MS&E 233
Game Theory, Data Science and AI
Lecture 14

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(by courtesy) Computer Science and Electrical Engineering

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Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- Basic Auctions and Learning to bid in auctions (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- Simple vs optimal mechanisms (T)
- *HW6: implement simple and optimal auctions, analyze revenue empirically*

6
- Basics of Statistical Learning Theory (T)
- **Optimizing Mechanisms from Samples (T)**
- *HW7: implement procedures to learn approximately optimal auctions from historical samples*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design and LLMs, Song Zuo, Google Research
- A/B testing in auction markets, Okke Schrijvers, Central Applied Science, Meta

All these designs required knowledge of distributions of values F_i!

What can we do if we only have data from F_i?

Learning Auctions from Samples

Learning from Samples

- We are given a set S of m samples of value profiles

$$S = \left\{ v^j = \left(v_1^j, \ldots, v_n^j \right) \right\}_{j=1}^{m}$$

- Each sample is drawn i.i.d. from the distribution of values

$$v_i^j \sim F_i, \qquad v^j \sim \boldsymbol{F} \stackrel{\text{def}}{=} F_1 \times \cdots \times F_n$$

- Samples can be collected from historical runs of truthful auction
- Bids of each bidder in each of the m historical runs of the auction

Desiderata

- Without knowledge of distributions F_i, we want to produce a mechanism M_S, that achieves good revenue on these distributions
- For some $\epsilon(m) \to 0$ as the number of samples grows:

$$\mathrm{Rev}(M_S) \stackrel{\text{def}}{=} E_{v \sim F}\left[\sum_i p_i^{M_S}(v)\right] \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon(m)$$

- Either in expectation over the draw of the samples, i.e.

$$E_S[\mathrm{Rev}(M_S)] \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon(m)$$

- Or with high-probability over the draw of the samples, i.e.

$$\text{w. p. } 1-\delta: \quad \mathrm{Rev}(M_S) \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon_\delta(m)$$

Easy Start: Pricing from Samples

Pricing from Samples

- Suppose we have only one bidder with $v \sim F$, for simplicity in $[0, 1]$
- Optimal mechanism is to post the monopoly reserve price
- The optimal price r is the one that maximizes

$$\mathrm{Rev}(r) = E_{v \sim F}[r \cdot 1\{v \geq r\}] = r \Pr(v \geq r) = r\left(1 - F(r)\right)$$

  which is the monopoly reserve price η that solves:

$$\eta - \frac{1 - F(\eta)}{f(\eta)} = 0$$

- Choosing η requires knowledge of the CDF F and the pdf f
- Can we optimize r if we have m samples of v?

The Obvious Algorithm

- We want to choose r that maximizes

$$\max_{r\in[0,1]} \mathrm{Rev}(r) \stackrel{\text{def}}{=} E_{v\sim F}[r \cdot 1\{v \geq r\}], \qquad \text{(population objective)}$$

- With m samples S, we can optimize average revenue on samples!

$$\max_{r\in[0,1]} \mathrm{Rev}_S(r) \stackrel{\text{def}}{=} \frac{1}{m}\sum_{j=1}^{m} r \cdot 1\{v^j \geq r\}, \qquad \text{(empirical objective)}$$

- This approach is called Empirical Reward Maximization (ERM)
- **Intuition.** Since each value is drawn from distribution F the empirical average over i.i.d. draws from F, by law of large numbers, should be very close to expected value

Same as Empirical Risk Minimization (ERM) in Machine Learning (loss vs reward)

A Potential Problem with ERM

- The Law of Large Numbers applies if we wanted to evaluate the revenue of a fixed reserve price, we had in mind using the samples
- If we optimize over a very large set of reserve prices, then by random chance, it could be that we find a reserve price that has a large revenue on the samples, but small on the distribution

- This behavior is called overfitting to the samples
- We need to argue that overfitting cannot arise when we optimize over the reserve price!

Basic Elements of Statistical Learning Theory

Uniform Convergence

- Uniform Convergence. Suppose that we show that, w.p. $1 - \delta$

$$\forall r \in [0,1]: \ |\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_\delta(m)$$

- Alert. Note that this is different than: $\forall r \in [0,1]$, w.p. $1 - \delta$

$$|\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_\delta(m)$$

- The first asks that with probability $1 - \delta$, the empirical revenue of all reserve prices is close to their population revenue
- The second asks that for a given reserve price, with probability $1 - \delta$ its empirical revenue is close to its population
- The second claims nothing about the probability of the *joint event* that this is satisfied for all prices simultaneously

Uniform Converges Suffices for No-Overfitting

- Uniform Convergence. Suppose that we show that, w.p. $1 - \delta$

$$\forall r \in [0,1]: \ |\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_\delta(m)$$

- Empirical Risk Maximization reserve:

$$r_S = \operatorname*{argmax}_{r \in [0,1]} \mathrm{Rev}_S(r)$$

Theorem. If uniform convergence holds then, w.p. $1 - \delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m) = \mathrm{OPT}(F) - 2\epsilon_\delta(m)$$

Uniform Converges Suffices for No-Overfitting

Theorem. If uniform convergence holds then, w.p. $1-\delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m) = \mathrm{OPT}(F) - 2\epsilon_\delta(m)$$

- By uniform convergence, with probability $1-\delta$:
$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}_S(r_S) - \epsilon_\delta(m)$$
- Since, r_S optimizes the empirical objective
$$\mathrm{Rev}_S(r_S) \geq \mathrm{Rev}_S(\eta)$$
- By uniform convergence:
$$\mathrm{Rev}_S(\eta) \geq \mathrm{Rev}(\eta) - \epsilon_\delta(m)$$
- Putting it all together:
$$\mathrm{Rev}(r_s) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m)$$

This is the no-overfitting property: It **cannot be** that we found a reserve price that has *large empirical revenue* but very *small population revenue*

The *monopoly reserve* is a **feasible** reserve price but **was not chosen** by ERM. So, it must have had smaller empirical average revenue.

LLN vs Uniform Convergence

Crucial Argument: with probability $1-\delta$: $\mathrm{Rev}(r_S) \geq \mathrm{Rev}_S(r_S) - \epsilon_\delta(m)$

- Cannot be argued solely using Law of Large Numbers: if we have i.i.d. X^j with mean $E[X]$

$$\left| \frac{1}{m} \sum_{j=1}^{m} X^j - E[X] \right| \to 0$$

- For reserve price r that is chosen before looking at the samples, define $X^j(r) = r \cdot \ 1\{v^j \geq r\}$

$$|\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| = \left| \frac{1}{m} \sum_{j} r \cdot 1\{v^j \geq r\} - E[r \cdot 1\{v \geq r\}] \right| \to 0$$

- Problem. The reserve price r_S was chosen by looking at all the samples in S
  - If I tell you r_S you learn something about the samples
  - Conditional on r_S the samples are no-longer i.i.d.

- Uniform convergence, essentially means "*what I learn about S from r_S is not that much…*"

Concentration Inequalities and Uniform Convergence

- Concentration inequalities give us a stronger version of LLN

- Chernoff-Hoeffding Bound. If we have i.i.d. $X^j \in [0,1]$ with mean $E[X]$, w.p. $1-\delta$:

$$\left|\frac{1}{m}\sum_{j=1}^{m} X^j - E[X]\right| \leq \epsilon_\delta(m) \stackrel{\text{def}}{=} \sqrt{\frac{\log(2/\delta)}{2m}}$$

- Crucial. The bound grows only logarithmically with $1/\delta$

Union Bound

- Suppose we had only K possible reserve prices $\left\{\frac{1}{K}, \frac{2}{K}, \frac{3}{K} \ldots, 1\right\}$
- For each reserve price r on the grid, for any probability δ', by Chernoff bound

$$\Pr\left((\text{Bad Event})_r\right) = \Pr\left(\left|\frac{1}{m}\sum_{j=1}^{m} X^j(r) - E[X(r)]\right| > \epsilon_{\delta'}(m)\right) \leq \delta'$$

- Union Bound. The probability of the union of events is at most the sum of the probabilities

$$\Pr\left(\cup_{r=1}^{K} (\text{Bad Event})_r\right) \leq \sum_{r=1}^{K} \Pr\left((\text{Bad Event})_r\right) \leq K \cdot \delta'$$

- Apply Chernoff bound with $\delta' = \delta/K$

$$\Pr\left(\cup_{r=1}^{K} (\text{Bad Event})_r\right) \leq \delta$$

- Probability(*exists reserve price whose empirical revenue is far from its population*) at most δ

Uniform Convergence via Union Bound

Theorem. Suppose we had K possible reserve prices $\text{Grid}_K \stackrel{\text{def}}{=} \left\{\frac{1}{K}, \frac{2}{K}, \frac{3}{K} \ldots, 1\right\}$

Then with probability at least $1 - \delta$

$$\forall r \in \text{Grid}_K: |\text{Rev}_S(r) - \text{Rev}(r)| \leq \epsilon_{\delta/K}(m) \stackrel{\text{def}}{=} \sqrt{\frac{\log(2K/\delta)}{2m}}$$

Problem. The optimal reserve η can potentially not be among these K reserves

Intuition. For a sufficiently large K, for any reserve price, we can find a reserve price on this discretized grid that achieves almost as good revenue

We don't lose much by optimizing over the grid!

Discretization

- For a reserve price r, pick largest reserve price below r on the grid
- Denote this discretization of r as r_K

- By doing so, you allocate to any value you used to allocate before
- For any such value you receive revenue at least $r - 1/K$

- Overall, you lose revenue at most $1/K$

$$\mathrm{Rev}(r_K) \geq \mathrm{Rev}(r) - 1/K$$

Discretized ERM

- Let's modify ERM to optimize only over the grid

$$r_S = \max_{r \in \text{Grid}_K} \text{Rev}_S(r)$$

- We can apply the uniform convergence over the grid

$$\text{Rev}(r_S) \geq \text{Rev}_S(r_S) - \epsilon_{\delta/K}(m)$$

We cannot overfit, when optimizing over the grid of reserves

- Since, r_S optimizes the empirical objective over the grid

$$\text{Rev}_S(r_S) \geq \text{Rev}_S(\eta_K)$$

The *discretized monopoly reserve* is a **feasible** reserve in the grid but **was not chosen** by ERM.

- By uniform convergence over the grid:

$$\text{Rev}_S(\eta_K) \geq \text{Rev}(\eta_K) - \epsilon_{\delta/K}(m)$$

- By the discretization error argument:

$$\text{Rev}(\eta_K) \geq \text{Rev}(\eta) - 1/K$$

Theorem. The revenue of the reserve price output by discretized ERM over the K-grid satisfies, with probability $1 - \delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{OPT}(F) - 2\sqrt{\frac{\log(2K/\delta)}{2m}} - \frac{1}{K}$$

Choosing $K = m$

$$\mathrm{Rev}(r_S) \geq \mathrm{OPT}(F) - 3\sqrt{\frac{\log(2m/\delta)}{2m}}$$

Desideratum satisfied!
$\epsilon_\delta(m) \to 0$ as m grows

The Limits of Discretization

- Do we really need to optimize over the discrete grid?
- What if we insist on optimizing over $[0,1]$. Can we still overfit?

- Now that we have infinite possible reserves, we cannot apply the union bound argument ($K = \infty$)!
- How do we argue about optima over continuous, infinite cardinality spaces?

Sneak Peek

- Would have been ideal if we only have to argue about behavior of our optimization space, *on the given set of samples*
- As opposed to the unknown distribution of values

- What if we can find a small set of reserves and argue that for all reserves there is an approximately equivalent one in the small set, in terms of revenue on the samples

- Maybe then it suffices to invoke the union bound over the smaller space, even though we optimize over the bigger space

Statistical Learning Theory

General Framework

- Given samples $S = \{v_1, \ldots, v_m\}$ that are i.i.d. from distribution F
- Given a hypothesis/function space H
- Given a reward function $r(v; h)$

- Goal is to maximize the expected reward over distribution F

$$R(h) = E_{v \sim F}[r(v; h)]$$

Desiderata

- Without knowledge of distribution F, we want to produce a hypothesis h_S, that achieves good reward on this distribution
- For some $\epsilon(m) \to 0$ as the number of samples grows:

$$R(h_S) \stackrel{\text{def}}{=} E_{v\sim F}[r(v;h)] \geq \max_{h\in H} R(h) - \epsilon(m)$$

- Either in expectation over the draw of the samples, i.e.

$$E_S[R(h_S)] \geq \max_{h\in H} R(h) - \epsilon(m)$$

- Or with high-probability over the draw of the samples, i.e.

$$\text{w. p. } 1-\delta: \quad R(h_S) \geq \max_{h\in H} R(h) - \epsilon_\delta(m)$$

Desiderata (Mechanism Design from Samples)

- Without knowledge of Distribution of value profiles F, we want to produce a hypothesis h_S, that achieves good Revenue on this distribution
- For some $\epsilon(m) \to 0$ as the number of samples grows:

$$R(h_S) \stackrel{\text{def}}{=} E_{v \sim F}\left[\sum_i p_i(v)\right] \geq \max_{h \in H} R(h) - \epsilon(m)$$

- Either in expectation over the draw of the samples, i.e.

$$E_S[R(h_S)] \geq \max_{h \in H} R(h) - \epsilon(m)$$

- Or with high-probability over the draw of the samples, i.e.

$$\text{w. p. } 1-\delta: \quad R(h_S) \geq \max_{h \in H} R(h) - \epsilon_\delta(m)$$

The Obvious Algorithm

- We want to choose r that maximizes

$$\max_{h \in H} R(h) \stackrel{\text{def}}{=} E_{v \sim F}[r(v; h)], \qquad \text{(population objective)}$$

