MS&E 233
Game Theory, Data Science and AI
Lecture 15

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(by courtesy) Computer Science and Electrical Engineering

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Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- Basic Auctions and Learning to bid in auctions (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- Simple vs optimal mechanisms (T)
- *HW6: implement simple and optimal auctions, analyze revenue empirically*

6
- Basics of Statistical Learning Theory (T)
- Optimizing Mechanisms from Samples (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples*

Further Topics

7
- **Econometrics in games and auctions (T+A)**
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Recap of Last Lecture

- Given i.i.d. samples of value profiles $v_1, \dots, v_m$ from unknown $F_1 \times \cdots \times F_n$
- We can learn personalized reserve prices $r = (r_1, \dots, r_n)$, such that:

$$\mathrm{Rev}(\mathrm{SPA} - \mathrm{r}) \geq \max_{\mathrm{r}} \mathrm{Rev}(\mathrm{SPA} - \mathrm{r}) - 4\sqrt{\frac{2n\log(2m)}{m}}$$

- We can learn virtual value functions $\hat{\phi} = (\hat{\phi}_1, \dots, \hat{\phi}_n)$, such that:

$$\mathrm{Rev}(\hat{\phi}) \gtrsim \mathrm{Myerson} - \left(\frac{n\log(m)}{m}\right)^{1/3}$$

Where do we get these samples from?

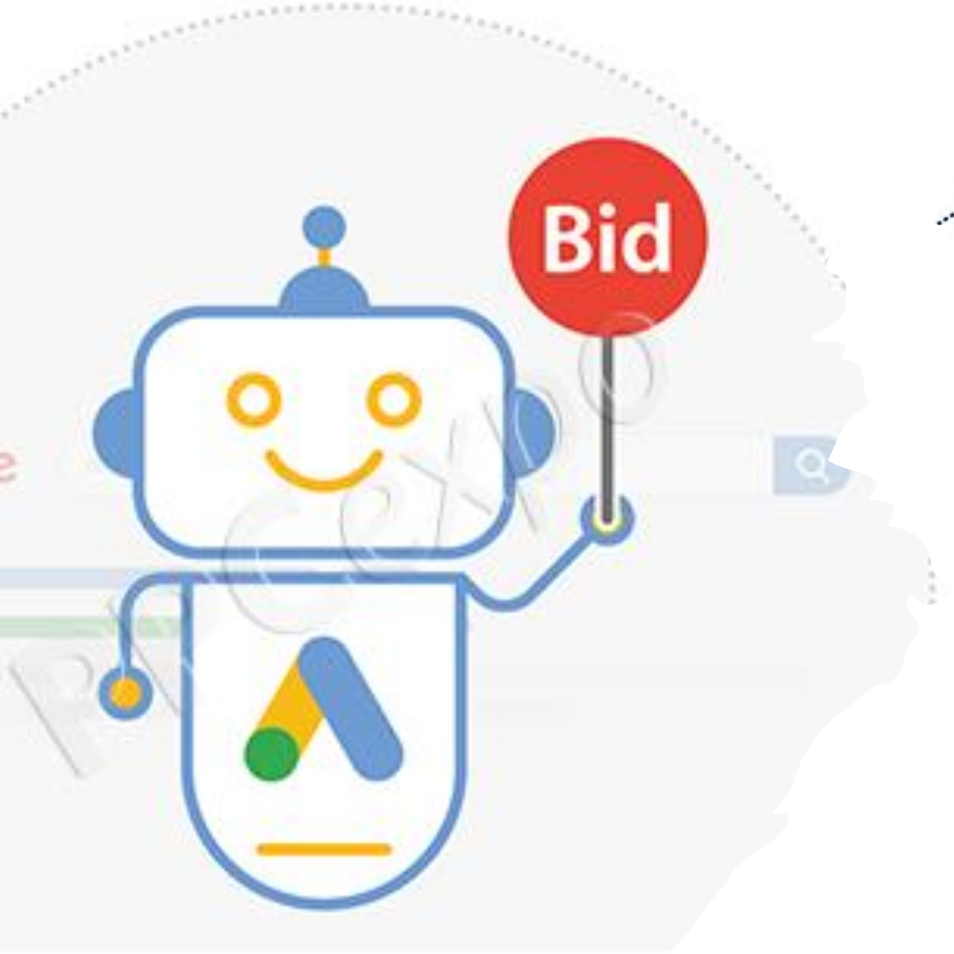
Typically, from historical executions of a truthful mechanism

Example: if we had run second price auctions in the past, we can use the bids of the players, in each of these historical auctions, as samples from their values

What if our auction platform is based on a non-truthful auctions?

Example: If we typically run a First Price Auction, now we have historical samples of bids in an FPA. These are not samples of values; bidders submit bids that are much lower than values in an FPA.

How do we go from bids to values?



Econometrics in Games and Auctions

Econometrics in Games and Auctions

- We are given data from actions of players in a game (and potentially auxiliary contextual information about the game)

- Multiple instances were players played the same type of game

- We don’t know the exact utilities of the players in the game

- We want to use the data to learn the parameters of the utilities of the players in the game or the distribution of these parameters

Example 1: Econometrics in Auctions

- Given bids of players in multiple instances of a First Price Auction

$$b_1, \dots, b_m, \qquad b_j = \left(b_{1j}, \dots, b_{nj}\right)$$

- Each bidder i has a value $v_{ij} \sim F_i$, independently across auctions
- Each bidder has a utility

$$u_{ij} = \left(v_{ij} - b_{ij}\right) \cdot 1\{\text{wins auction } j\}$$

- Find the distribution F_i of values for each i

Example 2: Econometrics in Entry Games

- Two firms deciding whether to enter a market
- Example: airline firms deciding whether to enter a particular route
- Observe entry decisions $y_i \in \{0, 1\}$ for different markets with characteristics x
- Each firm has profits from entering

$$\pi_1 = \underbrace{x^\top \beta_1}_{} + \underbrace{y_2 \delta_1}_{} + \underbrace{\epsilon_1}_{}$$
$$\pi_2 = x^\top \beta_2 + y_1 \delta_2 + \epsilon_2$$

effect of market characteristics

effect of competition

Private costs or payoff shocks $\epsilon_i \sim F_i$ **known only by player $\boldsymbol{i}$**

- Learn parameters β, δ

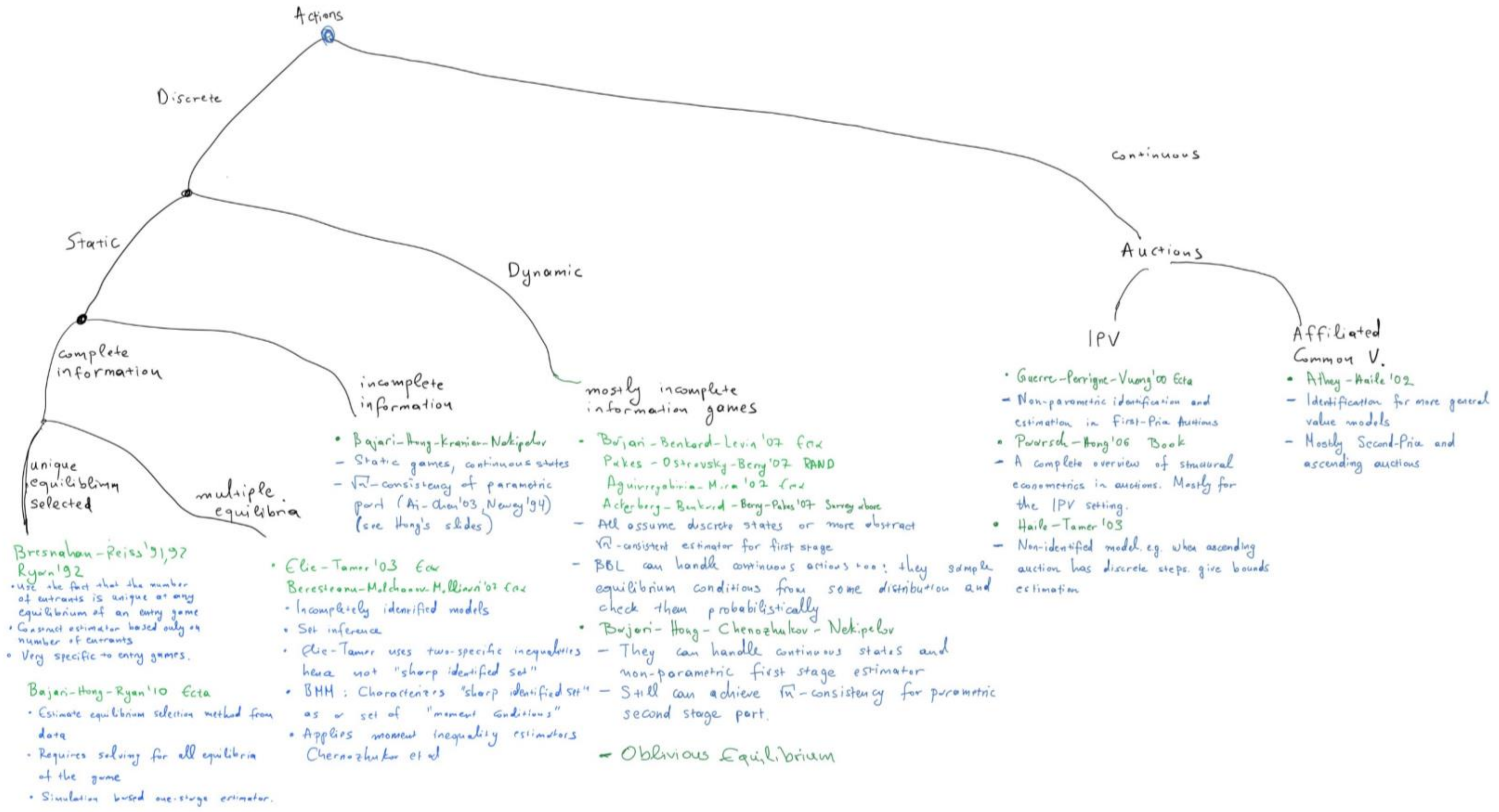
Why useful?

Scientific: economically meaningful quantities

Perform counter-factual analysis: what would happen if we change the game?

