MS&E 233
Game Theory, Data Science and AI
Lecture 10

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(by courtesy) Computer Science and Electrical Engineering

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Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- **Basic Auctions and Learning to bid in auctions (T)**
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- Simple vs optimal mechanisms (T)
- *HW6: calculate equilibria in simple auctions, implement simple and optimal auctions, analyze revenue empirically*

6
- Optimizing mechanisms from samples (T)
- Online optimization of auctions and mechanisms (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Sum: Auction Applications

- Traditionally, selling of luxury goods, art
- Digital auction markets for goods (eBay)
- Energy markets
- Digital ad markets (sponsored search, display ads, amazon ads)
- Spectrum auctions
- Government procurement auctions
- Web3.0 transaction protocols

Sum: First Price

- First Price is arguably the simplest auction rule
- It can be hard to strategize in such an auction
- The auction can lead to inefficient allocations

- Though approximately efficient
- Still used in practice in many settings (e.g. online advertising, government procurement)
- Primarily because it has very transparent rules

Sum: Second Price

- Second Price is arguably the simplest truthful auction rule
- It is very easy to strategize in such an auction (be truthful)
- Auction always leads to efficient allocations (highest value wins)
- Auction can be run very quickly (computationally efficient)

- Still not always the auction used in many places
- Primarily because it has not very transparent rules
- Susceptible to collusion and manipulations by the auctioneer

Sponsored Search Auctions

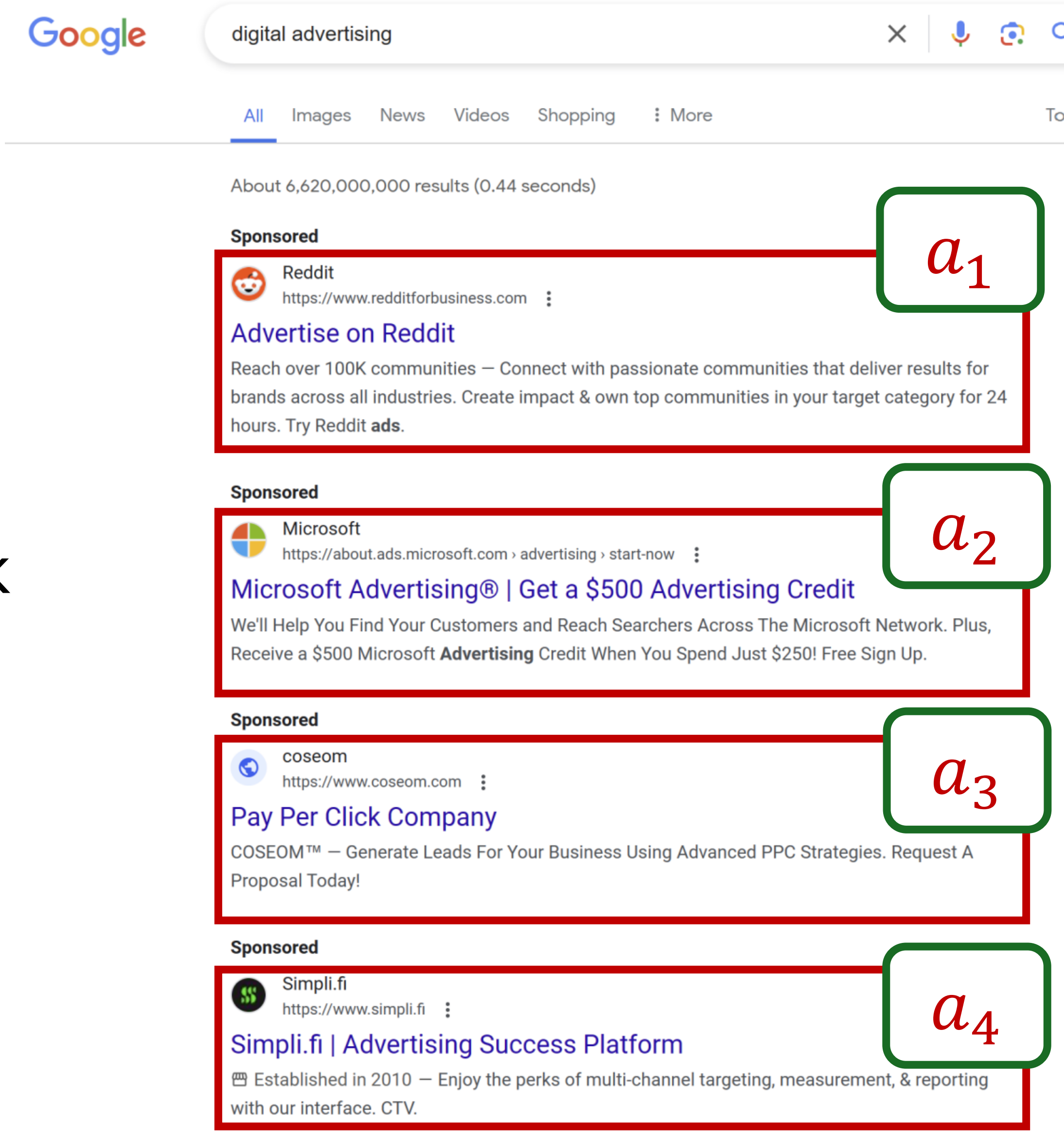
Sponsored Search Auctions

- Now we have many items to sell
- Slots on a web impressions

- Higher slots get more clicks!
- Each slot has some probability of click

$$a_1 > a_2 > \cdots > a_m$$

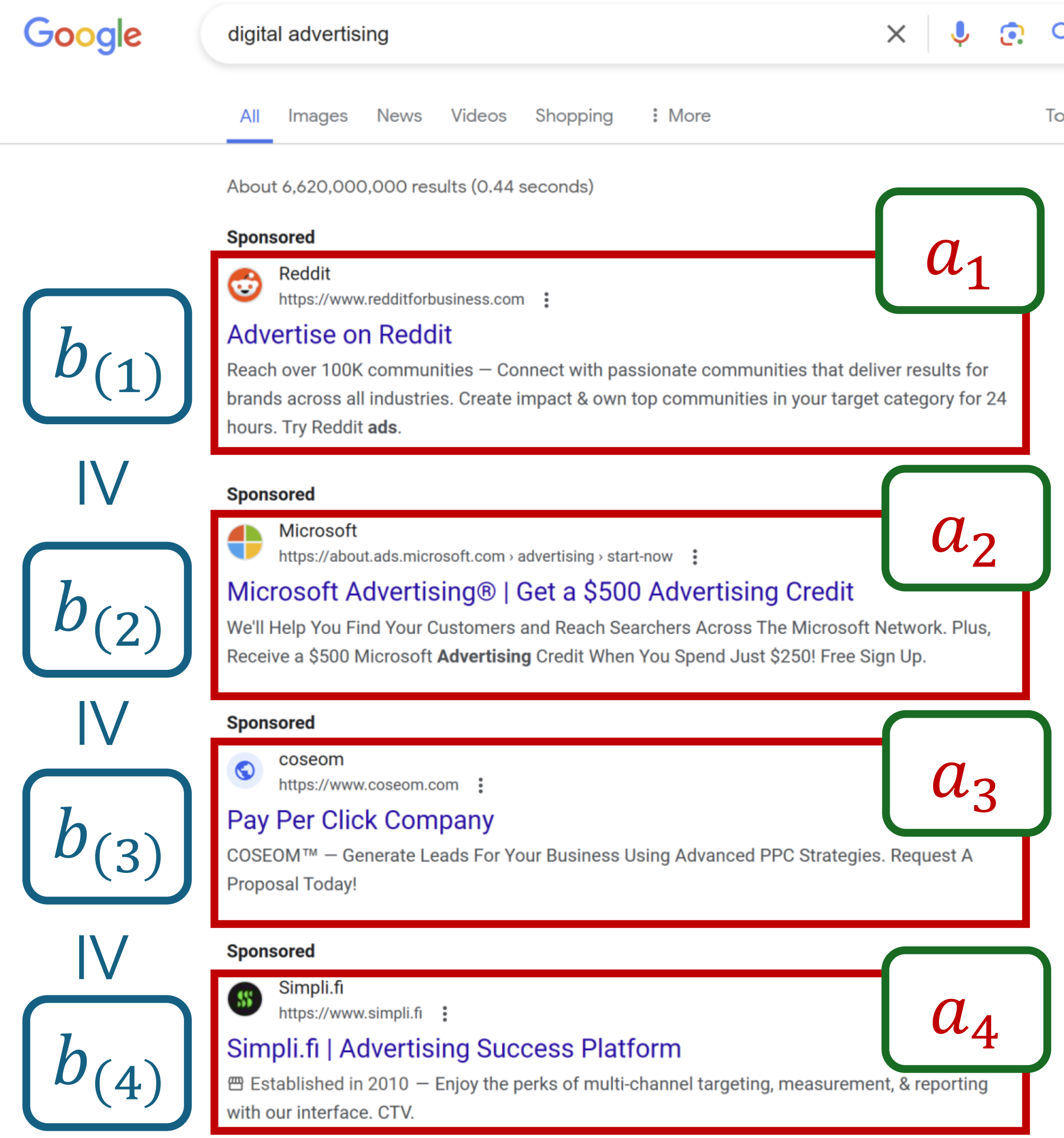
- Bidders have a value-per-click v_i



Generalized First Price (GFP) Auction

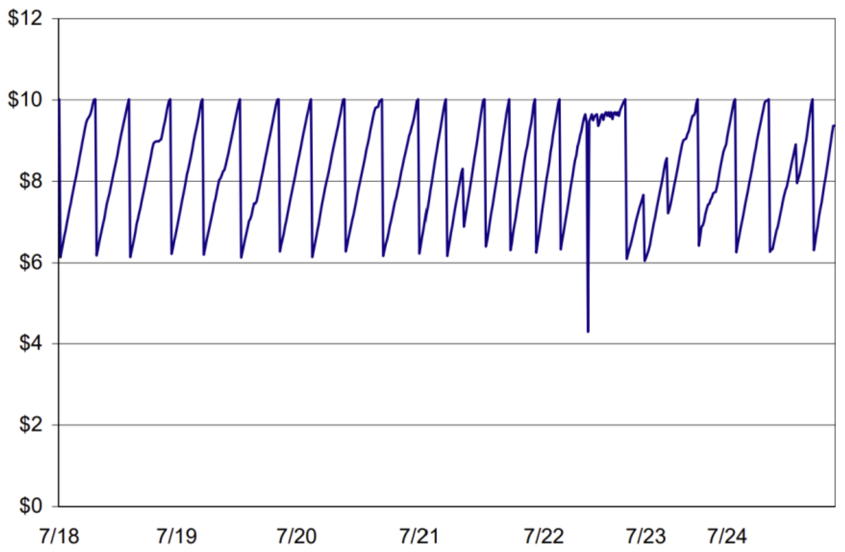
- Bidders submit a bid-per-click b_i
- Slots allocated in decreasing order of bids
- Bidder i is allocated slot $j_i(b)$
- Bidder pays their bid when clicked

