CME 307/MSE 311: Optimization

Acceleration, Stochastic Gradient Descent, and Variance Reduction

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Convergence of gradient descent

unconstrained minimization: find $x \in \mathbb{R}^n$ to solve

$$\text{minimize} \quad f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable

we analyzed gradient descent (GD) on this problem:

- a point x is ϵ-*suboptimal* if $f(x) - f^\star \leq \epsilon$
- when f is L-smooth and μ-PL (or μ-strongly convex), we showed GD converges to sub-optimality ϵ in at most

$$T = \mathcal{O}\left(\kappa \log\left(\frac{1}{\epsilon}\right)\right) \text{ iterations,}$$

where $\kappa := \frac{L}{\mu}$ is the condition number

Acceleration: motivation

Definition

a *first-order method* uses only a first-order oracle for $f : \mathbb{R}^n \to \mathbb{R}$ (*i.e.*, gradient and function evaluation) to minimize $f(x)$

GD $x \leftarrow x - \alpha \nabla f(x)$ is a first-order method

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GD $x \leftarrow x - \alpha \nabla f(x)$ is a first-order method

Q: is GD the best first-order method for L-smooth, μ-strongly convex functions?
A: no! Nemirovski and Yudin showed a *lower-bound* of

$$T_{\text{opt}} = \Omega\left(\sqrt{\kappa}\log\left(\frac{1}{\epsilon}\right)\right) \text{ iterations}$$

to find an ϵ-suboptimal point of *any* L-smooth, μ-strongly convex function
notice: same rate as CG if f is quadratic

A worst-case quadratic function

the lower bound can be obtained by constructing a particularly hard problem instance using quadratic functions

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- easier to work in the infinite dimensional-space $\ell^2(\mathbb{R})$, which consists of vectors x of infinite length, satisfying

$$\|x\|^2 = \sum_{j=1}^{\infty} x_j^2 < \infty$$

A worst-case quadratic function

the lower bound can be obtained by constructing a particularly hard problem instance using quadratic functions

- easier to work in the infinite dimensional-space $\ell^2(\mathbb{R})$, which consists of vectors x of infinite length, satisfying

$$\|x\|^2 = \sum_{j=1}^{\infty} x_j^2 < \infty$$

- following this setup, the evil quadratic function is then given by

$$f(x) = \frac{\mu(\kappa_f - 1)}{8}\left(x_1^2 + \sum_{j=1}^{\infty}(x_j - x_{j+1})^2 - 2x_1\right) + \frac{\mu}{2}\|x\|^2,$$

  where $\mu > 0$ and $\kappa_f > 1$
- above example actually gives a family of hard quadratic functions parametrized by μ, κ_f

source: Section 2.1, Nesterov, 2018

The lower bound

Using the family of quadratics on the preceding slide, the following theorem may be shown

Theorem (Nesterov Theorem 2.1.13)

Let $\mu > 0$, $\kappa_f > 1$. Suppose $\mathcal{M}$ is a first-order method such that for any input function f, $\mathcal{M}$ generates a sequence satisfying

$$x_k \in x_0 + \operatorname{span}\{\nabla f(x_0), \ldots, \nabla f(x_k)\}, \quad \forall k$$

Then there exists a L-smooth, μ-strongly convex function with $L/\mu = \kappa_f$ such that the sequence output by $\mathcal{M}$ applied to f satisfies

$$\|x_k - x_\star\|^2 \geq \left(\frac{\sqrt{\kappa_f} - 1}{\sqrt{\kappa_f} + 1}\right)^{2k} \|x_0 - x_\star\|^2,$$

$$f(x_k) - f(x_\star) \geq \frac{\mu}{2} \left(\frac{\sqrt{\kappa_f} - 1}{\sqrt{\kappa_f} + 1}\right)^{2k} \|x_0 - x_\star\|^2$$

Accelerated Gradient Descent

Nesterov's accelerated gradient method (AGD)

- a first-order method
- that matches the lower bound

so, converges faster than GD (esp. on ill-conditioned functions)

(one variant of) Nesterov's AGD:

