MS&E 233
Game Theory, Data Science and AI
Lecture 10

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(by courtesy) Computer Science and Electrical Engineering

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Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- **Basic Auctions and Learning to bid in auctions (T)**
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- Simple vs optimal mechanisms (T)
- *HW6: calculate equilibria in simple auctions, implement simple and optimal auctions, analyze revenue empirically*

6
- Optimizing mechanisms from samples (T)
- Online optimization of auctions and mechanisms (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Sum: Auction Applications

- Traditionally, selling of luxury goods, art

- Digital auction markets for goods (eBay)

- Energy markets

- Digital ad markets (sponsored search, display ads, amazon ads)

- Spectrum auctions

- Government procurement auctions

- Web3.0 transaction protocols

Sum: First Price

- First Price is arguably the simplest auction rule
- It can be hard to strategize in such an auction
- The auction can lead to inefficient allocations

- Though approximately efficient
- Still used in practice in many settings (e.g. online advertising, government procurement)
- Primarily because it has very transparent rules

Sum: Second Price

- Second Price is arguably the simplest truthful auction rule
- It is very easy to strategize in such an auction (be truthful)
- Auction always leads to efficient allocations (highest value wins)
- Auction can be run very quickly (computationally efficient)

- Still not always the auction used in many places
- Primarily because it has not very transparent rules
- Susceptible to collusion and manipulations by the auctioneer

Sponsored Search Auctions

Sponsored Search Auctions

- Now we have many items to sell
- Slots on a web impressions

- Higher slots get more clicks!
- Each slot has some probability of click

$$a_1 > a_2 > \cdots > a_m$$

- Bidders have a value-per-click v_i

Generalized First Price (GFP) Auction

- Bidders submit a bid-per-click b_i

- Slots allocated in decreasing order of bids

- Bidder i is allocated slot $j_i(b)$

- Bidder pays their bid when clicked

$$u_i(b; v_i) = a_{j_i(b)} \cdot (v_i - b_i)$$

Generalized First Price (GFP) Auction

- The first auction that was used by Overture in late 90s

- Lead to weird bidding patterns

(b) 1 week

$b_{(1)}$

$\lor$

$b_{(2)}$

$\lor$

$b_{(3)}$

$\lor$

$b_{(4)}$

a_1

a_2

a_3

a_4

Generalized Second Price (GSP) Auction

- Bidders submit a bid-per-click b_i

- Slots allocated in decreasing order of bids

- Bidder i is allocated slot $j_i(b)$

- Bidder pays the next highest bid when clicked

$$u_i(b; v_i) = a_{j_i(b)} \cdot \left(v_i - b_{(j_i(b)+1)} \right)$$

Generalized Second Price (GSP) Auction

- The auction of choice in current sponsored search systems
- Even though still not truthful

$v_1 = 7$

$u_1 = 1 \cdot (7 - 6) = 1$

$v_2 = 6$

$v_3 = 2$

7

IV

6

IV

2

$u_1' = .5 \cdot (7 - 2) = 2.5$

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0.5

How would you turn GSP truthful?

Right intuition, why Second-Price is truthful

- Second price is truthful **not because** we charge next highest bid
- Second price is truthful **not because** we charge smallest bid to maintain the same allocation

- Second price is truthful **because** we charged the winner their "externalities to the rest of society"

The Deep Reason why SP is Truthful

- When highest bidder exists, rest of players achieve reported welfare of 0
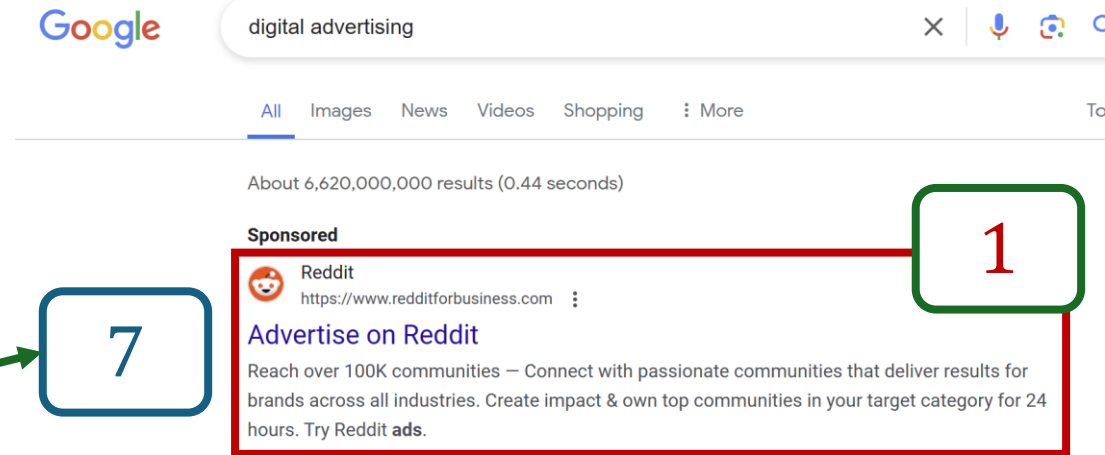
The Deep Reason why SP is Truthful