Performance measures: welfare, revenue

Testing game-theoretic models: if theory on estimated quantities predicts different behavior, then in trouble



Foundations:

- Large sample asymptotic theory:
  - Amemiya: Advanced Econometrics '85
  - Newey-McFadden: Large sample estimation and hypothesis testing '94
- Non-parametric Regression
  - Stone'82: Optimal rates
- Semiparametric Estimation
  - Chen'07: Large Sample Sieve Estimation
- Two-Stage Estimators
  - Hotz-Miller'93
  - Ai-Chen'00
  - Newey'94
  - Robinson'87
- Empirical Process Theory
  - Pollard
  - v. de Waard
- Set-Estimation
  - Molchanov: Random Sets
  - Chernozhukov-Hong-Tamer-Ecta'07

Books

- Amemiya
- HandBook of Econometrics Series
- Paarch-Hong: SEAD

Surveys

- Ackerberg-Benkard-Berry-Pakes'07
- Bajari-Hong-Nekipelov'10

Credits: https://vsyrgkanis.com/rough_map.jpg

Econometrics in First Price Auctions

Econometrics in First-Price Auctions

- Given bids of players in multiple instances of a First Price Auction

$$b_1, \dots, b_m, \qquad b_j = \left(b_{1j}, \dots, b_{nj}\right)$$

- Each bidder i has a value $v_{ij} \sim F_i$, independently across auctions
- Each bidder has a utility

$$u_{ij} = \left(v_{ij} - b_{ij}\right) \cdot 1\{\text{wins auction } j\}$$

- Find the distribution F_i of values for each i

First Question: how are bids related to values?

Reminder: Bayes-Nash Equilibrium

- Each bidder's value is drawn from some distribution

$$v_i \sim F_i, \qquad v = (v_1, \dots, v_n) \sim F = F_1 \times \cdots \times F_n$$

- Bidders submit a bid as a function of their value

$$s_i(v_i) = \text{Bid of player } i \text{ when their value is } v_i$$

Bayes-Nash Equilibrium. A bidding strategy profile $s = (s_1, \dots, s_n)$ is a Bayes-Nash equilibrium, if players cannot gain by deviating in expectation, assuming others follow their strategies

$$E_{v\sim F}[u_i(s(v); v_i)] \geq E_{v\sim F}[u_i(b_i', s_{-i}(v_{-i}); v_i)]$$

Behavioral Assumption: Bids are BNE

- Assume bids submitted according to a Bayes-Nash Equilibrium

$$(s_1, \dots, s_n)$$

Data generating process:

- Independently for each bidder i and auction j draw value $v_{ij} \sim F_i$
- Submit bid $b_{ij} = s_i(v_{ij})$

Gives rise to a bid distribution for each bidder $b_i \sim G_i$

The Identification Problem:
(*the Reverse Engineering problem*)

If I had infinite data, equivalently, *I know the distribution of bids* G_i, does that *uniquely determine the distribution of values* F_i

The Identification Problem

- When calculating BNE, we knew the distribution of values F_i and we wanted to calculated the bid b_i as a function of the value v_i
- Now we know the distribution of bids G_i and we want to calculate the value v_i as a function of the bid b_i!

- For simplicity, we will restrict to the ***symmetric*** bidder setting!

- All bidder values v_{ij} drawn from the same distribution F
- Equilibrium is symmetric and monotone, i.e., $b_{ij} = s(v_{ij})$

Identification for Symmetric Bidders

- At equilibrium s bidders don't benefit from *submitting another bid*!

- Consider bidder i with value v_i submitting bid b
- They win if $b \geq b_{i'}$ for all $i' \neq i$

- By independence of private values and independence of bids

$$\Pr(b \geq b_{i'}, \forall i' \neq i) = \prod_{i' \neq i} \Pr(b \geq b_{i'}) = G(b)^{n-1}$$

Identification for Symmetric Bidders

- At equilibrium s bidders don't benefit from *submitting another bid*!
- Consider bidder i with value v_i submitting a bid b

$$u_i(b; v_i) = (v_i - b) \cdot G(b)^{n-1}$$

- Since this is not beneficial

$$u_i(b_i; v_i) = \max_b u_i(b; v_i)$$

- Equilibrium bid $b_i = s(v_i)$ must satisfy the First Order Conditions

$$\partial_b u_i(b; v_i)\Big|_{z=b_i} = 0$$

Identification for Symmetric Bidders

- True value must satisfy the FOC

$$(n-1)(v_i - b_i) \cdot G(b_i)^{n-2}\, g(b_i) - G(b_i)^{n-1} = 0$$

- We can write value as function of equilibrium bid

$$v_i = \boxed{b_i} + \boxed{\frac{G(b_i)}{(n-1)\, g(b_i)}}$$

observed equilibrium bid

A function of the observed equilibrium distribution of bids

If I know the equilibrium bid distribution G, then whenever *I see a bid b_i*, I can *reverse engineer* and *uniquely determine the value* that led to such a bid

unobserved value

$$\boxed{v_i} = \boxed{b_i} + \boxed{\frac{G(b_i)}{(n-1)\, g(b_i)}}$$

observed equilibrium bid

amount by which the bidder reduced their value to determine their bid

If I know the equilibrium bid distribution G, then whenever *I see a bid b_i*, I can *reverse engineer* and *uniquely determine the value* that led to such a bid

$$\boxed{v_i} = \boxed{b_i} + \frac{1}{\boxed{(n-1)}\boxed{\frac{g(b_i)}{G(b_i)}}}$$

unobserved value

observed equilibrium bid

Reverse hazard ratio
of distribution of bids
"Probability that opponent bid is immediately below b_i given that it is below b_i"

More competition ⇒ less "value reduction"

Side Note (Asymmetric Bidders): If I know the equilibrium bid distributions G_i, then whenever *I see a bid b_i*, I can *reverse engineer* and *uniquely determine the value v_i* that led to such a bid

unobserved value

$$v_i = b_i + \frac{1}{\sum_{k \neq i} \frac{g_k(b_i)}{G_k(b_i)}}$$

observed equilibrium bid

Reverse hazard ratio of distribution of bids of k-th opponent
"Probability that opponent bid is immediately below b_i given that it is below b_i"

More competition ⇒ less "value reduction"

CDF of values uniquely determined by distribution of bids

$$F(z) = \Pr(v_i \leq z) = \Pr_{b_i \sim G}\left(b_i + \frac{G(b_i)}{(n-1)\, g(b_i)} \leq z \right)$$

The Estimation Problem:
(*Reverse Engineering with Finite Samples*)

If I have finite samples of bids, construct estimates $\widehat{F}_i$ *of the distributions of values* F_i *that converge to the true distribution as the number of samples grows*

Warm-up: Estimation with Truthful Bidding

- Given truthful bids of players in instances of Second Price Auction

$$v_1, \dots, v_m, \qquad v_j = (v_{1j}, \dots, v_{nj})$$

- Assuming $v_{ij} \sim F$, can approximate CDF by "empirical CDF"

$$F(z) \stackrel{\text{def}}{=} \Pr(v < z) = E[1\{v < z\}] \approx \frac{1}{n \cdot m} \sum_{i,j} 1\{v_{ij} < z\} \stackrel{\text{def}}{=} \hat{F}(z)$$

Probability that random draw from distribution lies below z

fraction of samples that lie below z

- By concentration inequalities and "*Rademacher complexity of threshold functions*" $\hat{F}$ is close to F, w.p. $1 - \delta$: $\sup_z \left| F(z) - \hat{F}(z) \right| \lesssim \sqrt{\frac{\log(m) + \log(1/\delta)}{m}}$

A slightly more refined variant of this is known as the DKW inequality

How do we “mimic” this approach, now that we only have samples from bids?

Plug-in Approach to Estimation

- Given bid vectors of players in multiple instances of a First Price Auction $b_1, \dots, b_m$
- We can write v_{ij} as a function of bid distribution G, g

$$v_{ij} = b_{ij} + \frac{G(b_{ij})}{(n-1)g(b_{ij})}$$

Plug-in paradigm. If we can construct estimates $\hat{G}, \hat{g}$ of G, g, then we can plug them in the above formula, to get an "estimated value"

$$\hat{v}_{ij} = b_{ij} + \frac{\hat{G}(b_{ij})}{(n-1)\,\hat{g}(b_{ij})}$$

We can pretend that $\hat{v}_{ij}$ are i.i.d. samples from values

$$\hat{F}(z) \stackrel{\text{def}}{=} \frac{1}{n \cdot m} \sum_{i,j} 1\{\hat{v}_{ij} < z\}$$

Constructing Estimates of Bid Distribution

- Estimate of the CDF is easy: use empirical CDF

$$\boxed{G(z) \stackrel{\text{def}}{=} \Pr(b < z)} = E[1\{b < z\}] \approx \boxed{\frac{1}{n \cdot m} \sum_{i,j} 1\{b_{ij} < z\} \stackrel{\text{def}}{=} \hat{G}(z)}$$

Probability that random draw of an equilibrium bid lies below z

fraction of bids that lie below z

- The density of the distribution is harder to learn
- Standard approach. Kernel density estimation

$$\boxed{g(z) = \partial_z G(z)} \approx \frac{G(z+h) - G(z-h)}{h} \approx \boxed{\frac{1}{n \cdot m} \sum_{i,j} \frac{1}{2h} 1\{|b_{ij} - z| \le h\} \stackrel{\text{def}}{=} \hat{g}(z)}$$

Probability mass that the distribution assigns in an infinitesimal region around z

Fraction of samples that lie within h from z, divided by region length

Constructing Estimates of Bid Distribution

- Estimate of the CDF is easy: use empirical CDF

$$\boxed{G(z) \stackrel{\text{def}}{=} \Pr(b < z)} = E[1\{b < z\}] \approx \boxed{\frac{1}{n \cdot m} \sum_{i,j} 1\{b_{ij} < z\} \stackrel{\text{def}}{=} \hat{G}(z)}$$

Probability that random draw of an equilibrium bid lies below z

fraction of bids that lie below z

- The density of the distribution is harder to learn
- Standard approach. Kernel density estimation

$$\boxed{g(z) = \partial_z G(z),} \qquad \boxed{\hat{g}(z) = \frac{1}{n \cdot m} \sum_{i,j} \frac{1}{h_n} K\left(\frac{b_{ij} - b}{h_n}\right)}$$

Probability mass that the distribution assigns in an infinitesimal region around z

For some smooth kernel function K that enjoys “nice properties” (e.g. Gaussian Kernel, Epanechnikov Kernel)

Formal Guarantees

- Suppose pdf f has R uniformly bounded continuous derivatives
- If we observed values then error rate of $\left(\frac{nm}{\log(nm)}\right)^{-\frac{R}{2R+1}}$ [Stone'82]
- Now that only bids are observed, [GPV'00] show that best achievable is: $\left(\frac{nm}{\log(nm)}\right)^{-\frac{R}{2R+3}}$
- The density f depends on the derivative of g

What if only winning bid is observed?

