Ingredients of a logic

Syntax ↖

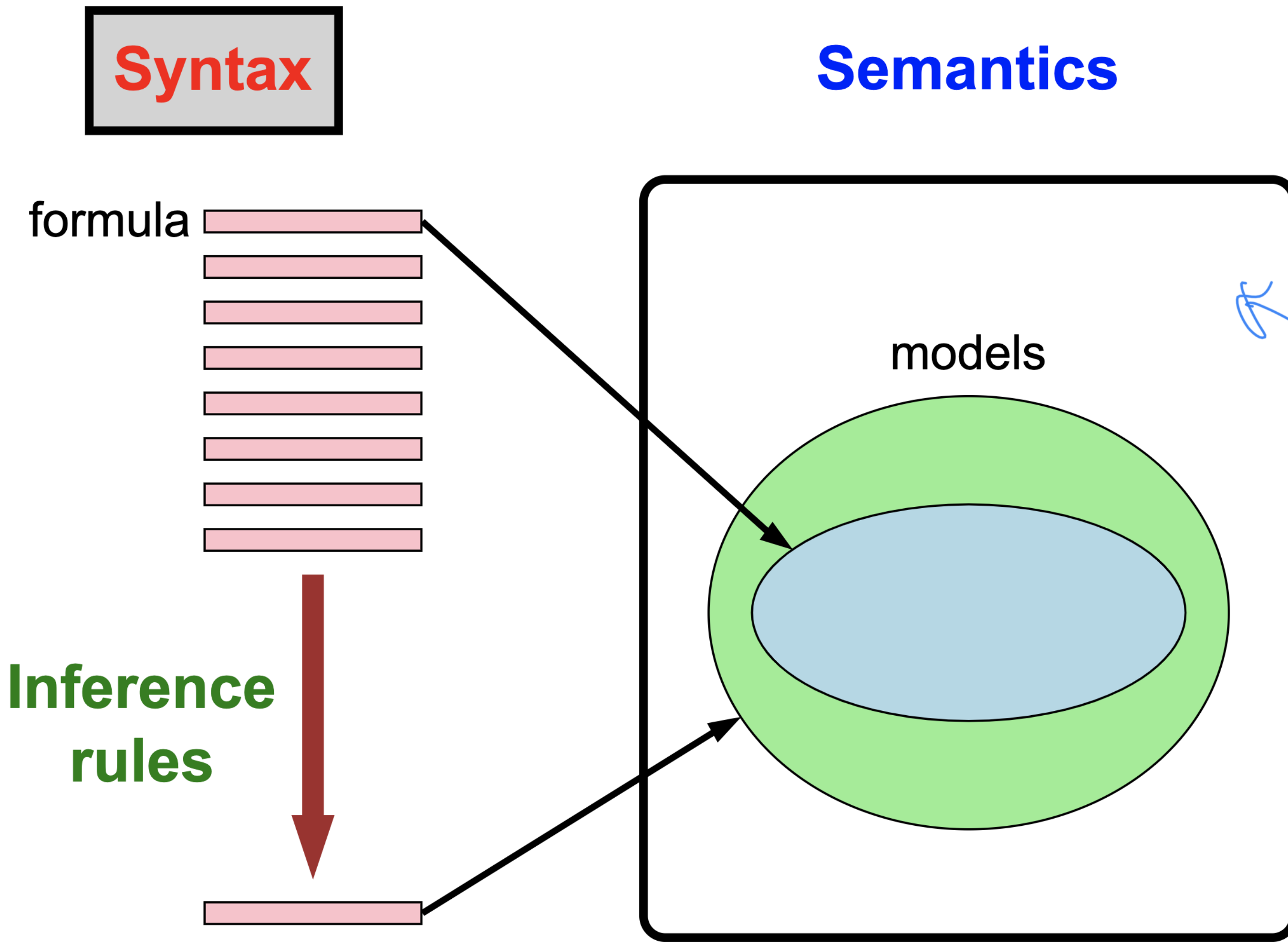
Semantics ↖

Inference ↖

Syntax: defines a set of valid **formulas** $(\mathrm{Formulas})$

Example: $\mathrm{Rain} \wedge \mathrm{Wet}$

Propositional logic



{wet:0, rain:5}
{wet:1, rain:0}

Model

Definition: model

A **model** w in propositional logic is an **assignment** of truth values to propositional symbols.