$$u_i(b; v_i) = a_{j_i(b)} \cdot (v_i - b_i)$$

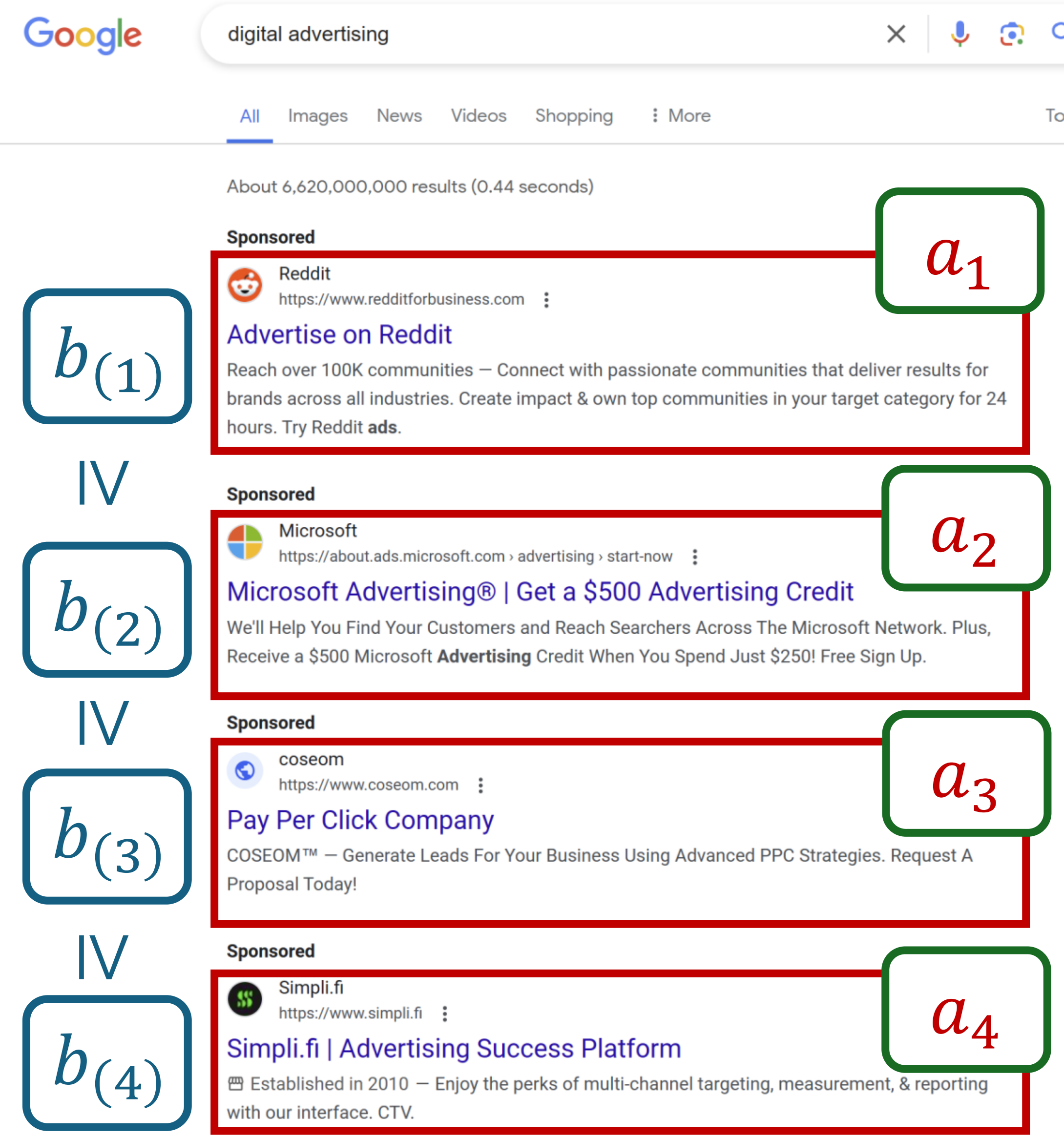


Generalized First Price (GFP) Auction

- The first auction that was used by Overture in late 90s
- Lead to weird bidding patterns



(b) 1 week

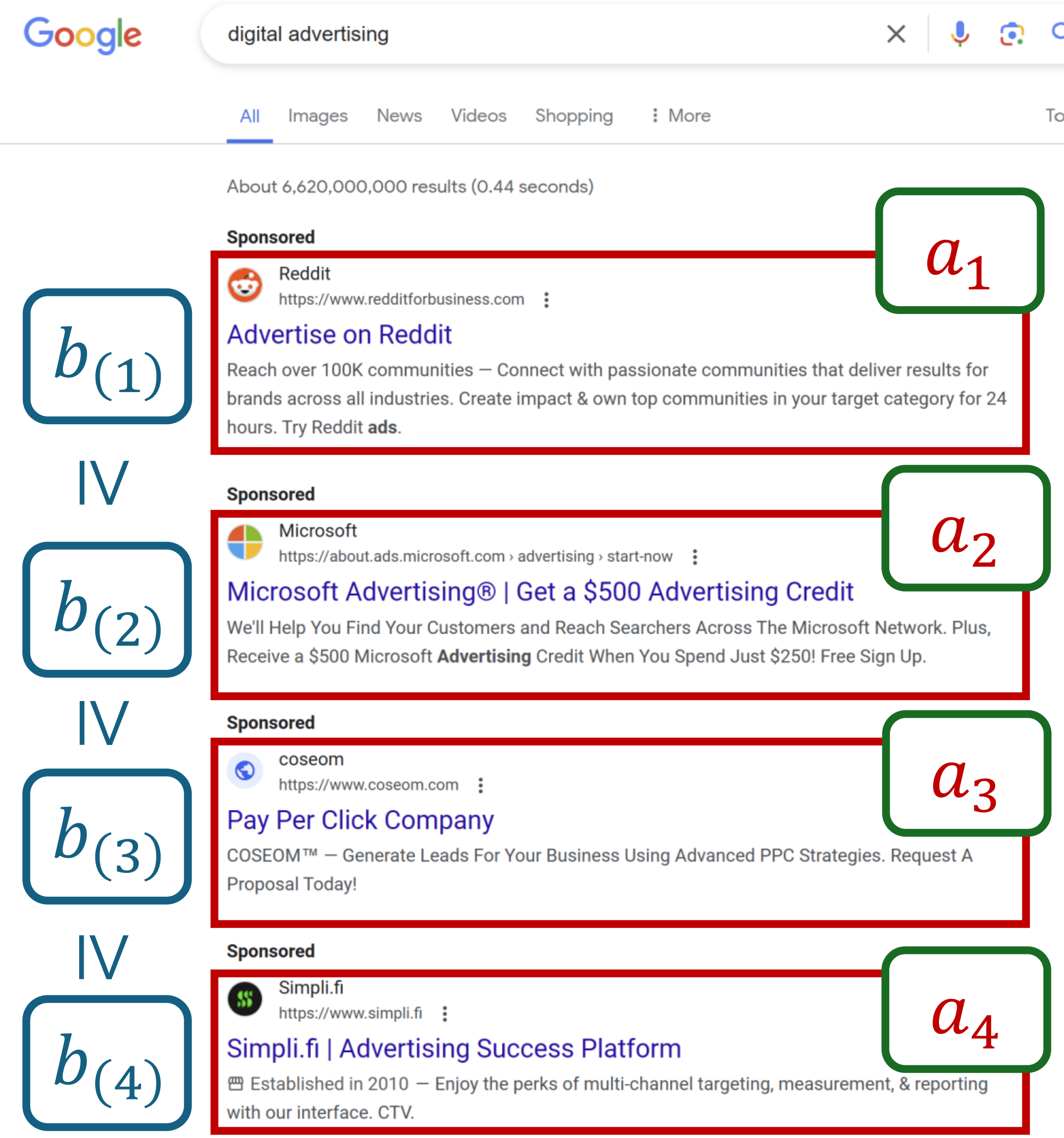


Credits: https://www.benedelman.org/publications/cycling-060703.pdf

Generalized Second Price (GSP) Auction

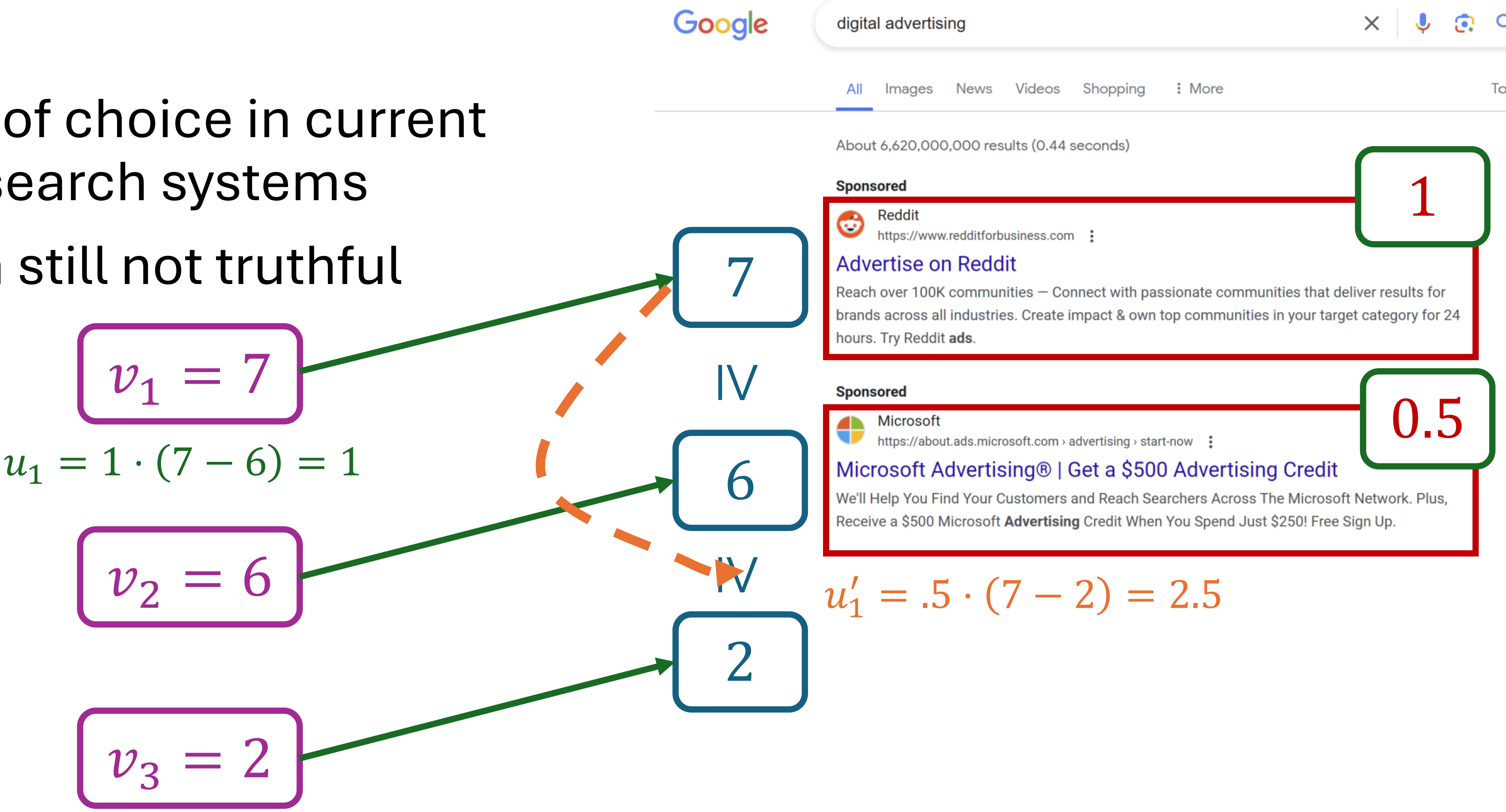
- Bidders submit a bid-per-click b_i
- Slots allocated in decreasing order of bids
- Bidder i is allocated slot $j_i(b)$
- Bidder pays the next highest bid when clicked

$$u_i(b; v_i) = a_{j_i(b)} \cdot \left(v_i - b_{(j_i(b)+1)}\right)$$



Generalized Second Price (GSP) Auction

- The auction of choice in current sponsored search systems
- Even though still not truthful



How would you turn GSP truthful?

