

# MS&E 233

# Game Theory, Data Science and AI

## Lecture 6

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(by courtesy) Computer Science and Electrical Engineering

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# Computational Game Theory for Complex Games

- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- 1 • *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

- Basics of extensive-form games
- 2 • **Solving extensive-form games via online learning (T)**
- *HW3: implement agents to solve very simple variants of poker*

- General games and equilibria (T)
- 3 • Online learning in general games, multi-agent RL (T+A)
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

## Data Science for Auctions and Mechanisms

- Basics and applications of auction theory (T+A)
- 4 • **Learning to bid in auctions via online learning (T)**
- *HW5: implement bandit algorithms to bid in ad auctions*

- Optimal auctions and mechanisms (T)
- 5 • **Simple vs optimal mechanisms (T)**
- *HW6: calculate equilibria in simple auctions, implement simple and optimal auctions, analyze revenue empirically*

- Optimizing mechanisms from samples (T)
- 6 • **Online optimization of auctions and mechanisms (T)**
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

## Further Topics

- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- 7 • *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

## Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

# Solving Extensive Form Games via No-Regret Learning

# ***Recap:*** No-Regret Learning in Sequence Form

- We have successfully turned imperfect information extensive form zero-sum games into a familiar object

$$\max_{\tilde{x} \in X} \min_{\tilde{y} \in Y} \tilde{x}^\top A \tilde{y}$$

- $X, Y$  are convex sets, i.e., sequence-form strategies
- We can invoke minimax theorem to prove existence of equilibria
- We can calculate equilibria via LP duality
- We can calculate equilibria via no-regret learning!

# Recap: Regret of FTRL

$$\text{(FTRL)} \quad x_t = \operatorname{argmax}_{x \in X} \underbrace{\sum_{\tau < t} \langle x, u_\tau \rangle}_{\substack{\text{Historical performance} \\ \text{of always choosing} \\ \text{strategy } x}} - \underbrace{\frac{1}{\eta} \mathcal{R}(x)}_{\substack{\text{1-strongly convex} \\ \text{function of } x \text{ that} \\ \text{stabilizes the maximizer}}}$$

**Theorem.** Assuming the utility function at each period  $f_t(x) = \langle x, u_t \rangle$

is  $L$ -Lipschitz with respect to some norm  $\|\cdot\|$  and the regularizer is 1-strongly convex with respect to the same norm then

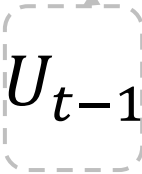
$$\text{Regret} - \text{FTRL}(T) \leq \underbrace{\eta L}_{\substack{\text{Average stability} \\ \text{induced by regularizer}}} + \underbrace{\frac{1}{\eta T} \left( \max_{x \in X} \mathcal{R}(x) - \min_{x \in X} \mathcal{R}(x) \right)}_{\substack{\text{Average loss distortion} \\ \text{caused by regularizer}}}$$

# **Recap:** Regularizer for the Treeplex Space $X$

- The only thing we are missing is a good Regularizer for  $X$

$$U_{t-1} = \sum_{\tau < t} u_{\tau}$$

- **Desiderata.** Be strongly convex in  $x$  within  $X$  and for the optimization problem to be fast to solve

$$\tilde{x}_t = \operatorname{argmax}_{\tilde{x} \in X} \sum_{\tau < t} \langle \tilde{x}, u_{\tau} \rangle - \frac{1}{\eta} \mathcal{R}(\tilde{x}) = \operatorname{argmax}_{\tilde{x} \in X} \langle \tilde{x}, U_{t-1} \rangle - \frac{1}{\eta} \mathcal{R}(\tilde{x})$$


- $X$  is no longer a “simplex”, so entropy is not a good Regularizer

# Dilated Entropy

- $X$  is a combination of *scaled simplices*, i.e.,  $\tilde{x} = (\tilde{x}^j)_{j \in \mathcal{J}_1}$
- $\tilde{x}^j = (\tilde{x}_a)_{a \in A_j}$ : sequence-form strategies for actions in info set  $j \in \mathcal{J}_1$

$$\tilde{x}^j \in \tilde{x}_{p_j} \cdot \Delta_j \quad \Leftrightarrow \quad \tilde{x}^j / \tilde{x}_{p_j} \in \Delta_j$$

- Consider a *weighted combination of local negative entropies*

$$\mathcal{R}(\tilde{x}) := \sum_j \beta_j \tilde{x}_{p_j} \text{H} \left( \tilde{x}^j / \tilde{x}_{p_j} \right), \quad \text{H}(u) = \sum_i u_i \log(u_i)$$

Lies in a simplex  $\Delta_j$ 
Negative Entropy

Equivalent to the behavioral strategy  $x^j$

- $\mathcal{R}(\tilde{x})$  is  $1/M$  strongly convex w.r.t.  $\ell_1$  norm, where  $M = \max_{\tilde{x} \in X} \|\tilde{x}\|_1$ , for appropriate choice of  $\beta_j$  based on game tree structure

# Solving the Optimization Problem

- Optimization problem decomposes into local simplex problems

$$\sum_{j \in \mathcal{J}_1} \left\langle \tilde{x}^j, U_{t-1}^j \right\rangle \underbrace{\left[ -\frac{1}{\eta} \beta_j \tilde{x}_{p_j} \right]}_{:= \frac{1}{\eta_j}} H \left( \frac{\tilde{x}^j}{\tilde{x}_{p_j}} \right) = \sum_{j \in \mathcal{J}_1} \tilde{x}_{p_j} \left\{ \left\langle \frac{\tilde{x}^j}{\tilde{x}_{p_j}}, U_{t-1}^j \right\rangle - \frac{1}{\eta_j} H \left( \frac{\tilde{x}^j}{\tilde{x}_{p_j}} \right) \right\}$$

- Quantity  $\frac{\tilde{x}^j}{\tilde{x}_{p_j}}$  is essentially the behavioral strategy  $x^j$  at info set  $j$

$$\sum_{j \in \mathcal{J}_1} \tilde{x}_{p_j} \left\{ \left\langle x^j, U_{t-1}^j \right\rangle - \frac{1}{\eta_j} H(x^j) \right\}$$

- Quantity  $x^j$  over simplex  $\Delta_j$  is independent of solution  $x_a$  for all ancestral actions and only appears in subsequent info sets



# Solving the Optimization Problem

- Decomposes in local max over behavioral strategies  $x^j$  solved bottom up

$$V^j = \max_{x^j \in \Delta_j} \left\langle x^j, U_{t-1}^j \right\rangle - \frac{1}{\eta_j} H(x^j) \Rightarrow \begin{cases} x^j \propto \exp(\eta_j U_{t-1}^j) \\ V^j = \log \sum_{a \in A_j} \exp(\eta_j U_{t-1}^a) = \text{softmax}_{\eta_j}(U_{t-1}^j) \end{cases}$$

- Value  $V^j$  multiplies  $\tilde{x}_{p_j}$ ; when solving for  $\tilde{x}_{p_j}$  we need to take it into account. If  $p_j \in A_k$

$$\max_{x^k \in \Delta_k} \left\langle \tilde{x}^k, U_{t-1}^k \right\rangle - \eta_k \tilde{x}_{p_k} H\left(\frac{\tilde{x}^k}{\tilde{x}_{p_k}}\right) + \tilde{x}_{p_j} V^j + \dots$$

- Add  $V^j$  to “cumulative utility”  $Q_{p_j}$  (initialized at  $U_{t-1,p_j}$ ) associated with  $p_j$

$$Q_{p_j} \leftarrow Q_{p_j} + V^j$$

# ***Sum:*** Nash via FTRL with Dilated Entropy

Each player chooses  $\tilde{x}_t, \tilde{y}_t$  based on FTRL with dilated entropy

- For x-player  $u_t = A\tilde{y}_t$  and  $U_t = U_{t-1} + u_t$  and initialize  $Q = U_t$
- Traverse the tree bottom-up; for each info set  $j \in \mathcal{J}_1$   
$$x_{t+1}^j \propto \exp(\eta_j Q^j), \quad V^j = \text{softmax}_{\eta_j}(Q^j), \quad Q_{p_j} \leftarrow Q_{p_j} + V^j$$
- Define sequence-form strategies top-down:  $\tilde{x}_{t+1}^j = \tilde{x}_{p_j} \cdot x_{t+1}^j$

Similarly, for y player

Return average of sequence-form strategies as equilibrium

# Interpreting utility vector

$$u_{t,a} = A\tilde{y}_t = \sum_{a' \in A_{P2}} A_{a,a'} \tilde{y}_{t,a'}$$

$A_{a,a'}$  is zero if the combination of  $a, a'$  does not lead to a leaf node

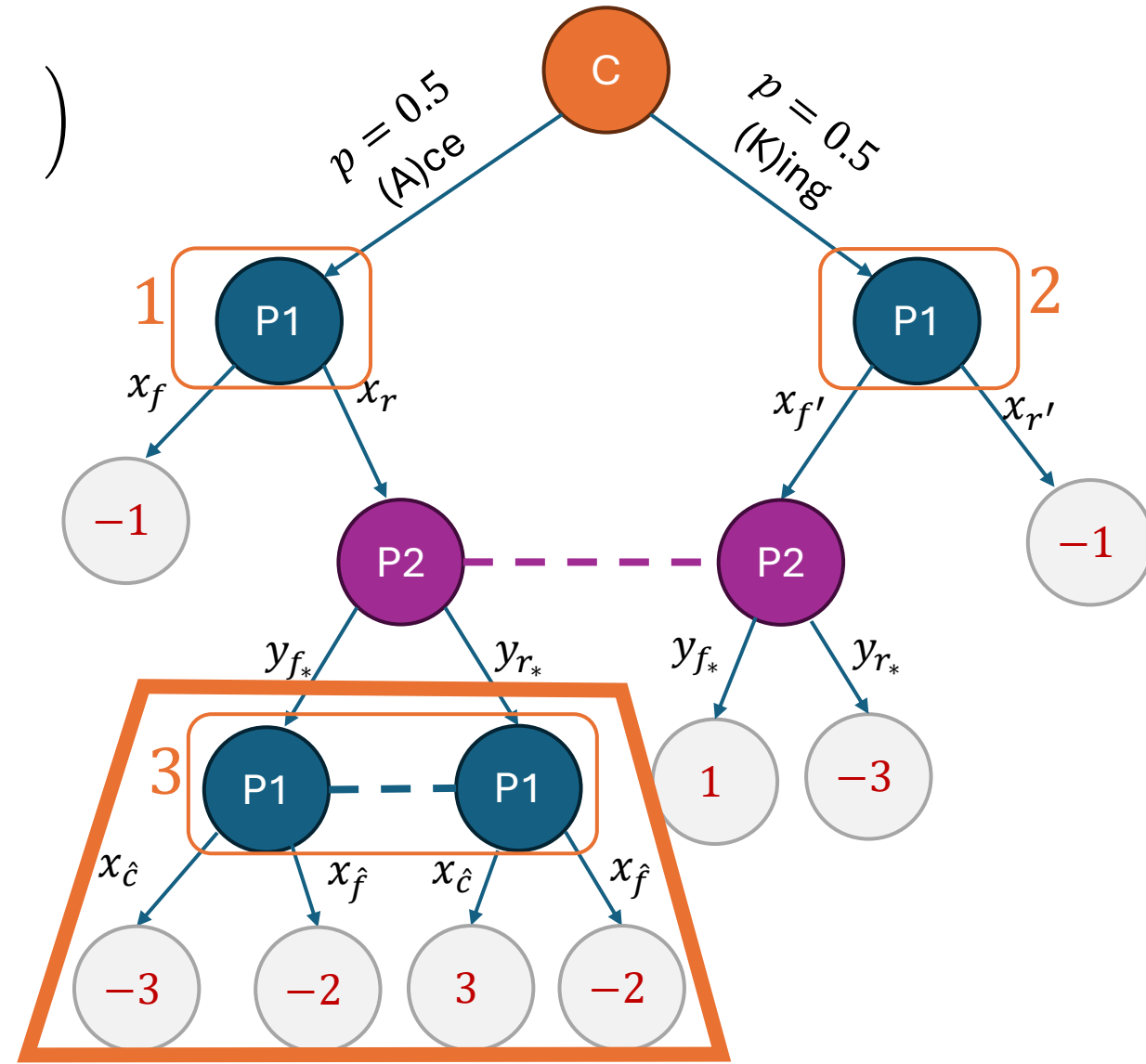
$$u_{t,a} = \sum_{\text{Leafs } z: \substack{a \text{ was last P1 action} \\ a' \text{ was last P2 action}}} u(z) \Pr \left( \begin{array}{c} \text{Chance chooses} \\ \text{sequence on} \\ \text{path to } z \end{array} \right) \Pr \left( \begin{array}{c} \text{P2 plays} \\ \text{sequence} \\ \text{leading to } a' \end{array} \right)$$

**Interpretation.** If I play with the intend to arrive at action  $a$  (i.e.  $\tilde{x}_a = 1$ ) *and then don't make any other moves*, what is the expected reward that I will collect, in expectation over the choices of my opponent and nature

# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

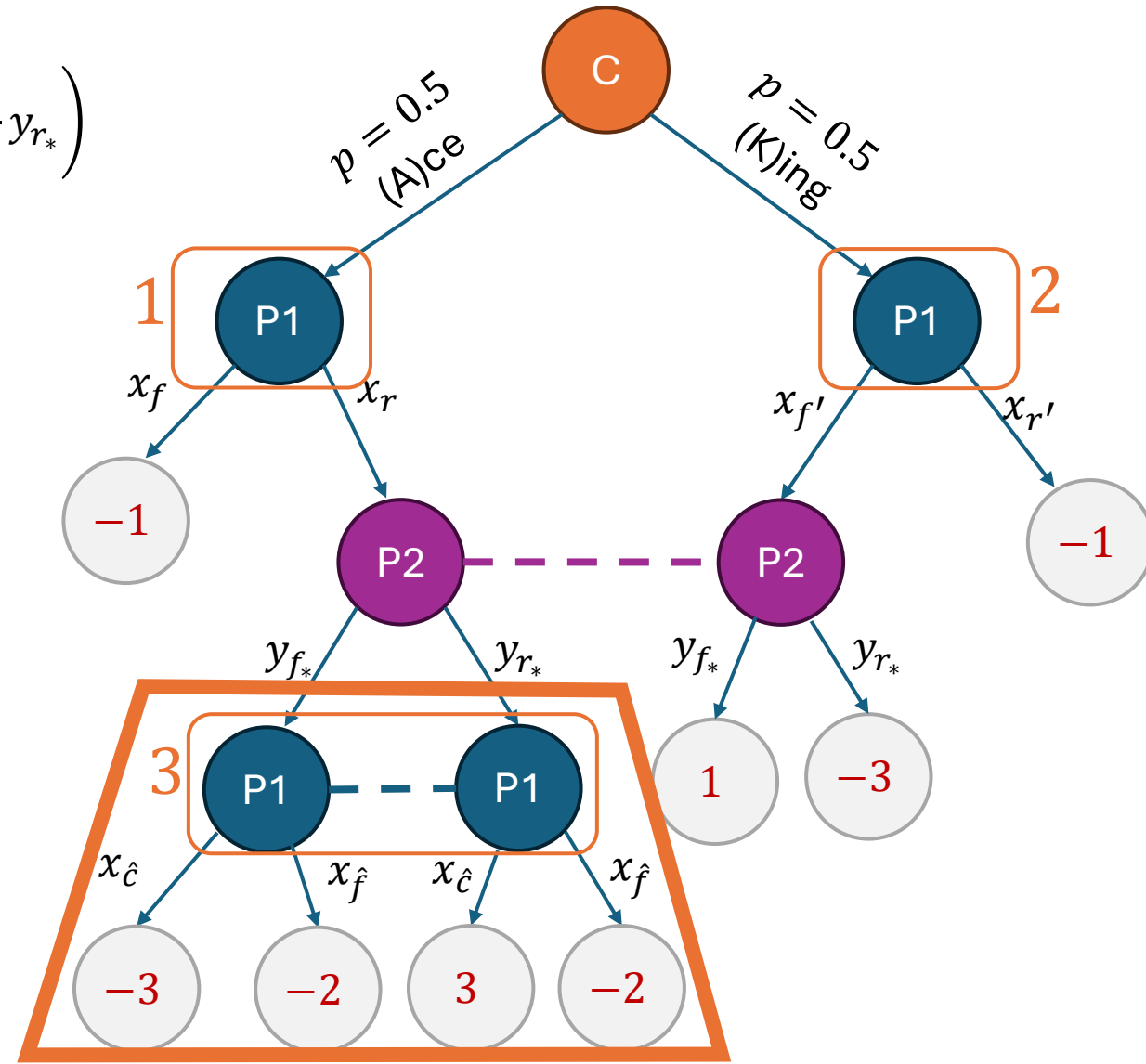
$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( \begin{array}{c} \end{array} \right)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( -3\frac{1}{2}y_{f_*} + 3\frac{1}{2}y_{r_*}, -2\frac{1}{2}y_{f_*} - 2\frac{1}{2}y_{r_*} \right)$$

