MS&E 233
Game Theory, Data Science and AI
Lecture 13

Vasilis Syrgkanis

Assistant Professor

Management Science and Engineering

(by courtesy) Computer Science and Electrical Engineering

Institute for Computational and Mathematical Engineering

Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- Basic Auctions and Learning to bid in auctions (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- Simple vs optimal mechanisms (T)
- *HW6: implement simple and optimal auctions, analyze revenue empirically*

6
- Basics of Statistical Learning Theory (T)
- Optimizing Mechanisms from Samples (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Summarizing Last Lecture

Myerson's Theorem. When valuations are independently distributed, for any BIC, NNT and IR mechanism (and any BNE of a non-truthful mechanism), the payment contribution of each player is their expected virtual value

$$E[\hat{p}_i(v_i)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)], \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Corollary. When valuations are independently distributed, for any Bayes-Nash equilibrium of any non-truthful mechanism, the payment contribution of each player is their expected virtual value

$$E[\hat{p}_i(v)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)], \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Myerson's Optimal Auction. Assuming that virtual value functions are monotone non-decreasing, the mechanism that maximizes virtual welfare, achieves the largest possible revenue among all possible mechanisms and Bayes-Nash

$$x(v) = \text{argmax}_{x \in X} \sum_i x \cdot \phi_i(v_i), \qquad p_i(v) = v_i x_i(v) - \int_0^{v_i} x_i(z, v_{-i})\, dz$$

$$\text{Rev} = E\left[\max_{x \in X} \sum_i x \cdot \phi_i(v_i)\right]$$

Optimal auction is
1) cumbersome, 2) hard to understand, 3) hard to explain, 4) does not always allocate to the highest value player, 5) discriminates a lot, 6) is many times counter-intuitive, 7) can seem unfair!

Second-Price with Player-Specific Reserves

- What if we simply run a second price auction but have different reserves for each bidder

- Each bidder i has a reserve price r_i
- Reject all bidders with bid below the reserve
- Among all bidders with value $v_i \geq r_i$, allocate to highest bidder
- Charge winner max of their reserve and the next highest surviving bid

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- Choose θ such that:

$$\Pr\left(\max_i \phi_i^+(v_i) \geq \theta\right) = 1/2$$

- Then set personalized reserve prices implied by:

$$\phi_i^+(v_i) \geq \theta \Leftrightarrow v_i \geq r_i$$

All these designs required knowledge of distributions of values F_i!

What can we do if we only have data from F_i?

Learning Auctions from Samples

Learning from Samples

- We are given a set S of m samples of value profiles

$$S = \left\{ v^j = \left(v_1^j, \ldots, v_n^j \right) \right\}_{j=1}^{m}$$

- Each sample is drawn i.i.d. from the distribution of values

$$v_i^j \sim F_i, \qquad v^j \sim \boldsymbol{F} \stackrel{\text{def}}{=} F_1 \times \cdots \times F_n$$

- Samples can be collected from historical runs of truthful auction
- Bids of each bidder in each of the m historical runs of the auction

Desiderata

- Without knowledge of distributions F_i, we want to produce a mechanism M_S, that achieves good revenue on these distributions
- For some $\epsilon(m) \to 0$ as the number of samples grows:

$$\mathrm{Rev}(M_S) \stackrel{\text{def}}{=} E_{v \sim F}\left[\sum_i p_i^{M_S}(v)\right] \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon(m)$$

- Either in expectation over the draw of the samples, i.e.

$$E_S[\mathrm{Rev}(M_S)] \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon(m)$$

- Or with high-probability over the draw of the samples, i.e.

$$\text{w. p. } 1-\delta: \quad \mathrm{Rev}(M_S) \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon_\delta(m)$$

Easy Start: Pricing from Samples

Pricing from Samples

- Suppose we have only one bidder with $v \sim F$, for simplicity in $[0, 1]$
- Optimal mechanism is to post the monopoly reserve price
- The optimal price r is the one that maximizes

$$\mathrm{Rev}(r) = E_{v \sim F}[r \cdot 1\{v \geq r\}] = r \Pr(v \geq r) = r\left(1 - F(r)\right)$$

  which is the monopoly reserve price η that solves:

$$\eta - \frac{1 - F(\eta)}{f(\eta)} = 0$$

- Choosing η requires knowledge of the CDF F and the pdf f
- Can we optimize r if we have m samples of v?

