MS&E 233
Game Theory, Data Science and AI
Lecture 11

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(by courtesy) Computer Science and Electrical Engineering

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Computational Game Theory for Complex Games

1

- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2

- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3

- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4

- Basics and applications of auction theory (T+A)
- Basic Auctions and Learning to bid in auctions (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5

- **Optimal auctions and mechanisms (T)**
- Simple vs optimal mechanisms (T)
- *HW6: calculate equilibria in simple auctions, implement simple and optimal auctions, analyze revenue empirically*

6

- Optimizing mechanisms from samples (T)
- Online optimization of auctions and mechanisms (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7

- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

No-Regret Learning with Bandit Feedback

At each period t

- Adversary chooses a loss vector $\ell_t \in [0,1]^N$
- I choose an action i_t (not knowing ℓ_t)
- I observe loss of my chosen action $\ell_t^{i_t}$

- I want to guarantee small expected regret with any fixed action:

$$\max_{i \in N} E\left[\frac{1}{T}\sum_{t=1}^{T} \ell_t^{i_t} - \ell_t^i\right] \le \epsilon(T)$$

Constructing Un-biased Estimates of Vector

- There is a hidden loss vector $\ell_t = (\ell_t^1, \ldots, \ell_t^N)$ (potential outcomes)
- At each period I choose action (treatment) j with probability p_t^j
- I learn the loss ℓ_t^j with probability p_t^j
- Remember: no-regret algorithms work well, even if we have unbiased proxies of the true losses (e.g. Monte Carlo CFR)

Question. Can I construct a random variable that guarantees that in expectation over the choice of actions?

$$E[\tilde{\ell}_t] = \ell_t \Leftrightarrow \forall j\colon\ E\left[\tilde{\ell}_t^j\right] = \ell_t^j$$

Constructing Un-biased Estimates of Vector

Question. Can I construct a random variable that guarantees that in expectation over the choice of actions?

$$E\left[\tilde{\ell}_t\right] = \ell_t \Leftrightarrow \forall j\colon\ E\left[\tilde{\ell}_t^j\right] = \ell_t^j$$

- Random variable can always depend on identity of chosen action j_t. When I choose j random variable can also depend on ℓ_t^j

$$\tilde{\ell}_t^j = 1\{j_t = j\} f_j\left(\ell_t^j\right) + 1\{j_t \neq j\} g_j(j_t)$$

- Let's make g_j zero, and f_j linear in ℓ_t^j

$$\tilde{\ell}_t^j = 1\{j_t = j\} a_j \ell_t^j \Rightarrow E\left[\tilde{\ell}_t^j\right] = p_t^j a_j \ell_t^j = \ell_t^j \Rightarrow a_j = \frac{1}{p_t^j}$$

Inverse Propensity Estimates

At each period t

- Consider the random variables

$$\tilde{\ell}_t^j = \frac{1\{j_t = j\}}{p_t^j} \ell_t^j$$

- The vector $\tilde{\ell}_t$ can always be calculated $\left(0, \dots, 0, \frac{\ell_t^{j_t}}{p_t^{j_t}}, 0, \dots, 0\right)$
- The vector $\tilde{\ell}_t$ is an unbiased proxy of the true loss vector:

$$E\left[\tilde{\ell}_t\right] = \ell_t$$

The EXP Algorithm with Bandit Feedback

```
Initialize pt to the uniform distribution
For t in 1..T
    Draw action jt based on distribution pt
    Observe loss of chosen action lt[jt]

    Construct un-biased proxy loss vector
      ltproxy[j] = 1(jt=j) * lt[jt] / pt[jt]

    Update probabilities based on EXP update
      pt = pt * exp(-eta * ltproxy)
      pt = pt / sum(pt)
```

Recap: Regret of FTRL

(FTRL) $$x_t = \operatorname*{argmin}_{x \in X} \boxed{\sum_{\tau < t} \langle x, \ell_\tau \rangle} + \boxed{\frac{1}{\eta} \mathcal{R}(x)}$$

1-strongly convex function of x that stabilizes the maximizer

Historical performance of always choosing strategy x

Theorem. Assuming the utility function at each period

$$f_t(x) = \langle x, \ell_t \rangle$$

is L-Lipschitz with respect to some norm $\|\cdot\|$ and the regularizer is 1-strongly convex with respect to the same norm then

$$\text{Regret} - \text{FTRL}(T) \leq \boxed{\eta L} + \boxed{\frac{1}{\eta T} \left(\max_{x \in X} \mathcal{R}(x) - \min_{x \in X} \mathcal{R}(x) \right)}$$

Average stability induced by regularizer

Average loss distortion caused by regularizer

Problem! The loss vector $\tilde{\ell}_t$ is not in $[0,1]$.

It can take huge values, as probability of an action goes to 0!

Intuition: if probability goes to 0, then this action is chosen very infrequently. The loss vector very rarely takes this large value, i.e., the *variance* of the loss should be small.

Variance of Loss Vector

- Variance is

$$E\left[\left(\tilde{\ell}_t^j\right)^2\right] - E\left[\tilde{\ell}_t^j\right]^2 = E\left[\left(\tilde{\ell}_t^j\right)^2\right] - E\left[\ell_t^j\right]^2$$

- Second term is in $[0, 1]$. We will focus on first term (call it “variance”)

$$E\left[\left(\tilde{\ell}_t^j\right)^2\right] = p_t^j\left(\frac{\ell_t^j}{p_t^j}\right)^2 = \frac{\left(\ell_t^j\right)^2}{p_t^j}$$

- And we collect this “variance” term only when end up choosing j

$$\text{Average "Variance"} = \sum_j p_t^j \cdot E\left[\left(\tilde{\ell}_t^j\right)^2\right] = \sum_j \left(\ell_t^j\right)^2 \le N$$

Recap: Regret of FTRL

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1-strongly convex function of x that stabilizes the maximizer

Historical performance of always choosing strategy x

Can we replace L with the Average "Variance"?

Theorem. Assuming the utility function at each period

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Update: Regret of EXP

(EXP) $$p_t = \operatorname*{argmin}_{p\in\Delta} \sum_{\tau<t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta}\mathcal{R}(p)} \begin{pmatrix}\text{Negative}\\ \text{Entropy}\end{pmatrix} \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

$$p_t \propto p_{t-1} \exp\left(-\eta\, \tilde{\ell}_{t-1}\right)$$

Theorem. Assuming $\tilde{\ell}_t$ are random proxies that, conditional on history, have expected value equal to true loss vector ℓ_t and $\tilde{\ell}_t \geq 0$, then regret of EXP is bounded as:

$$\text{Regret} - \text{EXP}(T) \leq \frac{\eta}{T} \sum_t E\left[\sum_j p_t^j \left(\tilde{\ell}_t^j\right)^2\right] + \frac{\log(N)}{\eta T}$$

See Lemma 20 and Theorem 2.22 and Corollary 4.2 of mal-018.dvi (huji.ac.il)

Update: Regret of EXP

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Expected Average
"Variance"?

Update: Regret of EXP

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For the inverse propensity proxies

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$$\text{Regret} - \text{EXP}(T) \leq \eta N + \frac{\log(N)}{\eta T} \Rightarrow \text{Regret} - \text{EXP}(T) \lesssim \sqrt{\frac{N \log(N)}{T}}$$

For $\eta \sim \sqrt{\frac{\log(N)}{NT}}$

See Lemma 20 and Theorem 2.22 and Corollary 4.2 of mal-018.dvi (huji.ac.il)

Back to Bandit Learning in Auctions

Bandit Learning in Auctions

- Want to choose my bids b_i^t, based on algorithm that guarantees

$$\frac{1}{T}\sum_{t=1}^{T} u_i\left(b^t\right) \geq \max_{b_i \in [N]} \frac{1}{T}\sum_{t=1}^{T} u_i\left(b_i, b^t\right) - \epsilon(T)$$

- We can apply EXP3 algorithm for each bidder
- We now have utilities, but EXP3 expects non-negative losses

$$\text{Maximizing utility } = \text{Minimizing (negative utility)}$$

- However, to ensure losses are non-negative, add a large enough offset

$$\text{loss } = \text{ H } - \text{ utility}$$

- If for instance we know that utility $\leq H$, we can choose this H above

Update: Regret of EXP

(EXP) $$p_t = \underset{p \in \Delta}{\operatorname{argmin}} \sum_{\tau < t} \langle p, \tilde{\ell}_\tau \rangle + \boxed{\frac{1}{\eta} \mathcal{R}(p)} \begin{pmatrix} \text{Negative} \\ \text{Entropy} \end{pmatrix} \quad \mathcal{R}(p) = \sum_{i=1}^{n} p_i \log(p_i)$$

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Sum: Vickrey-Clarke-Groves (VCG)

A universal welfare maximizing auction/mechanism!

For any mechanism design setting, it guarantees that:

1. It is dominant strategy truthful
2. It always chooses the welfare maximizing outcome/allocation
3. All bidders have non-negative utility
4. All payments are non-negative

For special case of single-item auction = Second-Price Auction

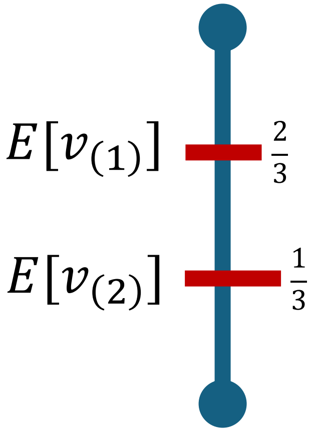
What if we want to maximize revenue?

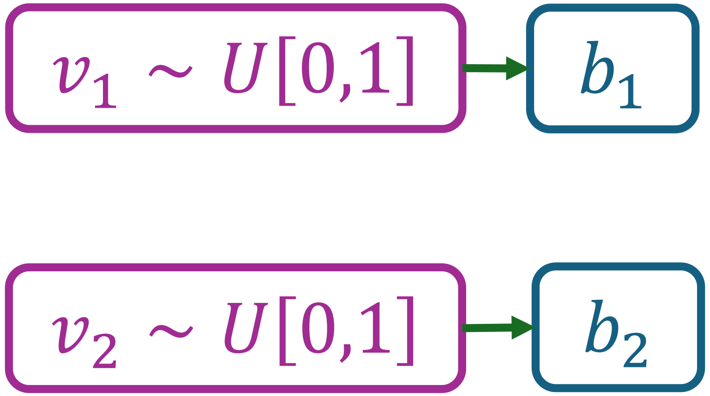
Let’s go back to basics: Single-Item Auction

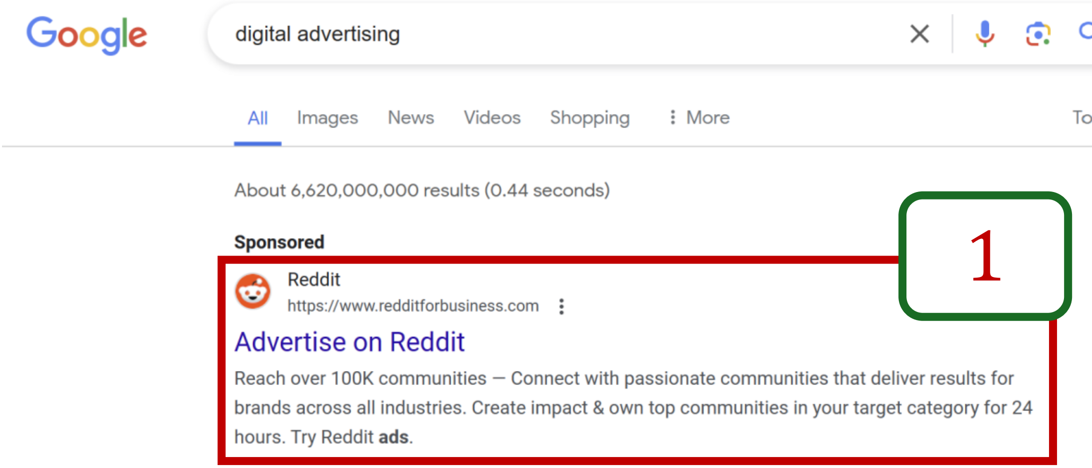
- How much revenue does the second-price auction achieve?

$$\text{Rev} = E\left[v_{(2)}\right] = E[\min(v_1, v_2)] = 1/3$$

- Can we do better?







