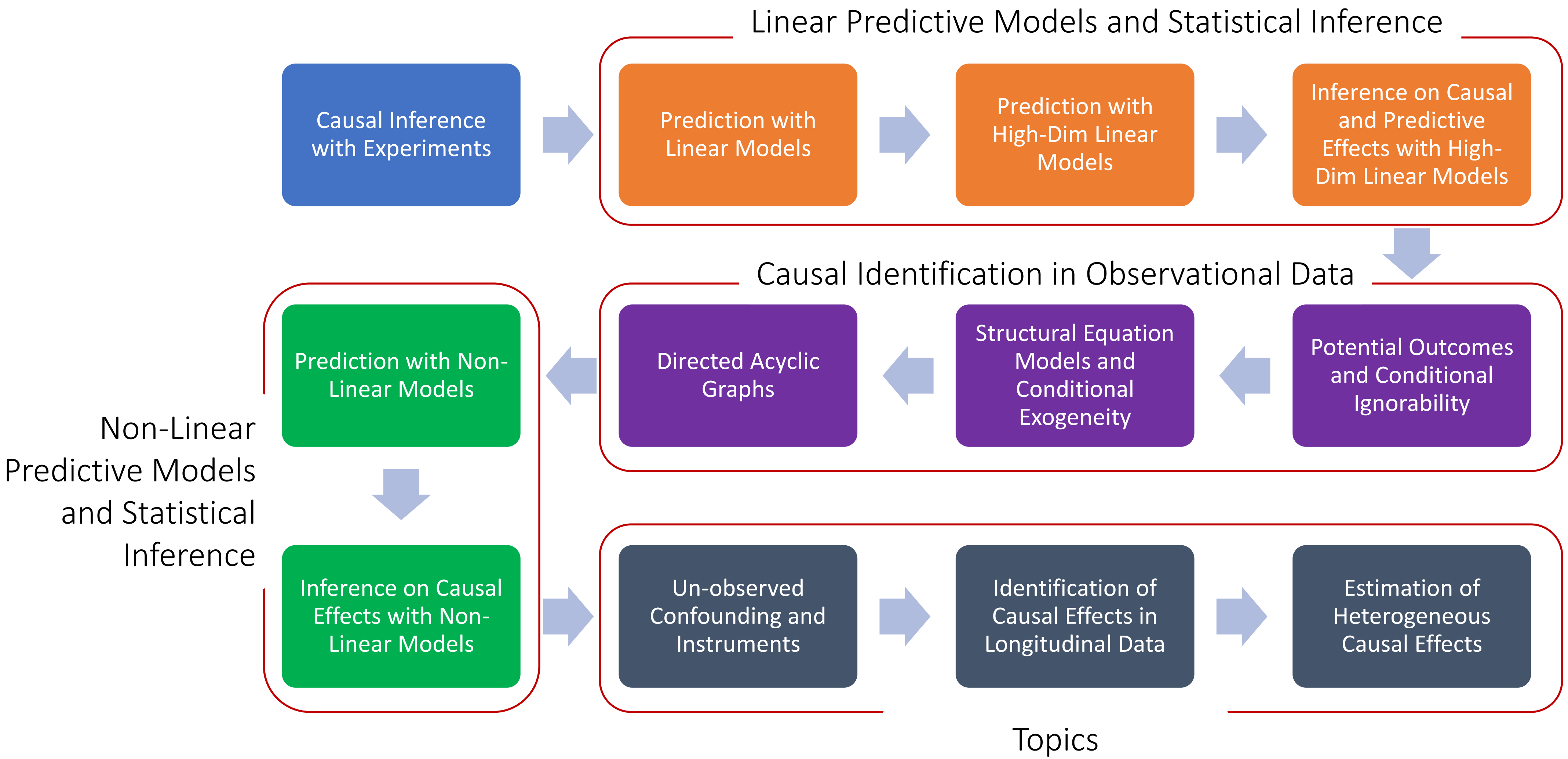
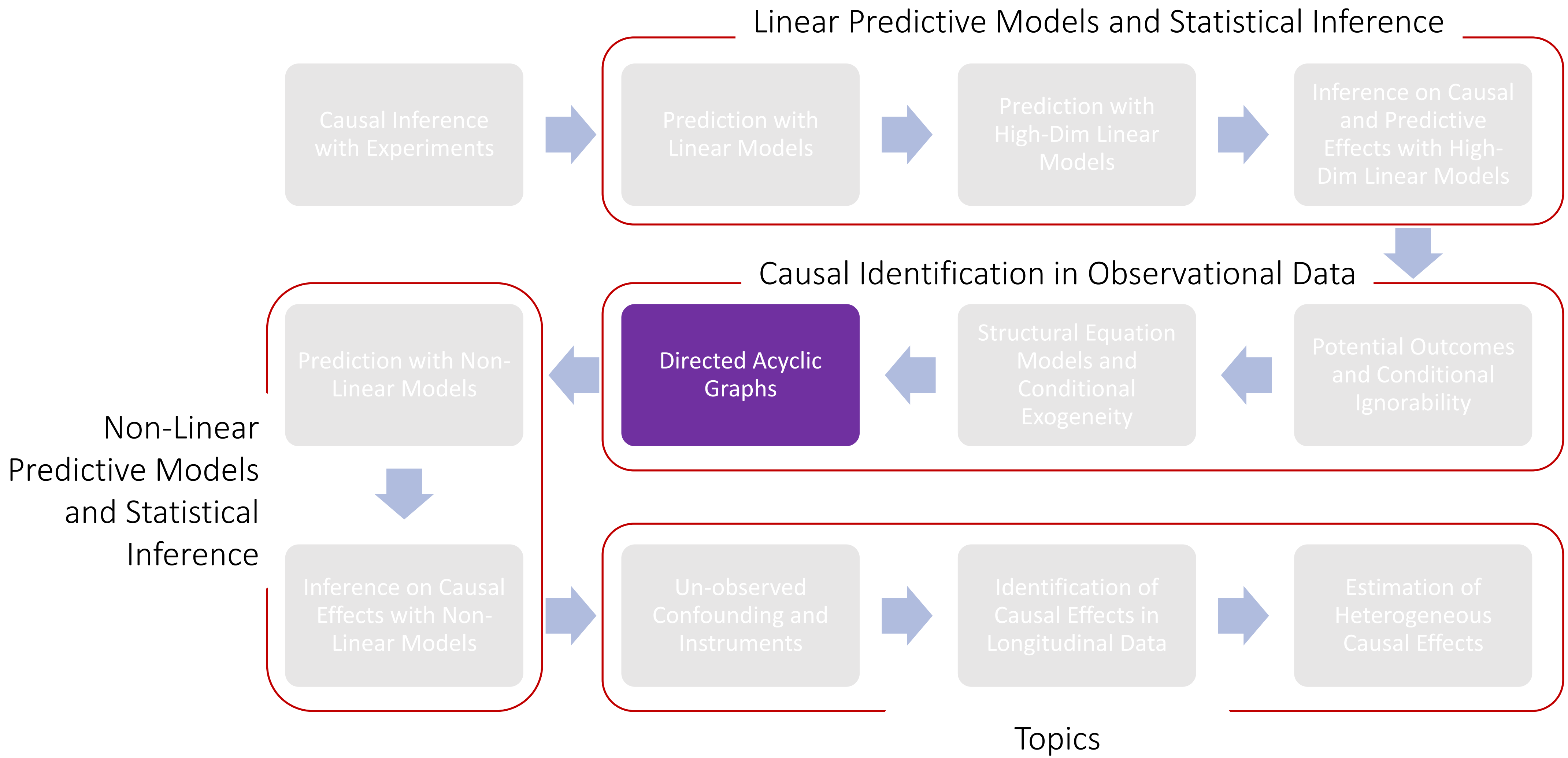
MS&E 228: Directed Acyclic Graphs and Non-Linear SEMs

