

MS&E 233

Game Theory, Data Science and AI

Lecture 11

Vasilis Syrgkanis

Assistant Professor

Management Science and Engineering

(by courtesy) Computer Science and Electrical Engineering

Institute for Computational and Mathematical Engineering

Computational Game Theory for Complex Games

- 1
 - Basics of game theory and zero-sum games (T)
 - Basics of online learning theory (T)
 - Solving zero-sum games via online learning (T)
 - *HW1: implement simple algorithms to solve zero-sum games*
 - Applications to ML and AI (T+A)
 - *HW2: implement boosting as solving a zero-sum game*

- 2
 - Basics of extensive-form games
 - Solving extensive-form games via online learning (T)
 - *HW3: implement agents to solve very simple variants of poker*

- 3
 - General games, equilibria and online learning (T)
 - Online learning in general games
 - *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

- 4
 - Basics and applications of auction theory (T+A)
 - Basic Auctions and Learning to bid in auctions (T)
 - *HW5: implement bandit algorithms to bid in ad auctions*

- 5
 - **Optimal auctions and mechanisms (T)**
 - **Simple vs optimal mechanisms (T)**
 - *HW6: calculate equilibria in simple auctions, implement simple and optimal auctions, analyze revenue empirically*

- 6
 - **Optimizing mechanisms from samples (T)**
 - **Online optimization of auctions and mechanisms (T)**
 - *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

- 7
 - **Econometrics in games and auctions (T+A)**
 - **A/B testing in markets (T+A)**
 - *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Sum: Vickrey-Clarke-Groves (VCG)

A universal welfare maximizing auction/mechanism!

For any mechanism design setting, it guarantees that:

1. It is dominant strategy truthful
2. It always chooses the welfare maximizing outcome/allocation
3. All bidders have non-negative utility
4. All payments are non-negative

For special case of single-item auction = Second-Price Auction

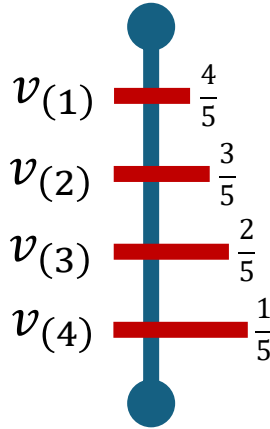
What if we want to maximize
revenue?

Let's go back to basics: Single-Item Auction

- How much revenue does the second-price auction achieve?

$$\text{Rev} = E[v_{(2)}] = E[\min(v_1, v_2)] = 1/3$$

- Can we do better?



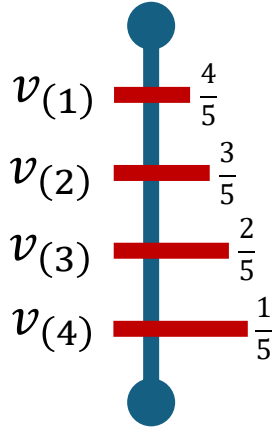
A screenshot of a Google search results page. The search bar contains 'digital advertising'. Below the search bar, there are tabs for 'All', 'Images', 'News', 'Videos', 'Shopping', and 'More'. The results show 'About 6,620,000,000 results (0.44 seconds)'. A sponsored ad for Reddit is highlighted with a red border. The ad includes the Reddit logo, the text 'Reddit', the URL 'https://www.redditforbusiness.com', and the heading 'Advertise on Reddit'. Below the heading, it says 'Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads.' A green box with the number '1' is placed to the right of the ad.

Let's go back to basics: Single-Item Auction

- What if we only had one bidder?

$$\text{Rev} = E[v_{(2)}] = 0$$

- Can we do better?



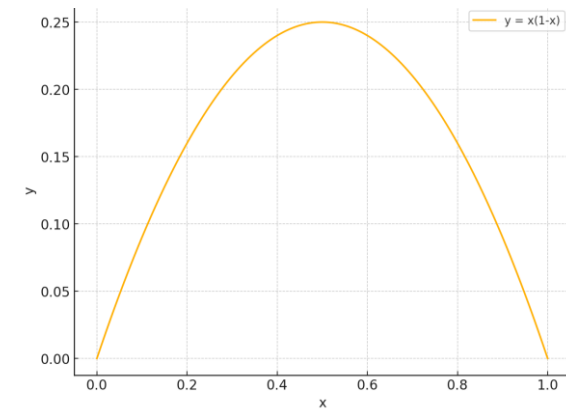
A screenshot of a Google search results page for the query "digital advertising". The search bar at the top shows the query and a "X" button. Below the search bar are tabs for "All", "Images", "News", "Videos", "Shopping", and "More". The results show "About 6,620,000,000 results (0.44 seconds)". A sponsored ad for "Reddit" is highlighted with a red border. The ad includes the Reddit logo, the text "Reddit", the URL "https://www.redditforbusiness.com", and the heading "Advertise on Reddit". The ad body text says: "Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads." A green box with the number "1" is placed to the right of the ad.

What if we post a reserve price?

Let's go back to basics: Single-Item Auction

- **Auctioneer:** “If you bid less than r , I will not accept your bid and not show any ad on the page! If you win you must pay r .”

$$\text{Rev}(r) = E[r \mathbf{1}(v \geq r)] = r(1 - r) \Rightarrow \text{Rev}(1/2) = 1/4$$



- Is the auction truthful?
- Is the auction efficient?

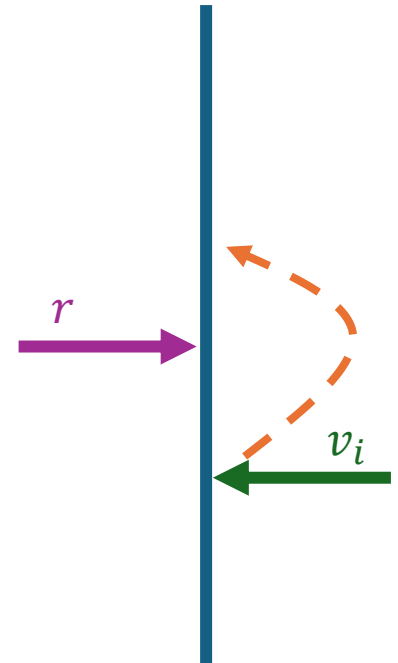


A screenshot of a Google search results page for the query "digital advertising". The search bar shows the query and the Google logo. Below the search bar are tabs for "All", "Images", "News", "Videos", "Shopping", and "More". The results show "About 6,620,000,000 results (0.44 seconds)". A sponsored ad for Reddit is highlighted with a red border. The ad includes the Reddit logo, the text "Reddit", the URL "https://www.redditforbusiness.com", and the heading "Advertise on Reddit". The ad text describes reaching over 100K communities and creating impact. A green box with the number "1" is placed to the right of the ad.

Truthfulness of Mechanism

Suppose I bid my value. Would I want to deviate?

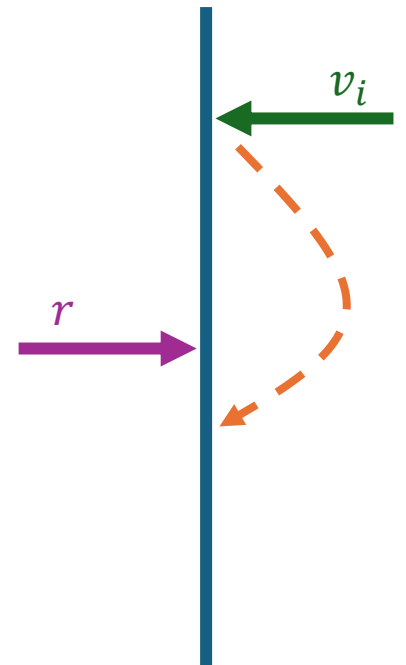
- **Case 1.** My value is below reserve price
- Only way to change anything is bid above
- But then I get negative utility as I pay more than value



Truthfulness of Mechanism

Suppose I bid my value. Would I want to deviate?

- **Case 2.** My value is above reserve price
- I get non-negative utility
- Only way to change anything is bid below
- But then I get zero utility as I lose



Is the mechanism efficient?

Yes

No

Is the mechanism efficient?

Yes

0%

No

0%

Is the mechanism efficient?