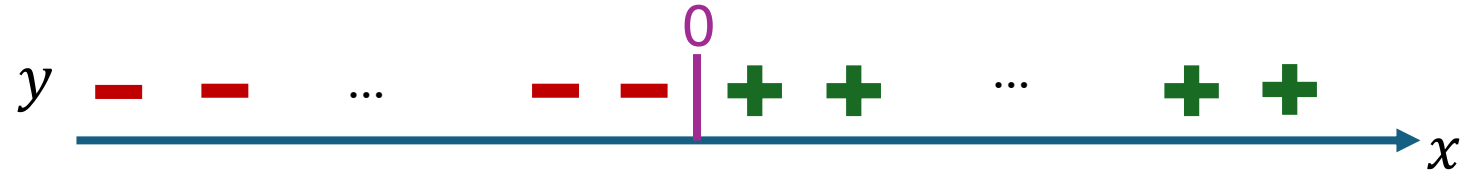
- With m samples, we can optimize average reward on samples!

$$\max_{h \in H} R_S(h) \stackrel{\text{def}}{=} \frac{1}{m} \sum_{j=1}^{m} r(v_j; h), \qquad \text{(empirical objective)}$$

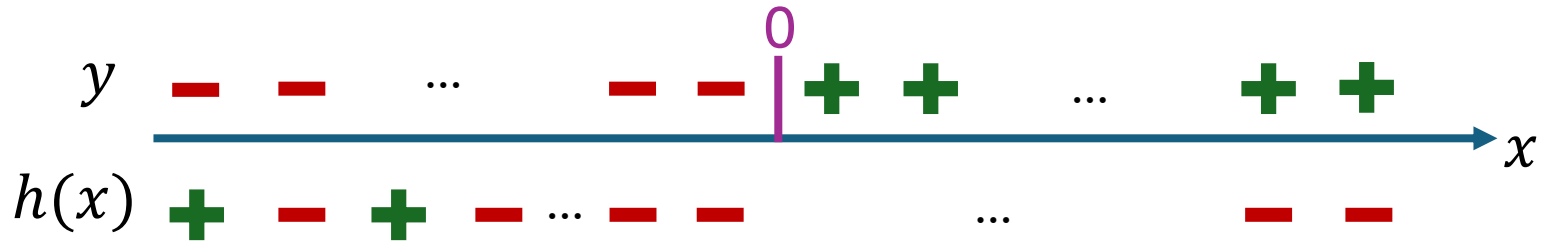
- This approach is called Empirical Reward Maximization (ERM)
- **Intuition.** Since each value is drawn from distribution F the empirical average over i.i.d. draws from F, by law of large numbers, should be very close to expected value

Standard Classification Example

- Suppose samples $v = (x, y)$ where $x \sim U[-1,1]$ and $y \in \{-1, 1\}$



- We want to choose a "labeling" function $h(x) \in \{-1,1\}$

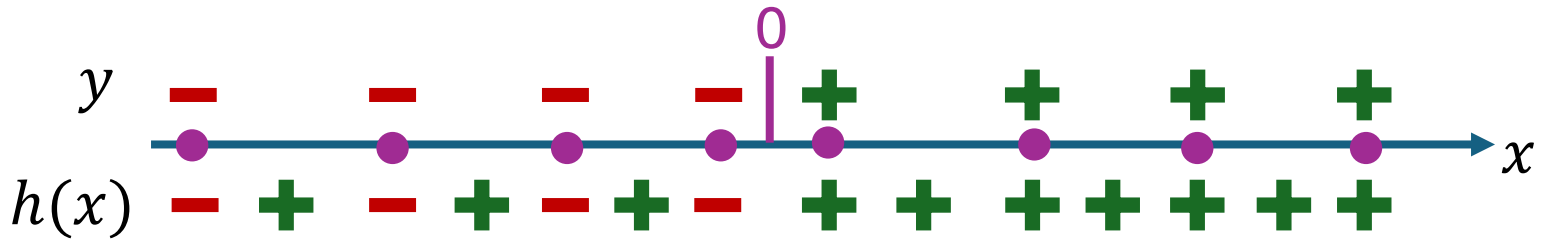


- That achieves good accuracy

$$r(v; h) = 1\{h(x) = y\}$$

ERM Gone Bad

- Suppose we choose the following h_S: label all samples correctly and predict $+1$ for any value that is not on the samples

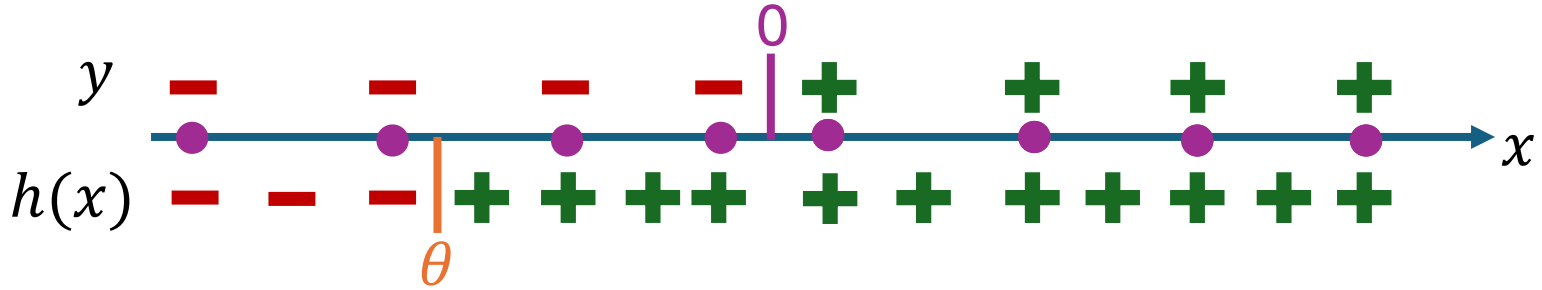


- The empirical average reward of this h_S is 1. The largest possible!
- The expected reward of this h_S is ½
- The discrepancy between the empirical reward of the ERM solution and its population reward never vanishes! Overifitting!

ERM Over Threshold Functions

- Suppose we restrict to optimizing over threshold functions
- Label every $x \geq \theta$ with $+1$ and every $x < \theta$ with -1

$$H = \{x \to 1(x \geq \theta) - 1(x < \theta) : \theta \in \Theta\}$$

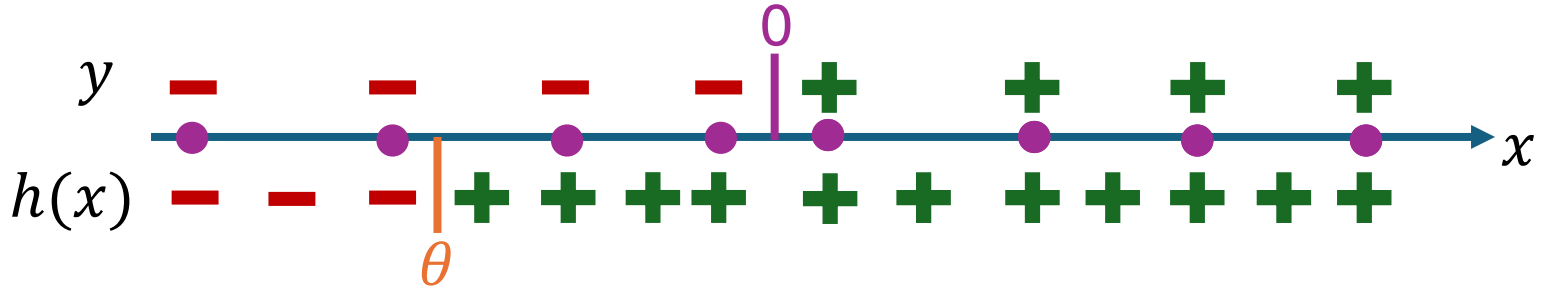


- Optimizing over such θ we will never be able to overfit
- How do we argue this?
- Discretization argument fails!
- No matter how we discretize, there exists a distribution of x that will have a very large discretization error

Sufficient Hypothesis Subspace on Samples

- Suppose we restrict to optimizing over threshold functions
- Label every $x \geq \theta$ with $+1$ and every $x < \theta$ with -1

$$H = \{x \to 1(x \geq \theta) - 1(x < \theta) : \theta \in \Theta\}$$



- Given the m samples, then on the samples there are at most $m + 1$ equivalent hypothesis: choose the threshold on the sample (or $\theta = 1$)
- Every other hypothesis produces the exact same labeling of the samples and achieves the same empirical reward
- Is there an argument that only takes union bound over this set?

Back to the General Framework

- We will try to argue the expected performance

$$E_S[R(h_S)] \geq \max_{h \in H} R(h) - \epsilon(m)$$

- Expected Sample Average *Representativeness*: suppose that

$$\text{Rep} = E_S\left[\sup_{h \in H} R_S(h) - R(h)\right] \leq \epsilon(m)$$

How good is the sample average in terms of **representing** the population expectation, in the worst case over H

- Then we can prove expected error of $\epsilon(m)$

$$E_S[R(h_S)] = E[R_S(h_S)] - E[R_S(h_S) - R(h_S)] \geq E[R_S(h_S)] - \epsilon(m)$$

- Since h_S optimizes $R_S(h)$ and $h_* = \text{argmax}_{h \in H} R(h)$ is feasible

$$E[R_S(h_S)] \geq E[R_S(h_*)] = R(h_*)$$

h_* does not depend on the samples

$$E[R_S(h_*)] \stackrel{\text{def}}{=} \frac{1}{m}\sum_j E\left[h\left(v^j; h_*\right)\right] = E[h(v; h_*)] = R(h_*)$$

If we can bound representativeness

$$\text{Rep} = E_S\left[\sup_h R_S(h) - R(h)\right] \leq \epsilon(m)$$

Then we can bound expected performance

$$E[R(h_S)] \geq E[R(h_*)] - \epsilon(m)$$

Bounding Error via Representativeness*

- We will try to argue the expected performance

$$E_S[R(h_S)] \geq \max_{h \in H} R(h) - \epsilon(m)$$

- Expected Sample Average *Representativeness*: suppose that

$$\text{Rep}_* = E_S[R_S(h_S) - R(h_S)] \leq \epsilon(m)$$

How good is the sample average in terms of **representing** the population expectation, **of the ERM hypothesis**

- Then we can prove expected error of $\epsilon(m)$

$$E_S[R(h_S)] = E[R_S(h_S)] - E[R_S(h_S) - R(h_S)] \geq E[R_S(h_S)] - \epsilon(m)$$

- Since h_S optimizes $R_S(h)$ and $h_* = \text{argmax}_{h \in H} R(h)$ is feasible

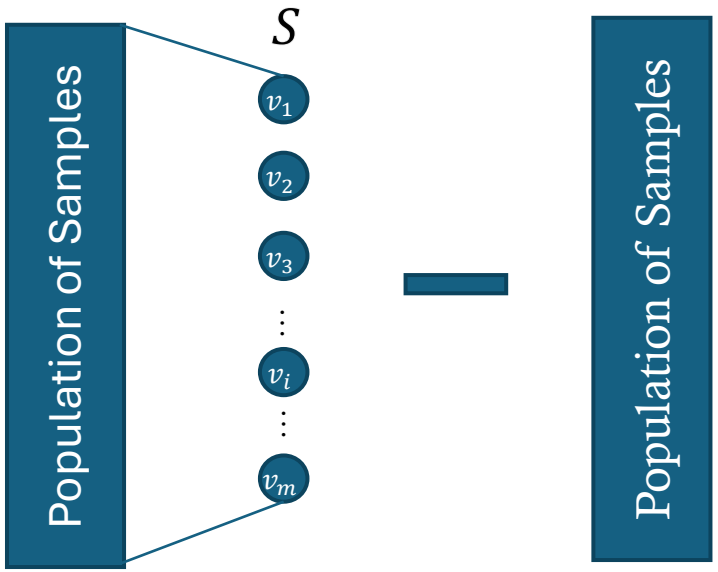
$$E[R_S(h_S)] \geq E[R_S(h_*)] = R(h_*)$$

h_* does not depend on the samples

$$E[R_S(h_*)] \stackrel{\text{def}}{=} \frac{1}{m} \sum_j E\left[h\left(v^j; h_*\right)\right] = E[h(v; h_*)] = R(h_*)$$

$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$

How good is the sample average in terms of **representing** the population expectation, **of the ERM hypothesis**



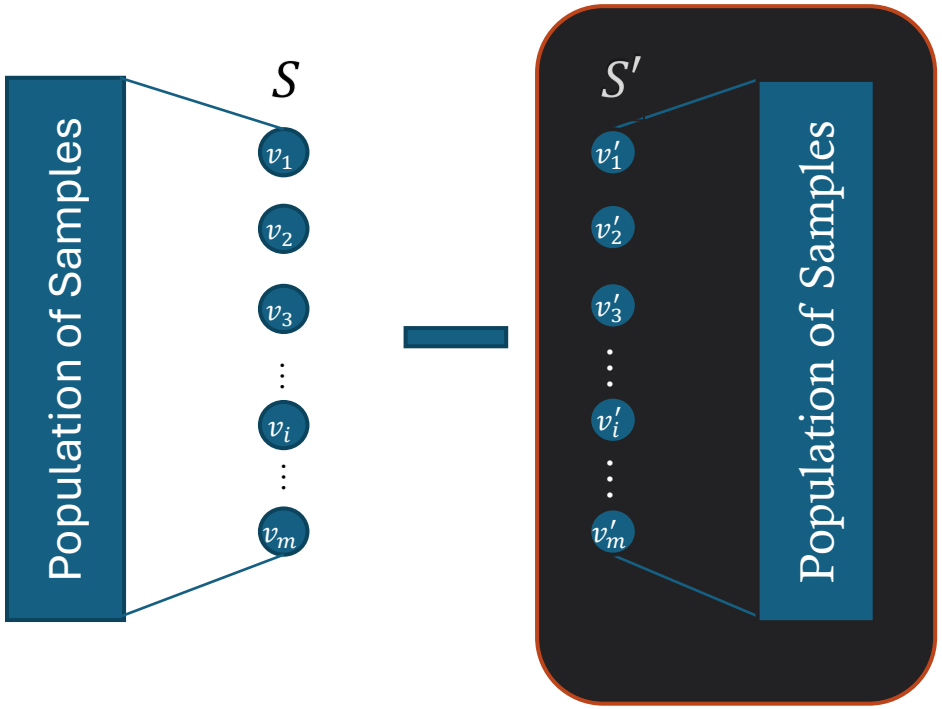
Choose $h_S \in H$ to maximize average reward on S