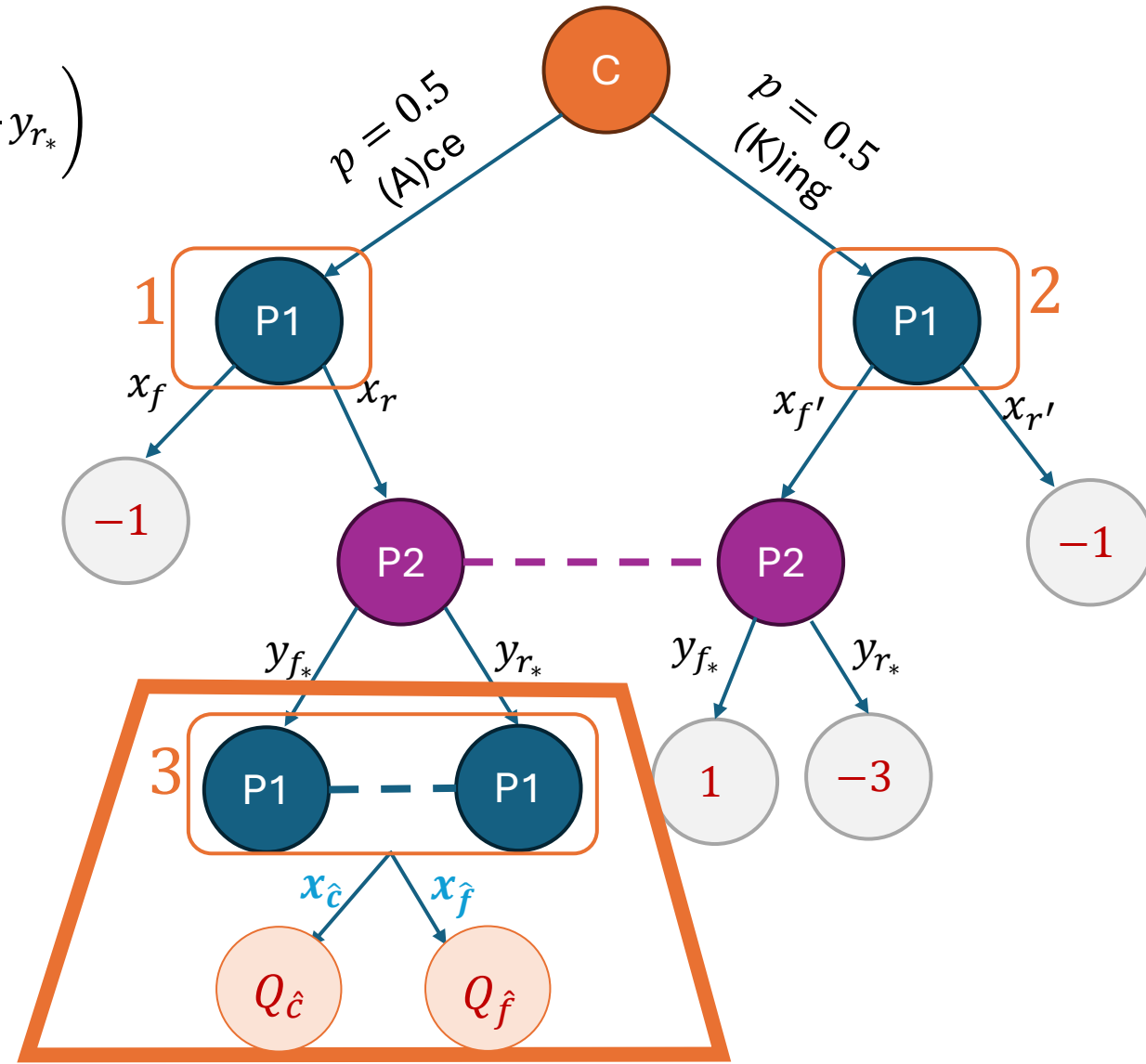


# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( -3\frac{1}{2}y_{f_*} + 3\frac{1}{2}y_{r_*}, -2\frac{1}{2}y_{f_*} - 2\frac{1}{2}y_{r_*} \right)$$

$$Q^3 = U^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 Q^3)$$



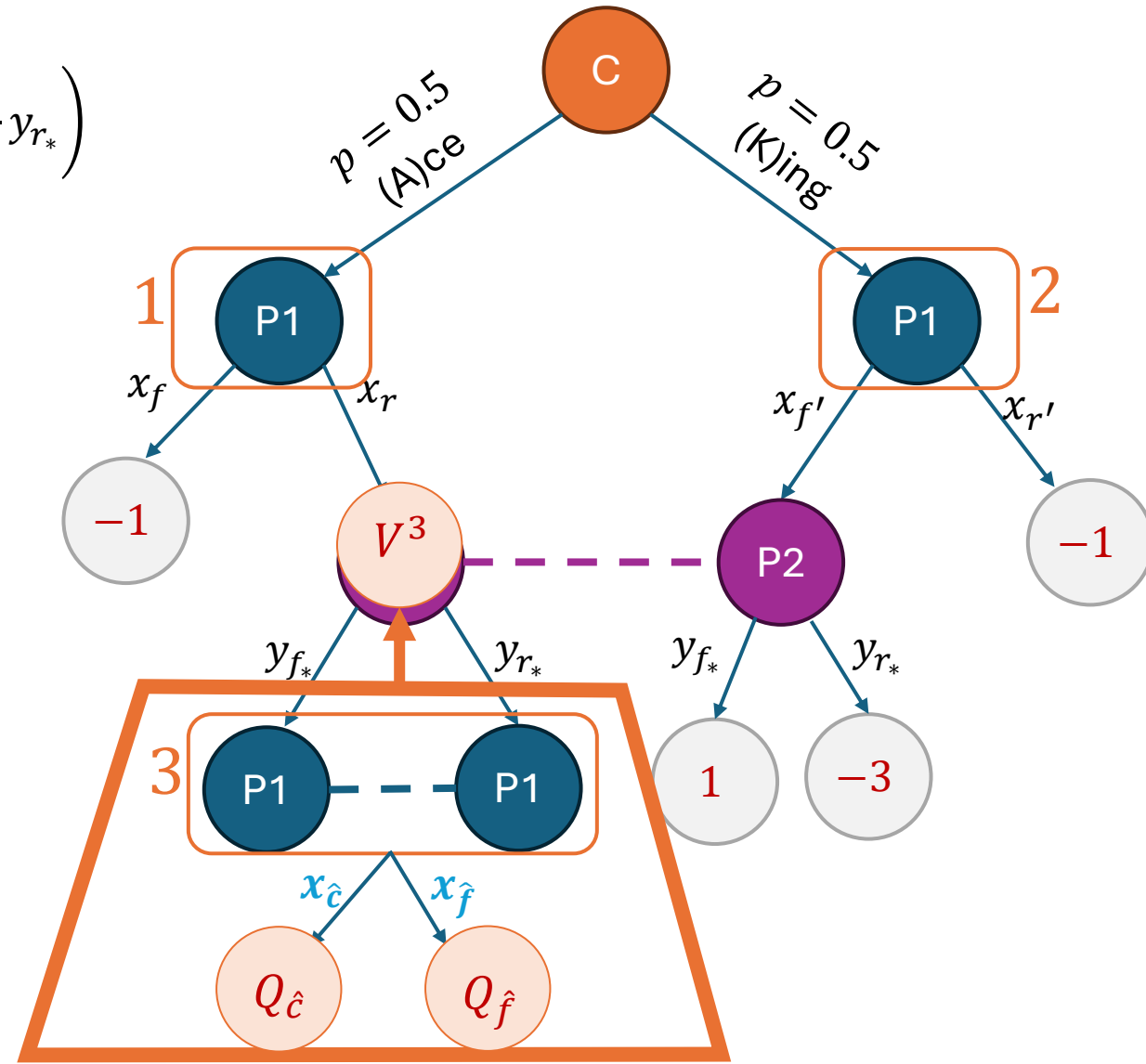
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$$Q^3 = U^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 Q^3)$$

$$V^3 = \text{softmax}(\eta_3 Q^3)$$



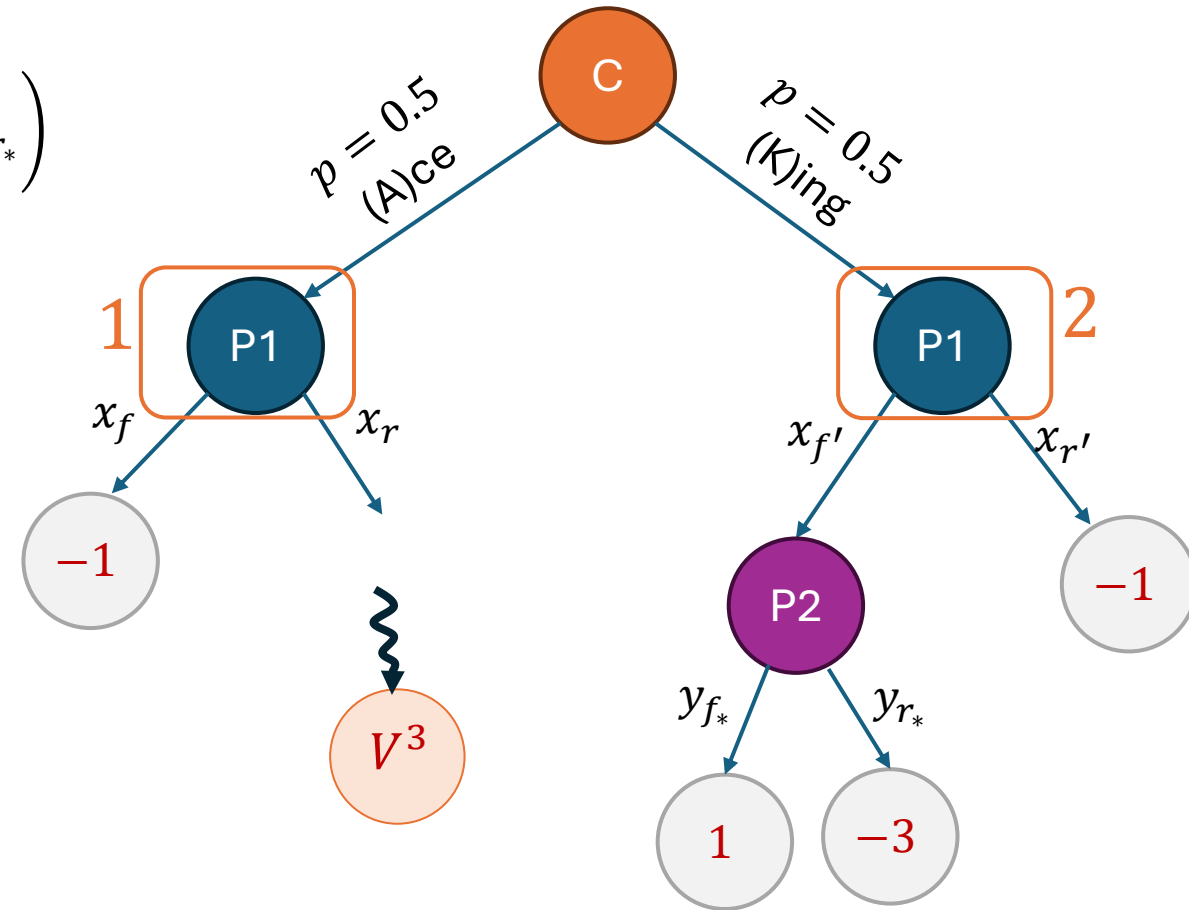
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# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

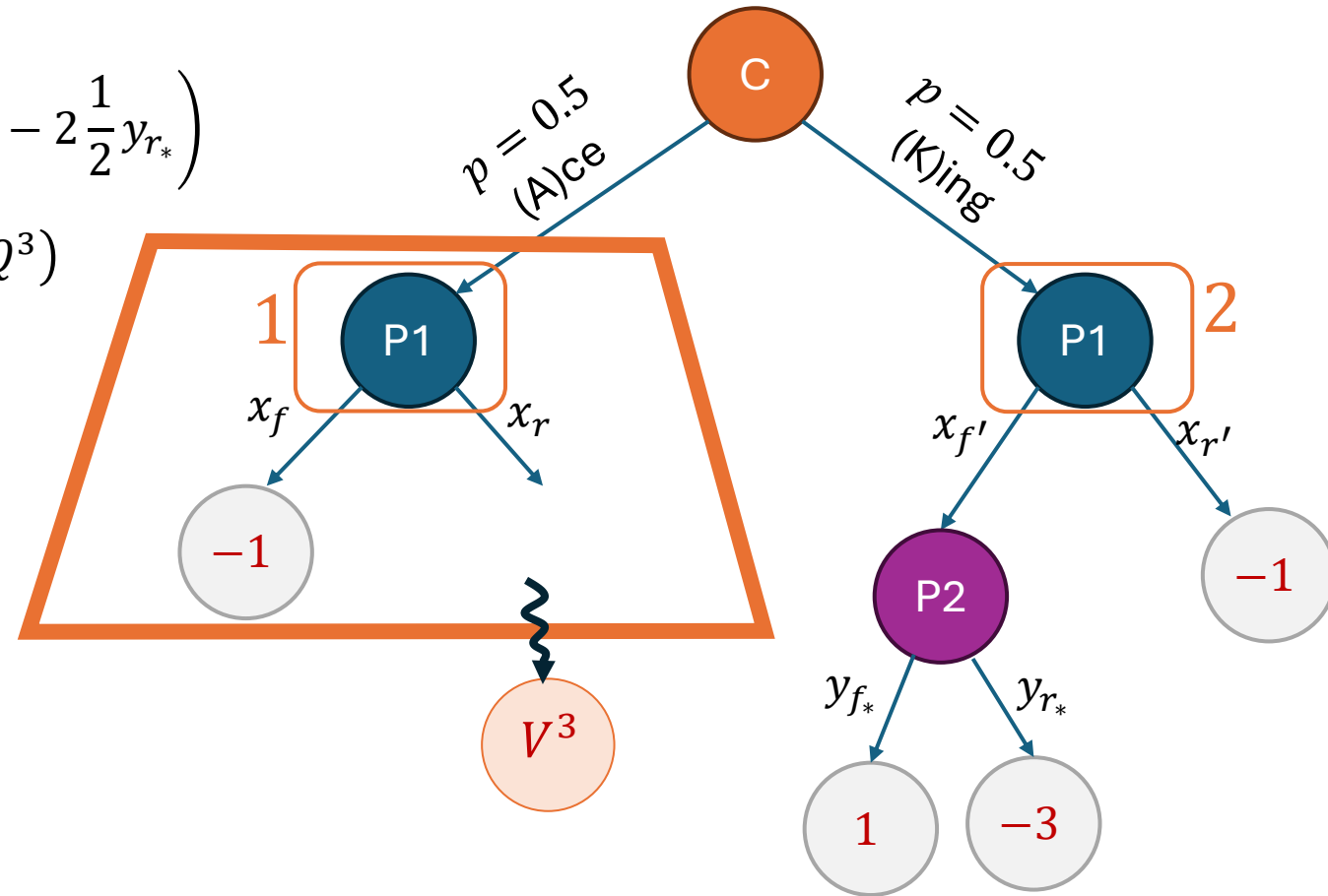
$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( -3\frac{1}{2}y_{f_*} + 3\frac{1}{2}y_{r_*}, -2\frac{1}{2}y_{f_*} - 2\frac{1}{2}y_{r_*} \right)$$

$$Q^3 = U^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 Q^3)$$

$$V^3 = \text{softmax}(\eta_3 Q^3)$$

- Go to **InfoSet 1**

$$U^1 += (u_f, u_r) = \left( \begin{array}{c} \\ \end{array} \right)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

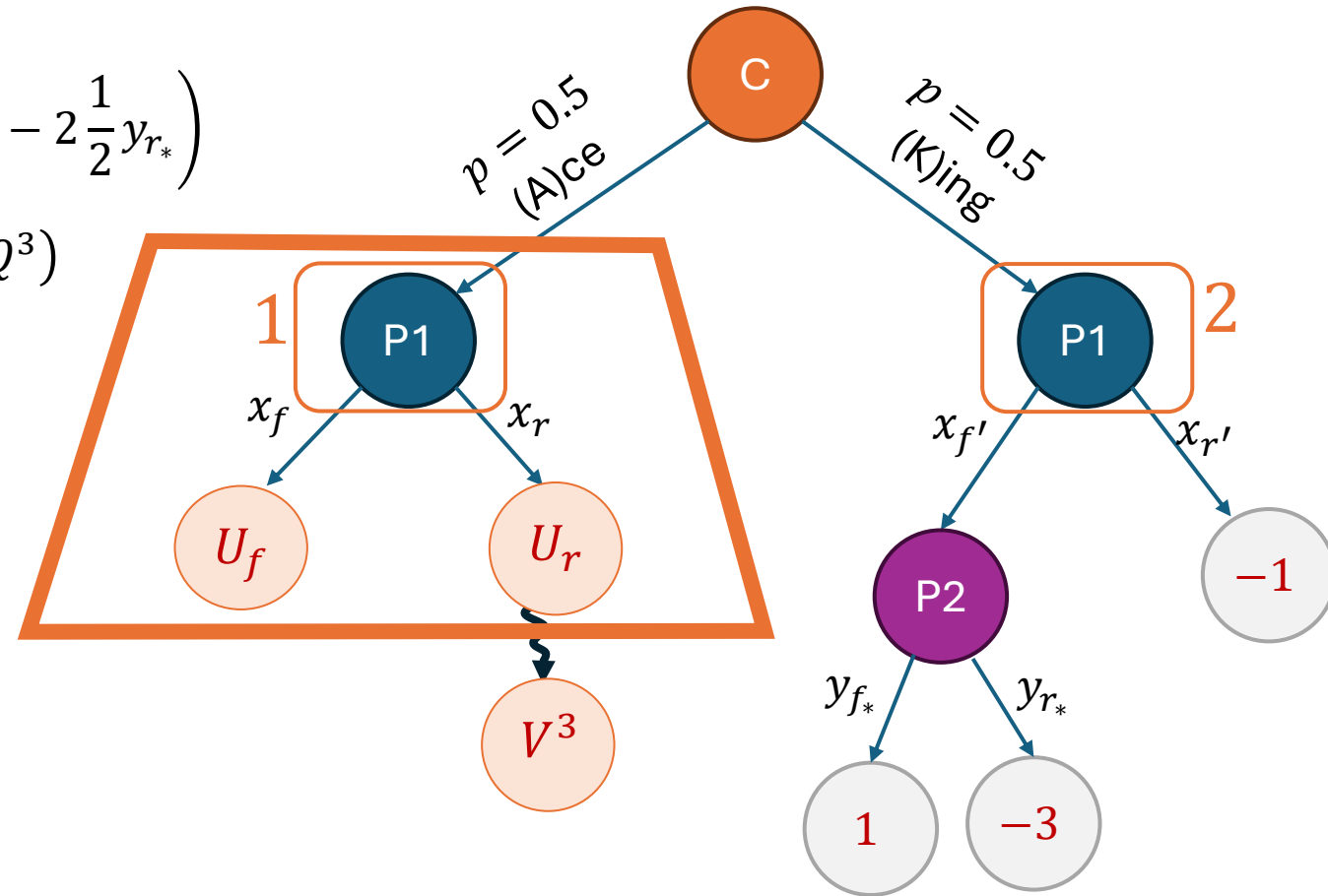
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$$V^3 = \text{softmax}(\eta_3 Q^3)$$