Vasilis Syrgkanis

MS&E, Stanford


Linear Predictive Models and Statistical Inference
Causal Inference with Experiments
Prediction with Linear Models
Prediction with High-Dim Linear Models
Inference on Causal and Predictive Effects with High-Dim Linear Models
Causal Identification in Observational Data
Potential Outcomes and Conditional Ignorability
Structural Equation Models and Conditional Exogeneity
Directed Acyclic Graphs
Non-Linear Predictive Models and Statistical Inference
Prediction with Non-Linear Models
Inference on Causal Effects with Non-Linear Models
Un-observed Confounding and Instruments
Identification of Causal Effects in Longitudinal Data
Estimation of Heterogeneous Causal Effects
Topics


Linear Predictive Models and Statistical Inference
Causal Inference with Experiments
Prediction with Linear Models
Prediction with High-Dim Linear Models
Inference on Causal and Predictive Effects with High-Dim Linear Models
Causal Identification in Observational Data
Prediction with Non-Linear Models
Directed Acyclic Graphs
Structural Equation Models and Conditional Exogeneity
Potential Outcomes and Conditional Ignorability
Non-Linear Predictive Models and Statistical Inference
Inference on Causal Effects with Non-Linear Models
Un-observed Confounding and Instruments
Identification of Causal Effects in Longitudinal Data
Estimation of Heterogeneous Causal Effects
Topics

Goals for Today

- Learn the “language” of Directed Acyclic Graphs (DAGs) and their associated non-linear structural equation models (SEMs)
- Introduce “intervention” concepts “do” and “fix”
- Introduce d-separation and conditional independence in DAGs
- Proof sketch of fundamental theorem d-separation ⇒ conditional ind.

Next lecture

- Graphical criteria for selection of adjustment set
- Crash course on good and bad “controls”

DAGs

Judea Pearl. 'Causal diagrams for empirical research'. In: *Biometrika* 82.4 (1995), pp. 669–688 (cited on page 30).

Trygve Haavelmo. 'The probability approach in econometrics'. In: *Econometrica: Journal of the Econometric Society* 12 (1944), pp. iii–vi+1–115 (cited on pages 30, 32).

James Heckman and Rodrigo Pinto. 'Causal analysis after Haavelmo'. In: *Econometric Theory* 31.1 (2015 (NBER 2013)), pp. 115–151 (cited on pages 30, 35).

James Robins. 'A new approach to causal inference in mortality studies with a sustained exposure period—application to control of the healthy worker survivor effect'. In: *Mathematical modelling* 7.9-12 (1986), pp. 1393–1512 (cited on page 54).

Thomas S. Richardson and James M. Robins. *Single world intervention graphs (SWIGs): A unification of the counterfactual and graphical approaches to causality*. Working Paper

Non-Linear SEMs

Non-Linear SEMs

- Last time we looked at linear SEMs
- The language of SEMs does not really rely on the linearity assumption

- For example, the Triangular Structural Equation (TSEM)

$$Y := \delta P + \beta' X + \epsilon_Y$$
$$P := \nu' X + \epsilon_P$$
$$X := \epsilon_X$$

Non-Linear SEMs

- Last time we looked at linear SEMs
- The language of SEMs does not really rely on the linearity assumption

- For example, the Triangular Structural Equation (TSEM)
- Can be made non-linear

$$Y \coloneqq f_Y(P, X, \epsilon_Y)$$
$$P \coloneqq f_P(X, \epsilon_P)$$
$$X \coloneqq f_X(\epsilon_X)$$

- While we still maintain that $\epsilon_Y, \epsilon_P, \epsilon_X$ are independent “exogenous” shocks

Non-Linear SEMs

- Non-Linear Triangular Structural Equation (TSEM)

Endogenously determined by the structural model

$$\boxed{Y} := f_Y(P, X, \boxed{\epsilon_Y})$$
$$\boxed{P} := f_P(X, \boxed{\epsilon_P})$$
$$\boxed{X} := f_X(\boxed{\epsilon_X})$$

Exogenously determined "outside" of the model

- $\epsilon_Y, \epsilon_P, \epsilon_X$ are independent "exogenous" shocks
- The functions f_Y, f_P, f_X are deterministic "structural functions"
- Instead of "structural parameters" we now have "structural functions"
- Moreover, the dimension of exogenous shocks is un-restricted
- Note that the TSEM implies: $\epsilon_Y \perp\!\!\!\perp P, X$ and $\epsilon_P \perp\!\!\!\perp X$