- For instance, in a Dutch auction (descending price auction)
- CDF of winning bid is simply:

$$G_w(b) = G(b)^n \Rightarrow G(b) = \left(G_w(b)\right)^{\frac{1}{n}}$$

- Hence, densities are related as:

$$g(b) = \frac{1}{n} g_w(b) \left(G_w(b)\right)^{\frac{1}{n}-1}$$

- Thus, G and g are identified from G_w and g_w
- Can apply previous argument and identify F and f

What if only winning bid is observed?

- Alternatively, we can identify value of winner as:

$$v_w = b_w + \frac{1}{n-1}\frac{G(b_w)}{g(b_w)} = b_w + \frac{n}{n-1}\frac{G_w(b_w)}{g_w(b_w)}$$

- Thus, we can identify distribution of highest value F_w and f_w
- Use $\mathrm{F}(v) = \left(F_w(v)\right)^{\frac{1}{n}}$ and $f(v) = \frac{1}{N} f_w(v)\left(F_w(v)\right)^{\frac{1}{n}-1}$ to identify F and f

- This also gives an estimation strategy (similar to case when all bids observed)

Notable Literature

- [Athey-Haile’02]
  - Identification in more complex than independent private values setting.
  - Primarily second price and ascending auctions
  - Mostly, winning price and bidder is observed
  - Most results in IPV or Common Value model
- [Haile-Tamer’03]
  - Incomplete data and partial identification
  - Prime example: ascending auction with large bid increments
  - Provides upper and lower bounds on the value distribution from necessary equilibrium conditions
- [Paarsch-Hong’06]
  - Complete treatment of structural estimation in auctions and literature review
  - Mostly presented in the IPV model

Main Take-Aways

- Closed form solutions of equilibrium bid functions in auctions
- Allows for non-parametric identification of value distribution

- Easy two-stage estimation strategy (similar to discrete incomplete information games)
- Estimation and Identification robust to what information is observed (winning bid, winning price)

- Rates for estimating density of value distribution are very slow

Econometrics of Entry Games (Discrete Choice Games)

High level idea

- At equilibrium agents have beliefs about other players actions and best respond

- If econometrician observes the same information about opponents as the player does, then:
  - Estimate these beliefs from the data in first stage
  - Use best-response inequalities to these estimated beliefs in the second stage and infer parameters of utility

Example 2: Econometrics in Entry Games

- Two firms deciding whether to enter a market
- Example: airline firms deciding whether to enter a particular route
- Observe entry decisions $y_i \in \{0, 1\}$ for different markets with characteristics x
- Each firm has profits from entering

$$\pi_1 = \underbrace{x^\top \beta_1}_{\text{effect of market characteristics}} + \underbrace{y_2 \delta_1}_{\text{effect of competition}} + \epsilon_1$$
$$\pi_2 = x^\top \beta_2 + y_1 \delta_2 + \epsilon_2$$

Private costs or payoff shocks $\epsilon_i \sim F_i$ **known only by player i**

- Learn parameters β, δ

Static Entry Game with Private Shocks

- BNE: Firms best-respond only in expectation
- Expected profits from entry:

$$\Pi_1 = x \cdot \beta_1 + \Pr[y_2 = 1|x]\,\delta_1 + \epsilon_1$$
$$\Pi_2 = x \cdot \beta_2 + \Pr[y_1 = 1|x]\,\delta_2 + \epsilon_2$$

- Let $\sigma_i(x) = \Pr[y_i = 1|x]$
- Then:

$$\sigma_1(x) = \Pr[x \cdot \beta_1 + \sigma_2(x)\delta_1 + \epsilon_1 > 0]$$
$$\sigma_2(x) = \Pr[x \cdot \beta_2 + \sigma_1(x)\delta_2 + \epsilon_2 > 0]$$

Static Entry Game with Private Shocks

- If ϵ_i is distributed according to an extreme value distribution:

$$\sigma_1(x) \propto \exp[x \cdot \beta_1 + \sigma_2(x)\delta_1]$$
$$\sigma_2(x) \propto \exp[x \cdot \beta_2 + \sigma_1(x)\delta_2]$$

- Non-linear system of simultaneous equations
- Computing fixed point is hard and fixed-point might not be unique

Key Idea: Two Stage Estimation

Two-Stage Estimation Approach

[Hotz-Miller'93, Bajari-Benkard-Levin'07, Pakes-Ostrovsky-Berry'07, Aguirregabiria-Mira'07, Bajari-Hong-Chernozhukov-Nekipelov'09]

1. Compute non-parametric estimate $\hat{\sigma}_i(x)$ of function $\sigma_i(x)$ from data

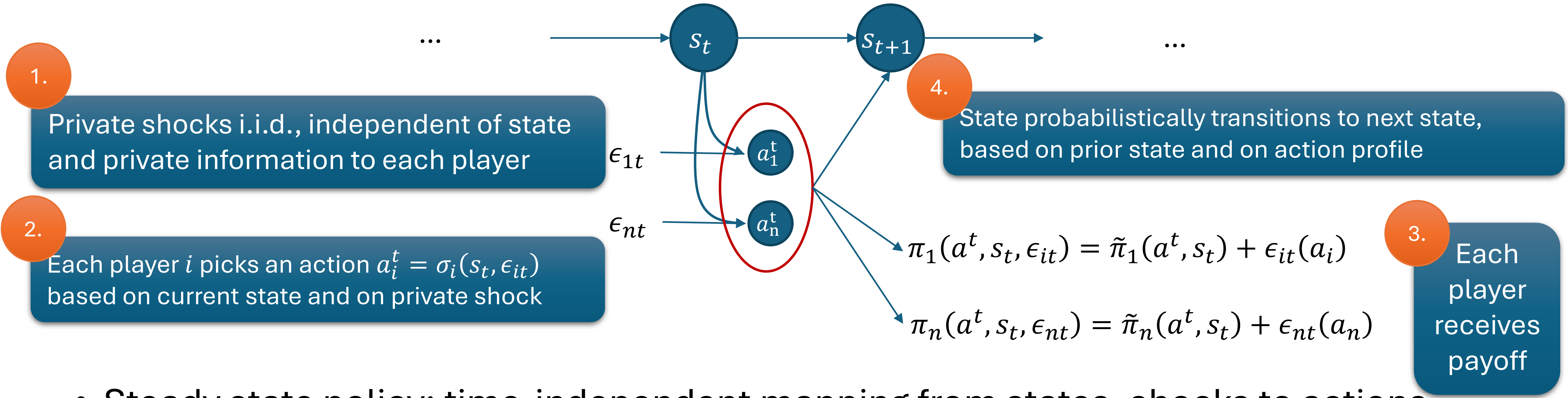
2. Run parametric regressions for each agent individually using that:

$$\sigma_i(x) \propto \exp[x \cdot \beta_i + \hat{\sigma}_{-i}(x)\ \delta_i]$$

3. The latter is a simple logistic regression for each player to estimate β_i, δ_i

Econometrics of Dynamics Discrete Choice Games

Steady-State Markovian Dynamic Games



- Steady state policy: time-independent mapping from states, shocks to actions

$$V_i(s;\sigma,\theta) = E\left[\sum_{t=0}^{T} \beta^t \pi_i(\sigma(s_t,\epsilon_t), s_t, \epsilon_{it}) \,\middle|\, s_0 = s;\theta\right] = \underbrace{v_i(\sigma(s,\epsilon_0), s)}_{\text{“shockless” discounted expected equilibrium payoff.}} + \epsilon_{i0}(\sigma(s,\epsilon_0))$$

- Markov-Perfect-Equilibrium: player chooses action a_i if:

$$v_i(a_i, s) + \epsilon_i(a_i) \geq v_i(a_i', s) + \epsilon_i(a_i')$$

Dynamic Games: First Stage

[Bajari-Benkard-Levin'07]

- Let $P_i(a_i|s)$: probability of playing action a_i conditional on state s
- Suppose ϵ_i are extreme value and $v_i(0,s) = 0$, then
$$\log P_i(a_i|s) - \log P_i(0|s) = v_i(a_i, s)$$
- Non-parametrically estimate $\widehat{P_i}(a_i|s)$
- Invert and get estimate $\hat{v}_i(a_i, s) = \log \hat{P}_i(a_i|s) - \log \hat{P}_i(0|s)$

- We have a non-parametric first-stage estimate of the policy function:
$$\hat{\sigma}_i(s, \epsilon_i) = \underset{a_i \in A_i}{\operatorname{argmax}}\, \hat{v}_i(a_i, s) - \epsilon_i(a_i)$$
- Combine with non-parametric estimate of state transition probabilities
- Compute a non-parametric estimate of discounted payoff for each policy, state, parameter tuple: $\hat{V}_i(\sigma, s; \theta)$, by forward simulation

Dynamic Games: First Stage

[Bajari-Benkard-Levin'07]

- If payoff is linear in parameters:

$$\pi_i(a, s, \epsilon_i; \theta) = \Psi_\mathrm{i}(a, s, \epsilon_i) \cdot \theta$$

- Then:

$$V_i(\sigma, s; \theta) = W_i(\sigma, s) \cdot \theta$$

- Suffices to do only simulation for each (policy, state) pair and not for each parameter, to get first stage estimates $\widehat{W}_i(\sigma, s)$

Dynamic Games: Second Stage

[Bajari-Benkard-Levin'07]

- We know by equilibrium:
$$g(i, s, \sigma_i'; \theta) = V_i(\sigma, s; \theta) - V_i(\sigma_i', \sigma_{-i}; \theta) \geq 0$$
- Can use an extremum estimator:
  - Definite a probability distribution over (player, state, deviation) triplets
  - Compute expected gain from [deviation]- under the latter distribution
$$Q(\theta) = E[\min\{g(i, s, \sigma_i'; \theta), 0\}]$$
  - By Equilibrium $Q(\theta_0) = 0 = \min_\theta Q(\theta)$
- Do empirical analogue with estimate $\hat{g}$:
$$\hat{g}(i, s, \sigma_i'; \theta) = \hat{V}_i(\hat{\sigma}, s; \theta) - \hat{V}_i(\sigma_i', \hat{\sigma}_{-i}; \theta)$$

  coming from first stage estimates
- Two sources of error:
  - Error of $\hat{\sigma}$ and $\hat{\mathrm{P}}(s'|s, a)$: $\sqrt{n}$-consistent, asymptotically normal, for discrete actions/states
  - Simulation error: can be made arbitrarily small by taking as many sample paths as you want

