MS&E 233
Game Theory, Data Science and AI
Lecture 13

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Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- Basic Auctions and Learning to bid in auctions (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- **Simple vs optimal mechanisms (T)**
- *HW6: implement simple and optimal auctions, analyze revenue empirically*

6
- Basics of Statistical Learning Theory (T)
- Optimizing Mechanisms from Samples (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design and LLMs, Song Zuo, Google Research
- A/B testing in auction markets, Okke Schrijvers, Central Applied Science, Meta

Summarizing Last Lecture

Myerson's Theorem. When valuations are independently distributed, for any BIC, NNT and IR mechanism (and any BNE of a non-truthful mechanism), the payment contribution of each player is their expected virtual value

$$E[\hat{p}_i(v_i)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)], \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Corollary. When valuations are independently distributed, for any Bayes-Nash equilibrium of any non-truthful mechanism, the payment contribution of each player is their expected virtual value

$$E[\hat{p}_i(v)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)], \qquad \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Myerson's Optimal Auction. Assuming that virtual value functions are monotone non-decreasing, the mechanism that maximizes virtual welfare, achieves the largest possible revenue among all possible mechanisms and Bayes-Nash

$$x(v) = \mathrm{argmax}_{x \in X} \sum_i x \cdot \phi_i(v_i)\,, \qquad p_i(v) = v_i x_i(v) - \int_0^{v_i} x_i(z, v_{-i})\, dz$$

$$\mathrm{Rev} = E\left[\max_{x \in X} \sum_i x \cdot \phi_i(v_i)\right]$$

Dissecting Myerson's Optimal Auction

Identically Distributed Bidders

- Single-item setting, with all bidder values are from same distribution $v_i \sim F$
- Virtual value function is the same for all bidders

$$\phi(v) = v - \frac{1 - F(v)}{f(v)}$$

- Assume that $\phi(v)$ is monotone non-decreasing (F is regular)
- Allocating to highest virtual value $\equiv$ allocating to highest value
- **Optimal auction.** Allocate to highest value, as long as $\phi\left(v_{(1)}\right) \geq 0$
- **Optimal auction.** allocate to highest value, as long as $v_1 \geq r_*$

$$r_*\colon r - \frac{1 - F(r)}{f(r)} = 0, \qquad \text{(monopoly reserve price)}$$

When bidders are independently and identically distributed according to a regular distribution, then the optimal single-item auction among all auctions is a Second-Price Auction with a Monopoly Reserve Price

Monopoly Reserve Price

- What if we had only one bidder (monopoly)
- Then optimal thing to do is post a reserve price r_*
- The revenue from that single bidder if we post a reserve r is

$$E[r\, 1\{v \geq r\}] = r\left(1 - F(r)\right)$$

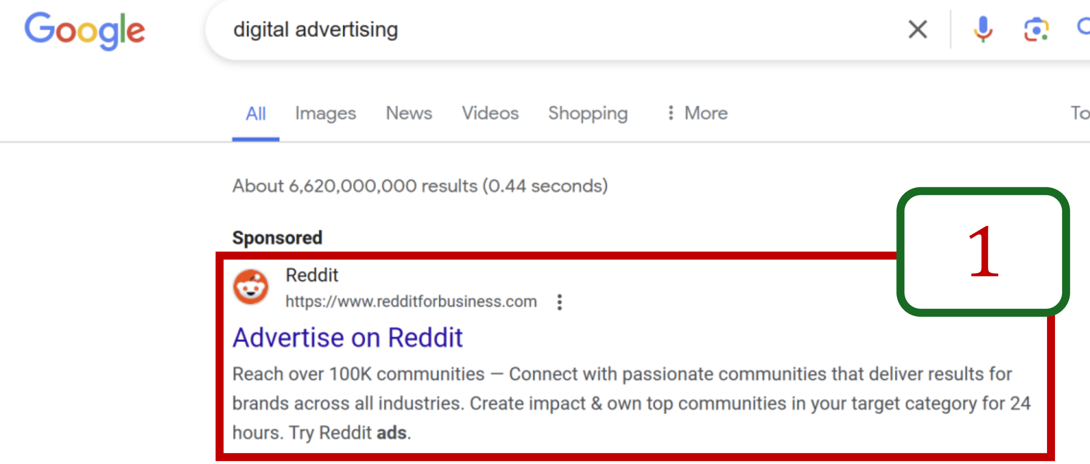
- The optimal reserve price is given by the first order condition

$$r_*\colon \left(1 - F(r)\right) - r\, f(r) = 0 \Rightarrow r - \frac{1 - F(r)}{f(r)} = 0$$

- Same as reserve price that we should be using with many bidders

Non-Identically Distributed Bidders

- What if you know ahead of time that one bidder tends to have higher values than the other bidder?
- Shouldn't you treat these bidders differently (price discrimination)?
- Shouldn't you try to extract more revenue from the bidder that tends to have a higher value?