Non-Linear SEMs

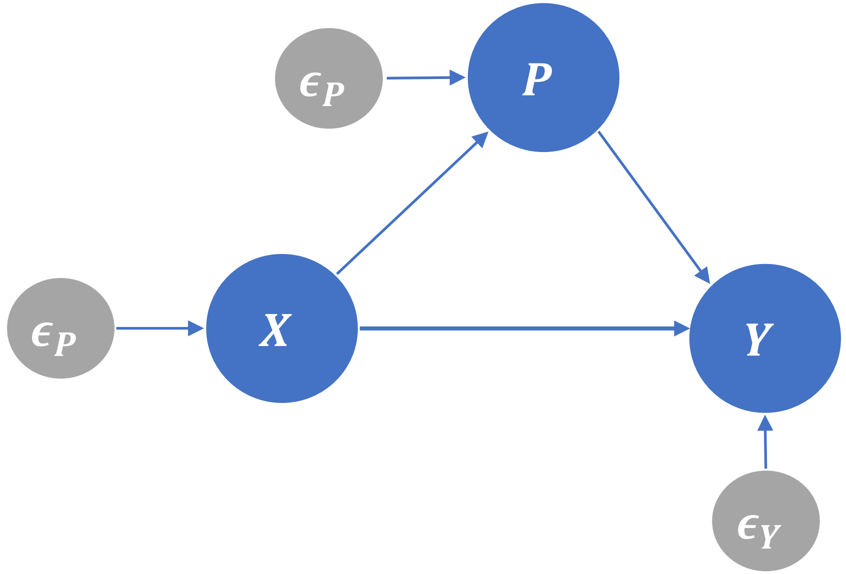
- Non-Linear Triangular Structural Equation (TSEM)

Endogenously determined by the structural model

$$\boxed{Y} := f_Y(P, X, \boxed{\epsilon_Y})$$
$$\boxed{P} := f_P(X, \boxed{\epsilon_P})$$
$$\boxed{X} := f_X(\boxed{\epsilon_X})$$

Exogenously determined "outside" of the model

- $\epsilon_Y, \epsilon_P, \epsilon_X$ are independent "exogenous" shocks



Non-Linear SEMs

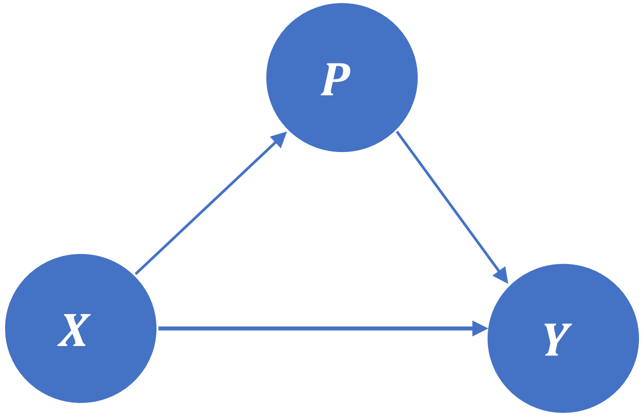
- Non-Linear Triangular Structural Equation (TSEM)

Endogenously determined by the structural model

$$\boxed{Y} := f_Y(P, X, \boxed{\epsilon_Y})$$
$$\boxed{P} := f_P(X, \boxed{\epsilon_P})$$
$$\boxed{X} := f_X(\boxed{\epsilon_X})$$

Exogenously determined “outside” of the model

- $\epsilon_Y, \epsilon_P, \epsilon_X$ are independent “exogenous” shocks; typically omitted from DAG visualization



Non-Linear SEMs

- Non-Linear Triangular Structural Equation (TSEM)

Endogenously determined by the structural model

$$\boxed{Y} := f_Y(P, X, \boxed{\epsilon_Y})$$
$$\boxed{P} := f_P(X, \boxed{\epsilon_P})$$
$$\boxed{X} := f_X(\boxed{\epsilon_X})$$

Exogenously determined “outside” of the model

- $\epsilon_Y, \epsilon_P, \epsilon_X$ are independent “exogenous” shocks; typically omitted from DAG visualization
- A TSEM is simply a statistical “generative” model that determines a distribution over observed random variables (*c.f. Neural-Causal Models in further reading)

Structural Form

- TSEM is "structural" in that it is endowed with the following properties
- Made up of a collection of stochastic potential outcome processes indexed by (p, x)

$$Y(p, x) \coloneqq f_Y(p, x, \epsilon_Y)$$
$$P(x) \coloneqq f_P(x, \epsilon_P)$$
$$X \coloneqq f_X(\epsilon_X)$$

- **Exogeneity:** $\epsilon_P, \epsilon_Y, \epsilon_X$ are independent "shock" variables generated outside of the model
- **Consistency:** endogenous variables (Y, P, X) generated by recursive substitutions

$$Y \coloneqq Y(P, X), \qquad P \coloneqq P(X), \qquad X \coloneqq \epsilon_X$$

- **Invariance:** structure remains invariant to changes of distributions of shocks

Link to Potential Outcomes

- Consider (for simplicity) binary treatments $p \in \{0,1\}$
- Suppose that potential outcomes are generated wlog as:

$$Y(p) \coloneqq g\big(p, X, \epsilon_Y(p)\big)$$

- This is equivalent to a SEM where we define $\epsilon_Y = \big(\epsilon_Y(0), \epsilon_Y(1)\big)$ and

$$Y(p) \coloneqq f_Y(p, X, \epsilon_Y)$$

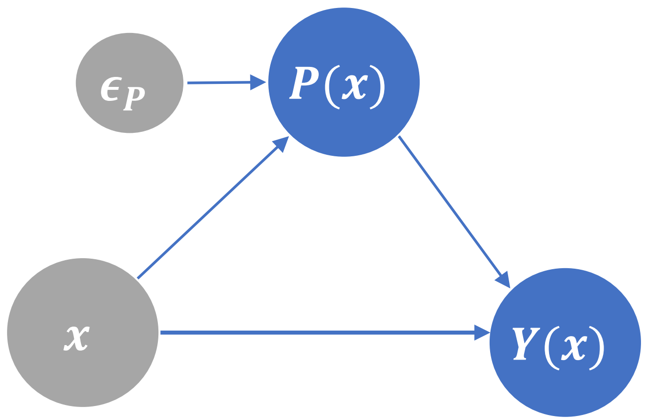
- where

$$f_Y(p, X, e) = p \cdot g\big(p, X, e(1)\big) + (1 - p) \cdot g\big(p, X, e(0)\big)$$

Identification of Structural Responses

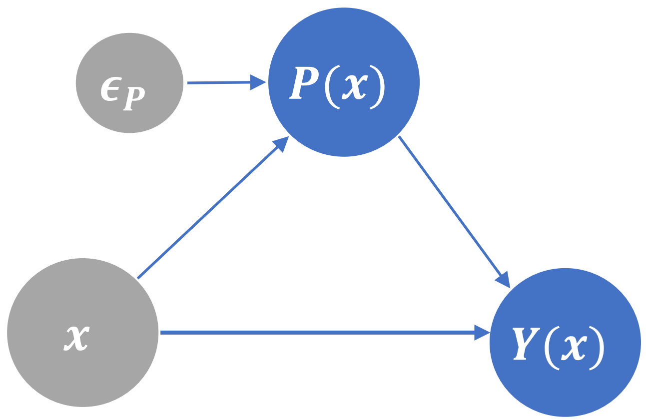
Identification by Regression Revisited

- If we condition on $X = x$ it's as if we're altering the graph and SEM
$$Y = f_Y(P(x), x, \epsilon_Y), \qquad \epsilon_Y \perp\!\!\!\perp P(x)$$
- Notice that remnant variation in $P(x)$, is ϵ_P, which is exogenous
- As if driven by a randomized trial process