Recap of main idea

- At equilibrium agents have beliefs about other players actions and best respond

- If econometrician observes the same information about opponents as the player does then:
  - Estimate these beliefs from the data in first stage
  - Use best-response inequalities to these estimated beliefs in the second stage and infer parameters of utility

Econometrics for Learning Agents

Econometrics for Learning Agents

[Nekipelov-Syrgkanis-Tardos'15]

- Analyze repeated strategic interactions
- Finite horizon $t \in \{1, \ldots, T\}$
- Players are learning over time
- Unlike stationary equilibrium, or stationary MPE, or static game

- Use no-regret notion of learning behavior:

$$\forall a_i' : \sum_t \pi_i(a_i^t, a_{-i}^t; \theta) \geq \sum_t \pi_i\left(a_i', a_{-i}^t; \theta\right) - \epsilon$$

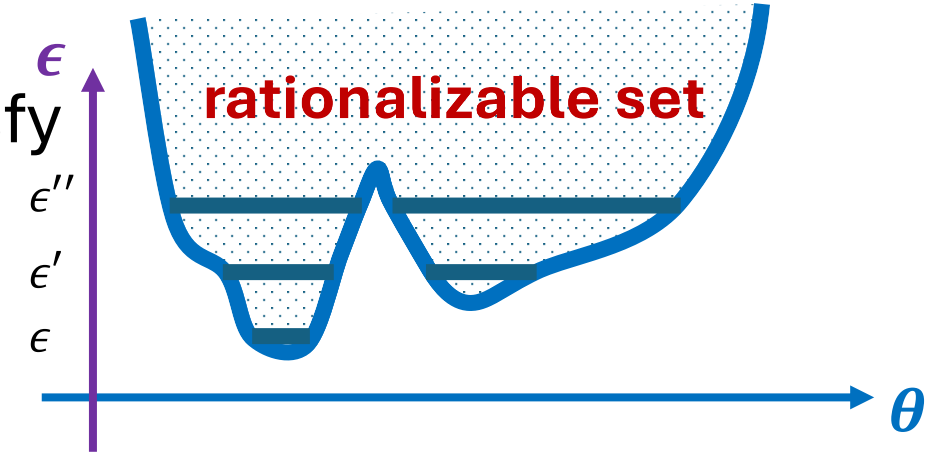
High-level approach

[Nekipelov-Syrgkanis-Tardos'15]

If we assume ϵ regret

$$\text{For all } a_i': \quad \underbrace{\frac{1}{T}\sum_t \pi_i(\boldsymbol{a}^t; \boldsymbol{\theta})}_{\text{Current average utility}} \geq \underbrace{\frac{1}{T}\sum_t \pi_i(a_i', \boldsymbol{a}_{-i}^t; \boldsymbol{\theta})}_{\text{Average deviating utility from fixed action}} \underbrace{- \epsilon}_{\text{Regret}}$$

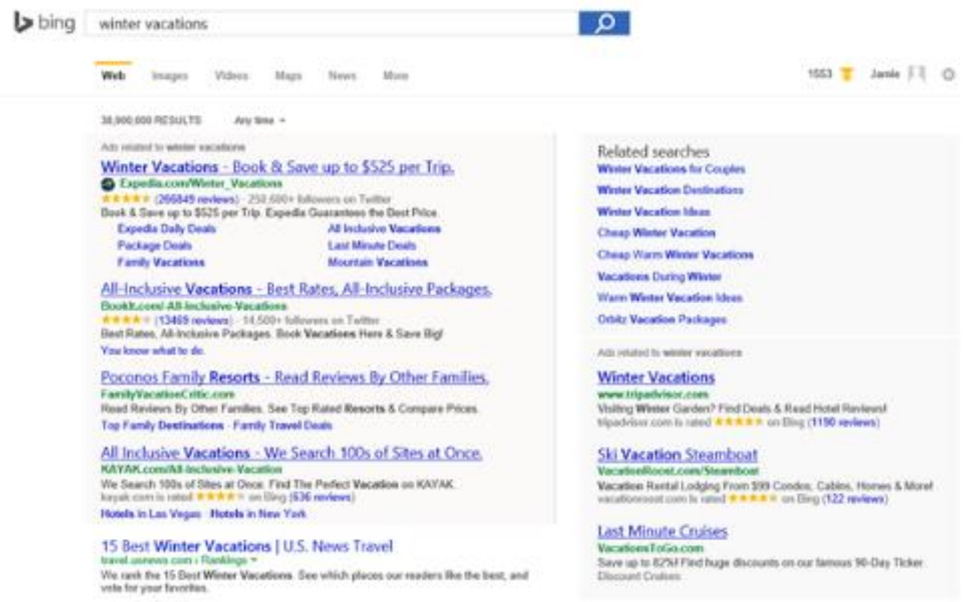
- Inequalities that unobserved $\boldsymbol{\theta}$ must satisfy
- Varying $\boldsymbol{\epsilon}$ we get the **rationalizable set of parameters**



Application: Online Ad Auction setting

[Nekipelov-Syrgkanis-Tardos’15]

- Each player has value-per-click v_i
- Bidders ranked according to a scoring rule
- Number of clicks and cost depends on position
- Quasi-linear utility

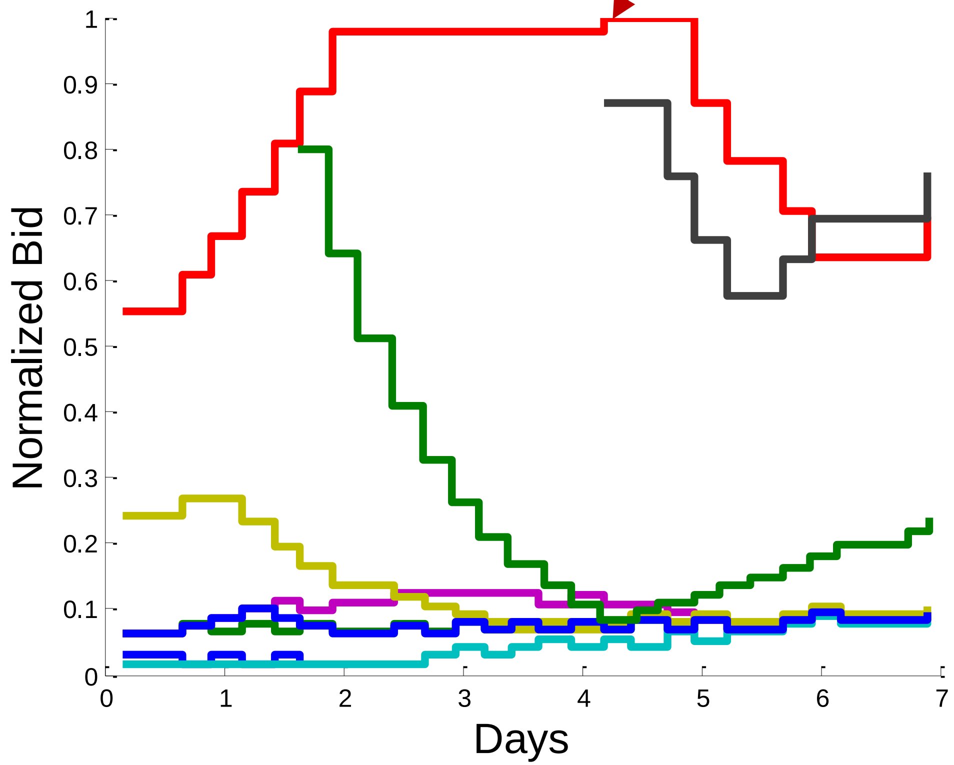


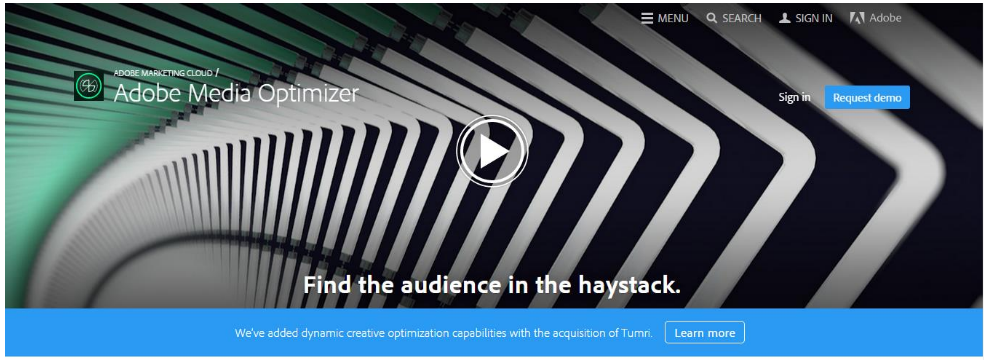
$$\pi_i(\boldsymbol{b}; \boldsymbol{v_i}) = \overbrace{\boldsymbol{v_i}}^{\text{Value-Per-Click}} \cdot \underbrace{x_i(\boldsymbol{b})}_{\text{Expected click probability}} - \overbrace{p_i(\boldsymbol{b})}^{\text{Expected Payment}}$$

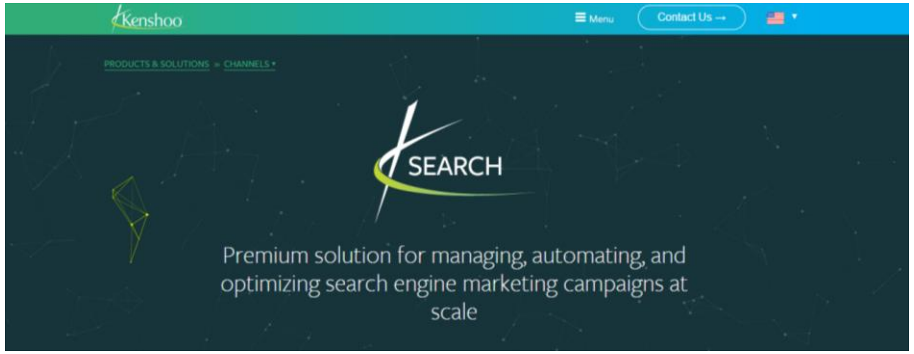
Data description

- 9 frequent bid changing advertisers
- Each advertiser has bids on many keywords or variants: few hundreds
- Studied auctions for a period of a week: Terabytes of auction data!
- Each keyword: from few hundreds to 100k auctions!