Yes

0%

No

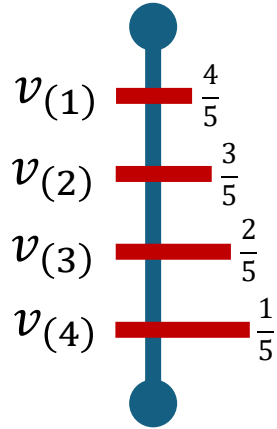
0%

Let's go back to basics: Single-Item Auction

- How much revenue does the second-price auction achieve?

$$\text{Rev} = E[v_{(2)}] = E[\min(v_1, v_2)] = 1/3$$

- Can we do better?



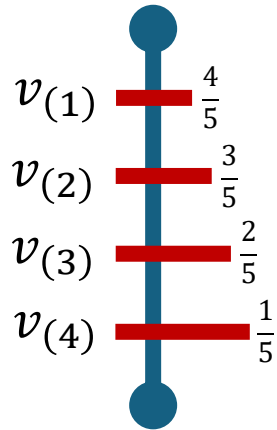
A screenshot of a Google search results page. The search bar contains 'digital advertising'. Below the search bar are tabs for 'All', 'Images', 'News', 'Videos', 'Shopping', and 'More'. The results show 'About 6,620,000,000 results (0.44 seconds)'. A sponsored ad for Reddit is highlighted with a red border. The ad includes the Reddit logo, the text 'Reddit', the URL 'https://www.redditforbusiness.com', and the heading 'Advertise on Reddit'. Below the heading is a paragraph: 'Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads.' A green box with the number '1' is placed to the right of the ad.

Let's go back to basics: Single-Item Auction

- **Auctioneer:** “If you bid less than r , I will not accept your bid!
If you win you must pay $\max(b_2, r)$.”

$$\text{Rev}(1/2) = E[\max(v_{(2)}, r) 1(v_{(1)} \geq r)] = 5/12$$

- Can we do better?



A screenshot of a Google search results page. The search bar contains "digital advertising". Below the search bar, there are tabs for "All", "Images", "News", "Videos", "Shopping", and "More". The results show "About 6,620,000,000 results (0.44 seconds)". A sponsored ad for Reddit is highlighted with a red border. The ad includes the Reddit logo, the text "Reddit", the URL "https://www.redditforbusiness.com", and the heading "Advertise on Reddit". Below the heading, it says "Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads." A green box with the number "1" is placed to the right of the ad.

How do we optimize over all possible mechanisms!

Single-Parameter Settings

- Each bidder has some value v_i for being allocated
- Bidders submit a reported value b_i (without loss of generality)
- Mechanism decides on an allocation $x \in X \subseteq \{0,1\}^n$
- Mechanism fixes a probabilistic allocation rule:

$$x(b) \in \Delta(X)$$

- **First question.** Given an allocation rule, when can we find a payment rule p so that the overall mechanism is truthful?
- If we can find such a payment, we will say that x is **implementable**

Some Shorthand Notation

- Let's fix bidder i and what other bidders bid b_{-i}
- For simplicity of notation, we drop index i and b_{-i}

- What properties does the function

$$x(v) \equiv x_i(v, b_{-i})$$

need to satisfy, so that x is implementable?

- Can we find a truthful payment function

$$p(v) \equiv p(v, b_{-i})$$

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

Yes

No

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

Yes

0%

No

0%

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

Yes

0%

No

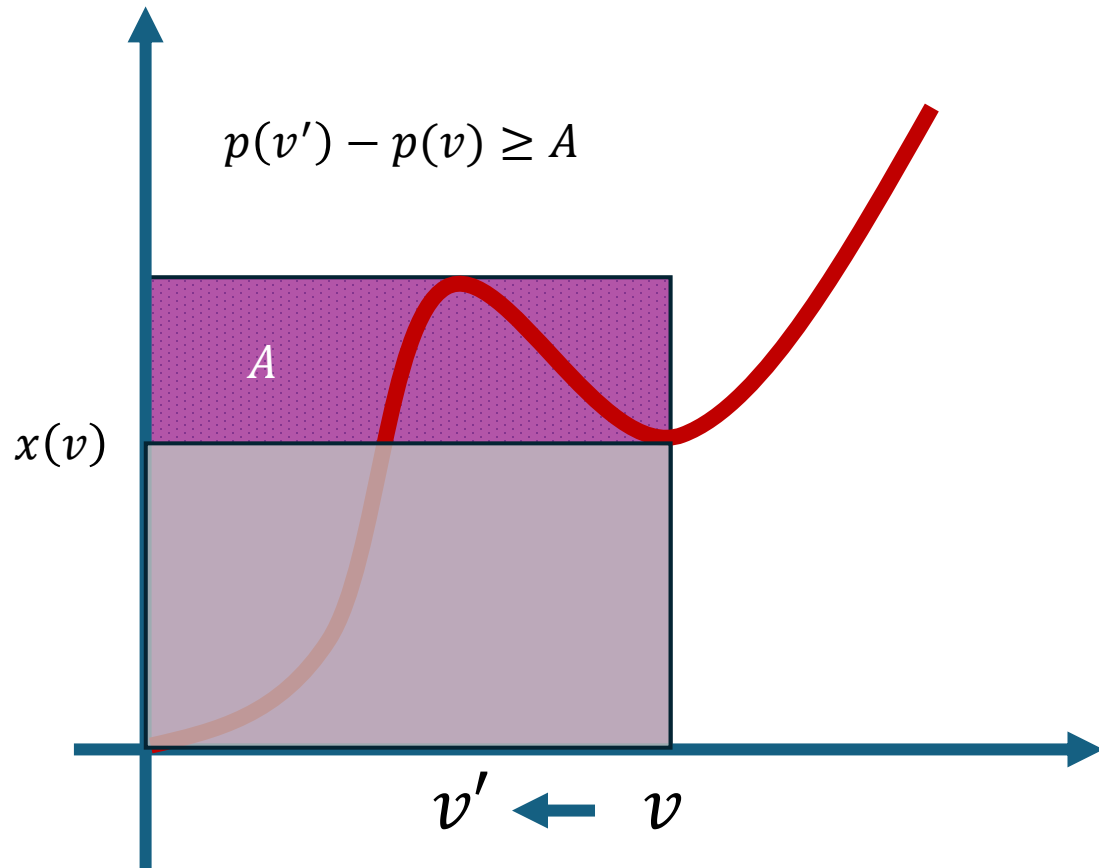
0%

Suppose it is possible

- Suppose that we both bid truthfully
- Suppose that I am the highest value bidder
- No matter what the payment rule is, I can always reduce my bid to the second highest bid minus ϵ
- By doing so, I am paying at most the second highest bid and I am winning deterministically

