MS&E 233
Game Theory, Data Science and AI
Lecture 12

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(by courtesy) Computer Science and Electrical Engineering

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Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- Basic Auctions and Learning to bid in auctions (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- **Simple vs optimal mechanisms (T)**
- *HW6: implement simple and optimal auctions, analyze revenue empirically*

6
- Optimizing mechanisms from samples (T)
- Online optimization of auctions and mechanisms (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Summarizing Last Lecture

What if we want to maximize revenue?

How do we optimize over all possible mechanisms!

Single-Parameter Settings

- Each bidder has some value v_i for being allocated
- Bidders submit a reported value b_i (without loss of generality)
- Mechanism decides on an allocation $x \in X \subseteq \{0,1\}^n$
- Mechanism fixes a probabilistic allocation rule:

$$x(b) \in \Delta(X)$$

- **First question.** Given an allocation rule, when can we find a payment rule p so that the overall mechanism is truthful?
- If we can find such a payment, we will say that x is implementable

Some Shorthand Notation

- Let's fix bidder i and what other bidders bid b_{-i}
- For simplicity of notation, we drop index i and b_{-i}

- What properties does the function

$$x(v) \equiv x_i(v, b_{-i})$$

need to satisfy, so that x is implementable?

- Can we find a truthful payment function

$$p(v) \equiv p(v, b_{-i})$$

Any implementable allocation rule must be monotone!

“If not allocated with value v, I should not be allocated if I report a lower value!”

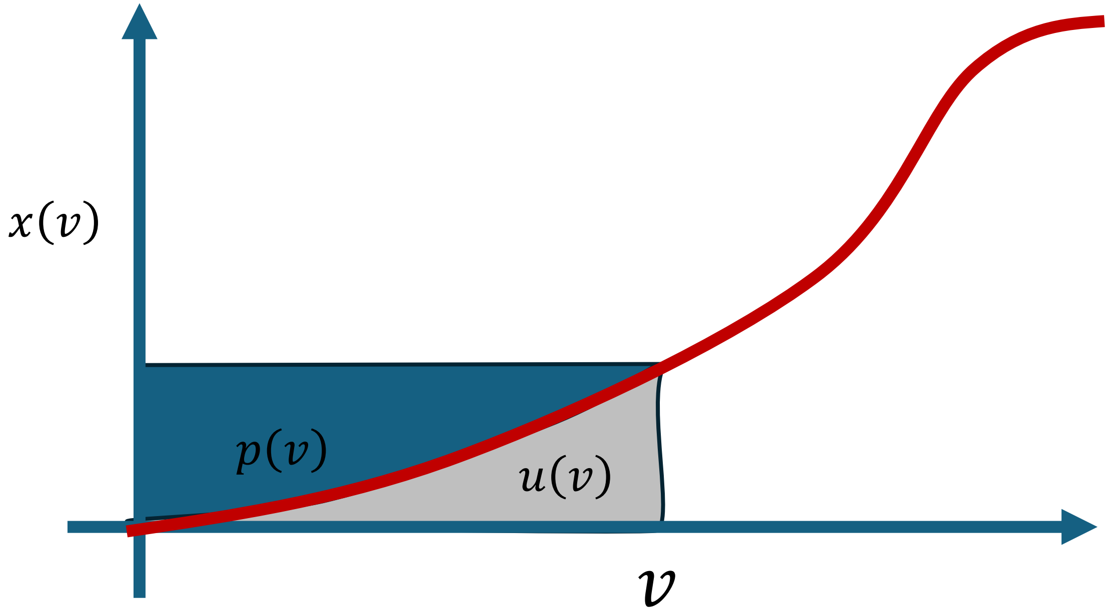
Under any truthful payment rule

$$u(v) = u(0) + \int_0^v x(z)\, dz$$

Under any truthful payment rule that satisfies NNT and IR

$$p(v) = v \cdot x(v) - \int_0^v x(z)\, dz$$

For any dominant-strategy truthful, NNT and IR mechanism, given an allocation rule, utility and payment are uniquely determined!



Myerson's Theorem. When valuations are independently distributed, for any dominant-strategy truthful, NNT and IR mechanism, the payment contribution of each player is their expected virtual value

$$E[p_i(v)] = E[x_i(v) \cdot \phi_i(v_i)], \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Myerson's Optimal Auction. Assuming that virtual value functions are monotone non-decreasing, the optimal mechanism is the mechanism that maximizes virtual welfare

$$x(v) = \mathrm{argmax}_{x \in X} \sum_i x \cdot \phi_i(v_i), \qquad p_i(v) = v_i x_i(v) - \int_0^{v_i} x_i(z, v_{-i})\, dz$$

$$\mathrm{Rev} = E\left[\max_{x \in X} \sum_i x \cdot \phi_i(v_i)\right]$$

Can non-truthful mechanisms generate higher revenue at some Bayes-Nash equilibrium?