1. Choose $x_0, y_0 \in \mathbb{R}^n$
2. for $k = 0, 1, \ldots, T$,

$$\begin{aligned} x_{k+1} &= y_k - \alpha \nabla f(y_k) \\ y_{k+1} &= x_{k+1} + \beta\left(x_{k+1} - x_k\right) \end{aligned}$$

3. Return x_{k+1}

achieves lower bound when $\alpha = \frac{1}{L}$, $\beta = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$

source: Section 2.2, Nesterov, 2018

GD vs. AGD: numerical example

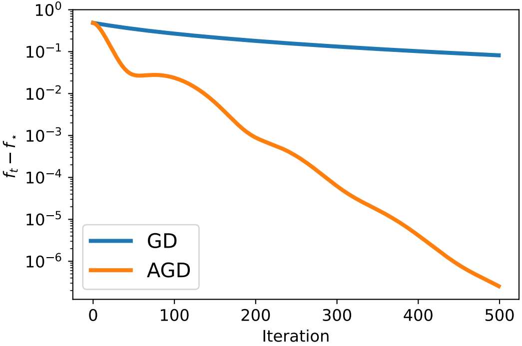
goal is to solve the logistic regression problem

$$\text{minimize} \quad \frac{1}{m}\sum_{i=1}^{m} \log\left(1+\exp\left(-b_i a_i^T x\right)\right) + \frac{1}{m}\|x\|^2$$

with variable x on rcv1 dataset, with data matrix $A \in \mathbb{R}^{20{,}242\times 47{,}236}$ and labels $b \in \mathbb{R}^{20{,}242}$

- GD and AGD both use theoretically-chosen stepsizes:
  - GD is run with stepsize $\frac{1}{L}$, which for this example equals 4
  - AGD is run with $\alpha = \frac{1}{L}$ and $\beta = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$
- here strong convexity $\mu = \frac{1}{m}$ from the regularizer

GD vs. AGD results



AGD summary and closing remarks

- AGD is theoretically optimal among first-order methods for L-smooth and μ-strongly convex functions
- converges to ϵ-suboptimality in at most

$$\mathcal{O}\left(\sqrt{\kappa}\log\left(\frac{1}{\epsilon}\right)\right) \text{ iterations}$$

- despite its elegance, AGD is rarely used in practice (quasi-Newton methods work better and are more stable)
- however, it forms the basis for more useful accelerated gradient methods like FISTA and Katyusha

Outline

Stochastic optimization

Finite sum minimization

Minimizing a sum

finite sum minimization: solve

$$\text{minimize} \quad \frac{1}{m}\sum_{i=1}^{m} f_i(x)$$

examples:

- least squares: $f_i(x) = (a_i^T x - b_i)^2$
- logistic regression: $f_i(x) = \log(1 + \exp\left(-b_i a_i^T x\right))$
- maximum likelihood estimation: $f_i(x)$ is -loglik of observation i given parameter x
- machine learning: f_i is misfit of model x on example i

Minimizing a sum

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with variable $x \in \mathbb{R}^n$

quandary:

- solving a problem with *more data* should be *easier*
- but complexity of algorithms increases with m!

goal: find algorithms that work *better* given *more* data
(or at least, not worse)

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(or at least, not worse)
idea: throw away data! (cleverly)

Minimizing an expectation

Stochastic optimization: solve

$$\text{minimize} \quad \mathbb{E}f(x) = \mathbb{E}_\omega f(x;\omega)$$

with variable $x \in \mathbb{R}^n$

- random loss function f
- or equivalently, function $f(\cdot;\omega)$ of random variable ω

Minimizing an expectation

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- or equivalently, function $f(\cdot;\omega)$ of random variable ω

examples: *data* $\omega = (a, b)$ is random

- least squares: $f(x;\omega) = (a^T x - b)^2$
- logistic regression: $f(x;\omega) = \log(1 + \exp\left(-ba^T x\right))$
- maximum likelihood estimation: $f(x;\omega)$ is -loglik of observation ω given parameter x
- machine learning: $f(x;\omega)$ is misfit of model x on example ω

minimize test loss, not just training loss

Stochastic optimization: more math

stochastic optimization problem

$$\begin{array}{ll} \text{minimize} & \mathbb{E}_{\omega \sim \mu_\Omega}\left[f(\omega, x)\right] \\ \text{variable} & x \in \mathbb{R}^n \end{array} \tag{2}$$

with $f(\omega, x) : \Omega \times \mathbb{R}^n$ convex, $\Omega \subseteq \mathbb{R}^n$, ω a random variable distributed according to probability measure μ_Ω