Learner

$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$

$$\text{Rep}_* = E\left[R_S(h_S) - E_{S'}[R_{S'}(h_S)]\right]$$

How good is the sample average in terms of **representing** the population expectation, **of the ERM hypothesis**



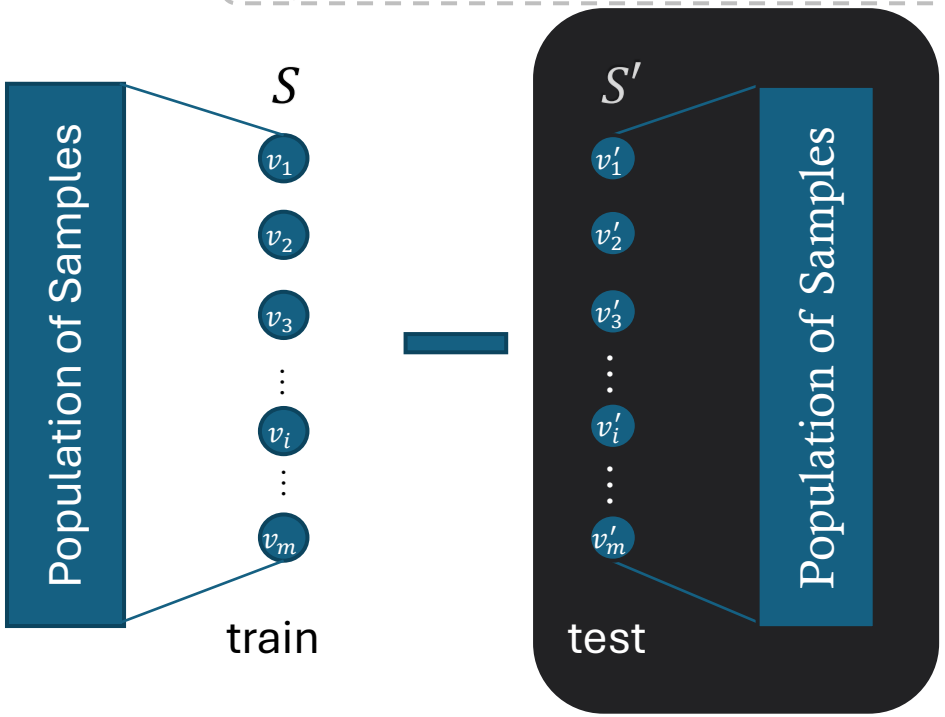
Choose $h_S \in H$ to maximize average reward on S

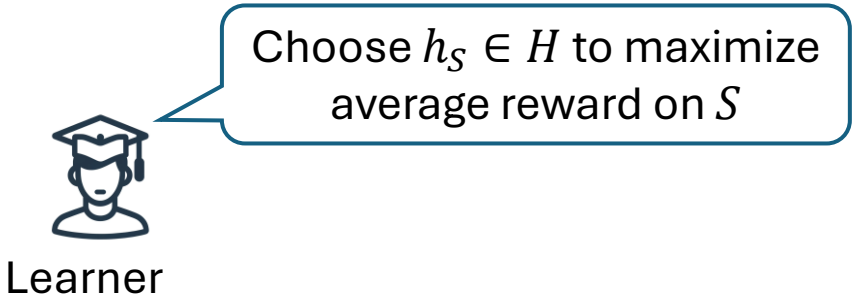
Learner

$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$

$$\text{Rep}_* = E\left[R_S(h_S) - E_{S'}\left[R_{S'}(h_S)\right]\right]$$

$$\text{Rep}_* = E_{S,S'}\left[R_S(h_S) - R_{S'}(h_S)\right]$$

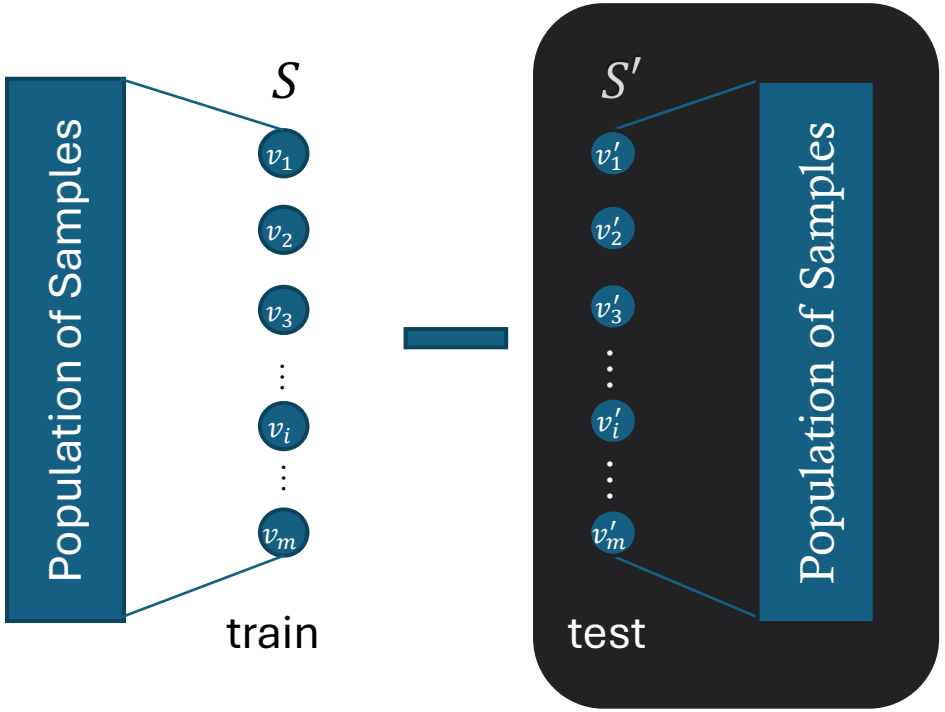




How good is the sample average in terms of **representing** the population expectation, **of the ERM hypothesis**

If we randomly split $2m$ samples into a **train and test split**, and choose a hypothesis based on train, **how different is the training reward from the test reward**

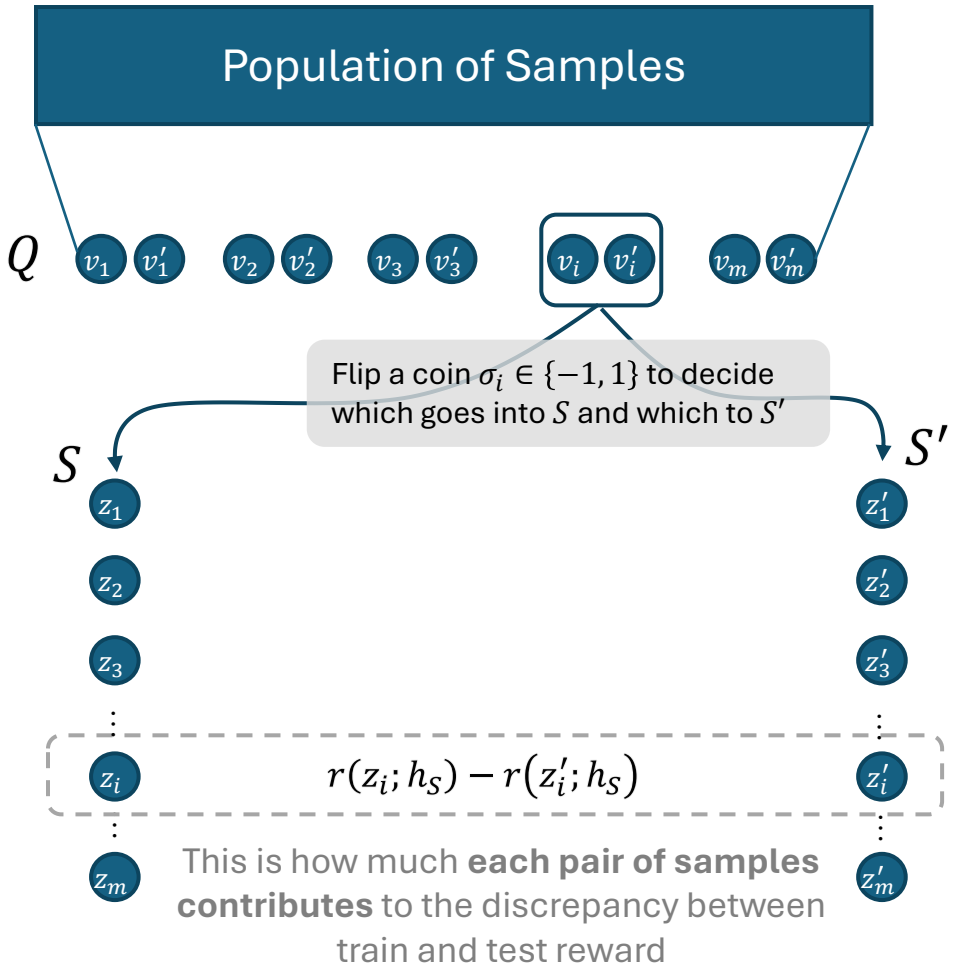
$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$
$$\text{Rep}_* = E\left[R_S(h_S) - E_{S'}\left[R_{S'}(h_S)\right]\right]$$
$$\text{Rep}_* = E_{S,S'}\left[R_S(h_S) - R_{S'}(h_S)\right]$$



Choose $h_S \in H$ to maximize average reward on S

Learner

Equivalent Process



This is how much **each pair of samples contributes** to the discrepancy between train and test reward

If we randomly split $2m$ samples into a **train and test split**, and choose a hypothesis based on train, **how different is the training reward from the test reward**

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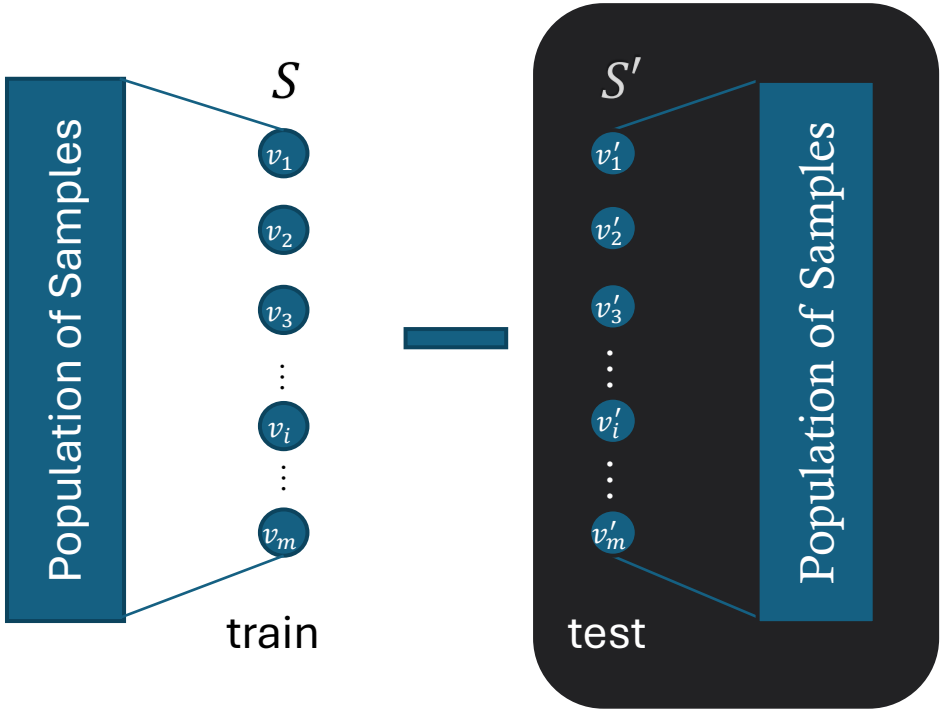
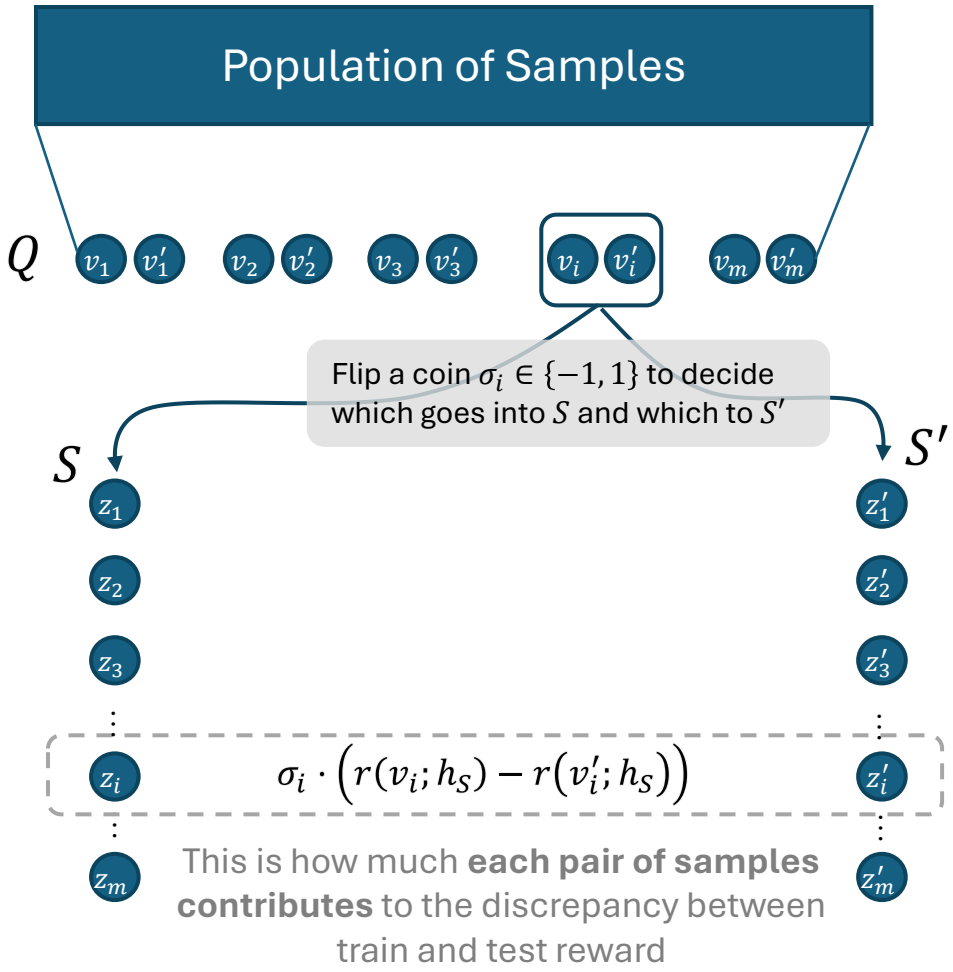