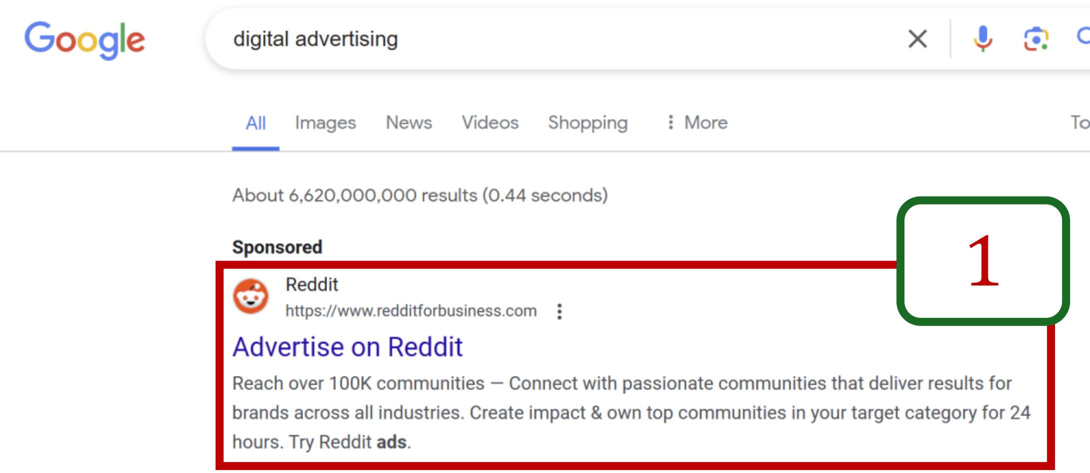
Let’s go back to basics: Single-Item Auction

- What if we only had one bidder?

$$\text{Rev} = E\left[v_{(2)}\right] = 0$$

- Can we do better?

$$\boxed{v_1 \sim U[0,1]} \rightarrow \boxed{b_1}$$

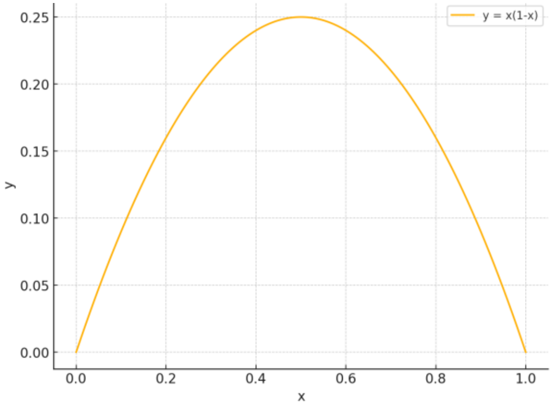


What if we post a reserve price?

Let's go back to basics: Single-Item Auction

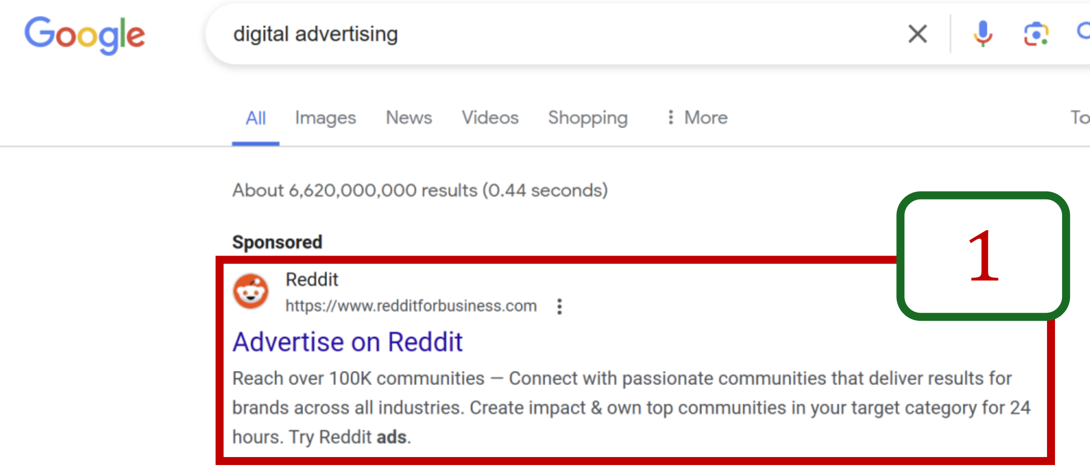
- **Auctioneer:** "If you bid less than r, I will not accept your bid and not show any ad on the page! If you win you must pay r."

  $$\mathrm{Rev(r)} = E[\mathrm{r}\,1(\mathrm{v} \geq \mathrm{r})] = r\,(1-r) \Rightarrow \mathrm{Rev}(1/2) = 1/4$$



- Is the auction truthful?
- Is the auction efficient?

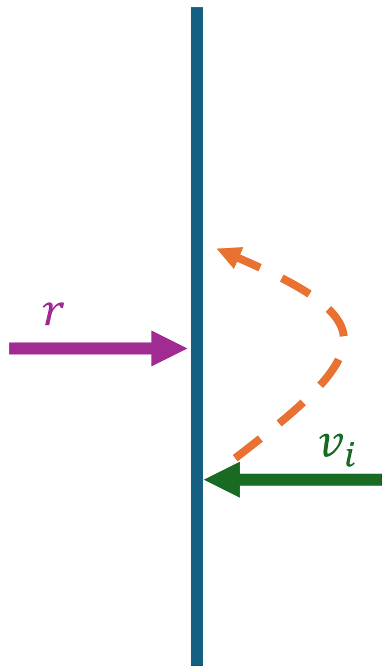
$$\boxed{v_1 \sim U[0,1]} \rightarrow \boxed{b_1}$$



Truthfulness of Mechanism

Suppose I bid my value. Would I want to deviate?

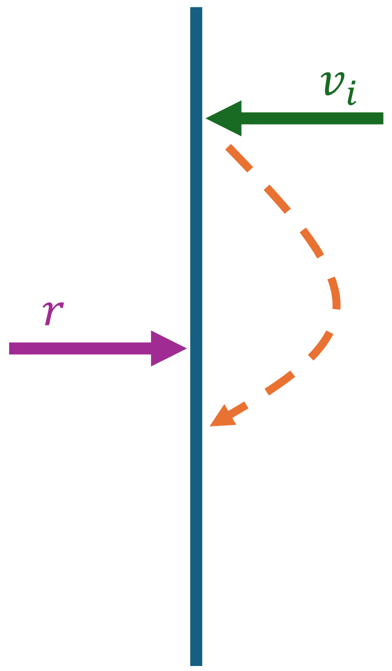
- **Case 1.** My value is below reserve price
- Only way to change anything is bid above
- But then I get negative utility as I pay more than value



Truthfulness of Mechanism

Suppose I bid my value. Would I want to deviate?

- **Case 2.** My value is above reserve price
- I get non-negative utility
- Only way to change anything is bid below
- But then I get zero utility as I lose



Is the mechanism efficient?

Yes

No

Is the mechanism efficient?

Yes 0%

No 0%

Is the mechanism efficient?

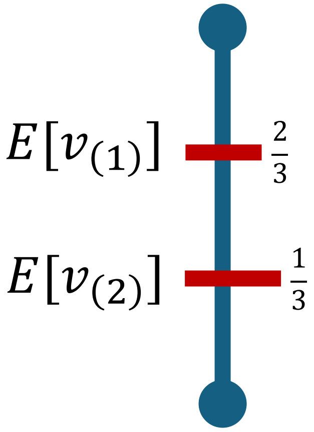
Yes	0%
No	0%

Let's go back to basics: Single-Item Auction

- How much revenue does the second-price auction achieve?

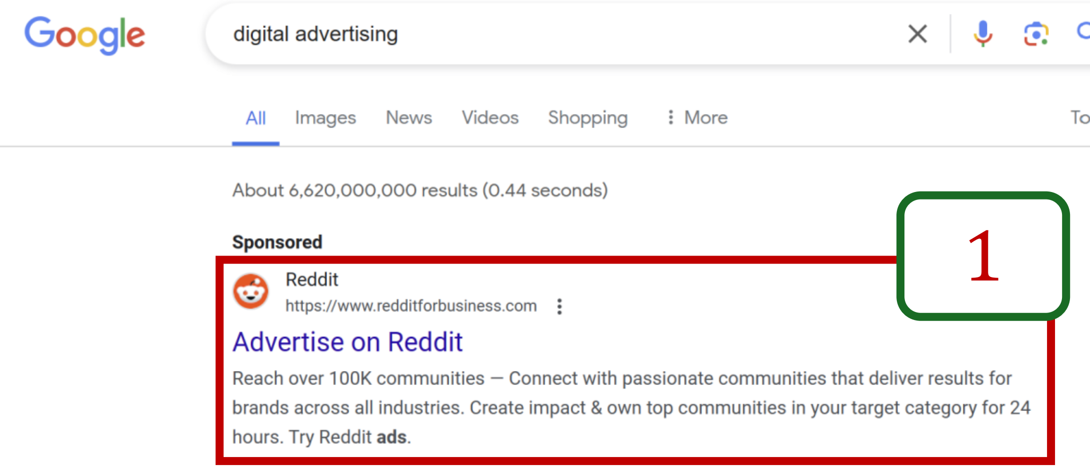
$$\text{Rev} = E\left[v_{(2)}\right] = E[\min(v_1, v_2)] = 1/3$$

- Can we do better?



$$\boxed{v_1 \sim U[0,1]} \rightarrow \boxed{b_1}$$

$$\boxed{v_2 \sim U[0,1]} \rightarrow \boxed{b_2}$$

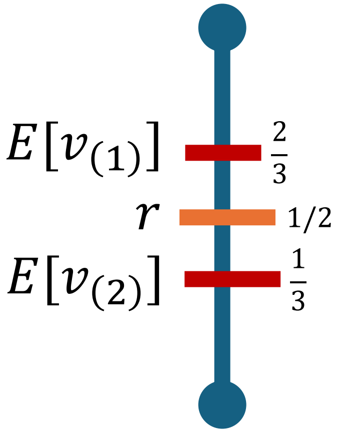


Let’s go back to basics: Single-Item Auction

- **Auctioneer:** “If you bid less than r, I will not accept your bid! If you win you must pay $\max(\mathrm{b}_2, r)$.”

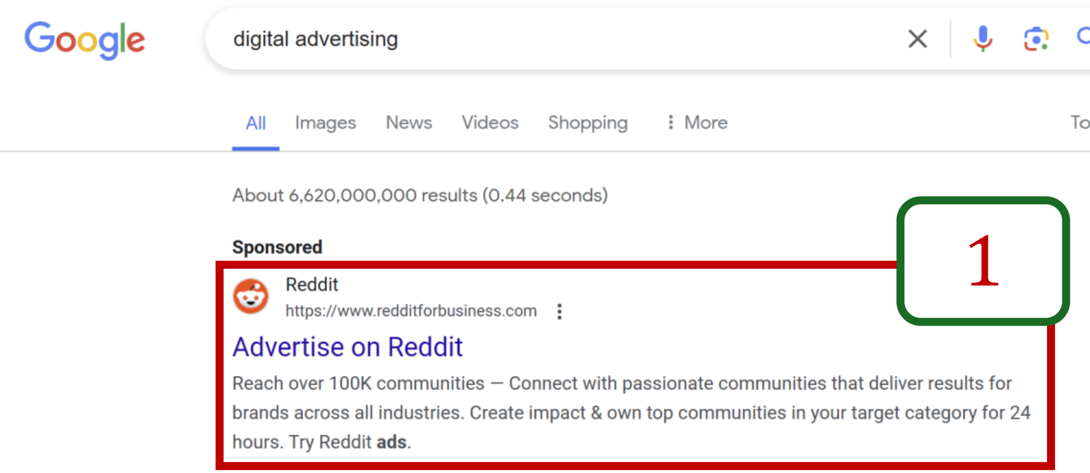
$$\mathrm{Rev}(1/2) = E\left[\max\left(v_{(2)}, r\right) 1\left(v_{(1)} \geq r\right)\right] = 5/12$$

- Can we do better?



