CME 307 / MS&E 311: Optimization

Interior Point Methods

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Management Science and Engineering
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November 18, 2024

slides developed with Prof. Luiz-Rafael Santos, UFSC https://lrsantos11.github.io/

Convex optimization

Convex optimization problem

$$\begin{array}{ll}\text{minimize} & f(x)\\ \text{subject to} & g(x) \leq 0\\ & Ax = b\end{array}$$

where $f : \mathbf{R}^n \to \mathbf{R}$, $g : \mathbf{R}^n \to \mathbf{R}^p$ are smooth and convex, $A \in \mathbf{R}^{m \times n}$ is full rank.

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KKT conditions:

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KKT conditions:

$$\begin{aligned} \nabla f(x) + A^T y + (\nabla g(x))^T s &= 0 \\ Ax &= b \\ g(x) &\leq 0 \\ s &\geq 0 \\ s_j g_j(x) &= 0, \quad j = 1, \ldots, p \end{aligned}$$

Outline

IPM for linear and quadratic programs

IPM for convex nonlinear programming

IPM for conic optimization

IPM for linear and quadratic programs

Linear/Quadratic Program

$$
\begin{array}{ll}
\text{minimize} & c^\top x + \frac{1}{2} x^\top Q x \\
\text{subject to} & Ax = b, \\
& x \geq 0,
\end{array}
$$

where $Q \in \mathbf{S}^n_+$, and $A \in \mathbf{R}^{m \times n}$ is full-rank.

IPM for linear and quadratic programs

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- $\mathcal{P} := \{x \in \mathbf{R}^n \mid Ax = b, x \geq 0\}$ is a polyhedron.

IPM for linear and quadratic programs

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IPM for linear and quadratic programs

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How to solve LP/QP problems?

IPM for linear and quadratic programs

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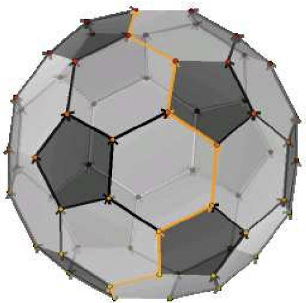
How to solve LP/QP problems?

Simplex: vertex to vertex
IPM: go through the middle!



IPM for linear and quadratic programs

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How to solve LP/QP problems?

Advantages of vertex solution vs interior solution?

Simplex: vertex to vertex
IPM: go through the middle!

Building blocks of IPM

Ingredients for Interior Point Method

- Duality theory: Lagrangian function; KKT (first order optimality) condition.
- Barrier function: logarithmic barrier.
- Newton's method (and a good linear solver)

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- Duality theory: Lagrangian function; KKT (first order optimality) condition.
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The reward: fantastic convergence properties!

- Theoretical: $O(\sqrt{n}\log(1/\varepsilon))$ iterations
- Practical: $O(\log n \log(1/\varepsilon))$ iterations

(but the per-iteration cost may be high due to the Newton solve: often $O(n^3)$)

IPM: algorithmic template

IPM procedure

- replace inequalities with log barriers;
- form the Lagrangian;
- write down the KKT conditions of the perturbed problem;
- find one (or more) directions using Newton's method on the KKT system;
- (decide how to combine the directions and) compute a stepsize.

Duality and KKT conditions

Primal-dual QPs

Primal problem

$$
\begin{aligned}
\text{minimize} \quad & c^\top x + \tfrac{1}{2} x^\top Q x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

Dual problem

$$
\begin{aligned}
\text{maximize} \quad & b^\top y - \tfrac{1}{2} x^\top Q x \\
\text{subject to} \quad & A^\top y + s - Qx = c \\
& s \geq 0
\end{aligned}
$$

Duality and KKT conditions

Primal-dual QPs

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KKT conditions

$$\begin{aligned} Ax &= b && \triangleright\text{(primal feasibility)} \\ A^\top y + s - Qx &= c && \triangleright\text{(dual feasibility)} \\ XSe &= 0 && \triangleright\text{(complementarity:} \quad x_i s_i = 0, i = 1, \ldots, n) \\ (x, s) &\geq 0 \end{aligned}$$

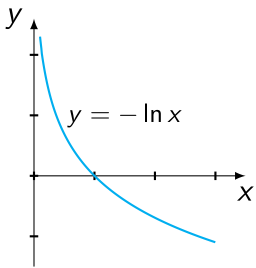
where $X = \mathbf{diag}(x_1, \ldots, x_n), S = \mathbf{diag}(s_1, \ldots, s_n) \in \mathbf{R}^{n \times n}$, and $e = (1, \ldots, 1) \in \mathbf{R}^n$.