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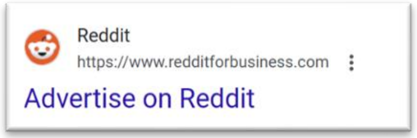
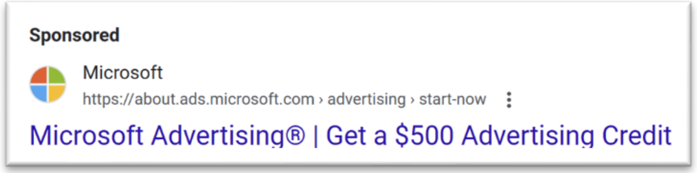


$v_1 \sim U[0,1]$



$v_2 \sim U[0,100]$

You are selling a single item to two bidders. One has values drawn U[0,1] the other U[0,100]. What is the optimal auction?

- Second-price with a reserve price
- Second-price where each bidder has a different reserve price
- First-price where each bidder has a different reserve price
- None of the above

You are selling a single item to two bidders. One has values drawn U[0,1] the other U[0,100]. What is the optimal auction?

Second-price with a reserve price	0%
Second-price where each bidder has a different reserve price	0%
First-price where each bidder has a different reserve price	0%
None of the above	0%

You are selling a single item to two bidders. One has values drawn U[0,1] the other U[0,100]. What is the optimal auction?

Second-price with a reserve price	0%
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First-price where each bidder has a different reserve price	0%
None of the above	0%

Non-Identically Distributed Bidders

- Suppose we have two bidders, $v_1 \sim U[0,1], v_2 \sim U[0,100]$
- Virtual value function for each bidder
- We should allocate to the bidder with the highest virtual value (if positive)!

Non-Identically Distributed Bidders

- Suppose we have two bidders, $v_1 \sim U[0,1], v_2 \sim U[0,100]$

- Virtual value function for each bidder

$$\phi_1(v) = v - \frac{1 - F_1(v)}{f_1(v)} = 2v - 1, \qquad \phi_2(v) = v - \frac{1 - \frac{v}{100}}{\frac{1}{100}} = 2v - 100$$

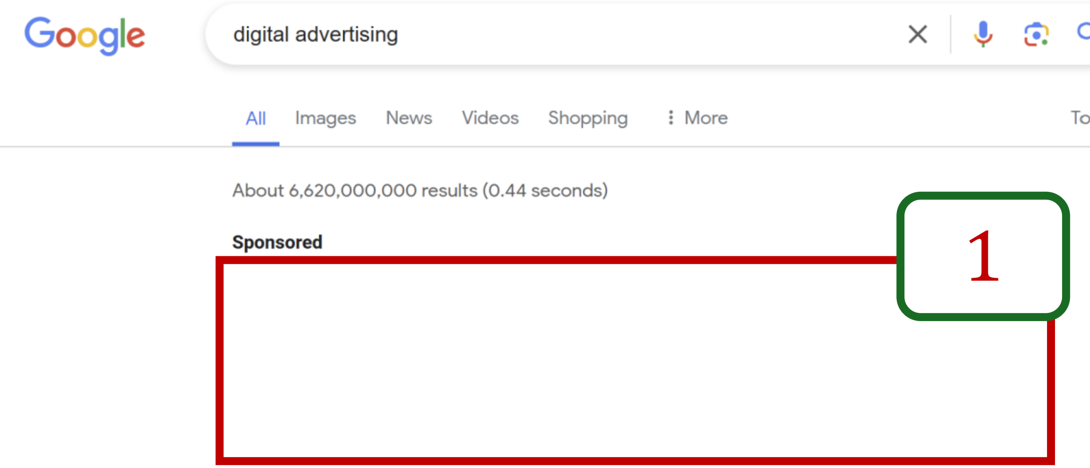
- We should allocate to the bidder with the highest virtual value (if positive)!