$v_1 \sim U[0,1] \rightarrow b_1$

$v_2 \sim U[0,1] \rightarrow b_2$



How do we optimize over all possible mechanisms!

Single-Parameter Settings

- Each bidder has some value v_i for being allocated
- Bidders submit a reported value b_i (without loss of generality)
- Mechanism decides on an allocation $x \in X \subseteq \{0,1\}^n$
- Mechanism fixes a probabilistic allocation rule:
$$x(b) \in \Delta(X)$$

- **First question.** Given an allocation rule, when can we find a payment rule p so that the overall mechanism is truthful?
- If we can find such a payment, we will say that x is implementable

Some Shorthand Notation

- Let's fix bidder i and what other bidders bid b_{-i}
- For simplicity of notation, we drop index i and b_{-i}

- What properties does the function

$$x(v) \equiv x_i(v, b_{-i})$$

need to satisfy, so that x is implementable?

- Can we find a truthful payment function

$$p(v) \equiv p(v, b_{-i})$$

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

Yes

No

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

Yes		0%
No		0%

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

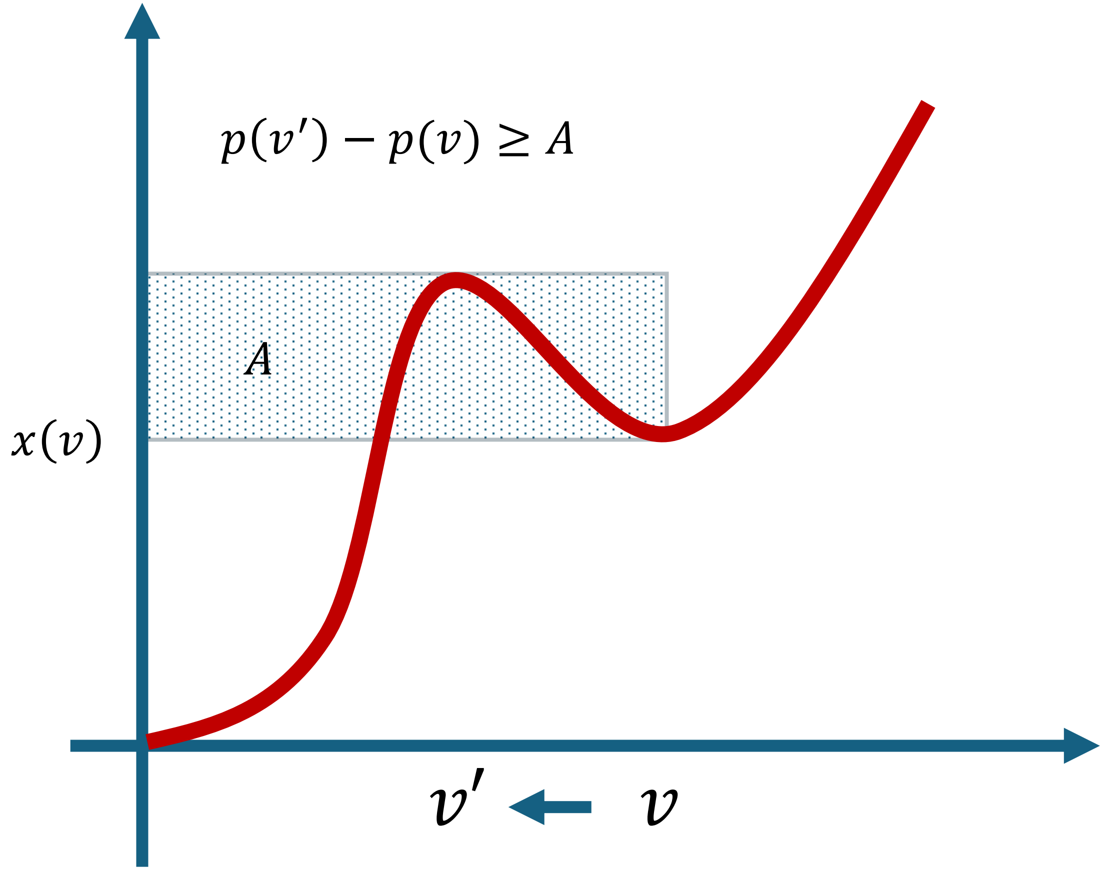
Yes	0%
No	0%

Suppose it is possible

- Suppose that we both bid truthfully
- Suppose that I am the highest value bidder
- No matter what the payment rule is, I can always reduce my bid to the second highest bid minus ϵ
- By doing so, I am paying at most the second highest bid and I am winning deterministically

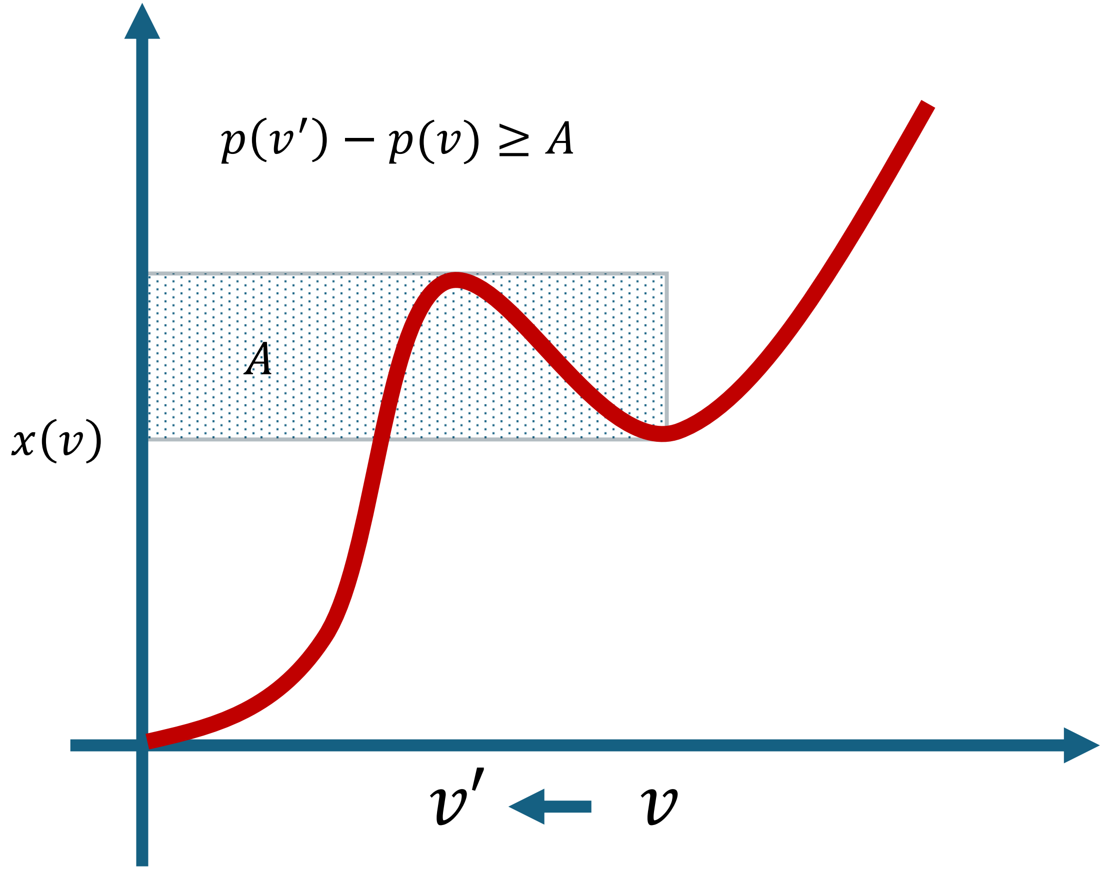
Implementable Rules are Monotone

$$v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$$



Implementable Rules are Monotone

$$p(v') - p(v) \geq v \cdot \left(x(v') - x(v)\right)$$



Implementable Rules are Monotone

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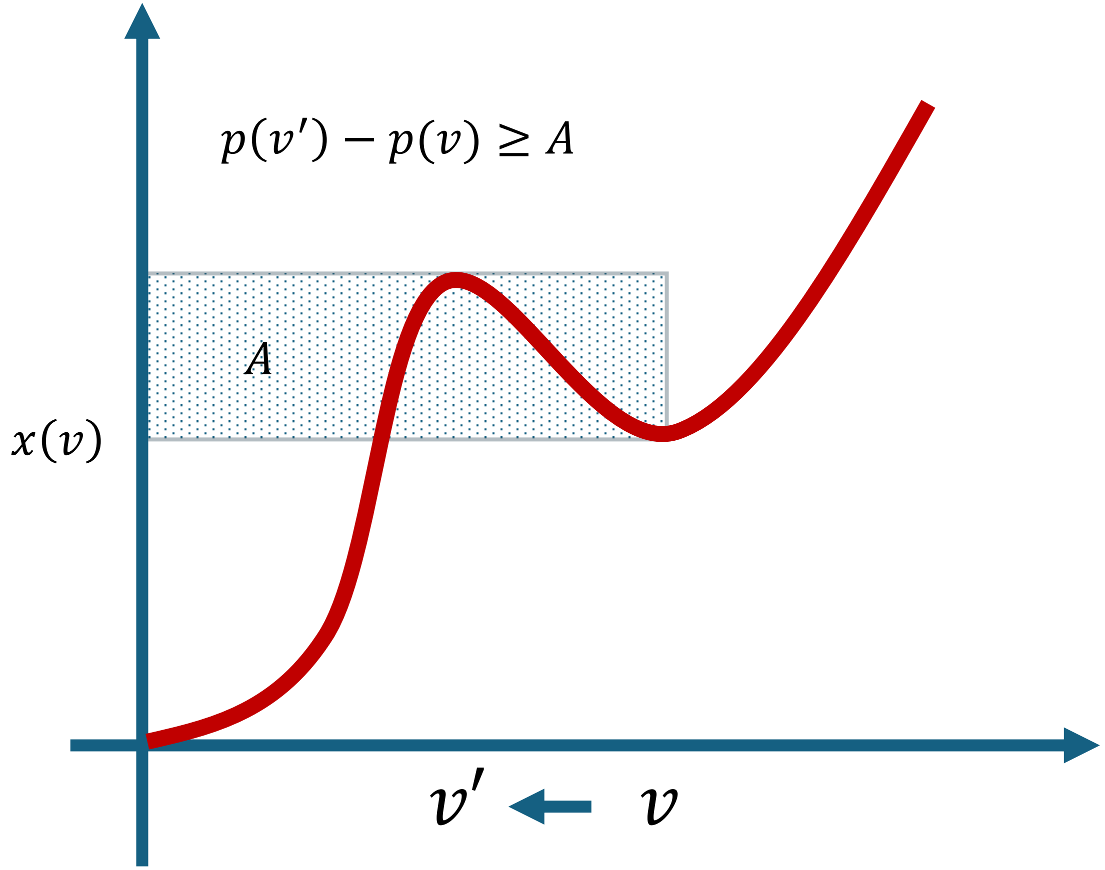
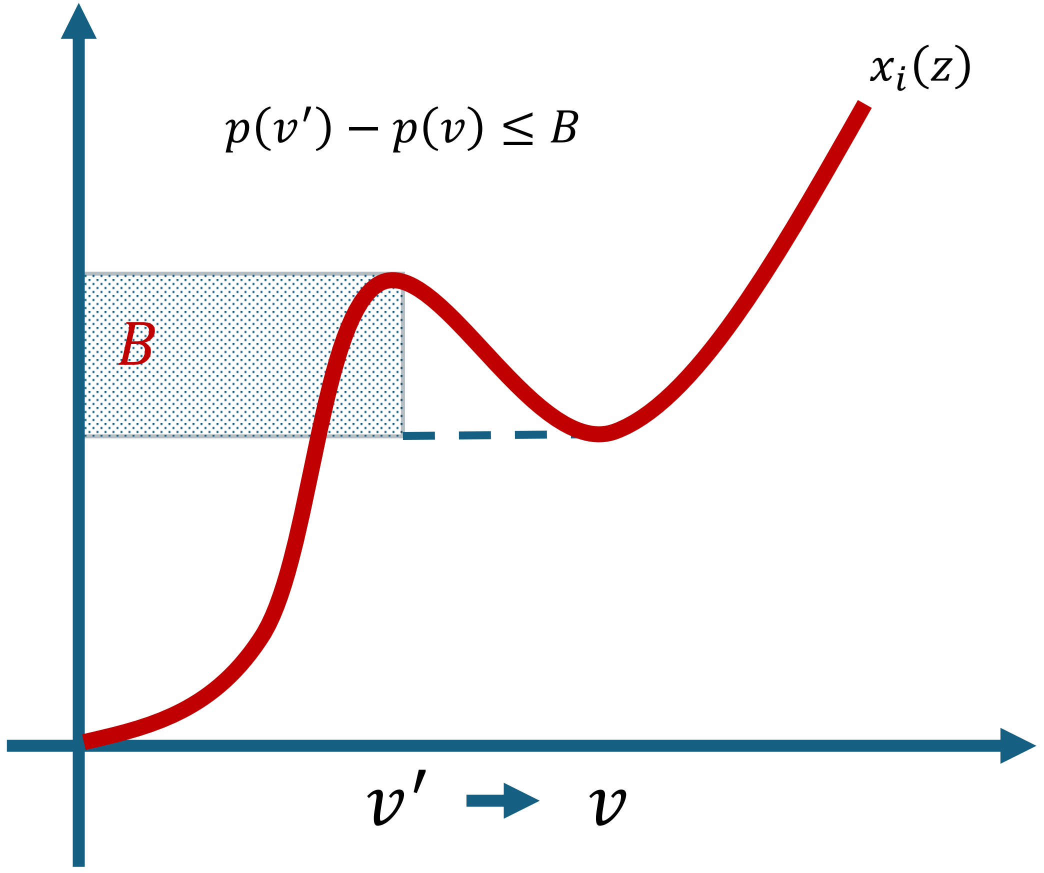
$$v' \cdot x(v') - p(v') \geq v' \cdot x(v) - p(v)$$