Implementable Rules are Monotone

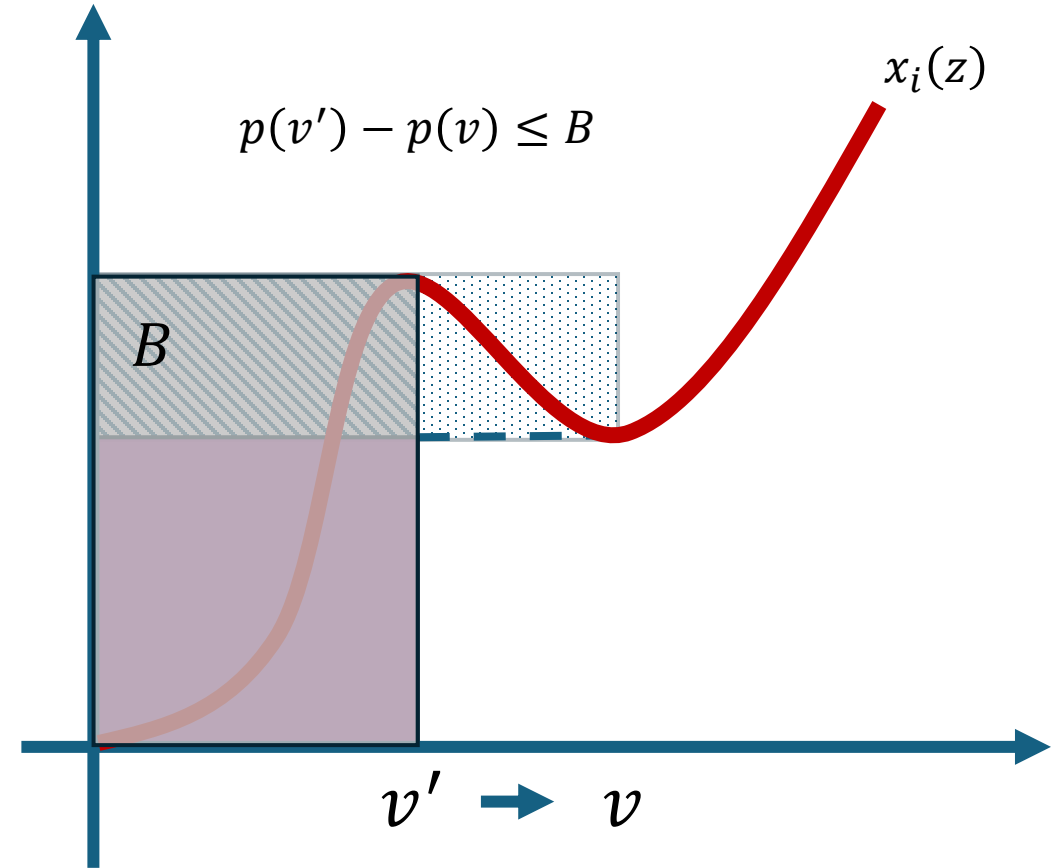
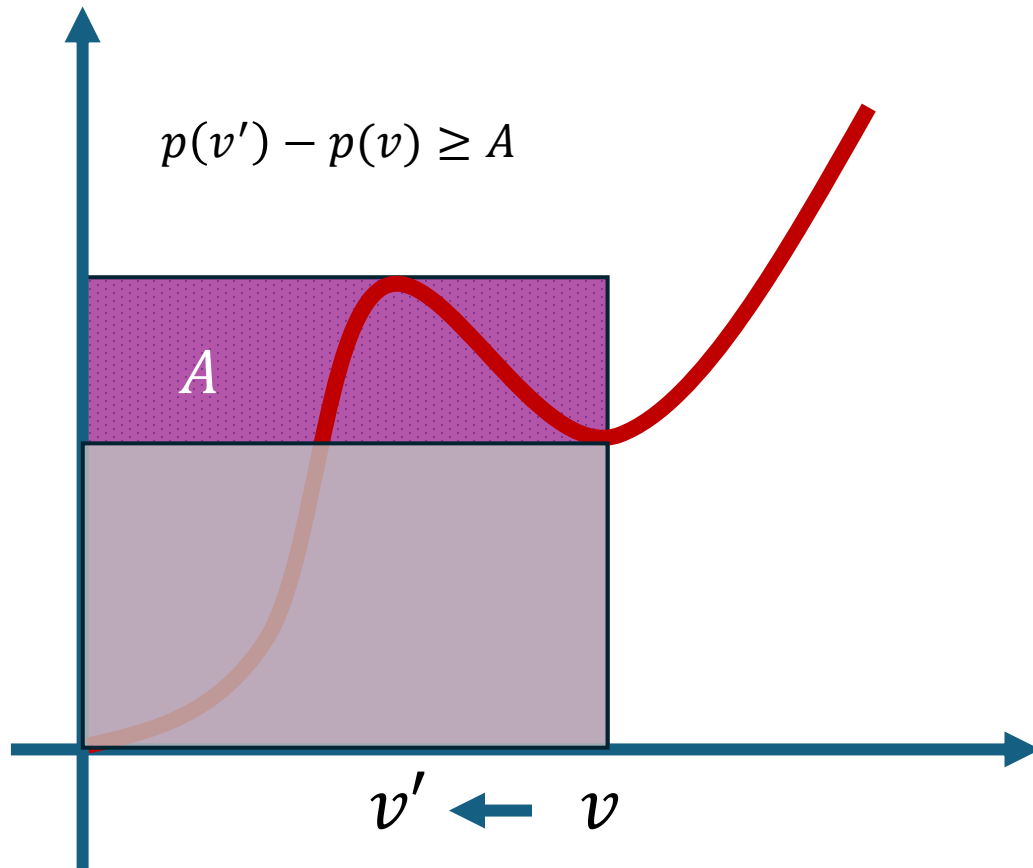
$$v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$$



Implementable Rules are Monotone

$$v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$$

$$v' \cdot x(v') - p(v') \geq v' \cdot x(v) - p(v)$$

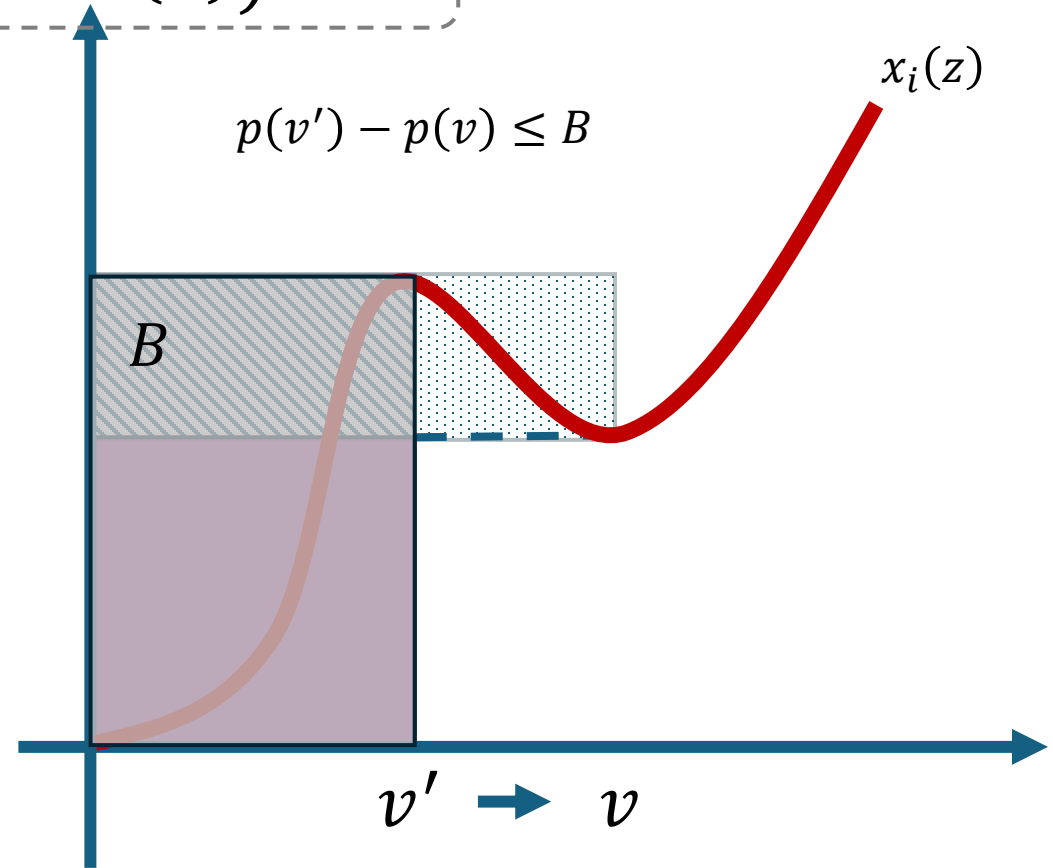
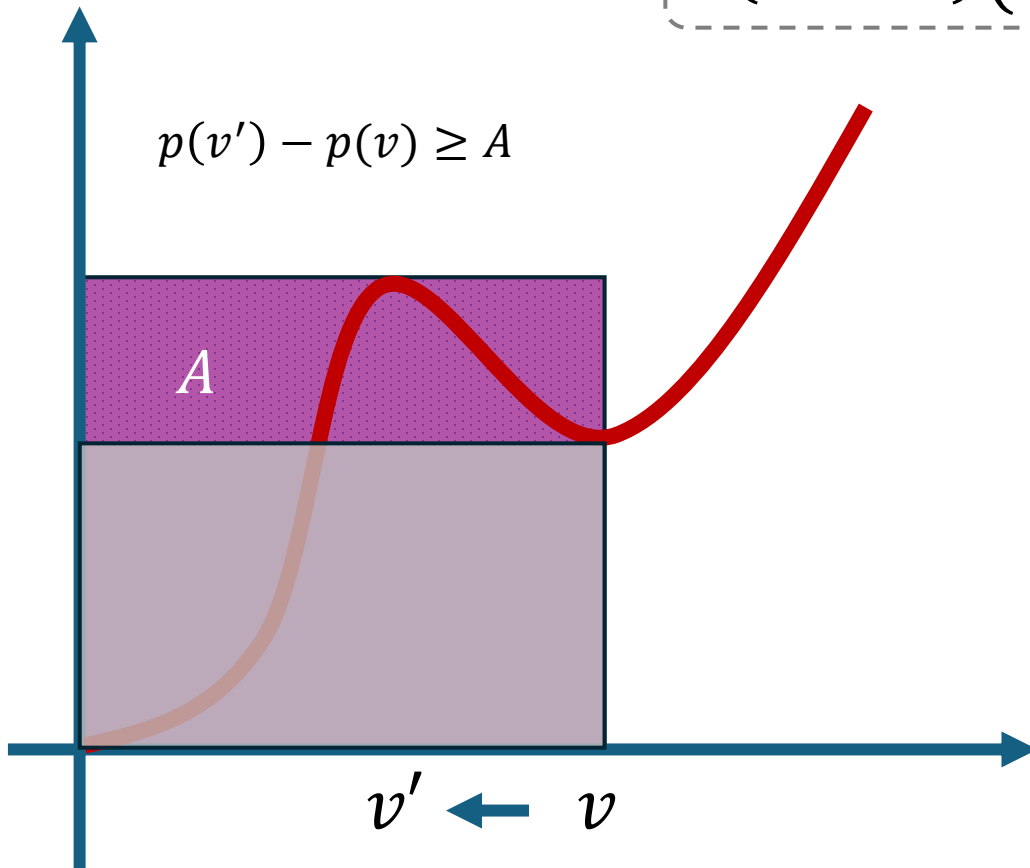