- Go to **InfoSet 1**

$$U^1 += (u_f, u_r) = \left( -1\frac{1}{2}, 0 \right)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

$$Q^3 = U^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 Q^3)$$

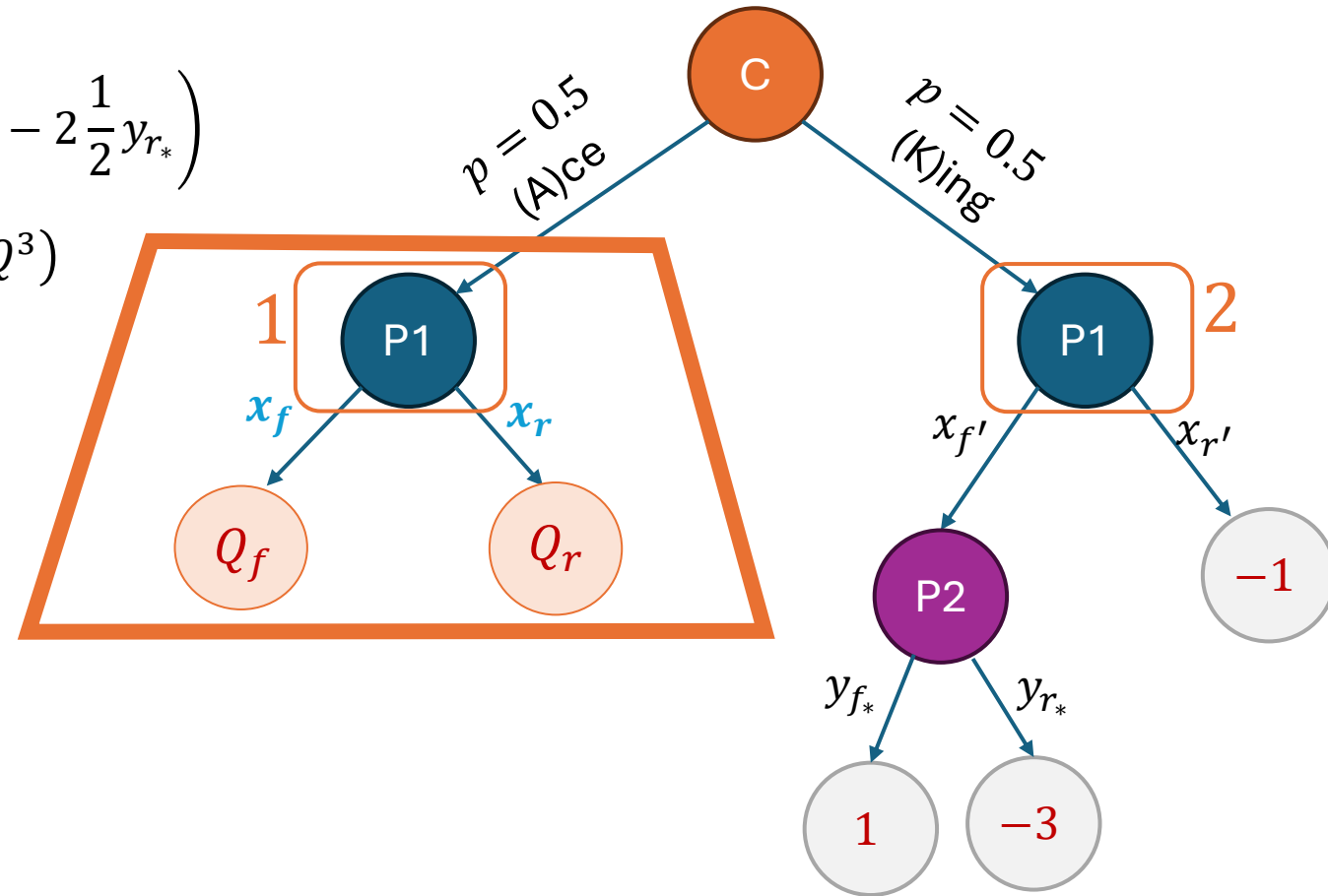
$$V^3 = \text{softmax}(\eta_3 Q^3)$$

- Go to **InfoSet 1**

$$U^1 += (u_f, u_r) = \left( -1\frac{1}{2}, 0 \right)$$

$$Q^1 = U^1 + (0, V^3) = \left( -1\frac{1}{2}, V^3 \right)$$

$$x^1 = (x_f, x_r) \propto \exp(\eta_1 Q^1)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

$$Q^3 = U^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 Q^3)$$

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- Go to **InfoSet 1**

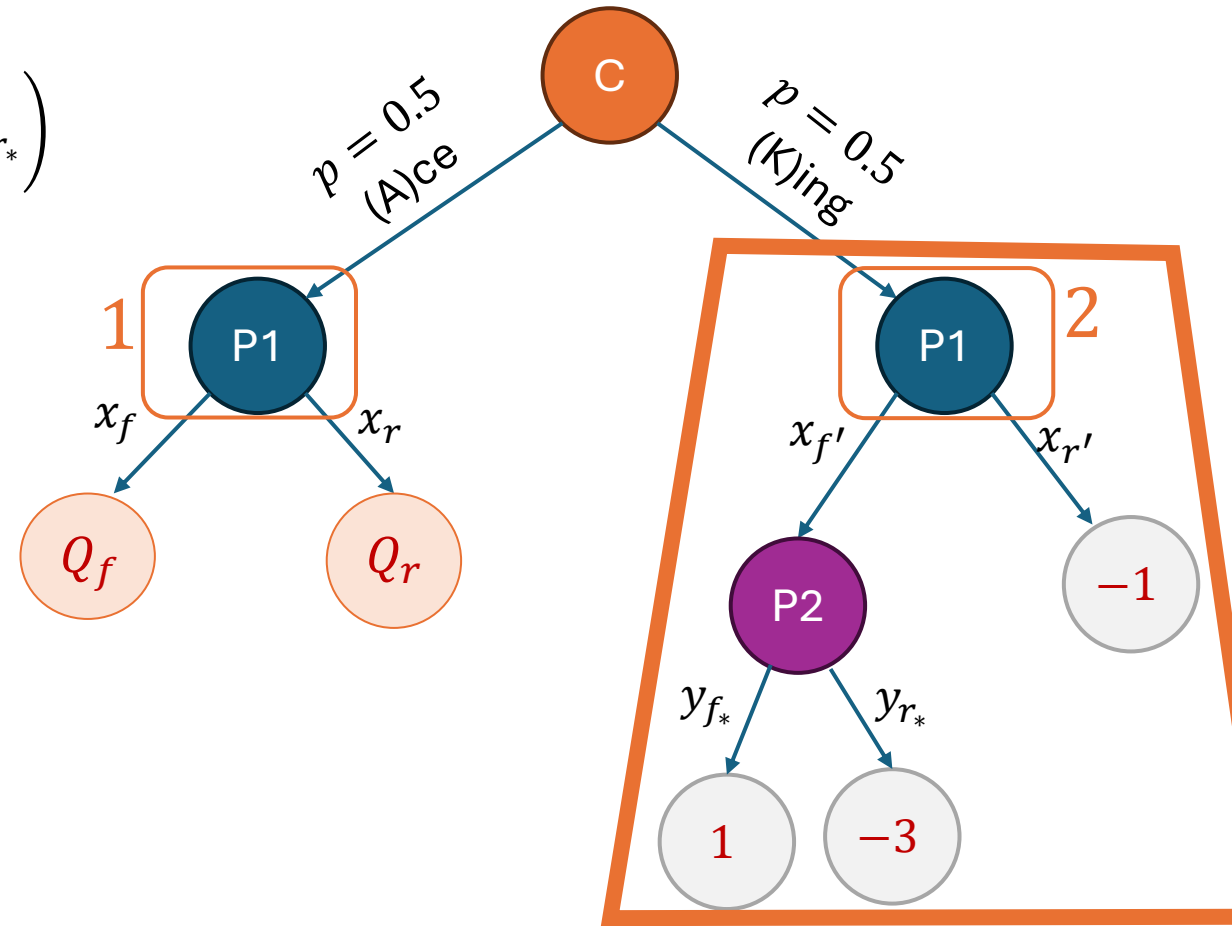
$$U^1 += (u_f, u_r) = \left( -1\frac{1}{2}, 0 \right)$$

$$Q^1 = U^1 + (0, V^3) = \left( -1\frac{1}{2}, V^3 \right)$$

$$x^1 = (x_f, x_r) \propto \exp(\eta_1 Q^1)$$

- Go to **InfoSet 2**

$$U^2 += (u_{f'}, u_{r'}) = \left( \begin{array}{c} \\ \end{array} \right)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( -3\frac{1}{2}y_{f_*} + 3\frac{1}{2}y_{r_*}, -2\frac{1}{2}y_{f_*} - 2\frac{1}{2}y_{r_*} \right)$$

$$Q^3 = U^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 Q^3)$$

$$V^3 = \text{softmax}(\eta_3 Q^3)$$

- Go to **InfoSet 1**

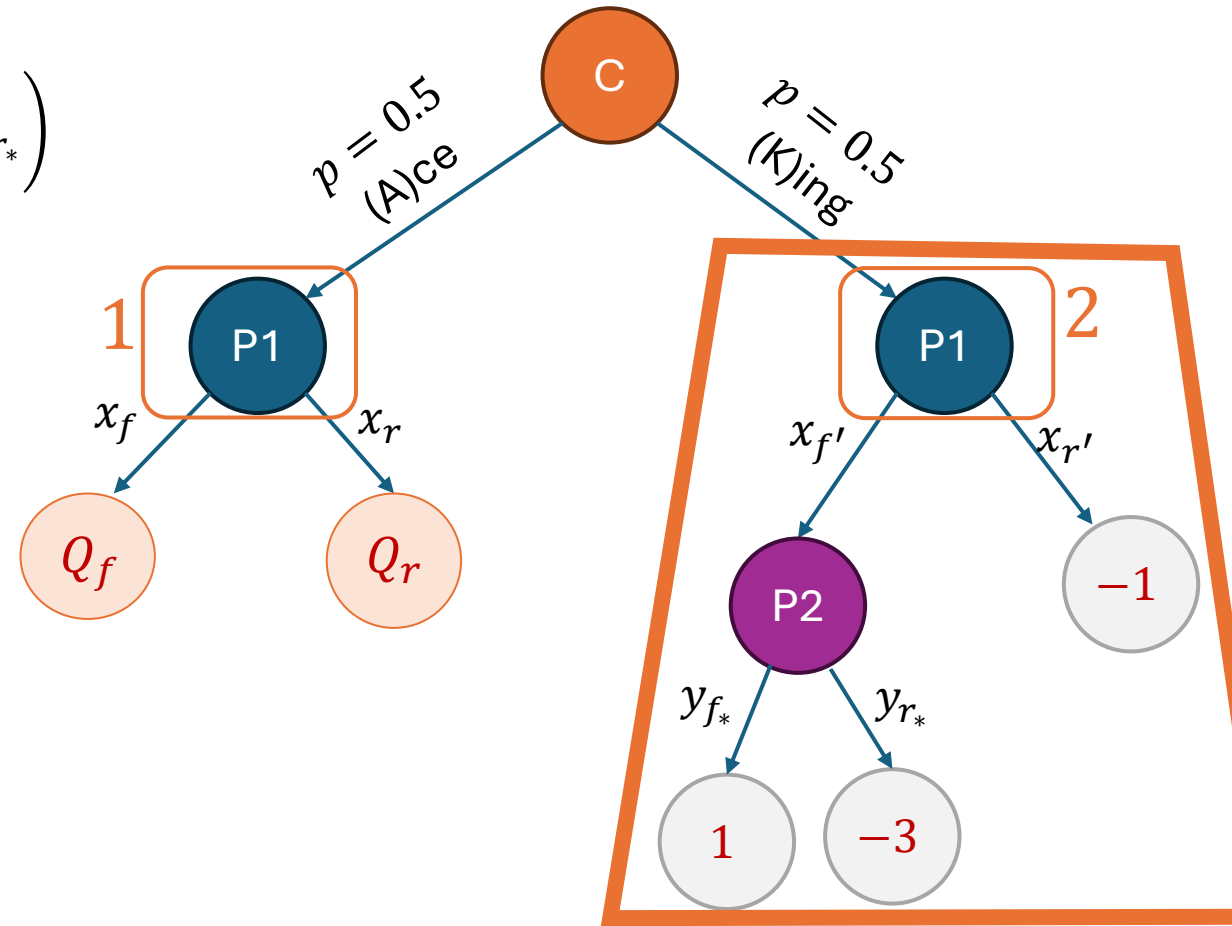
$$U^1 += (u_f, u_r) = \left( -1\frac{1}{2}, 0 \right)$$

$$Q^1 = U^1 + (0, V^3) = \left( -1\frac{1}{2}, V^3 \right)$$

$$x^1 = (x_f, x_r) \propto \exp(\eta_1 Q^1)$$

- Go to **InfoSet 2**

$$U^2 += (u_{f'}, u_{r'}) = \left( 1\frac{1}{2}y_{f_*} - 3\frac{1}{2}y_{r_*}, -1\frac{1}{2} \right)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$U^3 += (u_{\hat{c}}, u_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

$$Q^3 = U^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 Q^3)$$

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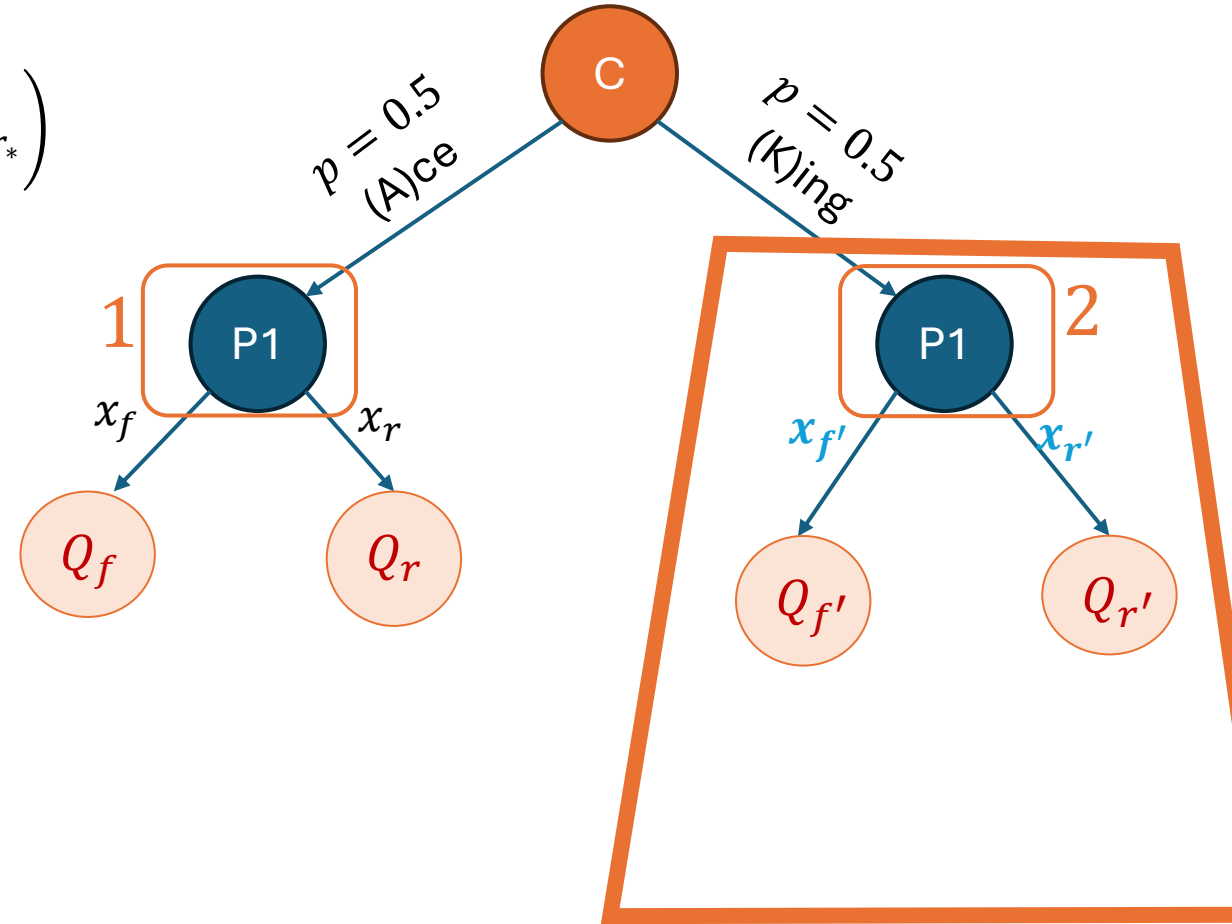
$$Q^1 = U^1 + (0, V^3) = \left( -1\frac{1}{2}, V^3 \right)$$

$$x^1 = (x_f, x_r) \propto \exp(\eta_1 Q^1)$$

- Go to **InfoSet 2**

$$U^2 += (u_{f'}, u_{r'}) = \left( 1\frac{1}{2}y_{f*} - 3\frac{1}{2}y_{r*}, -1\frac{1}{2} \right)$$

$$Q^2 = U^2, \quad x^2 = (x_{f'}, x_{r'}) \propto \exp(\eta_2 Q^2)$$



# ***Sum:*** Nash via FTRL with Dilated Entropy

Each player chooses  $\tilde{x}_t, \tilde{y}_t$  based on FTRL with dilated entropy

- For x-player  $u_t = A\tilde{y}_t$  and  $U_t = U_{t-1} + u_t$  and initialize  $Q = U_t$
- Traverse the tree bottom-up; for each info set  $j \in \mathcal{J}_1$   
$$x_{t+1}^j \propto \exp(\eta_j Q^j), \quad V^j = \text{softmax}_{\eta_j}(Q^j), \quad Q_{p_j} \leftarrow Q_{p_j} + V^j$$
- Define sequence-form strategies top-down:  $\tilde{x}_{t+1}^j = \tilde{x}_{p_j} \cdot x_{t+1}^j$

Similarly, for y player

Return average of sequence-form strategies as equilibrium

# ***Fast Rates***

**Theorem.** If we use Optimistic FTRL instead of FTRL then we get faster convergence to a Nash equilibrium at rate  $1/T$  instead of  $1/\sqrt{T}$ . Plus, we get last-iterate convergence instead of only average iterate convergence.