Implementable Rules are Monotone

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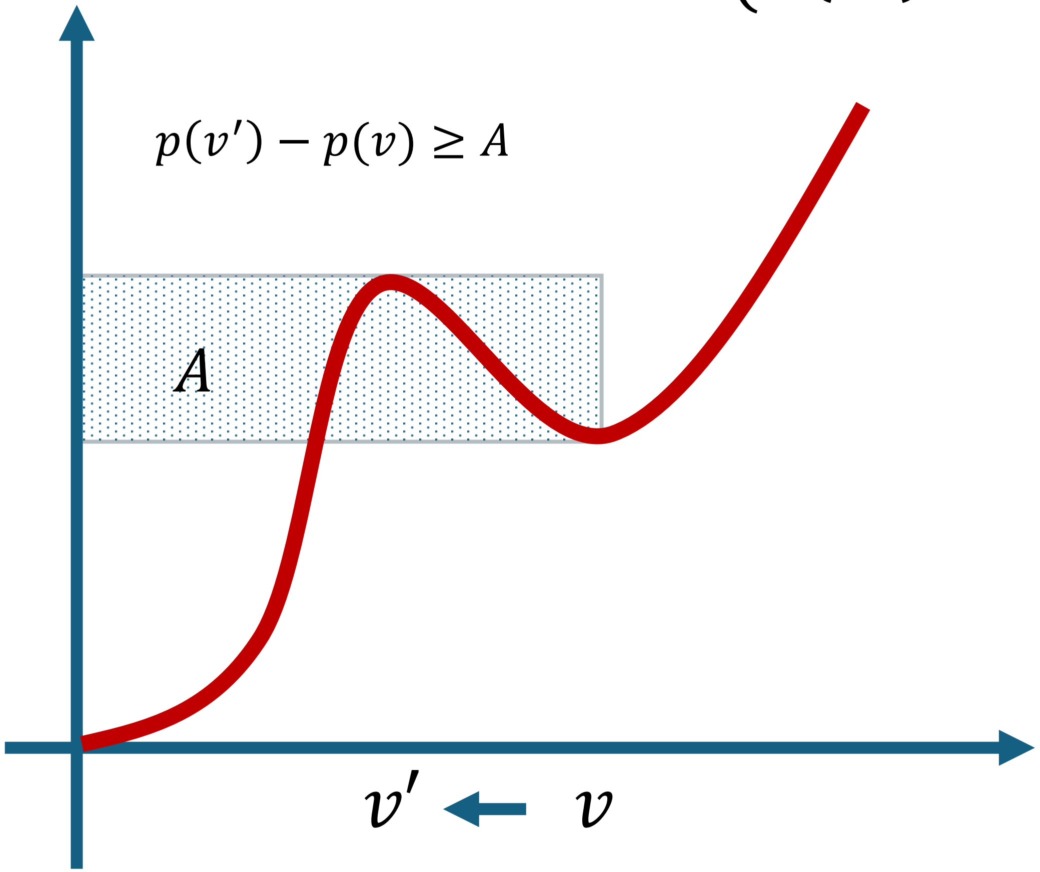
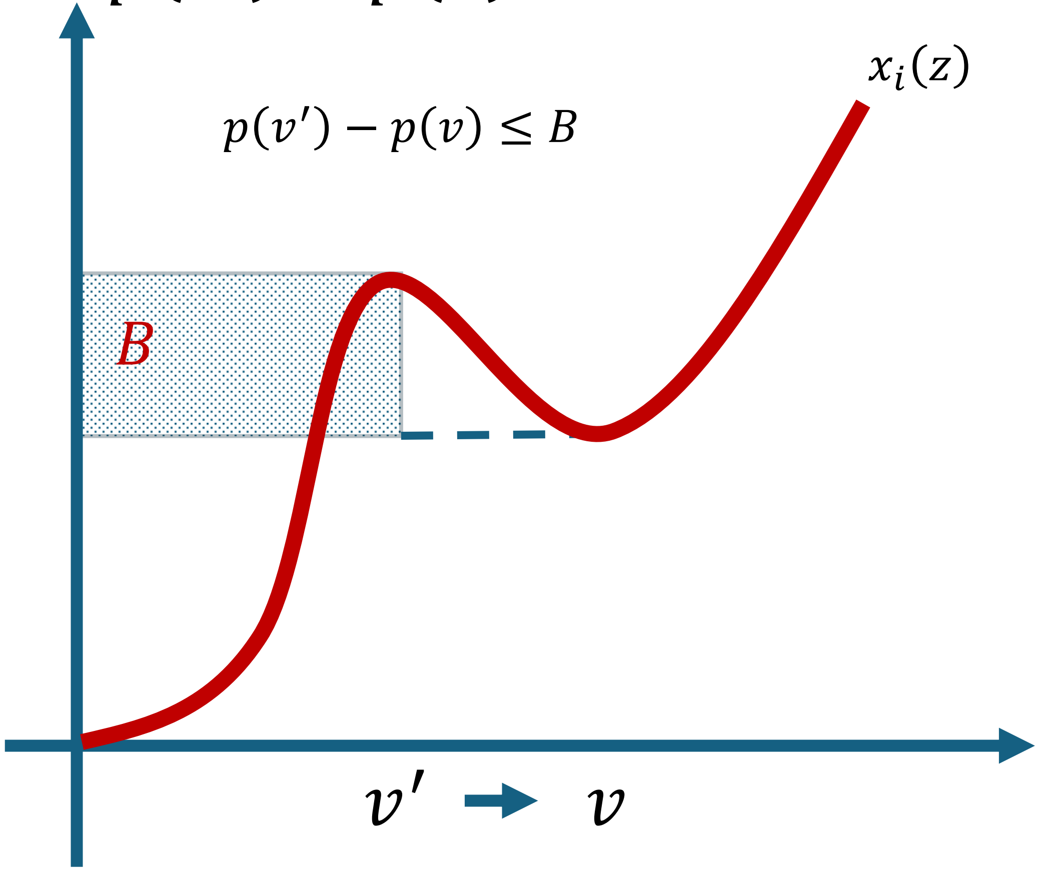