Non-Truthful Mechanism

- Consider any potentially non-truthful mechanism M
- Bidder i uses some strategy $s_i(v_i)$ to participate in the game
- This could even be a complicated action plan not just a number
- Strategies are Bayes-Nash equilibrium if for any other strategy s_i'

$$\boxed{E[u_i(s(v); v_i) \mid v_i]} \geq \boxed{E\left[u_i\left(s_i'(v_i), s_{-i}(v_{-i}); v_i\right) \mid v_i\right]}$$

Expected utility given my value, in expectation over other values

Expected utility given my value, **if I deviate**, in expectation over other values

- Mechanism implies expected allocation and payment for bidder i

$$\hat{x}_i(v_i) = \boxed{E_{v_{-i}}\left[x_i\left(s(v)\right)\right],} \qquad \hat{p}_i(v_i) = \boxed{E_{v_{-i}}\left[p_i\left(s(v)\right)\right]}$$

Expected allocation probability, given my value, in expectation over other values

Expected payment, given my value, in expectation over other values

Is there any mechanism M with some equilibrium s such that

$$\text{Rev} := \sum_i E[\hat{p}_i(v_i)] \geq \text{Myerson?}$$

Revelation Principle

- Consider the following "wrapper" mechanism $\widetilde{M}$
- The mechanism asks from bidders to each report their value
- Given value profile v, mechanism $\widetilde{M}$ simulates mechanism M, with strategies $s(v)$. Do bidders have incentive to not bid truthfully?
- Consider the deviation $s_i'(v_i) = s_i(v_i')$. By equilibrium properties

$$\boxed{E[u_i(s(v); v_i) \mid v_i]} \geq \boxed{E\left[u_i\left(s_i(v_i'), s_{-i}(v_{-i}); v_i\right) \mid v_i\right]}$$

$$E[\tilde{u}_i(v; v_i) \mid v_i] \qquad\qquad E\left[\tilde{u}_i\left(v_i', v_{-i}; v_i\right) \mid v_i\right]$$

- Mechanism $\widetilde{M}$ implies same expected allocation and payment

$$\hat{x}_i(v_i) = E_{v_{-i}}\left[\tilde{x}_i\left(s(v)\right)\right] = E_{v_{-i}}\left[x_i\left(s(v)\right)\right]$$

$$\hat{p}_i(v_i) = E_{v_{-i}}[\tilde{p}(v)] = E_{v_{-i}}\left[p_i\left(s(v)\right)\right]$$

Bayesian-Incentive Compatible Mechanism

- A **direct mechanism** elicits private values and comprises of an allocation function x and a payment function p
- **BIC.** bidders have no incentive to deviate from truthful reporting

$$E[u_i(v; v_i) \mid v_i] \geq E\left[u_i\left(v_i', v_{-i}; v_i\right) \mid v_i\right]$$

$$E[v_i x_i(v) - p(v) \mid v_i] \geq E\left[v_i x_i\left(v_i', v_{-i}\right) - p_i\left(v_i', v_{-i}\right) \mid v_i\right]$$

- Implies "interim" expected utility, allocation and payment for bidder i

$$\hat{u}_i(v_i) = E_{v_{-i}}[u_i(v)], \qquad \hat{x}_i(v_i) = E_{v_{-i}}[x_i(v)], \qquad \hat{p}_i(v_i) = E_{v_{-i}}[p_i(v)]$$

The interim allocation and payment function that is implied by an equilibrium of a non-truthful auction can always be implemented by a direct BIC mechanism

Properties of BIC Mechanisms

- Equilibrium constraints are
$$\forall v_i, v_i' \colon\ v_i \cdot \hat{x}_i(v_i) - \hat{p}_i(v_i) \geq v_i \cdot \hat{x}_i(v_i') - \hat{p}_i(v_i')$$
$$E[v_i x_i(v) - p(v) \mid v_i] \geq E\left[v_i x_i(v_i', v_{-i}) - p_i(v_i', v_{-i}) \mid v_i\right]$$
- Exact same constraints we used in the properties of dominant strategy truthful mechanisms
- Only thing that changes: now use the *interim* allocation and payment functions and not the *ex-post* functions, for each opponent bid/value profile
- We can prove the same properties!

For any BIC mechanism (and any BNE of a non-truthful mechanism) the interim allocation function $\hat{x}_i(v_i)$ is monotone non-decreasing in the player's value

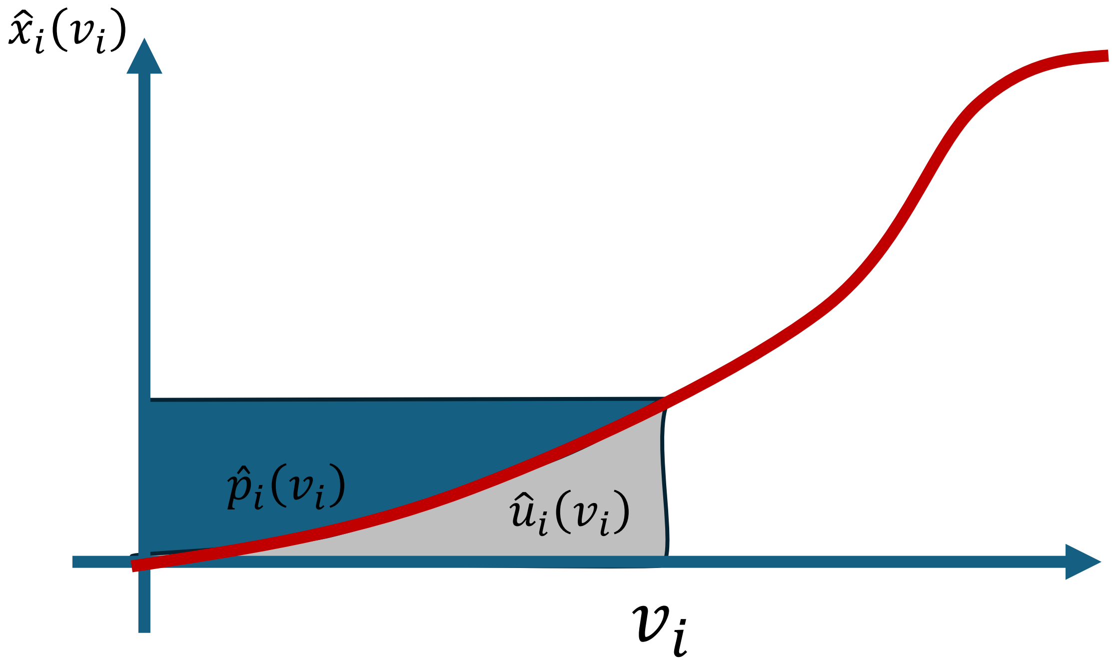
For any BIC mechanism (and any BNE of a non-truthful mechanism)

