Duality

Lecture 6

October 8, 2025

Quiz

Recap From Last Time & Today's Plan

Last time...

- **Separating Hyperplane Thm $\Rightarrow$ Farkas Lemma $\Rightarrow$ Strong duality**

Agenda for today:

- Two motivating applications
- Implications of strong duality
- Optimality conditions and primal/dual simplex
- Complementary slackness
- Global sensitivity & Shadow prices as marginal costs
- One more application: network revenue management

Polynomially-Sized CVaR Representation

- Recall homework: ensure CVaR of portfolio payoff exceeds a lower limit
- CVaR was defined as the average over the k-smallest values (for suitable integer k)
- If payoffs in the scenarios are $v_1, v_2, \ldots, v_n$, the key constraint is:

$$\sum_{i=1}^{k} v_{[i]} \geq b, \tag{1}$$

where $v_{[1]} \leq v_{[2]} \leq \cdots \leq v_{[n]}$ is the sorted vector of payoffs.

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  where $v_{[1]} \leq v_{[2]} \leq \cdots \leq v_{[n]}$ is the sorted vector of payoffs.
- Can write one constraint for each vector in $\{0, 1\}^n$ with exactly k values of 1.
- *How to formulate with a polynomial number of variables and constraints?*

Application in Robust Optimization

- Consider an LP with an uncertain constraint:

$$a^\top x \le b, \tag{2}$$

  where a satisfies $a \in \mathcal{A}$ and $\mathcal{A}$ is polyhedral

- We seek decisions x that are **robustly feasible**, i.e.,

$$a^\top x \le b, \; \forall a \in \mathcal{A} := \{a \in \mathbb{R}^n \; : \; Ca \le d\} \tag{3}$$

  Infinitely many constraints : "semi-infinite" LP. *Any ideas?*

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Strong Duality

Consider the following primal-dual pair:

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Theorem (Strong Duality)

If $(\mathcal{P})$ has an optimal solution, so does $(\mathcal{D})$, and their optimal values are equal.

Implications

Strong duality leaves only a few possibilities for a primal-dual pair:

		Dual		
		Finite Optimum	Unbounded	Infeasible
Primal	Finite Optimum	?	?	?
	Unbounded	?	?	?
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Primal	Finite Optimum	Possible	Impossible	Impossible
	Unbounded	Impossible	Impossible	Possible
	Infeasible	Impossible	Possible	?

Strong Duality and Theorems of Alternative

- Strong duality allows you to **prove** various "theorems of alternative"

Example (Farkas Lemma)

Prove that exactly one of the following is true:
(i) $\exists x \geq 0$ such that $Ax = b$,
(ii) $\exists \lambda$ such that $\lambda^\mathsf{T} A \geq 0$ and $\lambda^\mathsf{T} b < 0$.

Optimality for Standard-Form LPs

$$(\mathcal{P}) \ \min c^\mathsf{T} x \qquad\qquad (\mathcal{D}) \ \max \lambda^\mathsf{T} b$$
$$Ax = b, \quad x \geq 0 \qquad\qquad \lambda^\mathsf{T} A \leq c^\mathsf{T}$$

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  - Simplex algorithm: feasibility and optimality for $(\mathcal{P})$ are given by:

$$\text{Feasibility-}(\mathcal{P}) : \qquad x_B := A_B^{-1}b \geq 0 \tag{4a}$$

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- $(\mathcal{D})$: same basis B can also be used to determine **a dual vector** λ:

$$\lambda^{\mathsf{T}}A_i = c_i, \forall\, i \in B \quad \Rightarrow \quad \lambda^{\mathsf{T}} = c_B^{\mathsf{T}}A_B^{-1}, \forall\, i \in B.$$

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  - The dual objective value corresponding to λ is: $\lambda^\mathsf{T} b = c_B^\mathsf{T} A_B^{-1} b = c^\mathsf{T} x$
  - λ is feasible in the dual if and only if:

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Primal optimality $\Leftrightarrow$ Dual feasibility

Simplex terminates when finding a dual-feasible solution!