# Monte-Carlo Stochastic Approximation of Utilities

- Calculating utilities on all nodes of the tree can be very expensive
- In linear online learning it suffices that we use an unbiased estimate of the utility vector

$$\tilde{x}_t = \operatorname{argmax}_{x \in X} \sum_{\tau < t} \langle x, \hat{u}_\tau \rangle - \frac{1}{\eta} \mathcal{R}(x), \quad E[\hat{u}_\tau \mid F_\tau] = u_\tau$$

All random variables observed before period  $\tau$

- By standard martingale concentration inequality arguments, the error vanishes with the number of iterations (*we will see later*)
- In this setting, it suffices that we “sample a path for opponent” and that we “sample chance moves”

# Illustration: First Step of Dynamics

- Sample chance moves based on fixed distribution and opponent moves based on  $y_t$ ; Suppose, we sampled  $A$  and  $f_*$
- Go to **Infoset 3**

$$\hat{U}^3 += (\hat{u}_{\hat{c}}, \hat{u}_{\hat{f}}) = (-3, -2)$$

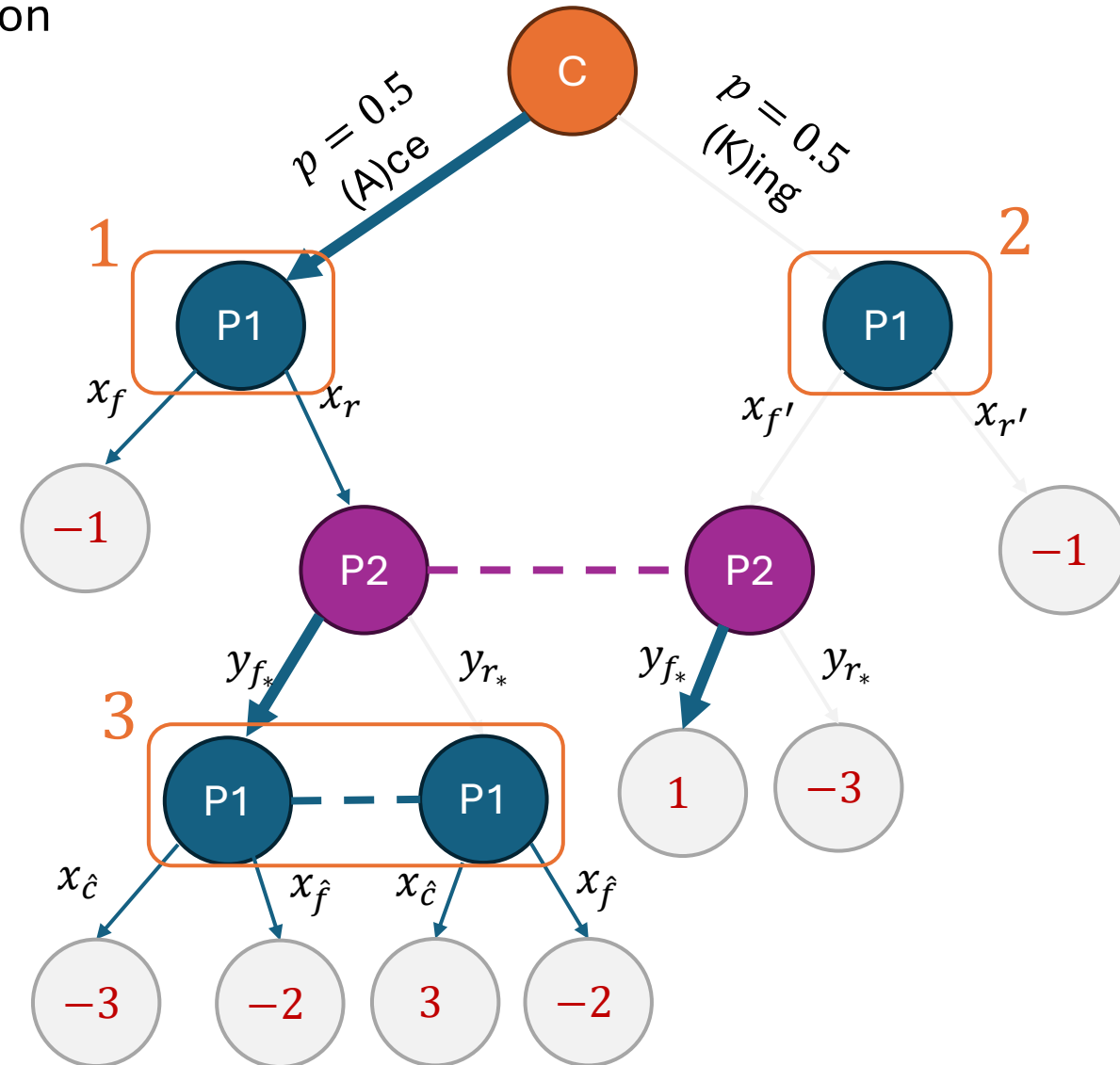
$$\hat{Q}^3 = \hat{U}^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 \hat{Q}^3)$$

$$\hat{V}^3 = \text{softmax}(\eta_3 \hat{Q}^3)$$

- Go to **Infoset 1**

$$\begin{aligned} \hat{U}^1 &+= (\hat{u}_f, \hat{u}_r) = (-1, 0) \\ \hat{Q}^1 &= \hat{U}^1 + (0, \hat{V}^3) = (-1, \hat{V}^3) \end{aligned}$$

$$x^1 = (x_f, x_r) \propto \exp(\eta_1 \hat{Q}^1)$$



# Illustration: First Step of Dynamics

- Equivalently top down and evaluate recursively

- Sample chance move (e.g. sampled A)

- Go to **Infoset 1**

$$\hat{U}_f += -1, \quad \hat{U}_r += 0, \quad \hat{Q}_f = \hat{U}_f, \quad \hat{Q}_r = \hat{U}_r$$

- Recursively go down tree after action  $r$

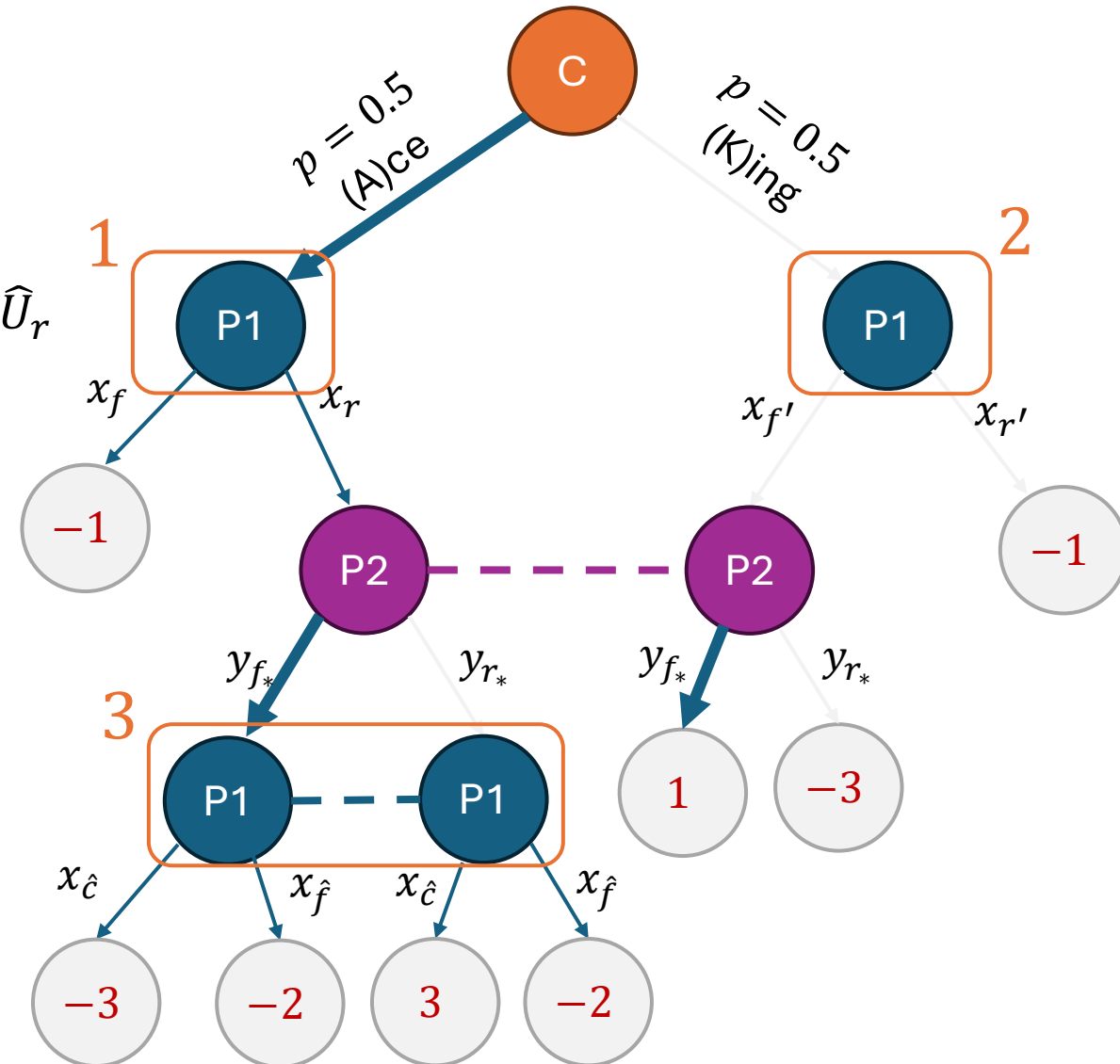
- Sample P2 move (e.g. sampled  $f_*$ )

- Go down to **Infoset 3**

$$\begin{aligned} \hat{U}^3 &+= (-3, -2) \\ \hat{Q}^3 &= \hat{U}^3, \quad x^3 = (x_{\hat{c}}, x_{\hat{f}}) \propto \exp(\eta_3 \hat{Q}^3) \\ \hat{Q}_r &+= \hat{V}^3 = \text{softmax}(\eta_3 \hat{Q}^3) \end{aligned}$$

- Go back up to **Infoset 1**;

$$x^1 = (x_f, x_r) \propto \exp(\eta_1 Q^1)$$



# Local Dynamics

- These dynamics seem to be doing “local updates” at each node
- They came out of a specific algorithm FTRL with Dilated Entropy
- Is this a general paradigm?
- Can we decompose the no-regret learning problem into local no-regret learners at each node?
- What feedback should each node receive from the learners in nodes below?
- What loss should each learner be optimizing?

# Counterfactual Regret Minimization (CRM)

# Re-interpreting Utilities

**Interpretation of  $u_a$ .** If I play with the intent to arrive at action  $a$  (i.e.  $\tilde{x}_a = 1$ ) *and then don't make any other moves*, what is the expected reward that I will collect, in expectation over the choices of my opponent and nature

**What if we now want to express:** If I play with the intent to arrive at action  $a$  (i.e.  $\tilde{x}_a = 1$ ) *and then continue playing based on some behavioral policy  $x$* , what is the expected reward that I will collect, in expectation over the choices of my opponent and nature

# Re-interpreting Utilities

**Interpretation of  $u_a$ .** If I play with the intent to arrive at action  $a$  (i.e.  $\tilde{x}_a = 1$ ) *and then don't make any other moves*, what is the expected reward that I will collect, in expectation over the choices of my opponent and nature

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- Let  $C_a$  be all infosets of the player that are reachable **as next infosets** after playing  $a$

$$\tilde{u}_a(x) = \boxed{u_a} + \sum_{k \in C_a} \boxed{V^k(x)}$$

*“Instantaneous  $E[\text{utility}]$ ”, if this is the last action I play*

*Continuation  $E[\text{utility}]$  from paths that pass through infoset  $k$ , if I continue playing based on behavioral strategy  $x$*

# Re-interpreting Utilities

**Interpretation of  $u_a$ .** If I play with the intend to arrive at action  $a$  (i.e.  $\tilde{x}_a = 1$ ) *and then don't make any other moves*, what is the expected reward that I will collect, in expectation over the choices of my opponent and nature

**What if we now want to express:** If I play with the intend to arrive at action  $a$  (i.e.  $\tilde{x}_a = 1$ ) *and then continue playing based on some behavioral policy  $x$* , what is the expected reward that I will collect, in expectation over the choices of my opponent and nature

- Let  $C_a$  be all infosets of the player that are reachable **as next infosets** after playing  $a$

$$\tilde{u}_a(x) = \underbrace{u_a}_{\text{"Instantaneous E[utility]", if this is the last action I play}} + \sum_{k \in C_a} \underbrace{V^k(x)}_{\text{"Continuation E[utility] from paths that pass through info set } k, \text{ if I continue playing based on behavioral strategy } x}$$

- Continuation utility  $V^j(x)$  from paths that pass through info set  $j$  recursively defined:

$$V^j(x) = \sum_{a \in A^j} x_a \tilde{u}_a(x) = \underbrace{\sum_{a \in A^j} x_a u_a}_{\text{"Instantaneous utility", if this is the last move I make}} + \underbrace{\sum_{a \in A^j} x_a \left( \sum_{k \in C_a} V^k(x) \right)}_{\text{"Continuation utility", if I continue playing based on } x}$$



# Re-interpreting Utilities

- Continuation utility  $V^j(x)$  from paths that pass through  $j$ , assuming I play to arrive deterministically at the parent action  $p_j$  (i.e.,  $\tilde{x}_{p_j} = 1$ )

$$V^j(x) = \sum_{a \in A^j} x_a \tilde{u}_a(x) = \sum_{a \in A^j} x_a \left( u_a + \sum_{k \in C_a} V^k(x) \right)$$

- Obviously  $V^{\text{root}}(x)$  is total expected utility from behavior strategy  $x$
- From equivalence of behavioral and sequence-form strategies

$$V^{\text{root}}(x) = \langle \tilde{x}, u \rangle$$

- The same also holds for regrets

$$R^{\text{root}}(x) = \max_{x'} V^{\text{root}}(x') - V^{\text{root}}(x) = \max_{\tilde{x}' \in X} \langle \tilde{x}', u \rangle - \langle \tilde{x}, u \rangle = R(\tilde{x})$$

# Local Regrets

- We can also define infoset regrets based on local utilities  $\tilde{u}_a$

$$R^j(x) = \max_{x'} V^j(x') - V^j(x) = \max_{x'} \sum_a x'_a \tilde{u}_a(x') - x_a \tilde{u}_a(x)$$

- Right-hand-side can be decomposed as:

$$\max_{x'} \left( \sum_a x'_a \tilde{u}_a(x) - x_a \tilde{u}_a(x) \right) + \left( \sum_a x'_a (\tilde{u}_a(x') - \tilde{u}_a(x)) \right)$$

*Fix continuation strategy to current strategy and only change the behavioral strategy at the current infoset*

*Weighted average of changes in continuation strategy*

# Local Regrets

- We can also define info set regrets based on local utilities  $\tilde{u}_a$

$$R^j(x) = \max_{x'} V^j(x') - V^j(x) = \max_{x'} \sum_a x'_a \tilde{u}_a(x') - x_a \tilde{u}_a(x)$$

- Right-hand-side can be decomposed as:

$$\max_{x'} \sum_a x'_a \tilde{u}_a(x) - x_a \tilde{u}_a(x) + \sum_a x'_a (\tilde{u}_a(x') - \tilde{u}_a(x))$$

- Maximum is upper bounded by the decoupled optima

$$\max_{x'} \sum_a x'_a \tilde{u}_a(x) - x_a \tilde{u}_a(x) + \sum_a \max_{x'} (\tilde{u}_a(x') - \tilde{u}_a(x))$$

**Local Regret:  $LR^j(x)$**

*Regret if you only change current info set behavioral strategy and keep continuation strategy*

# Recursive Bound of Local Regrets

- Infoset regrets are bounded by local regret plus continuation terms

$$R^j(x) \leq \text{LR}^j(x) + \sum_a \max_{x'} (\tilde{u}_a(x') - \tilde{u}_a(x))$$

- The continuation terms are recursive infoset regrets!

$$\tilde{u}_a(x') - \tilde{u}_a(x) = \cancel{u_a} + \sum_{k \in C_a} V^k(x') - \cancel{u_a} - \sum_{k \in C_a} V^k(x)$$

- Deriving the recursive upper bound

$$\begin{aligned} R^j(x) &\leq \text{LR}^j(x) + \sum_a \sum_{k \in C_a} \max_{x'} V^k(x') - V^k(x) \\ &\leq \text{LR}^j(x) + \sum_a \sum_{k \in C_a} R^k(x) \end{aligned}$$

# Recursive Bound of Local Regrets

- Deriving the recursive upper bound

$$R^j(x) \leq \text{LR}^j(x) + \sum_a \sum_{k \in C_a} R^k(x)$$

# Recursive Bound of Local Regrets

- Deriving the recursive upper bound

$$R^j(x) \leq LR^j(x) + \sum_a \sum_{k \in C_a} R^k(x)$$

**Theorem.** By induction:

$$R^j(x) \leq LR^j(x) + \sum_{k \text{ eventually reachable from } j} LR^k(x)$$

# Local Regrets Upper Bound Total Regret

- Deriving the recursive upper bound

$$R^j(x) \leq LR^j(x) + \sum_a \sum_{k \in C_a} R^k(x)$$

**Theorem.** By induction:

$$R^j(x) \leq LR^j(x) + \sum_{k \text{ eventually reachable from } j} LR^k(x)$$

**Main Corollary.** Regret is upper bounded by sum of local regrets

$$R(\tilde{x}) = R^{\text{root}}(x) \leq \sum_{k \in \mathcal{J}_1} LR^k(x)$$

# Regret over Time

Same inequalities can be followed for the average regret over time

$$R = \max_{\tilde{x}' \in X} \frac{1}{T} \sum_t \langle \tilde{x}', u_t \rangle - \langle \tilde{x}_t, u_t \rangle$$

$$LR^j = \max_{x^j} \frac{1}{T} \sum_t \langle x^j, \tilde{u}_t(x_t) \rangle - \langle x_t^j, \tilde{u}_t(x_t) \rangle$$

**Main CFR Theorem.** Regret is upper bounded by local regrets

$$R \leq \sum_{j \in \mathcal{L}_1} LR^j$$



# Achieving vanishing Local Regrets

$$\text{LR}^j(x) = \max_{x^j} \frac{1}{T} \sum_t \langle x^j, \tilde{u}_t(x_t) \rangle - \langle x_t^j, \tilde{u}_t(x_t) \rangle$$