Logarithmic barrier

$$-\ln x_j$$

replaces the inequality

$$x_j \geq 0$$



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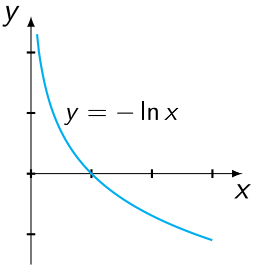

$$\text{minimize} \quad -\sum_{j=1}^{n} \ln x_j \quad \Longleftrightarrow \quad \text{maximize} \quad \prod_{1 \leq j \leq n} x_j$$

$\Longrightarrow$ keeps every entry of x away from 0.

Barrier primal QP

Step 1: replace inequality constraints by barrier

Replace the primal QP

$$\begin{array}{ll} \text{minimize} & c^\top x + \frac{1}{2}x^\top Q x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

with the barrier primal QP

$$\begin{array}{ll} \text{minimize} & c^\top x + \frac{1}{2}x^\top Q x - \mu \sum_{j=1}^{n} \ln x_j \\ \text{subject to} & Ax = b \end{array}$$

Logarithmic barrier and stationarity

Step 2: remove equality constraints using Lagrangian

$$\mathcal{L}(x, y, \mu) = c^\top x + \frac{1}{2} x^\top Q x - y^\top (Ax - b) - \mu \sum_{j=1}^{n} \ln x_j$$

Logarithmic barrier and stationarity

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$$\mathcal{L}(x, y, \mu) = c^\top x + \frac{1}{2}x^\top Qx - y^\top(Ax - b) - \mu \sum_{j=1}^{n} \ln x_j$$

A stationary point (x, y, μ) of the Lagrangian satisfies

$$\nabla_x \mathcal{L}(x, y, \mu) = 0 \qquad = c + Qx - A^\top y - \mu X^{-1} e$$

with $X^{-1} = \mathbf{diag}(x_1^{-1}, \ldots, x_n^{-1}) \in \mathbf{R}^{n \times n}, (x_j > 0)$.

KKT conditions for barrier problem

- Define $s := \mu X^{-1} e$, which implies $XSe = \mu e$, to get

KKT$_\mu$

$$\begin{aligned} Ax &= b \\ A^\top y + s - Qx &= c \\ XSe &= \mu e \\ (x, s) &> 0 \end{aligned}$$

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$$\text{KKT}_\mu \to \text{KKT as } \mu \to 0.$$

Central path (LP case)

- Parameter μ controls the distance to optimality

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- Analytic center (μ-center): unique point

$$(x(\mu), y(\mu), s(\mu)), \qquad x(\mu) > 0,\; s(\mu) > 0$$

that satisfies the KKT_μ conditions.

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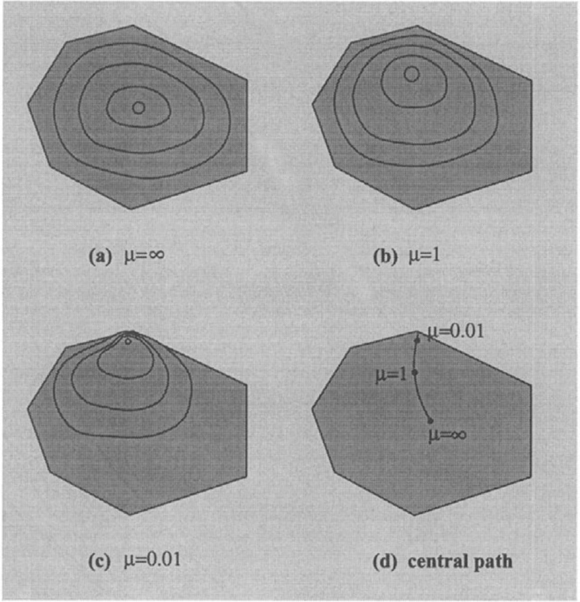
$$(x(\mu), y(\mu), s(\mu)), \qquad x(\mu) > 0,\ s(\mu) > 0$$

  that satisfies the KKT_μ conditions.
- The curve

$$\mathcal{C}_\mu = \{(x(\mu), y(\mu), s(\mu)) \mid \mu > 0\}$$

  is called the primal-dual central path.