$\{A: 0, B: 1, C: 0\}$

Interpretation function: definition

Base case:

- For a propositional symbol p (e.g., A, B, C):
  $\mathcal{I}(p, w) = w(p)$

$w = \{A: 0, B: 1, C: 0\}$ $p = B$

$I(p, w) = 1$

Knowledge base

Definition: Knowledge base

A **knowledge base** KB is a set of formulas representing their conjunction / intersection:

Let $\mathrm{KB} = \{\mathrm{Rain} \vee \mathrm{Snow}, \mathrm{Traffic}\}$. $\equiv$ $(\mathrm{Rain} \vee \mathrm{Snow}) \wedge (\mathrm{Traffic})$

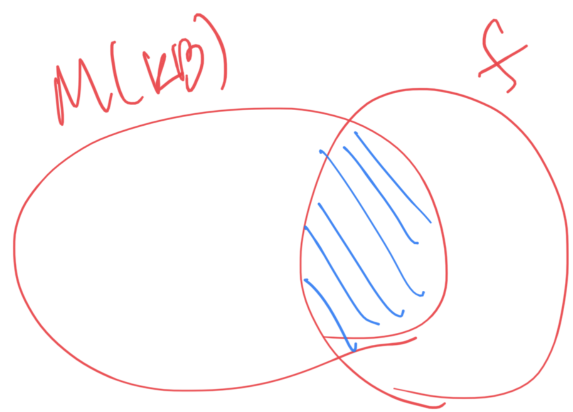
Adding to the knowledge base

Adding more formulas to the knowledge base:

$$\mathrm{KB} \longrightarrow \mathrm{KB} \cup \{f\}$$

Shrinks the set of models:

$$\mathcal{M}(\mathrm{KB}) \longrightarrow \mathcal{M}(\mathrm{KB}) \cap \mathcal{M}(f)$$



Adding to the knowledge base

Adding more formulas to the knowledge base:

$$\text{KB} \longrightarrow \text{KB} \cup \{f\}$$

Shrinks the set of models:

$= \emptyset$ $\neq \emptyset$ $\neq \mathcal{M}(\text{KB})$

$$\mathcal{M}(\text{KB}) \longrightarrow \mathcal{M}(\text{KB}) \cap \mathcal{M}(f) = \mathcal{M}(\text{KB})$$

How much does $\mathcal{M}(\text{KB})$ shrink?

Model checking

Checking satisfiability (SAT) in propositional logic is special case of solving CSPs!

Mapping:

A

propositional symbol $\Rightarrow$ variable

A

$\text{domain}(A) = \{0, 1\}$

Inference example

$KB \vdash Wet \wedge Slippery$

Example: Modus ponens inference

Starting point:

$$\text{KB} = \{\text{Rain}, \text{Rain} \to \text{Wet}, \text{Wet} \to \text{Slippery}\}$$

$KB \vdash Wet$

$KB \vdash Slippery$

Apply modus ponens to Rain and $\text{Rain} \to \text{Wet}$:

$$\text{KB} = \{\text{Rain}, \text{Rain} \to \text{Wet}, \text{Wet} \to \text{Slippery}, \text{Wet}\}$$

Apply modus ponens to Wet and $\text{Wet} \to \text{Slippery}$:

$$\text{KB} = \{\text{Rain}, \text{Rain} \to \text{Wet}, \text{Wet} \to \text{Slippery}, \text{Wet}, \text{Slippery}\}$$

Definite clauses

Definition: Definite clause

A **definite clause** has the following form:

$$(p_1 \wedge \cdots \wedge p_k) \to q$$

where $p_1, \dots, p_k, q$ are propositional symbols.

$\neg A \to B$

Intuition: if $p_1, \dots, p_k$ hold, then q holds.

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Example: $(\text{Rain} \wedge \text{Snow}) \to \text{Traffic}$

$() \Rightarrow \text{Traffic}$

Example: Traffic

Non-example: $\neg\text{Traffic}$

Non-example: $(\text{Rain} \wedge \text{Snow}) \to (\text{Traffic} \vee \text{Peaceful})$

Completeness of modus ponens

Theorem: Modus ponens on Horn clauses

Modus ponens is **complete** with respect to Horn clauses:

- Suppose KB contains only Horn clauses and p is an entailed propositional symbol.
- Then applying modus ponens will derive p.