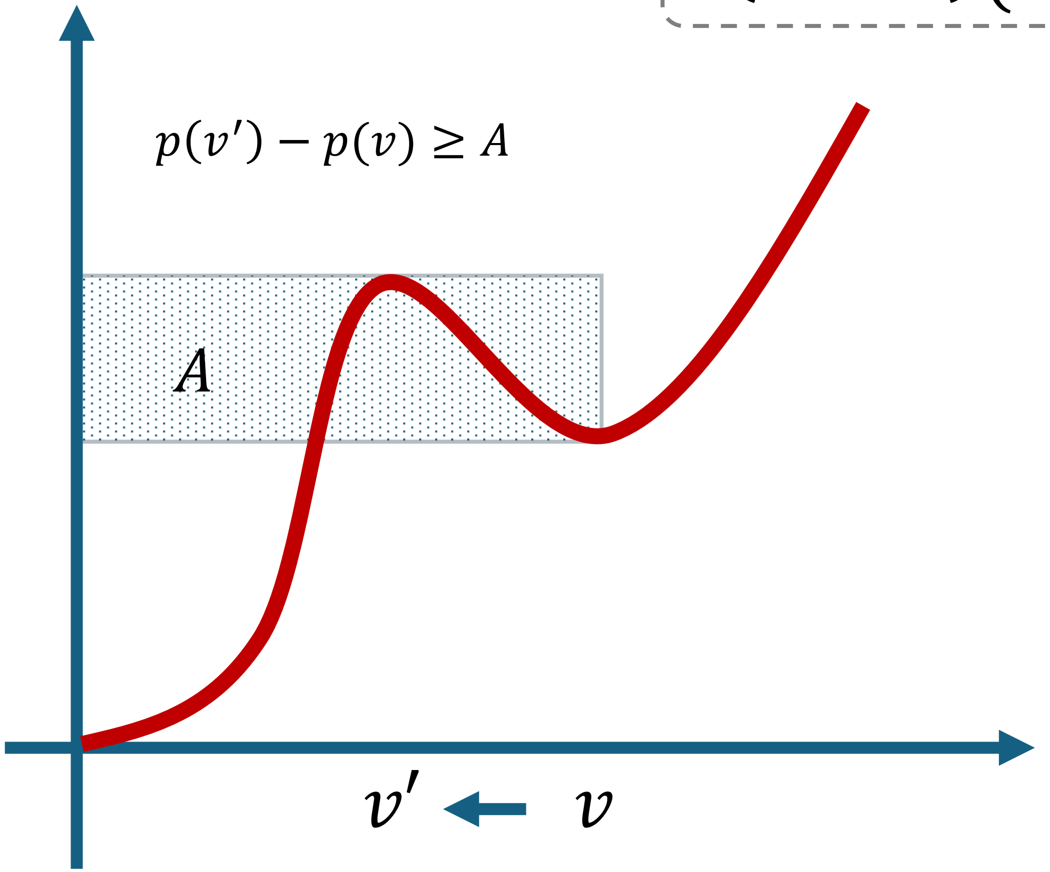


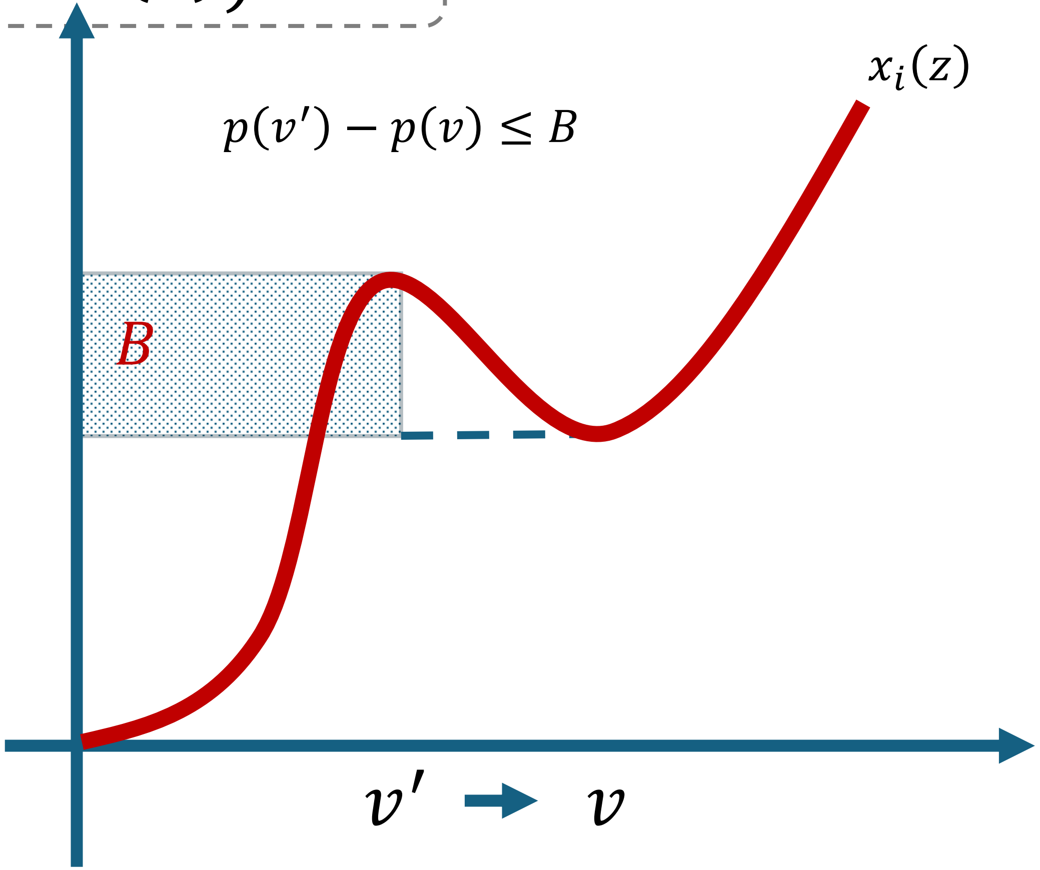
Implementable Rules are Monotone

$$A = v\left(x(v') - x(v)\right) \leq p(v') - p(v) \leq v'\left(x(v') - x(v)\right) = B$$

$$\boxed{(v' - v)\left(x(v') - x(v)\right) \geq 0}$$

Non-decreasing function





Any implementable allocation rule must be monotone!

"If not allocated with value v*, I should not be allocated if I report a lower value!"*

Uniqueness of Payment Rule

- I should not want to pretend to have value $v + \epsilon$, for infinitesimal ϵ

$$\begin{aligned} u(v) &\geq v \cdot x(v+\epsilon) - p(v+\epsilon) \\ &= (v+\epsilon) \cdot x(v+\epsilon) - p(v+\epsilon) - \epsilon \cdot x(v+\epsilon) \\ &= u(v+\epsilon) - \epsilon \cdot x(v+\epsilon) \end{aligned}$$

Dividing over by ϵ, restricts the rate of change of utility

$$\frac{u(v+\epsilon) - u(v)}{\epsilon} \leq x(v+\epsilon)$$

Uniqueness of Payment Rule

- I should not want to pretend to have value $v - \epsilon$, for infinitesimal ϵ

$$\begin{aligned} u(v) &\geq v \cdot x(v-\epsilon) - p(v-\epsilon) \\ &= (v-\epsilon) \cdot x(v-\epsilon) - p(v-\epsilon) + \epsilon \cdot x(v-\epsilon) \\ &= u(v-\epsilon) + \epsilon \cdot x(v-\epsilon) \end{aligned}$$

Dividing over by ϵ, restricts the rate of change of utility

$$\frac{u(v) - u(v-\epsilon)}{\epsilon} \geq x(v-\epsilon)$$

Uniqueness of Payment Rule

- I should not want to deviate locally up or down infinitesimally

$$\frac{u(v+\epsilon)-u(v)}{\epsilon} \leq x(v+\epsilon)$$
$$\frac{u(v)-u(v-\epsilon)}{\epsilon} \geq x(v-\epsilon)$$

- If u was differentiable, then taking the limit of the above as $\epsilon \to 0$

$$x(v) \leq u'(v) \leq x(v) \Rightarrow u'(v) = x(v)$$

- Under any truthful payment rule, utility is uniquely determined by allocation

$$u(v) - u(0) = \int_0^v x(z)\, dz$$

Under any truthful payment rule

$$u(v) = u(0) + \int_0^v x(z)\,dz$$

Discontinuity of Allocation Rule

- Even though allocation rule can be discontinuous, because it is monotone, it is Riemann integrable

$$\int_0^v x(z)\,dz = \lim_{\epsilon\to 0}\sum_{k=0}^{v/\epsilon} x\big(\epsilon\cdot(k+1)\big)\cdot\epsilon$$

$$\geq \lim_{\epsilon\to 0}\sum_{k=0}^{v/\epsilon} u\big(\epsilon\cdot(k+1)\big) - u(\epsilon\cdot k) = u(v) - u(0)$$

$$\int_0^v x(z)\,dz = \lim_{\epsilon\to 0}\sum_{k=0}^{v/\epsilon} x\big(\epsilon\cdot(k-1)\big)\cdot\epsilon$$

$$\leq \lim_{\epsilon\to 0}\sum_{k=0}^{v/\epsilon} u(\epsilon\cdot k) - u\big(\epsilon\cdot(k-1)\big) \ = u(v) - u(0)$$

Under any truthful payment rule

$$u(v) = u(0) + \int_0^v x(z)\,dz$$

What does that imply about payments

- Since utility is value minus payment

$$v\,x(v) - p(v) = -p(0) + \int_0^v x(z)\,dz$$

- Non-Negative-Transfers (NNT). We never have negative payments

$$p(0) \geq 0$$

- Individually Rational (IR). We never give bidders negative utility

$$p(0) \leq 0$$

- Thus, payment at 0 should be zero!

Under any truthful payment rule that satisfies NNT and IR

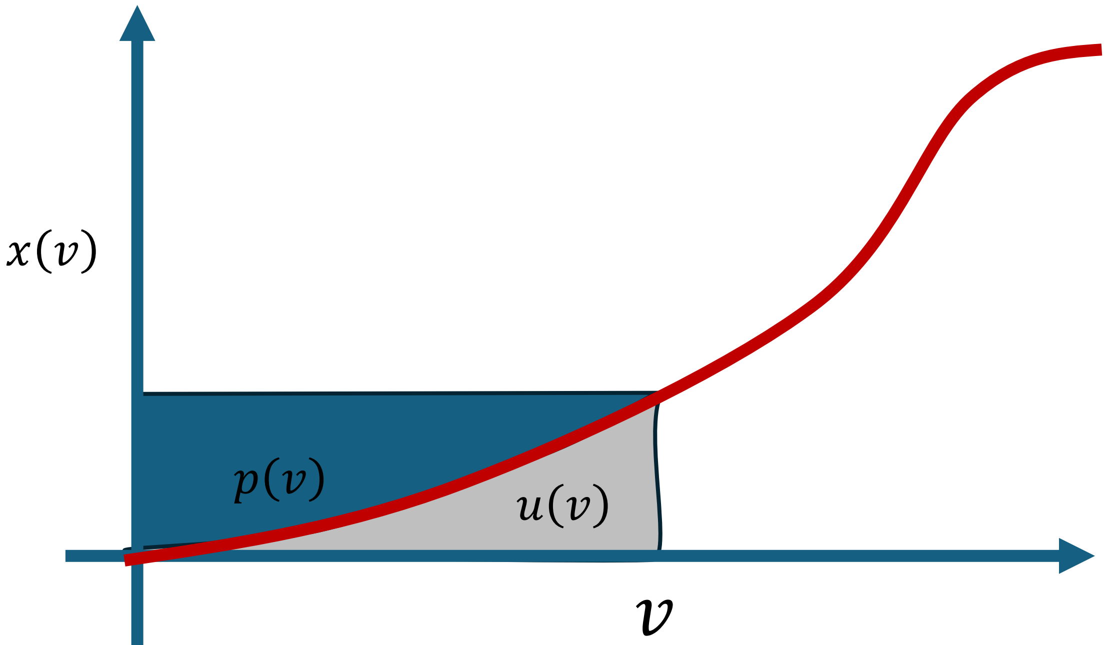
$$p(v) = v \cdot x(v) - \int_0^v x(z)\, dz$$

Given an allocation rule, the payment is uniquely determined!

Visualizing Utility

- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z)\,dz \qquad\qquad p(v) = v \cdot x(v) - \int_0^v x(z)\,dz$$