Bid on some keyword

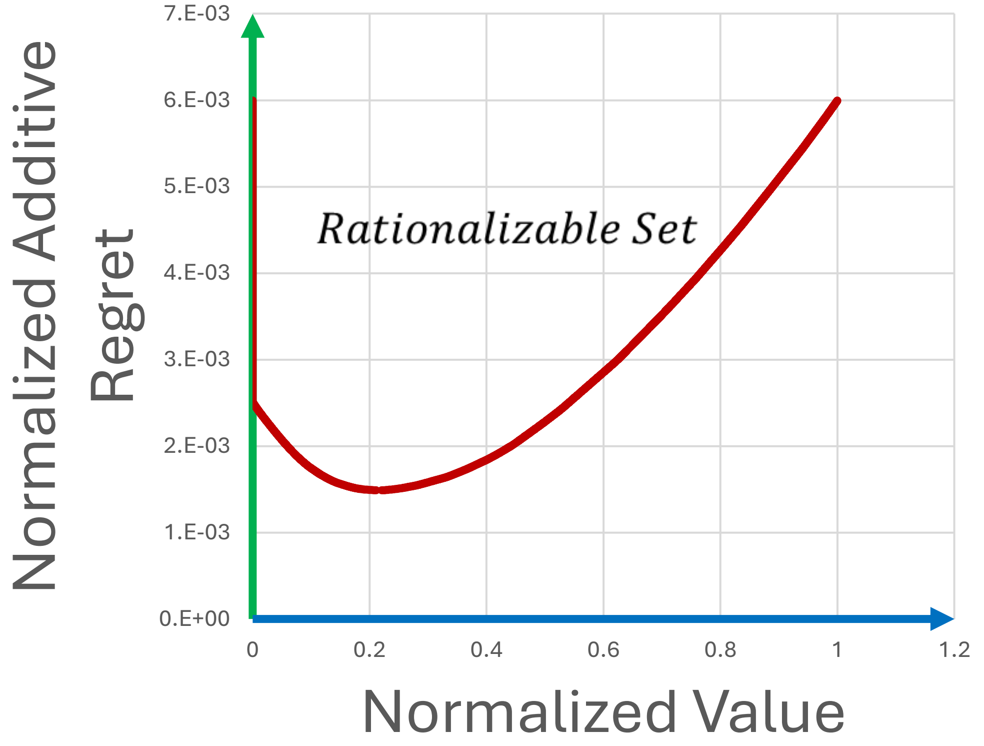


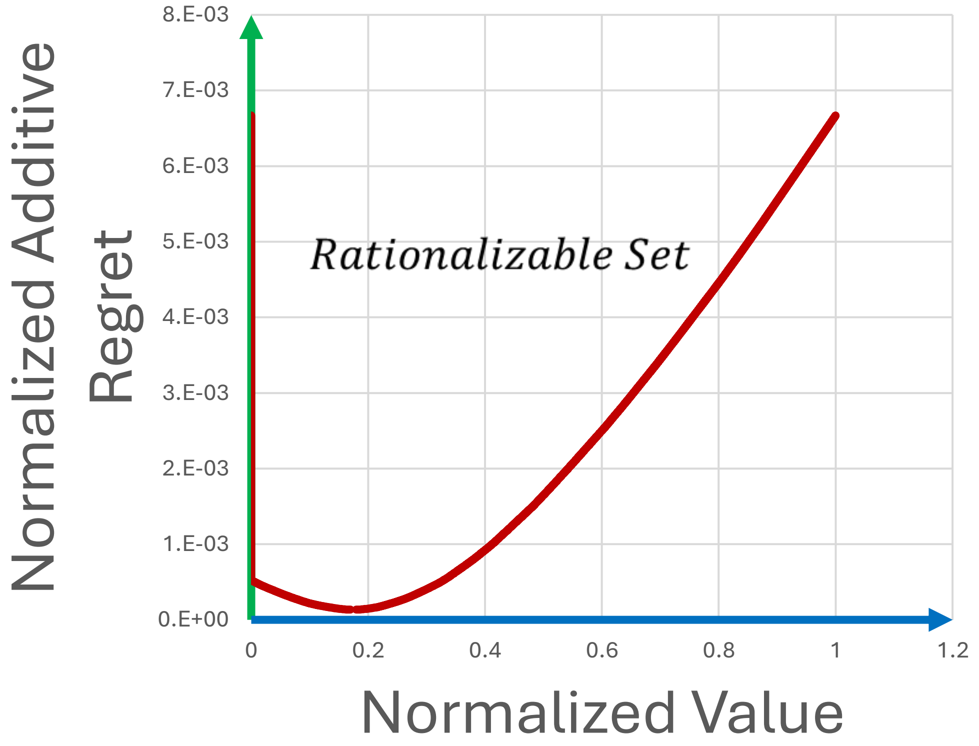




Data description

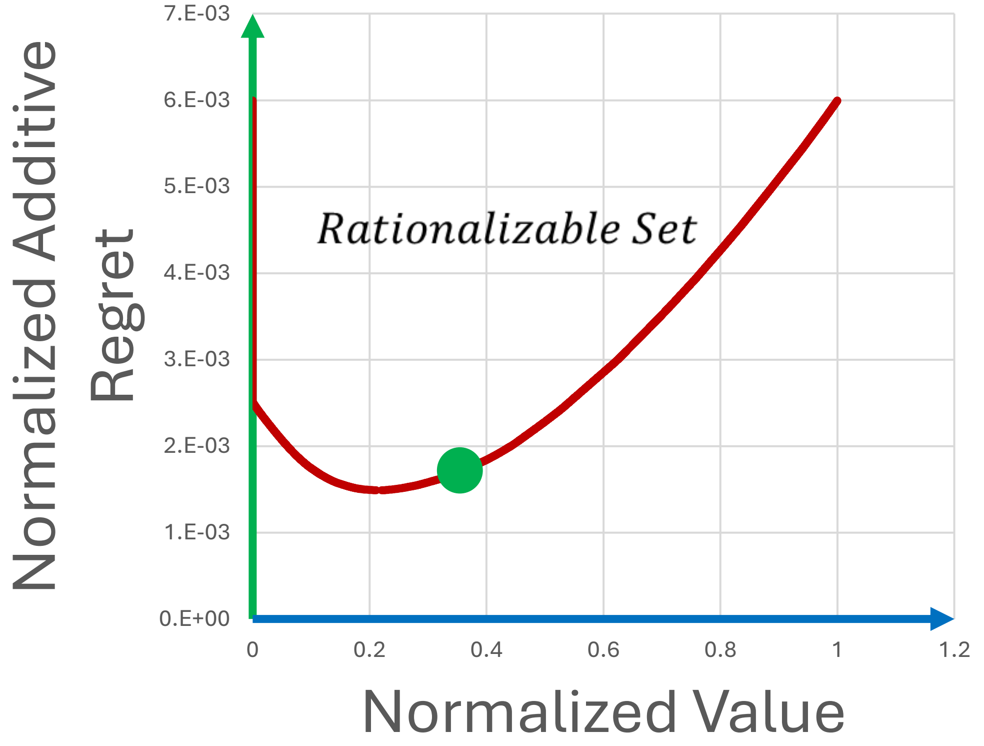
- Applied inference method to each (keyword,bid) pair of each advertiser