Identification by Regression Revisited

- If we condition on $X = x$ it's as if we're altering the graph and SEM
$$Y = f_Y(P(x), x, \epsilon_Y), \qquad \epsilon_Y \perp\!\!\!\perp P(x)$$
- Notice that remnant variation in $P(x)$, is ϵ_P, which is exogenous
- As if driven by a randomized trial process
- If we further condition on $P(x) = p$, we learn $E[f_Y(p, x, \epsilon_Y)]$



Identification by Regression Revisited

- The conditional expectation function $E[Y|P = p, X = x]$ recovers the conditional average structural response function $E[f_Y(p, x, \epsilon_Y)]$

$$\begin{aligned} E[Y|P = p, X = x] &= E[\, f_Y(P, X, \epsilon_Y) \mid P = p, X = x \,] \\ &= E[\, f_Y(p, x, \epsilon_Y) \mid P = p, X = x \,] \\ &= E[f_Y(p, x, \epsilon_Y)] \end{aligned}$$

- Average structural response = Expected outcome when P, X are exogenously set (outside of the model) to take values (p, x)
- It is useful for generating counterfactual predictions; what "would happen on average if" we intervene and set $(P, X) \leftarrow (p, x)$
- For TSEM: counterfactual predictions $\equiv$ predictions

Identification by Regression Re-stated

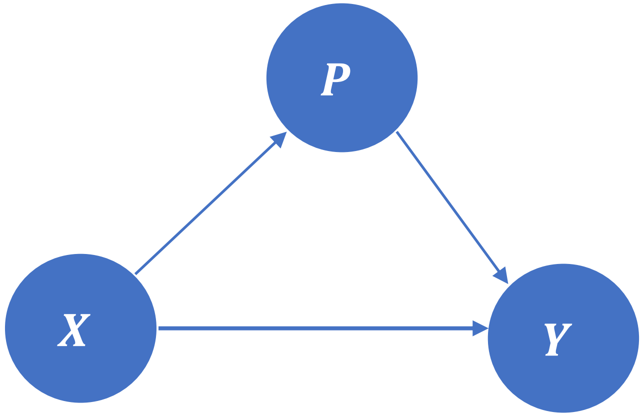
- For TSEM, the conditional average structural causal effect coincides with the conditional average predictive effect
$$E[f_Y(p_1, x, \epsilon_Y)] - E[f_Y(p_0, x, \epsilon_Y)]$$
$$= E[Y|P = p_1, X = x] - E[Y|P = p_0, X = x]$$
- Left hand side is a structural hypothetical quantity: what would happen if we intervene and change P from p_0 to p_1, at $X = x$
- Right hand side is a statistical quantity that can be calculated from observed random variables Y, P, X
- **Identification:** Mapping of “structural hypothetical quantities” to “measurable quantities” from data

Formalizing the Language of Interventions

Do Interventions: $do(P = p)$

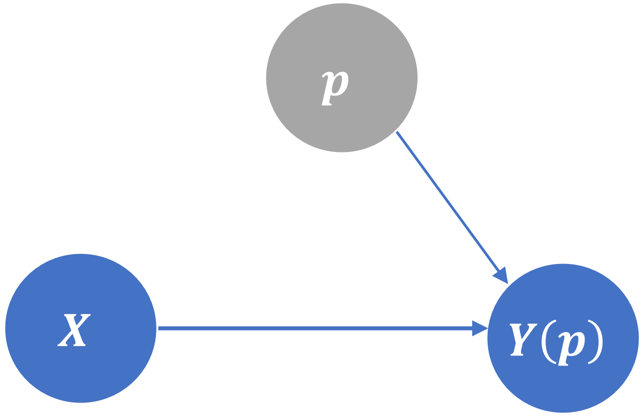
Original Data Generative Model

$$
\begin{aligned}
Y &:= f_Y(P, X, \epsilon_Y) \\
P &:= f_P(X, \epsilon_P) \\
X &:= \epsilon_X
\end{aligned}
$$



Data Generative Model under $do(P = p)$

$$
\left.\begin{matrix} Y \\ P \\ X \end{matrix}\right| do(P = p) \quad
\begin{aligned}
&:= f_Y(p, X, \epsilon_Y) \\
&\quad\; p \\
&:= \epsilon_X
\end{aligned}
$$



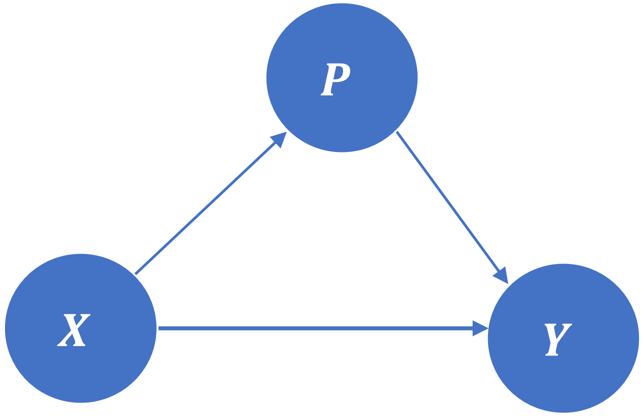
Interventions

- Do-interventions is only one way of defining counterfactuals
- We can define any type of counterfactual by simply changing one of the equations to something else
- Wright in his seminal work in ‘28 defined an intervention where the demand equation was replaced by another one that reflects a tax hike
- We can also define “soft-interventions”: increase price by 10% of its current value
- Another useful variant of do-interventions does not replace the treatment equation are “fix” interventions

Fix Interventions: $\text{fix}(P = p)$

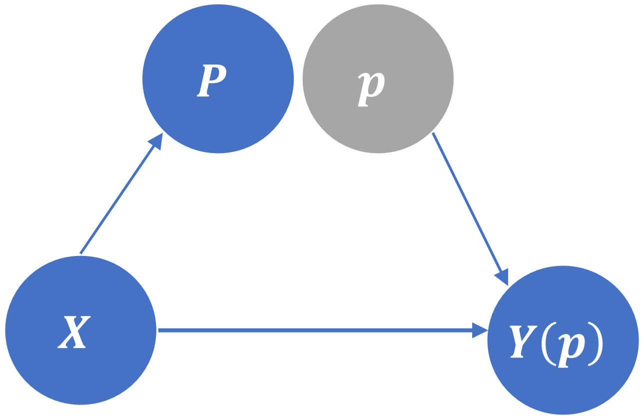
Original Data Generative Model

$$
\begin{aligned}
Y &:= f_Y(P, X, \epsilon_Y) \\
P &:= f_P(X, \epsilon_P) \\
X &:= \epsilon_X
\end{aligned}
$$