(a) μ=∞
(b) μ=1
μ=0.01
μ=1
μ=∞
(c) μ=0.01
(d) central path

Recall Newton's method for nonlinear equation

- For $F : \mathbf{R}^n \to \mathbf{R}^n$ smooth, solve $F(x) = 0$.

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- Newton's method: define Jacobian $J_F(x)$ so $J_F(x)_{ij} = \frac{\partial F_i}{\partial x_j}$, and iterate

$$x^{k+1} = x^k - \alpha_k J_F(x^k)^{-1} F(x^k)$$

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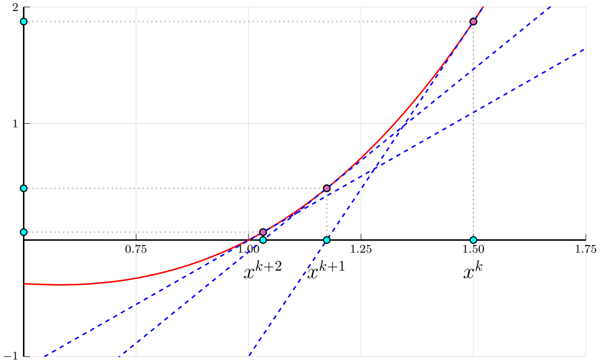

Apply Newton Method to KKT$_\mu$

The first order optimality conditions for the barrier problem form a large system of nonlinear equations:

$$F(x, y, s) = 0,$$

where $F : \mathbf{R}^{2n+m} \mapsto \mathbf{R}^{2n+m}$ is defined as

$$F(x, y, s) = \begin{bmatrix} Ax & -b \\ A^\top y + s - Qx & -c \\ XSe & -\mu e \end{bmatrix}$$

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- The first two blocks are linear.
- The last block, corresponding to the complementarity condition, is nonlinear.
- Jacobian is

$$J_F(x, y, s) = \begin{bmatrix} A & 0 & 0 \\ -Q & A^\top & I \\ S & 0 & X \end{bmatrix}$$

Interior-point QP Algorithm

IPM Framework

Fix the barrier parameter μ and make *one* (damped) Newton step towards the solution of KKT_μ. Then reduce the barrier parameter μ and repeat.

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- For $k = 1, 2, \ldots$

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  - $\mu_k = \sigma \mu_{k-1}$, where $\sigma \in (0, 1)$

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- For $k = 1, 2, \ldots$
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  - Find Newton direction $(\Delta x^k, \Delta y^k, \Delta s^k)$ by solving

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^\top & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} b - Ax^k \\ c - A^\top y^k - s^k + Qx^k \\ \mu_k e - X^k S^k e \end{bmatrix}$$

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  - Find step length α_k so $(x^k + \alpha_k \Delta x^k, y^k + \alpha_k \Delta y^k, s^k + \alpha_k \Delta s^k)$ is feasible.
  - Make step $(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k + \alpha_k \Delta x^k, y^k + \alpha_k \Delta y^k, s^k + \alpha_k \Delta s^k)$.

Path-following algorithm

- **Short-step path-following method:** $\mathcal{O}(\sqrt{n})$ complexity result

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Theorem ([Gondzio, 2012, Thm. 3.1])

Given $\epsilon > 0$, *suppose that a feasible starting point* $(x^0, y^0, s^0) \in \mathcal{N}_2(0.1)$ *satisfies*

$$(x^0)^\top s^0 = n\mu^0, \text{ where } \mu^0 \leq 1/\epsilon^\kappa,$$

for some positive constant κ. *Then for some* $K = \mathcal{O}(\sqrt{n}\ln(1/\epsilon))$, *the optimality gap is bounded by* ϵ *after at most* K *iterations:*

$$\mu^k \leq \epsilon, \quad \forall k \geq K$$

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$$\mu^k \leq \epsilon, \quad \forall k \geq K$$

- θ-neighborhood of the central path:
  $\mathcal{N}_2(\theta) := \{(x, y, s) \in \mathcal{F}^0 \mid \|XSe - \mu e\| \leq \theta\mu\}$, with $\mu = \frac{1}{n} x^\top s$.