- objective is expected cost under the randomness due to ω:

$$F(x) = \mathbb{E}_{\omega \sim \mu_\Omega}\left[f(\omega, x)\right] = \int_\Omega f(\omega; x) d\mu_\Omega(\omega)$$

Stochastic optimization: examples

1. $n = 1, \Omega = \mathbb{R}$, and $f(\omega, x) = (x - \omega)^2$. (2) becomes

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   then $x_\star =$ the median of $\mu_{\mathbb{R}}$
3. $\Omega = \mathbb{R}^n$, $\mu_{\mathbb{R}^n} = \frac{1}{m} \sum_{i=1}^m \delta_{\omega_i}$. (2) becomes the finite sum minimization problem

$$\text{minimize} \quad \frac{1}{m} \sum_{i=1}^m f(\omega_i, x).$$

Stochastic gradient oracle

Definition

a *stochastic gradient oracle* $\mathcal{G}$, when queried at $x \in \mathbb{R}^n$ produces $g(\omega; x) \in \mathbb{R}^n$ satisfying

$$\mathbb{E}_{\omega \sim \mu_\Omega}\left[g(\omega; x)\right] = \nabla F(x)$$

i.e., $\mathcal{G}$ produces an unbiased estimate of the true gradient $\nabla F(x)$

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Q: examples of stochastic gradient oracle?
A: minibatch gradient

$$\frac{1}{|S|} \sum_{\omega \in S} \nabla f_i(\omega, x)$$

notation: use $\hat{\nabla} f(x)$ to denote stochastic gradient at x

Stochastic gradient descent (SGD)

SGD:

1. Choose $x_0 \in \mathbb{R}^n$
2. for $k = 0, 1, \ldots$
   i. query $\mathcal{G}$ at x_k to obtain $g(\omega_k, x_k)$
   ii. compute update:

$$x_{k+1} = x_k - \eta_k g(\omega_k, x_k)$$

- SGD is not a descent method!
- SGD exactly the same as GD, except that it uses a stochastic gradient $g(\omega_k, x_k)$ rather than the true gradient
- selection of stepsize η_k is challenging!

A typical convergence result

Theorem (General SGD convergence)

Consider (2) with smooth and strongly convex f and stochastic gradient oracle satisfying

$$\mathbb{E}_\omega \|g(\omega, x)\|^2 \leq M_1 + M_2 \|\nabla F(\omega, x)\|^2.$$

1. *for an appropriate fixed stepsize $\eta_k = O(1)$,*

$$\lim_{k \to \infty} \mathbb{E}[f(\omega_k, x_k)] - f_\star = O(1)$$

2. *for decreasing stepsizes $\eta_k = O(1/k)$,*

$$\mathbb{E}[f(\omega_k, x_k)] - f_\star = O(1/k)$$

SGD convergence: discussion

- with fixed stepsize, the algorithm converges to ϵ-sublevel set
- convergence of SGD requires a decreasing stepsize $\implies$ slow!

contrast to GD, which converges to the exact optimum even with fixed stepsize

analysis is tight: there is a matching lower bound.
Agarwal et al., 2012 shows that for strongly convex problems, any algorithm using a stochastic gradient oracle must make at least $\Omega(1/\epsilon)$ queries to obtain an ϵ-suboptimal point

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don't despair: add more assumptions!

Outline

Stochastic optimization

Finite sum minimization

Finite-sum minimization

return to finite sum problem:

$$\text{minimize} \quad \frac{1}{m}\sum_{i=1}^{m} f_i(x), \tag{3}$$

where each f_i is L_i-smooth and convex

why use SGD for finite sum minimization?