Implementable Rules are Monotone

$$A = v(x(v') - x(v)) \leq v'(x(v') - x(v)) = B$$

$$(v' - v)(x(v') - x(v)) \geq 0$$

Non-decreasing function



Any implementable allocation rule must be monotone!

“If not allocated with value v , I should not be allocated if I report a lower value!”

Uniqueness of Payment Rule

- I should not want to deviate locally up or down infinitesimally

$$u(v) \geq v \cdot x(v + \epsilon) - p(v + \epsilon) = u(v + \epsilon) - \epsilon \cdot x(v + \epsilon)$$

$$u(v) \geq v \cdot x(v - \epsilon) - p(v - \epsilon) = u(v - \epsilon) + \epsilon \cdot x(v - \epsilon)$$

- Dividing over by ϵ , restricts the rate of change of utility

$$\frac{u(v + \epsilon) - u(v)}{\epsilon} \leq x(v + \epsilon)$$

$$\frac{u(v) - u(v - \epsilon)}{\epsilon} \geq x(v - \epsilon)$$

- If u was differentiable, then taking the limit of the above as $\epsilon \rightarrow 0$

$$x(v) \leq u'(v) \leq x(v) \Rightarrow u'(v) = x(v) \Rightarrow u(v) - u(0) = \int_0^v x(z) dz$$

Under any truthful payment rule

$$u(v) = u(0) + \int_0^v x(z) dz$$

Discontinuity of Allocation Rule

- Even though allocation rule can be discontinuous, because it is monotone, it is Riemann integrable

$$\int_0^v x(z) dz = \lim_{\epsilon \rightarrow 0} \sum_k x(z + \epsilon) \cdot \epsilon \geq \lim_{\epsilon \rightarrow 0} \sum_k u(z + \epsilon) - u(z) = u(v) - u(0)$$

$$\int_0^v x(z) dz = \lim_{\epsilon \rightarrow 0} \sum_k x(z - \epsilon) \cdot \epsilon \leq \lim_{\epsilon \rightarrow 0} \sum_k u(z) - u(z - \epsilon) = u(v) - u(0)$$

Under any truthful payment rule

$$u(v) = u(0) + \int_0^v x(z) dz$$

What does that imply about payments

- Since utility is value minus payment

$$v x(v) - p(v) = -p(0) + \int_0^v x(z) dz$$

- Non-Negative-Transfers (NNT). We never have negative payments

$$p(0) \geq 0$$

- Individually Rational (IR). We never give bidders negative utility

$$p(0) \leq 0$$

- Thus, payment at 0 should be zero!

Under any truthful payment rule that satisfies NNT and IR

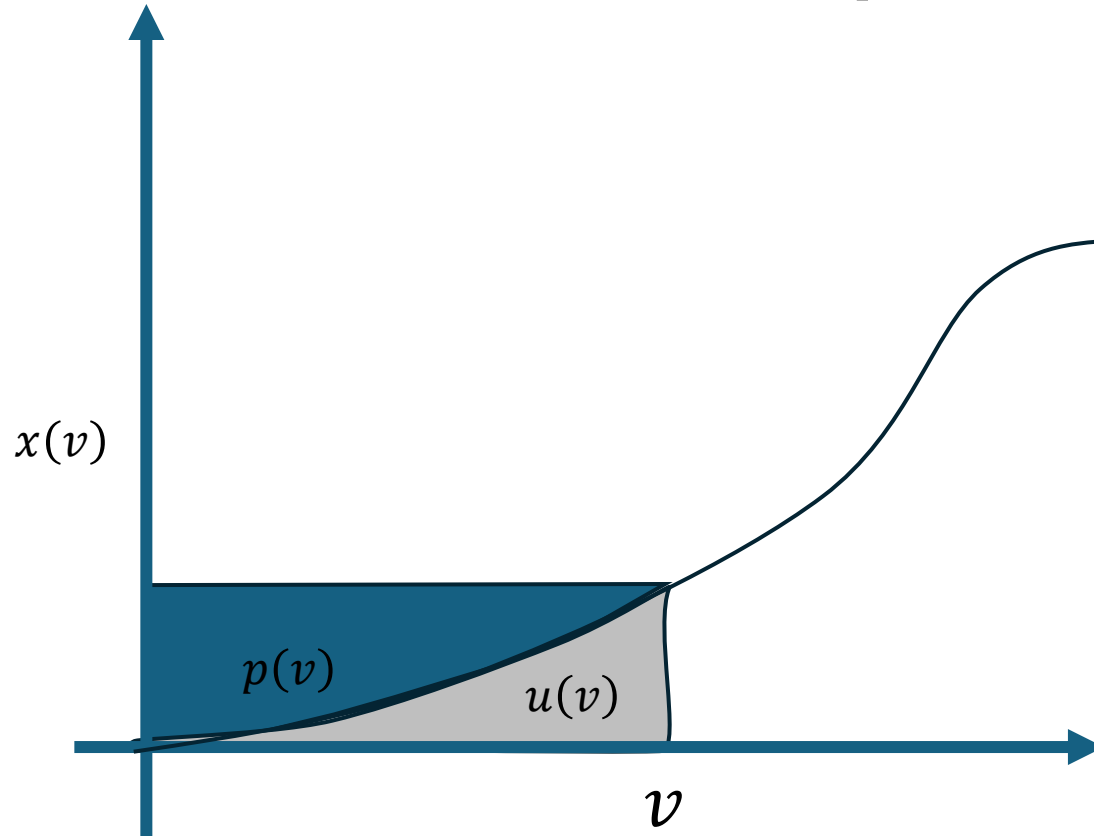
$$p(v) = v \cdot x(v) - \int_0^v x(z) dz$$