Solve $(\mathcal{P})$ or $(\mathcal{D})$?

$$(\mathcal{P}) \ \min \ c^\mathsf{T} x \qquad Ax = b, \quad x \geq 0$$

$$(\mathcal{D}) \ \max \ \lambda^\mathsf{T} b \qquad \lambda^\mathsf{T} A \leq c^\mathsf{T}$$

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Primal simplex

- maintain a **basic feasible solution**
- basis $B \subset \{1, \dots, n\}$
- stopping criterion: dual feasibility

Dual simplex

- maintain a dual feasible solution
- stopping criterion: primal feasibility
- different from primal simplex: works with an LP with inequalities

- How to choose $(\mathcal{P})$ or $(\mathcal{D})$?
- Suppose we have $x^\star$, $\lambda^\star$ and must now solve a **larger** problem, i.e., with extra decisions or extra constraints.
- *Any preference between primal and dual simplex?*
- Modern solvers include primal and dual simplex and allow concurrent runs

Optimality Conditions and Complementary Slackness

Primal-Dual Pair of Problems

$$
\begin{array}{llrl}
(\mathcal{P}) & \underset{x}{\text{minimize}} & c^{\mathsf{T}} x & \\
& & Ax & \substack{\le \\ = \\ \ge} \; b \\
& & x & \substack{\ge \\ \le} \; 0 \\
& \text{variables} & x & \in \mathbb{R}^n
\end{array}
\qquad \Bigg| \qquad
\begin{array}{llrl}
(\mathcal{D}) & \underset{\lambda}{\text{maximize}} & \lambda^{\mathsf{T}} b & \\
& & \lambda & \substack{\le \\ = \\ \ge} \; 0 \\
& & \lambda^{\mathsf{T}} A & \substack{\le \\ \ge} \; c^{\mathsf{T}} \\
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$$

Consider $x \in P, \lambda \in D$ (each feasible). How to check if they are **optimal**?

Optimality Conditions and Complementary Slackness

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Theorem (Complementary Slackness)

$x \in P$ and $\lambda \in D$ are **optimal** *solutions for $(\mathcal{P})$ and $(\mathcal{D})$, respectively,* **if and only if**:

$$\lambda_i(a_i^{\mathsf{T}} x - b_i) = 0, \; i = 1, \ldots, m$$

$$(\lambda^{\mathsf{T}} A_j - c_j) x_j = 0, \; j = 1, \ldots, n.$$

- Follows from primal/dual feasibility and $c^{\mathsf{T}} x = b^{\mathsf{T}} \lambda$

Optimality Conditions and Complementary Slackness

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$$(\mathcal{P}) \quad \underset{x}{\text{minimize}} \quad c^\mathsf{T} x \qquad\qquad (\mathcal{D}) \quad \underset{\lambda}{\text{maximize}} \quad \lambda^\mathsf{T} b$$

$$\begin{array}{rcl|rcl} Ax & \substack{\leq \\ \geq \\ =} & b & \lambda & \substack{\leq \\ \geq \\ \gtrless} & 0 \\ x & \substack{\geq \\ \leq \\ \gtrless} & 0 & \lambda^\mathsf{T} A & \substack{\leq \\ \geq \\ =} & c^\mathsf{T} \\ \text{variables} \quad x & \in & \mathbb{R}^n & \text{variables} \quad \lambda & \in & \mathbb{R}^m. \end{array}$$

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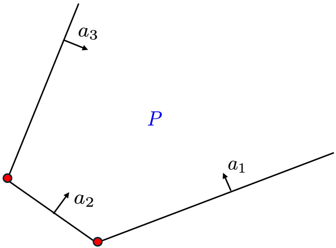
- Follows from primal/dual feasibility and $c^\mathsf{T} x = b^\mathsf{T} \lambda$
- Interesting insight: **non-binding constraint** $\Rightarrow$ dual variable is **zero**

Representation of Polyhedra

Important consequence of duality: alternative representation of all polyhedra

Definition

Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$. Then:



Representation of Polyhedra

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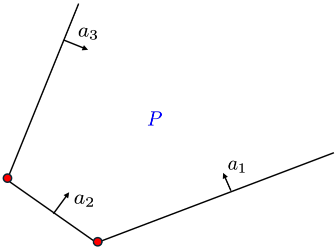
Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$. Then:

1. $\mathcal{C} := \{d \in \mathbb{R}^n : Ad \geq 0\}$ is called the **recession cone** of P.
2. Any $d \in \mathcal{C}$ with $d \neq 0$ is called a **ray** of P.