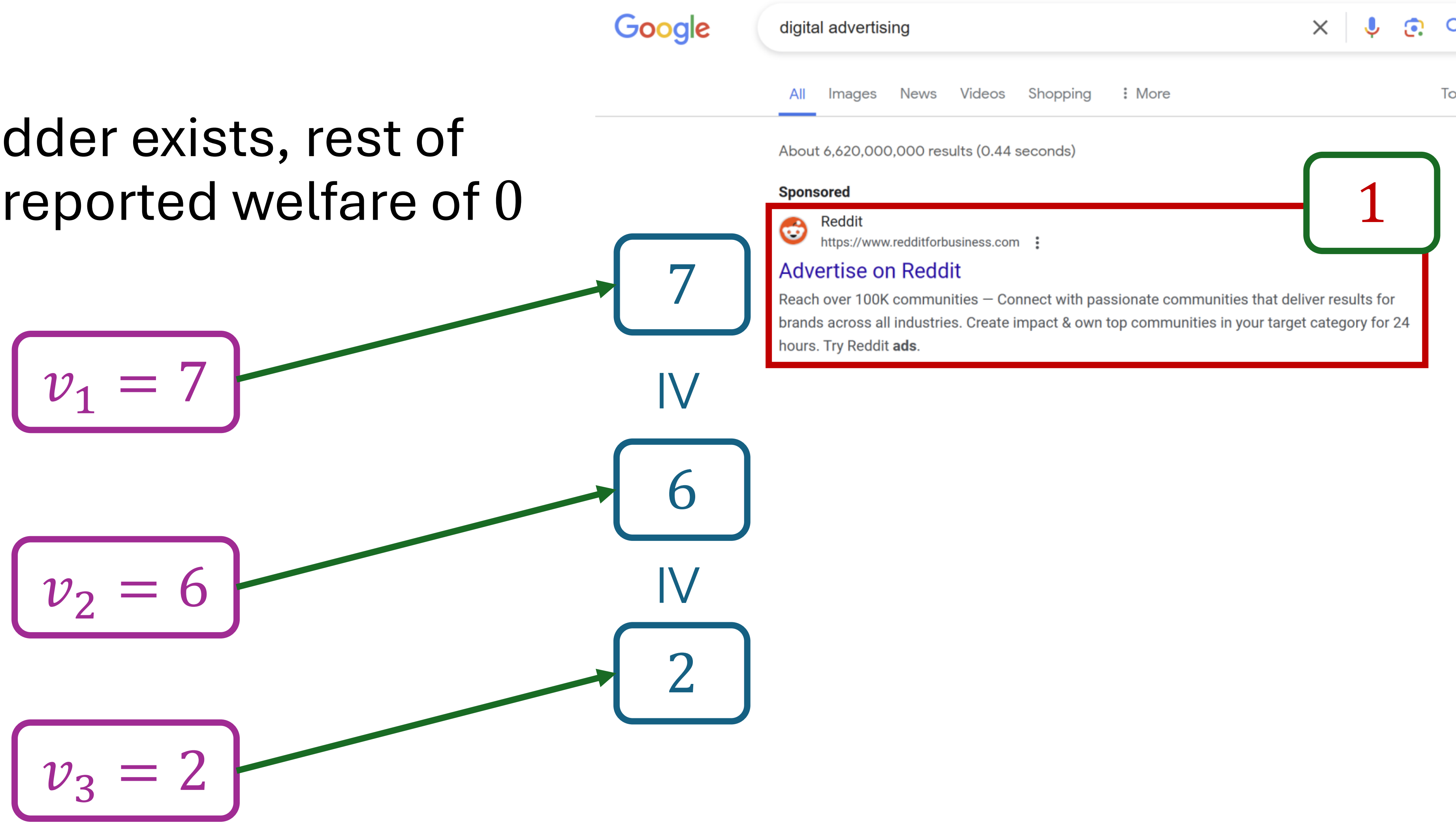
Right intuition, why Second-Price is truthful

- Second price is truthful **not because** we charge next highest bid
- Second price is truthful **not because** we charge smallest bid to maintain the same allocation

- Second price is truthful **because** we charged the winner their “externalities to the rest of society”

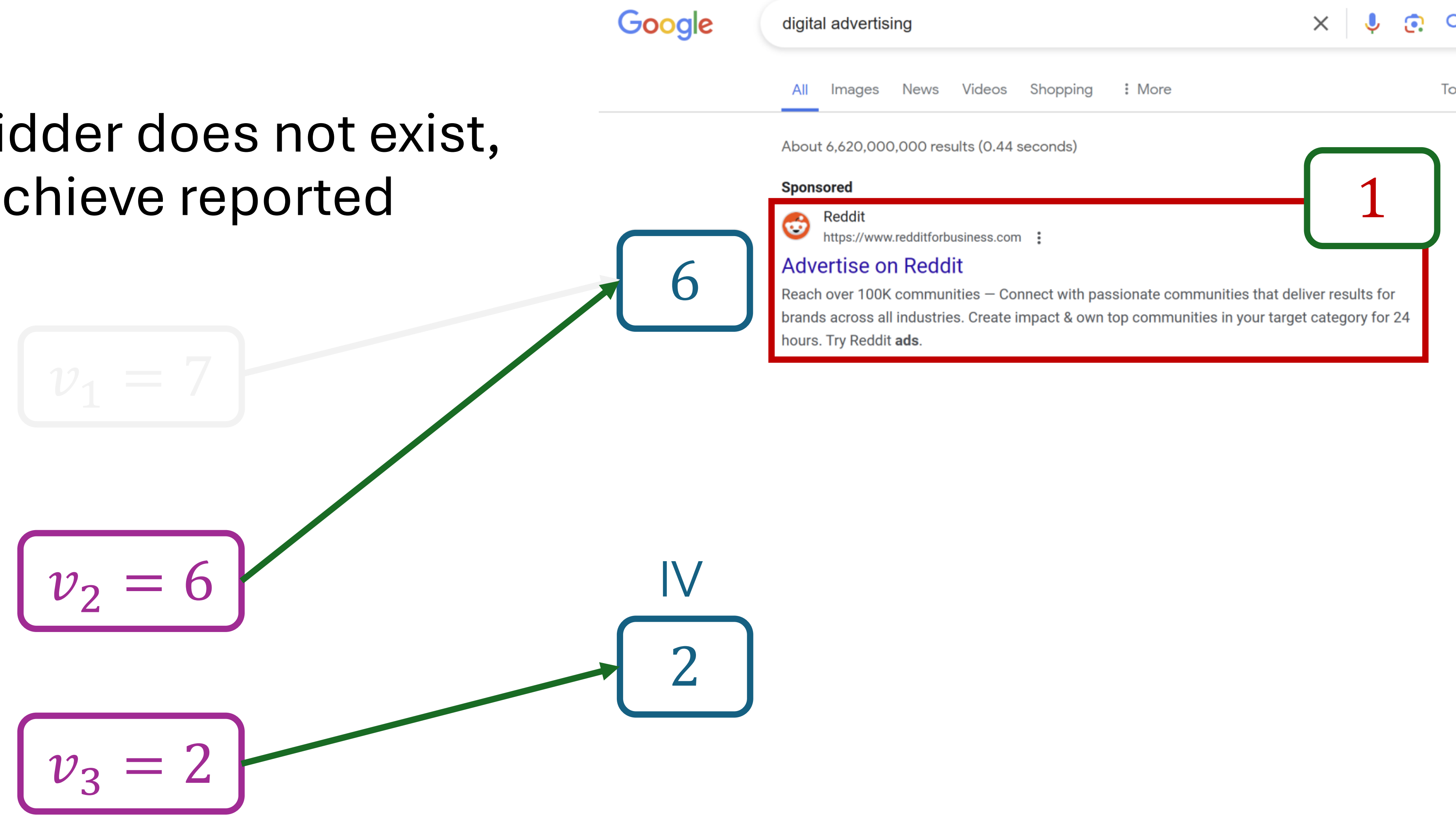
The Deep Reason why SP is Truthful

- When highest bidder exists, rest of players achieve reported welfare of 0



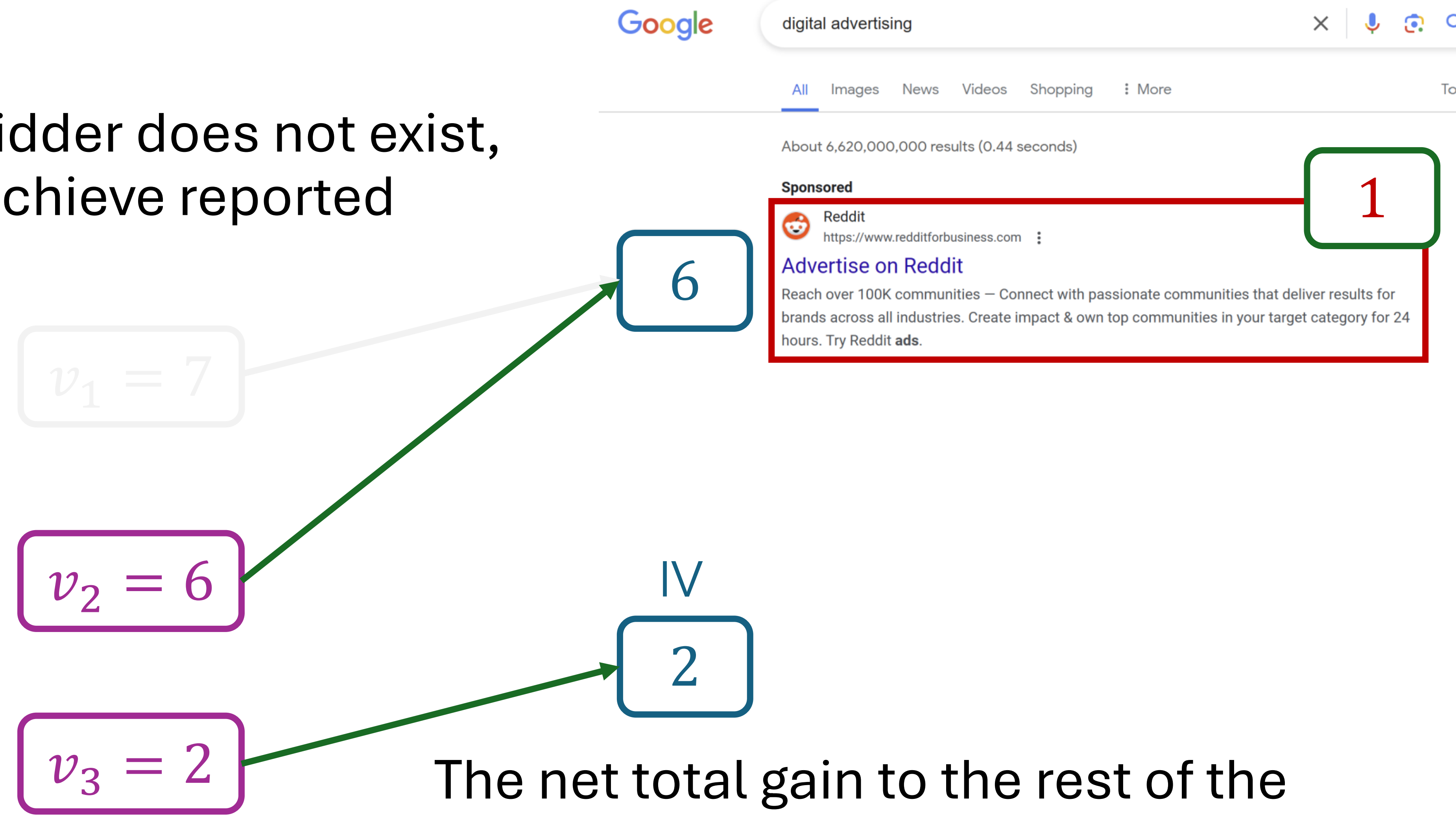
The Deep Reason why SP is Truthful

- When highest bidder does not exist, rest of players achieve reported welfare of 6