Choose $h_S \in H$ to maximize average reward on S

Learner

Equivalent Process



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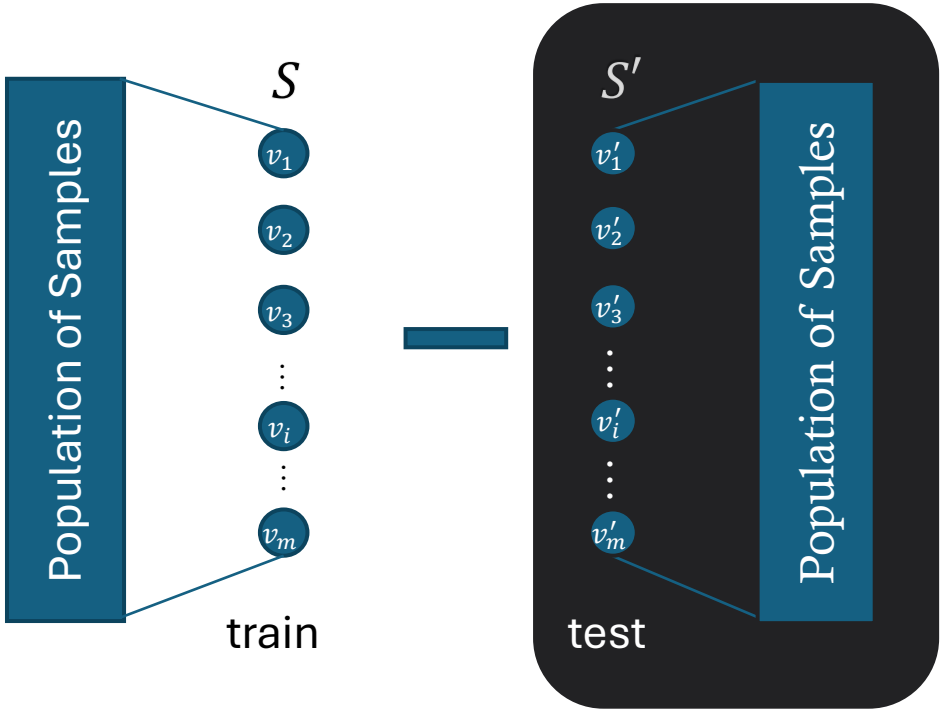
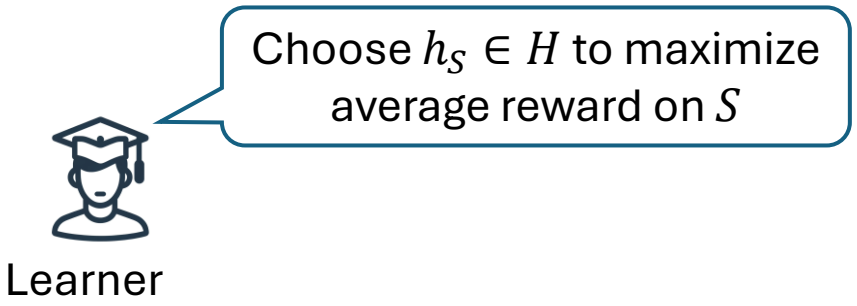
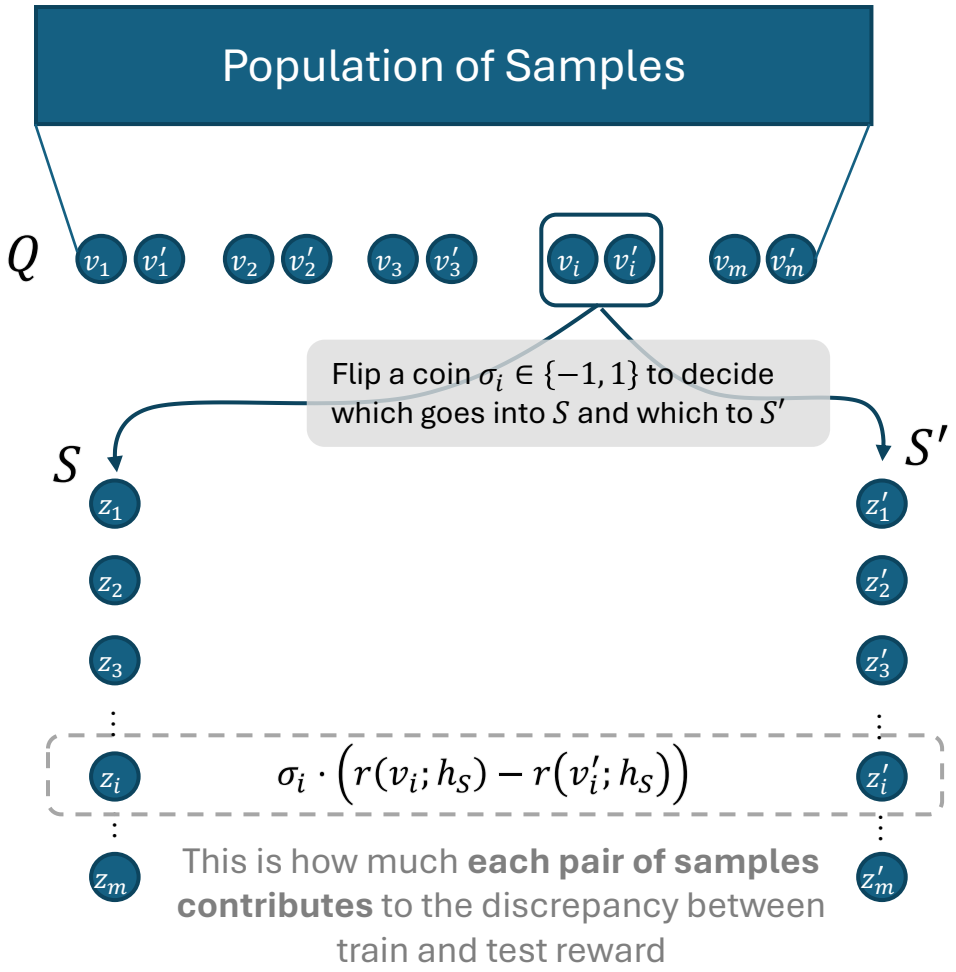




Equivalent Process



Important. If h was not chosen based on S but was some fixed $h \in H$. Then this contribution is mean zero in expectation over the random split σ_i, even when we condition on the sample values Q

$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$
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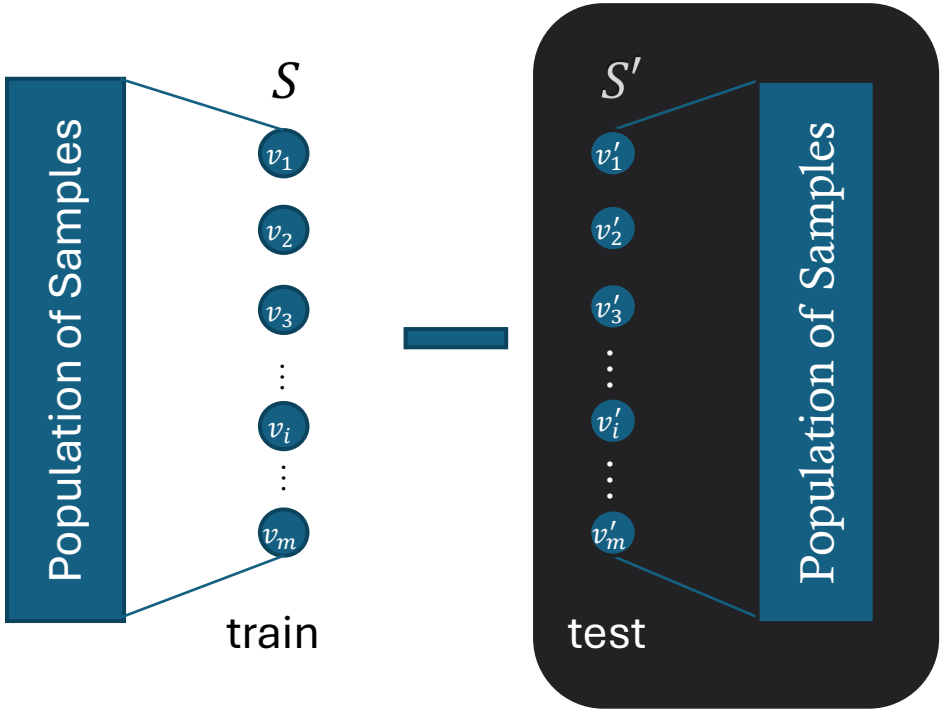
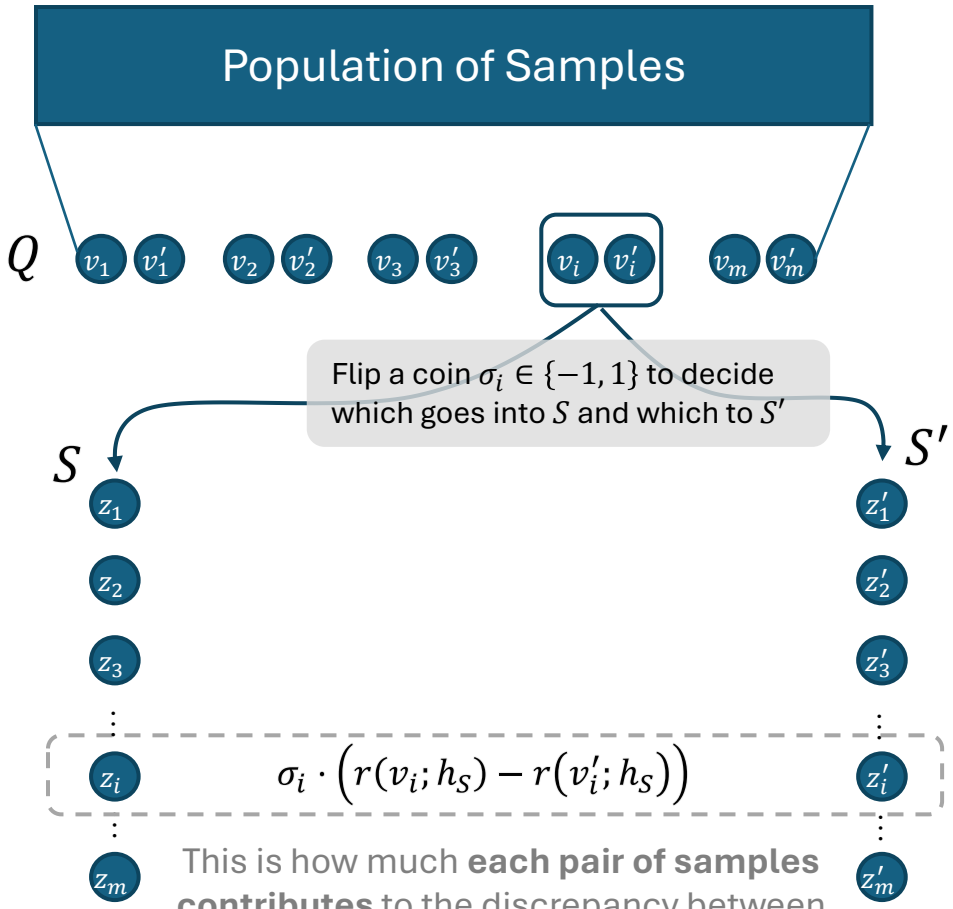