- evaluating minibatch gradient is cheaper per iteration
- converges faster than GD b/c each iteration is faster

Convergence of SGD

prove SGD minimizes finite sum (3):

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$$
\begin{aligned}
\|x_{k+1} - x_\star\|^2 &= \|x_k - x_\star - \eta \widehat{\nabla} f(x_k)\|^2 \\
&= \|x_k - x_\star\|^2 - 2\eta \langle x_k - x_\star, \widehat{\nabla} f(x_k)\rangle + \eta^2 \|\widehat{\nabla} f(x_k)\|^2.
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$$

take expectation wrt $\hat{\nabla} f(x_k)$:

$$
\begin{aligned}
\mathbb{E}_k \|x_{k+1} - x_\star\|^2 &= \|x_k - x_\star\|^2 - 2\eta \langle x_k - x_\star, \nabla f(x_k)\rangle + \eta^2 \mathbb{E}_k \|\widehat{\nabla} f(x_k)\|^2 \\
&\leq (1 - \eta\mu)\|x_k - x_\star\|^2 - 2\eta\left(f(x_k) - f(x_\star)\right) \\
&\quad + \eta^2 \mathbb{E}_k \|\widehat{\nabla} f(x_k)\|^2
\end{aligned}
$$

using strong convexity:

$$
f(x_\star) \geq f(x_k) + \nabla f(x_k)^T (x_\star - x_k) + \frac{\mu}{2}\|x_\star - x_k\|^2.
$$

One-step lemma

we have shown the following progress bound for one step of SGD

Lemma

at iteration k of SGD,

$$
\begin{aligned}
&\mathbb{E}_k\|x_{k+1} - x_\star\|^2 \\
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to show convergence, we must bound $\mathbb{E}_k\|\widehat{\nabla} f(x_k)\|^2$

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we will follow the approach of Gower et al., 2019

Expected smoothness

Definition (Expected smoothness)

f satisfies *L-expected smoothness* (L-ES) if $\exists L > 0$ such that

$$\mathbb{E}\|\widehat{\nabla} f(x) - \widehat{\nabla} f(x_\star)\|^2 \leq 2L(f(x) - f(x_\star))$$

reduces to L-smoothness if we replace $\widehat{\nabla}$ by ∇:

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Corollary

define $\sigma^2 := \mathbb{E}\|\widehat{\nabla} f(x_\star)\|^2$*. then*

$$\mathbb{E}\|\widehat{\nabla} f(x)\|^2 \leq 4L(f(x) - f(x_\star)) + 2\sigma^2, \qquad \forall x$$

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under ES, gradient variance is controlled by suboptimality and variance of the gradient at the optimum

L-ES condition for smooth convex functions

Theorem (special case of Gower et al., 2019)

Suppose each f_i is L_i-smooth and convex. Consider mini-batch stochastic gradients $\widehat{\nabla} f = \frac{1}{|S|} \sum_{i \in S} f_i(x)$ with batch-size $b_g = |S|$. Then

$$\mathbb{E}\|\widehat{\nabla} f(x)\|^2 \leq 4L(f(x) - f(x_\star)) + 2\sigma^2,$$

with

$$L = \frac{m(b_g - 1)}{b_g(m-1)} \frac{1}{m} \sum_{i=1}^{m} L_i + \frac{m - b_g}{b_g(m-1)} \max_{1 \leq i \leq m} L_i$$

and

$$\sigma^2 = \frac{m - b_g}{b_g(m-1)} \frac{1}{m} \sum_{i=1}^{m} \|\nabla f_i(x_\star)\|^2$$

sanity check: $\sigma^2 \to 0$ as $b_g \to n$

Back to SGD convergence

using the one-step lemma with μ-strong convexity and L-ES, we find

$$\mathbb{E}_k\|x_{k+1} - x_\star\|^2 \le (1 - \eta\mu)\|x_k - x_\star\|^2 + 2\eta(2\eta L - 1)\left(f(x_k) - f(x_\star)\right) + \eta^2 2\sigma^2$$

so, choosing stepsize $\eta \le \frac{1}{2L}$,

$$\mathbb{E}_k\|x_{k+1} - x_\star\|^2 \le (1 - \eta\mu)\|x_k - x_\star\|^2 + \eta^2 2\sigma^2$$