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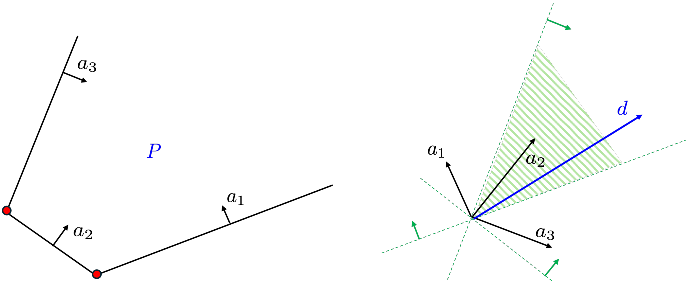

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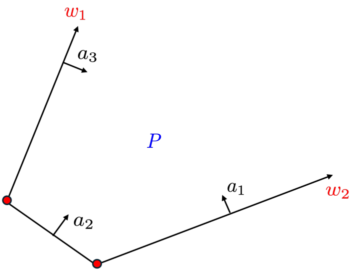
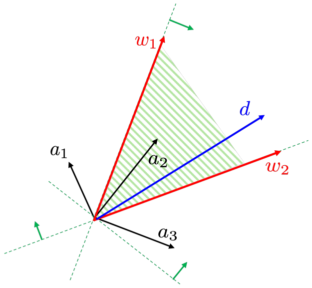
3. Any ray d that satisfies $a_i^\mathsf{T} d = 0$ for $n-1$ linearly independent a_i is called an **extreme ray** of P.





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Theorem (Resolution Theorem)

Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ *be a non-empty polyhedron,* $x^1, x^2, \ldots, x^k$ *be its* **extreme points**, *and* $w^1, w^2, \ldots, w^r$ *be its* **extreme rays**. *Then,*

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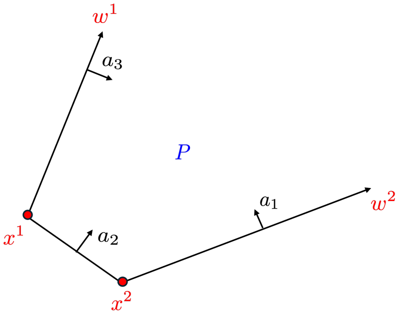
$$P = \mathrm{conv}\big(\{x^1, \dots, x^k\}\big) + \mathrm{cone}\big(\{w^1, \dots, w^r\}\big)$$

$$= \left\{ \sum_{i=1}^{k} \mu_i x^i + \sum_{j=1}^{r} \theta_j w^j \;:\; \mu \geq 0,\; e^{\mathsf{T}}\mu = 1,\; \theta \geq 0 \right\}.$$



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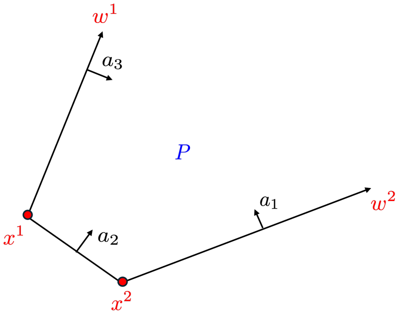

Note: It is **not** "easy" (i.e., poly-time) to switch between these representations

Dual Variables As Marginal Costs

$$
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Ax = b, \quad x \geq 0 \qquad\qquad \lambda^{\mathsf{T}} A \leq c^{\mathsf{T}}
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- Solved the LP and obtained $x^\star$ and $\lambda^\star$
- Want to show that $\lambda^\star$ is the **gradient of the optimal cost with respect to** b "almost everywhere"
- Related to **sensitivity analysis**
  How do the optimal value and solution depend on problem data A, b, c?