$$\hat{u}_i(v_i) = \hat{u}_i(0) + \int_0^{v_i} \hat{x}_i(z)\, dz$$

For any BIC mechanism (and any BNE of a non-truthful mechanism) that satisfies NNT and BIR

$$\hat{p}_i(v) = v_i \cdot \hat{x}_i(v_i) - \int_0^{v_i} \hat{x}_i(z)\, dz$$

For any BIC, NNT and BIR mechanism (and any BNE of a non-truthful mechanism), given the interim allocation rule, utility and payment are uniquely determined!



Myerson's Theorem. When valuations are independently distributed, for any BIC, NNT and IR mechanism (and any BNE of a non-truthful mechanism), the payment contribution of each player is their expected virtual value

$$E[\hat{p}_i(v_i)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)], \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Corollary. When valuations are independently distributed, for any Bayes-Nash equilibrium of any non-truthful mechanism, the payment contribution of each player is their expected virtual value

$$E[\hat{p}_i(v)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)], \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Myerson's Optimal Auction. Assuming that virtual value functions are monotone non-decreasing, the mechanism that maximizes virtual welfare, achieves the largest possible revenue among all possible mechanisms and Bayes-Nash

$$x(v) = \mathrm{argmax}_{x \in X} \sum_i x \cdot \phi_i(v_i), \qquad p_i(v) = v_i x_i(v) - \int_0^{v_i} x_i(z, v_{-i})\, dz$$

$$\mathrm{Rev} = E\left[\max_{x \in X} \sum_i x \cdot \phi_i(v_i)\right]$$

Side-Note: Best among BIC is DSIC

Even though we optimized over the bigger space of Bayes-Nash equilibria and Bayesian Incentive Compatible auctions, the optimal revenue is achievable by a dominant strategy truthful mechanism!

Side-Note: Revenue Equivalence

- For any equilibrium of any mechanism
$$E[\hat{p}_i(v)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)]$$

Corollary. If two mechanisms and two equilibria have the same *interim* allocation function $\hat{x}_i(v_i)$, as a function of the bidder's value, for each bidder, then they generate the same revenue

Example. Consider a Second-Price auction and a First-Price auction, when bidders have the same distribution and use a symmetric strategy. In both auctions the allocation is efficient, highest *value* bidder wins.

$\hat{x}_i(v_i)$ is the same for both auctions $\Rightarrow$ they generate the same revenue

Dissecting Myerson’s Optimal Auction

Identically Distributed Bidders

- Single-item setting, with all bidder values are from same distribution $v_i \sim F$
- Virtual value function is the same for all bidders

$$\phi(v) = v - \frac{1 - F(v)}{f(v)}$$

- Assume that $\phi(v)$ is monotone non-decreasing (F is regular)
- Allocating to highest virtual value $\equiv$ allocating to highest value
- **Optimal auction.** Allocate to highest value, as long as $\phi\left(v_{(1)}\right) \geq 0$
- **Optimal auction.** allocate to highest value, as long as $v_1 \geq r_*$

$$r_*\!: r - \frac{1 - F(r)}{f(r)} = 0, \qquad \text{(monopoly reserve price)}$$

When bidders are independently and identically distributed according to a regular distribution, then the optimal single-item auction among all auctions is a Second-Price Auction with a Monopoly Reserve Price

Monopoly Reserve Price

- What if we had only one bidder (monopoly)
- Then optimal thing to do is post a reserve price r_*
- The revenue from that single bidder if we post a reserve r is