Choose $h_S \in H$ to maximize average reward on S

Learner

Equivalent Process



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$$\text{Rep}_* = E_{Q,\sigma}\left[\frac{1}{m}\sum_{i=1}^{m} \sigma_i \cdot \left(r(v_i; h_S) - r\left(v'_i; h_S\right)\right)\right]$$

$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$
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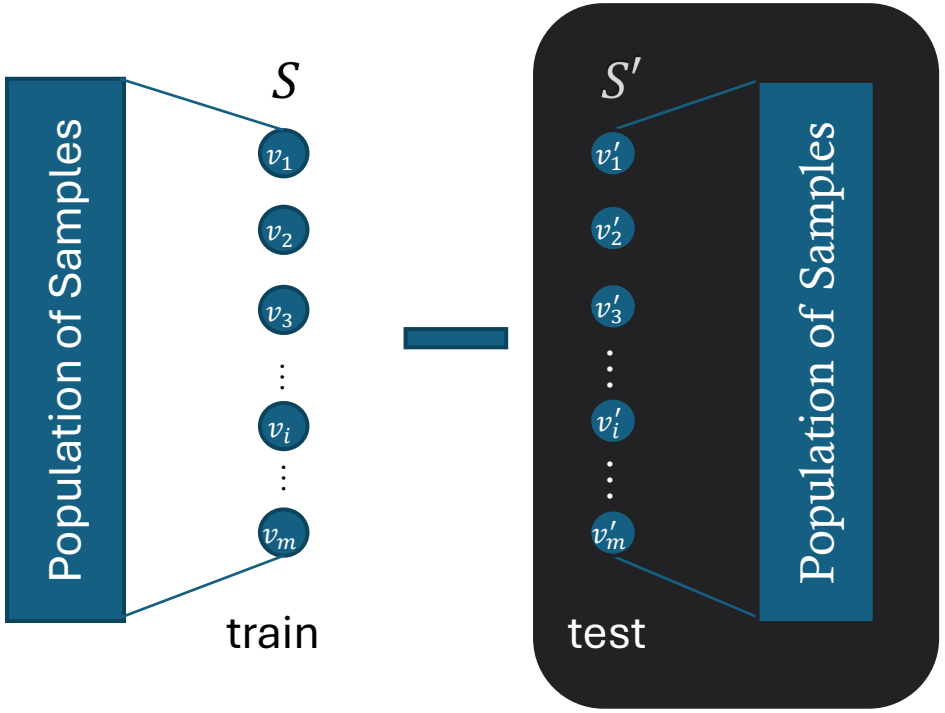

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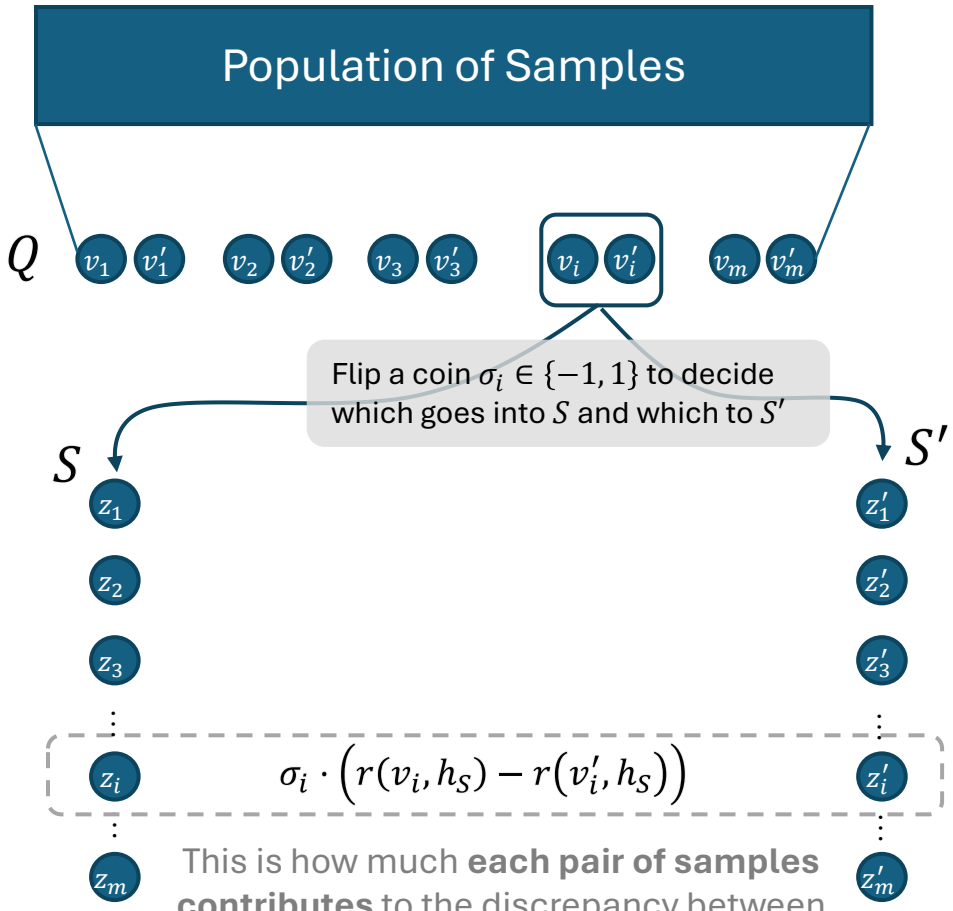
Equivalent Process



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Since it is hard to argue about the "inner-workings" of ERM, let's be pessimistic here and replace h_S **with an adversarial choice**

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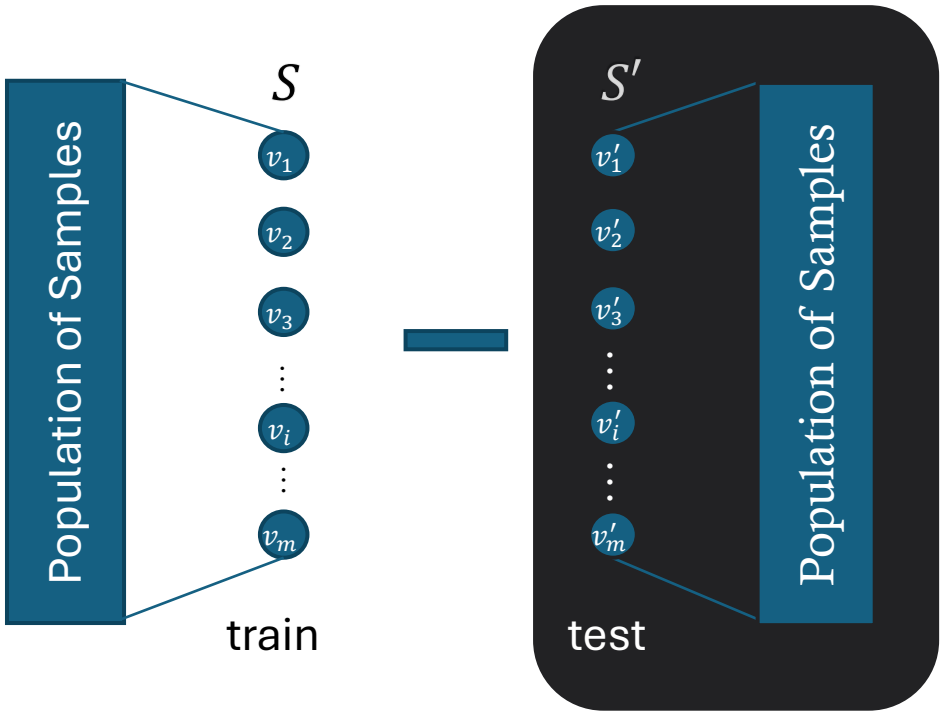

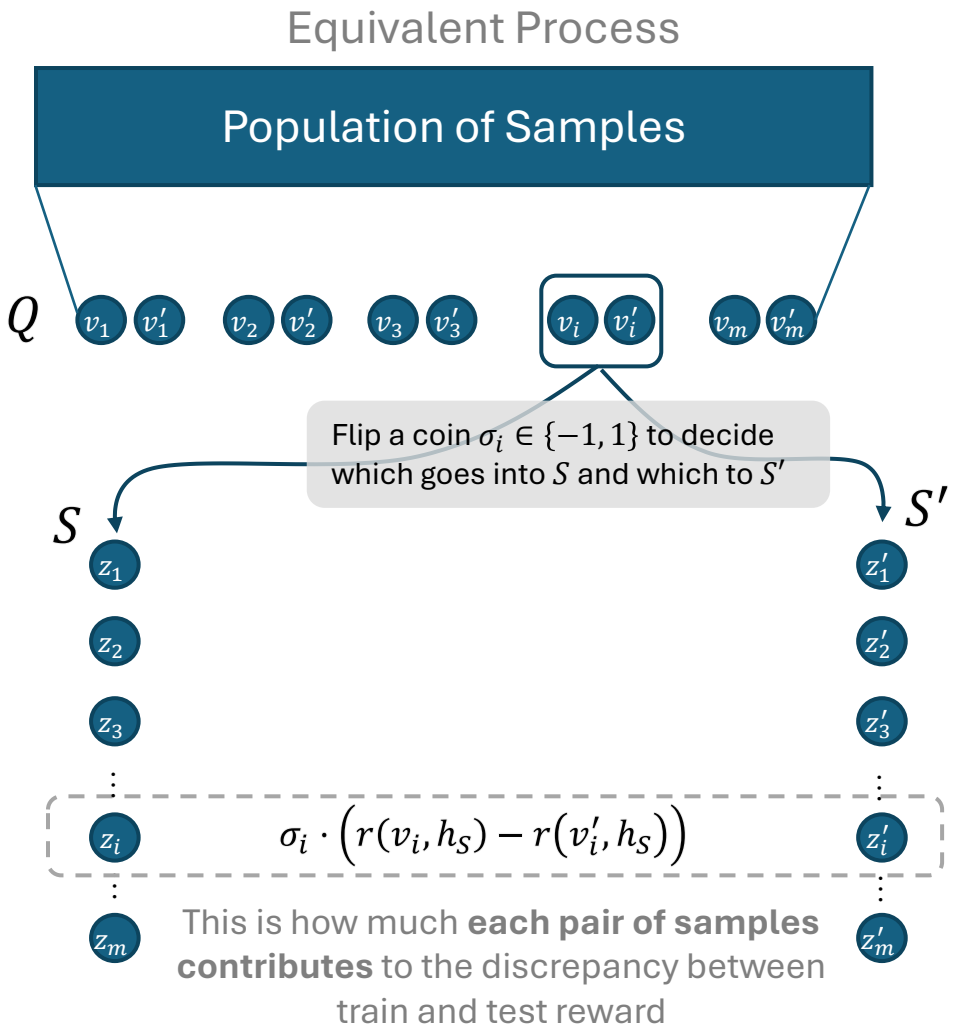

Learner

Choose $h_S \in H$ to maximize average reward on S

Adversary

For any Q, σ, choose $h \in H$ to maximize the average discrepancy

$$\text{Rep}_* \leq E_{Q,\sigma}\left[\max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot \left(r(v_i; h) - r(v'_i; h)\right)\right]$$

Symmetrization and Rademacher Complexity

- We can upper bound representativeness by

$$\mathrm{Rep}_* \leq E_{S,S',\sigma}\left[\max_{h\in H}\frac{1}{m}\sum_{j=1}^{m}\sigma_j\left(r(v_j;h)-r\left(v_j';h\right)\right)\right]$$

- We can upper bound by splitting the max into the two separate

$$\mathrm{Rep}_* \leq E_{S,\sigma}\left[\max_{h\in H}\frac{1}{m}\sum_{j=1}^{m}\sigma_j r(v_j;h)\right]+E_{S',\sigma}\left[\max_{h\in H}-\frac{1}{m}\sum_{j=1}^{m}\sigma_j r\left(v_j';h\right)\right]$$

- But these two quantities are the same

$$\mathrm{Rep} \leq 2\,E_{S,\sigma}\left[\max_{h\in H}\frac{1}{m}\sum_{j=1}^{m}\sigma_j r(v_j;h)\right]$$

Rademacher Complexity of Hypothesis Space H

Empirical Rademacher Complexity

Empirical Rademacher Complexity of hypothesis space H on samples S:

$$\mathrm{Rad}(S, H) \coloneqq 2E_{\sigma}\left[\max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h)\right]$$

Theorem. We have thus proven that:

$$E[R(h_S)] \geq R(h_*) - E_S[\mathrm{Rad}(S, H)]$$

Bounding Empirical Rademacher Complexity

- We have now conditioned on the value of the samples $S = \{v_1, \ldots, v_m\}$
- For a fixed h, contribution of each sample v_i, in expectation over the random split σ_i, is zero

$$E_\sigma \left[\frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h) \right] = 0$$

- We have reduced to arguing how large this "almost mean zero" quantity can be for a fixed set of samples $\{v_1, \ldots, v_m\}$
- Requires arguing properties of the hypothesis space H on the samples S and not on the whole unknown support of the unknown distribution F

Simple Case

- Suppose that H was finite, i.e. $H = \{h_1, \dots, h_K\}$ and reward bounded in $[0,1]$
- By Hoeffding concentration inequality and the mean-zero property, for each h_t, w.p. $1 - \delta$

$$\left| \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h_t) \right| \leq \sqrt{\frac{\log(2/\delta)}{2m}}$$

- By the union bound, w.p. $1 - \delta$

$$\max_{t=1}^{K} \left| \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h_t) \right| \leq \sqrt{\frac{\log(2K/\delta)}{2m}}$$