The Obvious Algorithm

- We want to choose r that maximizes

$$\max_{r\in[0,1]} \mathrm{Rev}(r) \stackrel{\text{def}}{=} E_{v\sim F}[r \cdot 1\{v \geq r\}], \qquad \text{(population objective)}$$

- With m samples S, we can optimize average revenue on samples!

$$\max_{r\in[0,1]} \mathrm{Rev}_S(r) \stackrel{\text{def}}{=} \frac{1}{m}\sum_{j=1}^{m} r \cdot 1\{v^j \geq r\}, \qquad \text{(empirical objective)}$$

- This approach is called Empirical Reward Maximization (ERM)
- **Intuition.** Since each value is drawn from distribution F the empirical average over i.i.d. draws from F, by law of large numbers, should be very close to expected value

Same as Empirical Risk Minimization (ERM) in Machine Learning (loss vs reward)

A Potential Problem with ERM

- The Law of Large Numbers applies if we wanted to evaluate the revenue of a fixed reserve price, we had in mind using the samples
- If we optimize over a very large set of reserve prices, then by random chance, it could be that we find a reserve price that has a large revenue on the samples, but small on the distribution

- This behavior is called overfitting to the samples
- We need to argue that overfitting cannot arise when we optimize over the reserve price!

Basic Elements of Statistical Learning Theory

Uniform Convergence

- Uniform Convergence. Suppose that we show that, w.p. $1-\delta$

$$\forall r \in [0,1]: \ |\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_\delta(m)$$

- Alert. Note that this is different than: $\forall r \in [0,1]$, w.p. $1-\delta$

$$|\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_\delta(m)$$

- The first asks that with probability $1-\delta$, the empirical revenue of all reserve prices is close to their population revenue
- The second asks that for a given reserve price, with probability $1-\delta$ its empirical revenue is close to its population
- The second claims nothing about the probability of the *joint event* that this is satisfied for all prices simultaneously

Uniform Converges Suffices for No-Overfitting

- Uniform Convergence. Suppose that we show that, w.p. $1-\delta$

$$\forall r \in [0,1]: \ |\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_\delta(m)$$

- Empirical Risk Maximization reserve:

$$r_S = \operatorname*{argmax}_{r \in [0,1]} \mathrm{Rev}_S(r)$$

Theorem. If uniform convergence holds then, w.p. $1-\delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m) = \mathrm{OPT}(F) - 2\epsilon_\delta(m)$$

Uniform Converges Suffices for No-Overfitting

Theorem. If uniform convergence holds then, w.p. $1-\delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m) = \mathrm{OPT}(F) - 2\epsilon_\delta(m)$$

- By uniform convergence, with probability $1-\delta$:
$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}_S(r_S) - \epsilon_\delta(m)$$
- Since, r_S optimizes the empirical objective
$$\mathrm{Rev}_S(r_S) \geq \mathrm{Rev}_S(\eta)$$
- By uniform convergence:
$$\mathrm{Rev}_S(\eta) \geq \mathrm{Rev}(\eta) - \epsilon_\delta(m)$$
- Putting it all together:
$$\mathrm{Rev}(r_s) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m)$$

This is the no-overfitting property: It **cannot be** that we found a reserve price that has *large empirical revenue* but very *small population revenue*

The *monopoly reserve* is a **feasible** reserve price but **was not chosen** by ERM. So, it must have had smaller empirical average revenue.

LLN vs Uniform Convergence

Crucial Argument: with probability $1-\delta$: $\mathrm{Rev}(r_S) \geq \mathrm{Rev}_S(r_S) - \epsilon_\delta(m)$

- Cannot be argued solely using Law of Large Numbers: if we have i.i.d. X^j with mean $E[X]$

$$\left| \frac{1}{m} \sum_{j=1}^{m} X^j - E[X] \right| \to 0$$

- For reserve price r that is chosen before looking at the samples, define $X^j(r) = r \cdot \ 1\{v^j \geq r\}$

$$|\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| = \left| \frac{1}{m} \sum_{j} r \cdot 1\{v^j \geq r\} - E[r \cdot 1\{v \geq r\}] \right| \to 0$$

- Problem. The reserve price r_S was chosen by looking at all the samples in S
  - If I tell you r_S you learn something about the samples
  - Conditional on r_S the samples are no-longer i.i.d.