$$E[r\ 1\{v \geq r\}] = r\left(1 - F(r)\right)$$

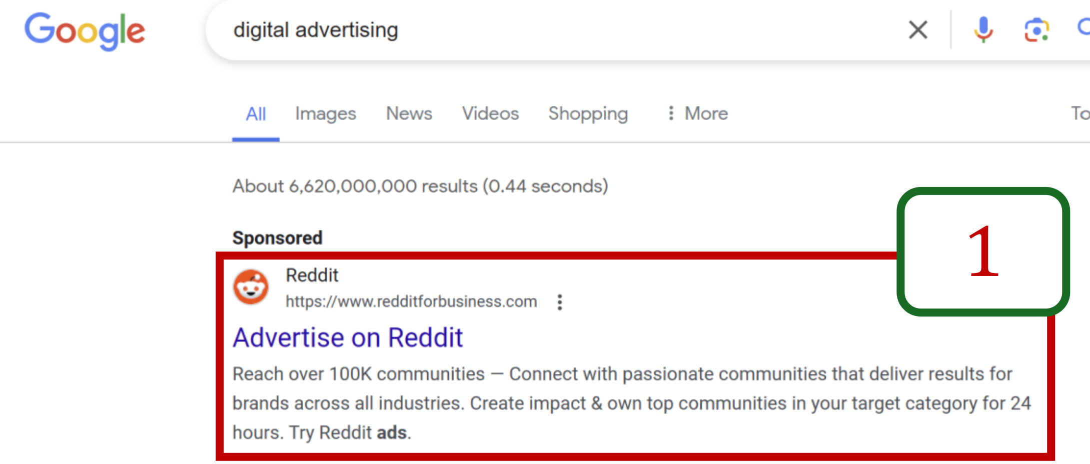
- The optimal reserve price is given by the first order condition

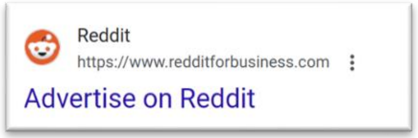
$$r_*\colon \left(1 - F(r)\right) - r\, f(r) = 0 \Rightarrow r - \frac{1 - F(r)}{f(r)} = 0$$

- Same as reserve price that we should be using with many bidders

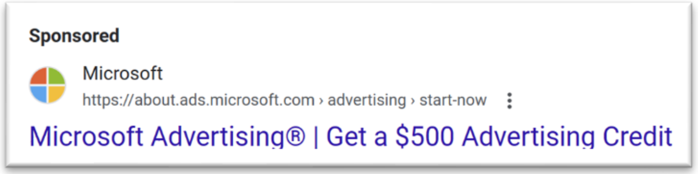
Non-Identically Distributed Bidders

- What if you know ahead of time that one bidder tends to have higher values than the other bidder?
- Shouldn't you treat these bidders differently (price discrimination)?
- Shouldn't you try to extract more revenue from the bidder that tends to have a higher value?





$v_1 \sim U[0,1]$



$v_2 \sim U[0,100]$

You are selling a single item to two bidders. One has values drawn U[0,1] the other U[0,100]. What is the optimal auction?

- Second-price with a reserve price
- Second-price where each bidder has a different reserve price
- First-price where each bidder has a different reserve price
- None of the above

You are selling a single item to two bidders. One has values drawn U[0,1] the other U[0,100]. What is the optimal auction?

You are selling a single item to two bidders. One has values drawn U[0,1] the other U[0,100]. What is the optimal auction?

Non-Identically Distributed Bidders

- Suppose we have two bidders, $v_1 \sim U[0,1], v_2 \sim U[0,100]$

- Virtual value function for each bidder

$$\phi_1(v) = v - \frac{1 - F_1(v)}{f_1(v)} = 2v - 1, \qquad \phi_2(v) = v - \frac{1 - \frac{v}{100}}{\frac{1}{100}} = 2v - 100$$

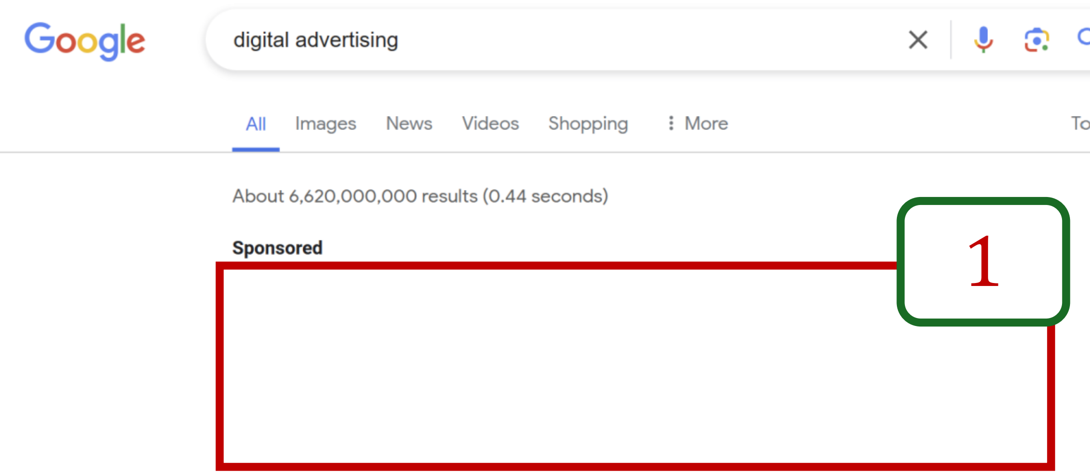
- We should allocate to the bidder with the highest virtual value (if positive)!