The Deep Reason why SP is Truthful

- When highest bidder does not exist, rest of players achieve reported welfare of 6



The net total gain to the rest of the bidders, from bidder 1 vanishing is 6

Right intuition, why Second-Price is truthful

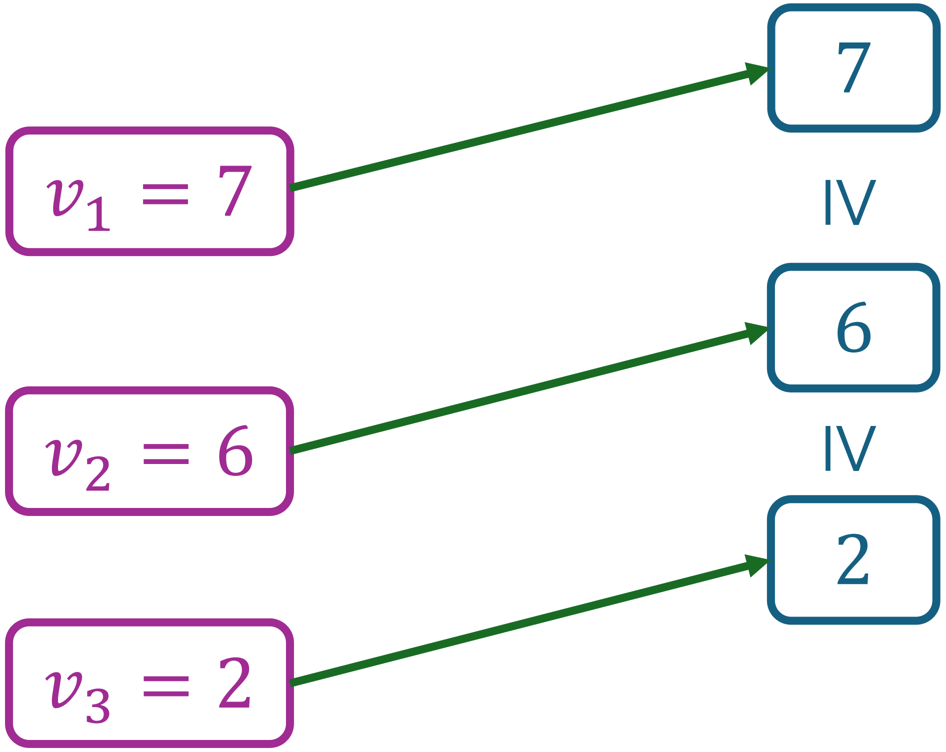
- Second price is truthful **because** we charged the winner their "externalities to the rest of society"

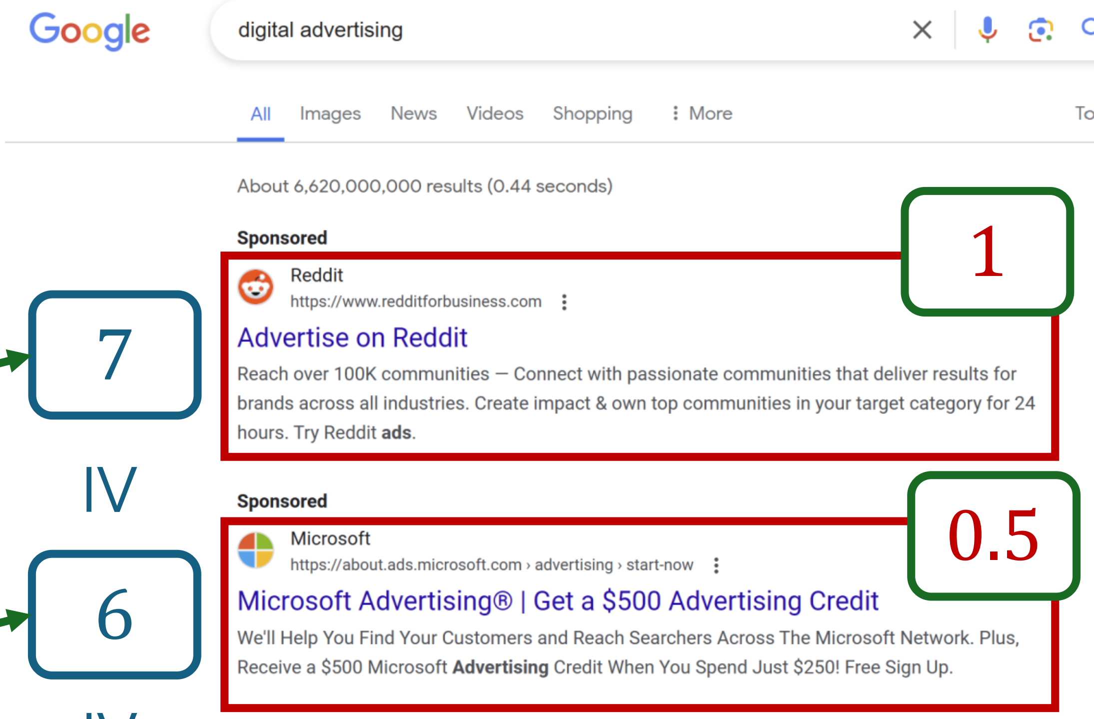
- When highest bidder exists, rest of players achieve reported welfare 0
- When highest bidder vanishes, rest of players achieve reported welfare
$$b_{(2)} = \text{second highest bid}$$
- The net total gain to the rest of the bidders, from bidder 1 vanishing is
$$b_{(2)} = \text{second highest bid}$$

- That's what we should charge the winner!

Let’s repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of ...?