# Counterfactual Regret Minimization

- Device local regret algorithms for local regret

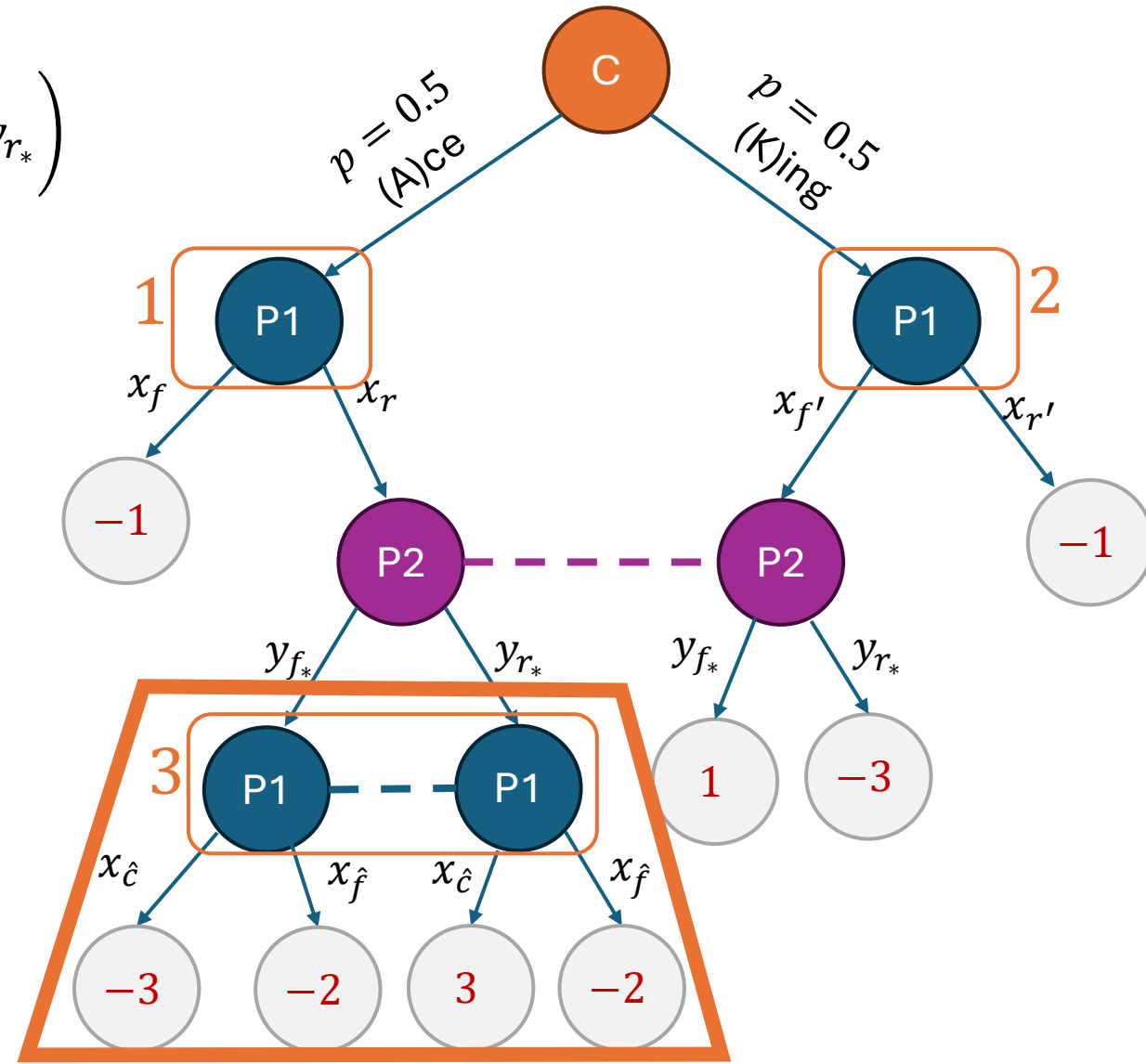
$$\text{LR}^j(x) = \max_{x^j} \frac{1}{T} \sum_t \langle x^j, \tilde{u}_t(x_t) \rangle - \left\langle x_t^j, \tilde{u}_t(x_t) \right\rangle$$

- Standard  $n$ -action no-regret problem: reward vector at period  $t$  is  $\tilde{u}^j(x_t)$  and reward for choice  $x^j$  is  $\langle x^j, \tilde{u}^j(x_t) \rangle$
- At period  $t$  run bottom-up recursion to calculate  $\tilde{u}^j(x_t)$  for  $j \in \mathcal{J}_1$
- Update probabilities  $x_{t+1}^j$  using reward vectors  $\tilde{u}^j(x_t)$  for  $j \in \mathcal{J}_1$

# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

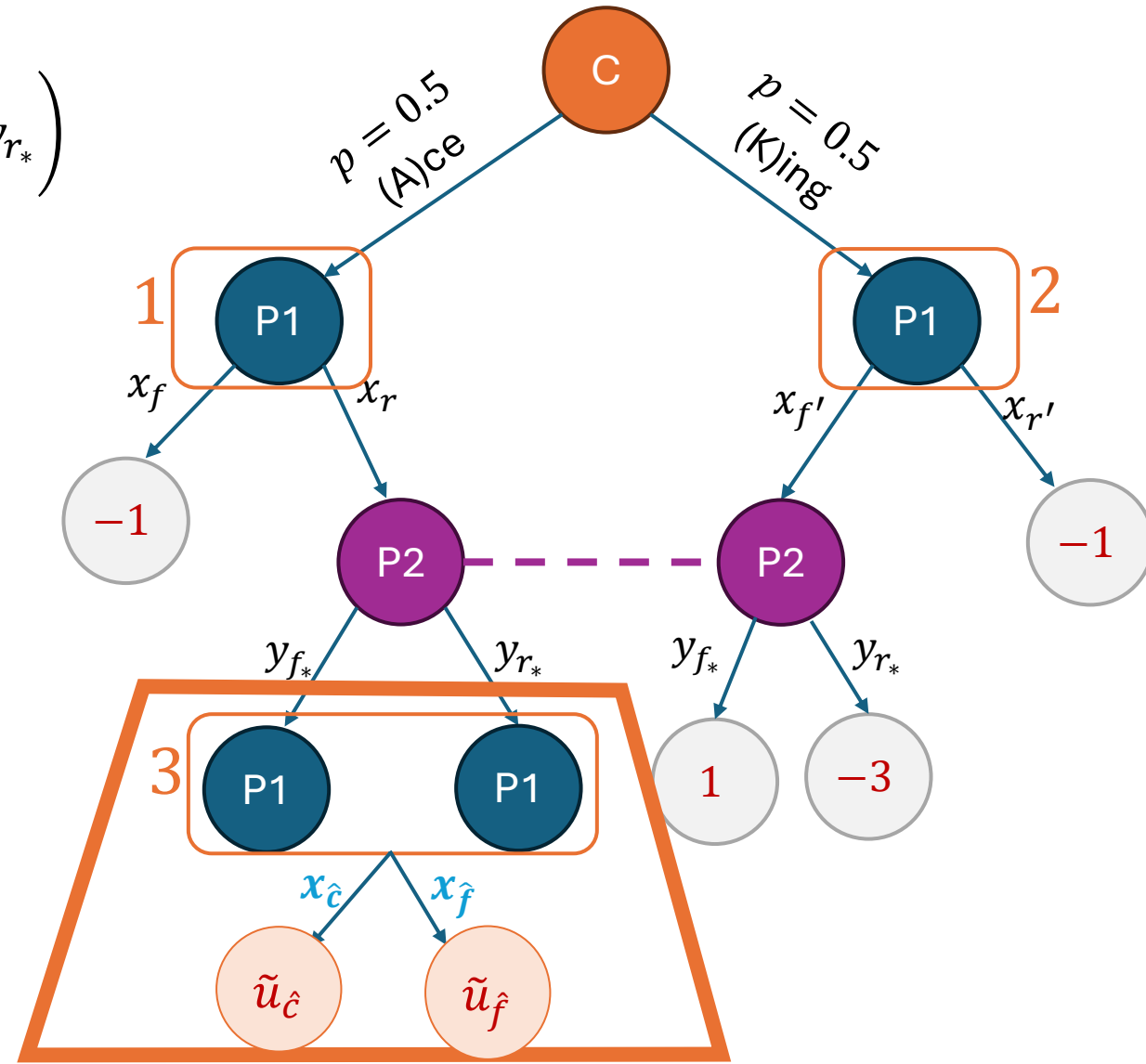
$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

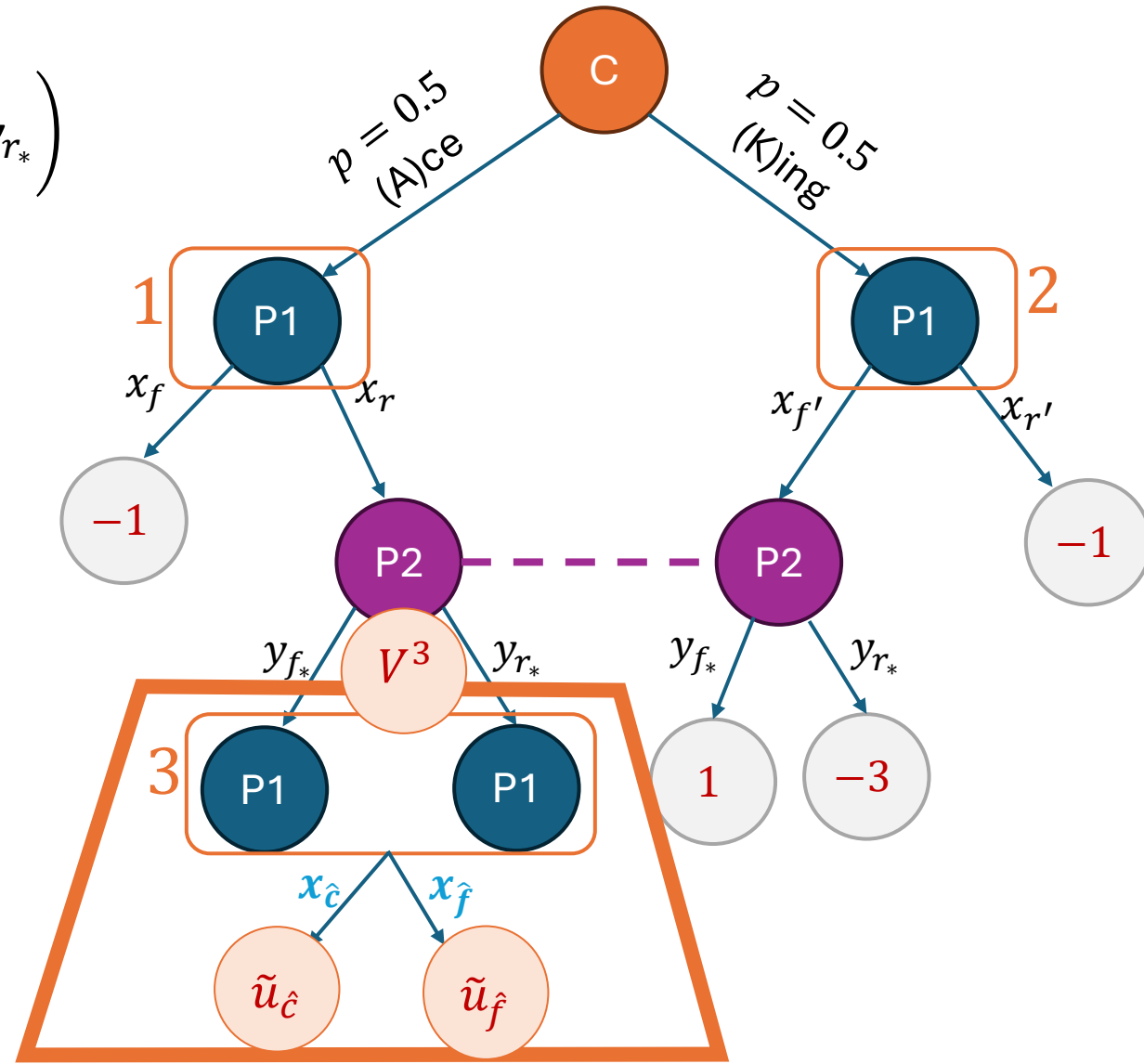


# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

$$V^3 \leftarrow x_{\hat{c}}\tilde{u}_{\hat{c}} + x_{\hat{f}}\tilde{u}_{\hat{f}}$$



# Illustration: First Step of Dynamics

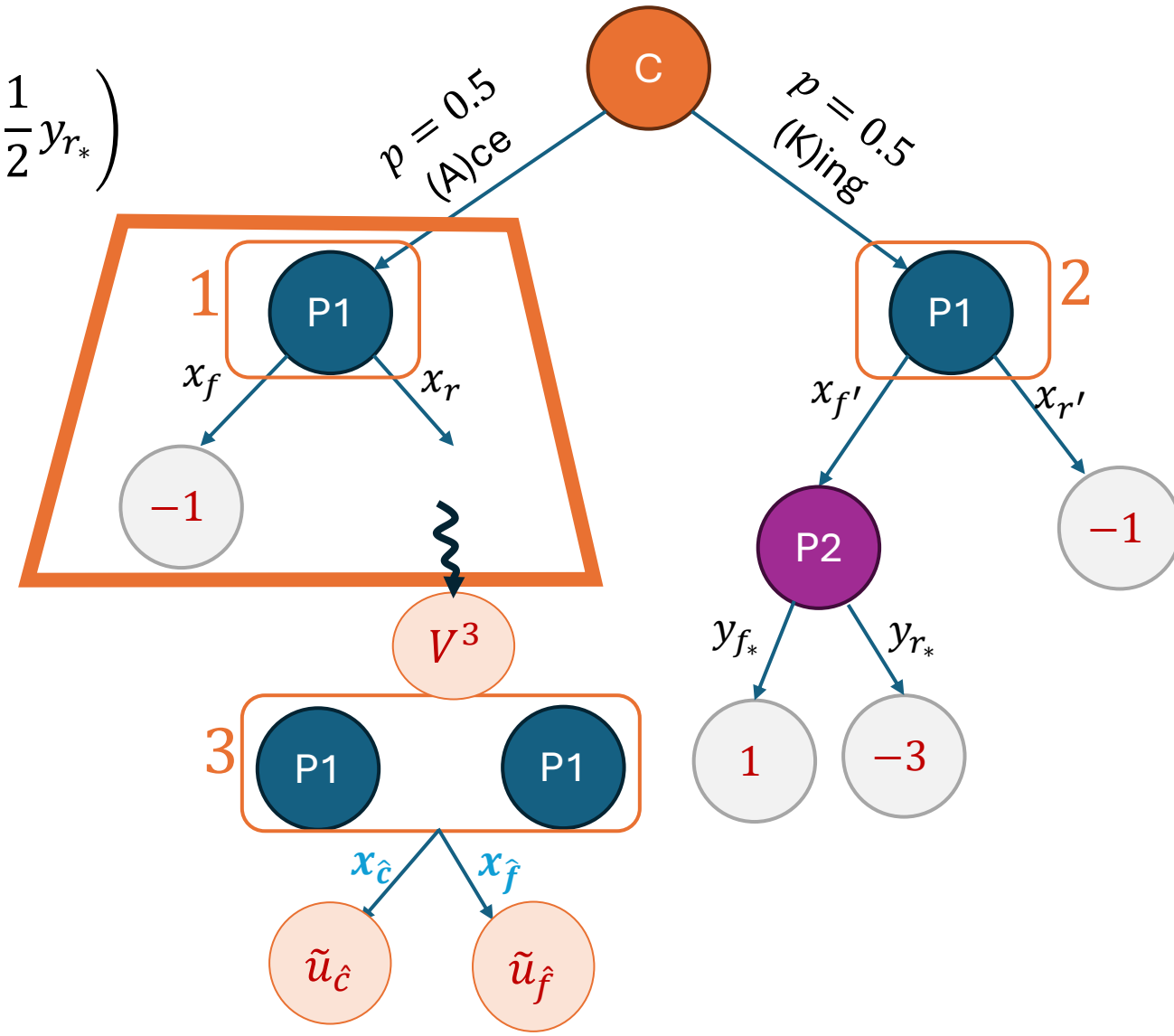
- Go to **InfoSet 3**

$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

$$V^3 \leftarrow x_{\hat{c}}\tilde{u}_{\hat{c}} + x_{\hat{f}}\tilde{u}_{\hat{f}}$$

- Go to **InfoSet 1**

$$(\tilde{u}_f, \tilde{u}_r) = \left( -1\frac{1}{2}, V^3 \right)$$



# Illustration: First Step of Dynamics

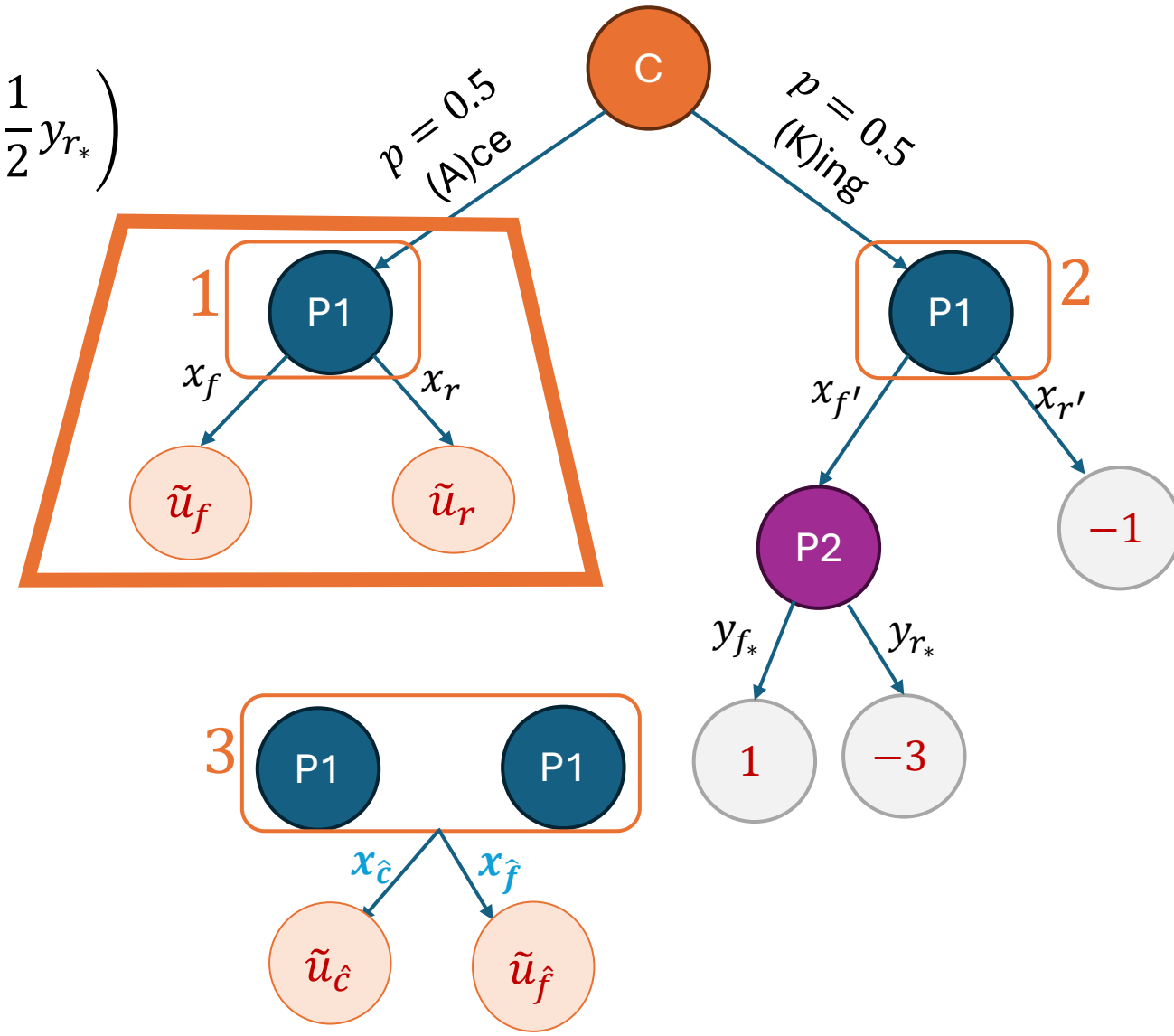
- Go to **InfoSet 3**

$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

$$V^3 \leftarrow x_{\hat{c}}\tilde{u}_{\hat{c}} + x_{\hat{f}}\tilde{u}_{\hat{f}}$$