- Uniform convergence, essentially means "*what I learn about S from r_S is not that much…*"

Concentration Inequalities and Uniform Convergence

- Concentration inequalities give us a stronger version of LLN

- Chernoff-Hoeffding Bound. If we have i.i.d. $X^j \in [0,1]$ with mean $E[X]$, w.p. $1-\delta$:

$$\left|\frac{1}{m}\sum_{j=1}^{m} X^j - E[X]\right| \leq \epsilon_\delta(m) \stackrel{\text{def}}{=} \sqrt{\frac{\log(2/\delta)}{2m}}$$

- Crucial. The bound grows only logarithmically with $1/\delta$

Union Bound

- Suppose we had only K possible reserve prices $\left\{\frac{1}{K}, \frac{2}{K}, \frac{3}{K} \ldots, 1\right\}$
- For each reserve price r on the grid, for any probability δ', by Chernoff bound

$$\Pr\left((\text{Bad Event})_r\right) = \Pr\left(\left|\frac{1}{m}\sum_{j=1}^{m} X^j(r) - E[X(r)]\right| > \epsilon_{\delta'}(m)\right) \leq \delta'$$

- Union Bound. The probability of the union of events is at most the sum of the probabilities

$$\Pr\left(\cup_{r=1}^{K} (\text{Bad Event})_r\right) \leq \sum_{r=1}^{K} \Pr\left((\text{Bad Event})_r\right) \leq K \cdot \delta'$$

- Apply Chernoff bound with $\delta' = \delta/K$

$$\Pr\left(\cup_{r=1}^{K} (\text{Bad Event})_r\right) \leq \delta$$

- Probability(*exists reserve price whose empirical revenue is far from its population*) at most δ

Uniform Convergence via Union Bound

Theorem. Suppose we had K possible reserve prices $\text{Grid}_K \stackrel{\text{def}}{=} \left\{\frac{1}{K}, \frac{2}{K}, \frac{3}{K} \ldots, 1\right\}$

Then with probability at least $1 - \delta$

$$\forall r \in \text{Grid}_K: |\text{Rev}_S(r) - \text{Rev}(r)| \leq \epsilon_{\delta/K}(m) \stackrel{\text{def}}{=} \sqrt{\frac{\log(2K/\delta)}{2m}}$$

Problem. The optimal reserve η can potentially not be among these K reserves

Intuition. For a sufficiently large K, for any reserve price, we can find a reserve price on this discretized grid that achieves almost as good revenue

We don't lose much by optimizing over the grid!

Discretization

- For a reserve price r, pick largest reserve price below r on the grid
- Denote this discretization of r as r_K

- By doing so, you allocate to any value you used to allocate before
- For any such value you receive revenue at least $r - 1/K$

- Overall, you lose revenue at most $1/K$

$$\mathrm{Rev}(r_K) \geq \mathrm{Rev}(r) - 1/K$$

Discretized ERM

- Let's modify ERM to optimize only over the grid

$$r_S = \max_{r \in \text{Grid}_K} \text{Rev}_S(r)$$

- We can apply the uniform convergence over the grid

$$\text{Rev}(r_S) \geq \text{Rev}_S(r_S) - \epsilon_{\delta/K}(m)$$

We cannot overfit, when optimizing over the grid of reserves

- Since, r_S optimizes the empirical objective over the grid

$$\text{Rev}_S(r_S) \geq \text{Rev}_S(\eta_K)$$

The *discretized monopoly reserve* is a **feasible** reserve in the grid but **was not chosen** by ERM.

- By uniform convergence over the grid:

$$\text{Rev}_S(\eta_K) \geq \text{Rev}(\eta_K) - \epsilon_{\delta/K}(m)$$

- By the discretization error argument:

$$\text{Rev}(\eta_K) \geq \text{Rev}(\eta) - 1/K$$

Theorem. The revenue of the reserve price output by discretized ERM over the K-grid satisfies, with probability $1 - \delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{OPT}(F) - 2\sqrt{\frac{\log(2K/\delta)}{2m}} - \frac{1}{K}$$

Choosing $K = 1/m$

$$\mathrm{Rev}(r_S) \geq \mathrm{OPT}(F) - 3\sqrt{\frac{\log(2m/\delta)}{2m}}$$

Desideratum satisfied!
$\epsilon_\delta(m) \to 0$ as m grows

The Limits of Discretization

- Do we really need to optimize over the discrete grid?
- What if we insist on optimizing over $[0,1]$. Can we still overfit?