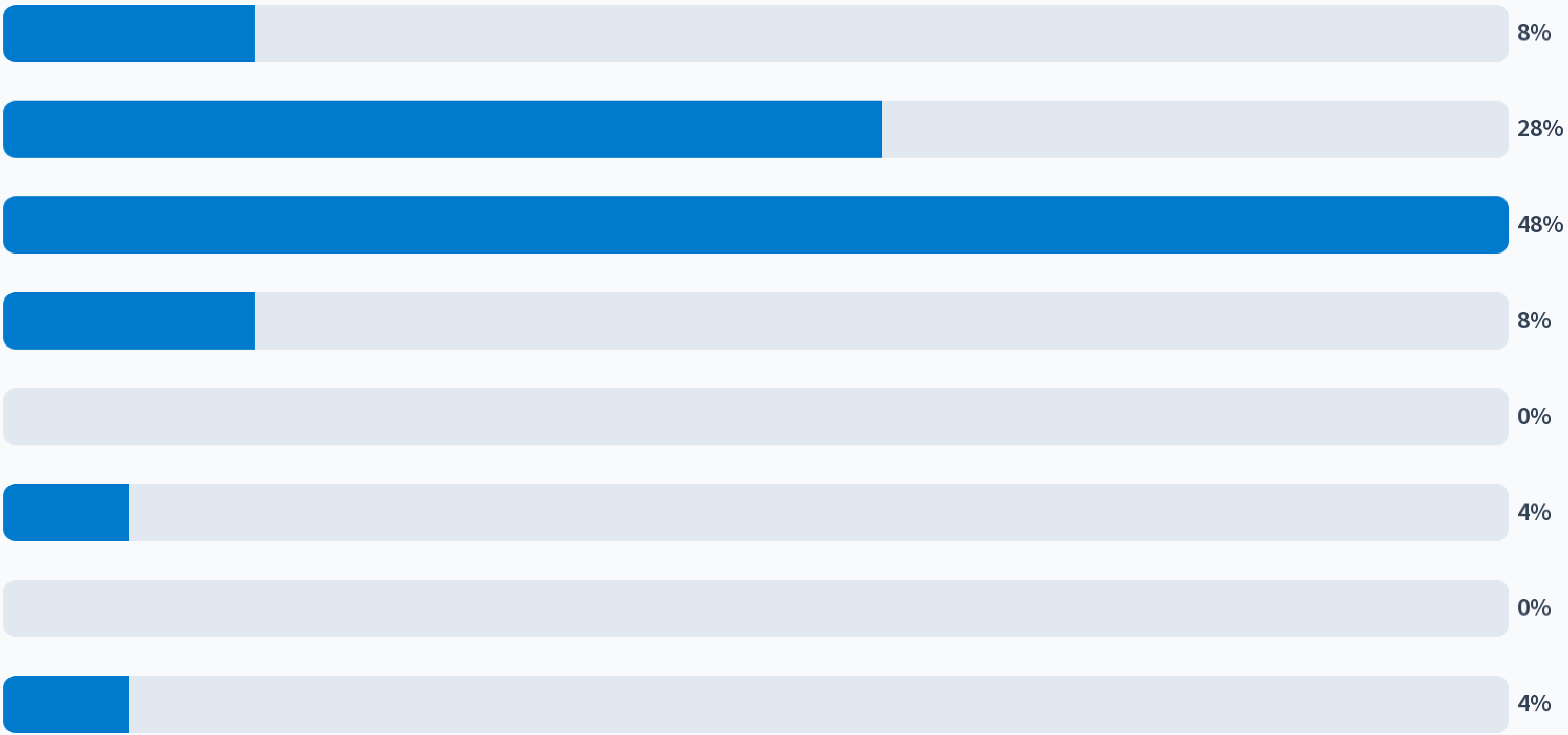




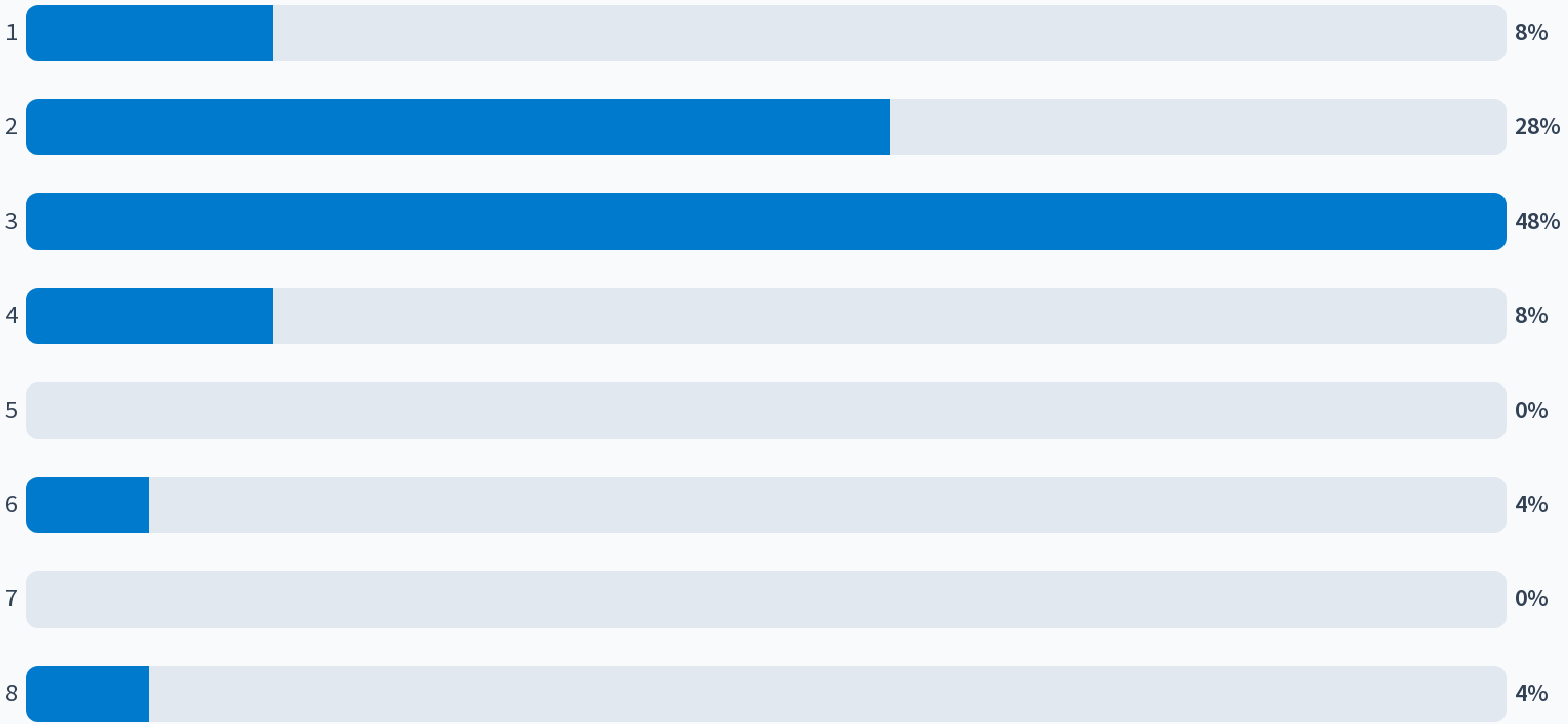
When the highest value bidder exists the rest of the players get a reported welfare of

- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8

When the highest value bidder exists the rest of the players get a reported welfare of

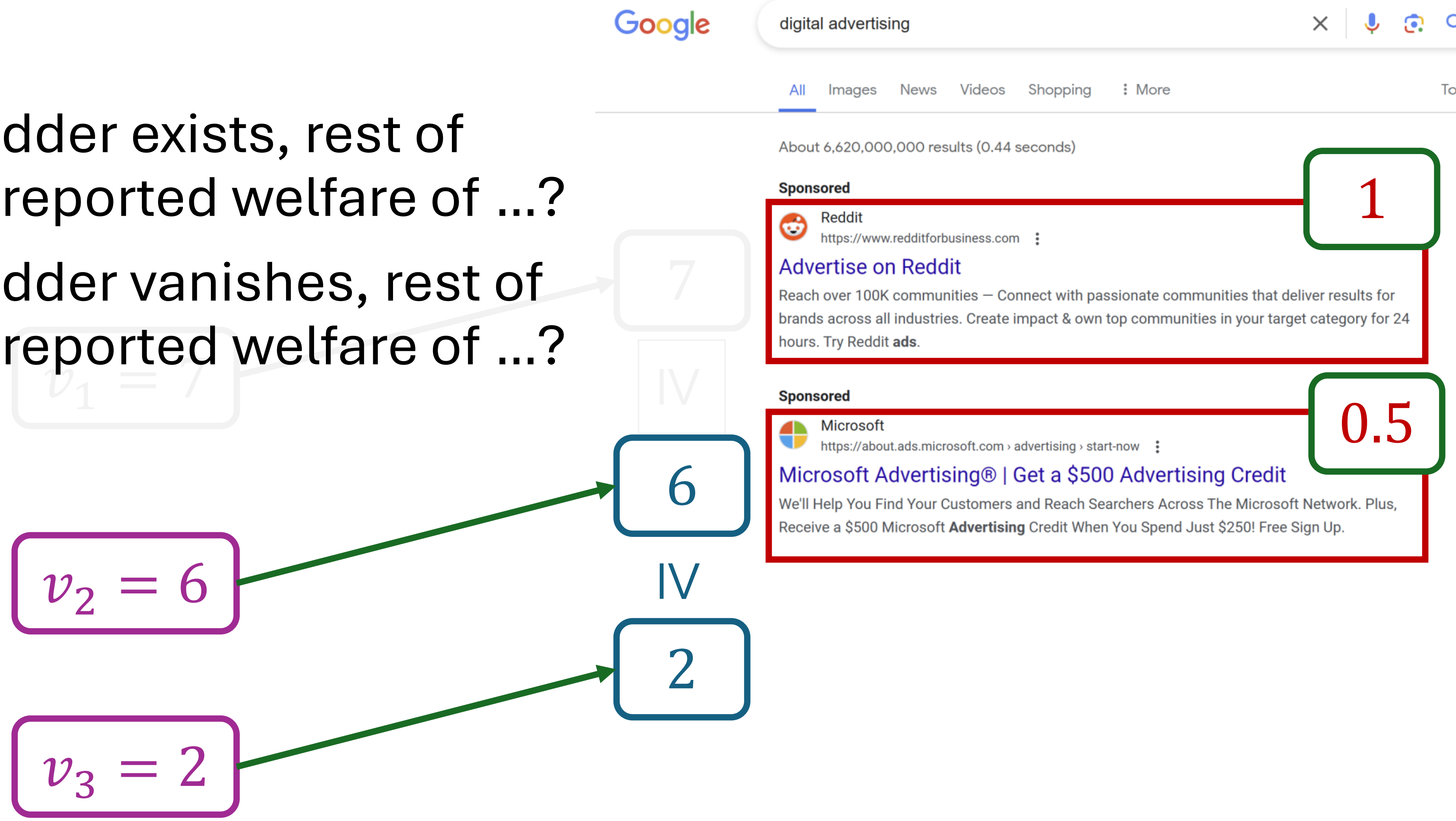


When the highest value bidder exists the rest of the players get a reported welfare of



Let’s repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of …?
- When highest bidder vanishes, rest of players achieve reported welfare of …?



When the highest value bidder vanishes the rest of the players get a reported welfare of