Global Dependency On b, c

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- What to show that the **optimal value** (when finite) **as a function of** b is
- What to show that the **optimal value** (when finite) **as a function of** c is

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- What to show that the **optimal value** (when finite) **as a function of** b is piecewise linear and **convex**

- What to show that the **optimal value** (when finite) **as a function of** c is piecewise linear and **concave**

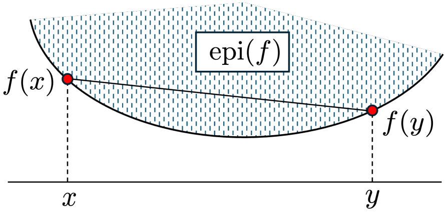
Convex and Concave Functions

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **convex** if X is a convex set and

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in X \;\text{ and } \lambda \in [0,1]. \tag{6}$$

A function is **concave** if $-f$ is convex.



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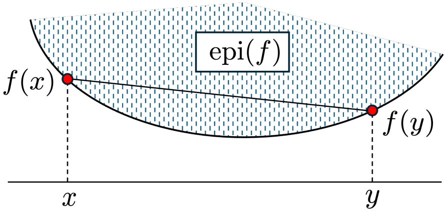

Equivalent definition in terms of **epigraph**:

$$\text{epi}(f) = \{(x,t) \in X \times \mathbb{R} \,:\, t \geq f(x)\} \tag{7}$$

f is convex if and only if epi(f) is a convex set.

Global Dependency On b

- Let $P(b) := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ denote the feasible set of the primal
- Let $S := \{b \in \mathbb{R}^m : P(b) \neq \emptyset\}$: right-hand-side values that yield a feasible primal
- Let $p^\star(b)$ denote the optimal objective; assume $p^\star(b) > -\infty$ (i.e., dual is feasible)

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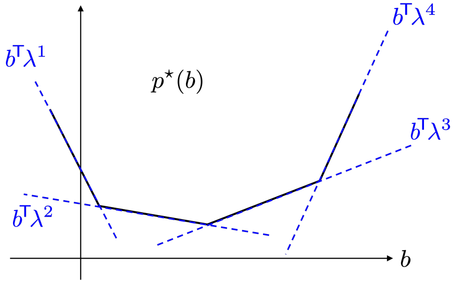
- If $\lambda^1, \lambda^2, \ldots, \lambda^r$ are the extreme points of D, then: $p^\star(b) = \max_{i=1,\ldots,r} b^\mathsf{T}\lambda^i$, $\forall\, b \in S$



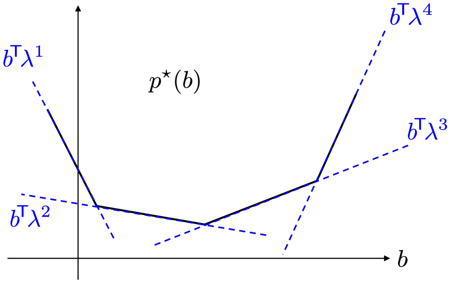
Global Dependency On b

- Let $P(b) := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ denote the feasible set of the primal
- Let $S := \{b \in \mathbb{R}^m : P(b) \neq \emptyset\}$: right-hand-side values that yield a feasible primal
- Let $p^\star(b)$ denote the optimal objective; assume $p^\star(b) > -\infty$ (i.e., dual is feasible)

- To argue: $p^\star : S \to \mathbb{R}$ is a **piecewise linear and convex** function of b

Proof. *Is S a convex set?*

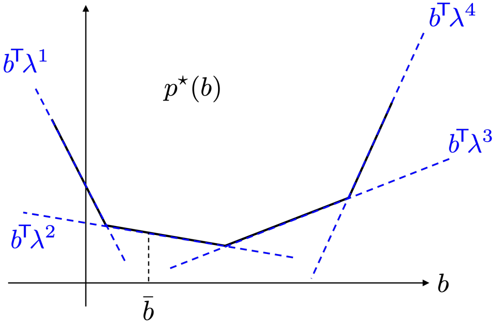
- Strong duality: $p^\star(b) = \min\left\{c^\mathsf{T}x : Ax = b,\ x \geq 0\right\} = \max\left\{\lambda^\mathsf{T}b : \lambda^\mathsf{T}A \leq c^\mathsf{T}\right\}$
- If $\lambda^1, \lambda^2, \ldots, \lambda^r$ are the extreme points of D, then: $p^\star(b) = \max_{i=1,\ldots,r} b^\mathsf{T}\lambda^i$, $\forall\, b \in S$



How to prove $p^\star(b)$ convex?