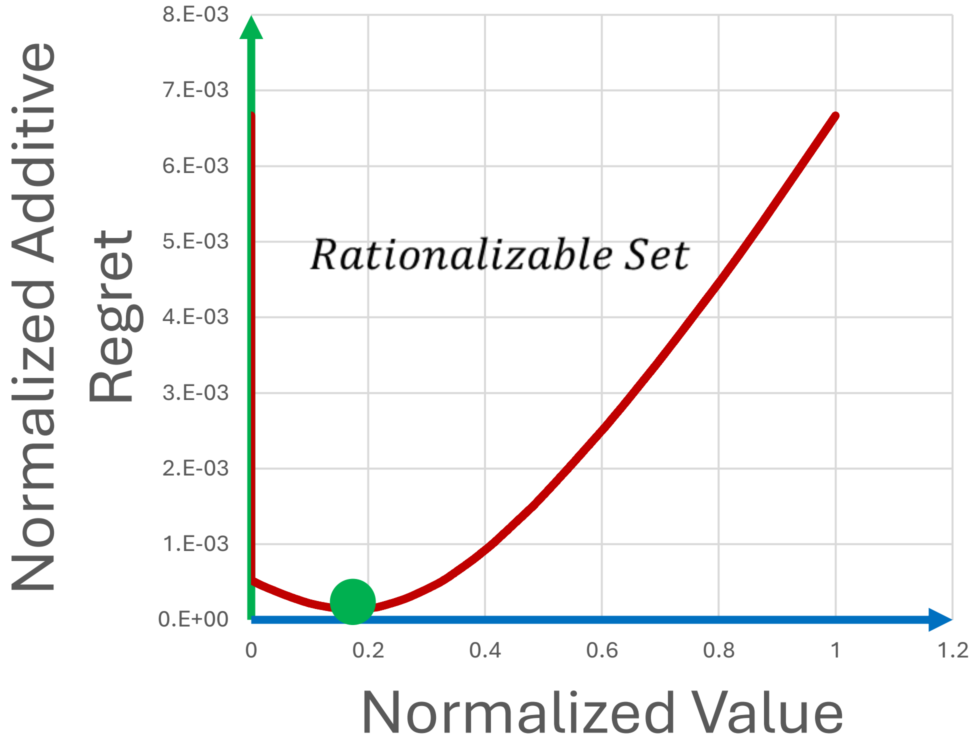




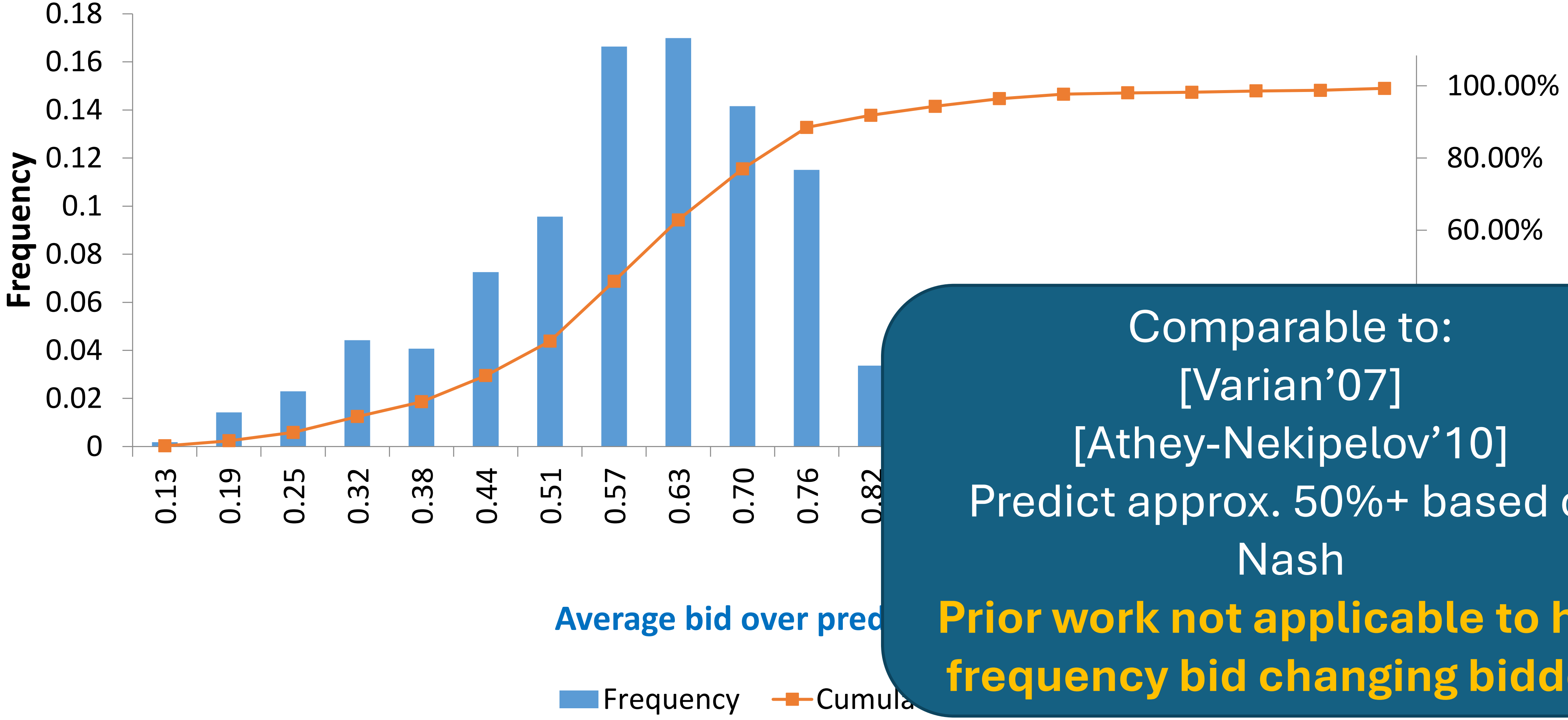
Point prediction method

- Applied inference method to each (keyword,bid) pair of each advertiser





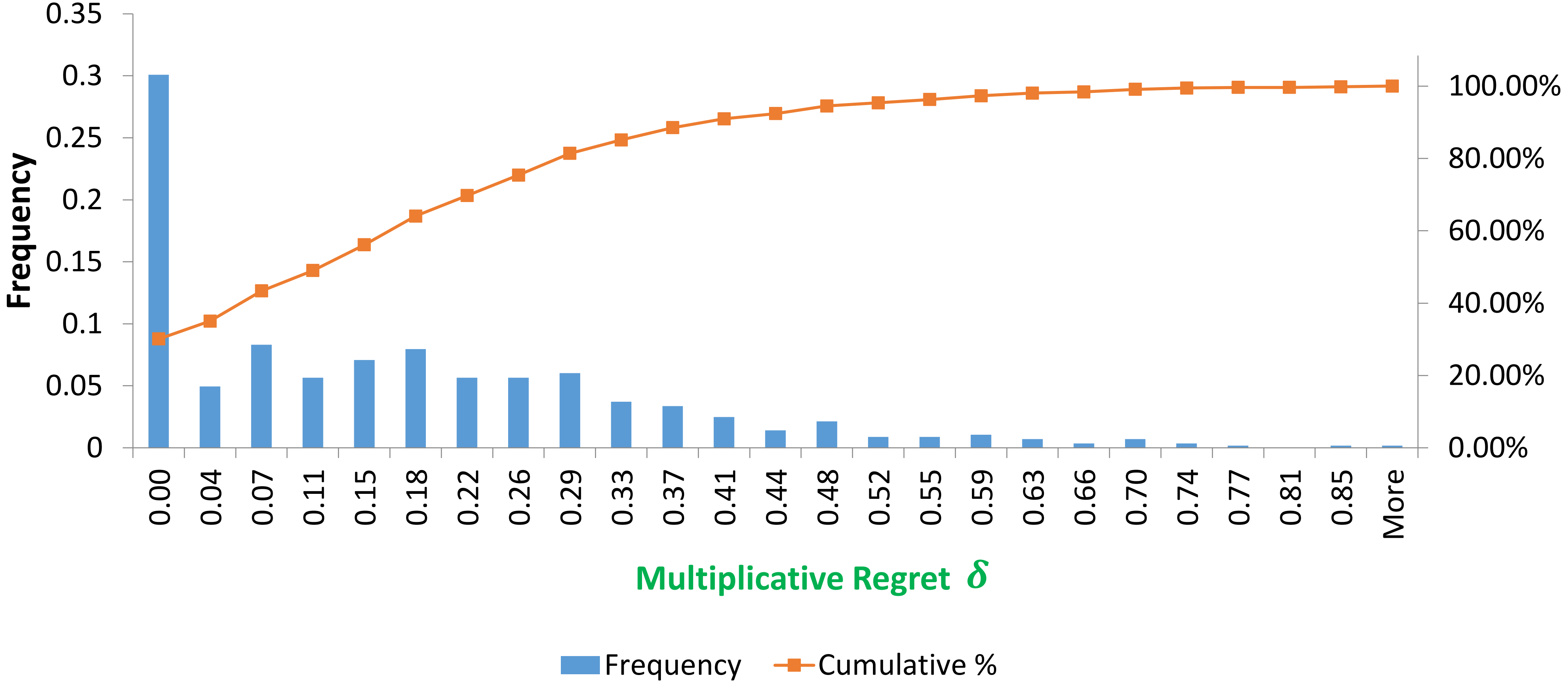
Distribution of bid shading: Average bid / Predicted Value



Comparable to:
[Varian'07]
[Athey-Nekipelov'10]
Predict approx. 50%+ based on Nash

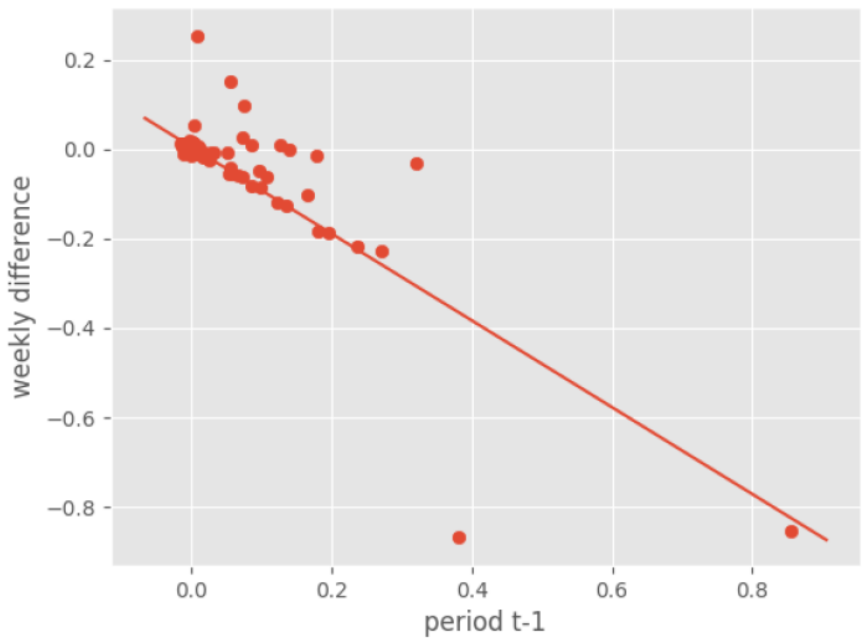
Prior work not applicable to high frequency bid changing bidders

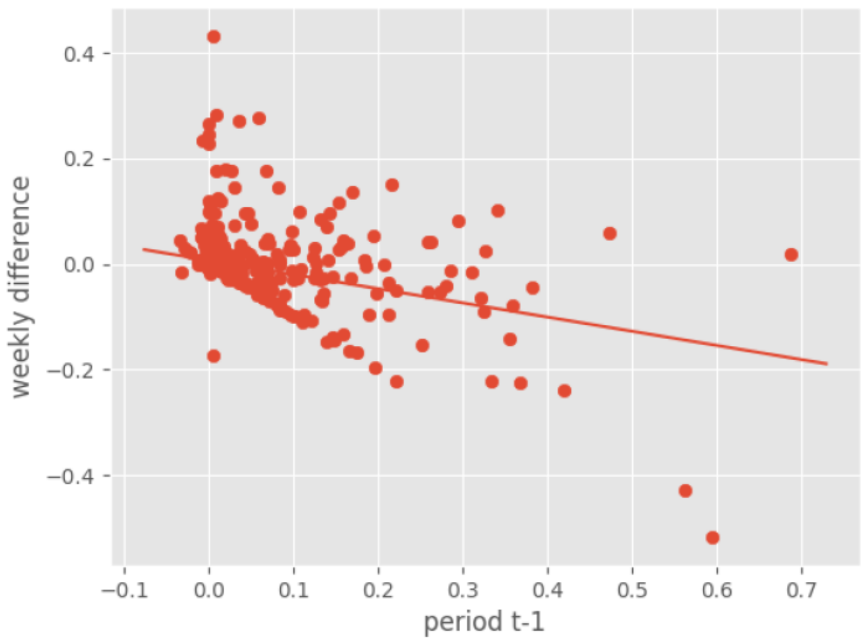
Distribution of smallest rationalizable multiplicative regret