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# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

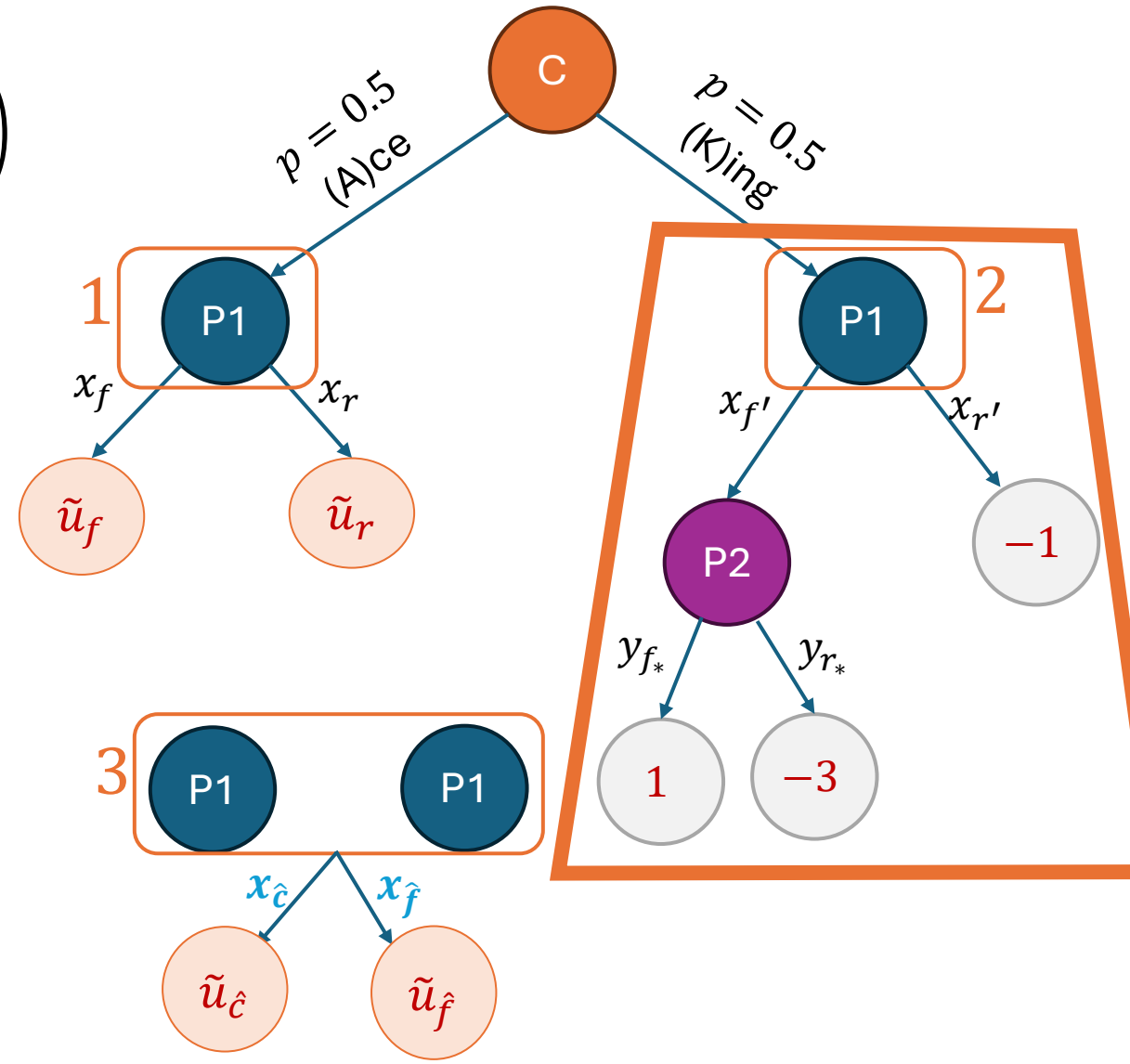
$$V^3 \leftarrow x_{\hat{c}}\tilde{u}_{\hat{c}} + x_{\hat{f}}\tilde{u}_{\hat{f}}$$

- Go to **InfoSet 1**

$$(\tilde{u}_f, \tilde{u}_r) = \left( -1\frac{1}{2}, V^3 \right)$$

- Go to **InfoSet 2**

$$(\tilde{u}_{f'}, \tilde{u}_{r'}) = \left( 1\frac{1}{2}y_{f*} - 3\frac{1}{2}y_{r*}, -1\frac{1}{2} \right)$$





# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f_*} + 3\frac{1}{2}y_{r_*}, -2\frac{1}{2}y_{f_*} - 2\frac{1}{2}y_{r_*} \right)$$

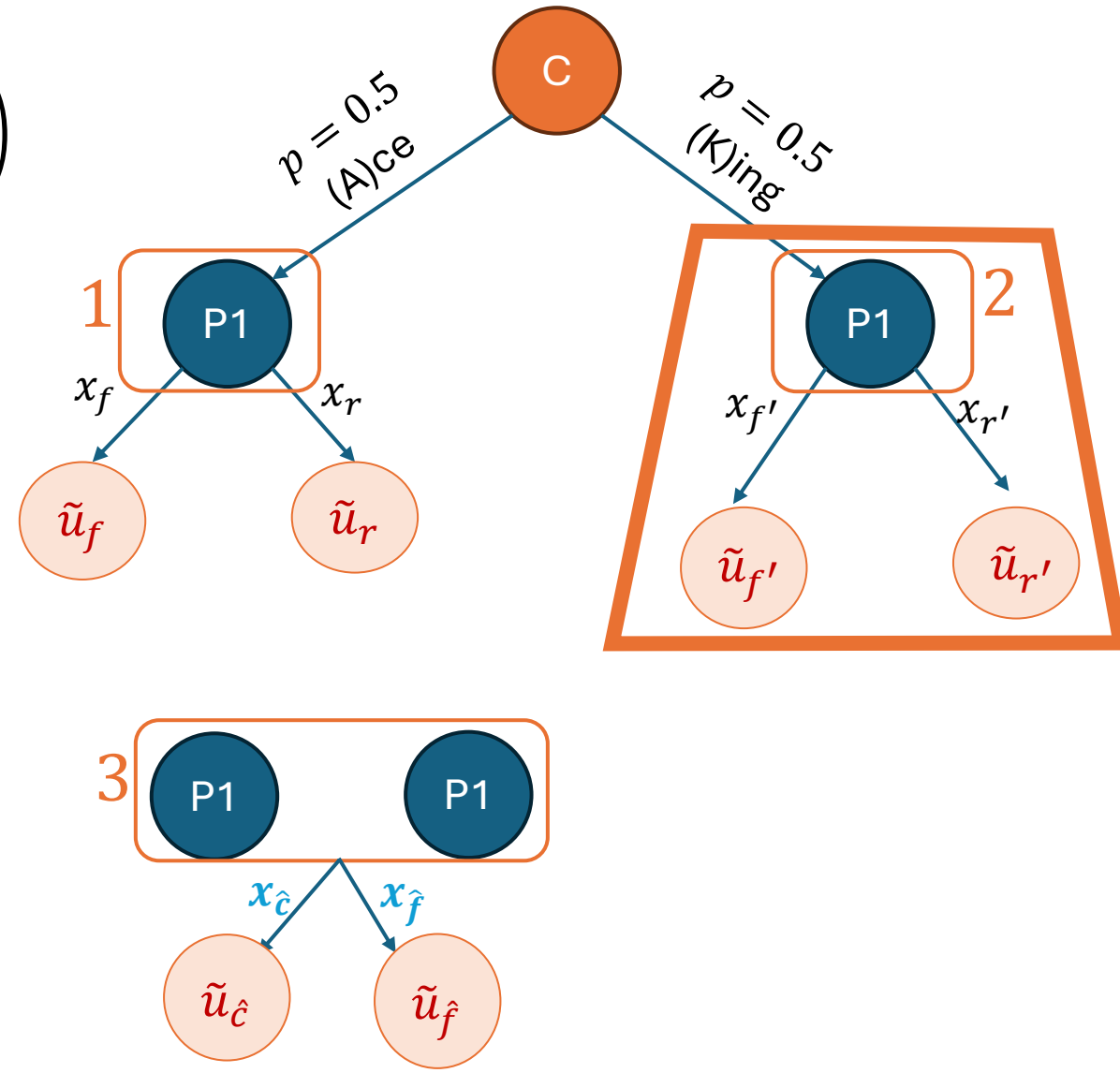
$$V^3 \leftarrow x_{\hat{c}}\tilde{u}_{\hat{c}} + x_{\hat{f}}\tilde{u}_{\hat{f}}$$

- Go to **InfoSet 1**

$$(\tilde{u}_f, \tilde{u}_r) = \left( -1\frac{1}{2}, V^3 \right)$$

- Go to **InfoSet 2**

$$(\tilde{u}_{f'}, \tilde{u}_{r'}) = \left( 1\frac{1}{2}y_{f_*} - 3\frac{1}{2}y_{r_*}, -1\frac{1}{2} \right)$$



# Illustration: First Step of Dynamics

- Go to **InfoSet 3**

$$(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}}) = \left( -3\frac{1}{2}y_{f*} + 3\frac{1}{2}y_{r*}, -2\frac{1}{2}y_{f*} - 2\frac{1}{2}y_{r*} \right)$$

$$\tilde{u}_r \leftarrow x_{\hat{c}}\tilde{u}_{\hat{c}} + x_{\hat{f}}\tilde{u}_{\hat{f}}$$

- Go to **InfoSet 1**

$$(\tilde{u}_f, \tilde{u}_r) = \left( -1\frac{1}{2}, \tilde{u}_r \right)$$

- Go to **InfoSet 2**

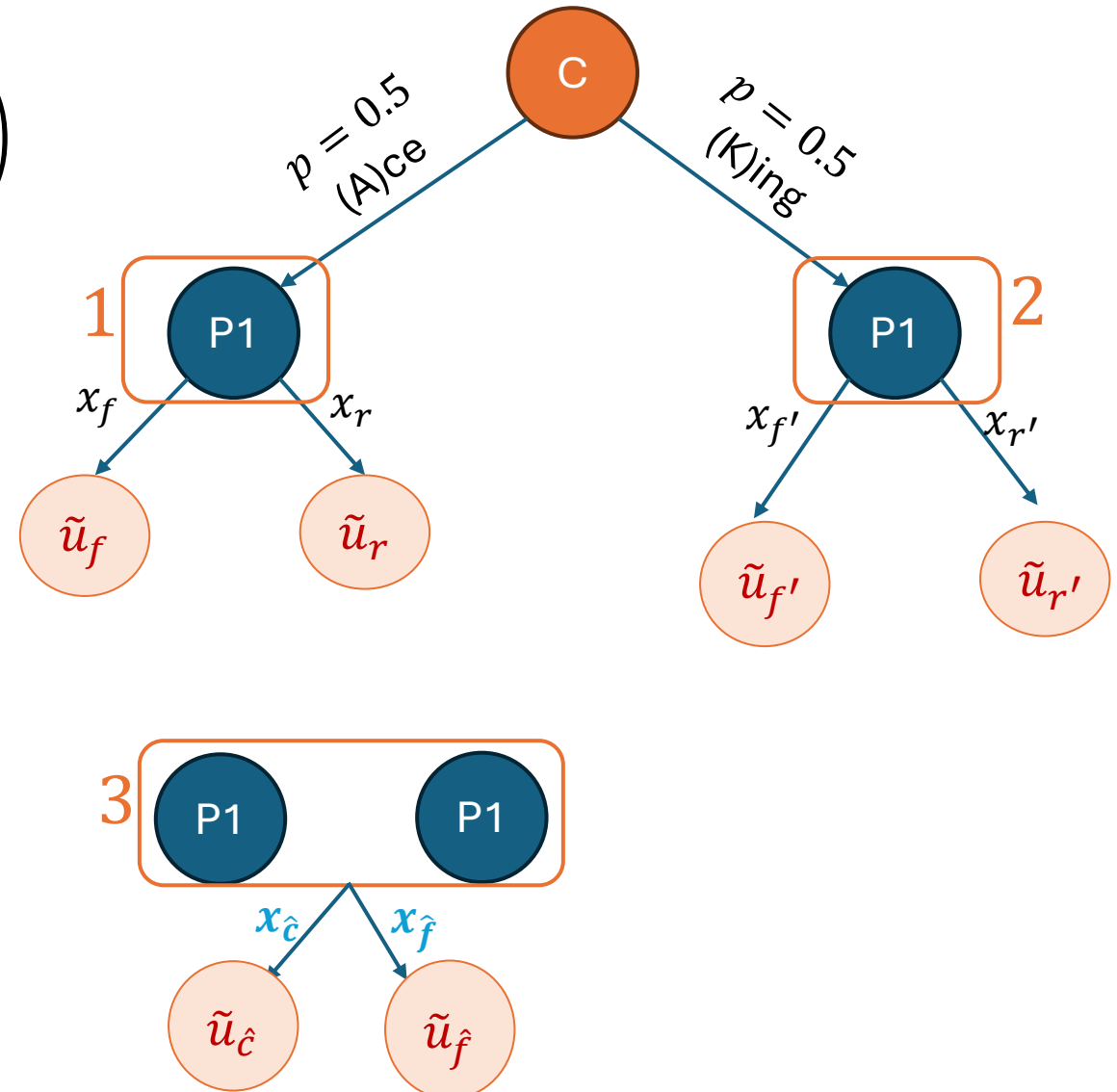
$$(\tilde{u}_{f'}, \tilde{u}_{r'}) = \left( 1\frac{1}{2}y_{f*} - 3\frac{1}{2}y_{r*}, -1\frac{1}{2} \right)$$

- Update probabilities**

$$(x_f, x_r) \leftarrow \text{Update}(\tilde{u}_f, \tilde{u}_r)$$

$$(x_{f'}, x_{r'}) \leftarrow \text{Update}(\tilde{u}_{f'}, \tilde{u}_{r'})$$

$$(x_{\hat{c}}, x_{\hat{f}}) \leftarrow \text{Update}(\tilde{u}_{\hat{c}}, \tilde{u}_{\hat{f}})$$



# Recursive Algorithm

Value(ActionHistory  $h$ , AccOtherProb  $\pi_{-1}$ )

Let  $I$  be info set corresponding to  $h$

**if**  $I$  is terminal node  $z$  return  $\pi_{-1} \cdot u(z)$

**if** Player( $I$ ) = chance

Return  $\sum_{a \in A_I} \text{Value}(ha, \pi_{-1} \pi_a^C)$

**if** Player( $I$ ) = 2

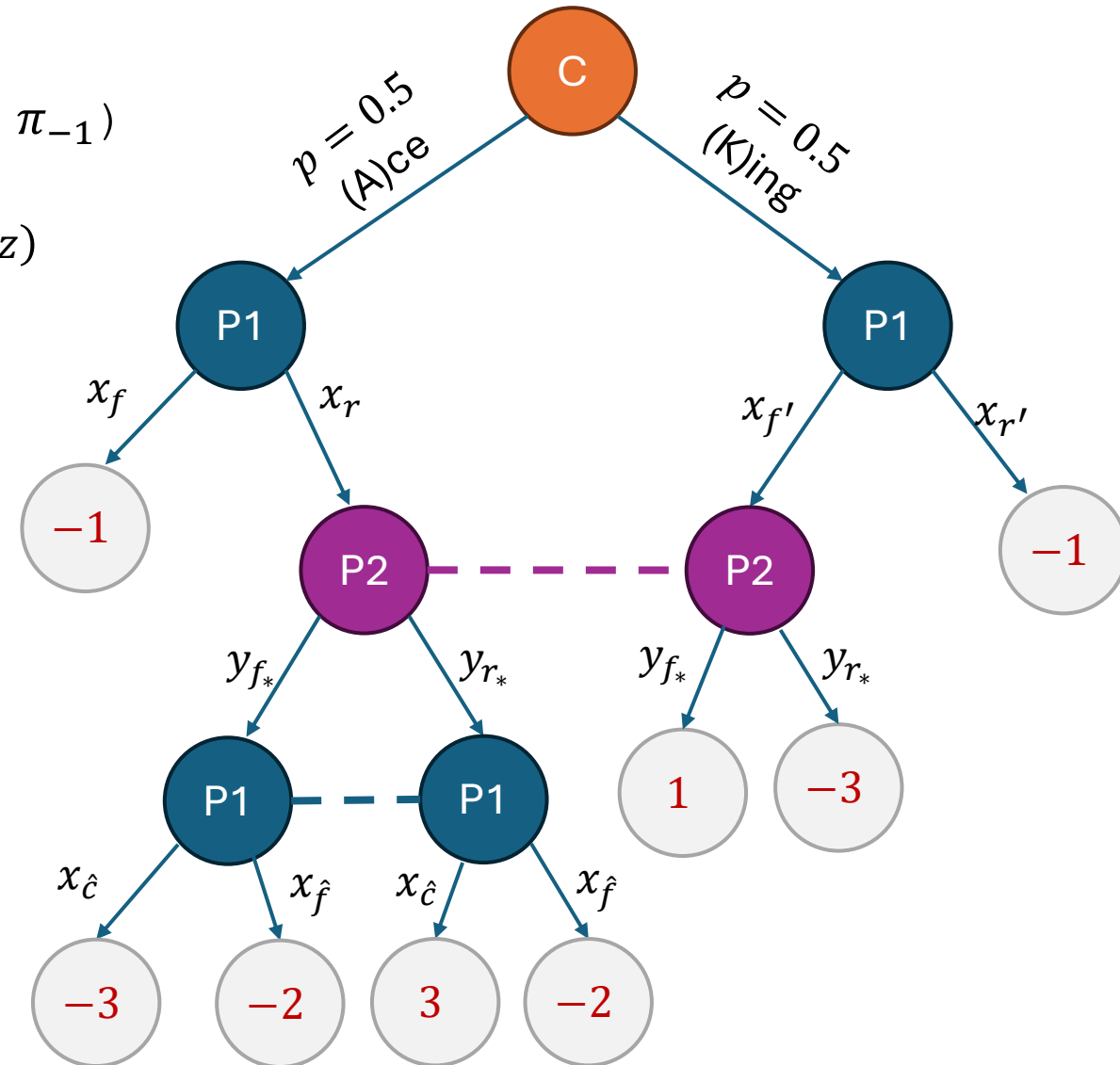
Return  $\sum_{a \in A_I} \text{Value}(ha, \pi_{-1} y_a)$