- When highest bidder does not exist, rest of players achieve reported welfare of 6

$v_1 = 7$

6

$v_2 = 6$

IV

2

$v_3 = 2$

The Deep Reason why SP is Truthful

- When highest bidder does not exist, rest of players achieve reported welfare of 6

$v_1 = 7$

$v_2 = 6$

$v_3 = 2$

6

IV

2

1
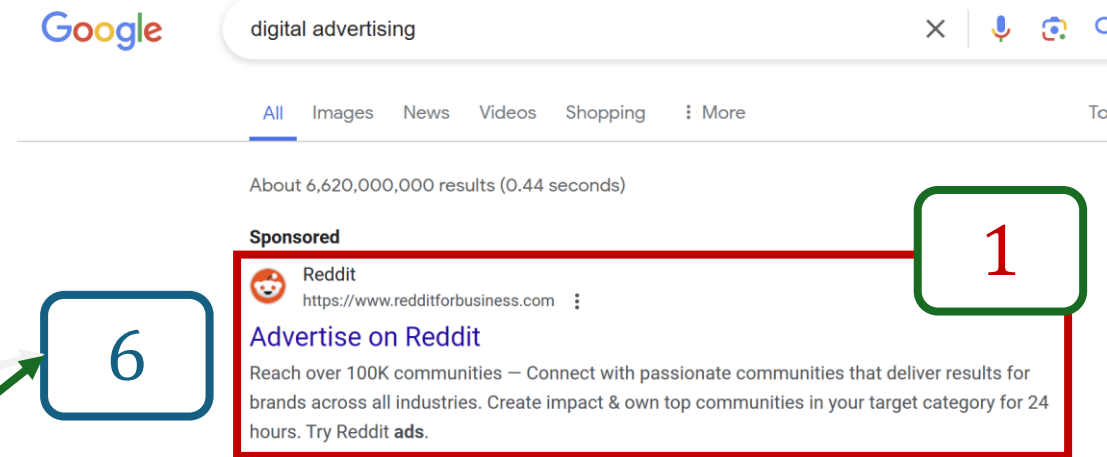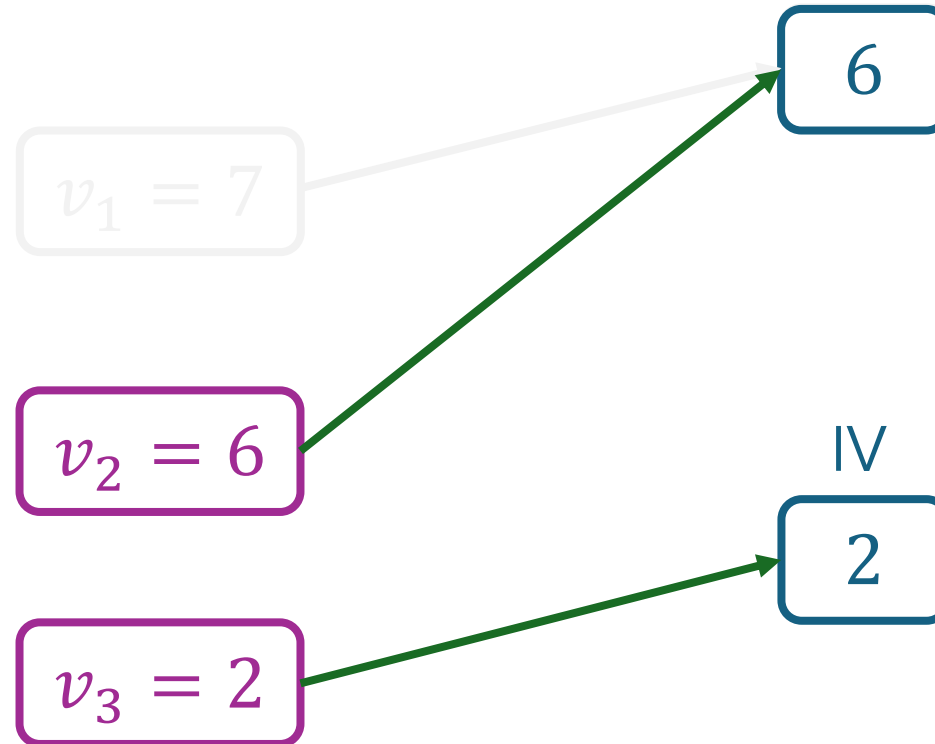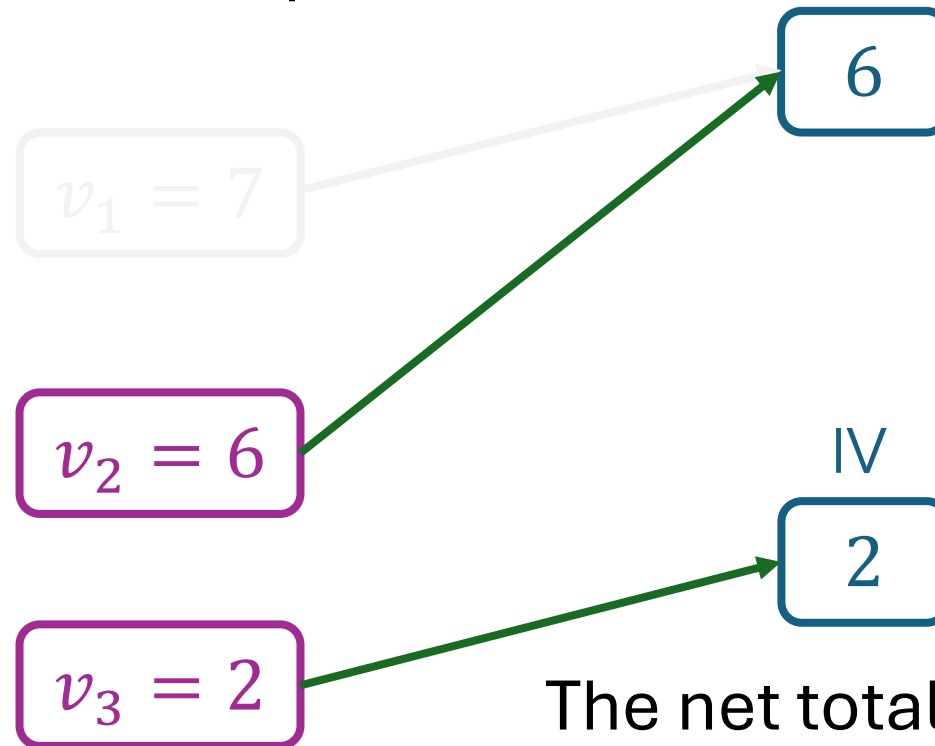
The net total gain to the rest of the bidders, from bidder 1 vanishing is 6
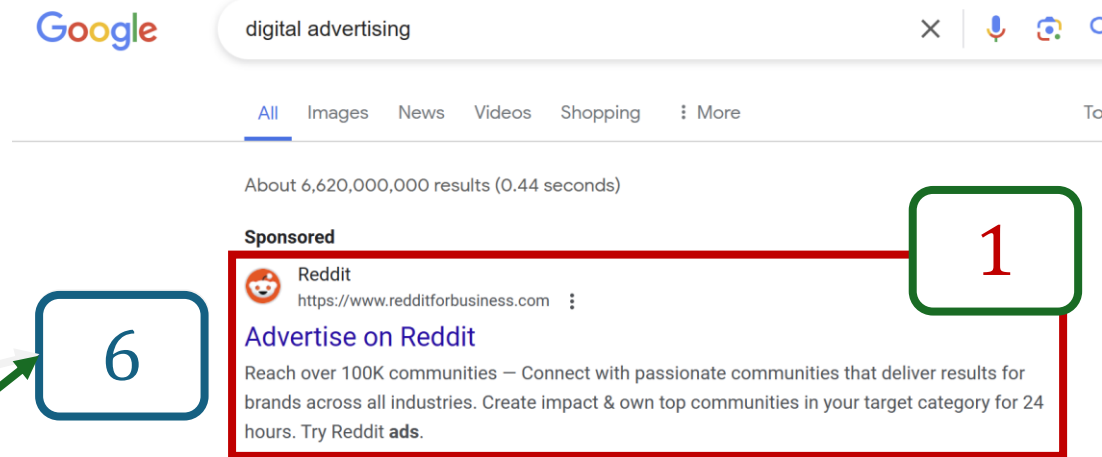
Right intuition, why Second-Price is truthful

- Second price is truthful **because** we charged the winner their "externalities to the rest of society"

- When highest bidder exists, rest of players achieve reported welfare 0

- When highest bidder vanishes, rest of players achieve reported welfare
$$b_{(2)} = \text{second highest bid}$$

- The net total gain to the rest of the bidders, from bidder 1 vanishing is
$$b_{(2)} = \text{second highest bid}$$

- That's what we should charge the winner!