Distribution of smallest rationalizable multiplicative regret

Regret over time





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Appendix

Primer on Econometric Theory

Econometric Theory

- Given a sequence of i.i.d. data points $Z_1, \dots, Z_n$
- Each Z_i is the outcome of some structural model

$$Z_i \sim D(\theta_0), \text{with } \theta_0 \in \Theta$$

- Parameter space Θ can be:
  - Finite dimensional (e.g. R^d): parametric model
  - Infinite dimensional (e.g. function): non-parametric model
  - Mixture of finite and infinite:
    - If we are interested only in parametric part: Semi-parametric
    - If we are interested in both: Semi-nonparametric

Main Goals

- Identification: If we knew "population distribution" $D(\theta_0)$ then can we pin-point θ_0?
- Estimation: Devise an algorithm that outputs an estimate $\hat{\theta}_n$ of θ_0 when having n samples

Estimator Properties of Interest

- Finite Sample Properties of Estimators:
  - Bias $= E\left[\hat{\theta}_n\right] - \theta_0 = 0?$
  - Variance: $\mathrm{Var}(\hat{\theta}_n)$?
  - Mean-Squared-Error (MSE): $E\left[\left(\hat{\theta}_n - \theta_0\right)^2\right] = \mathrm{Variance} + \mathrm{Bias}^2$
- Large Sample Properties: $n \to \infty$
  - Consistency: $\hat{\theta}_n \to \theta_0$?
  - Asymptotic Normality: $a_n\left(\hat{\theta}_n - \theta_0\right) \to N(0, V)$?
  - $\sqrt{n}$-consistency: $a_n = \sqrt{n}$?
  - Efficiency: is limit variance V information theoretically optimal? (typically achieved by MLE estimator)

General Classes of Estimators

- Generalized Method of Moments (GMM): suppose in population $E[m(z,\theta)] = 0$. Then $\hat{\theta}_n$ is solution to:

$$\frac{1}{n}\sum_i m\left(z_i, \hat{\theta}_n\right) = 0$$

- Example. Linear regression: $y = z \cdot \theta + \epsilon$. Then: $E[z(y - z \cdot \theta)] = 0$
- Empirical analogue:

$$\left(\frac{1}{n}\sum_i z_i \cdot z_i^T\right)\hat{\theta}_n = \frac{1}{n}\sum_i z_i \cdot y_i \Leftrightarrow \hat{\theta}_n = (Z \cdot Z^T)^{-1} Z \cdot y$$

Where $Z = [z_1 \ \ldots z_n]$ (matrix with columns z_i vectors, i.e. (OLS estimate)

General Classes of Estimators

- Extremum Estimator: Suppose we know that $\theta_0 = \operatorname{argmin}_{\theta \in \Theta} Q_0(\theta)$

$$\hat{\theta}_n = \operatorname{argmin}_{\theta \in \Theta} Q_n(\theta)$$

- Examples
  - MLE: $Q_n(\theta) = -\frac{1}{n}\sum_{i=1}^{n} \ln f(z_i; \theta)$
  - Overidentified GMM Estimator: suppose in population $E[m(z,\theta)] = 0$. Then: $\theta_0 = \operatorname{argmin}_\theta \|E[m(z,\theta)]\|_W = E[m(z,\theta)]^T W\, E[m(z,\theta)]$, for some W positive definite

$$Q_n(\theta) = \left[\frac{1}{n}\sum_i m(z_i,\theta)\right]^T W \left[\frac{1}{n}\sum_i m(z_i,\theta)\right]$$

Consistency of Extremum Estimators

Consistency Theorem. If there is a function $Q_0(\theta)$ s.t.:

1. $Q_0(\theta)$ is uniquely maximized at θ_0
2. $Q_0(\theta)$ is continuous
3. $Q_n(\theta)$ converges uniformly in probability to $Q_0(\theta)$, i.e. $\sup_\theta |Q_n(\theta) - Q_0(\theta)| \to_p 0$

Then $\hat{\theta} \to_p \theta_0$

- If $Q_n(\theta) = \frac{1}{n}\sum_i g(z_i, \theta)$ and $Q_0(\theta) = E[g(z, \theta)]$, then (2.,3.) will be satisfied if
  - $g(z, \theta)$ is continuous
  - $g(z, \theta) \leq d(z)$ with $E[d(z)] \leq \infty$
- Typically referred to as “regularity conditions”

Asymptotic Normality

- Under "regularity conditions" asymptotic normality of extremum estimators follows by ULLN, CLT, Slutzky thm and consistency
- Roughly: consider case $Q_n(\theta) = \frac{1}{n}\sum_i g(z_i, \theta)$
  - Take first order condition
$$\frac{1}{n}\sum_i \nabla_\theta g(z_i, \hat{\theta}) = 0$$
  - Linearize around θ_0 by mean value theorem
$$\frac{1}{n}\sum_i \nabla_\theta g(z_i, \theta_0) + \left[\frac{1}{n}\sum_i \nabla_{\theta\theta} g(z_i, \bar{\theta})\right](\hat{\theta} - \theta_0) = 0$$
  - Re-arrange:
$$\sqrt{n}(\hat{\theta} - \theta_0) = \underbrace{\left[\frac{1}{n}\sum_i \nabla_{\theta\theta} g(z_i, \bar{\theta})\right]^{-1}}_{\to_p E[\nabla_{\theta\theta} g(z, \theta_0)]} \cdot \underbrace{\frac{1}{\sqrt{n}}\sum_i \nabla_\theta g(z_i, \theta_0)}_{\to_d N\left(0, Var\left(\nabla_\theta g(z, \theta_0)\right)\right)} \quad \to_d N(0, U)$$

In practice, typically variance is computed via Bootstrap [Efron'79]:
Re-sample from your samples with replacement and compute empirical variance

Modern Econometric Theory for Entry Games

Simple case: finite discrete states

- If there are d states, then σ_i are d-dimensional parameter vectors
- Easy $\sqrt{n}$-consistent first-stage estimators $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2)$ of $\sigma = (\sigma_1, \sigma_2)$, i.e.:
$$\sqrt{n}(\hat{\sigma}_i - \sigma) \to N(0, V)$$
- Suppose for second stage we do generalized method of moment estimator:
  - Let $\hat{\theta} = \left(\hat{\beta}_1, \hat{\beta}_2, \hat{\delta}_1, \hat{\delta}_2\right)$ and $\theta_0 = (\beta_1, \beta_2, \delta_2, \delta_2)$
  - Let $y_t = (y_{1t}, y_{2t})$ and $\Gamma(x, \sigma, \theta) = \left(\Gamma_1(x, \sigma, \theta), \Gamma_2(x, \sigma, \theta)\right)$ with $\Gamma_i(x, \sigma, \theta) = \frac{e^{x\cdot\beta_i + \sigma_{-i}\delta}}{1+e^{x\cdot\beta_i + \sigma_{-i}\delta}}$
  - Then second stage estimator $\hat{\theta}$ is the solution to:
$$\frac{1}{n}\sum_{t=1}^{n} A(x_t)\cdot\left(y_t - \Gamma\left(x_t, \hat{\sigma}, \hat{\theta}\right)\right) = 0$$
- Does first stage error affect second stage variance and how?
- This is a general question about two stage estimators

How to approach: easy case

- Standard linearization for asymptotic normality: linearize moment equation around θ_0, leading to

$$\frac{1}{n}\sum_t m\left(Z_t, \theta_0, \hat{h}(X_t)\right) - \frac{1}{n}\sum_t \nabla_\theta m\left(Z_t, \bar{\theta}, \hat{h}(X_t)\right)(\theta - \theta_0) = 0$$

  For some point $\bar{\theta}$ in the line between θ and θ_0 (by MVT).

- Now re-arrange:

$$\sqrt{n}(\theta - \theta_0) = \left[\frac{1}{n}\sum_t \nabla_\theta m\left(Z_t, \bar{\theta}, \hat{h}(X_t)\right)\right]^{-1} \frac{1}{\sqrt{n}}\sum_t m\left(Z_t, \theta_0, \hat{h}(X_t)\right)$$

Converges to $E\left[\nabla_\theta m\left(Z, \theta_0, h_0(X)\right)\right]$ assuming that $\hat{\theta}, \hat{h}$ are consistent by Uniform Law of Large Numbers

Suffices to show that this term is asymptotically normal

How to approach: easy case

- Suppose that nuisance parameter was finite dimensional, i.e. $m(Z_t, \theta_0, h_0) = 0$ and $h_0 \in R^k$
- Then we need to argue that: $\frac{1}{\sqrt{n}} \sum_t m(Z_t, \theta_0, \hat{h}) \to N(0, \Sigma)$
- Linearize the term around the nuisance parameter:

$$\frac{1}{\sqrt{n}} \sum_t m(Z_t, \theta_0, \hat{h}) = \underbrace{\frac{1}{\sqrt{n}} \sum_t m(Z_t, \theta_0, h_0)}_{\substack{\text{This term is a sum of i.i.d.} \\ \text{quantities divided by} \\ \text{sqrt(n). So by CLT it is} \\ \text{asymptotically normal}}} + \underbrace{\frac{1}{\sqrt{n}} \sum_t \nabla_h m(Z_t, \theta_0, \bar{h}) \left(\hat{h} - h_0\right)}_{\text{This can be re-written as:}}$$

This can be re-written as:

$$\underbrace{\left(\frac{1}{n} \sum_t \nabla_h m(Z_t, \theta_0, \bar{h})\right)}_{\substack{\text{Converges to a} \\ \text{constant by ULLN}}} \cdot \underbrace{\left(\sqrt{n}\,(\hat{h} - h_0)\right)}_{\substack{\text{Converges to} \\ \text{independent } \mathrm{N}(0, \mathrm{V})!}}$$

Hard Case: Continuous State Space $x \in R^d$

[Bajari-Hong-Kranier-Nekipelov'12]

- Then there is no $\sqrt{n}$-consistent first stage non-parametric estimator $\hat{\sigma}(\cdot)$ for function $\sigma(\cdot) = E[y|x]$
- Remarkably: still $\sqrt{n}$-consistency for second stage estimate $\hat{\theta}$!!
- Intuition:
    - We can add a bias correction term to our moment, that will make the impact of the first stage error on the second stage estimate be of "second-order"
    - This property is called (Neyman) "orthogonality of the moment"

How to approach: hard case

- Standard linearization for asymptotic normality: linearize moment equation around θ_0, leading to

$$\frac{1}{n}\sum_t m\left(Z_t, \theta_0, \hat{h}(X_t)\right) - \frac{1}{n}\sum_t \nabla_\theta m\left(Z_t, \bar{\theta}, \hat{h}(X_t)\right)(\theta - \theta_0) = 0$$

For some point $\bar{\theta}$ in the line between θ and θ_0 (by MVT).