SGD convergence contd

apply induction + take total expectation to get

$$\mathbb{E}\|x_{k+1} - x_\star\|^2 \le (1-\eta\mu)^{k+1}\|x_0 - x_\star\|^2 + \left(\sum_{j=0}^{k}(1-\eta\mu)^j\right)\eta^2 2\sigma^2$$

$$\le (1-\eta\mu)^{k+1}\|x_0 - x_\star\|^2 + \frac{\eta 2\sigma^2}{\mu}$$

by summing the geometric series. choose $\eta \le \frac{\mu\epsilon}{4\sigma^2}$, so

$$\mathbb{E}\|x_{k+1} - x_\star\|^2 \le (1-\eta\mu)^{k+1}\|x_0 - x_\star\|^2 + \frac{\epsilon}{2}$$

we can solve for k to find how many iterations are needed to reach error $\frac{\epsilon}{2}$:

$$k \ge (\eta\mu)^{-1}\log\left(\frac{2(f(x_0) - f(x_\star))}{\epsilon}\right)$$

SGD convergence with fixed stepsize

we have shown

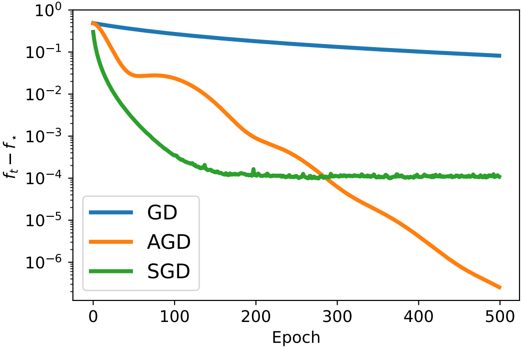
Theorem

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is μ-strongly convex, with an L-ES stochastic gradient oracle. Run SGD with batchsize b_g and fixed stepsize $\eta = \min\left\{\frac{1}{2L}, \frac{\epsilon\mu}{4\sigma^2}\right\}$. Then for $k \geq (\eta\mu)^{-1} \log\left(\frac{2(f(x_0)-f(x_\star))}{\epsilon}\right)$ iterations,

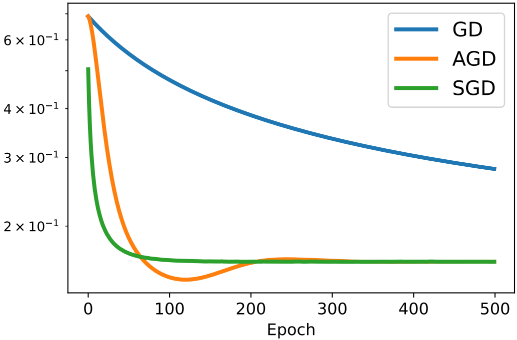
$$\mathbb{E}\|x_k - x_\star\|^2 \leq \epsilon$$

- same convergence rate as we'd get with decreasing stepsize sequence $\eta = \mathcal{O}(1/k)$
- but motivates variance reduction, which will give linear convergence!

Results: Optimization error



Results: Test error



The gradient is too noisy!

the expected smoothness condition shows the gradient is noisy,

$$\mathbb{E}\|\widehat{\nabla} f(x)\|^2 \leq 4L(f(x) - f(x_\star)) + 2\sigma^2,$$

even at $x_\star$

- good news: $f(x) - f^\star \to 0$ as $x \to x_\star$
- bad news: $\sigma^2 > 0$ even near $x_\star$

can we design an algorithm that eliminates this noise as $x \to x_\star$?