Visualizing Utility

- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z) dz$$

$$p(v) = v \cdot x(v) - \int_0^v x(z) dz$$

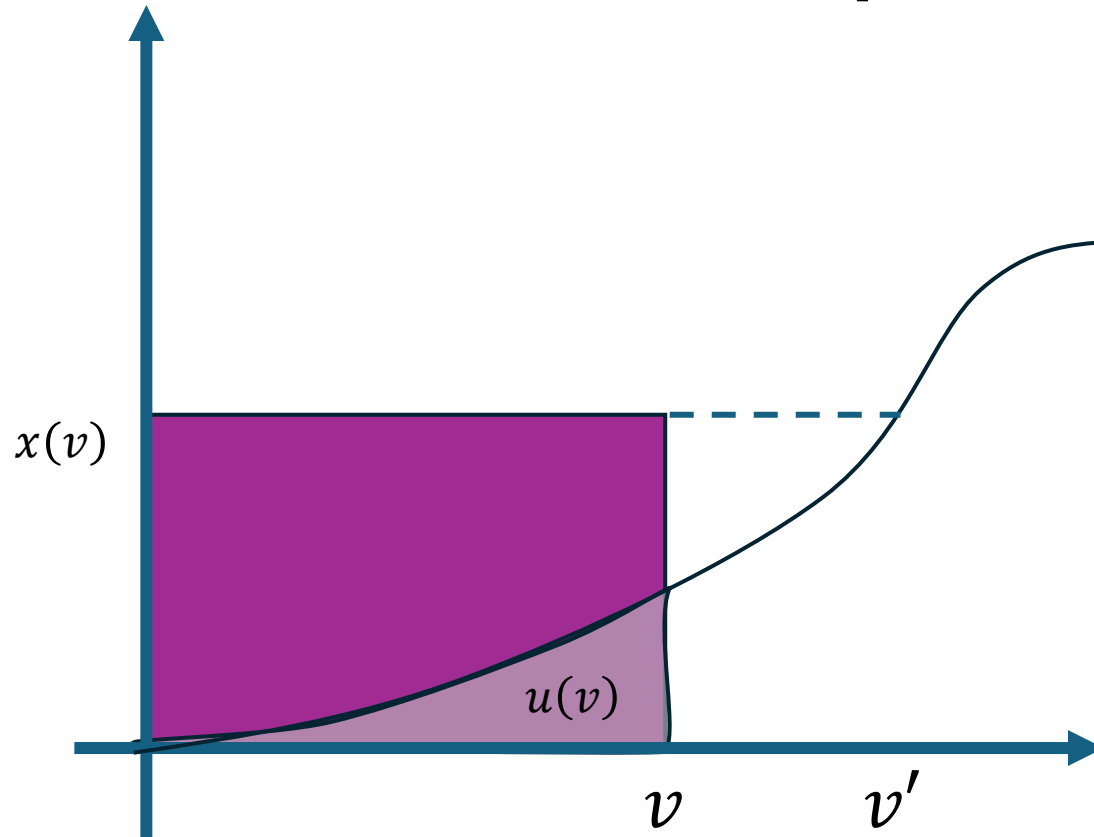


Visualizing Utility

- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z) dz$$

$$p(v) = v \cdot x(v) - \int_0^v x(z) dz$$

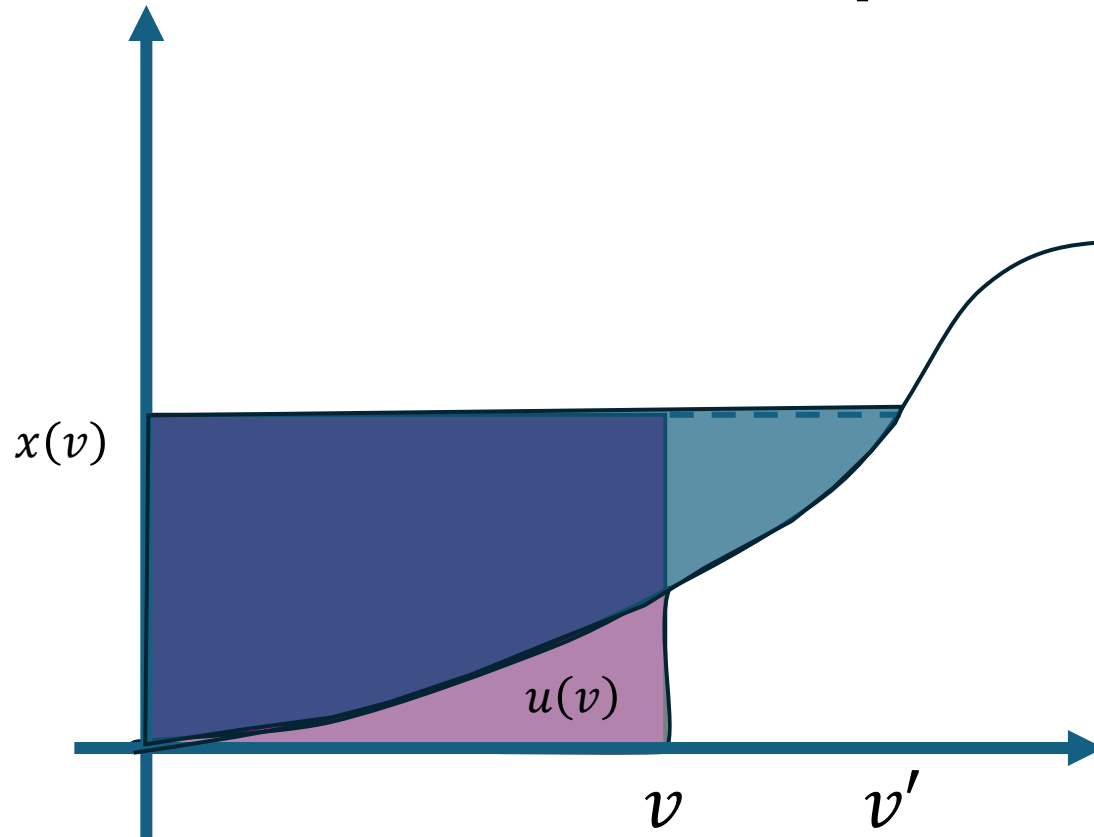


Visualizing Utility

- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z) dz$$

$$p(v) = v \cdot x(v) - \int_0^v x(z) dz$$

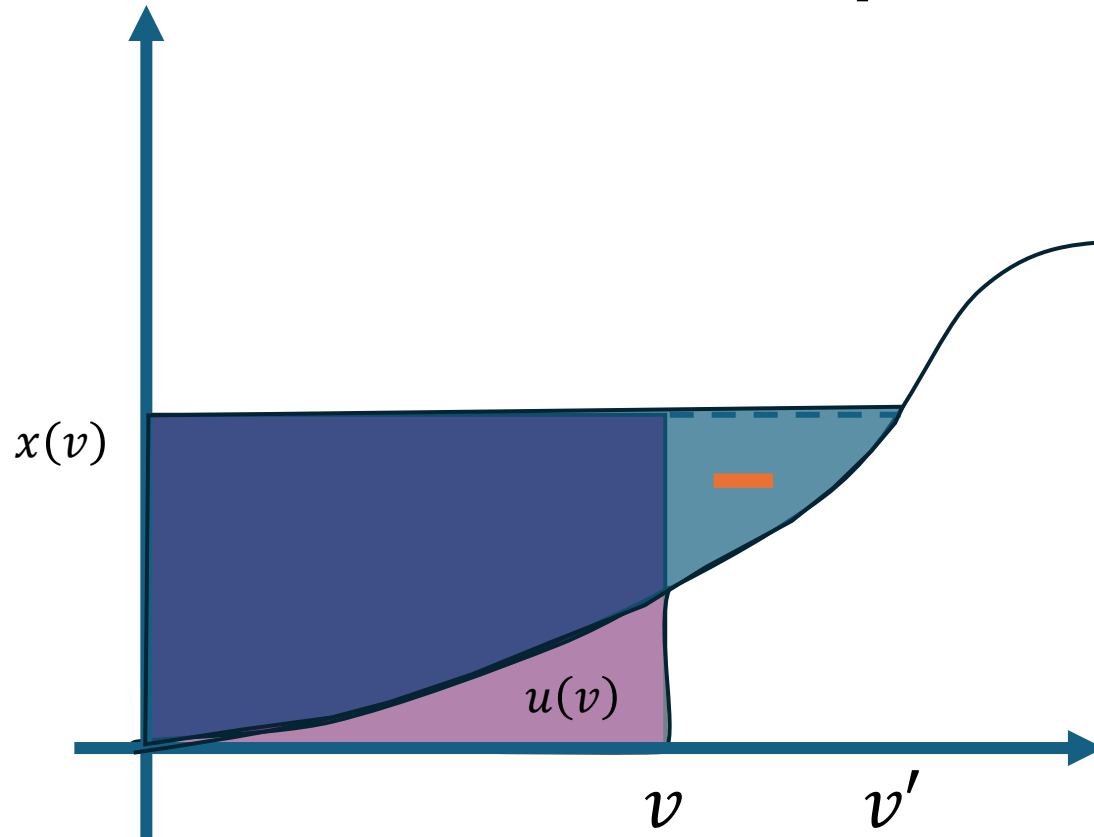


Visualizing Utility

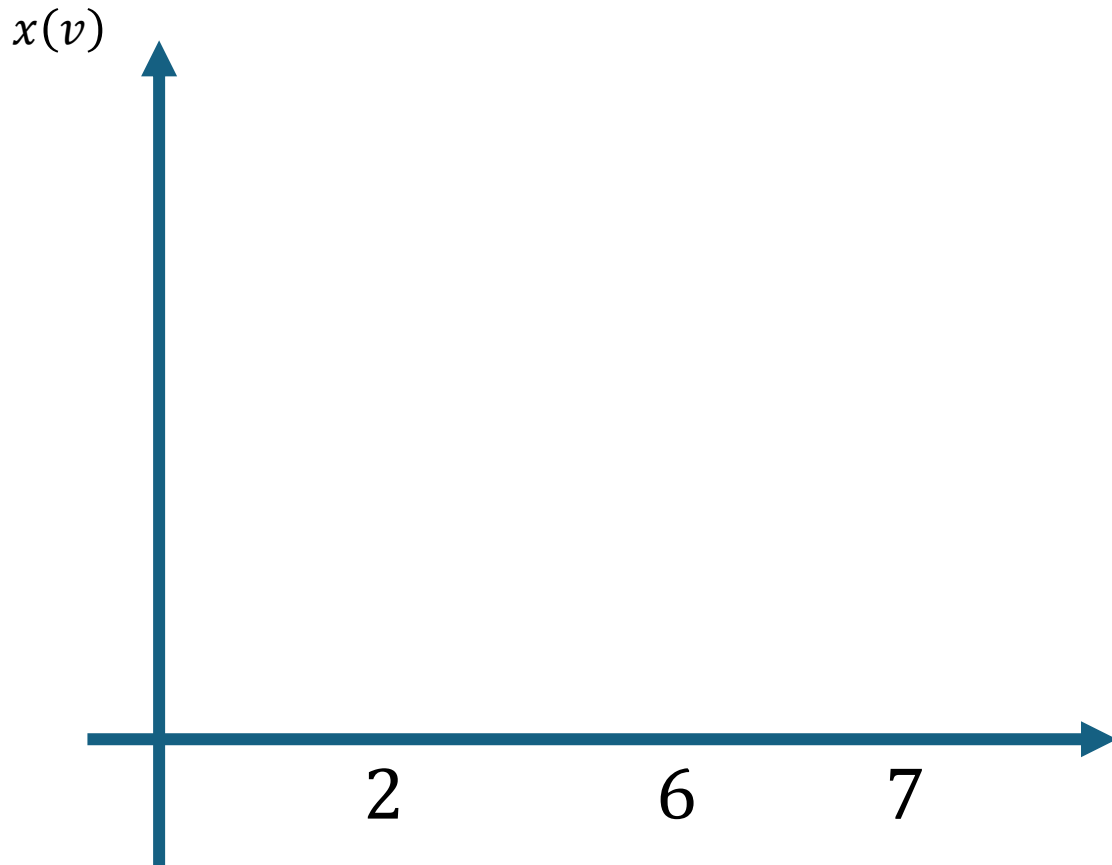
- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z) dz$$

$$p(v) = v \cdot x(v) - \int_0^v x(z) dz$$



Applying this to Sponsored Search



Google

digital advertising

All Images News Videos Shopping More

About 6,620,000,000 results (0.44 seconds)

Sponsored



Reddit