Data Generative Model under $\text{fix}(P = p)$

$$
\left.\begin{array}{l} Y \\ P \\ X \end{array}\right| \text{fix}(P = p) \quad
\begin{array}{l} := f_Y(p, X, \epsilon_Y) \\ \quad\; f_P(X, \epsilon_P) \\ := \epsilon_X \end{array}
$$



Fix Interventions

- A fix intervention is a form of “localized” do intervention
- We are only fixing the value of P in the structural equation for Y
- The random variables generated by the fix intervention are the triplets
$$(Y(p), P, X)$$
- The intervention does not affect the P, X equations nor the distribution of the exogenous shock ϵ_Y in the outcome equation

Single World Intervention Graphs

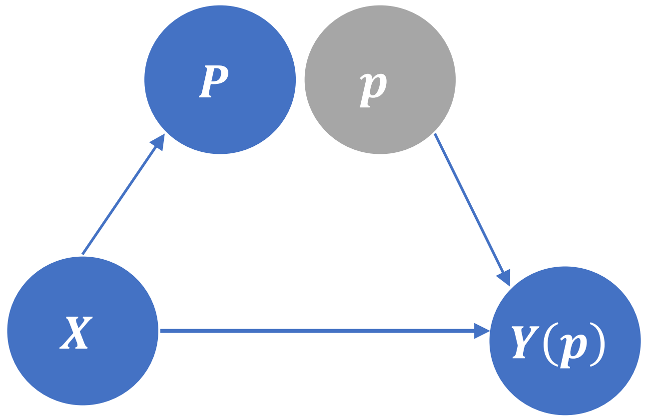
- The graphs that represent the generative model under a fix intervention

- Easy to verify visually that
$$Y(p) \perp\!\!\!\perp P \mid X$$

- Then we can do identification based on conditional ignorability

Data Generative Model under $\text{fix}(P = p)$

$$\left.\begin{array}{l} Y \\ P \\ X \end{array}\right| \begin{array}{lll} & := & f_Y(p, X, \epsilon_Y) \\ \text{fix}(P = p) & & f_P(X, \epsilon_P) \\ & := & \epsilon_X \end{array}$$



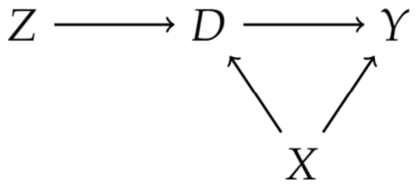
Single World Intervention Graph

Testable Implications of a DAG

D-Separation and Conditional Independence

DAGs Encode Factorization of Probability

- Graph implies factorization of the probability law
$$p(y,d,x,z) = p(y|x,d)p(d|x,z)p(x)p(z)$$
- By repeated application of Bayes rule
$$p(y,d,x,z) = p(y|d,x,z)p(d,x,z)$$
- From graph
$$p(y|d,x,z) = p(y|d,x)$$
- Further Bayes rule
$$p(d,x,z) = p(d|x,z)p(x,z)$$
- From independence: $p(x,z) = p(x)p(z)$



General DAGs

- For any DAG, we can write the ASEM
$$X_j \coloneqq f_j(\text{Parents}_j, \epsilon_j) = f_j(\text{Pa}_j, \epsilon_j)$$
- Shocks ϵ_j are jointly independent and independent of $\{X_j\}$
- And the corresponding structural response functions
$$X_j(pa_j) \coloneqq f_j(pa_j, \epsilon_j)$$
- Where pa_j are potential values of the parent nodes that index the stochastic potential outcome processes
- Consistency: variables X_j are generated by generating the shocks and then solving repeatedly the structural response functions

General DAGs and Factorization

- The probability law factorizes as:

$$p(\{x_\ell\}_{\ell \in V}) = \prod_{\ell \in V} p(x_\ell | pa_\ell)$$

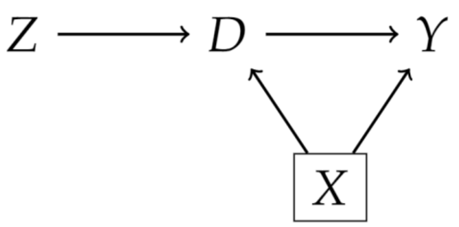
DAGs Encode Conditional Independencies

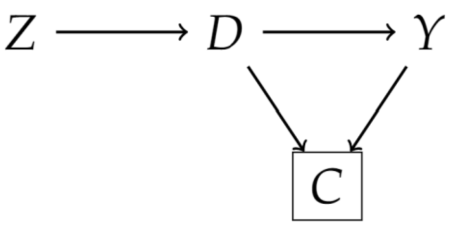
- Any two variables X, Y are independent conditional on a set S if they are D(irected)-separated in the graph
$$\left(X \perp\!\!\!\perp_{d} Y \mid S \right)_{G} \quad \Rightarrow \quad X \perp\!\!\!\perp Y \mid S$$

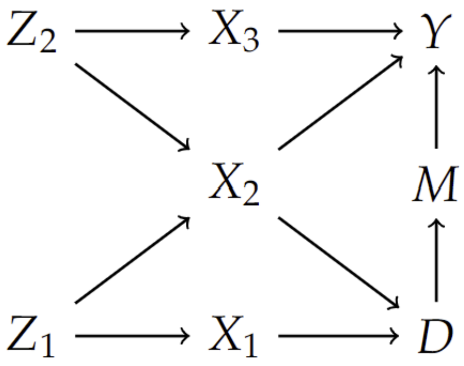
- Need to define the concept of D-separation

Some Graph Definitions

- A path π in a graph is blocked by a set of nodes S if
  - Either π contains a chain $i \to m \to j$ or a fork $i \leftarrow m \to j$ and $m \in S$
  - Or π contains a collider $i \to m \leftarrow j$ and neither m nor its descendants are in S







D-Separation

- In a DAG G, two nodes X, Y are D-separated by s set of nodes S if S blocks all paths between X and Y