**if** Player( $I$ ) = 1

For  $a \in A_I$ :  $\tilde{u}_a += \text{Value}(ha, \pi_{-1})$

Return  $\sum_{a \in A_I} x_a \cdot \text{Value}(ha, \pi_{-1})$

Value( $\emptyset$ , 1)



# Recursive Algorithm

Value(ActionHistory  $h$ , AccOtherProb  $\pi_{-1}$ )

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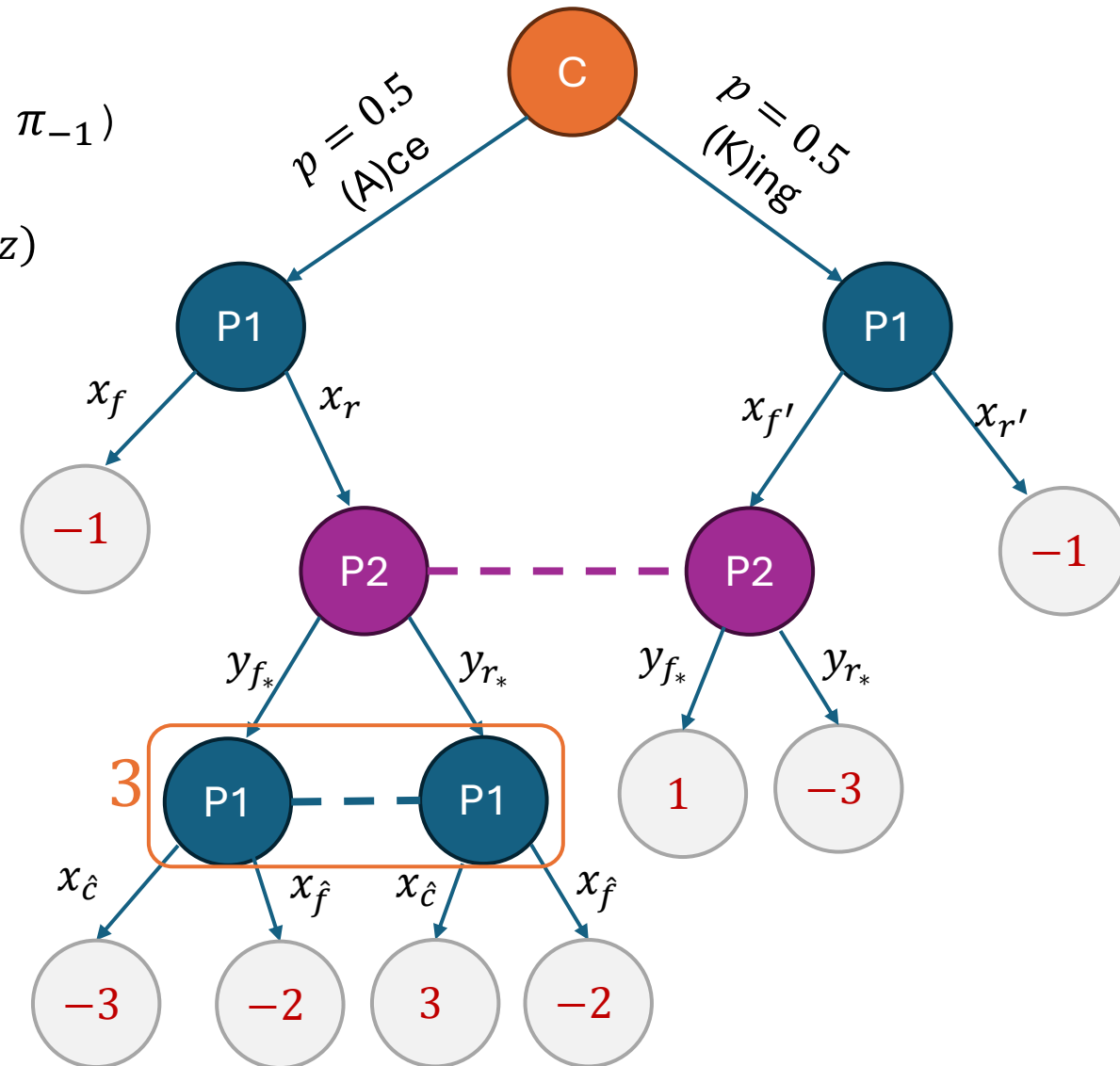
**if** Player( $I$ ) = 1

**For**  $a \in A_I$ :  $\tilde{u}_a += \text{Value}(ha, \pi_{-1})$

Return  $\sum_{a \in A_I} x_a \cdot \text{Value}(ha, \pi_{-1})$

We arrive at the same info set  $I$  multiple times, once for each node in the set;  $\tilde{u}_a$  accumulates continuation utility from taking action  $a$  from all these possible “arrival paths”.

**Example.** In info set 3 we arrive once on the left node and add  $-3 \frac{1}{2} y_{f*}$  and once on the right node and add  $3 \frac{1}{2} y_{r*}$  to  $u_{\hat{c}}$



# Recursive Algorithm

Value(ActionHistory  $h$ , AccOtherProb  $\pi_{-1}$ )

Let  $I$  be info set corresponding to  $h$

**if**  $I$  is terminal node  $z$  return  $\pi_{-1} \cdot u(z)$

**if** Player( $I$ ) = chance

Return  $\sum_{a \in A_I} \text{Value}(ha, \pi_{-1} \pi_a^C)$

**if** Player( $I$ ) = 2

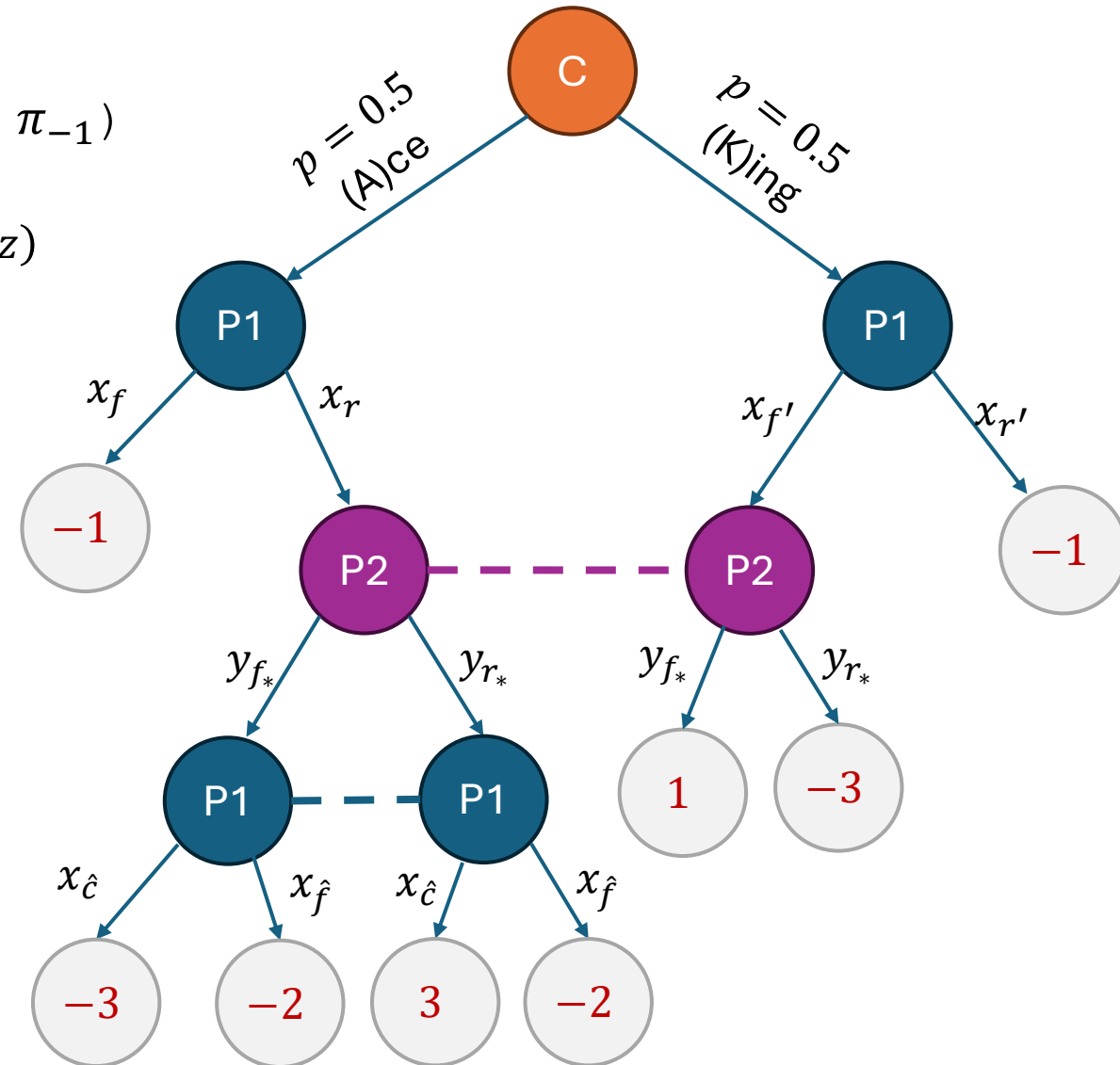
Return  $\sum_{a \in A_I} \text{Value}(ha, \pi_{-1} y_a)$

**if** Player( $I$ ) = 1

For  $a \in A_I$ :  $\tilde{u}_a += \text{Value}(ha, \pi_{-1})$

Return  $\sum_{a \in A_I} x_a \cdot \text{Value}(ha, \pi_{-1})$

Value( $\emptyset$ , 1)



# Equivalent Recursive Algorithm

$\text{CValue}(\text{ActionHistory } h, \text{AccOtherProb } \pi_{-1})$

Let  $I$  be info set corresponding to  $h$

**if**  $I$  is terminal node  $z$  return  ~~$\pi_{-1}$~~   $\cdot u(z)$

**if**  $\text{Player}(I) = \text{chance}$

Return  $\sum_{a \in A_I} \pi_a^C \cdot \text{CValue}(ha, \pi_{-1} \pi_a^C)$

**if**  $\text{Player}(I) = 2$

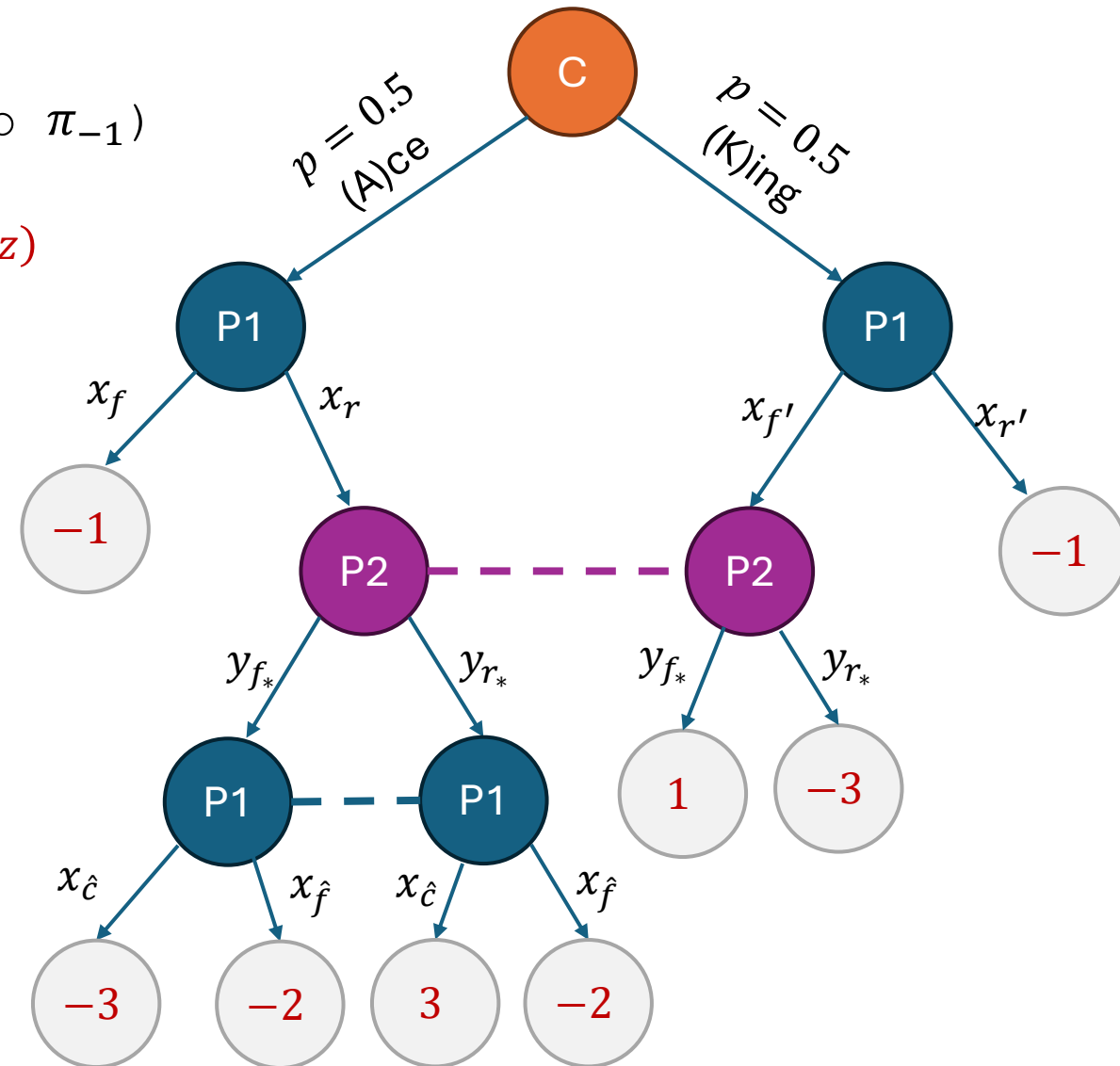
Return  $\sum_{a \in A_I} y_a \cdot \text{CValue}(ha, \pi_{-1} y_a)$

**if**  $\text{Player}(I) = 1$

For  $a \in A_I$ :  $\tilde{u}_a += \pi_{-1} \cdot \text{CValue}(ha, \pi_{-1})$

Return  $\sum_{a \in A_I} x_a \cdot \text{CValue}(ha, \pi_{-1})$

$\text{CValue}(\emptyset, 1)$



# The Typical CRM Algorithm Implementation

$\text{CValue}(\text{ActionHistory } h, \text{AccOtherProb } \pi_{-1})$

Let  $I$  be info set corresponding to  $h$

**if**  $I$  is terminal node  $z$  return  $u(z)$

**if**  $\text{Player}(I) = \text{chance}$

Return  $\sum_{a \in A_I} \pi_a^C \cdot \text{CValue}(ha, \pi_{-1} \pi_a^C)$

**if**  $\text{Player}(I) = 2$

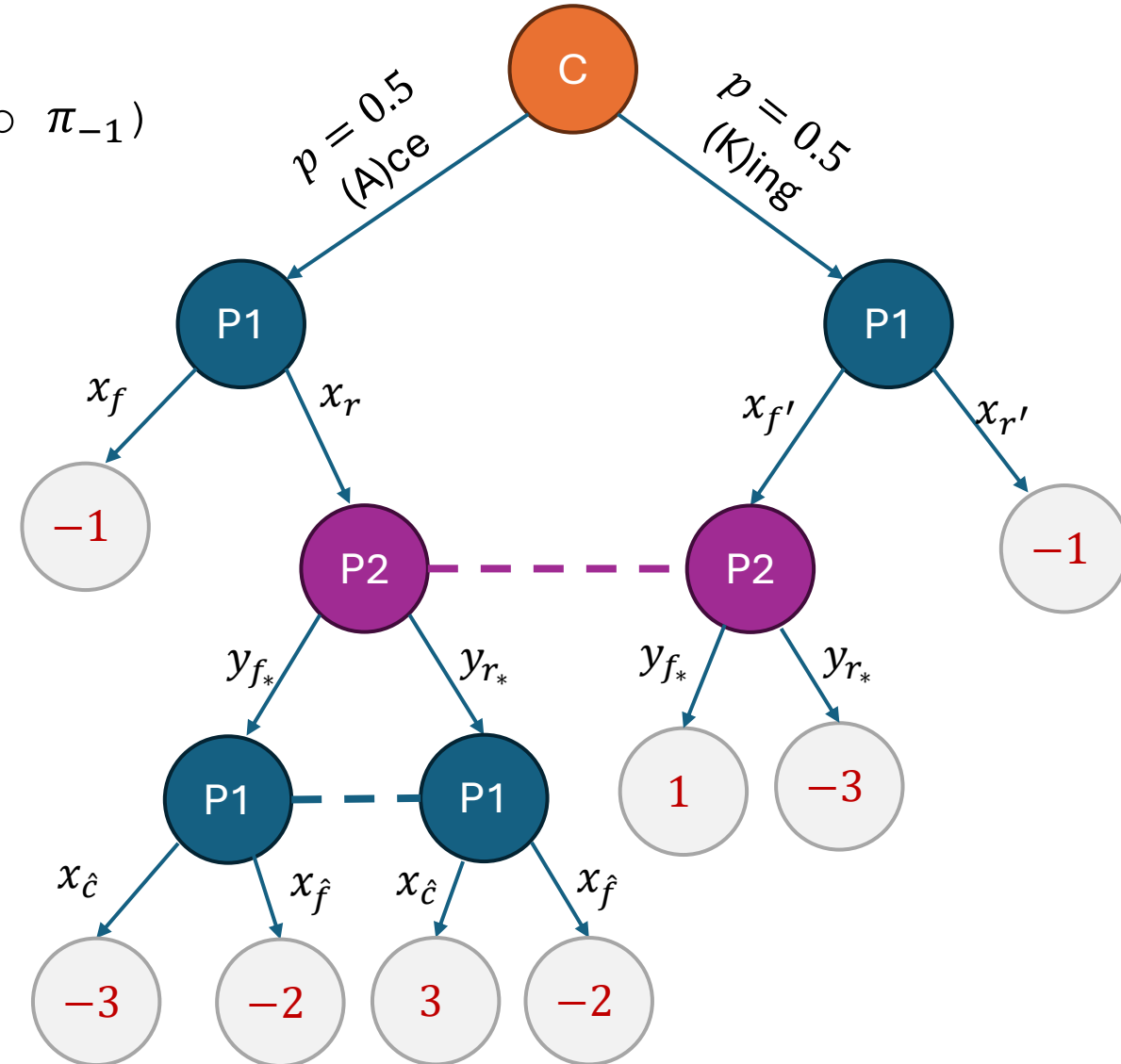
Return  $\sum_{a \in A_I} \gamma_a \cdot \text{CValue}(ha, \pi_{-1} \gamma_a)$

**if**  $\text{Player}(I) = 1$

For  $a \in A_I$ :  $\tilde{u}_a += \pi_{-1} \cdot \text{CValue}(ha, \pi_{-1})$

Return  $\sum_{a \in A_I} x_a \cdot \text{CValue}(ha, \pi_{-1})$

$\text{CValue}(\emptyset, 1)$



# Recovering Equilibrium from CRM Dynamics



We have run CRM dynamics generating behavioral strategies  $x_t, y_t$  for  $T$  periods.