- Now that we have infinite possible reserves, we cannot apply the union bound argument ($K = \infty$)!
- How do we argue about optima over continuous, infinite cardinality spaces?

Sneak Peek

- Would have been ideal if we only have to argue about behavior of our optimization space, *on the given set of samples*
- As opposed to the unknown distribution of values

- What if we can find a small set of reserves and argue that for all reserves there is an approximately equivalent one in the small set, in terms of revenue on the samples

- Maybe then it suffices to invoke the union bound over the smaller space, even though we optimize over the bigger space

Statistical Learning Theory

General Framework

- Given samples $S = \{v_1, \ldots, v_m\}$ that are i.i.d. from distribution F
- Given a hypothesis/function space H
- Given a reward function $r(v; h)$

- Goal is to maximize the expected reward over distribution F

$$R(h) = E_{v \sim F}[r(v; h)]$$

Desiderata

- Without knowledge of distribution F, we want to produce a hypothesis h_S, that achieves good reward on this distribution
- For some $\epsilon(m) \to 0$ as the number of samples grows:

$$R(h_S) \stackrel{\text{def}}{=} E_{v\sim F}[r(v;h)] \geq \max_{h\in H} R(h) - \epsilon(m)$$

- Either in expectation over the draw of the samples, i.e.

$$E_S[R(h_S)] \geq \max_{h\in H} R(h) - \epsilon(m)$$

- Or with high-probability over the draw of the samples, i.e.

$$\text{w. p. } 1-\delta: \quad R(h_S) \geq \max_{h\in H} R(h) - \epsilon_\delta(m)$$

Desiderata (Mechanism Design from Samples)

- Without knowledge of Distribution of value profiles F, we want to produce a hypothesis h_S, that achieves good Revenue on this distribution
- For some $\epsilon(m) \to 0$ as the number of samples grows:

$$R(h_S) \stackrel{\text{def}}{=} E_{v \sim F}\left[\sum_i p_i(v)\right] \geq \max_{h \in H} R(h) - \epsilon(m)$$

- Either in expectation over the draw of the samples, i.e.

$$E_S[R(h_S)] \geq \max_{h \in H} R(h) - \epsilon(m)$$

- Or with high-probability over the draw of the samples, i.e.

$$\text{w. p. } 1-\delta: \quad R(h_S) \geq \max_{h \in H} R(h) - \epsilon_\delta(m)$$

The Obvious Algorithm

- We want to choose r that maximizes

$$\max_{h \in H} R(h) \stackrel{\text{def}}{=} E_{v \sim F}[r(v; h)], \qquad \text{(population objective)}$$

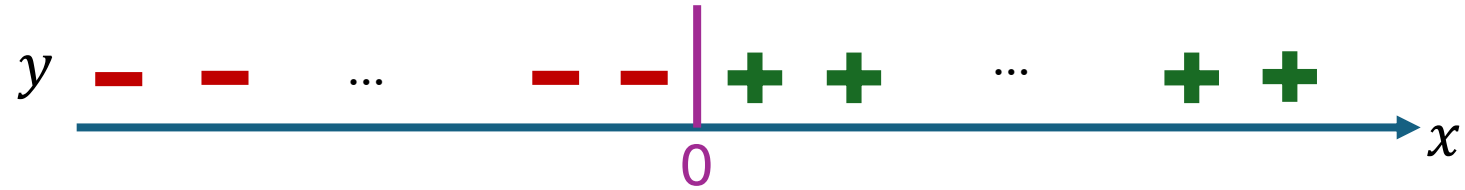
- With m samples, we can optimize average reward on samples!

$$\max_{h \in H} R_S(h) \stackrel{\text{def}}{=} \frac{1}{m} \sum_{j=1}^{m} r(v_j; h), \qquad \text{(empirical objective)}$$