Let's repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of …?

$$v_1 = 7$$

$$v_2 = 6$$

$$v_3 = 2$$

7

IV

6

IV

2

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0.5

When the highest value bidder exists the rest of the players get a reported welfare of

1

2

3

4

5

6

7

8

When the highest value bidder exists the rest of the players get a reported welfare of

8%

28%

48%

8%

0%

4%

0%

4%

When the highest value bidder exists the rest of the players get a reported welfare of

1		8%
2		28%
3		48%
4		8%
5		0%
6		4%
7		0%
8		4%

Let's repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of …?
- When highest bidder vanishes, rest of players achieve reported welfare of …?

$v_1 = 7$

7

IV

$v_2 = 6$

6

IV

$v_3 = 2$

2

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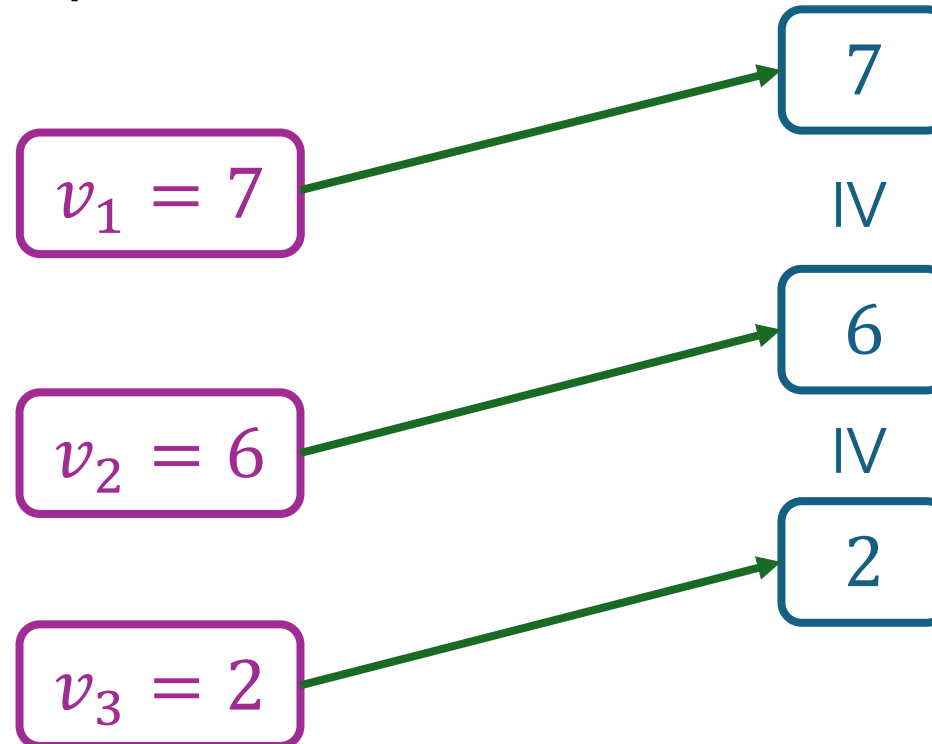
0.5

When the highest value bidder vanishes the rest of the players get a reported welfare of

1

2

3

4

5

6

7

8

When the highest value bidder vanishes the rest of the players get a reported welfare of

1   8%

2   4%

3   0%

4   13%

5   8%

6   21%

7   46%

8   0%

When the highest value bidder vanishes the rest of the players get a reported welfare of

1   8%

2   4%

3   0%

4   13%

5   8%

6   21%

7   46%

8   0%

Let's repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of …?
- When highest bidder vanishes, rest of players achieve reported welfare of …?

$v_1 = 7$

7

IV

$v_2 = 6$

6

IV

$v_3 = 2$

2

The net total gain to the rest of the bidders, from bidder 1 vanishing is …

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What about the second highest bidder?

- When second highest bidder exists, rest of players achieve reported welfare of 7

- When second highest bidder vanishes, rest of players achieve reported welfare of $7 + 1$

$v_1 = 7$

$v_3 = 2$

7

IV

6

IV

2

I should charge a total price of 1 (equivalently a price-per-bid of 2)

Bidders now don't have incentive to deviate

- The auction of choice in current sponsored search systems
- Even though still not truthful

$v_1 = 7$

$u_1 = 1 \cdot (7 - 4) = 3$

$v_2 = 6$

$u_2 = .5 \cdot (6 - 2) = 2.5$

$v_3 = 2$

7

IV

6

IV

2

$u_1' = .5 \cdot (7 - 2) = 2.5$

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1

0.5

How much utility do bidders receive?