- This implies that the expected value of this quantity is of the same order

$$\mathrm{Rad}(S, H) \coloneqq 2E_\sigma \left[\max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h) \right] \approx \sqrt{\frac{\log(2K)}{2m}}$$

Massart's lemma. For any finite hypothesis space H:

$$\mathrm{Rad}(S, H) \leq 2\sqrt{\frac{2\log(|H|)}{m}}$$

Beyond Simple Case

- Suppose we can find a finite subspace $\widetilde{H}_S \subseteq H$ such that every $h \in H$ has a **representative** $\tilde{h} \in \widetilde{H}_S$ that has the exact same behavior on the samples S

$$\forall v_i \in S: r(v_i; h) = r(v_i; \tilde{h})$$

- Then Empirical Rademacher Complexity of H is upper bounded by that of $\widetilde{H}_S$

$$\mathrm{Rad}(S, H) \coloneqq 2E_\sigma \left[\max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h) \right]$$

$$\leq 2E_\sigma \left[\max_{h \in \widetilde{H}_S} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h) \right] \leq 2\sqrt{\frac{2\log(|\widetilde{H}_S|)}{m}}$$

Beyond Simple Case

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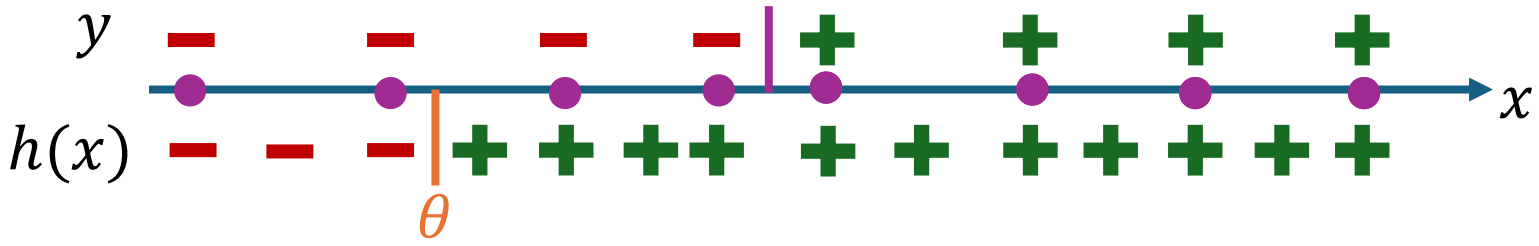
- Classification Example with Threshold Functions: Data $v = (x, y)$, with $x \in [-1, 1]$ and $y \in \{-1,1\}$. Label every $x \geq \theta$ with $+1$ and every $x < \theta$ with -1.

$$H = \{x \to 1(x \geq \theta) - 1(x < \theta) : \theta \in \Theta\}$$



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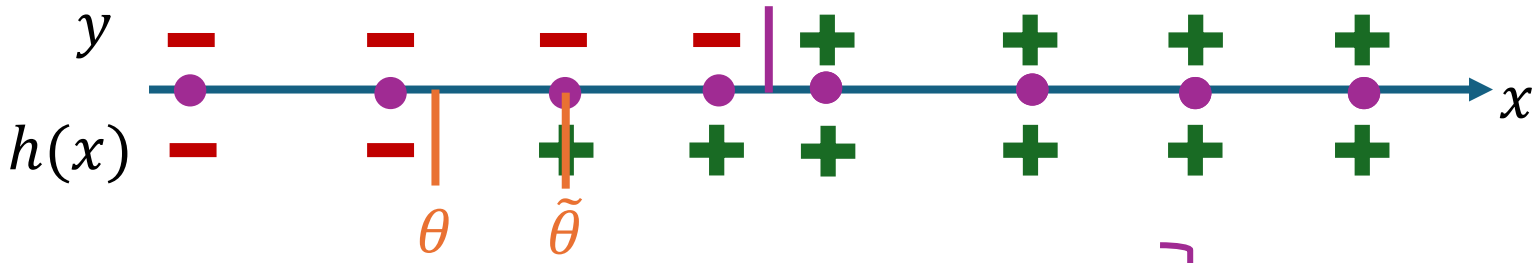


- We only look at the behavior on the samples
- Suffices to look at thresholds equal to a sample or 1

$$\mathrm{Rad}(S, H) \leq 2\sqrt{\frac{2\log(m+1)}{m}}$$

Growth Rate of Function Space

- Suppose we can find a finite subspace $\widetilde{H}_S \subseteq H$ such that every $h \in H$ has a **representative** $\tilde{h} \in \widetilde{H}_S$ that has the exact same behavior on the samples S

$$\forall v_i \in S: r(v_i; h) = r(v_i; \tilde{h})$$

- Empirical Rademacher Complexity of H is upper bounded by that of $\widetilde{H}_S$
- Growth Rate $\tau(m, H)$: the size of the smallest $\widetilde{H}_S$ that satisfies the above property, in the worst case over sample dataset of size m
- Example. For threshold classifiers $\tau(m, H) = m + 1$

Theorem. For any hypothesis H

$$\mathrm{Rad}(S, H) \leq 2\sqrt{\frac{2\log(\tau(m, H))}{m}}$$

SideNote For classification, a seminal notion is the Vapnik-Chervonenkis **(VC) dimension**: size d of largest dataset that the hypothesis can assign labels in all possible manners

Cannot be assigned by threshold classifiers $\Rightarrow d = 2$

Sauer's Lemma. If has VC-dim $\leq d$ then $\tau(m, H) \lesssim 2^d \Rightarrow \mathrm{Rad}(S, H) \lesssim \sqrt{d/m}$

Discretization on Samples

- Suppose we can find a finite subspace $\widetilde{H}_{S,\epsilon} \subseteq H$ such that every $h \in H$ has a **representative** $\tilde{h} \in \widetilde{H}_{S,\epsilon}$ that has approximately the same behavior on the samples S

$$\forall v_i \in S: \left| r(v_i; h) - r\left(v_i; \tilde{h}\right) \right| \leq \epsilon$$

- Empirical Rademacher Complexity of H upper bounded approximately by $\widetilde{H}_{S,\epsilon}$

$$\mathrm{Rad}(S, H) \coloneqq 2E_\sigma \left[\max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h) \right]$$

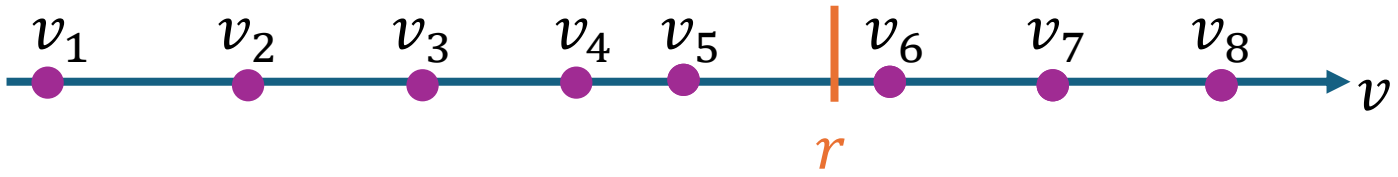
$$\leq 2E_\sigma \left[\max_{h \in \widetilde{H}_{S,\epsilon}} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h) \right] + 2\epsilon \leq 2\sqrt{\frac{2\log\left(\left|\widetilde{H}_{S,\epsilon}\right|\right)}{m}} + 2\epsilon$$

Back to Mechanism Design from Samples

Example: Pricing from Samples

- Suppose we are given a set of samples S of a bidder's value
- We optimize over the space of posted prices
- For every price r we want to find a price $\tilde{r}$ that achieves almost the same revenue as r for every value in the samples

$$\forall v_i \in S\colon \left| r \cdot 1\{v_i \geq r\} - \tilde{r} \cdot 1\{v_i \geq \tilde{r}\} \right| \leq \epsilon$$



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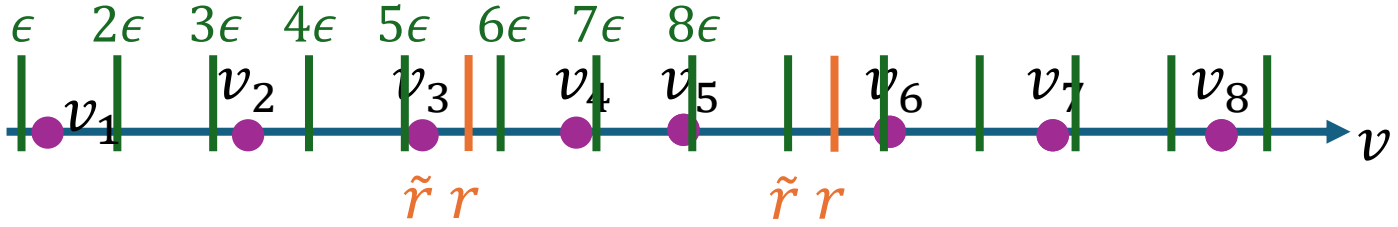

- For every r, pick maximum of {largest multiple of ϵ below r, largest sampled value below r}. At most $m + 1/\epsilon$ prices.

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- For every r, pick maximum of {largest multiple of ϵ below r, largest sampled value below r}. At most $m + 1/\epsilon$ prices.

$$\mathrm{Rad}(S, H) \leq 2\sqrt{\frac{2\log\left(m + \frac{1}{\epsilon}\right)}{m}} + 2\epsilon \leq 4\sqrt{\frac{2\log(2m)}{m}}$$

$\epsilon = 1/m$

Second Price with a Reserve

- Suppose we are given a set of samples S of n bidder value profiles
- Optimize over the space of Second-Price Auctions with a Reserve
- For every price r we want to find a price $\tilde{r}$ that achieves almost the same revenue as r for every value in the samples

$$\forall v_i = (v_{i1}, \dots, v_{in}) \in S\colon\ |\mathrm{rev}(v_i; r) - \mathrm{rev}(v_i; \tilde{r})| \leq \epsilon$$

- For every r, pick maximum of {largest multiple of ϵ below r, largest sampled value below r}. At most $m \cdot n + 1/\epsilon$ prices.

$$\mathrm{Rad}(S, H) \leq 2\sqrt{\frac{2\log(m \cdot n + 1/\epsilon)}{m}} + 2\epsilon \leq 4\sqrt{\frac{2\log(2m \cdot n)}{m}}$$

$\epsilon = 1/m$

Second Price with Player-Specific Reserves

- Suppose we are given a set of samples S of n bidder value profiles
- Optimize over the space of Second-Price with Player-Specific Reserves
- For every price vector $r = (r_1, \dots, r_n)$ we want to find a vector $\tilde{r}$ that achieves almost the same revenue as r for every value in the samples

$$\forall v_i = (v_{i1}, \dots, v_{in}) \in S\colon\ |\mathrm{rev}(v_i; r) - \mathrm{rev}(v_i; \tilde{r})| \leq \epsilon$$

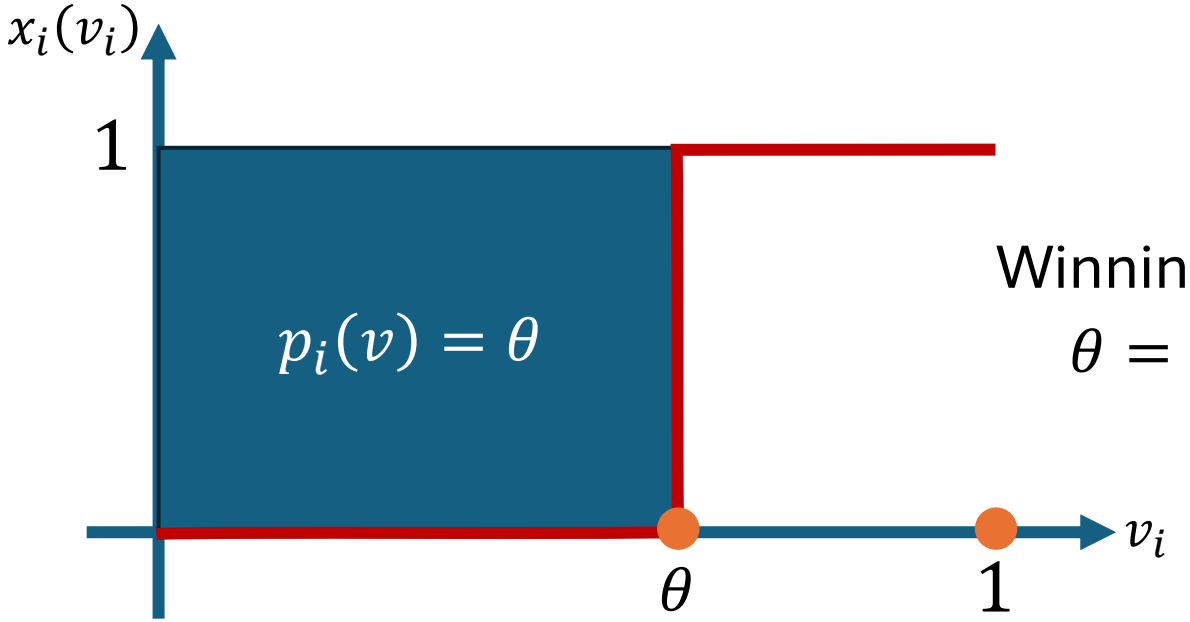
- For every r_j, pick maximum of {largest multiple of ϵ below r, largest sampled value for bidder j below r}. At most $(m + 1/\epsilon)^n$ prices.

$$\mathrm{Rad}(S, H) \leq 2\sqrt{\frac{2n\log(m + 1/\epsilon)}{m}} + 2\epsilon \leq 4\sqrt{\frac{2n\log(2m)}{m}}$$