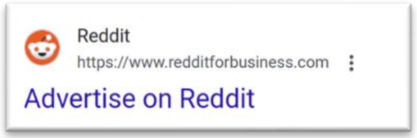
$$\text{argmax}\{0, \phi_1(v_1), \phi_2(v_2)\} = \text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Non-Identically Distributed Bidders

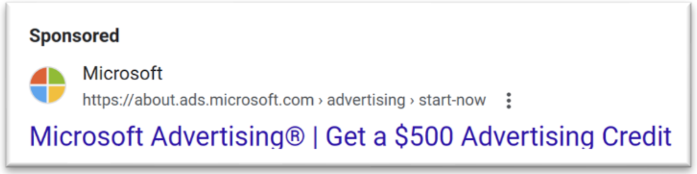
- Allocate to highest virtual value (if positive)!

$$\operatorname{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 =$ → $\phi_1 =$

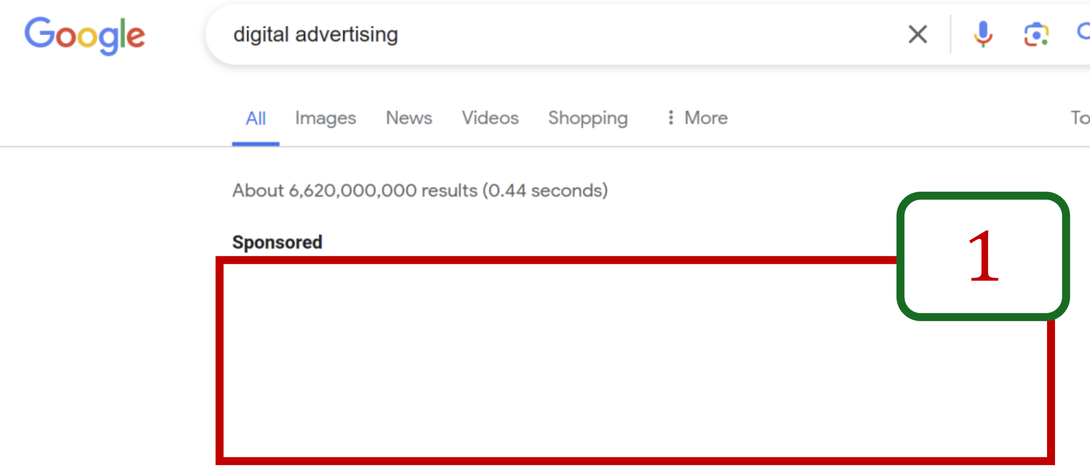


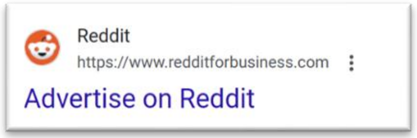
$v_2 \sim U[0,100]$ → $v_2 =$ → $\phi_2 =$

Non-Identically Distributed Bidders

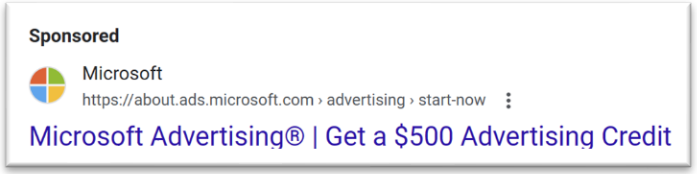
- Allocate to highest virtual value (if positive)!