1

2

3

4

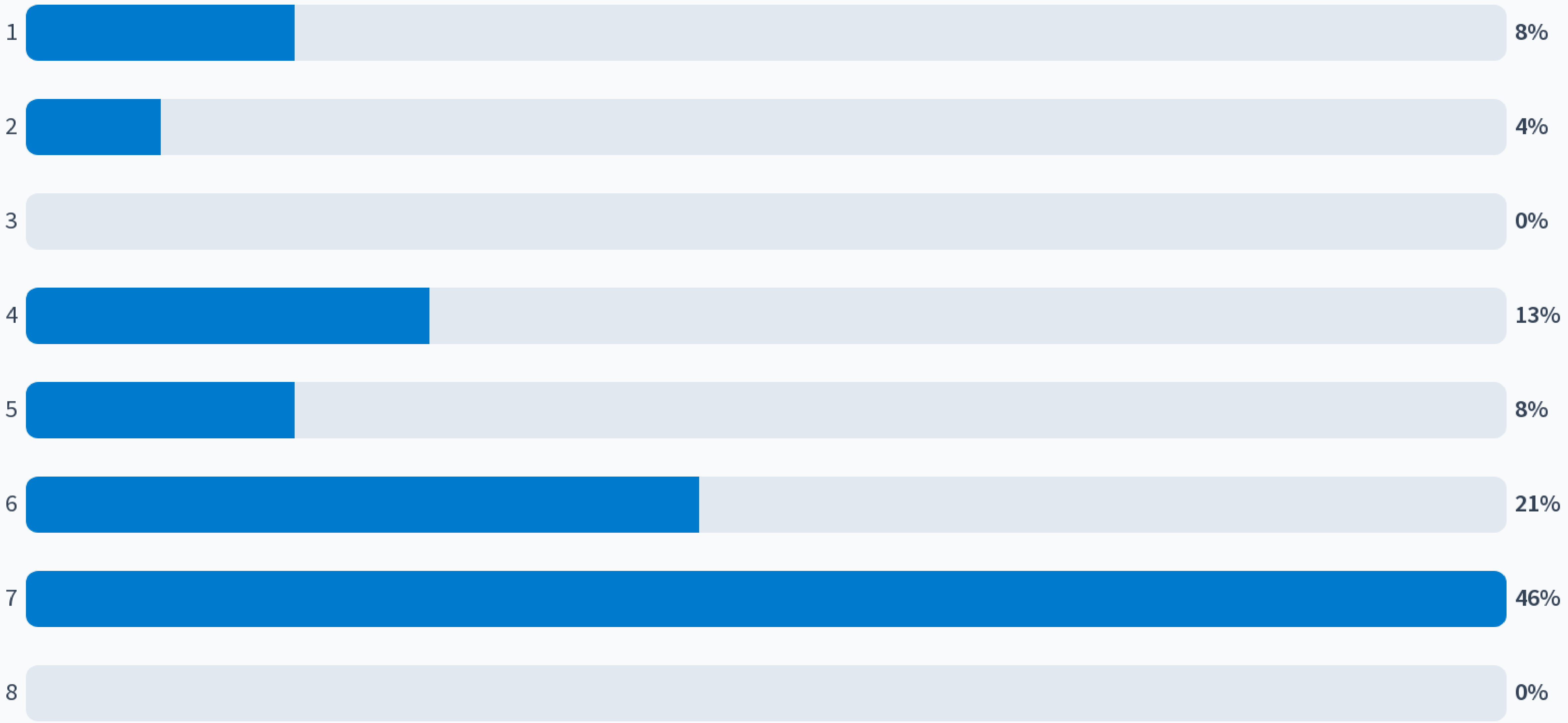
5

6

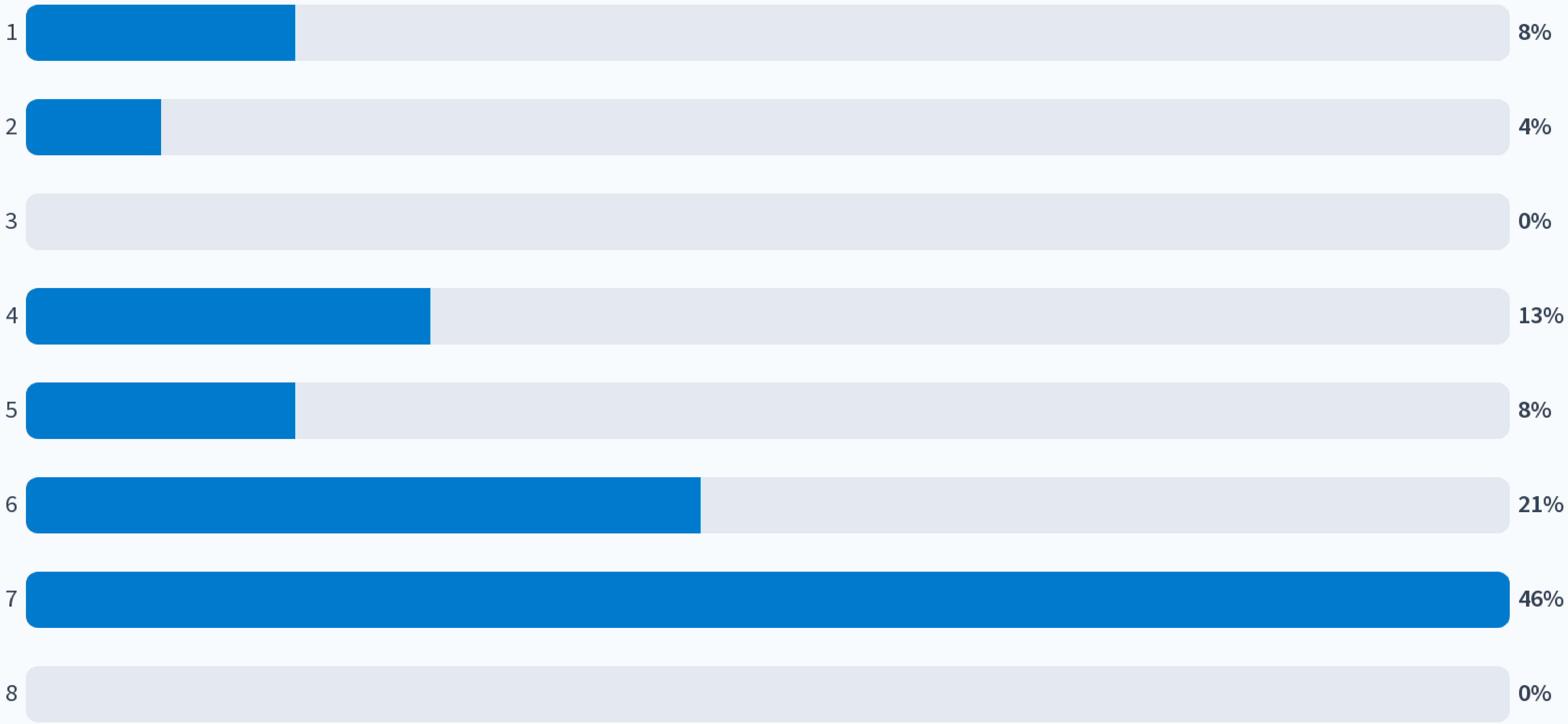
7

8

When the highest value bidder vanishes the rest of the players get a reported welfare of

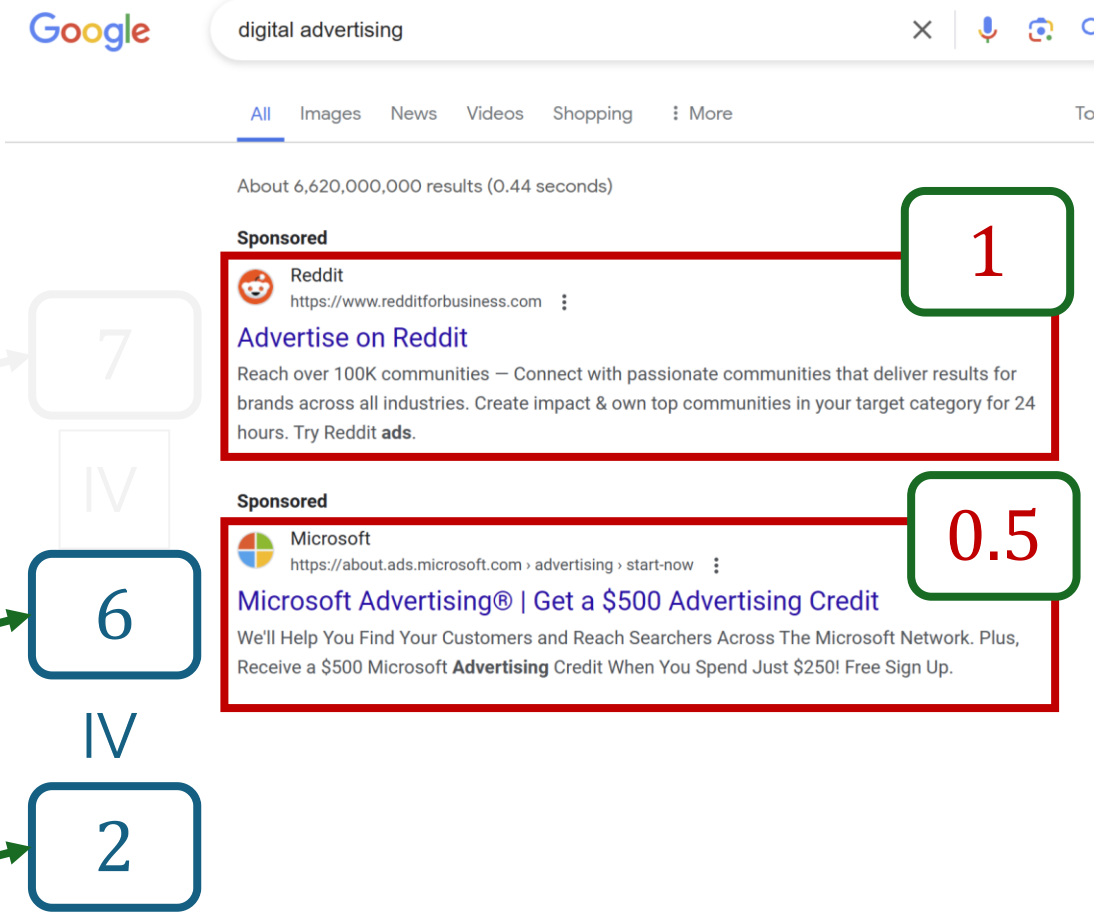


When the highest value bidder vanishes the rest of the players get a reported welfare of



Let's repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of ...?
- When highest bidder vanishes, rest of players achieve reported welfare of ...?



The net total gain to the rest of the bidders, from bidder 1 vanishing is ...

What about the second highest bidder?

- When second highest bidder exists, rest of players achieve reported welfare of 7

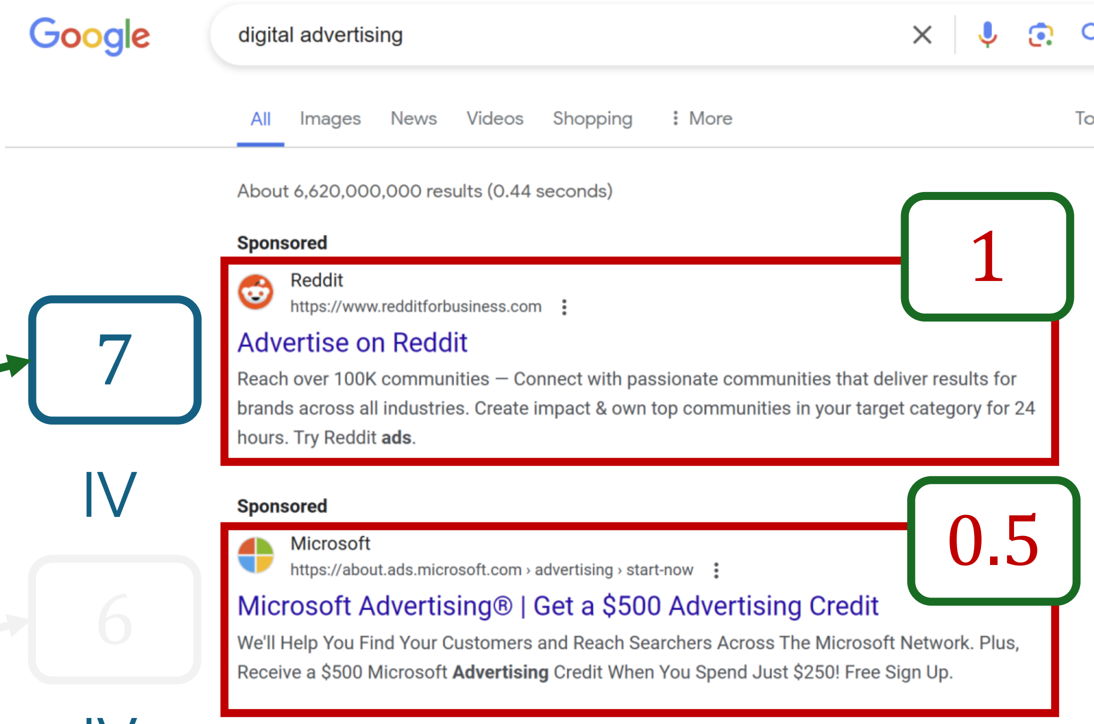
$v_1 = 7$ → 7

$\geq$

$\geq$

- When second highest bidder vanishes, rest of players achieve reported welfare of $7 + 1$

$v_3 = 2$ → 2



I should charge a total price of 1 (equivalently a price-per-bid of 2)

Bidders now don't have incentive to deviate

- The auction of choice in current sponsored search systems
- Even though still not truthful

$v_1 = 7$

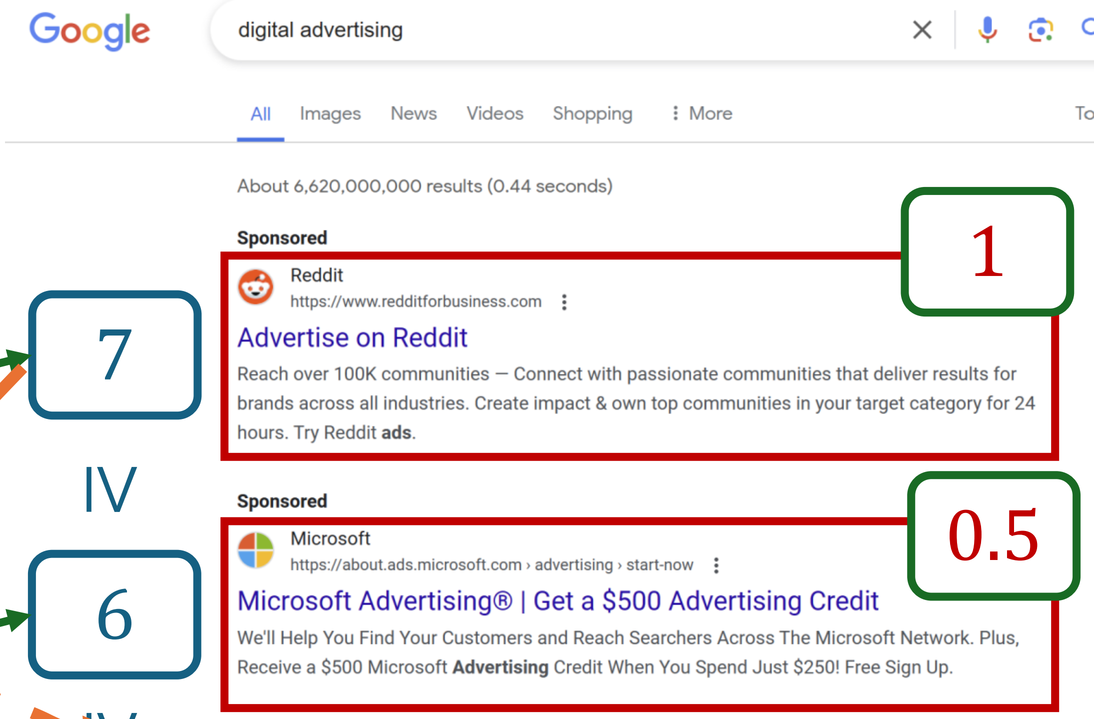
$u_1 = 1 \cdot (7 - 4) = 3$

$v_2 = 6$

$u_2 = .5 \cdot (6 - 2) = 2.5$

$v_3 = 2$

$7 \geq 6 \geq 2$



1

0.5

$u_1' = .5 \cdot (7 - 2) = 2.5$

How much utility do bidders receive?

$$\textbf{Externality} = \text{RWelfare of Others without me} - \overset{\text{Reported Welfare}}{\boxed{\text{RWelfare}}}\text{ of Others with me}$$

$$\textbf{Utility} = \text{Value of my Allocation} - \text{Payment}$$

How much utility do bidders receive?