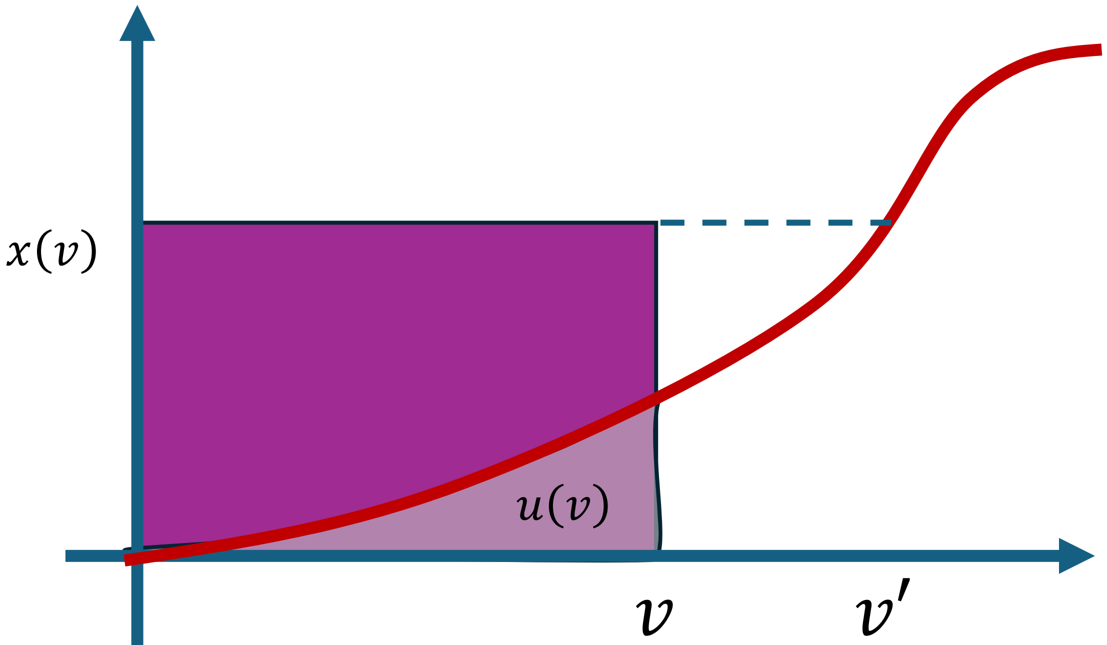


Visualizing Utility

- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z)\, dz$$

$$p(v) = v \cdot x(v) - \int_0^v x(z)\, dz$$

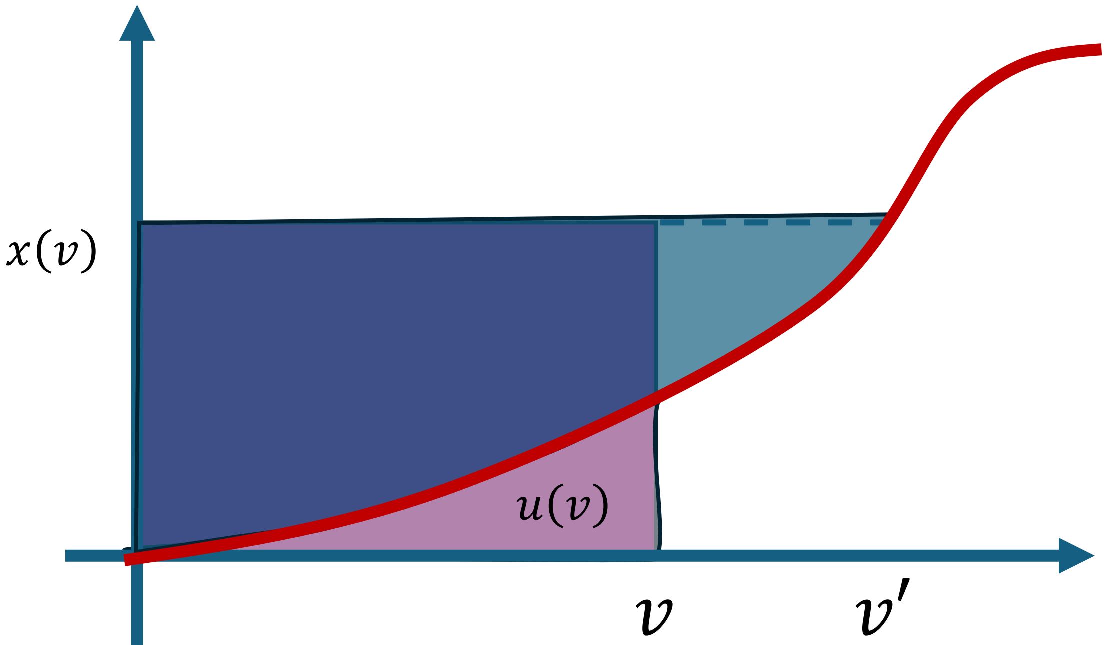


Visualizing Utility

- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z)\,dz$$

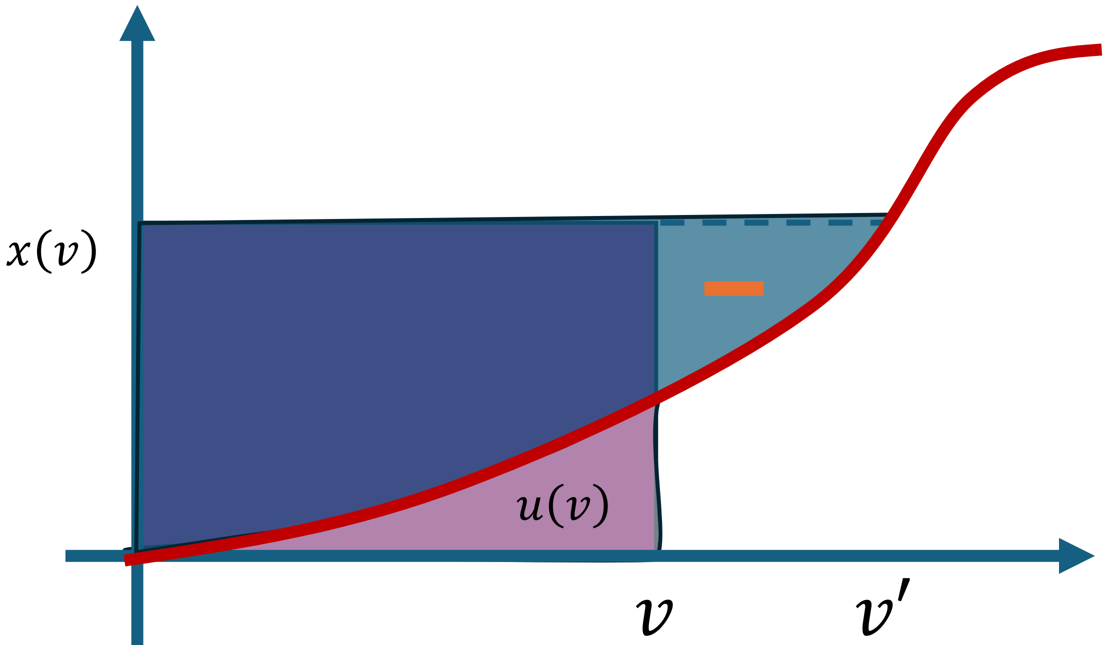
$$p(v) = v \cdot x(v) - \int_0^v x(z)\,dz$$



Visualizing Utility

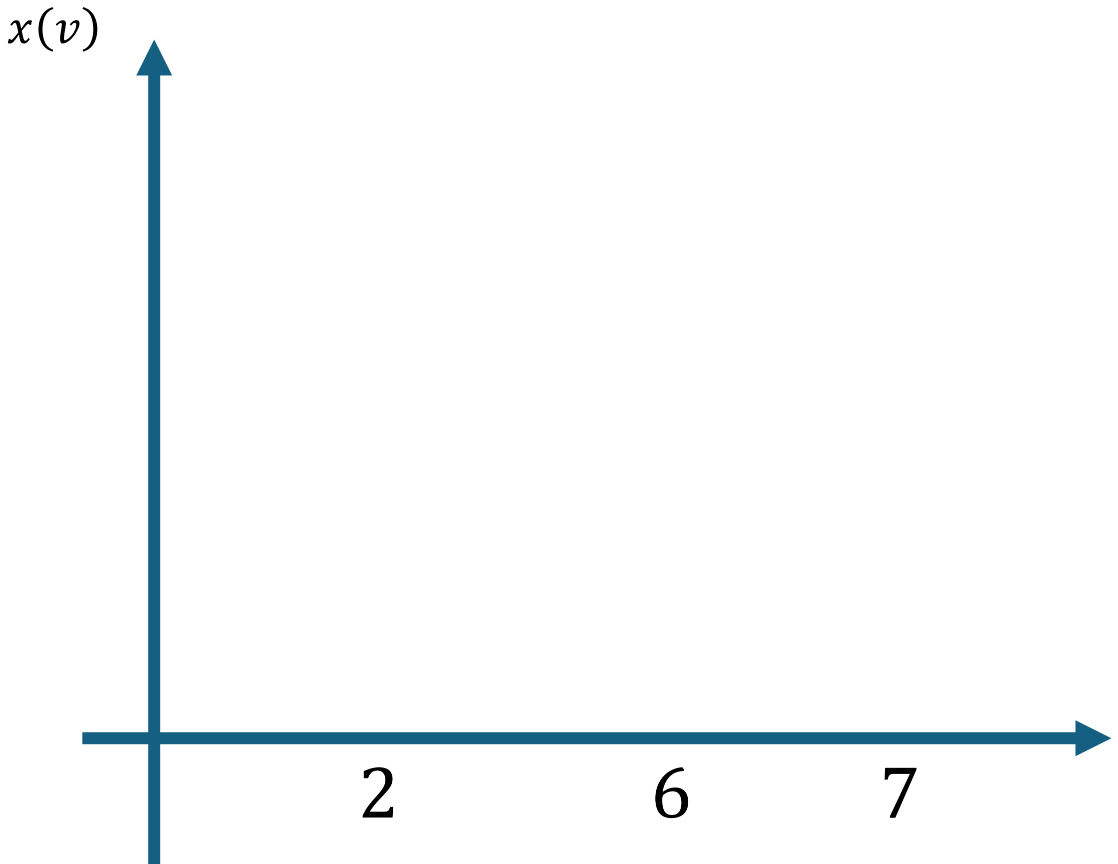
- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z)\, dz \qquad\qquad p(v) = v \cdot x(v) - \int_0^v x(z)\, dz$$