How do we calculate the behavioral strategies  $x^*, y^*$  that are an approximate Nash equilibrium?

# Recovering Nash Equilibrium

- We need to translate the behavioral strategies into sequence-form

$$\forall a \in A_j: \tilde{x}_{t,a} = \tilde{x}_{t,p_j} \cdot x_t$$

- Then average the sequence-form strategies

$$\bar{\tilde{x}} = \frac{1}{T} \sum_{t=1}^T \tilde{x}_t$$

Product of probabilities of actions of player P1 on path to info set of action  $i$

- Then translate back to equilibrium behavioral strategies  $x^*$

$$\forall a \in A_j: x_a^* = \frac{\bar{\tilde{x}}_a}{\bar{\tilde{x}}_{p_j}}$$

# Recovering Nash Equilibrium

- We need to translate the behavioral strategies into sequence-form

$$\forall a \in A_j: \tilde{x}_{t,a} = \tilde{x}_{t,p_j} \cdot x_t$$

- Then average the sequence-form strategies

$$\bar{\tilde{x}} = \frac{1}{T} \sum_{t=1}^T \tilde{x}_t = \frac{1}{T} \sum_{t=1}^T \tilde{x}_{t,p_j} \cdot x_t$$

Product of probabilities of actions of player P1 on path to info set of action  $i$

- Then translate back to equilibrium behavioral strategies  $x^*$

$$\forall a \in A_j: x_a^* = \frac{\bar{\tilde{x}}_a}{\bar{\tilde{x}}_{p_j}} = \frac{\sum_{t=1}^T \tilde{x}_{t,p_j} \cdot x_{t,a}}{\sum_{t=1}^T \tilde{x}_{t,p_j}}$$

# The Typical CRM Algorithm Implementation

CValue(ActionHistory  $h$ , AccOtherProb  $\pi_{-1}$ , AccProb  $\pi_1$ )

Let  $I$  be info set corresponding to  $h$

**if**  $I$  is terminal node  $z$  return  $u(z)$

**if** Player( $I$ ) = chance

Return  $\sum_{a \in A_I} \pi_a^c \cdot \text{CValue}(ha, \pi_{-1} \pi_a^c, \pi_1)$

**if** Player( $I$ ) = 2

Return  $\sum_{a \in A_I} y_a \cdot \text{CValue}(ha, \pi_{-1} y_a, \pi_1)$

**if** Player( $I$ ) = 1

For  $a \in A_I$ :  $\tilde{u}_a += \pi_{-1} \cdot \text{CValue}(ha, \pi_{-1}, \pi_1 x_a)$

Set  $q(I) = \pi_1$

Return  $\sum_{a \in A_I} x_a \cdot \text{CValue}(ha, \pi_{-1}, \pi_1 x_a)$

This is the product of the probabilities of prior actions of player P1 before arriving at info set  $I$

**Note.** Due to perfect recall this product is the same every time we visit the info set; irrespective of which node of the info set we arrived at.

CValue( $\emptyset$ , 1)

# The Overall Equilibrium Algorithm with CRM

After each period  $t$ :

- With last period behavior strategies  $x_t, y_t$  call CValue( $\emptyset, 1, 1$ )
- Store  $\tilde{u}_{t,a}$  and  $q_t(I)$  for each action  $a$  and info set  $I$  of P1
- For each info set  $j \in \mathcal{J}_1$ :  
$$x_{t+1} \leftarrow \text{Update}(\tilde{u}_t)$$
- Symmetrically, do so for player P2

$$\forall I \in \mathcal{J}_1 \forall a \in A_I: x_a^* = \frac{\sum_t q_t(I) x_{t,a}}{\sum_t q_t(I)}$$
$$\forall I \in \mathcal{J}_2 \forall a \in A_I: y_a^* = \frac{\sum_t q_t(I) y_{t,a}}{\sum_t q_t(I)}$$

Approximate Equilibrium Strategies

What algorithm to use for local regret updates?

# Recursive Value Calculation

After each period  $t$ :

- With last period behavior strategies  $x_t, y_t$  call  $\text{CValue}(\emptyset, 1, 1)$
- Store  $\tilde{u}_{t,a}$  and  $q_t(I)$  for each action  $a$  and info set  $I$  of P1
- For each info set  $j \in \mathcal{J}_1$ :  
 $x_{t+1} \leftarrow \text{Update}(\tilde{u}_t)$
- Symmetrically, do so for player P2

Any no-regret algorithm for the  $n$ -action no-regret problem can be used, e.g. FTRL, OFTRL, EXP, etc.

What performs well in practice is what is known as Regret Matching!

$$\forall I \in \mathcal{J}_1 \forall a \in A_I: x_a^* = \frac{\sum_t q_t(I) x_{t,a}}{\sum_t q_t(I)}$$
$$\forall I \in \mathcal{J}_2 \forall a \in A_I: y_a^* = \frac{\sum_t q_t(I) y_{t,a}}{\sum_t q_t(I)}$$

Approximate Equilibrium Strategies

# Regret Matching and Regret Matching+

- Consider the  $n$  action no-regret learning setting; at each period we choose  $x_t \in \Delta(n)$ , observe utility vector  $u_t$  and get utility  $\langle x_t, u_t \rangle$
- At each period  $t$  calculate regret of not playing action  $a$

$$r_{t,a} = u_{t,a} - \langle u_t, x_t \rangle$$

- Calculate cumulative regret of not playing action  $a$

$$R_{t,a} = \sum_{\tau \leq t} r_{\tau,a} = R_{t-1,a} + r_{t,a}$$

- Choose next distribution, proportional to positive part of regret

$$x_{t+1,a} \propto [R_{t,a}]^+ := \max\{R_{t,a}, 0\}$$

- People typically refer to CFR with RegretMatching as simply “CFR”



# Regret Matching+

- Consider the  $n$  action no-regret learning setting; at each period we choose  $x_t \in \Delta(n)$ , observe utility vector  $u_t$  and get utility  $\langle x_t, u_t \rangle$
- At each period  $t$  calculate regret of not playing action  $a$

$$r_{t,a} = u_{t,a} - \langle u_t, x_t \rangle$$

- Continuously clip above zero, as you accumulate regret of  $a$

$$R_{t,a} = [R_{t-1,a} + r_{t,a}]^+$$

- Choose next distribution, proportional to  $R_{t,a}$

$$x_{t+1,a} \propto R_{t,a}$$

- Regret Matching and Regret Matching+ achieve  $\text{Regret} \leq \sqrt{n/T}$

# Extra Tricks for Empirical Improvement

# Monte-Carlo Stochastic Approximation of Utilities

- Sample chance move (e.g. sampled A)

- Go to **InfoSet 1**

$$\hat{u}_f = -1, \quad \hat{u}_r = 0$$

- Go down tree the  $r$  path

- Sample P2 move (e.g. sampled  $f_*$ )

- Go down to **InfoSet 3**

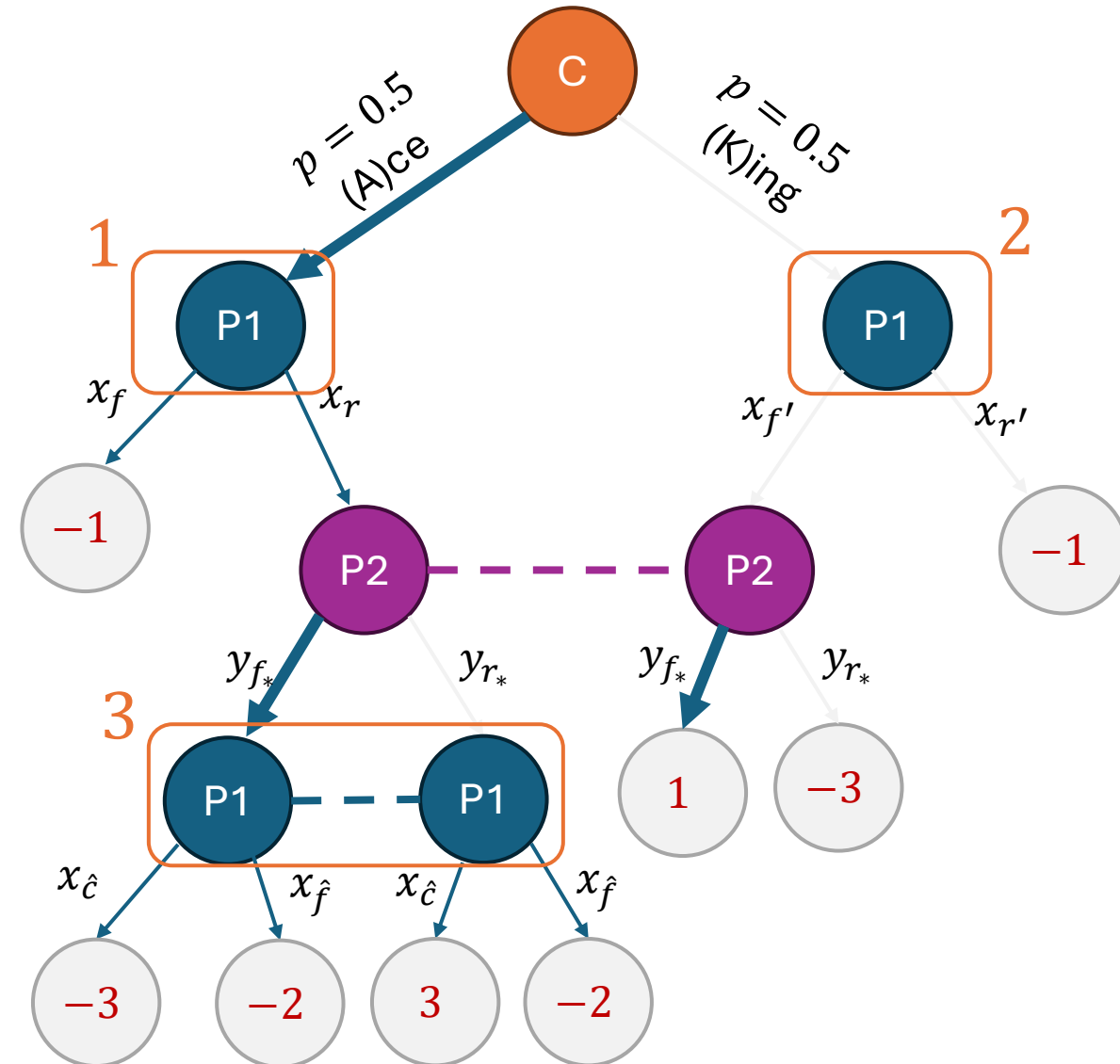
$$\hat{u}_{\hat{c}} = -3, \quad \hat{u}_{\hat{f}} = -1$$

$$\hat{u}_r += x_{\hat{c}} \hat{u}_{\hat{c}} + x_{\hat{f}} \hat{u}_{\hat{f}}$$

- Update probabilities of visited infosets

$$(x_f, x_r) \leftarrow \text{Update}(\hat{u}_f, \hat{u}_r)$$

$$(x_{\hat{c}}, x_{\hat{f}}) \leftarrow \text{Update}(\hat{u}_{\hat{c}}, \hat{u}_{\hat{f}})$$



# Typical Monte Carlo Algorithm Implementation

MCCValue(ActionHistory  $h$ , AccProb  $\pi_1$ )

Let  $I$  be info set corresponding to  $h$

**if**  $I$  is terminal node  $z$  return  $u(z)$

**if** Player( $I$ ) = chance

Sample  $a \sim \pi^C$

Return MCCValue( $ha, \pi_1$ )

**if** Player( $I$ ) = 2

Sample  $a \sim y^I$

Return MCCValue( $ha, \pi_1$ )

**if** Player( $I$ ) = 1

For  $a \in A_I$ :  $\tilde{u}_a += \text{MCCValue}(ha, \pi_1 \cdot x_a)$

Set  $q(I) = \pi_1$

Return  $\sum_{a \in A_I} x_a \cdot \text{MCCValue}(ha, \pi_1 \cdot x_a)$

Value( $\emptyset$ , 1)

# Can Combine with Update Step in One Pass

MCCValue(ActionHistory  $h$ , AccProb  $\pi_1$ )

Let  $I$  be info set corresponding to  $h$

**if**  $I$  is terminal node  $z$  return  $u(z)$

**if** Player( $I$ ) = chance

Sample  $a \sim \pi^C$

Return MCCValue( $ha, \pi_1$ )

**if** Player( $I$ ) = 2

Sample  $a \sim y^I$

Return MCCValue( $ha, \pi_1$ )

**if** Player( $I$ ) = 1

For  $a \in A_I$ :  $\tilde{u}_a += \text{MCCValue}(ha, \pi_1 \cdot x_a)$

Set  $q(I) = \pi_1$

Update  $x_{\text{next}}^I \leftarrow \text{Update}(\tilde{u}^I)$

Return  $\sum_{a \in A_I} x_a \cdot \text{MCCValue}(ha, \pi_1 \cdot x_a)$

# Alternation

After each period  $t$ :

- If  $t$  is odd then update the strategy of the  $x$ -player
- If  $t$  is even then update strategy of the  $y$ -player

For most natural algorithms, alternation can only help in terms of reducing the violation of best response constraints!

Can converge faster to equilibrium

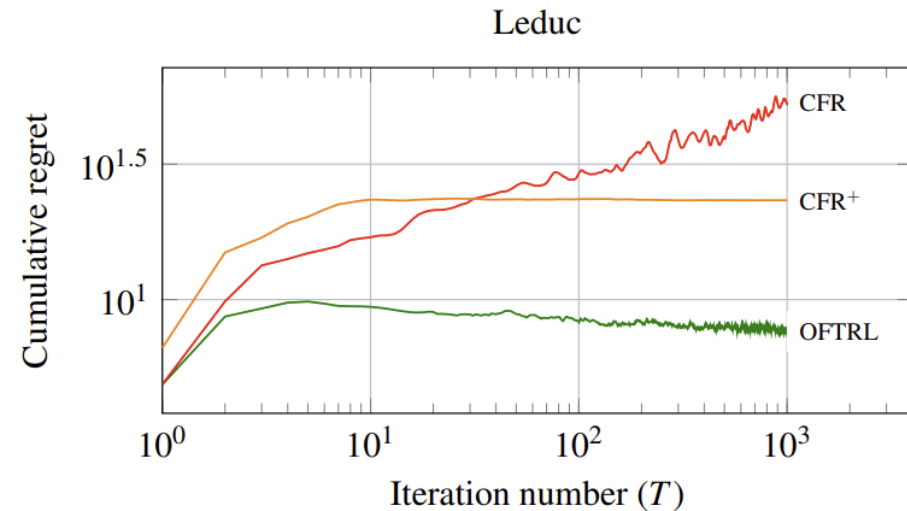
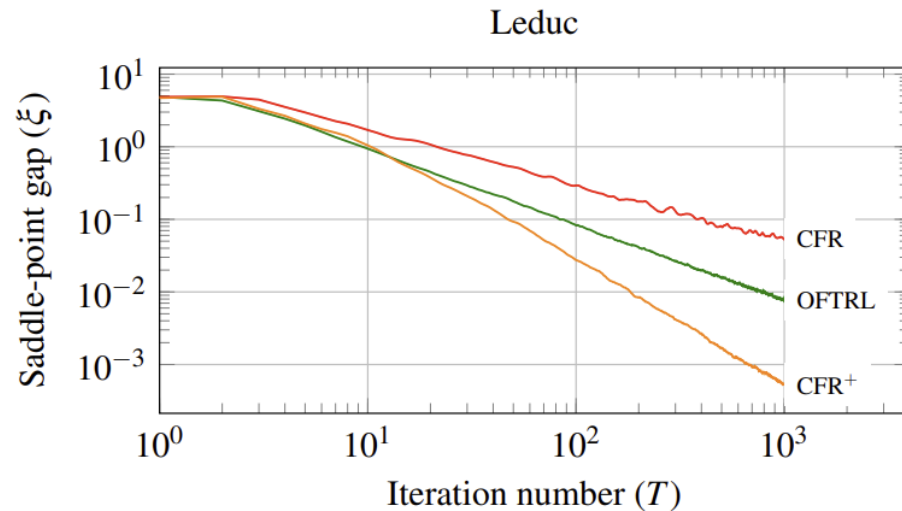
# Weighted Averaging

- Instead of uniformly weighting all rounds, put more weight on more recent rounds of play

$$\frac{1}{\sum_t t^\alpha} \sum_t t^\alpha \tilde{x}_t$$

- Typically, one uses linear averaging (i.e.,  $\alpha = 1$ )
- The CFR algorithm that uses RegretMatching+, alternation and linear averaging is typically referred to as “CFR+”

# Empirical Comparisons



Violations of best response

$$\text{Regret}_y(x_*, y_*) + \text{Regret}_x(x_*, y_*) := \max_y x_*^\top A y - x_*^\top A y_* + x_*^\top A y_* - \min_x x^\top A y_* = \max_y x_*^\top A y - \min_x x^\top A y_*$$

$$[R_y + R_x] = \max_y \bar{x}^\top A y - \frac{1}{T} \sum_t x_t^\top A y_t + \frac{1}{T} \sum_t x_t^\top A y_t - \min_x x^\top A \bar{y} = \max_y \bar{x}^\top A y - \min_x x^\top A \bar{y}$$

Sum of learning  
algorithm regrets

saddle-point gap  
of average strategies  $\bar{x}, \bar{y}$



# Elements of the Libratus AI

- The first agent to achieve superhuman performance in two player No-Limit Texas Hold'em poker ( $10^{161}$  decision points)
- Prior best was Limit Texas Hold'em ( $10^{13}$  decision points); solution is basically “run CFR+”
- For No-Limit Texas Hold'em game is too big for this approach!

# Elements of Libratus AI