Externality $=$ RWelfare of Others without me $-$ RWelfare of Others with me

Utility $=$ Value of my Allocation $-$ Payment

How much utility do bidders receive?

Reported Welfare

Externality $=$ RWelfare of Others without me $-$ RWelfare of Others with me

Utility $=$ Value of my Allocation $-$ Payment

If we set payment $=$ externality

Value of my Allocation $-$ RWelfare of Others without me $+$ RWelfare of Others with me

How much utility do bidders receive?

Reported Welfare

Externality $=$ RWelfare of Others without me $-$ RWelfare of Others with me

Utility $=$ Value of my Allocation $-$ Payment

If we set payment $=$ externality

Value of my Allocation $-$ RWelfare of Others without me $+$ RWelfare of Others with me

When I'm truthful:

Value of my Allocation $+$ RWelfare of Others with me $=$ Total RWeflare with me

How much utility do bidders receive?

Reported Welfare

Externality = RWelfare of Others without me − RWelfare of Others with me

Utility = Value of my Allocation − Payment

If we set payment = externality

Value of my Allocation − RWelfare of Others without me + RWelfare of Others with me

When I'm truthful:

Value of my Allocation + RWelfare of Others with me = Total RWeflare with me

When I'm truthful my utility is as simple as:

Utility = Total RWeflare with me − Total RWelfare without me

Can we ever charge bidders more than value?

- If we set payment = externality, and bidder is truthful

Utility = Total RWeflare with me − Total RWelfare without me

- If the auction always chooses the outcome that maximizes the reported welfare, then

Total RWeflare with me ≥ Total RWelfare without me

Why is the mechanism truthful?

- If we set payment = externality, and bidder is truthful

  Utility = Total RWeflare with me − Total RWelfare without me

- My bid does not affect the Total RWelfare without me!
- RWelfare only depends on the chosen allocation, not payments
- Trying to choose a bid b_i that leads to allocation x that maximizes

  Total RWeflare with me(x)

Intuition: Why is the mechanism truthful?

- If we set payment = externality, and bidder is truthful

    Utility = Total RWeflare with me − Total RWelfare without me

- My bid does not affect the Total RWelfare without me!
- RWelfare only depends on the chosen allocation, not payments

- If I'm truthful the auctioneer chooses the allocation that maximizes exactly this quantity and hence that maximizes my utility.

The Vickrey-Clarke-Groves (VCG) Mechanism

General Auction (Mechanism Design) Setting

- Auctioneer (Designer) wants to choose among set of outcomes O
- Each bidder i has some value for each outcome $v_i(o) \in R$
- The value function v_i is called the **type** of player i
- Designer elicits **types/bids** from players $b = (b_1, \dots, b_n)$
- Designer chooses allocation that maximizes the reported welfare

$$x(b) = \operatorname*{argmax}_{o \in O} RW(o; b) := \sum_{i=1}^{n} b_i(o)$$

Total Reported Welfare

Let's repeat this exercise with two slots

- When highest bidder exists, rest of players achieve reported welfare of …?

$v_1 = 7$ → 7

IV

$v_2 = 6$ → 6

IV

$v_3 = 2$ → 2

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General Auction (Mechanism Design) Setting

- Designer chooses allocation that maximizes the reported welfare

$$x(b) = \operatorname*{argmax}_{o \in O} RW(o; b) := \sum_{i=1}^{n} b_i(o)$$

- Charges to each player their externalities as payment

$$p_i(b) = \max_{o \in O} \sum_{j \neq i} b_j(o) - \sum_{j \neq i} b_j\big(x(b)\big) \geq 0$$

RWelfare of others without me   RWelfare of others with me

Why?

How much utility do bidders receive?

- The utility of bidder i for reporting b_i when others report b_{-i}

$$U_i(b) = v_i\big(x(b)\big) - p(b)$$

My value   My payment

- If payment=externality

$$U_i(b) = v_i\big(x(b)\big) - \max_{o \in O} \sum_{j \neq i} b_j(o) + \sum_{j \neq i} b_j\big(x(b)\big)$$

My value   RWelfare of others without me   RWelfare of others with me

What is the optimal bid?