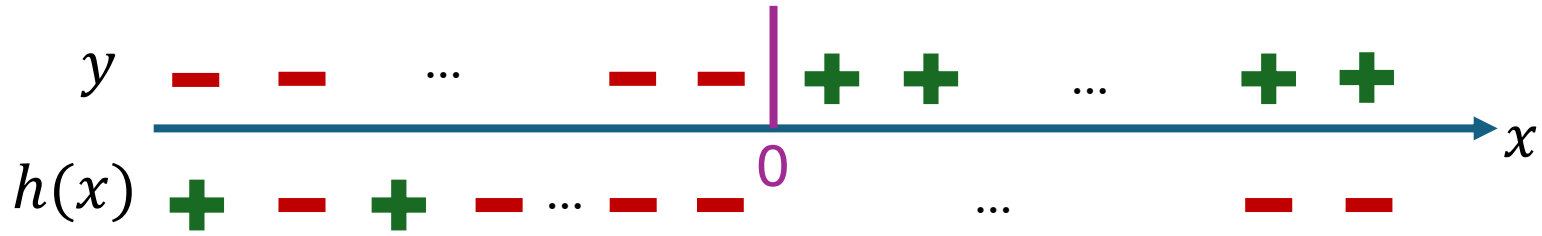
- This approach is called Empirical Reward Maximization (ERM)
- **Intuition.** Since each value is drawn from distribution F the empirical average over i.i.d. draws from F, by law of large numbers, should be very close to expected value

Standard Classification Example

- Suppose samples $v = (x, y)$ where $x \sim U[-1,1]$ and $y \in \{-1, 1\}$



- We want to choose a "labeling" function $h(x) \in \{-1,1\}$

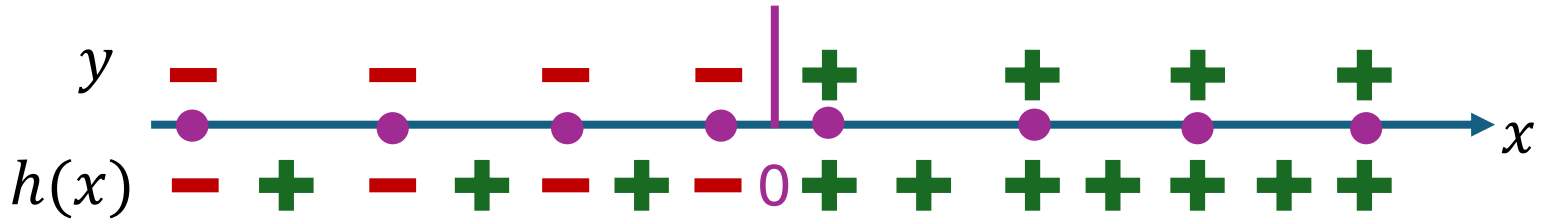


- That achieves good accuracy

$$r(v; h) = 1\{h(x) = y\}$$

ERM Gone Bad

- Suppose we choose the following h_S: label all samples correctly and predict $+1$ for any value that is not on the samples

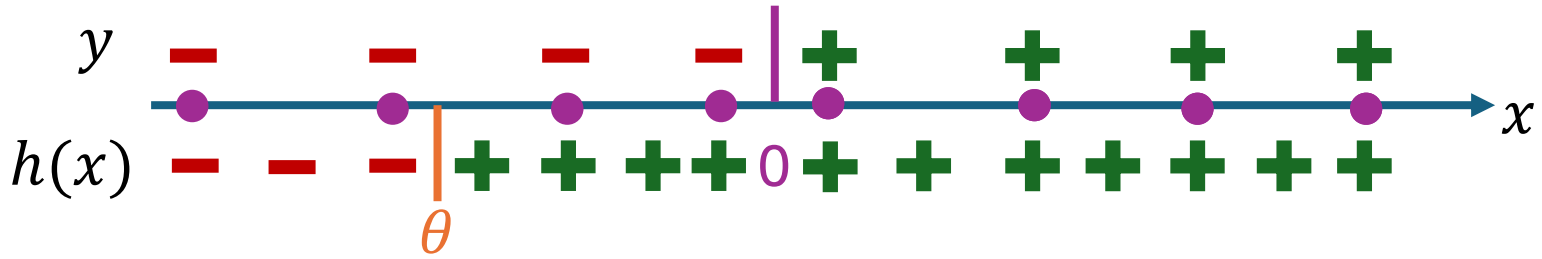


- The empirical average reward of this h_S is 1. The largest possible!
- The expected reward of this h_S is ½
- The discrepancy between the empirical reward of the ERM solution and its population reward never vanishes! Overifitting!

