalent primals")

If we transform a primal P_1 into an equivalent formulation P_2 by:

- *replacing a free variable x_i with $x_i = x_i^+ - x_i^-$,*
- *replacing an inequality with an equality by introducing a slack variable,*
- *removing linearly dependent rows $a_i^\intercal$ for a* **feasible** *LP in standard form,*

then the duals of (P_1) and (P_2) are **equivalent**, *i.e., they are either both infeasible or they have the same optimal objective.*

Weak duality

Primal $(\mathcal{P})$			**Dual** $(\mathcal{D})$		
minimize_x	$c^\mathsf{T} x$		maximize_λ	$\lambda^\mathsf{T} b$	
$(\lambda_i \to)$	$a_i^\mathsf{T} x \geq b_i,$	$\forall i \in I_{\text{ge}},$		$\lambda_i \geq 0,$	$\forall i \in I_{\text{ge}},$
$(\lambda_i \to)$	$a_i^\mathsf{T} x \leq b_i,$	$\forall i \in I_{\text{le}},$		$\lambda_i \leq 0,$	$\forall i \in I_{\text{le}},$
$(\lambda_i \to)$	$a_i^\mathsf{T} x = b_i,$	$\forall i \in I_{\text{eq}},$		λ_i free,	$\forall i \in I_{\text{eq}},$
	$x_j \geq 0,$	$\forall j \in J_p,$	$(x_j \to)$	$\lambda^\mathsf{T} A_j \leq c_j,$	$\forall j \in J_p,$
	$x_j \leq 0,$	$\forall j \in J_n,$	$(x_j \to)$	$\lambda^\mathsf{T} A_j \geq c_j,$	$\forall j \in J_n,$
	x_j free,	$\forall j \in J_f.$	$(x_j \to)$	$\lambda^\mathsf{T} A_j = c_j,$	$\forall j \in J_f.$

Weak duality

$$
\begin{array}{lll|lll}
& \textbf{Primal } (\mathcal{P}) & & & \textbf{Dual } (\mathcal{D}) & \\
\text{minimize}_x & c^\mathsf{T} x & & \text{maximize}_\lambda & \lambda^\mathsf{T} b & \\
(\lambda_i \rightarrow) & a_i^\mathsf{T} x \geq b_i, & \forall i \in I_{\text{ge}}, & & \lambda_i \geq 0, & \forall i \in I_{\text{ge}}, \\
(\lambda_i \rightarrow) & a_i^\mathsf{T} x \leq b_i, & \forall i \in I_{\text{le}}, & & \lambda_i \leq 0, & \forall i \in I_{\text{le}}, \\
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\end{array}
$$

Theorem (Weak duality)

If x is feasible for $(\mathcal{P})$ and λ is feasible for $(\mathcal{D})$, then $\lambda^\mathsf{T} b \leq c^\mathsf{T} x$.

Proof. Trivially true from our construction – omitted.

Implications of Weak Duality

Corollary

The following results hold:

(c) and (d) provide (sub)optimality certificates, but...

How do we know that the gaps in (c) are not very large?

How do we know that x and λ satisfying (d) even exist?

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(c) If $x \in P$ *and* $\lambda \in D$*, then:*

$$c^\mathsf{T} x - p^\star \leq c^\mathsf{T} x - \lambda^\mathsf{T} b \ \ \textbf{and} \ \ d^\star - \lambda^\mathsf{T} b \leq c^\mathsf{T} x - \lambda^\mathsf{T} b.$$

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(d) If $x \in P$, $\lambda \in D$, and $\lambda^\mathsf{T} b = c^\mathsf{T} x$, then x **optimal** *for $(\mathcal{P})$ and λ* **optimal** *for $(\mathcal{D})$.*

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