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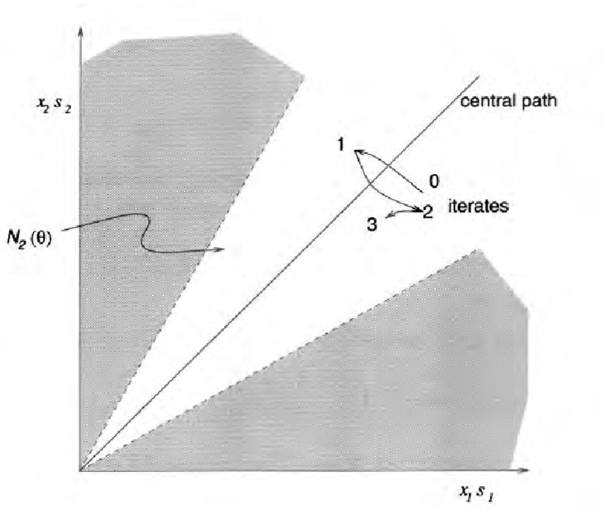
- Slow progress towards optimality



$x_2 s_2$
central path
1
0
iterates
2
3
$N_2(\theta)$
$x_1 s_1$

Augmented system

Newton direction

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^\top & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^\top y - s + Qx \\ \mu_k e - XSe \end{bmatrix} =: \begin{bmatrix} \xi_p \\ \xi_d \\ \xi_\mu \end{bmatrix}$$

use last (complementarity) block to solve for Δs as a function of Δx.

Augmented system

Define $\Theta = XS^{-1}$ (ill-conditioned!). Then Δx and Δy solve the Newton system $\iff$

$$\begin{bmatrix} -Q - \Theta^{-1} & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \xi_d - X^{-1}\xi_\mu \\ \xi_p \end{bmatrix}$$

- Newton system is nonsymmetric.
- Augmented system is symmetric but indefinite.

Normal equations

Augmented system

$$\begin{bmatrix} -\Theta^{-1} & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \xi_d - X^{-1}\xi_\mu \\ \xi_p \end{bmatrix} =: \begin{bmatrix} g \\ \xi_p \end{bmatrix}$$

Normal equations

Eliminate Δx to arrive at the *Normal equations*

$$(A\Theta A^\top)\Delta y = A\Theta g + \xi_p$$

Normal equations

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Normal equations

Eliminate Δx to arrive at the *Normal equations*

$$(A\Theta A^\top)\Delta y = A\Theta g + \xi_p$$

- $A\Theta A^\top$ is symmetric and positive semidefinite. (Finally!)
- Normal equations in QP $(A(Q + \Theta)A^\top)\Delta y = g$ are generally nearly dense, even when A and Q are sparse.
- LP: Normal equations are often used.
- QP: usually use the indefinite augmented system.

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IPM for NLP

- Convex NLP

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \le 0 \end{array} \iff \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) + z = 0, \quad z \ge 0 \end{array}$$

IPM for NLP

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- Replace inequality $z \ge 0$ with logarithmic barrier

$$\text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(z_i) \quad \text{subject to} \quad g(x) + z = 0$$

IPM for NLP

- Convex NLP

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \end{array} \iff \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) + z = 0, \quad z \geq 0 \end{array}$$

- Replace inequality $z \geq 0$ with logarithmic barrier

$$\text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(z_i) \quad \text{subject to} \quad g(x) + z = 0$$

- Write out Lagrangian

$$L(x, y, z, \mu) = f(x) + y^\top (g(x) + z) - \mu \sum_{i=1}^{m} \ln(z_i)$$

IPM for NLP

- Write conditions for stationary point

$$
\begin{aligned}
\nabla_x L(x, z, y) &= \nabla f(x) + J_g(x)^\top y = 0 \\
\nabla_y L(x, z, y) &= g(x) + z = 0 \\
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$$

- Write KKT system

$$
\begin{aligned}
\nabla f(x) + J_g(x)^\top y = 0, \\
g(x) + z = 0 \\
YZe = \mu e
\end{aligned}
$$

Newton for KKT of NLP

- Apply Newton method for KKT system

Newton for KKT of NLP

- Apply Newton method for KKT system
- Jacobian matrix of KKT system

$$J_F(x, z, y) = \begin{bmatrix} Q(x, y) & J_g(x)^\top & 0 \\ J_g(x) & 0 & I \\ 0 & Z & Y \end{bmatrix}$$

where $Q(x, y) = \nabla^2 f(x) + \sum_{i=1}^m y_i \nabla^2 g_i(x)$ is the Hessian of L

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- Newton step for KKT system

$$\begin{bmatrix} Q(x, y) & J_g(x)^\top & 0 \\ J_g(x) & 0 & I \\ 0 & Z & Y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - J_g(x)^\top y \\ -g(x) - z \\ \mu e - YZe \end{bmatrix}$$

From QP to NLP

- Newton direction for NLP

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- Augmented system for NLP

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- Need to compute $Q(x,y)$ and $J_g(x)$ at each iteration
- Caveat: use trust region method to choose stepsize as Hessian may be indefinite.