$\epsilon = 1/m$

Competing with the Myerson Auction

- Want to optimize over virtual welfare maximizing mechanisms
- For each bidder i, we assign a monotone virtual value function ϕ_i
- Allocate to the bidder with highest positive virtual value $\phi_i(v_i)$
- Charge dominant strategy truthful payments

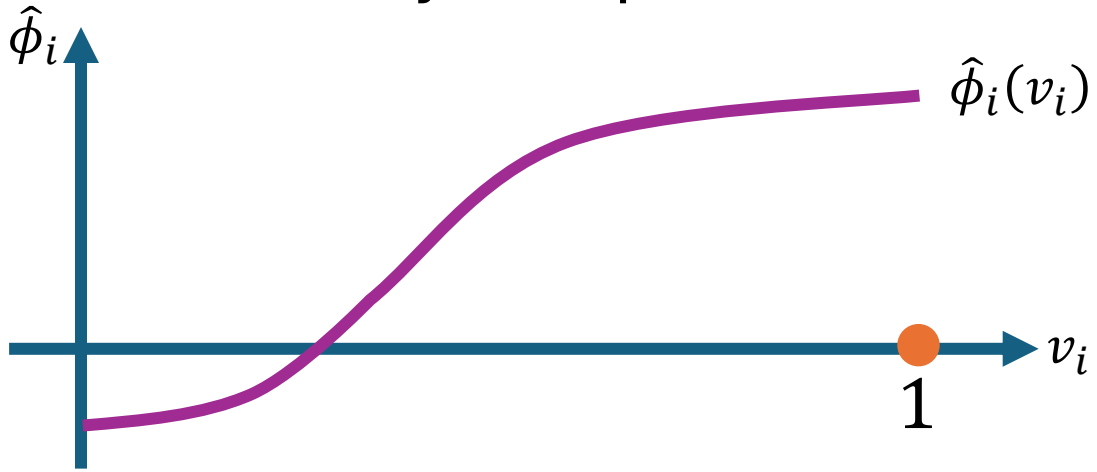


Winning threshold:

$$\theta = \phi_i^{-1}\left(\max_{j \neq i} \phi_j(v_j), \eta_i\right)$$

Optimizing over Virtual Value Functions

- ERM optimizes over all monotone functions for each bidder
- This space is infinite and a bit harder to discretize
- We will see that monotonicity is important!



- We introduce a variant of Rademacher complexity analysis that will help us in the analysis of ERM over virtual welfare maximizers

Back to Statistical Learning Theory

$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$
$$\text{Rep}_* = E\left[R_S(h_S) - E_{S'}\left[R_{S'}(h_S)\right]\right]$$
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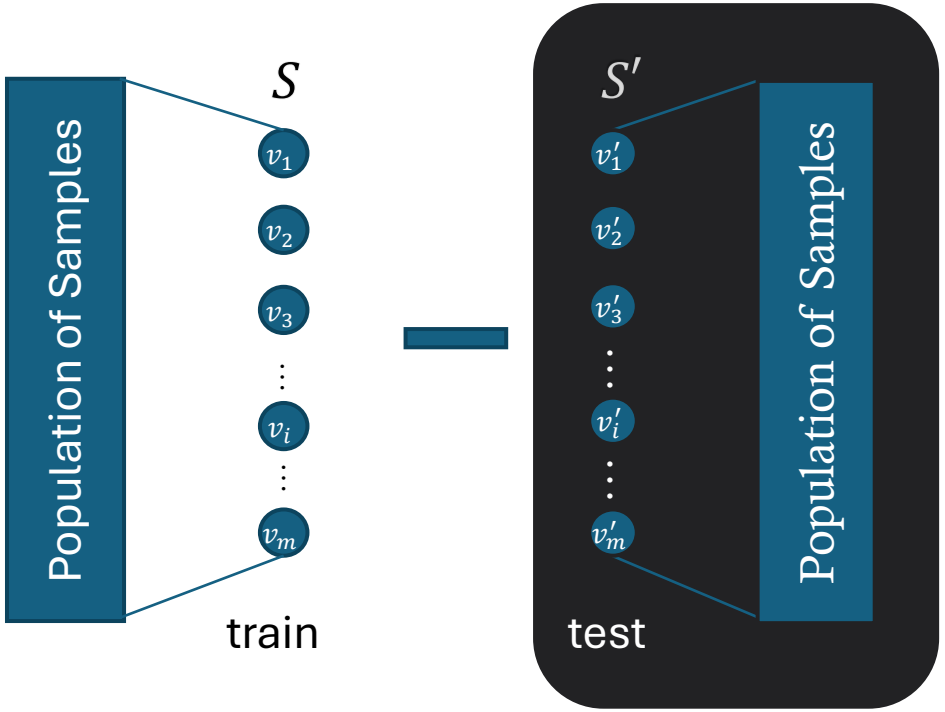
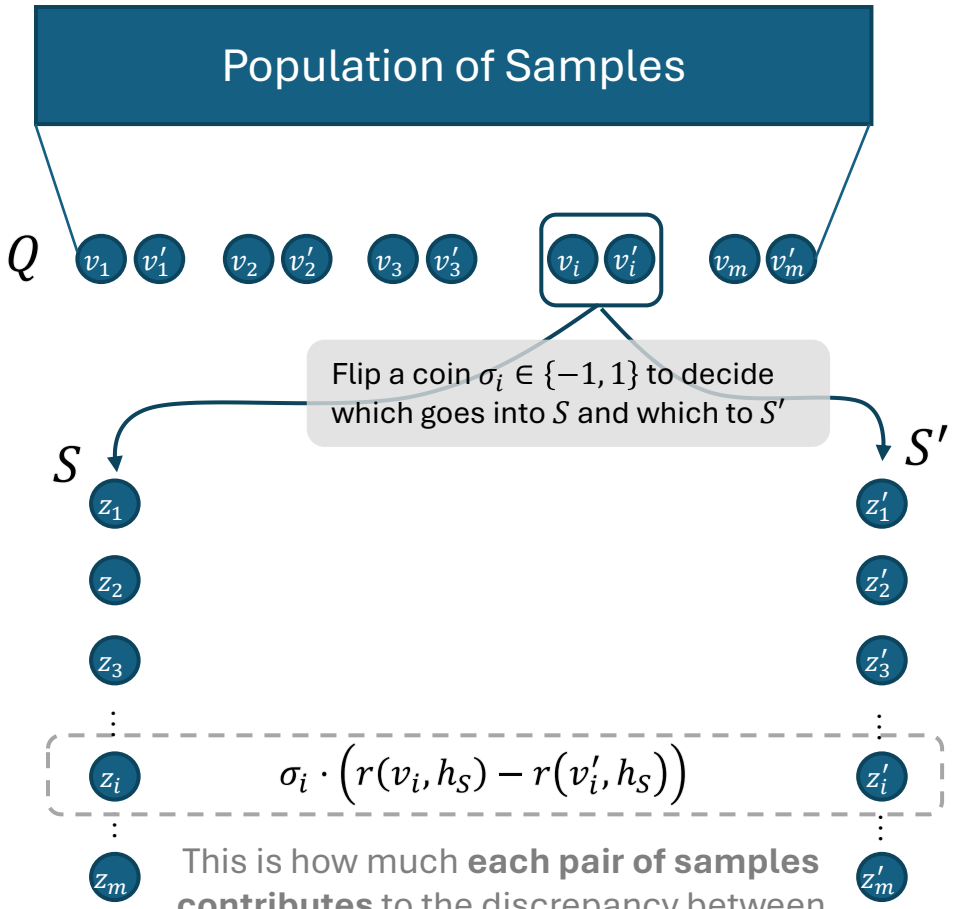


Learner

Choose $h_S \in H$ to maximize average reward on S

Equivalent Process

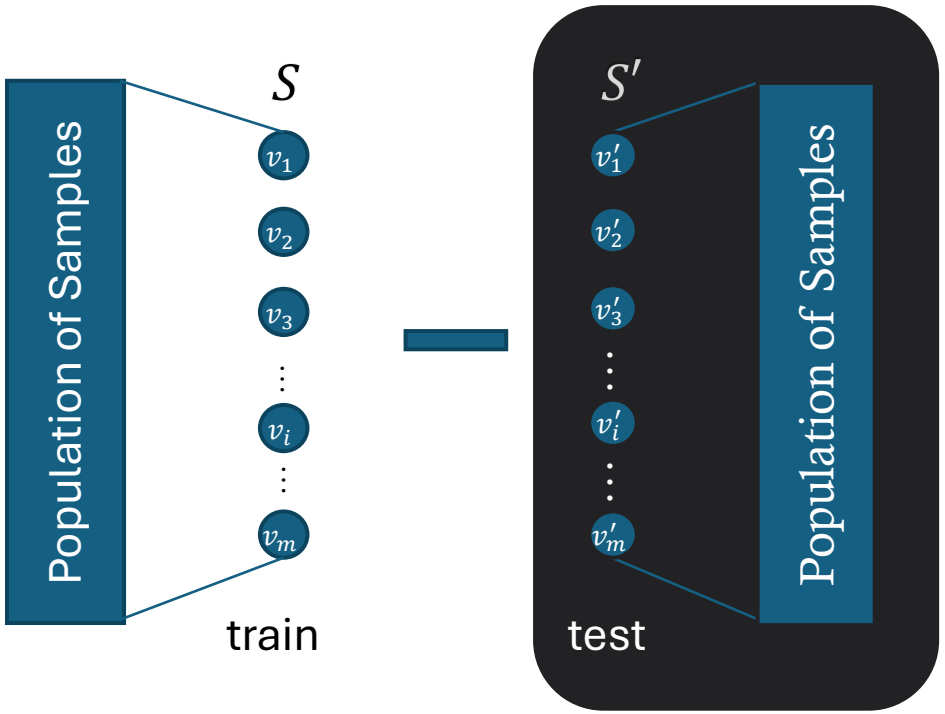


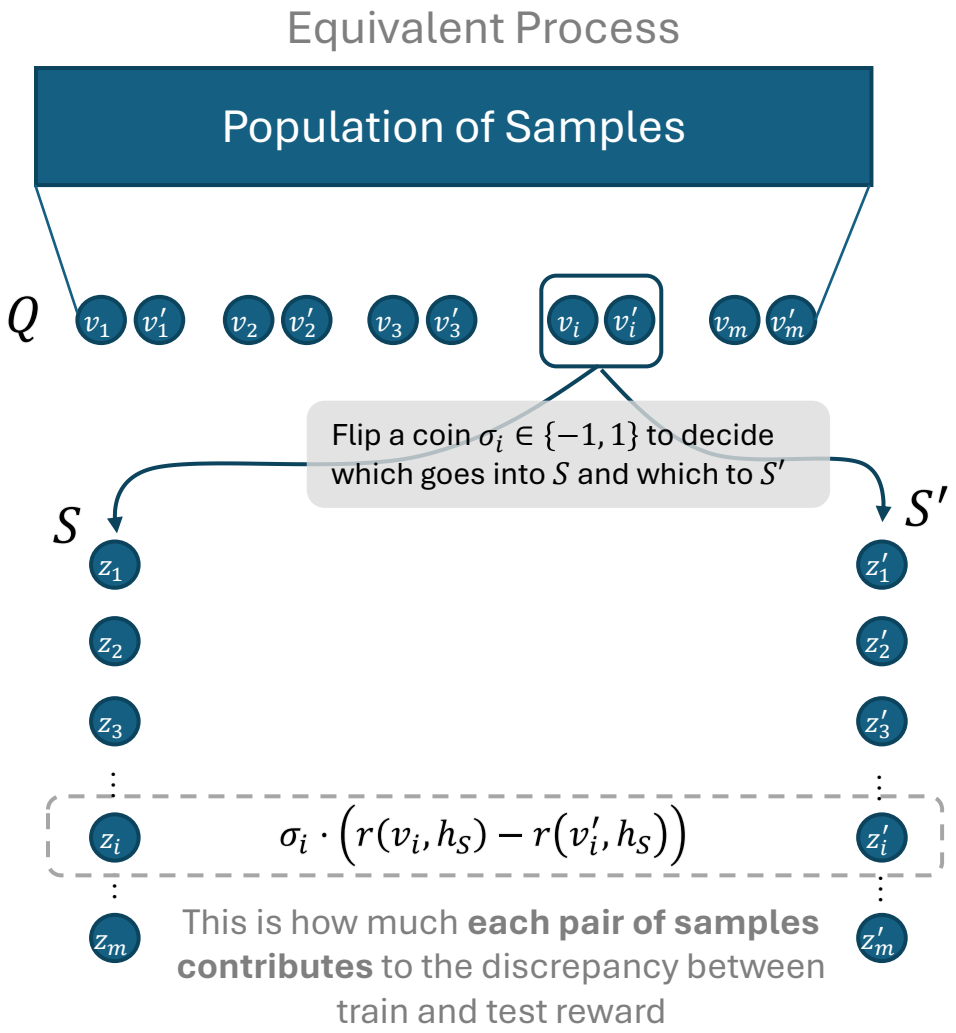
This is how much **each pair of samples contributes** to the discrepancy between train and test reward

$$\text{Rep}_* = E_{Q,\sigma}\left[\frac{1}{m}\sum_{i=1}^{m} \sigma_i \cdot \left(r(v_i; h_S) - r(v'_i; h_S)\right)\right]$$

Since it is hard to argue about the "inner-workings" of ERM, let's be pessimistic here and replace h_S **with an adversarial choice over all possible outputs of ERM on a half-sample of Q**

$$\text{Rep}_* = E[R_S(h_S) - R_D(h_S)]$$
$$\text{Rep}_* = E\left[R_S(h_S) - E_{S'}[R_{S'}(h_S)]\right]$$
$$\text{Rep}_* = E_{S,S'}\left[R_S(h_S) - R_{S'}(h_S)\right]$$