Reported Welfare

$$\textbf{Externality} = \text{RWelfare of Others without me} - \boxed{\text{RWelfare}}\text{ of Others with me}$$

$$\textbf{Utility} = \text{Value of my Allocation} - \text{Payment}$$

If we set payment = externality

$$\text{Value of my Allocation} - \text{RWelfare of Others without me} + \text{RWelfare of Others with me}$$

How much utility do bidders receive?

$$\textbf{Externality} = \text{RWelfare of Others without me} - \overset{\text{Reported Welfare}}{\boxed{\text{RWelfare}}}\ \text{of Others with me}$$

$$\textbf{Utility} = \text{Value of my Allocation} - \text{Payment}$$

If we set payment = externality

$$\text{Value of my Allocation} - \text{RWelfare of Others without me} + \text{RWelfare of Others with me}$$

When I'm truthful:

$$\text{Value of my Allocation} + \text{RWelfare of Others with me} = \text{Total RWeflare with me}$$

How much utility do bidders receive?

$$\textbf{Externality} = \text{RWelfare of Others without me} - \overset{\text{Reported Welfare}}{\boxed{\text{RWelfare}}}\ \text{of Others with me}$$

$$\textbf{Utility} = \text{Value of my Allocation} - \text{Payment}$$

If we set payment = externality

$$\text{Value of my Allocation} - \text{RWelfare of Others without me} + \text{RWelfare of Others with me}$$

When I’m truthful:

$$\text{Value of my Allocation} + \text{RWelfare of Others with me} = \text{Total RWeflare with me}$$

When I’m truthful my utility is as simple as:

$$\textbf{Utility} = \text{Total RWeflare with me} - \text{Total RWelfare without me}$$

Can we ever charge bidders more than value?

- If we set payment = externality, and bidder is truthful

$$\textbf{Utility} = \text{Total RWeflare with me} - \text{Total RWelfare without me}$$

- If the auction always chooses the outcome that maximizes the reported welfare, then

$$\text{Total RWeflare with me} \geq \text{Total RWelfare without me}$$

Why is the mechanism truthful?

- If we set payment = externality, and bidder is truthful

$$\textbf{Utility} = \text{Total RWeflare with me} - \text{Total RWelfare without me}$$

- My bid does not affect the Total RWelfare without me!
- RWelfare only depends on the chosen allocation, not payments
- Trying to choose a bid b_i that leads to allocation x that maximizes

$$\text{Total RWeflare with me(x)}$$

Intuition: Why is the mechanism truthful?

- If we set payment = externality, and bidder is truthful

$$\textbf{Utility} = \text{Total RWeflare with me} - \text{Total RWelfare without me}$$

- My bid does not affect the Total RWelfare without me!
- RWelfare only depends on the chosen allocation, not payments

- If I’m truthful the auctioneer chooses the allocation that maximizes exactly this quantity and hence that maximizes my utility.

The Vickrey-Clarke-Groves (VCG) Mechanism

General Auction (Mechanism Design) Setting

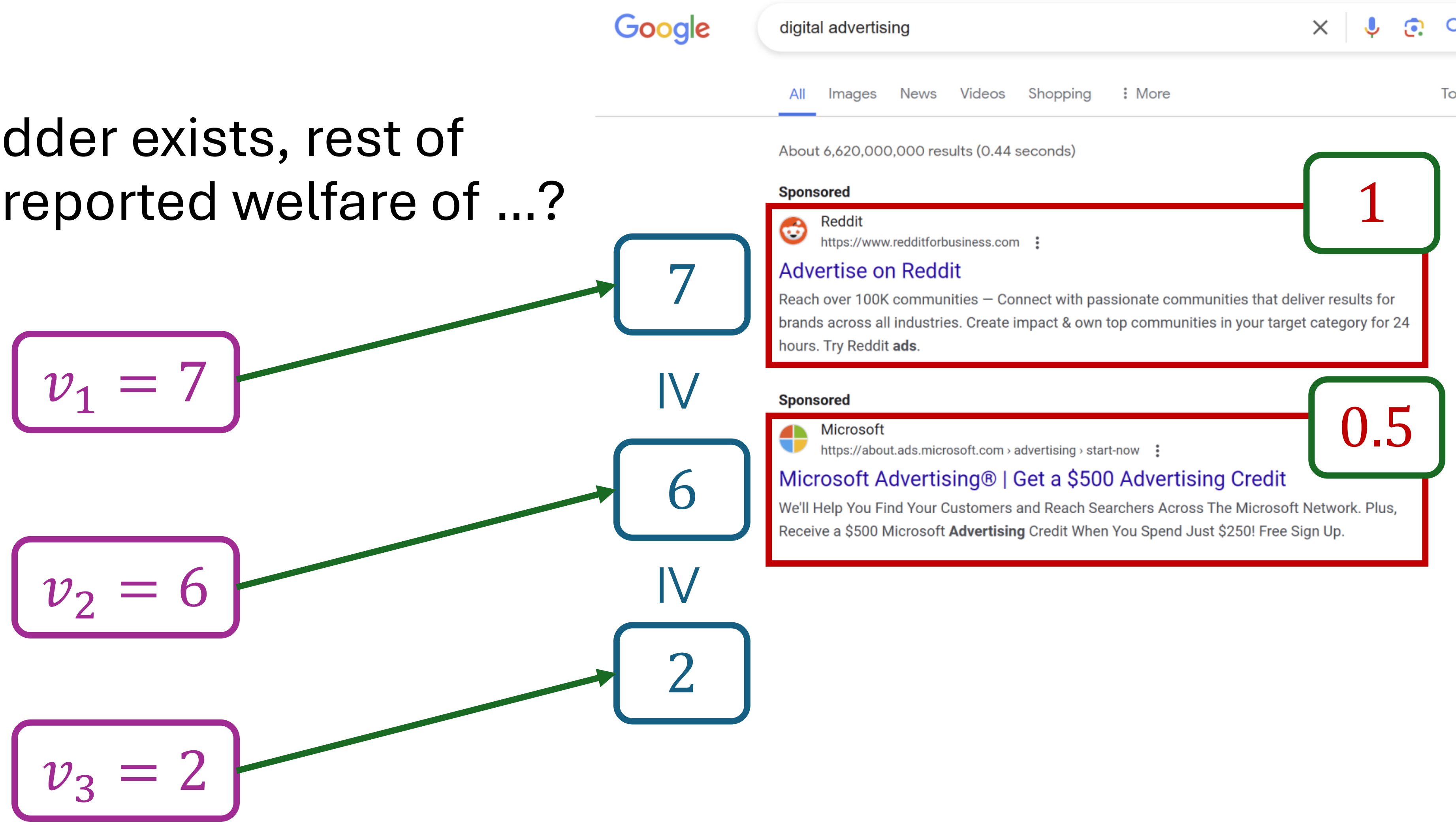
- Auctioneer (Designer) wants to choose among set of outcomes O
- Each bidder i has some value for each outcome $v_i(o) \in R$
- The value function v_i is called the **type** of player i
- Designer elicits **types/bids** from players $b = (b_1, \dots, b_n)$
- Designer chooses allocation that maximizes the reported welfare

$$x(b) = \operatorname*{argmax}_{o \in O} RW(o; b) \coloneqq \sum_{i=1}^{n} b_i(o)$$

Total Reported Welfare

Let’s repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of …?



General Auction (Mechanism Design) Setting

- Designer chooses allocation that maximizes the reported welfare

$$x(b) = \underset{o \in O}{\operatorname{argmax}}\, RW(o; b) \coloneqq \sum_{i=1}^{n} b_i(o)$$

- Charges to each player their externalities as payment

$$p_i(b) = \underbrace{\max_{o \in O} \sum_{j \neq i} b_j(o)}_{\text{RWelfare of others without me}} - \underbrace{\sum_{j \neq i} b_j\big(x(b)\big)}_{\text{RWelfare of others with me}} \boxed{\geq 0}$$

Why?

How much utility do bidders receive?

- The utility of bidder i for reporting b_i when others report b_{-i}

$$U_i(b) = \underbrace{v_i\big(x(b)\big)}_{\text{My value}} - \underbrace{p(b)}_{\text{My payment}}$$

- If payment=externality

$$U_i(b) = \underbrace{v_i\big(x(b)\big)}_{\text{My value}} - \underbrace{\max_{o \in O} \sum_{j \neq i} b_j(o)}_{\text{RWelfare of others without me}} + \underbrace{\sum_{j \neq i} b_j\big(x(b)\big)}_{\text{RWelfare of others with me}}$$

What is the optimal bid?

- If payment=externality

$$U_i(b) = v_i(x(b)) + \sum_{j \neq i} b_j(x(b)) - \max_{o \in O} \sum_{j \neq i} b_j(o)$$

My value

RWelfare of others with me

RWelfare of others without me

- I want to choose a bid b_i that optimizes my utility

$$\max_{b_i} v_i(x(b)) + \sum_{j \neq i} b_j(x(b)) - \cancel{\max_{o \in O} \sum_{j \neq i} b_j(o)}$$

Does not depend on my bid

What is the optimal bid?

- I want to choose a bid b_i that optimizes my utility

$$\max_{b_i} v_i(x(b)) + \sum_{j \neq i} b_j(x(b))$$

My value RWelfare of others with me

- This only depends on the chosen allocation $x(b)$
- Want to choose a bid that leads to an allocation x that maximizes

$$v_i(x) + \sum_{j \neq i} b_j(x)$$

What is the optimal bid?

- Want to choose a bid that leads to an allocation x that maximizes

$$v_i(x) + \sum_{j \neq i} b_j(x)$$

My value RWelfare of others with me

- Designer chooses allocation that maximizes reported welfare

$$b_i(x) + \sum_{j \neq i} b_j(x)$$

My bid RWelfare of others with me

What is the optimal bid? My true value

- Want to choose a bid that leads to an allocation x that maximizes

$$\underbrace{v_i(x)}_{\text{My value}} + \underbrace{\sum_{j \neq i} b_j(x)}_{\text{RWelfare of others with me}}$$

- Designer chooses allocation that maximizes reported welfare

$$\underbrace{b_i(x)}_{\text{My bid}} + \underbrace{\sum_{j \neq i} b_j(x)}_{\text{RWelfare of others with me}}$$

- If I'm **truthful** then auctioneer chooses the allocation that I want

What is my utility under truthful reporting

- If payment=externality

$$U_i(b) = v_i(x(b)) + \sum_{j\neq i} b_j(x(b)) - \max_{o\in O} \sum_{j\neq i} b_j(o)$$

Total RWelfare with me

RWelfare of others without me

- Since auctioneer optimizes reported welfare:

$$U_i(b) = \max_{o\in O} v_i(o) + \sum_{j\neq i} b_j(o) - \max_{o\in O} \sum_{j\neq i} b_j(o) \boxed{\geq 0}$$

Why?

Total RWelfare with me

RWelfare of others without me

Credits: https://rethinkmedia.org/blog/facebook-instagram-advertising-what-you-need-know-get-started

Learning in Non-Truthful Auctions

Non-Truthful Auctions

- Despite the universality of VCG, non-truthful auctions are frequently used

- More transparent and credible* rules

- The mechanism used in government procurement and display ads

*Credible Auctions: A Trilemma (stanford.edu)

Learning how to bid in auctions

- Given the complexity of digital auction markets
- Given the hardness of strategizing in non-truthful auctions
- Many of these auctions are repeated!