Stochastic Variance Reduced Gradient

Stochastic Variance Reduced Gradient (SVRG) uses a different stochastic gradient

$$g(x) = \widehat{\nabla} f(x) - \widehat{\nabla} f(x_s) + \nabla f(x_s)$$

where

- $\widehat{\nabla}$ still denotes the minibatch gradient
- $x_s \in \mathbb{R}^n$ is a reference point
- $\nabla f(x_s) - \widehat{\nabla} f(x_s)$ is a control variate introduced to reduce variance

$g(x) \in \mathbb{R}^n$ is a stochastic gradient at $x \in \mathbb{R}^n$:

$$\mathbb{E}[g(x)] = \nabla f(x) - \nabla f(x_s) + \nabla f(x_s) = \nabla f(x),$$

Some useful identities

recall the following two identities for random variables X, Y:

1. $\mathbb{E}\|X + Y\|^2 \leq 2\mathbb{E}\|X\|^2 + 2\mathbb{E}\|Y\|^2$
2. $\mathbb{E}\|X - \mathbb{E}[X]\|^2 \leq \mathbb{E}\|X\|^2$

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(exercise: prove these!)

SVRG reduces variance

variance of $g(x)$ depends on suboptimality of x and x_s

$$
\begin{aligned}
\mathbb{E}\|g(x)\|^2 &= \mathbb{E}\|g(x) - \widehat{\nabla} f(x_\star) + \widehat{\nabla} f(x_\star)\|^2 \\
&= \mathbb{E}\|\widehat{\nabla} f(x) - \widehat{\nabla} f(x_\star) + \widehat{\nabla} f(x_\star) - \widehat{\nabla} f(x_s) + \nabla f(x_s)\|^2 \\
&\leq 2\mathbb{E}\|\widehat{\nabla} f(x) - \widehat{\nabla} f(x_\star)\|^2 \\
&\quad +2\mathbb{E}\|\widehat{\nabla} f(x_s) - \widehat{\nabla} f(x_\star) - \nabla f(x_s)\|^2 \\
&= 2\mathbb{E}\|\widehat{\nabla} f(x) - \widehat{\nabla} f(x_\star)\|^2 \\
&\quad +2\mathbb{E}\|\widehat{\nabla} f(x_s) - \widehat{\nabla} f(x_\star) - \mathbb{E}[\widehat{\nabla} f(x_s) - \widehat{\nabla} f(x_\star)]\|^2 \\
&= 2\mathbb{E}\|\widehat{\nabla} f(x) - \widehat{\nabla} f(x_\star)\|^2 + 2\mathbb{E}\|\widehat{\nabla} f(x_s) - \widehat{\nabla} f(x_\star)\|^2 \\
&= 4L[f(x) - f(x_\star) + f(x_s) - f(x_\star)]
\end{aligned}
$$

hence $\mathrm{Var}(g(x)) \to 0$ as $f(x) \to f_\star$, $f(x_s) \to f_\star$

How to select x_s?

to ensure x, $x_s \to x_\star$ (and so $\mathrm{Var}(g(x)) \to 0$)

- update x_s as we make progress (so $f(x_s) \to f(x_\star)$)
- don't update too often, as computing $\nabla f(x_s)$ is expensive

SVRG algorithm

1. initialize at x_0 and set $x_s = x_0$
2. for $s = 0, \ldots, S$
   2.1 compute and store $\nabla f(x_s)$
   2.2 for $k = 0, \ldots, m-1$

   $$x_{k+1}^{(s)} = x_k^{(s)} - \eta \left(\widehat{\nabla} f(x_k^{(s)}) - \widehat{\nabla} f(x_s) + \nabla f(x_s) \right)$$

   2.3 select x_{s+1} by uniformly sampling at random from $\{x_0^{(s)}, \ldots, x_{m-1}^{(s)}\}$
   2.4 set $x_0^{(s+1)} = x_{s+1}$
3. return x_S

- notice that $\mathbb{E} f_{s+1} = \frac{1}{m} \sum_{i=1}^{m} f(x_i^{(s)})$ (needed for proof)
- in practice, fine to set $f_{s+1} = f(x_m^{(s)})$ (last iterate)

SVRG convergence

Theorem

Run SVRG with $S = \mathcal{O}\left(\log\left(\frac{1}{\epsilon}\right)\right)$ outer iterations, $m = O(\kappa)$ inner iterations, and fixed stepsize $\eta = O(1/L)$. Then

$$\mathbb{E}[f(x_S)] - f(x_\star) \leq \epsilon.$$

The number of gradient oracle calls is bounded by

$$\mathcal{O}\left((n + \kappa b_g)\log\left(\frac{1}{\epsilon}\right)\right).$$

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- unlike SGD, SVRG converges linearly to the optimum
- when $\kappa = \mathcal{O}(n)$, SVRG makes only $\widetilde{\mathcal{O}}(nb_g)$ oracle calls, while GD makes $\widetilde{\mathcal{O}}(n^2)$ calls. so SVRG reduces the number of calls by n/b_g!