- If payment=externality

$$U_i(b) = v_i\big(x(b)\big) + \sum_{j \neq i} b_j\big(x(b)\big) - \max_{o \in O} \sum_{j \neq i} b_j(o)$$

My value   RWelfare of others with me   RWelfare of others without me

- I want to choose a bid b_i that optimizes my utility

$$\max_{b_i} v_i\big(x(b)\big) + \sum_{j \neq i} b_j\big(x(b)\big) - \max_{o \in O} \sum_{j \neq i} b_j(o)$$

Does not depend on my bid

What is the optimal bid?

- I want to choose a bid b_i that optimizes my utility

$$\max_{b_i} v_i\big(x(b)\big) + \sum_{j \neq i} b_j\big(x(b)\big)$$

My value   RWelfare of others with me

- This only depends on the chosen allocation $x(b)$

- Want to choose a bid that leads to an allocation x that maximizes

$$v_i(x) + \sum_{j \neq i} b_j(x)$$

What is the optimal bid?

- Want to choose a bid that leads to an allocation x that maximizes

$$v_i(x) + \sum_{j \neq i} b_j(x)$$

My value   RWelfare of others with me

- Designer chooses allocation that maximizes reported welfare

$$b_i(x) + \sum_{j \neq i} b_j(x)$$

My bid   RWelfare of others with me

What is the optimal bid? My true value

- Want to choose a bid that leads to an allocation x that maximizes

$$v_i(x) + \sum_{j \neq i} b_j(x)$$

My value   RWelfare of others with me

- Designer chooses allocation that maximizes reported welfare

$$b_i(x) + \sum_{j \neq i} b_j(x)$$

My bid   RWelfare of others with me

- If I'm **truthful** then auctioneer chooses the allocation that I want

What is my utility under truthful reporting

- If payment=externality

$$U_i(b) = v_i\big(x(b)\big) + \sum_{j \neq i} b_j\big(x(b)\big) - \max_{o \in O} \sum_{j \neq i} b_j(o)$$

Total RWelfare with me   RWelfare of others without me

- Since auctioneer optimizes reported welfare:

$$U_i(b) = \max_{o \in O} v_i(o) + \sum_{j \neq i} b_j(o) - \max_{o \in O} \sum_{j \neq i} b_j(o) \geq 0$$

Why?

Total RWelfare with me   RWelfare of others without me

Learning in Non-Truthful Auctions

Non-Truthful Auctions

- Despite the universality of VCG, non-truthful auctions are frequently used

- More transparent and credible* rules

- The mechanism used in government procurement and display ads

Learning how to bid in auctions

- Given the complexity of digital auction markets
- Given the hardness of strategizing in non-truthful auctions
- Many of these auctions are repeated!

- It makes sense to study learning over time, to decide how to bid

- How do we learn over time when we repeatedly participate in an auction? Can we compete with the best fixed bid in hindsight?

No-Regret Learning in Auctions

At each period $t \in \{1, \dots, T\}$

- An auction among n bidders takes place (GFP, GSP, FP)
- Each bidder i submits bid b_i from discrete set of N bids $\{\epsilon, 2\epsilon \dots, 1\}$
- Each bidder learns their allocation and payment

$$x_i^t, p_i^t = x_i(b^t), p_i(b^t)$$

- e.g. in a first price auction, learn whether I won
- e.g. in a second price auction, learn whether I won and when I win, I learn the next highest bid.

No-Regret Learning

- Want to choose my bids b_i^t, based on algorithm that guarantees

$$\frac{1}{T}\sum_{t=1}^{T} u_i(b^t) \geq \max_{b_i \in [N]} \frac{1}{T}\sum_{t=1}^{T} u_i(b_i, b^t) - \epsilon(T)$$

- for some $\epsilon(T) \to 0$

What algorithm should I use?

EXP

Optimistic EXP

Online Gradient Descent

None of the above

What algorithm should I use?

EXP **8%**

Optimistic EXP **46%**

Online Gradient Descent **42%**

None of the above **4%**

What algorithm should I use?

EXP — 8%

Optimistic EXP — 46%

Online Gradient Descent — 42%

None of the above — 4%

No-Regret Learning with Limited Feedback

- Want to choose my bids b_i^t, based on algorithm that guarantees

$$\frac{1}{T} \sum_{t=1}^{T} u_i(b^t) \geq \max_{b_i \in [N]} \frac{1}{T} \sum_{t=1}^{T} u_i(b_i, b^t) - \epsilon(T)$$

- Seems like a standard N action no-regret problem

- What's the catch! I don't receive after each period the utility for all my actions. Only the utility for action I took!

- Limited Feedback. I cannot calculate how much I would have gotten with any other bid (e.g. in an FP, solely knowing whether I won or not).