- We denote it as:

$$\left(X \perp\!\!\!\perp_d Y \mid S \right)_G$$

D-separation implies conditional independency

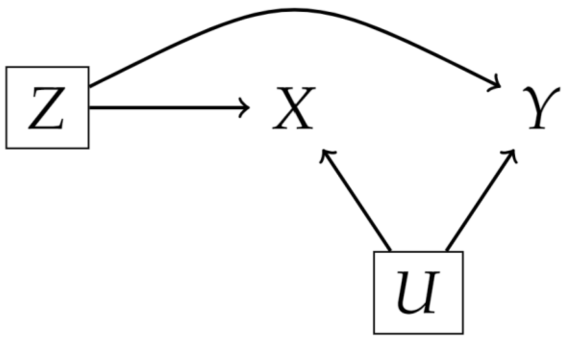
$$\left(X \perp\!\!\!\perp_d Y \mid S \right)_G \quad \Rightarrow \quad X \perp\!\!\!\perp Y \mid S,$$

$$(\text{Verma}, \text{Pearl},'\ 88)$$

Examples

- By factorization property and Bayes rule

$$p(y, x|z, u) = p(y|x, z, u)\, p(x|z, u) = p(y|z, u)\, p(x|z, u)$$



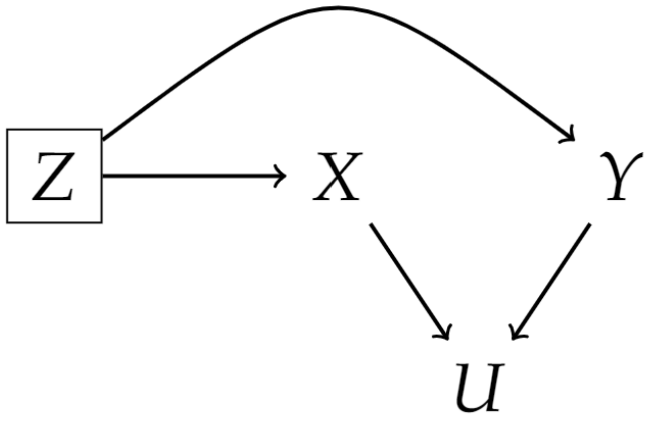
Examples

- By factorization property and Bayes rule

$$p(y, x|z) = p(y|x, z)\, p(x|z) = p(y|z)\, p(x|z)$$

- If we had included u, then

$$p(y, x|z, u) = \frac{p(y, x, u|z)}{p(u|z)} = \frac{p(u|x, y, z)p(y|x, z)p(x|z)}{p(u|z)}$$
$$= \frac{p(u|x, y)p(y|z)p(x|z)}{p(u|z)}$$



- We cannot write it as the product of two functions

$$p(y, x|z, u) = f(x, z, u) \cdot g(y, z, u)$$

Proving the Main Theorem!

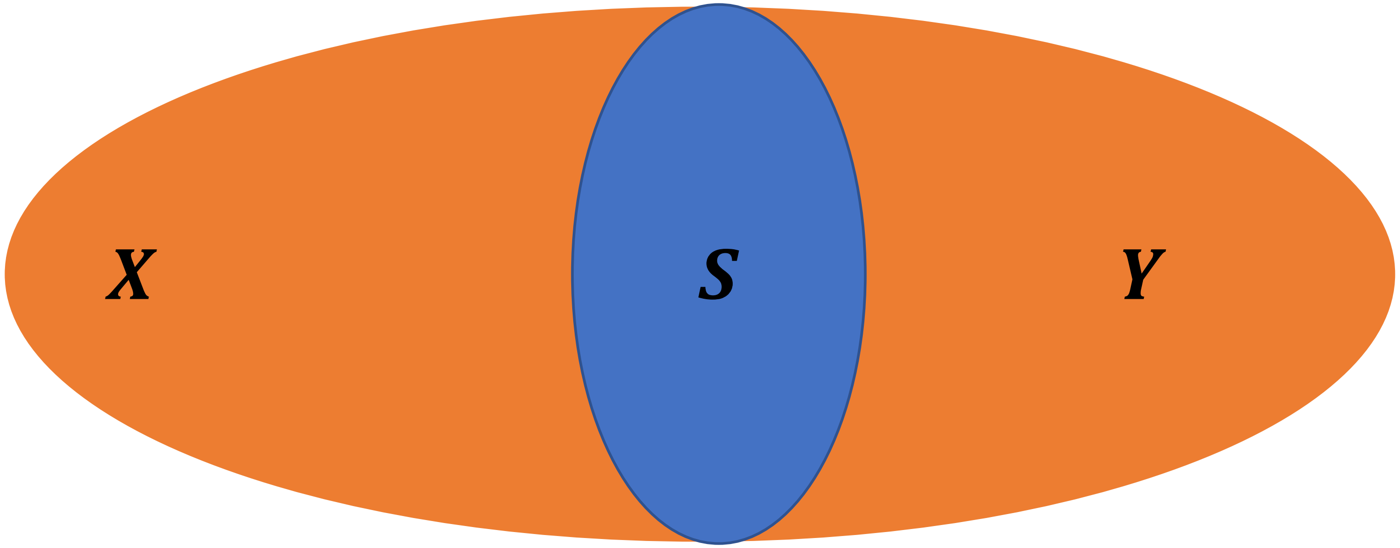
Proof Step 1

- A set of nodes $\boldsymbol{X}$ is called ancestral if all ancestors of $\boldsymbol{X}$ are in $\boldsymbol{X}$

- Removing all nodes outside of an ancestral set and looking at the resulting graph and ASEM, the probability law is the same as the probability law of $\boldsymbol{X}$ in the original graph (exercise)