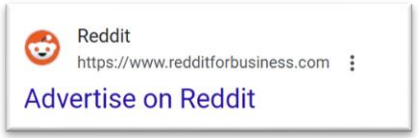
$$\text{argmax}\{0, \phi_1(v_1), \phi_2(v_2)\} = \text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Non-Identically Distributed Bidders

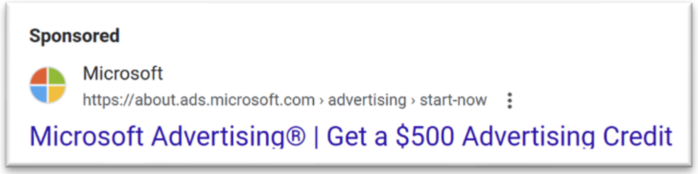
- Allocate to highest virtual value (if positive)!

$$\operatorname{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 = 1$ → $\phi_1 = 1$

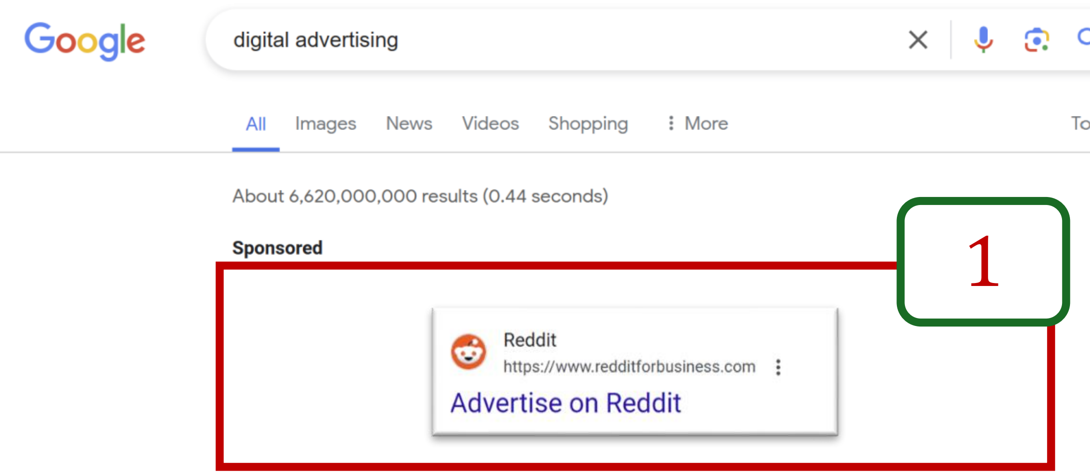


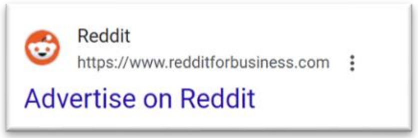
$v_2 \sim U[0,100]$ → $v_2 = 20$ → $\phi_2 = -60$

Non-Identically Distributed Bidders

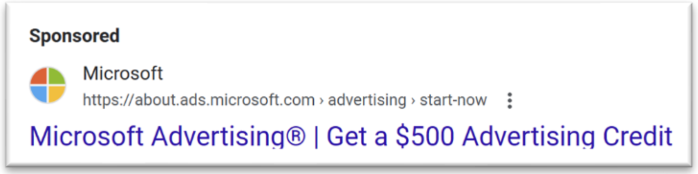
- Allocate to highest virtual value (if positive)!

$$\text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 = 1$ → $\phi_1 = 1$

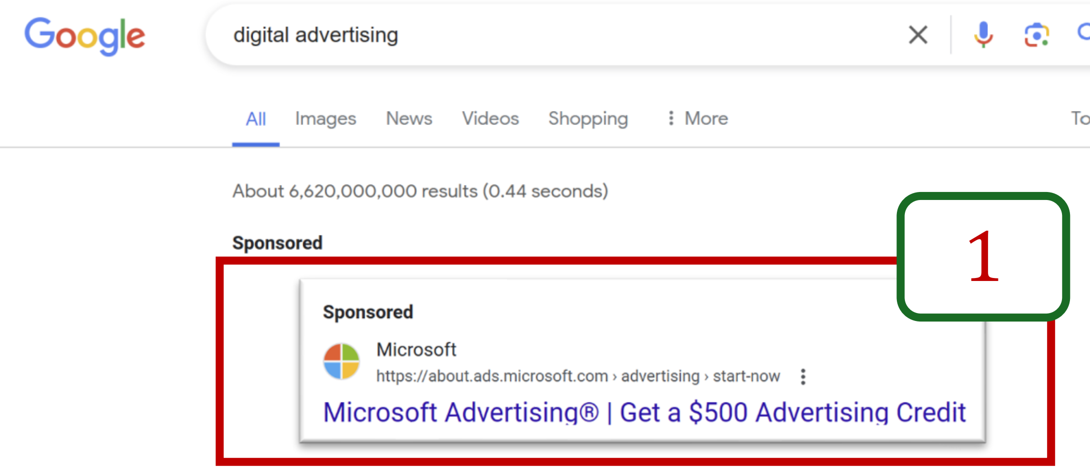


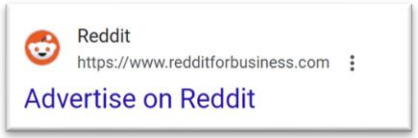
$v_2 \sim U[0,100]$ → $v_2 = 20$ → $\phi_2 = -60$

Non-Identically Distributed Bidders

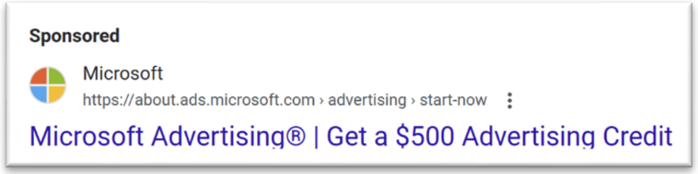
- Allocate to highest virtual value (if positive)!