ERM Over Threshold Functions

- Suppose we restrict to optimizing over threshold functions
- Label every $x \geq \theta$ with $+1$ and every $x \leq \theta$ with -1

$$H = \{x \to 1(x > \theta) - 1(x \leq \theta) : \theta \in \Theta\}$$

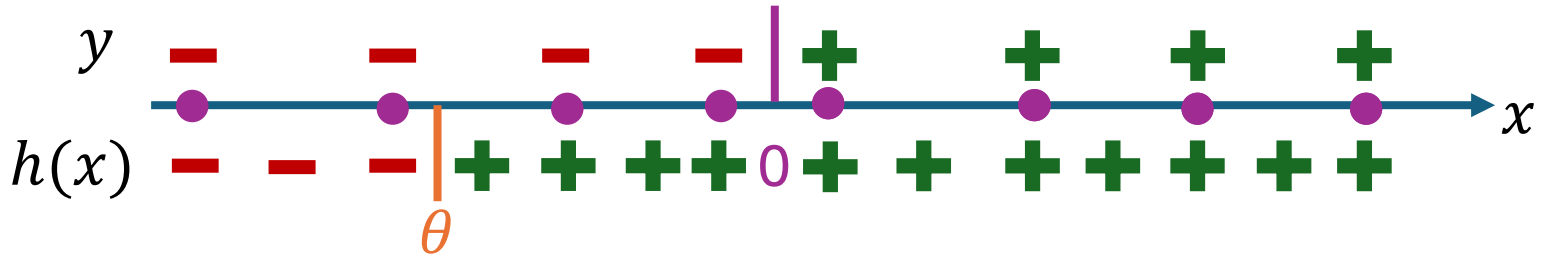


- Optimizing over such θ we will never be able to overfit
- How do we argue this?
- Discretization argument fails!
- No matter how we discretize, there exists a distribution of x that will have a very large discretization error

Sufficient Hypothesis Subspace on Samples

- Suppose we restrict to optimizing over threshold functions
- Label every $x \geq \theta$ with $+1$ and every $x < \theta$ with -1

$$H = \{x \to 1(x \geq \theta) - 1(x < \theta) : \theta \in \Theta\}$$



- Given the m samples, then on the samples there are at most $m + 1$ equivalent hypothesis: choose the threshold on the sample (or $\theta = 1$)
- Every other hypothesis produces the exact same labeling of the samples and achieves the same empirical reward
- Is there an argument that only takes union bound over this set?

Back to the General Framework

- We will try to argue the expected performance

$$E_S[R(h_S)] \geq \max_{h \in H} R(h) - \epsilon(m)$$

- Expected Sample Average *Representativeness*: suppose that

$$\text{Rep} = E_S\left[\sup_{h \in H} R_S(h) - R(h)\right] \leq \epsilon(m)$$

How good is the sample average objective in terms of representing the population expectation, in the worst case over H

- Then we can prove expected error of $\epsilon(m)$

$$E_S[R(h_S)] = E[R_S(h_S)] - E[R_S(h_S) - R(h_S)] \geq E[R_S(h_S)] - \epsilon(m)$$

- Since h_S optimizes $R_S(h)$ and $h_* = \text{argmax}_{h \in H} R(h)$ is feasible

$$E[R_S(h_S)] \geq E[R_S(h_*)] = R(h_*)$$

h_* does not depend on the samples

$$E[R_S(h_*)] \stackrel{\text{def}}{=} \frac{1}{m}\sum_j E\left[h\left(v^j; h_*\right)\right] = E[h(v; h_*)] = R(h_*)$$

If we can bound representativeness

$$\text{Rep} = E_S\left[\sup_h R_S(h) - R(h)\right] \leq \epsilon(m)$$

Then we can bound expected performance

$$E[R(h_S)] \geq E[R(h_*)] - \epsilon(m)$$