<https://www.redditforbusiness.com>

Advertise on Reddit

Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads.

1

Sponsored



Microsoft

<https://about.ads.microsoft.com> > advertising > start-now

Microsoft Advertising® | Get a \$500 Advertising Credit

We'll Help You Find Your Customers and Reach Searchers Across The Microsoft Network. Plus, Receive a \$500 Microsoft Advertising Credit When You Spend Just \$250! Free Sign Up.

0.5

$$v_1 = 7$$

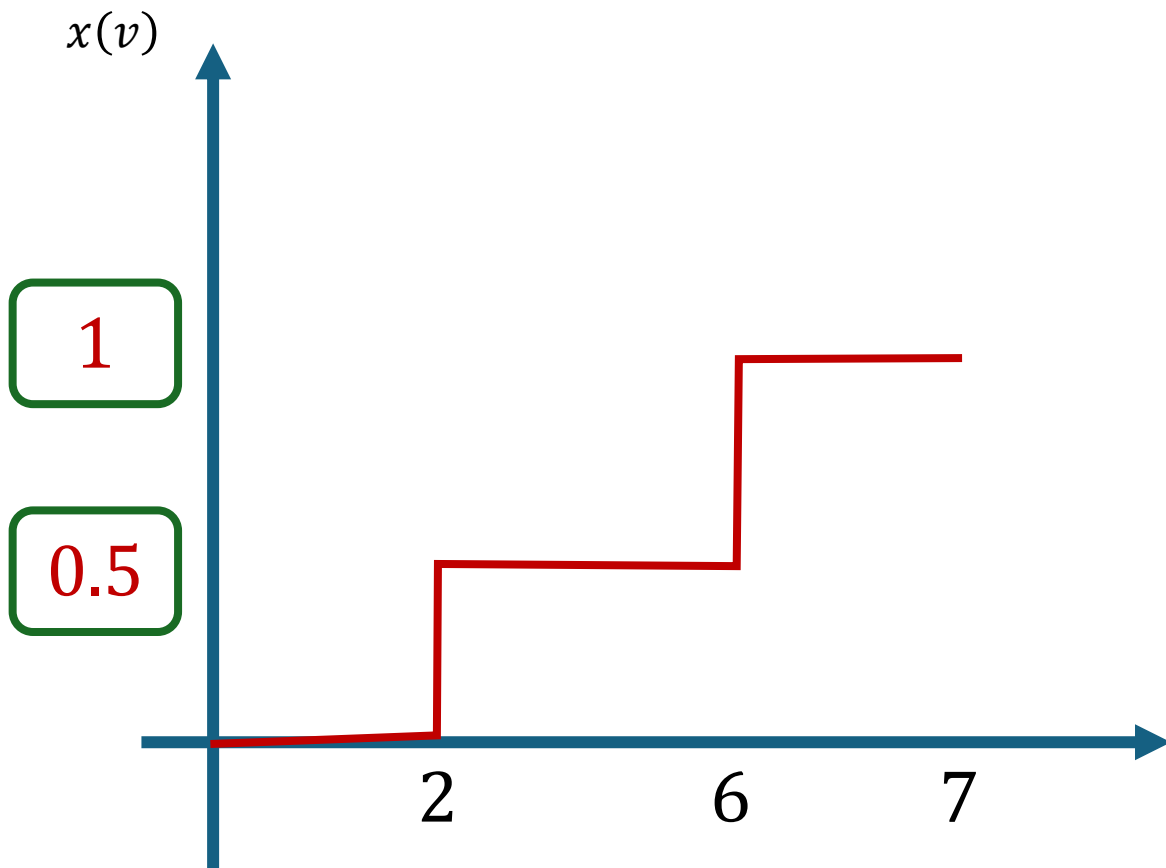
$$u_1 = 1 \cdot (7 - 4) = 3$$

$$v_2 = 6$$

$$u_2 = .5 \cdot (6 - 2) = 2$$

$$v_3 = 2$$

Applying this to Sponsored Search



Google

digital advertising

All Images News Videos Shopping More

About 6,620,000,000 results (0.44 seconds)

Sponsored

Reddit
<https://www.redditforbusiness.com>

Advertise on Reddit

Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads.