Proof of SVRG convergence

the argument may be broken down into two lemmas. We begin with the following one-step progress bound for outer-iteration s

Lemma (One-step lemma)

Suppose we are at iteration k of outer-iteration s. Then

$$\mathbb{E}_k \|x_{k+1}^{(s)} - x_\star\|^2 \le \|x_k^{(s)} - x_\star\|^2 + 2\eta\,(2\eta L - 1)\,[f(x_k^{(s)}) - f(x_\star)] + 4\eta^2 L[f(x_s) - f(x_\star)]$$

Proof of One-step lemma

$$
\begin{aligned}
\mathbb{E}_k\|x_{k+1}^{(s)} - x_\star\|^2 = \\
&\|x_k^{(s)} - x_\star\|^2 - 2\eta\langle\nabla f(x_k), x_k - x_\star\rangle + \eta^2\mathbb{E}_k\| \, g(x_k)\|^2 \\
&\leq \|x_k^{(s)} - x_\star\|^2 - 2\eta\,(f(x_k) - f(x_\star)) + \eta^2\mathbb{E}_k\| \, g(x_k)\|^2 \\
&\leq \|x_k^{(s)} - x_\star\|^2 - 2\eta\,(f(x_k) - f(x_\star)) + \\
&4\eta^2 L[f(x) - f(x_\star) + f(x_s) - f(x_\star),]
\end{aligned}
$$

where the first inequality uses convexity

$$
f(x_k) - f(x_\star) \leq \langle\nabla f(x_k), x_k - x_\star\rangle
$$

so, after rearranging

$$
\begin{aligned}
\mathbb{E}_k\|x_{k+1}^{(s)} - x_\star\|^2 \leq \|x_k^{(s)} - x_\star\|^2 + 2\eta\,(2\eta L - 1)\,[f(x_k^{(s)}) - f(x_\star)] \\
+ 4\eta^2 L[f(x_s) - f(x_\star)]
\end{aligned}
$$

Outer iteration contraction

the next step is show to the follow contraction result for the outer-iterations.

Lemma (Outer iteration contraction)

Suppose we are in outer iteration s. Then

$$\mathbb{E}_{0:s-1}[f(x_s)] - f(x_\star) \leq \left[\frac{1}{\eta\mu(1-2\eta L)m} + \frac{2}{1-2\eta L}\right](f(x_{s-1}) - f(x_\star)),$$

where $\mathbb{E}_{0:s-1}$ denotes the expectation conditioned on outer-iterations 0 through $s-1$.

Proof of outer iteration contraction

summing the inequality in the one-step lemma from $k = 0, \ldots, m-1$,

$$\sum_{k=1}^{m} \mathbb{E}_k \|x_{k+1}^{(s)} - x_\star\|^2 \leq \sum_{k=0}^{m-1} \|x_k^{(s)} - x_\star\|^2 +$$

$$2\eta m \left(2\eta L - 1\right) \frac{1}{m} \sum_{k=0}^{m-1} [f(x_k^{(s)}) - f(x_\star)] + 4m\eta^2 [f(x_{s-1}) - f(x_\star)].$$

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$$2\eta m \left(2\eta L - 1\right) \frac{1}{m} \sum_{k=0}^{m-1} [f(x_k^{(s)}) - f(x_\star)] + 4m\eta^2 [f(x_{s-1}) - f(x_\star)].$$

taking the expectation over all inner-iterations conditioned on outer-iterations 0 through $s-1$ + cancellation, yields

$$\mathbb{E}_{0:s-1} \|x_m^{(s)} - x_\star\|^2 \leq \|x_{s-1} - x_\star\|^2 +$$
$$+ 2\eta m \left(2\eta L - 1\right) \left(\mathbb{E}_{0:s-1} \left[f(x_s)\right] - f(x_\star)\right) + 4m\eta^2 L [f(x_{s-1}) - f(x_\star)].$$