Choose $h_S \in H$ to maximize average reward on S

Learner

Benign Adversary

For any Q, σ, choose h that could be the output of ERM on some half-sample of Q

$$\text{Rep}_* \leq E_{Q,\sigma}\left[\max_{h \in H_Q} \frac{1}{m}\sum_{i=1}^{m} \sigma_i \cdot \left(r(v_i; h) - r(v'_i; h)\right)\right]$$

Refined Rademacher Complexity

- We can upper bound representativeness by

$$\mathrm{Rep}_* \leq E_{S,S',\sigma}\left[\max_{h\in H_{S\cup S'}} \frac{1}{m}\sum_{j=1}^{m}\sigma_j\left(r(v_j;h)-r\left(v_j';h\right)\right)\right]$$

- We can upper bound by splitting the max into the two separate

$$\mathrm{Rep}_* \leq E_{S,S',\sigma}\left[\max_{h\in H_{S\cup S'}} \frac{1}{m}\sum_{j=1}^{m}\sigma_j r(v_j;h)\right] + E_{S,S',\sigma}\left[\max_{h\in H_{S\cup S'}} -\frac{1}{m}\sum_{j=1}^{m}\sigma_j r\left(v_j';h\right)\right]$$

- But these two quantities are the same

$$\mathrm{Rep} \leq 2\, E_{S,S',\sigma}\left[\max_{h\in H_{S\cup S'}} \frac{1}{m}\sum_{j=1}^{m}\sigma_j r(v_j;h)\right]$$

Empirical Rademacher Complexity

Empirical Rademacher Complexity of hypothesis space H on samples S:

$$\mathrm{Rad}(S, H) \coloneqq 2E_{\sigma}\left[\max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i \cdot r(v_i; h)\right]$$

Theorem. We have thus proven that:

$$E[R(h_S)] \geq R(h_*) - E_{S,S'}\left[\mathrm{Rad}\left(S, H_{S \cup S'}\right)\right]$$

$H_{S \cup S'} \stackrel{\text{def}}{=}$ all possible outputs of ERM on some subset of $S \cup S'$ of size m

Back to Mechanism Design from Samples

Example: Pricing from Samples

- Suppose we are given a set of samples S of a bidder's value
- We optimize over the space of posted prices
- On any subset of size m of a set of samples Q of size $2m$, ERM will return a reserve price that is equal to one of the $2m$ values!

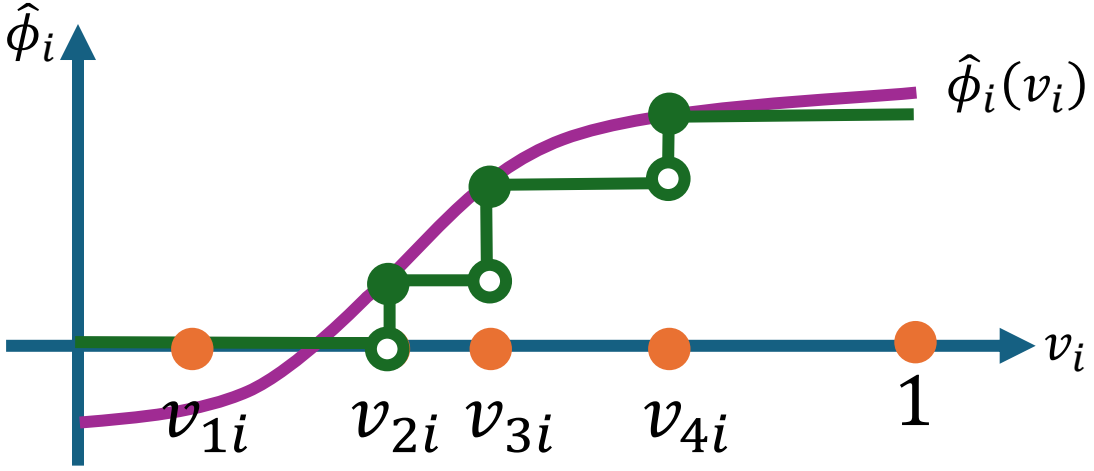
$$H_Q \subseteq \{r : r = v_i \text{ for some } v_i \in Q\}$$

- By Masart's Lemma

$$\mathrm{Rad}\left(S, H_{S \cup S'}\right) \leq 2\sqrt{\frac{2\log(2m)}{m}}$$

Optimizing over Virtual Value Functions

- ERM optimizes over all monotone functions for each bidder
- For any monotone function, we receive strictly larger payment had we used step-function on the samples (threshold to win is higher)!



Samples of bidder i values

- H_Q contains only monotone step functions that change on one of the $2m$ samples for each bidder

Equivalent Representation

- These mechanisms can be thought as follows
- Construct a set of $n \cdot m$ ranked positions
- For each sampled value of a bidder, assign a rank, in a manner that it is monotone across the values of the bidder
- Assign the item to the bidder with the highest rank

How Many are these Mechanisms?

- By monotonicity, each assignment of values to ranks, can be described by:

  "for each rank r, specify the smallest of the $2m$ sampled values for which the rank of the bidder goes above r"

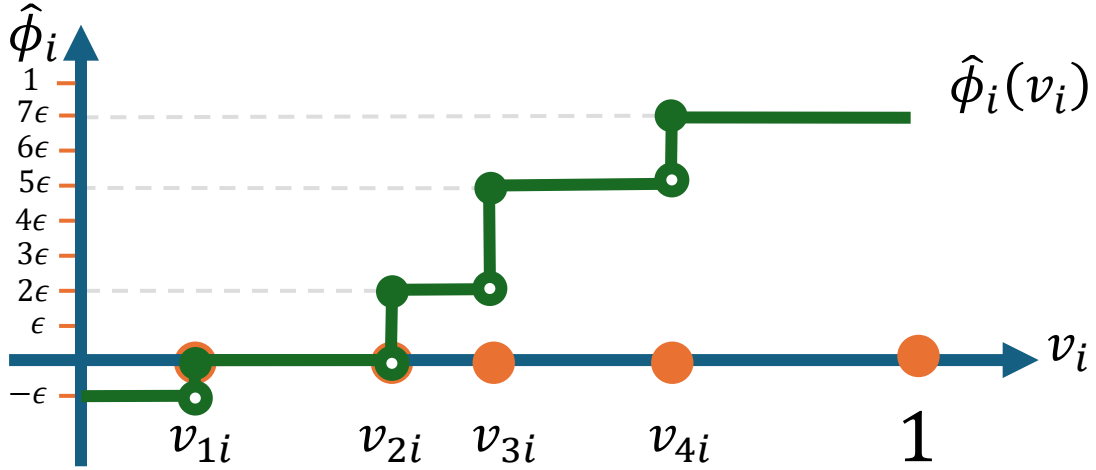
- Roughly $m^{n \cdot m}$ such combinations

$$\mathrm{Rad}\left(S, H_{S \cup S'}\right) \lesssim \sqrt{\frac{n \cdot m \cdot \log(m \cdot n)}{m}} = \sqrt{n \cdot m \cdot \log(m \cdot n)} \to \infty$$

Coarsen Space of Mechanisms we Optimize

- Consider only virtual value functions that take values on an ϵ-grid

$$\phi_i(v_i) \in \{-\epsilon, 0, \epsilon, \dots, 1\}$$



- These step functions in H_Q can be described by

  "for each value r on the grid, specify the smallest of the $2m$ sampled values for which the rank of the bidder goes above r"

- These are $\approx (2m)^{\frac{1}{\epsilon}}$ combinations for each player

Coarsen Space of Mechanisms we Optimize

- Optimize over virtual value functions that takes values on the grid
- On any subset of size m of a set of samples Q of size $2m$, ERM will return a monotone step function that takes these values and changes only on the $2m$ sampled points for each bidder
- These are $\approx (2m)^{\frac{1}{\epsilon}}$ functions for each bidder
- By Masart's Lemma

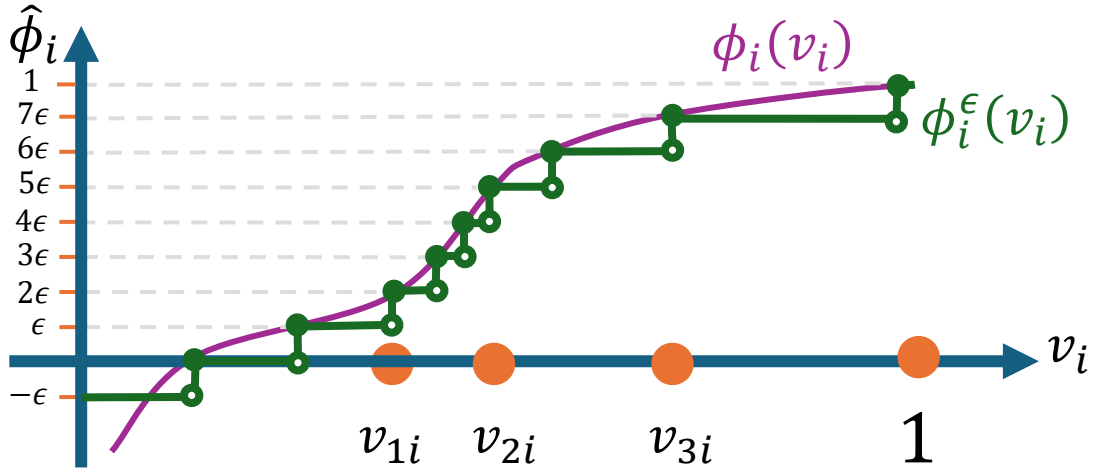
$$\mathrm{Rad}\left(S, H_{S \cup S', \epsilon}\right) \lesssim 2\sqrt{\frac{2n\log(4m)}{\epsilon \cdot m}}$$

Approximation Error

- By optimizing over ϵ-grid virtual values we don't lose more than $\approx \epsilon$
- Let h_* be optimal mechanism and h_ϵ the mechanism where each player's virtual value is rounded down to the closest multiple of ϵ. Let i_ϵ the winner in h_ϵ

$$\mathrm{Rev}(h_\epsilon) = E\left[\left[\phi_{i_\epsilon}(v_{i_\epsilon})\right]_+\right] \geq E\left[\left[\phi^\epsilon_{i_\epsilon}(v_{i_\epsilon})\right]_+\right] = E\left[\max_i\left[\phi^\epsilon_i(v_i)\right]_+\right]$$

$$\geq E\left[\max_i[\phi_i(v_i)]_+ - \epsilon\right] = \mathrm{Rev}(h_*) - \epsilon$$



Putting it all together

- If we output the mechanism h_S that optimizes the empirical revenue among all monotone virtual welfare maximizers, with virtual value functions taking values in an ϵ-grid

$$E_S[\mathrm{Rev}(h_\epsilon)] \gtrsim \mathrm{Rev}(h_*) - \sqrt{\frac{2\,\mathrm{nlog}(2m)}{\epsilon \cdot m}} - \epsilon$$

- For $\epsilon = \left(\frac{2n\,\log(2m)}{m}\right)^{\frac{1}{3}}$

$$E_S[\mathrm{Rev}(h_\epsilon)] \gtrsim \mathrm{Rev}(h_*) - 2\left(\frac{2n\log(2m)}{m}\right)^{\frac{1}{3}}$$

Zooming out

Setting	Lower Bound	Upper Bound
Regular	$\Omega(n\epsilon^{-3})$	$\tilde{O}(n\epsilon^{-3})$
MHR	$\tilde{\Omega}(n\epsilon^{-2})$	$\tilde{O}(n\epsilon^{-2})$
$[1, H]$	$\Omega(nH\epsilon^{-2})$	$\tilde{O}(nH\epsilon^{-2})$
$[0, 1]$-additive	$\Omega(n\epsilon^{-2})$	$\tilde{O}(n\epsilon^{-2})$

Table II. Sample complexity bounds in [Guo et al. 2019]

- Study initiated by [Cole, Roughgarden, 2014]
- Approaches either
  - Discretize the value space and use revenue monotonicity arguments
  - Discretize the virtual value space and use statistical learning theory arguments
  - Try to learn the virtual value functions and the CDF functions and use bounds on learning CDFs (DKW inequality and more elaborate inequalities that better control errors in the extreme quantiles)

- Statistical learning theory approaches side-step the computational question
  - Several papers on computionally efficient algorithms [Devanur et al, 2016], [Gonczarowski and Nisan 2017], [Roughgarden and Schrijvers 2016], [Guo et al, 19]

Multi-Item Auctions

- n bidders and m items and additive valuations
- A very active research area
- One well-explored direction: Sample complexity of simple mechanisms
  - Characterize simple mechanisms with constant-factor approximate revenue [Chawla et al. (2007, 2010, 2015); Hart and Nisan (2012); Babaioff et al. (2014); Rubinstein and Weinberg (2015); Yao (2015); Cai et al. (2016); Chawla and Miller (2016); Cai and Zhao (2017)]
  - Show that learning such simple mechanisms can be done with polynomial sample complexity [Morgenstern and Roughgarden (2016); Balcan et al. (2016, 2018); Cai and Daskalakis (2017); Syrgkanis (2017)]
- Optimal auctions do not have a simple characterization as virtual welfare maximizers
- One key property (revenue monotonicity) used in single-dimensional does not hold
- [Gonczarowski, Weinberg,'18] Prove a sample complexity result in this general setting with independent valuations, even without the need to use some characterization of how the optimal auction looks like