$$\text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 = 1$ → $\phi_1 = 1$

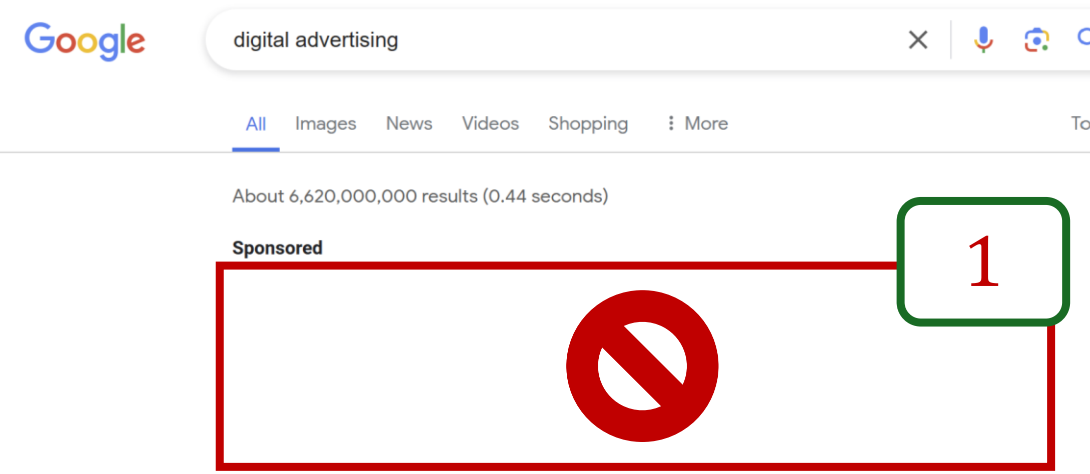


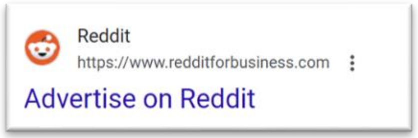
$v_2 \sim U[0,100]$ → $v_2 = 51$ → $\phi_2 = 2$

Non-Identically Distributed Bidders

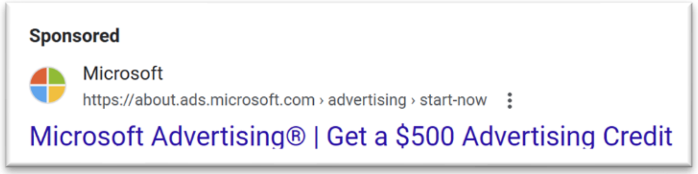
- Allocate to highest virtual value (if positive)!

$$\operatorname{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 = .49$ → $\phi_1 = -.02$



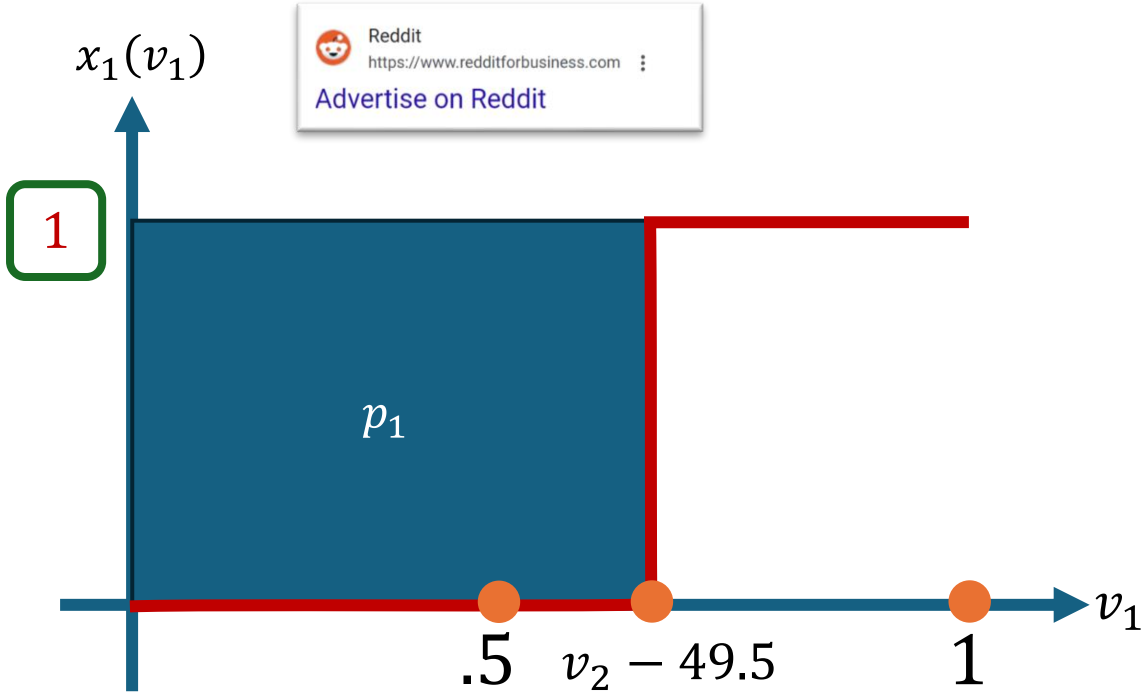
$v_2 \sim U[0,100]$ → $v_2 = 49$ → $\phi_2 = -2$