1

Sponsored

Microsoft
<https://about.ads.microsoft.com/advertising/start-now>

Microsoft Advertising® | Get a \$500 Advertising Credit

We'll Help You Find Your Customers and Reach Searchers Across The Microsoft Network. Plus, Receive a \$500 Microsoft Advertising Credit When You Spend Just \$250! Free Sign Up.

0.5

$$v_1 = 7$$

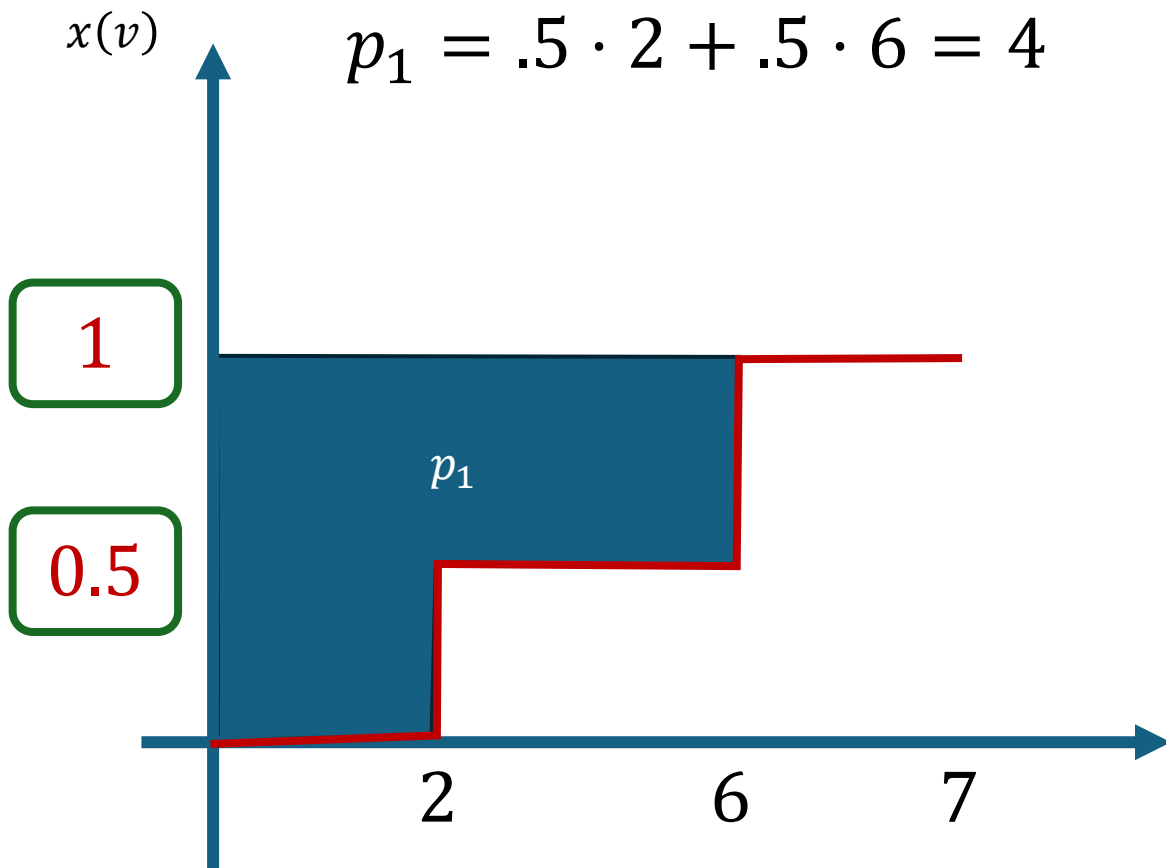
$$u_1 = 1 \cdot (7 - 4) = 3$$

$$v_2 = 6$$

$$u_2 = .5 \cdot (6 - 2) = 2$$

$$v_3 = 2$$

Applying this to Sponsored Search



Google

digital advertising

All Images News Videos Shopping More

About 6,620,000,000 results (0.44 seconds)

Sponsored

Reddit
<https://www.redditforbusiness.com>

Advertise on Reddit

Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads.

1

Sponsored

Microsoft
<https://about.ads.microsoft.com/advertising/start-now>

Microsoft Advertising® | Get a \$500 Advertising Credit

We'll Help You Find Your Customers and Reach Searchers Across The Microsoft Network. Plus, Receive a \$500 Microsoft Advertising Credit When You Spend Just \$250! Free Sign Up.

0.5

$$v_1 = 7$$

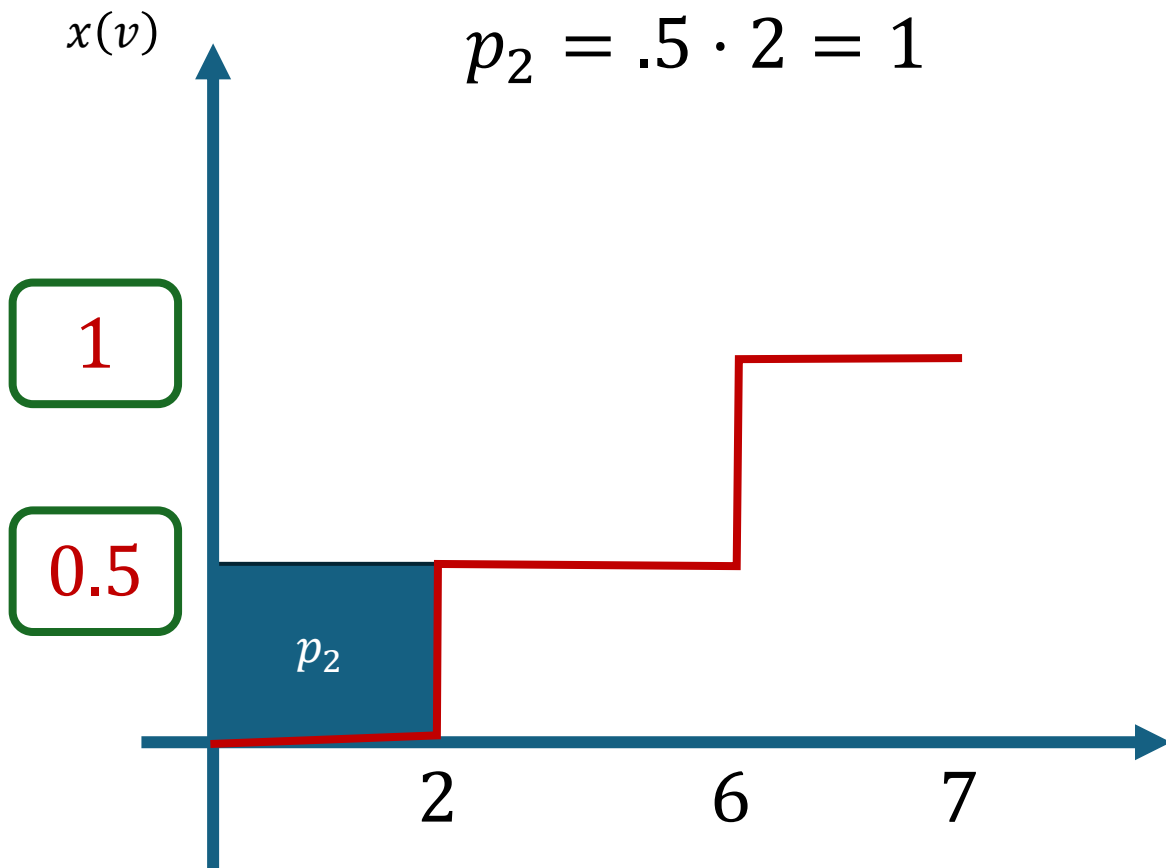
$$u_1 = 1 \cdot (7 - 4) = 3$$

$$v_2 = 6$$

$$u_2 = .5 \cdot (6 - 2) = 2$$

$$v_3 = 2$$

Applying this to Sponsored Search



Google

digital advertising

All Images News Videos Shopping More

About 6,620,000,000 results (0.44 seconds)

Sponsored



Reddit

<https://www.redditforbusiness.com>

[Advertise on Reddit](#)

Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads.