Global Dependency On b - Implications

$$p^\star(b) = \min\{c^\mathsf{T} x : Ax = b,\ x \geq 0\} = \max\{\lambda^\mathsf{T} b \ : \ \lambda^\mathsf{T} A \leq c^\mathsf{T}\}$$



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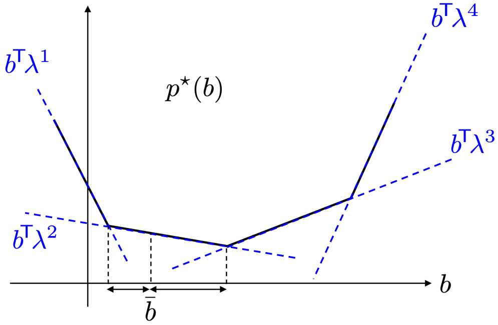

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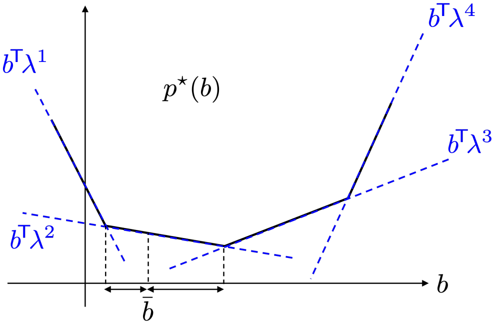

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- λ_i allows estimating **exact change in $p^\star$ in a range around $\bar{b}_i$**
- Modern solvers give **direct access to $\lambda_i^\star$ and the range**

  Gurobipy: for constraint **c**, the attribute **c.Pi** is $\lambda_i^\star$ and the range is from **c.SARHSLow** to **c.SARHSUp**

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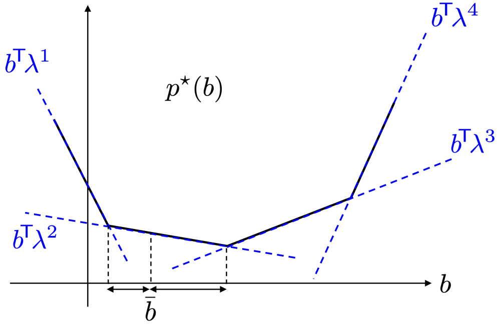

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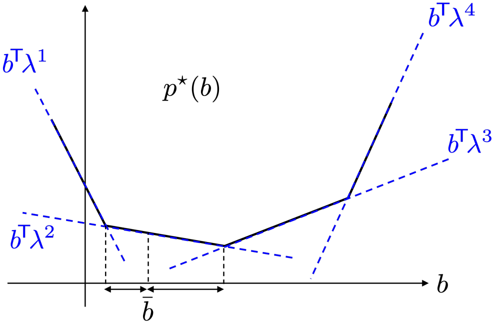

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Definition (Subgradient.)

$f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ convex function. A vector $g \in \mathbb{R}^n$ is a **subgradient** of f at $\bar{x} \in S$ if

$$f(x) \ge f(\bar{x}) + g^\mathsf{T}(x - \bar{x}), \quad \forall x \in S.$$

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- The optimal primal solution $x^\star$ **is a shadow price for the dual constraints**
- $x^\star$ remains optimal for a range of change in each objective coefficient c_j
- Modern solvers also allow obtaining the range directly

  Gurobipy: attributes **SAObjLow** and **SAObjUp** for each decision variable

Signs of Dual Variables Revisited

- There is a direct connection between:
  - **the optimization problem** (max/min)
  - the **constraint type** ($\leq$, $\geq$)
  - **the signs of the shadow prices**
- Given two of these, can figure out the third one!
- *What is the sign of the shadow price for a ...*

  $\leq$ constraint in a **minimization** problem ?
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- *What is the dependency of the optimal objective on the r.h.s. of a ...*

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- Requirements: $A \in \{0,1\}^{F \cdot I}$ with $A_{f,i} = 1 \Leftrightarrow$ itinerary i needs seat on flight leg f

Resource matrix A :

	Itinerary 1	Itinerary 2	...	Itinerary $\|I\|$
Flight leg 1	1	0	...	1
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- Goal: decide how many itineraries of each type to sell to maximize revenue

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