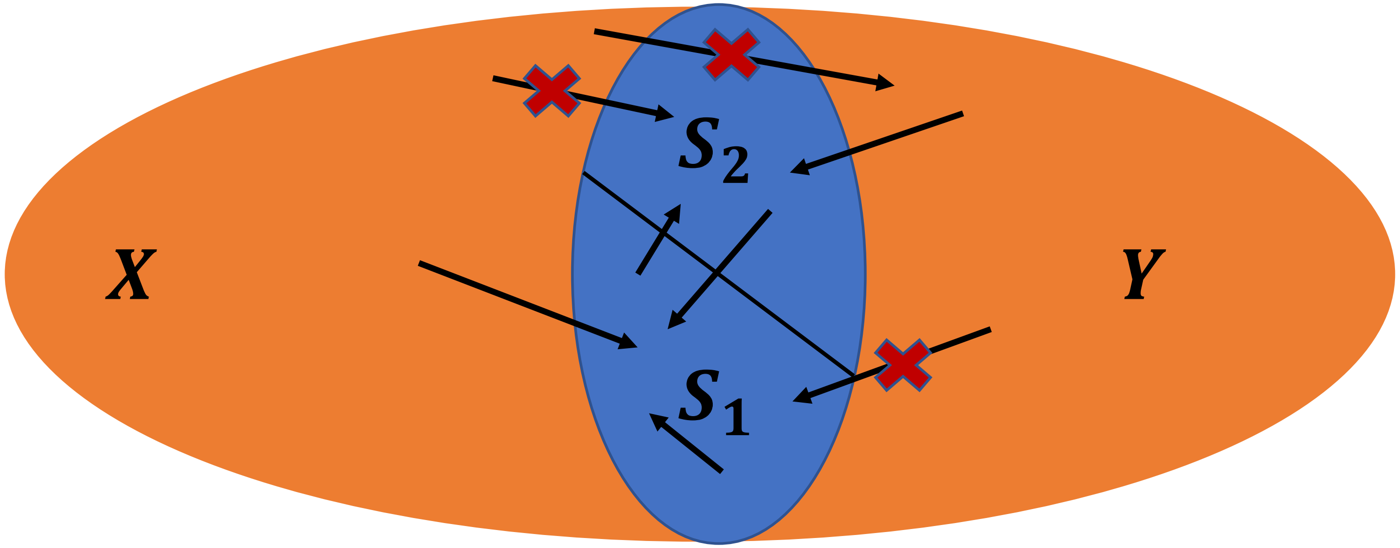
Proof Step 2

- Suppose that a set of nodes $\boldsymbol{X}$ is D-separated from a set of nodes $\boldsymbol{Y}$ by a set of nodes $\boldsymbol{S}$
- And that $\boldsymbol{X} \cup \boldsymbol{Y} \cup \boldsymbol{S}$ is the set of all nodes



Proof Step 2

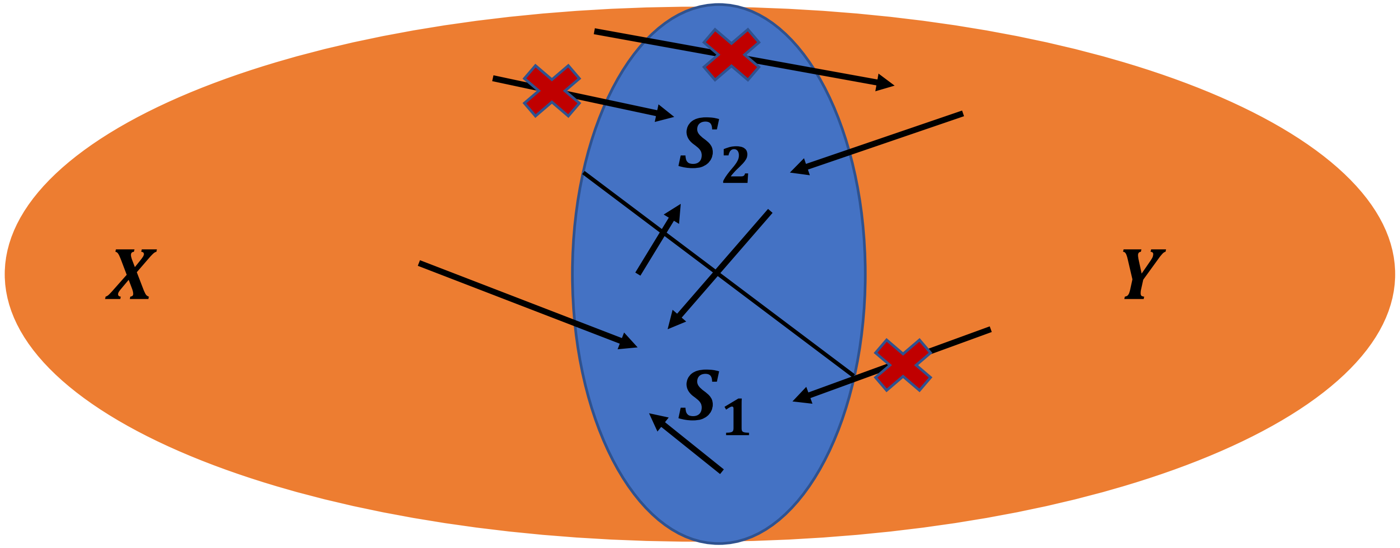
- Let S_1 the subset of S that has a parent in X. Let S_2 the remainder.
- It has to be that $Pa(X \cup S_1) \in X \cup S$
- It has to be that $Pa(Y \cup S_2) \in Y \cup S$



Proof Step 2

- We can factorize:

$$p(x,y,s) = \prod_{W \in X \cup S_1} p(w|pa_W) \prod_{W \in Y \cup S_2} p(w|pa_W) = f(x,s_1)g(y,s_2)$$



Proof Step 2

- We can factorize:

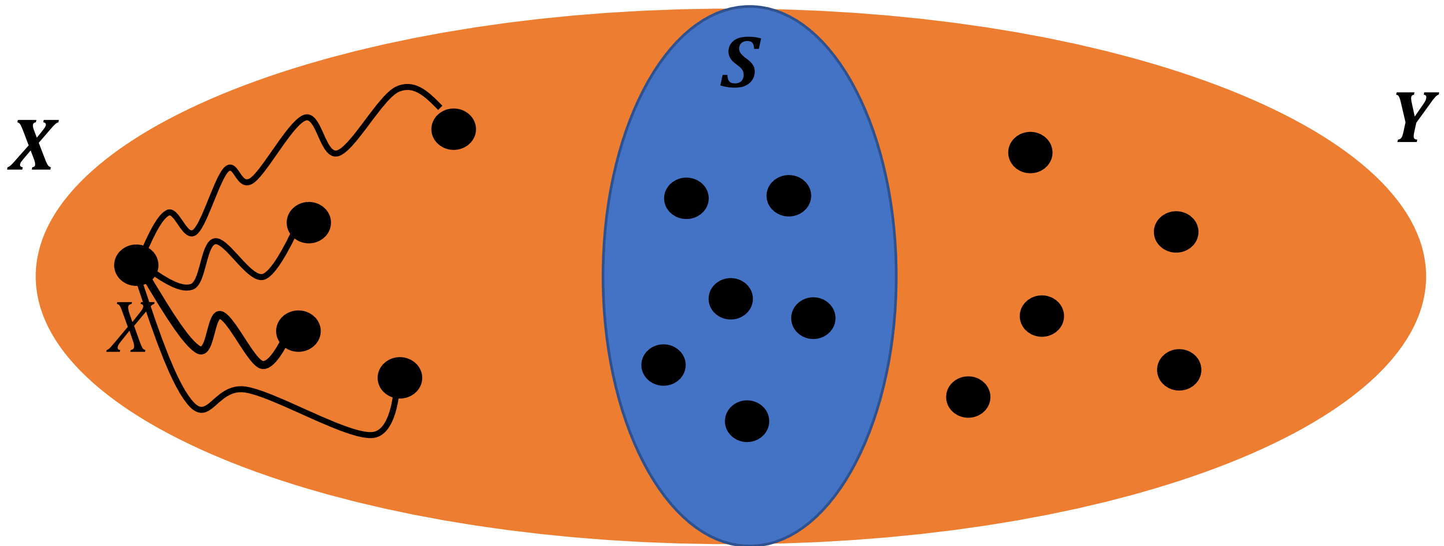
$$p(x, y, s) = \prod_{W \in X \cup S_1} p(w|pa_W) \prod_{W \in Y \cup S_2} p(w|pa_W) = f(x, s_1) g(y, s_2)$$