Back to Sponsored Search

Applying this to Sponsored Search



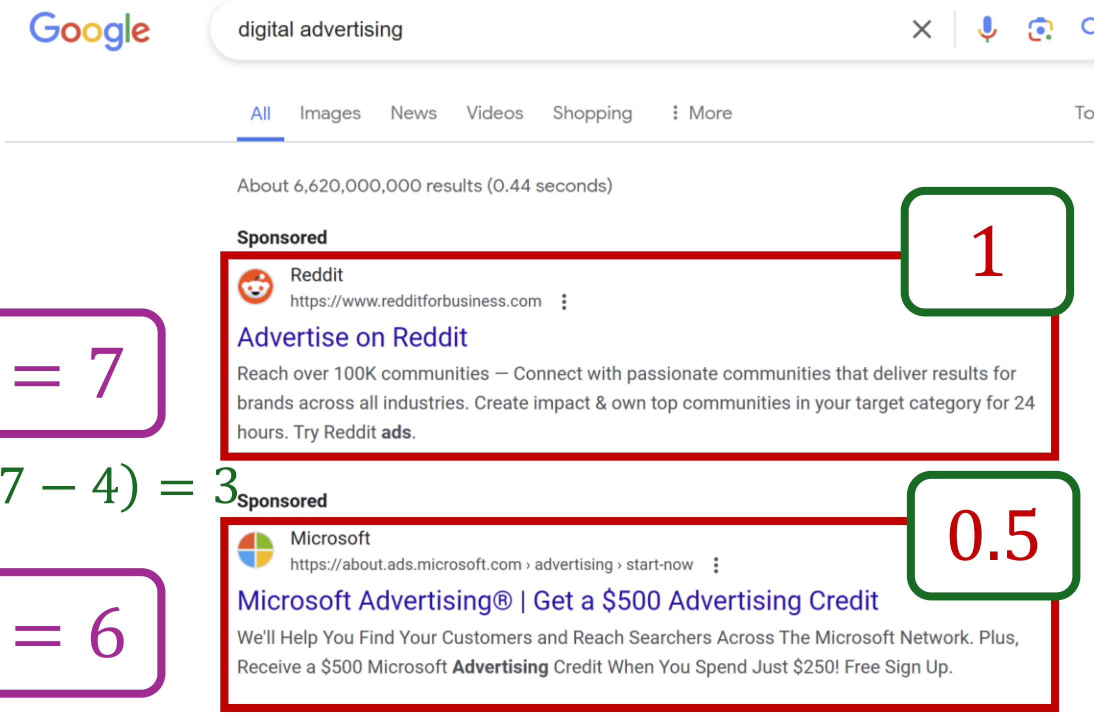
$v_1 = 7$

$u_1 = 1 \cdot (7 - 4) = 3$

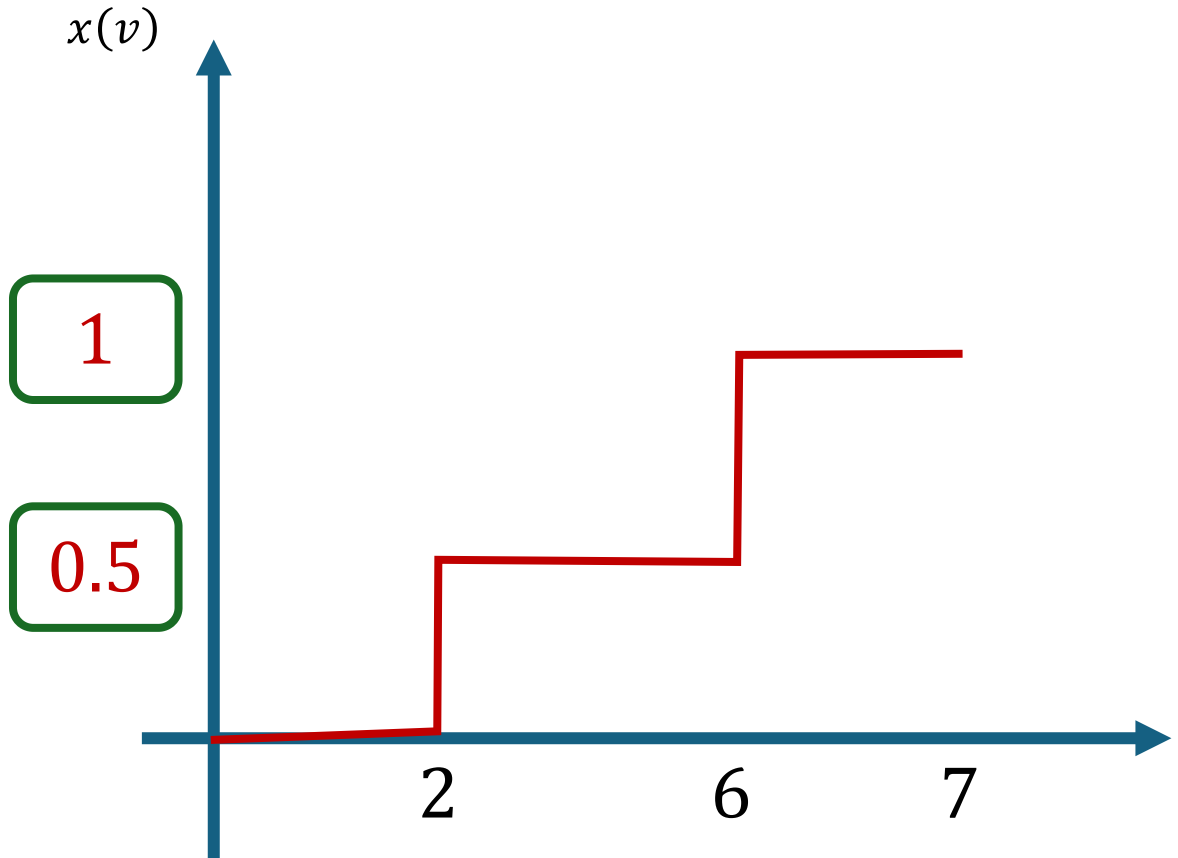
$v_2 = 6$

$u_2 = .5 \cdot (6 - 2) = 2$

$v_3 = 2$



Applying this to Sponsored Search



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$u_1 = 1 \cdot (7 - 4) = 3$

$v_2 = 6$

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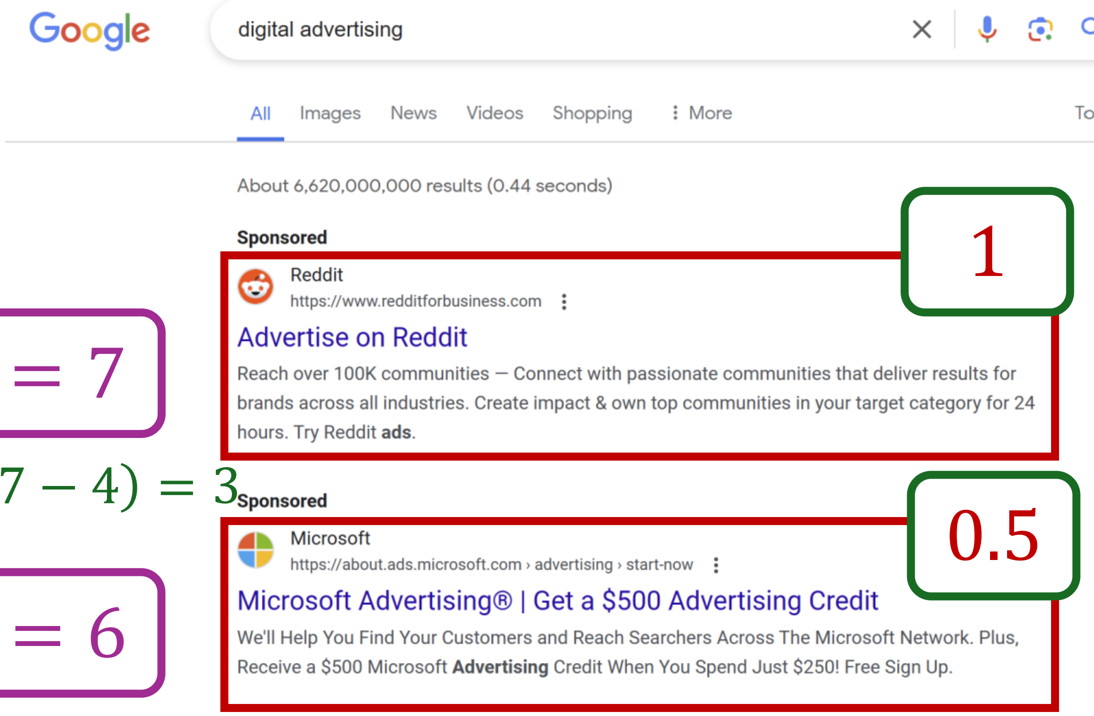
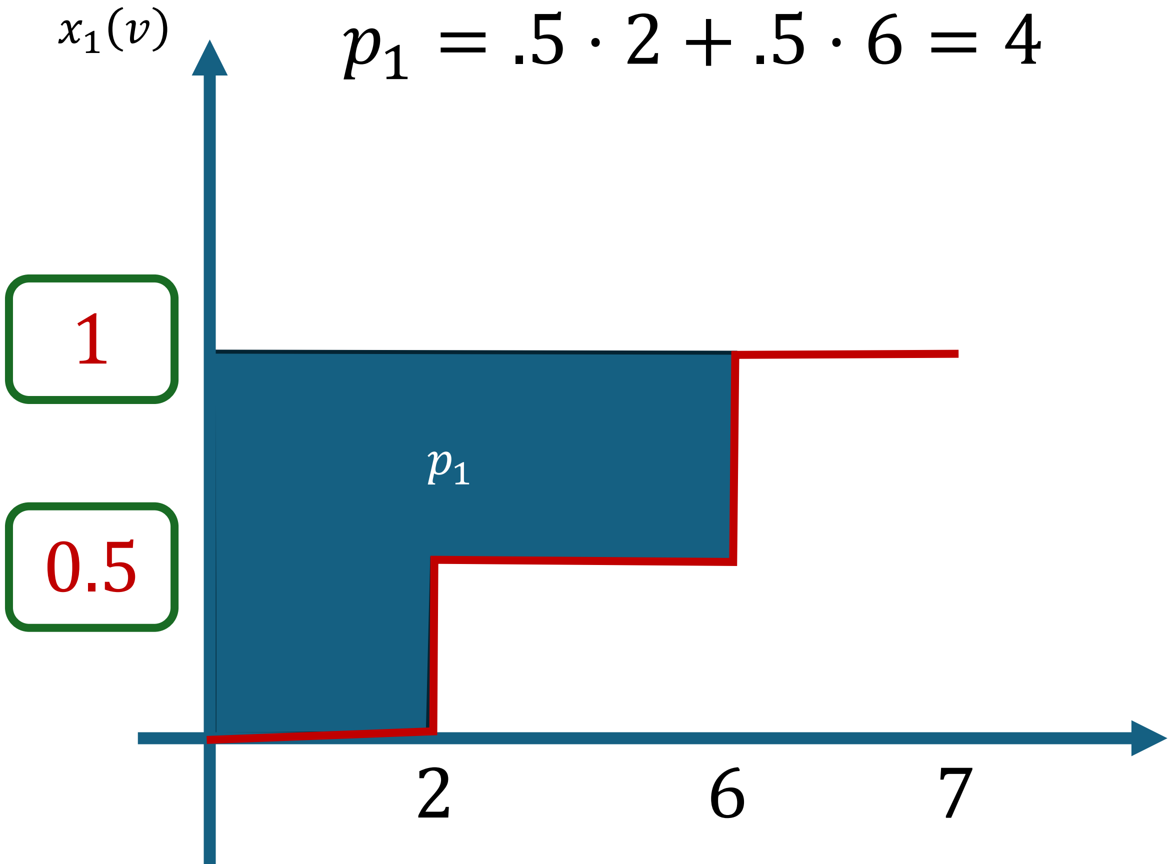
$v_3 = 2$