- It makes sense to study learning over time, to decide how to bid

- How do we learn over time when we repeatedly participate in an auction? Can we compete with the best fixed bid in hindsight?

No-Regret Learning in Auctions

At each period $t \in \{1, \dots, T\}$

- An auction among n bidders takes place (GFP, GSP, FP)
- Each bidder i submits bid b_i from discrete set of N bids $\{\epsilon, 2\epsilon \dots, 1\}$
- Each bidder learns their allocation and payment

$$x_i^t, p_i^t = x_i(b^t), p_i(b^t)$$

- e.g. in a first price auction, learn whether I won
- e.g. in a second price auction, learn whether I won and when I win, I learn the next highest bid.

No-Regret Learning

- Want to choose my bids b_i^t, based on algorithm that guarantees

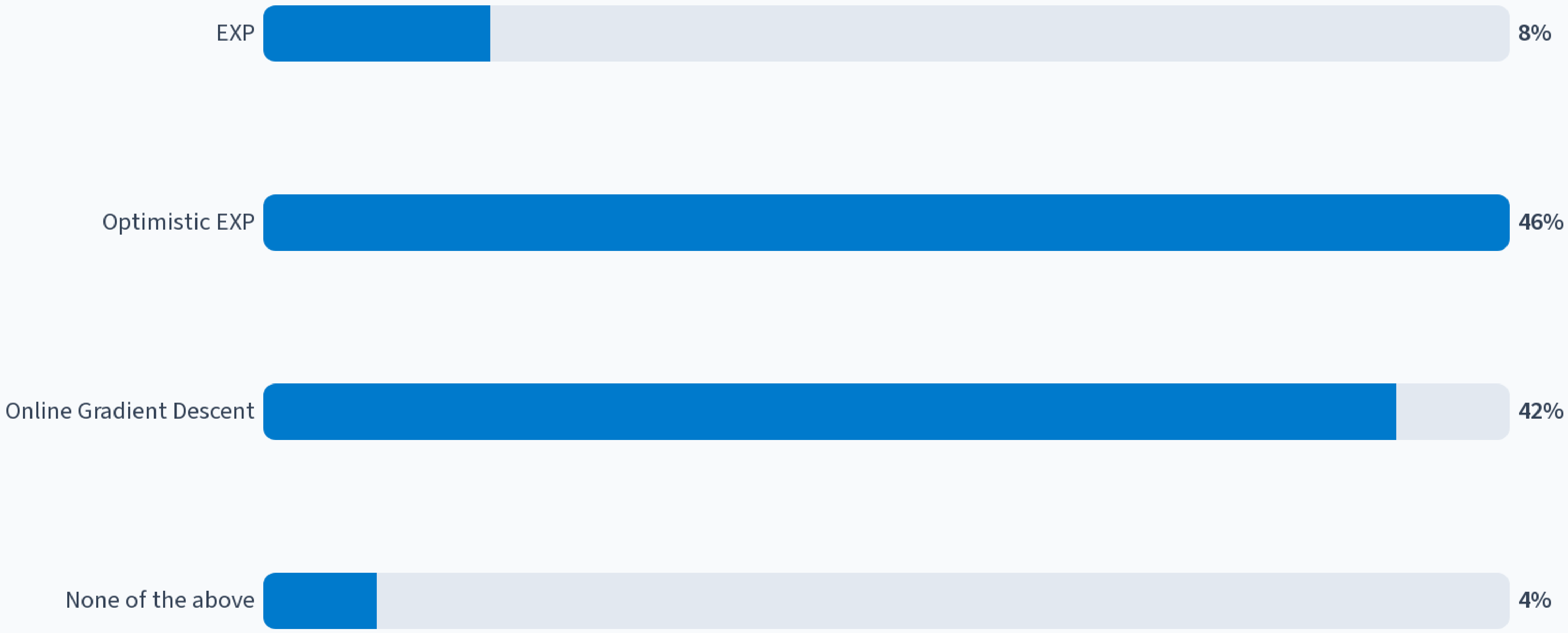
$$\frac{1}{T}\sum_{t=1}^{T} u_i\left(b^t\right) \geq \max_{b_i \in [N]} \frac{1}{T}\sum_{t=1}^{T} u_i\left(b_i, b^t\right) - \epsilon(T)$$

- for some $\epsilon(T) \to 0$

What algorithm should I use?

- EXP
- Optimistic EXP
- Online Gradient Descent
- None of the above

What algorithm should I use?



What algorithm should I use?

No-Regret Learning with Limited Feedback

- Want to choose my bids b_i^t, based on algorithm that guarantees

$$\frac{1}{T}\sum_{t=1}^{T} u_i(b^t) \geq \max_{b_i \in [N]} \frac{1}{T}\sum_{t=1}^{T} u_i(b_i, b^t) - \epsilon(T)$$

- Seems like a standard N action no-regret problem

- What's the catch! I don't receive after each period the utility for all my actions. Only the utility for action I took!
- Limited Feedback. I cannot calculate how much I would have gotten with any other bid (e.g. in an FP, solely knowing whether I won or not).

No-Regret Learning with Bandit Feedback

At each period t

- Adversary chooses a loss vector $\ell_t \in [0,1]^N$
- I choose an action i_t (not knowing ℓ_t)
- I observe loss of my chosen action $\ell_t^{i_t}$

- I want to guarantee small expected regret with any fixed action:

$$\max_{i \in N} E\left[\frac{1}{T}\sum_{t=1}^{T} \ell_t^{i_t} - \ell_t^{i}\right] \leq \epsilon(T)$$

Constructing Un-biased Estimates of Vector

- There is a hidden loss vector $\ell_t = \left(\ell_t^1, \dots, \ell_t^N\right)$ (potential outcomes)
- At each period I choose action (treatment) j with probability p_t^j
- I learn the loss ℓ_t^j with probability p_t^j
- Remember: no-regret algorithms work well, even if we have unbiased proxies of the true losses (e.g. Monte Carlo CFR)

Question. Can I construct a random variable that guarantees that in expectation over the choice of actions?

$$E\left[\tilde{\ell}_t\right] = \ell_t \Leftrightarrow \forall j\colon\ E\left[\tilde{\ell}_t^j\right] = \ell_t^j$$

Constructing Un-biased Estimates of Vector

Question. Can I construct a random variable that guarantees that in expectation over the choice of actions?

$$E\left[\tilde{\ell}_t\right] = \ell_t \Leftrightarrow \forall j:\ E\left[\tilde{\ell}_t^j\right] = \ell_t^j$$

- Random variable can always depend on identity of chosen action j_t. When I choose j random variable can also depend on ℓ_t^j

$$\tilde{\ell}_t^j = 1\{j_t = j\} f_j\left(\ell_t^j\right) + 1\{j_t \neq j\} g_j(j_t)$$

- Let's make g_j zero, and f_j linear in ℓ_t^j

$$\tilde{\ell}_t^j = 1\{j_t = j\} a_j \ell_t^j \Rightarrow E\left[\tilde{\ell}_t^j\right] = p_t^j a_j \ell_t^j = \ell_t^j \Rightarrow a_j = \frac{1}{p_t^j}$$

Inverse Propensity Estimates

At each period t

- Consider the random variables

$$\tilde{\ell}_t^j = \frac{1\{j_t = j\}}{p_t^j} \ell_t^j$$

- The vector $\tilde{\ell}_t$ can always be calculated $\left(0, \ldots, 0, \frac{\ell_t^{j_t}}{p_t^{j_t}}, 0, \ldots, 0\right)$
- The vector $\tilde{\ell}_t$ is an unbiased proxy of the true loss vector:

$$E\left[\tilde{\ell}_t\right] = \ell_t$$

The EXP Algorithm with Bandit Feedback

```
Initialize pt to the uniform distribution
For t in 1..T
    Draw action jt based on distribution pt
    Observe loss of chosen action lt[jt]

    Construct un-biased proxy loss vector
      ltproxy[j] = 1(jt=j) * lt[jt] / pt[jt]

    Update probabilities based on EXP update
      pt = pt * exp(-eta * ltproxy)
      pt = pt / sum(pt)
```

Recap: Regret of FTRL

(FTRL) $$x_t = \operatorname*{argmin}_{x \in X} \boxed{\sum_{\tau < t} \langle x, \ell_\tau \rangle} + \boxed{\frac{1}{\eta} \mathcal{R}(x)}$$

1-strongly convex function of x that stabilizes the maximizer

Historical performance of always choosing strategy x

Theorem. Assuming the utility function at each period

$$f_t(x) = \langle x, \ell_t \rangle$$

is L-Lipschitz with respect to some norm $\|\cdot\|$ and the regularizer is 1-strongly convex with respect to the same norm then

$$\text{Regret} - \text{FTRL}(T) \le \boxed{\eta L} + \boxed{\frac{1}{\eta T}\left(\max_{x \in X} \mathcal{R}(x) - \min_{x \in X} \mathcal{R}(x)\right)}$$

Average stability induced by regularizer

Average loss distortion caused by regularizer

Problem! The loss vector $\tilde{\ell}_t$ is not in [0,1].

It can take huge values, as probability of an action goes to 0!

Intuition: if probability goes to 0, then this action is chosen very infrequently. The loss vector very rarely takes this large value, i.e., the *variance* of the loss should be small.

Variance of Loss Vector

- Variance is

$$E\left[\left(\tilde{\ell}_t^j\right)^2\right] - E\left[\tilde{\ell}_t^j\right]^2 = E\left[\left(\tilde{\ell}_t^j\right)^2\right] - E\left[\ell_t^j\right]^2$$

- Second term is in $[0, 1]$. We will focus on first term (call it "variance")

$$E\left[\left(\tilde{\ell}_t^j\right)^2\right] = p_t^j \left(\frac{\ell_t^j}{p_t^j}\right)^2 = \frac{\left(\ell_t^j\right)^2}{p_t^j}$$

- And we collect this "variance" term only when end up choosing j

$$\text{Average "Variance"} = \sum_j p_t^j \cdot E\left[\left(\tilde{\ell}_t^j\right)^2\right] = \sum_j \left(\ell_t^j\right)^2 \leq N$$

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Can we replace L with the Average "Variance"?

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Average stability induced by regularizer

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Update: Regret of EXP

(EXP) $$p_t = \operatorname*{argmin}_{p \in \Delta} \sum_{\tau < t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta} \mathcal{R}(p)} \begin{pmatrix} \text{Negative} \\ \text{Entropy} \end{pmatrix} \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

$$p_t \propto p_{t-1} \exp\left(-\eta\, \tilde{\ell}_{t-1}\right)$$

Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \frac{\eta}{T} \sum_t E\left[\sum_j p_t^j \left(\tilde{\ell}_t^j\right)^2\right] + \frac{\log(N)}{\eta T}$$

See Lemma 20 and Theorem 2.22 and Corollary 4.2 of mal-018.dvi (huji.ac.il)

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Expected Average
“Variance”?

Update: Regret of EXP

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For the inverse propensity proxies

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Update: Regret of EXP

$$\text{(EXP)} \qquad p_t = \underset{p\in\Delta}{\operatorname{argmin}} \sum_{\tau<t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta}\mathcal{R}(p)} \quad \left(\begin{array}{c}\text{Negative}\\ \text{Entropy}\end{array}\right) \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

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Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \eta N + \frac{\log(N)}{\eta T} \Rightarrow \text{Regret} - \text{EXP}(T) \lesssim \sqrt{\frac{N\log(N)}{T}}$$

For $\eta \sim \sqrt{\frac{\log(N)}{NT}}$

See Lemma 20 and Theorem 2.22 and Corollary 4.2 of mal-018.dvi (huji.ac.il)