No-Regret Learning with Bandit Feedback

At each period t

- Adversary chooses a loss vector $\ell_t \in [0, 1]^N$

- I choose an action i_t (not knowing ℓ_t)

- I observe loss of my chosen action $\ell_t^{i_t}$

- I want to guarantee small expected regret with any fixed action:

$$\max_{i \in N} E\left[\frac{1}{T}\sum_{t=1}^{T} \ell_t^{i_t} - \ell_t^i\right] \leq \epsilon(T)$$

Constructing Un-biased Estimates of Vector

- There is a hidden loss vector $\ell_t = (\ell_t^1, \ldots, \ell_t^N)$ (potential outcomes)

- At each period I choose action (treatment) j with probability p_t^j

- I learn the loss ℓ_t^j with probability p_t^j

- Remember: no-regret algorithms work well, even if we have unbiased proxies of the true losses (e.g. Monte Carlo CFR)

Question. Can I construct a random variable that guarantees that in expectation over the choice of actions?

$$E\left[\tilde{\ell}_t\right] = \ell_t \Leftrightarrow \forall j: E\left[\tilde{\ell}_t^j\right] = \ell_t^j$$

Constructing Un-biased Estimates of Vector

Question. Can I construct a random variable that guarantees that in expectation over the choice of actions?

$$E[\tilde{\ell}_t] = \ell_t \Leftrightarrow \forall j: E\left[\tilde{\ell}_t^j\right] = \ell_t^j$$

- Random variable can always depend on identity of chosen action j_t. When I choose j random variable can also depend on ℓ_t^j

$$\tilde{\ell}_t^j = 1\{j_t = j\}f_j\left(\ell_t^j\right) + 1\{j_t \neq j\}g_j(j_t)$$

- Let's make g_j zero, and f_j linear in ℓ_t^j

$$\tilde{\ell}_t^j = 1\{j_t = j\}a_j\ell_t^j \Rightarrow E\left[\tilde{\ell}_t^j\right] = p_t^j a_j \ell_t^j = \ell_t^j \Rightarrow a_j = \frac{1}{p_t^j}$$

Inverse Propensity Estimates

At each period t

- Consider the random variables

$$\tilde{\ell}_t^j = \frac{1\{j_t = j\}}{p_t^j} \ell_t^j$$

- The vector $\tilde{\ell}_t$ can always be calculated $\left(0, \dots, 0, \frac{\ell_t^{j_t}}{p_t^{j_t}}, 0, \dots, 0 \right)$

- The vector $\tilde{\ell}_t$ is an unbiased proxy of the true loss vector:

$$E[\tilde{\ell}_t] = \ell_t$$

The EXP Algorithm with Bandit Feedback

```
Initialize pt to the uniform distribution
For t in 1..T
    Draw action jt based on distribution pt
    Observe loss of chosen action lt[jt]


    Construct un-biased proxy loss vector
        ltproxy[j] = 1(jt=j) * lt[jt] / pt[jt]


    Update probabilities based on EXP update
        pt = pt * exp(-eta * ltproxy)
        pt = pt / sum(pt)
```

Recap: Regret of FTRL

(FTRL) $\qquad x_t = \underset{x \in X}{\operatorname{argmin}} \left[\sum_{\tau < t} \langle x, \ell_\tau \rangle \right] + \left[\frac{1}{\eta} \mathcal{R}(x) \right]$

1-strongly convex function of x that stabilizes the maximizer

Historical performance of always choosing strategy x

Theorem. Assuming the utility function at each period

$$f_t(x) = \langle x, \ell_t \rangle$$

is L-Lipschitz with respect to some norm $\|\cdot\|$ and the regularizer is 1-strongly convex with respect to the same norm then

$$\text{Regret} - \text{FTRL}(T) \leq \boxed{\eta L} + \boxed{\frac{1}{\eta T} \left(\max_{x \in X} \mathcal{R}(x) - \min_{x \in X} \mathcal{R}(x) \right)}$$

Average stability induced by regularizer

Average loss distortion caused by regularizer

Problem! The loss vector $\widetilde{\ell}_t$ is not in [0,1].

It can take huge values, as probability of an action goes to 0!

Intuition: if probability goes to 0, then this action is chosen very infrequently. The loss vector very rarely takes this large value, i.e., the *variance* of the loss should be small.