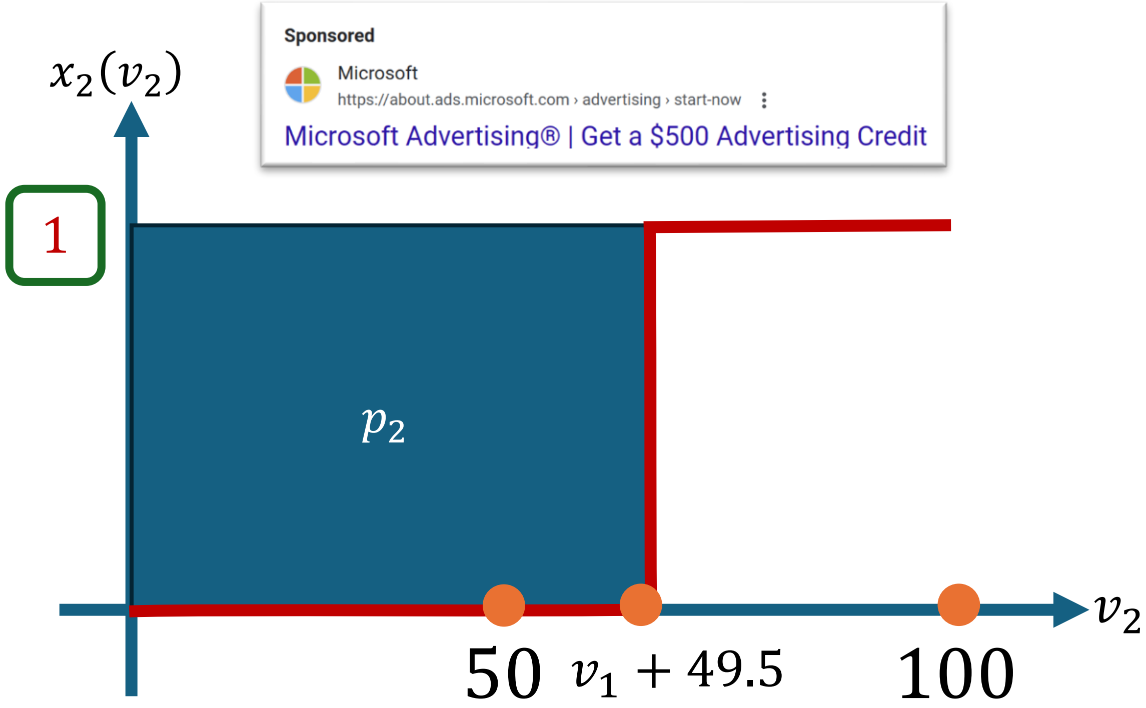
Non-Identically Distributed Bidders

- Allocate to highest virtual value (if positive)!

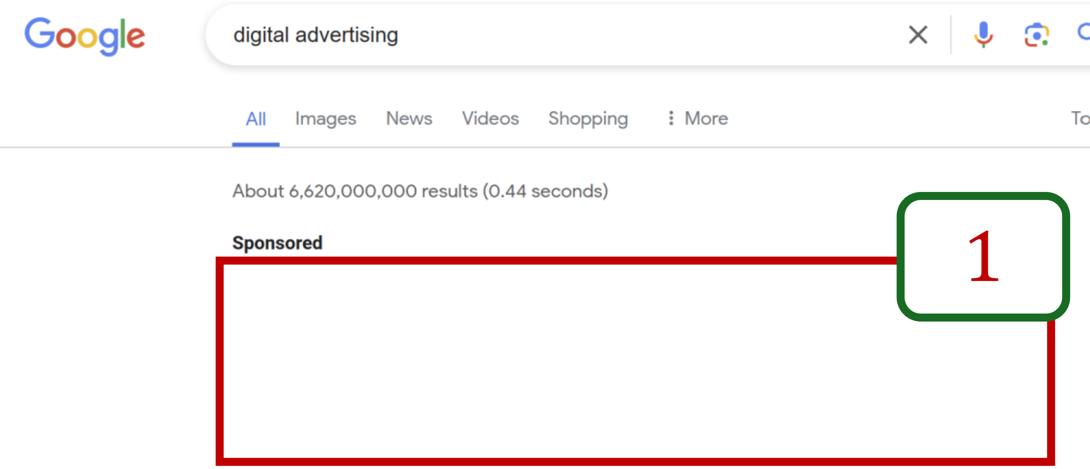
$$\mathrm{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Bidder 1 wins if: $2v_1 - 1 \geq 2v_2 - 100 \Rightarrow v_1 \geq v_2 - \frac{99}{2}$





Non-Identically Distributed Bidders

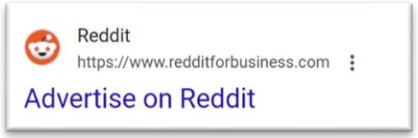


- Allocate to highest virtual value (if positive)!