- Implies that:

$$\boldsymbol{X} \perp\!\!\!\perp \boldsymbol{Y} \mid S$$

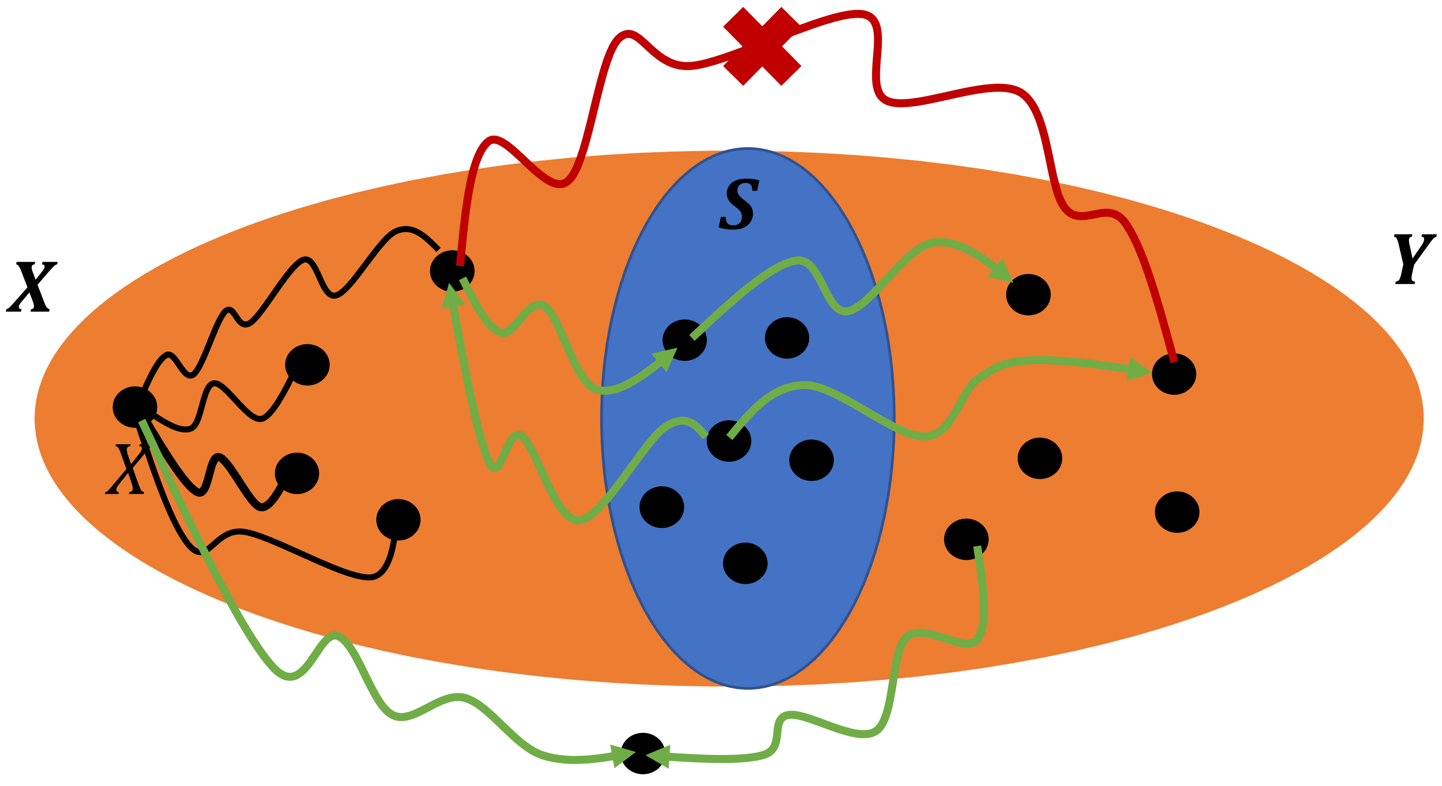
Final Step

- By first step, we can restrict to ancestral set of $X \cup Y \cup S$
- Does not change conditional independence relations (exercise)
- Does not change d-separation relations (exercise)
- Define $\boldsymbol{X}$ nodes in ancestral set of $X \cup Y \cup S$ not d-separated from X
- Define $\boldsymbol{Y}$ the remainder of nodes in ancestral set not in $\boldsymbol{X}, \boldsymbol{S}$.



Final Step

- By definition of d-separation, S must d-separate $\boldsymbol{X}$ from $\boldsymbol{Y}$ (exercise)
- We can invoke previous critical lemma



Final Step

- By marginalization

$$p(x, y, \boldsymbol{s}) = \int \int p(x, \boldsymbol{x}', y, \boldsymbol{y}', \boldsymbol{s}) d\boldsymbol{x}' \, d\boldsymbol{y}'$$

- By step 2

$$p(x, y, \boldsymbol{s}) = \int \int f(x, \boldsymbol{x}', \boldsymbol{s}) \, g(y, \boldsymbol{y}', \boldsymbol{s}) \, d\boldsymbol{x}' \, d\boldsymbol{y}'$$

- We can split integrals

$$p(x, y, \boldsymbol{s}) = \int f(x, \boldsymbol{x}', \boldsymbol{s}) d\boldsymbol{x}' \int g(y, \boldsymbol{y}', \boldsymbol{s}) \, d\boldsymbol{y}'$$

- Thus

$$p(x, y, \boldsymbol{s}) = \bar{f}(x, \boldsymbol{s}) \bar{g}(y, \boldsymbol{s}) \Rightarrow X \perp\!\!\!\perp Y \mid S$$