- Now re-arrange:

$$\sqrt{n}(\theta - \theta_0) = \left[\frac{1}{n}\sum_t \nabla_\theta m\left(Z_t, \bar{\theta}, \hat{h}(X_t)\right)\right]^{-1} \frac{1}{\sqrt{n}}\sum_t m\left(Z_t, \theta_0, \hat{h}(X_t)\right)$$

Converges to $E\left[\nabla_\theta m\left(Z, \theta_0, h_0(X)\right)\right]$ assuming that $\hat{\theta}$, $\hat{h}$ are consistent by Uniform Law of Large Numbers

Suffices to show that this term is asymptotically normal

How to approach: hard case

- Suppose that nuisance parameter was finite dimensional, i.e. $m(Z_t, \theta_0, h_0) = 0$ and $h_0 \in R^k$
- Then we need to argue that: $\frac{1}{\sqrt{n}} \sum_t m\left(Z_t, \theta_0, \hat{h}\right) \to N(0, \Sigma)$
- Linearize the term around the nuisance parameter:

$$\frac{1}{\sqrt{n}} \sum_t m\left(Z_t, \theta_0, \hat{h}\right) = \underbrace{\frac{1}{\sqrt{n}} \sum_t m(Z_t, \theta_0, h_0)} + \underbrace{\frac{1}{\sqrt{n}} \sum_t \nabla_h m\left(Z_t, \theta_0, \bar{h}\right)\left(\hat{h} - h_0\right)}$$

This term is a sum of i.i.d. quantities divided by sqrt(n). So by CLT it is asymptotically normal

This can be re-written as:

$$\underbrace{\left(\frac{1}{n} \sum_t \nabla_h m\left(Z_t, \theta_0, \bar{h}\right)\right)}_{\text{Converges to a constant by ULLN}} \cdot \underbrace{\left(\sqrt{n}\left(\hat{h} - h_0\right)\right)}_{\text{Converges to independent } \mathrm{N}(0, \mathrm{V}) \text{ if } \hat{h} \text{ was } \sqrt{n}\text{-consistent!}}$$

How do we make this term vanish when $\hat{h}$ is not $\sqrt{n}$-consistent?

The hard case

- Let's take a second order Taylor expansion of the crucial quantity around h_0

We are still left with this first order term

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_t m\left(Z_t,\theta_0,\hat{h}(X_t)\right)\\
&= \frac{1}{\sqrt{n}}\sum_t m(Z_t,\theta_0,h_0(X_t)) + \frac{1}{\sqrt{n}}\sum_t \nabla_h m(Z_t,\theta_0,h_0(X_t)\,)\left(\hat{h}(X_t)-h_0(X_t)\right)\\
&+ \frac{1}{2\sqrt{n}}\sum_t \nabla_{hh}\, m\left(Z_t,\theta_0,\bar{h}(X_t)\right)\left(\hat{h}(X_t)-h_0(X_t)\right)^2
\end{aligned}
$$

Assuming Hessian of moment is uniformly bounded and assuming that:

$$\frac{1}{\sqrt{n}}\sum_t \left(\hat{h}(X_t)-h_0(X_t)\right)^2 \to_{\mathrm{P}} 0$$

Then this term vanishes. Essentially this is a condition that

$$\sqrt{n}\cdot MSE\left(\hat{h}\right)\to 0$$

Or that square root mean squared error converges at a rate faster than $n^{-\frac{1}{4}}$

Dealing with the first order term

Question. Under which conditions does the first order term vanish?

$$A = \frac{1}{\sqrt{n}} \sum_t \nabla_h m(Z_t, \theta_0, h_0(X_t)\,)\left(\hat{h}(X_t) - h_0(X_t)\right)$$

- Notational convenience: since $\nabla_h m\big(Z_t, \theta_0, h_0(X_t)\big)$ contains true parameters, denote it with $\nabla_h m_0(Z_t)$
- Reminder on notation. X_t is a subset of Z_t

Observation. If both the variance and the mean of this quantity go to zero then it is $o_P(1)$

- Let's look at the mean conditional on auxiliary dataset:

$$E_Z[A] = \frac{1}{\sqrt{n}} \sum_t E_{Z_t}\left[\nabla_h m_0(Z_t) \cdot \left(\hat{h}(X_t) - h_0(X_t)\right)\right] = \sqrt{n}\, E_Z\left[\nabla_h m_0(Z) \cdot \left(\hat{h}(X) - h(X)\right)\right]$$

Orthogonality condition. For any estimator $\hat{h}$ that could arise from the first stage, my moments satisfy:

$$E_Z\left[\nabla_h m_0(Z) \cdot \left(\hat{h}(X) - h(X)\right)\right]$$

Dealing with the first order term

Question. Under which conditions does the first order term vanish?

$$A = \frac{1}{\sqrt{n}} \sum_t \nabla_h m\big(Z_t, \theta_0, h_0(X_t)\big) \left(\hat{h}(X_t) - h_0(X_t)\right)$$

Orthogonality condition. For any estimator $\hat{h}$ that could arise from the first stage, my moments satisfy:

$$E_Z\left[\nabla_h m_0(Z) \cdot \left(\hat{h}(X) - h(X)\right)\right]$$

Main Lemma. If the moments satisfy the orthogonality condition then $A \to_P 0$

Proof. We will show that both mean and variance of A go to 0 conditional on auxiliary dataset

- Bias is easy:

$$E_Z[A] = \sqrt{n}\, E_Z\left[\nabla_h m_0(Z) \cdot \left(\hat{h}(X) - h(X)\right)\right] = 0$$

- Variance slightly more involved. Main intuition: cross terms are zero

$$Var_Z[A] = \mathrm{E}[\mathrm{A}^2] = \frac{1}{\mathrm{n}} \sum_{t,t'} \mathrm{E}\left[\nabla_h m_0(Z_t)\left(\hat{h}(X_t) - h_0(X_t)\right) \cdot \nabla_h m_0(Z_{t'})\left(\hat{h}(X_{t'}) - h_0(X_{t'})\right)\right]$$

$$= \frac{1}{n} \sum_{t \neq t'} \mathrm{E}\left[\nabla_h m_0(Z_t)\left(\hat{h}(X_t) - h_0(X_t)\right)\right] \cdot E\left[\nabla_h m_0(Z_{t'})\left(\hat{h}(X_{t'}) - h_0(X_{t'})\right)\right] + \underbrace{\frac{1}{n} \sum_t E\left[\left(\nabla_h m_0(Z_{t'})\left(\hat{h}(X_{t'}) - h_0(X_{t'})\right)\right)^2\right]}_{= E_Z\left[\left(\nabla_h m_0(Z)\right)^2 \left(\hat{h}(X) - h_0(X)\right)^2\right] \to 0}$$

Both are zero, by orthogonality

by consistency of $\hat{h}$

When is orthogonality satisfied?

Conditional moment models. Suppose we have conditional moment equations of the form:

$$E\left[m\left(Z,\theta_0,g(X)\right)\middle|X\right]=0$$

Importantly. Conditional on the variables that go into the nuisance function, my expected moment is still zero.

Example. Partially linear model of treatment effects from the first slide.

Conditional orthogonality. Suppose that my moments satisfy:

$$E_Z\left[\nabla_h m\left(Z,\theta_0,h_0(X)\right)\mid X\right]\ =0$$

Lemma. Conditional orthogonality implies orthogonality

- By law of iterated expectations

$$E_Z\left[\nabla_h m_0(Z)\cdot\left(\hat{h}(X)-h(X)\right)\right]=E_Z\left[E\left[\nabla_h m_0(Z)\cdot\left(\hat{h}(X)-h(X)\right)\middle|X\right]\right]=E_Z\left[\cancelto{0}{E[\nabla_h m_0(Z)|X]}\cdot\left(\hat{h}(X)-h(X)\right)\right]$$

Orthogonal Moment for Games

Conditional moment models. Suppose we have conditional moment equations of the form:

$$E\left[m\left(Z,\theta_0,g(X)\right)\middle|X\right]=0$$

Importantly. Conditional on the variables that go into the nuisance function, my expected moment is still zero.

Example. Partially linear model of treatment effects from the first slide.

Conditional orthogonality. Suppose that my moments satisfy:

$$E_Z\left[\nabla_h m\left(Z,\theta_0,h_0(X)\right)\,\middle|\,X\right]\;=0$$

Lemma. Conditional orthogonality implies orthogonality

- By law of iterated expectations

$$E_Z\left[\nabla_h m_0(Z)\cdot\left(\hat{h}(X)-h(X)\right)\right]=E_Z\left[E\left[\nabla_h m_0(Z)\cdot\left(\hat{h}(X)-h(X)\right)\middle|X\right]\right]=E_Z\left[\underbrace{E[\nabla_h m_0(Z)|X]}_{0}\cdot\left(\hat{h}(X)-h(X)\right)\right]$$

Creating orthogonal moments more generally

Moment formulation for Games.

$$E\left[\Gamma\left(\theta \cdot X + \delta\sigma(X)\right) - y \middle| X\right] = 0$$

$$E[y_{-i} - \sigma(X)|X] = 0$$

The moment of the logistic regression is the gradient of the logistic loss with respect to the params

$$E[m(y, X, \sigma(X); \theta, \delta)] = E\left[\left(\Gamma\left(\theta \cdot X + \delta\sigma(X)\right) - y\right) \cdot \begin{pmatrix} X \\ \sigma(X) \end{pmatrix}\right] = 0$$

Is not orthogonal:

$$E[\nabla_\sigma m(y, X, \sigma_0(X); \theta_0, \delta_0)|X] = \overbrace{\delta E\left[\Gamma'\left(\theta \cdot X + \delta\sigma(X)\right)\middle|\mathrm{X}\right]}^{\coloneqq h(X)\text{: Non-zero}} \cdot \begin{pmatrix} X \\ \sigma(X) \end{pmatrix}$$

We can orthogonalize by subtracting a mean zero quantity that removes the first order effect

$$E\left[\left(\Gamma\left(\theta \cdot X + \delta\sigma(X)\right) - y\right) \cdot \begin{pmatrix} X \\ \sigma(X) \end{pmatrix} + h(X)\left(y_{-i} - \sigma(X)\right) \cdot \begin{pmatrix} X \\ \sigma(X) \end{pmatrix}\right] = 0$$