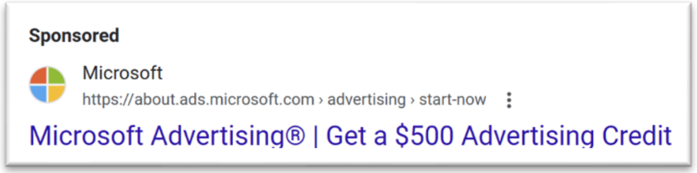
$$\operatorname{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Optimal auction rules

- If $v_1 > .5, v_2 < 50$, allocate to 1, charge .5
- If $v_1 < .5, v_2 > 50$, allocate to 2, charge 50
- If $.5 \leq v_1 < v_2 - 49.5$, allocate to 2, charge $v_1 + 49.5$
- If $50 \leq v_2 < v_1 + 49.5$, allocate to 1, charge $v_2 - 49.5$



$\sim U[0,1]$



$\sim U[0,100]$

At the optimal auction, we are giving a huge advantage to the weaker bidder! We roughly add 49.5$ to their bid!

We expect more from stronger bidders and make it harder for them to win, to incentivize them to pay more.

Optimal auction is
1) cumbersome, 2) hard to understand, 3) hard to explain, 4) does not always allocate to the highest value player, 5) discriminates a lot, 6) is many times counter-intuitive, 7) can seem unfair!

Are there simpler auctions that always achieve almost as good revenue?

Simple vs. Optimal Auctions

Second-Price with Player-Specific Reserves

- What if we simply run a second price auction but have different reserves for each bidder

- Each bidder i has a reserve price r_i
- Among all bidders with value $v_i \geq r_i$, allocate to highest bidder
- Charge winner: maximum of their reserve and the next highest bid

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- Revenue of the optimal auction is the maximum virtual welfare

$$\mathrm{OPT} = E\left[\max_i \phi_i^+(v_i)\right], \qquad \phi_i^+(v_i) = \max\{0, \phi_i(v_i)\}$$

- Assume that reserve prices are at least the monopoly reserves
- Revenue of the second-price with player specific reserves (SP-r)

$$\mathrm{Rev} = E\left[\sum_i x_i(v)\phi_i^+(v_i)\right]$$

- Can we guarantee that the auction collects a $\phi_i^+(v_i)$ that, in expectation, is at least half of the maximum $\phi_i^+(v_i)$?

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- Can we guarantee that the auction collects a $\phi_i^+(v_i)$ that, in expectation, is at least half of the maximum $\phi_i^+(v_i)$?
- Since the auction allocates to some player with $v_i \geq r_i$
- Since ϕ_i^+ are monotone: to some player with $\phi_i^+(v_i) > \theta_i$
- We can think of $\phi_i^+(v_i)$ as non-negative prizes Π_i

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- We can think of $\phi_i^+(v_i)$ as non-negative prizes Π_i
- The optimal auction gets revenue that corresponds to the expected maximum prize $E[\max_i \Pi_i]$
- The SP-r auction gets revenue that corresponds to some price Π_τ that satisfies that it is above some threshold θ_τ
- Is there a threshold rule for collecting prizes that guarantees at least half of the expected maximum prize?

Parenthesis: (Optimal Stopping Problems)

- There are n stages
- In each stage i, we are offered a prize $\Pi_i \sim G_i$
- Distributions G_i are known ahead of time
- Realized prize Π_i only revealed at stage i
- At each stage, we can choose to accept Π_i and end the game or discard the prize and continue opening prizes

Question. Is there a strategy to play the game that guarantees at least half of what an oracle who knows all the prizes ahead of time would achieve?

Parenthesis: (Optimal Stopping Problems)

Question. Is there a strategy to play the game that guarantees at least half of what an "prophet" who knows all the prizes ahead of time would achieve?

Theorem (Prophet Inequality). There exists a threshold strategy APX that accepts the first prize that passes a threshold θ, such that:

$$E[\Pi_\tau] \geq \frac{1}{2} E\left[\max_i \Pi_i\right]$$

τ is the random stopping time induced by the threshold policy.

Parenthesis: (Proof of Prophet Inequality)

- Let's be generous with the optimal benchmark

$$E[\Pi_*] = E\left[\max_i \Pi_i\right] \le E[\theta + [\Pi_* - \theta]_+] \le \theta + \sum_i \boxed{E[[\Pi_i - \theta]_+]}^{A_i}$$

- APX gets θ if there exists some prize above, i.e., $\Pi_* \ge \theta$
- On top of that, we also collect some ***excess*** $[\Pi_\tau - \theta]_+$
- ***Excess*** is A_i, when all rewards other than i is $\le \theta$

$$\text{Excess} \ge \sum_i A_i \Pr\left(\forall j \ne i : \Pi_j < \theta\right) \ge \sum_i A_i \Pr(\Pi_* < \theta)$$

Overall: $\text{APX} \ge \theta \Pr(\Pi_* \ge \theta) + \Pr(\Pi_* < \theta) \sum_i A_i$

Choosing $\Pr(\Pi_* \ge \theta) = 1/2$: $\text{APX} \ge \frac{1}{2}\left(\theta + \sum_i A_i\right) \ge \frac{1}{2} E[\Pi_*]$

Parenthesis: (Optimal Stopping Problems)

Question. Is there a strategy to play the game that guarantees at least half of what an "prophet" who knows all the prizes ahead of time would achieve?

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τ is the random stopping time induced by the threshold policy.

Policy. Simply choose θ such that $\Pr\left(\max_i \Pi_i \geq \theta\right) = 1/2$

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- Choose θ such that:

$$\Pr\left(\max_i \phi_i^+(v_i) \geq \theta\right) = 1/2$$

- Then set personalized reserve prices implied by:

$$\phi_i^+(v_i) \geq \theta \Leftrightarrow v_i \geq r_i$$