Proof contd.

rearranging gives,

$$\mathbb{E}_{0:s-1}\|x_s - x_\star\|^2 + 2\eta m\,(1 - 2\eta L)\,(\mathbb{E}_{0:s-1}[f(x_s)] - f(x_\star))$$
$$\leq 2\left(\frac{1}{\mu} + 2m\eta^2 L\right)[f(x_{s-1}) - f(x_\star)],$$

where we used strong convexity of f

$$\|x_{s-1} - x_\star\|^2 \leq \frac{2}{\mu}\,(f(x_{s-1}) - f(x_\star))$$

hence (dropping $\mathbb{E}_{0:s-1}\|x_s - x_\star\|^2 \geq 0$)

$$2\eta m\,(1 - 2\eta L)\,(\mathbb{E}_{0:s-1}[f(x_s)] - f(x_\star))$$
$$\leq 2\left(\frac{1}{\mu} + 2m\eta^2 L\right)[f(x_{s-1}) - f(x_\star)],$$

and so the claim follows by rearrangement

Finishing the proof

$$\mathbb{E}_{0:s-1}[f(x_{s+1})] - f(x_\star) \leq \left[\frac{1}{\eta\mu(1-2\eta L)m} + \frac{2}{1-2\eta L}\right](f(x_s) - f(x_\star))$$

setting $\eta = \frac{1}{10L}$ and $m = 20\frac{\mathcal{L}}{\mu}$, we find

$$\mathbb{E}_{0:s-1}[f(x_s)] - f(x_\star) \leq \frac{1}{2}(f(x_{s-1}) - f(x_\star))$$

now taking expectations over all outer iterations and recursing,

$$\mathbb{E}[f(x_s)] - f(x_\star) \leq \left(\frac{1}{2}\right)^s (f(x_0) - f(x_\star)),$$

which gives the theorem after setting $s = O(\log(1/\epsilon))$

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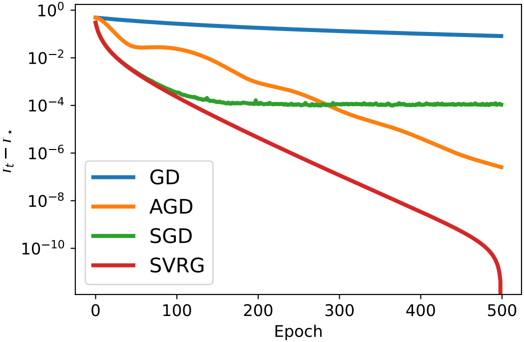
Q: how to choose step-size η?
A: in general, there is no good way, the theoretical step-size can be conservative
Q: does SVRG work for non-convex problems like deep learning?
A: generally, not without modification. For deep learning specifically, variance reduction hasn't been useful, in fact it can make things worse!

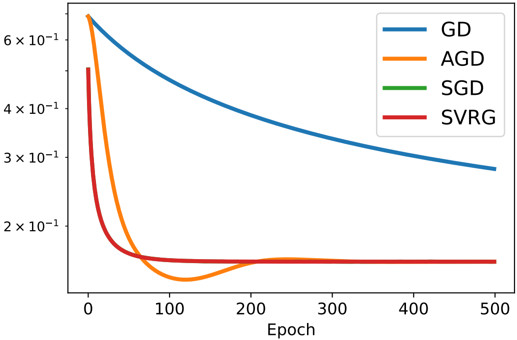
SVRG numerical performance

we once again visit the logistic regression example considered previously. SVRG is run with step-size $\eta = 4$ and the snapshote is updated every epoch

Results: Optimization error



Results: Test loss



SVRG: Final comments

- variance reduction has proven to be a powerful tool for convex finite-sum optimization, as it delivers linear convergence
- SVRG has motivated the development of better (usually) variance reduced algorithms such as SAGA and Katyusha
- outside of finite-sum convex optimization, variance reduction hasn't proven to be terribly useful