Outline

IPM for linear and quadratic programs

IPM for convex nonlinear programming

IPM for conic optimization

Self-concordant function

Definition

Function f is *self-concordant* if for some constant $M_f \geq 0$, the inequality

$$\nabla^3 f(x)[u, u, u] \leq M_f \|u\|^{3/2}_{\nabla^2 f(x)}$$

holds for any $x \in \mathrm{dom}\, f$ and $u \in R^n$.

- A self-concordant function is always well approximated by a quadratic model because the error of such an approximation can be bounded by the $\|u\|^{3/2}_{\nabla^2 f(x)}$

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Theorem ([Boyd and Vandenberghe, 2004, Section 11.5])

Newton's method with line search finds an ε approximate solution in less than $T := \text{constant} \times (f(x_0) - f^\star) + \log_2 \log_2 \frac{1}{\varepsilon}$ iterations.

Log-barrier is self-concordant

Theorem

The barrier function $-\ln(x)$ *is self-concordant in* $\mathbf{R}_+$.

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Consider $f(x) = -\ln(x)$. This function is self-concordant in $\mathbf{R}_+$ because

$$f'(x) = -\frac{1}{x}, \quad f''(x) = \frac{1}{x^2}, \quad f'''(x) = -\frac{2}{x^3}$$

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Compute and check that self-concordance holds with $M_f = 2$. $\square$

- $-\ln(1/x^\alpha)$, with $\alpha \in (0, \infty)$ is not self-concordant in $\mathbf{R}_+$.

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- $\exp(1/x)$ is not self-concordant in $\mathbf{R}_+$.

Conic optimization

- Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \in K \end{array}$$

where K is a convex closed cone.

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$$c^\top x - b^\top y = x^\top (c - A^\top y) = x^\top s \geq 0$$

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- Conic optimization can be solved in polynomial time with IPMs

Second-order conic optimization

- $K = \mathbb{L} \coloneqq \{(x, t) \mid x \in \mathbf{R}^{n-1}, t \in \mathbf{R}, \|x\|_2 \leq t, t \geq 0\}$ (Second-order cone)

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- $K = \mathbb{L} := \{(x, t) \mid x \in \mathbf{R}^{n-1}, t \in \mathbf{R}, \|x\|_2 \leq t, t \geq 0\}$ (Second-order cone)
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$$f(x, t) = \begin{cases} -\ln(t^2 - \|x\|_2^2) & \text{if } \|x\| < t \\ +\infty & \text{otherwise} \end{cases}$$

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Exercise: Prove in case $n = 2$.

Semidefinite programming

- Variable now is a symmetric matrix $X \in K = \mathbf{S}^n$

SDP and its dual

$$\begin{array}{ll} \text{minimize} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, i = 1, \ldots, m \\ & X \succeq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^\top y \\ \text{subject to} & \sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0 \end{array}$$

A_i, $C \in \mathbf{S}^n$ and $b \in \mathbf{R}^m$ given, and $X, S \in \mathbf{S}^n$ and $y \in \mathbf{R}^m$ unknown.

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Theorem (Weak duality for SDP)

If X is primal feasible and (y, S) is dual feasible, then

$$C \bullet X - b^\top y = X \bullet S \geq 0$$

Logarithmic barrier for SDP

- Logarithmic barrier function for the semi-definite cone

$$f(X) = \begin{cases} -\ln(\det(X)) & \text{if } X \succ 0 \\ +\infty & \text{otherwise} \end{cases}$$

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  - $\ln(1 + t\,\mathrm{tr}(U)) \approx t\,\mathrm{tr}(U)$
- Let $X \succ 0$ and $H \in \mathbf{S}^n$ be given. Then

$$\begin{aligned} f(X + tH) &= -\ln(\det(X + tH)) = -\ln(\det(X(I + tX^{-1}H))) \\ &= -\ln(\det(X)) - \ln(\det(I + tX^{-1}H)) \\ &= -\ln(\det(X)) - \ln(1 + t\,\mathrm{tr}(X^{-1}H) + \mathcal{O}(t^2)) \\ &= f(X) - tX^{-1} \bullet H + \mathcal{O}(t^2) \end{aligned}$$

Derivatives of Logarithmic barrier for SDP

- First derivative of $f(X)$

$$f'(X) = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t} = -X^{-1}$$

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So $Df(X)[H] = -X^{-1} \bullet H$.