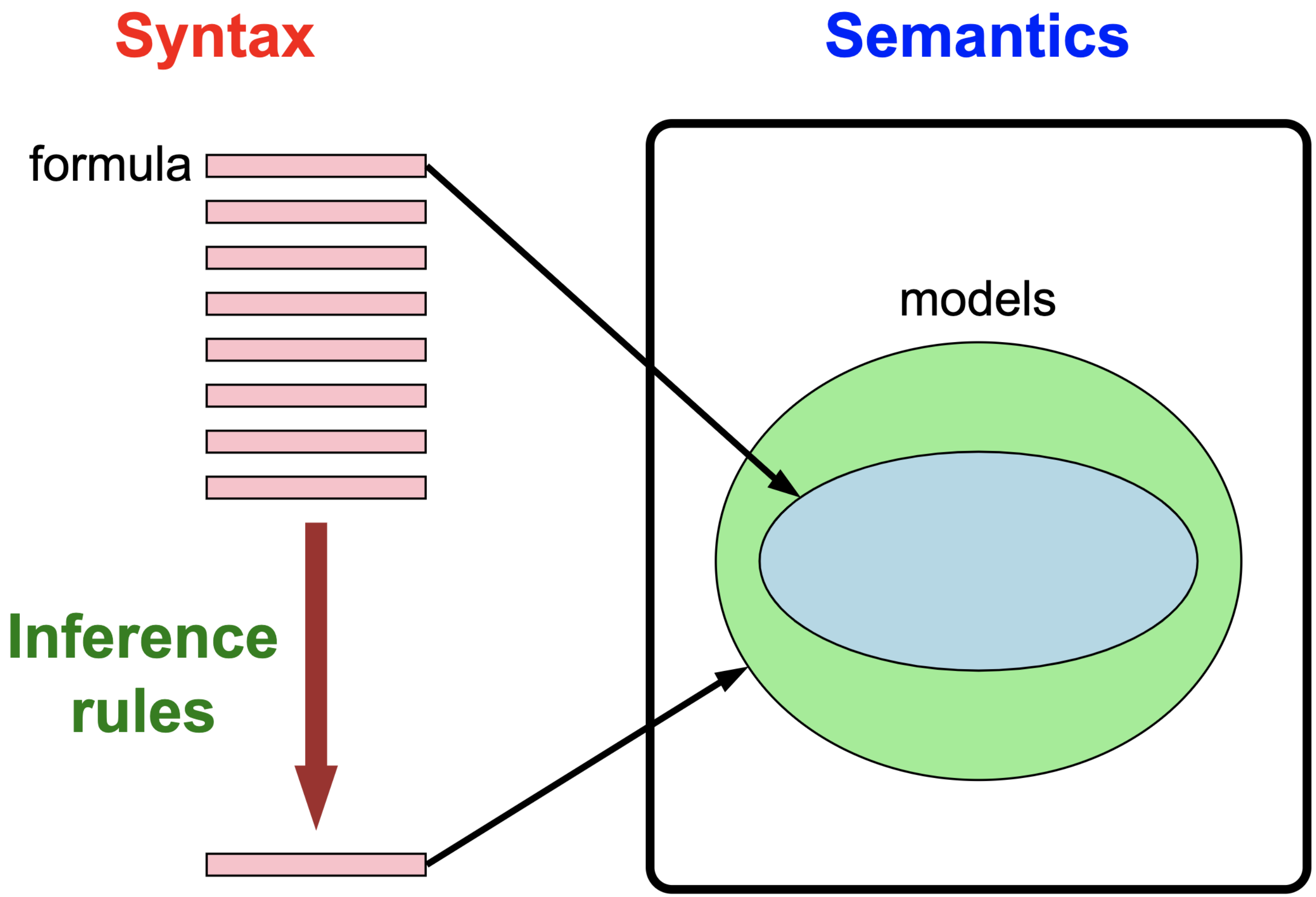
Model:
any p that was derived : 1
anything else : 0.

Upshot:

$\mathrm{KB} \models p$ (entailment) is the same as $\mathrm{KB} \vdash p$ (derivation)!

$\underbrace{KB \to P_1 \quad KB \cup \{P_1\} \to P_2 \quad \cancel{\to} \quad \ldots}$

Summary



Soundness: whatever we produce using inference rules is entailed.

Completeness: whatever formula f is true given KB, we can derive it.