Variance of Loss Vector

- Variance is

$$E\left[\left(\tilde{\ell}_t^j\right)^2\right] - E\left[\tilde{\ell}_t^j\right]^2 = E\left[\left(\tilde{\ell}_t^j\right)^2\right] - E\left[\ell_t^j\right]^2$$

- Second term is in $[0, 1]$. We will focus on first term (call it "variance")

$$E\left[\left(\tilde{\ell}_t^j\right)^2\right] = p_t^j\left(\frac{\ell_t^j}{p_t^j}\right)^2 = \frac{\left(\ell_t^j\right)^2}{p_t^j}$$

- And we collect this "variance" term only when end up choosing j

$$\text{Average "Variance"} = \sum_j p_t^j \cdot E\left[\left(\tilde{\ell}_t^j\right)^2\right] = \sum_j \left(\ell_t^j\right)^2 \leq N$$

Recap: Regret of FTRL

(FTRL)

$$x_t = \underset{x \in X}{\text{argmin}} \boxed{\sum_{\tau < t} \langle x, \ell_\tau \rangle} + \boxed{\frac{1}{\eta} \mathcal{R}(x)}$$

Historical performance of always choosing strategy x

1-strongly convex function of x that stabilizes the maximizer

Theorem. Assuming the utility function at each period

$$f_t(x) = \langle x, \ell_t \rangle$$

~~is L-Lipschitz with respect to some norm $\|\ \|$~~ and the regularizer is 1-strongly convex with respect to the same norm then

$$\text{Regret} - \text{FTRL}(T) \leq \boxed{\eta \textcolor{red}{\boldsymbol{L}}} + \boxed{\frac{1}{\eta T} \left(\max_{x \in X} \mathcal{R}(x) - \min_{x \in X} \mathcal{R}(x) \right)}$$

Can we replace L with the Average "Variance"?

Average stability induced by regularizer

Average loss distortion caused by regularizer

Update: Regret of EXP

$$\text{(EXP)} \qquad p_t = \operatorname*{argmin}_{p \in \Delta} \sum_{\tau < t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta} \mathcal{R}(p)} \begin{pmatrix} \text{Negative} \\ \text{Entropy} \end{pmatrix} \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

$$p_t \propto p_{t-1} \exp(-\eta \, \tilde{\ell}_{t-1})$$

Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \frac{\eta}{T} \sum_t E\left[\sum_j p_t^j \left(\tilde{\ell}_t^j \right)^2 \right] + \frac{\log(N)}{\eta T}$$

See Lemma 20 and Theorem 2.22 and Corollary 4.2 of mal-018.dvi (huji.ac.il)

Update: Regret of EXP

(EXP) $\quad p_t = \underset{p \in \Delta}{\text{argmin}} \sum_{\tau < t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\dfrac{1}{\eta} \mathcal{R}(p)} \left(\begin{array}{c} \text{Negative} \\ \text{Entropy} \end{array} \right) \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$

$$p_t \propto p_{t-1} \exp\left(-\eta \, \tilde{\ell}_{t-1}\right)$$

Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \frac{\eta}{T} \sum_t E\left[\sum_j p_t^j \, E\left[\left(\tilde{\ell}_t^j\right)^2 \right] \right] + \frac{\log(N)}{\eta T}$$

Expected Average
"Variance"?

Update: Regret of EXP

(EXP) $$p_t = \operatorname*{argmin}_{p \in \Delta} \sum_{\tau < t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta} \mathcal{R}(p)} \begin{pmatrix} \text{Negative} \\ \text{Entropy} \end{pmatrix} \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

$$p_t \propto p_{t-1} \exp\left(-\eta\, \tilde{\ell}_{t-1}\right)$$

Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \frac{\eta}{T} \sum_t N + \frac{\log(N)}{\eta T}$$

For the inverse
propensity proxies

Update: Regret of EXP

$$\text{(EXP)} \qquad p_t = \operatorname*{argmin}_{p \in \Delta} \sum_{\tau < t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta} \mathcal{R}(p)} \begin{pmatrix} \text{Negative} \\ \text{Entropy} \end{pmatrix} \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

$$p_t \propto p_{t-1} \exp\left(-\eta \, \tilde{\ell}_{t-1}\right)$$

Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \eta N + \frac{\log(N)}{\eta T}$$

For the inverse propensity proxies

Update: Regret of EXP

(EXP)
$$p_t = \underset{p \in \Delta}{\arg\min} \sum_{\tau < t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta} \mathcal{R}(p)} \begin{pmatrix} \text{Negative} \\ \text{Entropy} \end{pmatrix} \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

$$p_t \propto p_{t-1} \exp(-\eta \, \tilde{\ell}_{t-1})$$

Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \eta N + \frac{\log(N)}{\eta T} \Rightarrow \text{Regret} - \text{EXP}(T) \lesssim \sqrt{\frac{N \log(N)}{T}}$$

For $\eta \sim \sqrt{\frac{\log(N)}{NT}}$