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so $f''(X)[H] = X^{-1}HX^{-1}$ and $D^2 f(X)[H,G] = X^{-1}HX^{-1} \bullet G$.

- $f'''(X)[H,G] = -X^{-1}HX^{-1}GX^{-1} - X^{-1}GX^{-1}HX^{-1}$

Characterization of self-concordance for SDP

Theorem

The function $f(X) = -\ln\det X$ *is a convex barrier for* $\mathbf{S}^n_+$.

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Proof sketch.

Let $\varphi(t) = F(X + tH)$. Then, prove that $\varphi''(t) \geq 0$ for $t > 0$ such that $X + tH \succ 0$. Therefore, when $X \succ 0$ approaches a singular matrix, its determinant approaches zero, and the function $f(X) \to +\infty$. $\square$

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Theorem ([Nestervov and Nemirovskii, 1994])

The barrier function $f(X) = -\ln \det X$ is self-concordant on $\mathbf{S}^n_+$.

Solving SDPs with IPMs

- Replace the primal SDP

$$\begin{aligned} \text{minimize} \quad & C \bullet X \\ \text{subject to} \quad & \mathcal{A}X = b, \\ & X \succeq 0, \end{aligned}$$

with the primal barrier SDP

$$\begin{aligned} \text{minimize} \quad & C \bullet X + \mu f(X) \\ \text{subject to} \quad & \mathcal{A}X = b, \end{aligned}$$

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  (with a barrier parameter $\mu \geq 0$).
- Formulate the Lagrangian

$$L(X, y, S) = C \bullet X + \mu f(X) - y^T(\mathcal{A}X - b),$$

  with $y \in \mathcal{R}^m$, and write the first order conditions (FOC) for a stationary point of L:

$$C + \mu f'(X) - \mathcal{A}^* y = 0$$

Solving SDPs with IPMs (cont'd)

- Use $f(X) = -\ln\det X$ and $f'(X) = -X^{-1}$ to obtain

$$C - \mu X^{-1} - \mathcal{A}^* y = 0$$

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- Use $f(X) = -\ln \det X$ and $f'(X) = -X^{-1}$ to obtain

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- Denote $S = \mu X^{-1}$, i.e., $XS = \mu I$. Then, the FOC can be written as

$$\begin{aligned} \mathcal{A}X &= b \\ A^* y + S &= C \\ XS &= \mu I \end{aligned}$$

with $X, S \in \mathbf{S}^n_{++}$.

Newton direction

Differentiating this system is hard! The Newton direction solves:

$$\begin{bmatrix} \mathcal{A} & 0 & 0 \\ 0 & \mathcal{A}^* & \mathcal{I} \\ \mu\left(X^{-1} \odot X^{-1}\right) & 0 & \mathcal{I} \end{bmatrix} \cdot \begin{bmatrix} \Delta X \\ \Delta y \\ \Delta S \end{bmatrix} = \begin{bmatrix} \xi_b \\ \xi_C \\ \xi_\mu \end{bmatrix}.$$

We define the Kronecker product $P \odot Q$ for $P,\, Q \in \mathbf{R}^{n \times n}$, which yields a linear operator from $\mathbf{S}^n$ to $\mathbf{S}^n$ given by

$$(P \odot Q)U = \frac{1}{2}\left(PUQ^T + QUP^T\right).$$

Summary

- IPM for SOCP and SDP with self-concordant barrier:

source: [Gondzio, 2012: Interior Point Methods 25 Years Later]

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  - from LP via QP to NLP, SOCP and SDP
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- problematic for SDP because solving a problem of size n involves linear algebra operations in dimension n^2

source: [Gondzio, 2012: Interior Point Methods 25 Years Later]

Summary

- IPM for SOCP and SDP with self-concordant barrier:
  - polynomial complexity (predictable behaviour)
- Unified algorithm with fast convergence
  - from LP via QP to NLP, SOCP and SDP
- efficient for LP, QP, SOCP
- problematic for SDP because solving a problem of size n involves linear algebra operations in dimension n^2
  - and this requires n^6 flops!

source: [Gondzio, 2012: Interior Point Methods 25 Years Later]