1

Sponsored



Microsoft

<https://about.ads.microsoft.com/advertising/start-now>

[Microsoft Advertising® | Get a \\$500 Advertising Credit](#)

We'll Help You Find Your Customers and Reach Searchers Across The Microsoft Network. Plus, Receive a \$500 Microsoft Advertising Credit When You Spend Just \$250! Free Sign Up.

0.5

$$v_1 = 7$$

$$u_1 = 1 \cdot (7 - 4) = 3$$

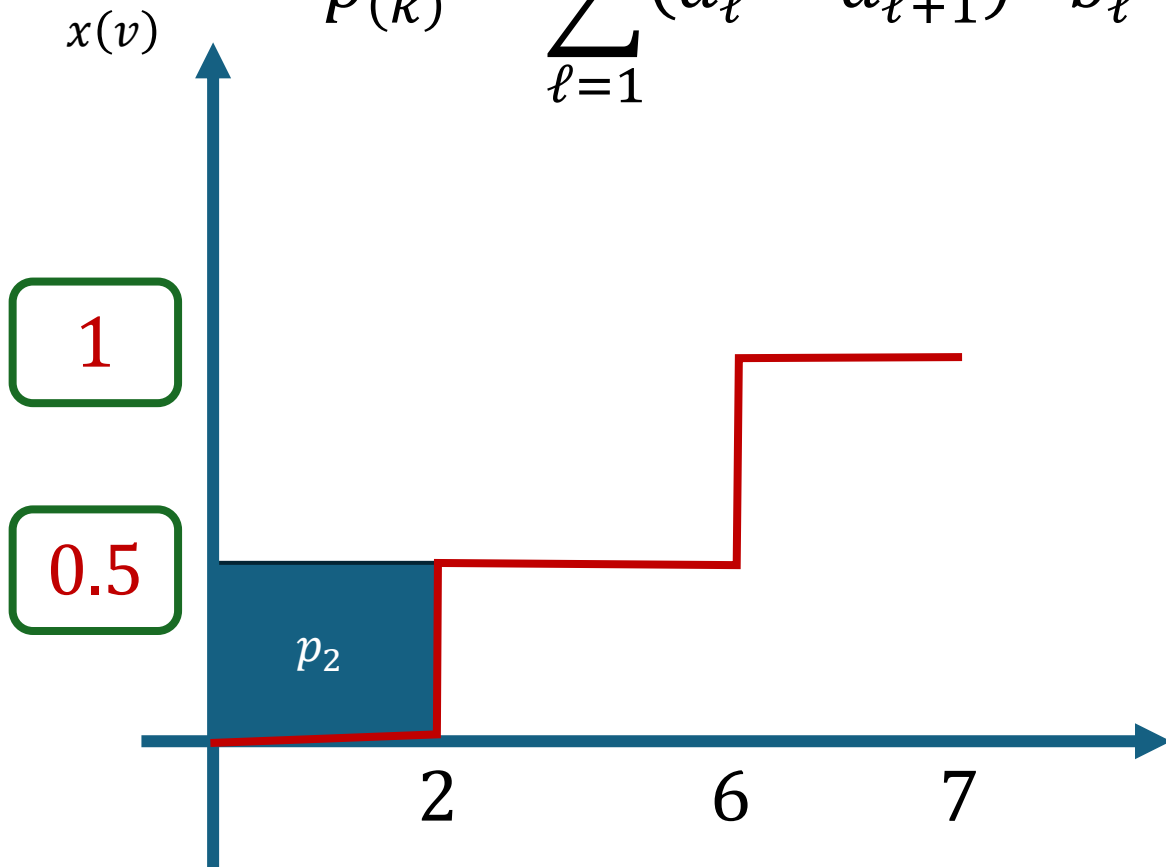
$$v_2 = 6$$

$$u_2 = .5 \cdot (6 - 2) = 2$$

$$v_3 = 2$$

Applying this to Sponsored Search

$$p_{(k)} = \sum_{\ell=1}^k (a_{\ell} - a_{\ell+1}) \cdot b_{\ell}$$



Google

digital advertising

All Images News Videos Shopping More

About 6,620,000,000 results (0.44 seconds)

Sponsored

Reddit
https://www.redditforbusiness.com

Advertise on Reddit

Reach over 100K communities — Connect with passionate communities that deliver results for brands across all industries. Create impact & own top communities in your target category for 24 hours. Try Reddit ads.

1

$$v_1 = 7$$

$$u_1 = 1 \cdot (7 - 4) = 3$$

Sponsored

Microsoft
https://about.ads.microsoft.com > advertising > start-now

Microsoft Advertising® | Get a \$500 Advertising Credit

We'll Help You Find Your Customers and Reach Searchers Across The Microsoft Network. Plus, Receive a \$500 Microsoft Advertising Credit When You Spend Just \$250! Free Sign Up.

0.5

$$v_2 = 6$$

$$u_2 = .5 \cdot (6 - 2) = 2$$

$$v_3 = 2$$

Given an allocation rule, the
payment is uniquely determined!

Optimizing over allocation rules

Myerson's Theorem

- Let x, p be any DSIC mechanism
- Suppose each value $v_i \sim F_i$ independently and let $v = (v_1, \dots, v_n)$
$$E[p_i(v)] = E[v_i \phi_i(v_i)]$$
where $\phi_i(v_i)$ is bidder i 's "virtual value".
- Letting F_i the CDF and f_i the density:
$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$
- Assuming $\phi_i(v_i)$ is monotone non-decreasing, then the optimal DSIC mechanism is the mechanism that allocates to the highest virtual value bidder (or none if highest virtual value is negative)

Back to Uniform Example

- If $v_i \sim U[0,1]$ then $F(v) = v$ and $f(v) = 1$
- Virtual value simplifies to
$$\phi_i(v_i) = v_i - (1 - v_i) = 2v_i - 1$$
- We should allocate to the highest virtual value player, as long as the highest virtual value is non-negative
$$v_i \geq 1/2$$
- Since all virtual value functions are the same, allocating to the highest virtual value is the same as allocating to the highest value
- Simply: Second Price with a reserve price of $1/2$!

Myerson's Theorem

- Consider the revenue contribution of a single bidder i and drop other bids and index from notation

$$E[p(v)] = E \left[v x(v) - \int_0^v x(z) dz \right] = E \left[v \hat{x}(v) - \int_0^v \hat{x}(z) dz \right]$$

- Allocation $\hat{x}(z)$ is the expected allocation over other bidder values

$$\hat{x}(z) = E_{v_{-i}}[x(z, v_{-i})]$$

- We can do an exchange of the integrals:

$$\begin{aligned} E \left[\int_0^v \hat{x}(z) dz \right] &= \int_{v=0}^{\infty} \int_{z=0}^v \hat{x}(z) dz f(v) dv = \int_{z=0}^{\infty} \hat{x}(z) \int_{v=z}^{\infty} f(v) dv dz \\ &= \int_{z=0}^{\infty} \hat{x}(z) (1 - F(z)) dz = E \left[\hat{x}(v) \frac{1 - F(v)}{f(v)} \right] \end{aligned}$$

Myerson's Theorem

$$E[p_i]$$

Appendix: Deriving the Optimal Reserve

- Bidders are symmetric. Revenue is twice the revenue we collect from each bidder

$$\begin{aligned}\text{Rev}_1(r) &= E[\max(v_2, r) 1(v_1 \geq \max(v_2, r))] \\ &= E[v_2 \mid v_2 \in [r, v_1]] \Pr(v_2 \in [r, v_1] \mid v_1 \geq r) \Pr(v_1 \geq r) + r \Pr(v_2 \leq r) \Pr(v_1 \geq r) \\ &= \int_r^1 \frac{v+r}{2} (v-r) dv + r^2(1-r) \\ &= \int_r^1 \frac{v^2 - r^2}{2} dv + r^2(1-r) \\ &= \left(\frac{1-r^3}{6} - \frac{r^2}{2} (1-r) + r^2(1-r) \right) \\ &= \frac{1-r^3}{6} + \frac{r^2(1-r)}{2} = \frac{1-r^3+3r^2-3r^3}{6} = \frac{1+3r^2-4r^3}{6}\end{aligned}$$

- The first order condition

$$(\text{Rev}_1(r))' = r(1-2r) = 0 \Rightarrow r = 1/2$$