Applying this to Sponsored Search

$$p_1 = .5 \cdot 2 + .5 \cdot 6 = 4$$



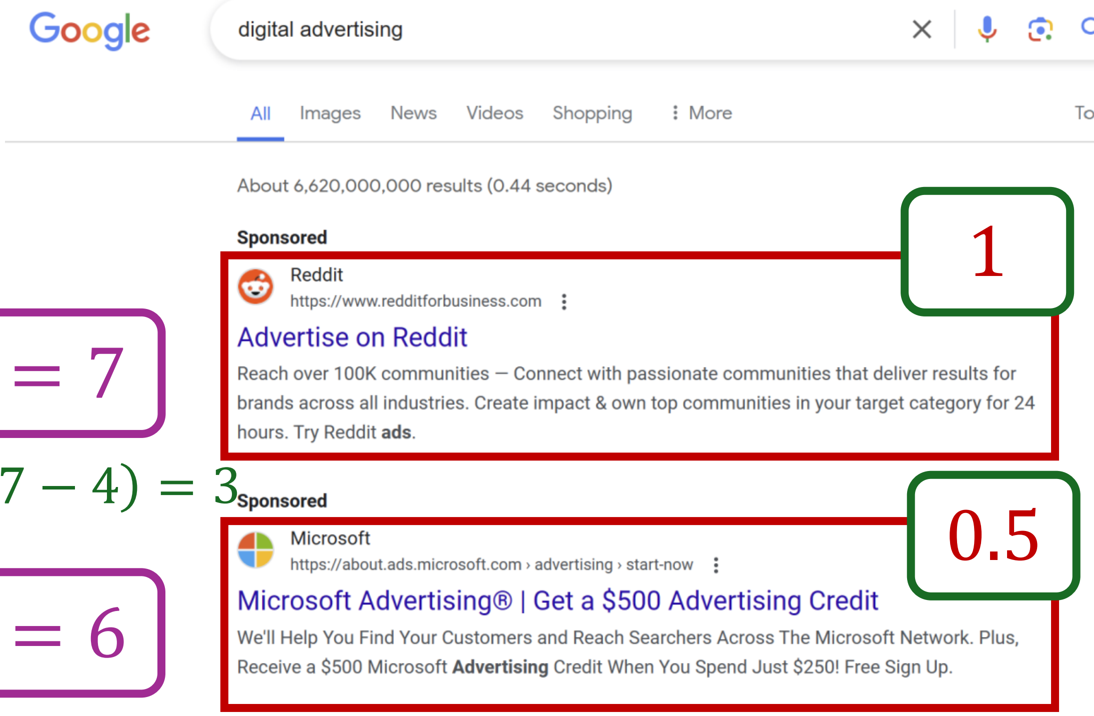
$$v_1 = 7$$

$$u_1 = 1 \cdot (7 - 4) = 3$$

$$v_2 = 6$$

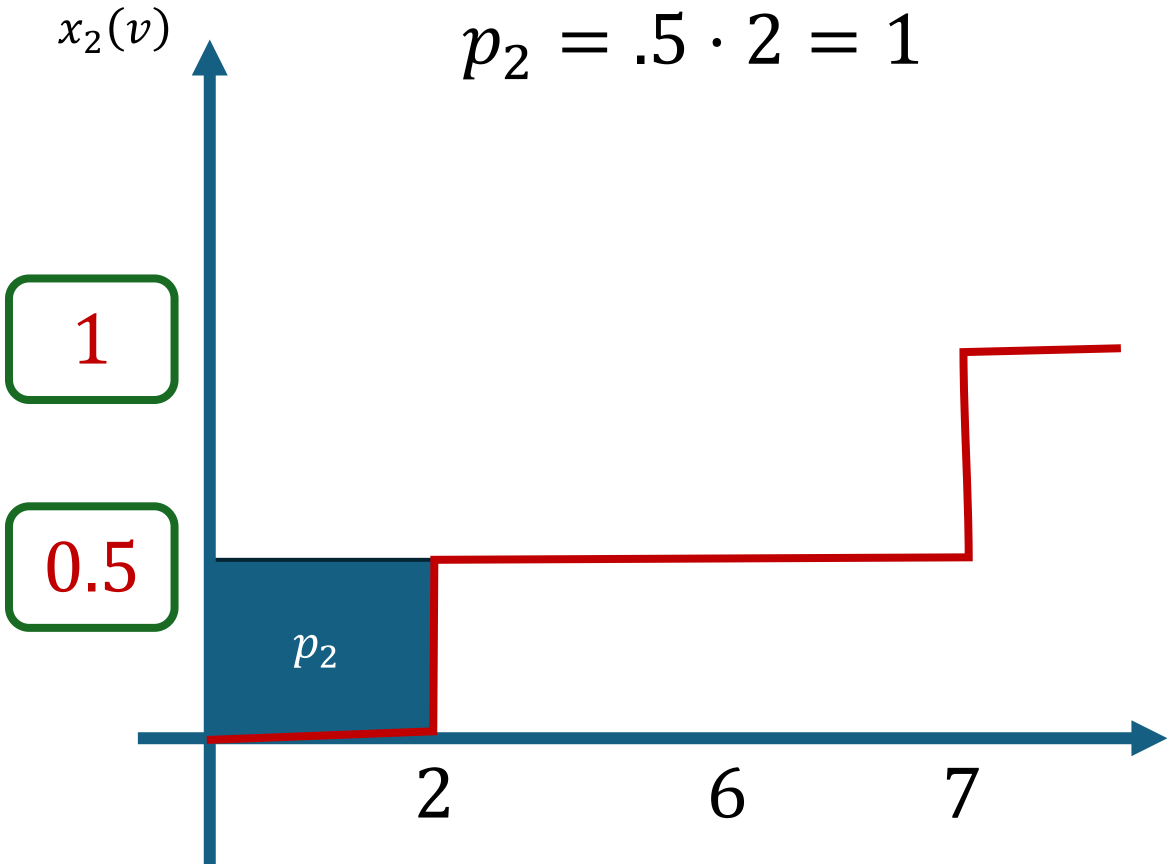
$$u_2 = .5 \cdot (6 - 2) = 2$$

$$v_3 = 2$$



Applying this to Sponsored Search

$$p_2 = .5 \cdot 2 = 1$$



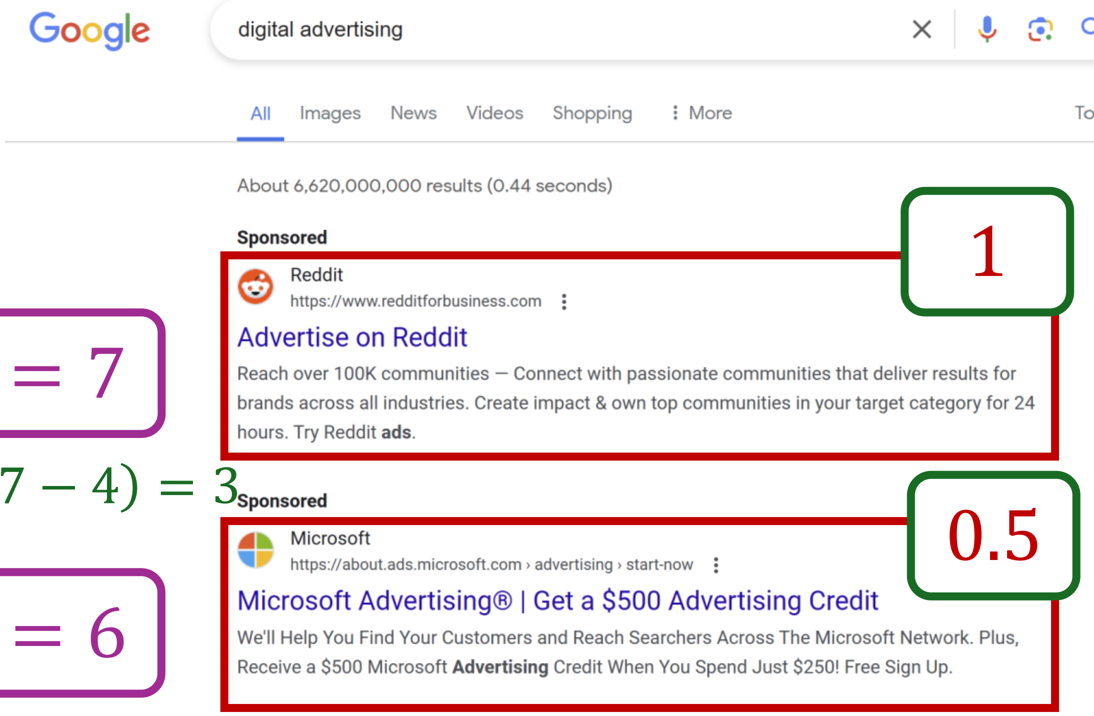
$\boxed{v_1 = 7}$

$u_1 = 1 \cdot (7 - 4) = 3$

$\boxed{v_2 = 6}$

$u_2 = .5 \cdot (6 - 2) = 2$

$\boxed{v_3 = 2}$



Applying this to Sponsored Search

With an arbitrary number of slots, payment of bidder in slot j is:

$$p_{(j)} = \sum_{\ell=j}^{k} (a_\ell - a_{\ell+1}) \cdot b_{(\ell)}$$

where $b_{(\ell)}$ is the bid of the player allocated in slot ℓ

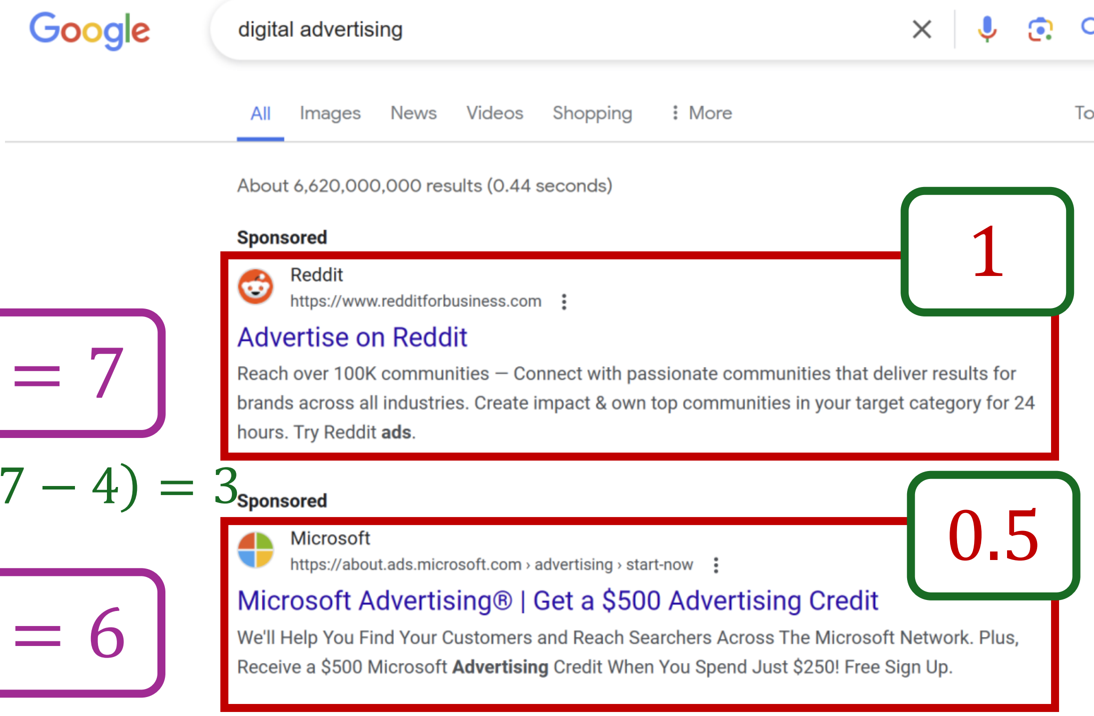
$v_1 = 7$

$u_1 = 1 \cdot (7 - 4) = 3$

$v_2 = 6$

$u_2 = .5 \cdot (6 - 2) = 2$

$v_3 = 2$



Optimizing over allocation rules

Myerson’s Theorem

- Let x, p be any DSIC mechanism
- Suppose each value $v_i \sim F_i$ independently and let $v = (v_1, \dots, v_n)$
$$E[p_i(v)] = E[x_i(v) \cdot \phi_i(v_i)]$$
where $\phi_i(v_i)$ is bidder i’s “virtual value”.
- Letting F_i the CDF and f_i the density:
$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$
- Assuming $\phi_i(v_i)$ is monotone non-decreasing, then the optimal DSIC mechanism is the mechanism that allocates to the highest virtual value bidder (or none if highest virtual value is negative)

Back to Uniform Example

- If $v_i \sim U[0,1]$ then $F(v) = v$ and $f(v) = 1$
- Virtual value simplifies to
$$\phi_i(v_i) = v_i - (1 - v_i) = 2v_i - 1$$
- We should allocate to the highest virtual value player, as long as the highest virtual value is non-negative
$$v_i \geq 1/2$$
- Since all virtual value functions are the same, allocating to the highest virtual value is the same as allocating to the highest value
- Simply: Second Price with a reserve price of 1/2!

Myerson's Theorem

- Consider the revenue contribution of a single bidder i and drop other bids and index from notation

$$E[p(v)] = E\left[v\, x(v) - \int_0^v x(z) dz\right] = E\left[v\, \hat{x}(v) - \int_0^v \hat{x}(z)\, dz\right]$$

- Allocation $\hat{x}(z)$ is the expected allocation over other bidder values

$$\hat{x}(z) = E_{v_{-i}}[x(z, v_{-i})]$$

- We can do an exchange of the integrals:

$$E\left[\int_0^v \hat{x}(z)\, dz\right] = \int_{v=0}^{\infty} \int_{z=0}^{v} \hat{x}(z)\, dz\, f(v) dv = \int_{z=0}^{\infty} \hat{x}(z) \int_{v=z}^{\infty} f(v)\, dv\, dz$$

$$= \int_{z=0}^{\infty} \hat{x}(z)\big(1 - F(z)\big) dz = E\left[\hat{x}(v) \frac{1 - F(v)}{f(v)}\right]$$

Myerson's Theorem (cont'd)

- Consider the revenue contribution of a single bidder i and drop other bids and index from notation

$$E[p(v)] = E\left[\hat{x}(v)\left(v - \hat{x}(v)\frac{1-F(v)}{f(v)}\right)\right] = E[\hat{x}(v)\,\phi(v)]$$

- Re-introducing the bidder index:

$$E[p_i(v)] = E[\hat{x}_i(v_i)\cdot\phi_i(v_i)] = E[x_i(v)\cdot\phi_i(v_i)]$$

- Summing across bidders we get:

$$\sum_i E[p_i(v)] = \sum_i E[x_i(v)\cdot\phi_i(v_i)] = E\left[\sum_i x(v)\cdot\phi_i(v_i)\right]$$

Myerson's Optimal Auction. The optimal mechanism is the mechanism that maximizes virtual welfare (and charges the corresponding payments that make this truthful)

$$x(v) = \text{argmax}_{x \in X} \sum_i x \cdot \phi_i(v_i)\,, \qquad p_i(v) = v_i x_i(v) - \int_0^{v_i} x_i(z, v_{-i})\, dz$$

$$\text{Rev} = E\left[\max_{x \in X} \sum_i x \cdot \phi_i(v_i)\right]$$

Appendix: Deriving the Optimal Reserve

- Bidders are symmetric. Revenue is twice the revenue we collect from each bidder

$$
\begin{aligned}
\mathrm{Rev}_1(r) &= E[\max(v_2, r)\, 1(v_1 \geq \max(v_2, r))] \\
&= E\big[v_2 \mid v_2 \in [r, v_1]\big] \Pr(v_2 \in [r, v_1] | v_1 \geq r) \Pr(v_1 \geq r) + r \Pr(v_2 \leq r) \Pr(v_1 \geq r) \\
&= \int_r^1 \frac{v + r}{2} (v - r) dv \; + r^2(1 - r) \\
&= \int_r^1 \frac{v^2 - r^2}{2} dv + r^2(1 - r) \\
&= \left(\frac{1 - \mathrm{r}^3}{6} - \frac{\mathrm{r}^2}{2} (1 - \mathrm{r}) + \mathrm{r}^2 (1 - r) \right) \\
&= \frac{1 - \mathrm{r}^3}{6} + \frac{r^2(1 - r)}{2} = \frac{1 - r^3 + 3r^2 - 3r^3}{6} = \frac{1 + 3r^2 - 4r^3}{6}
\end{aligned}
$$

- The first order condition

$$
\big(\mathrm{Rev}_1(r)\big)' = r(1 - 2r) = 0 \Rightarrow r = 1/2
$$