$$\text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 = 1$ → $\phi_1 = 1$

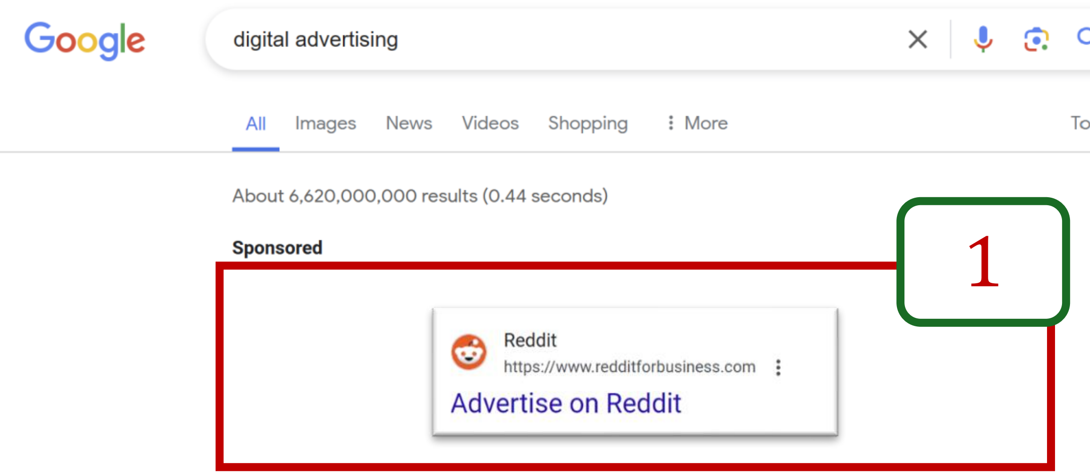


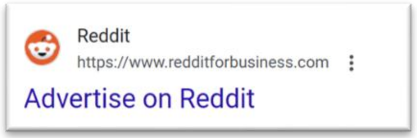
$v_2 \sim U[0,100]$ → $v_2 = 20$ → $\phi_2 = -60$

Non-Identically Distributed Bidders

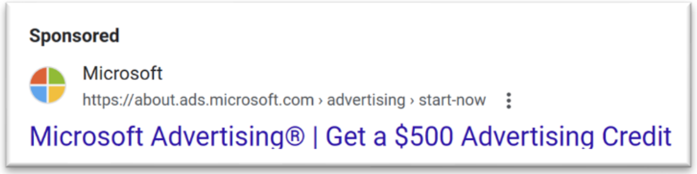
- Allocate to highest virtual value (if positive)!

$$\mathrm{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 = 1$ → $\phi_1 = 1$

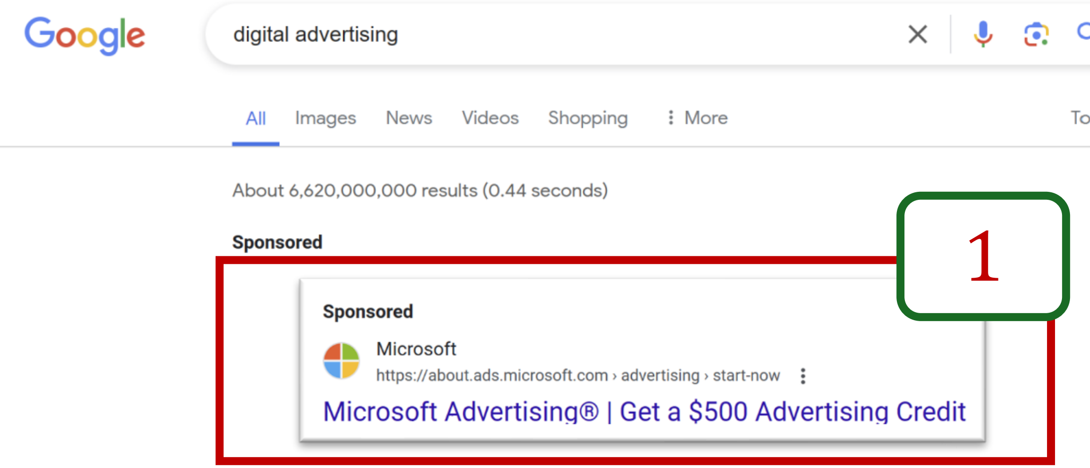


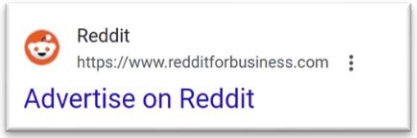
$v_2 \sim U[0,100]$ → $v_2 = 20$ → $\phi_2 = -60$

Non-Identically Distributed Bidders

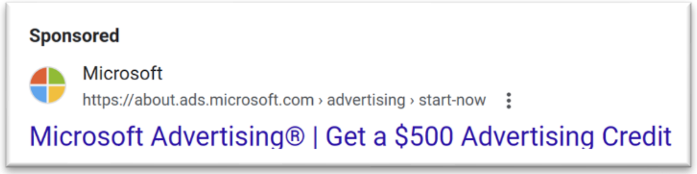
- Allocate to highest virtual value (if positive)!

$$\text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$v_1 \sim U[0,1]$ → $v_1 = 1$ → $\phi_1 = 1$

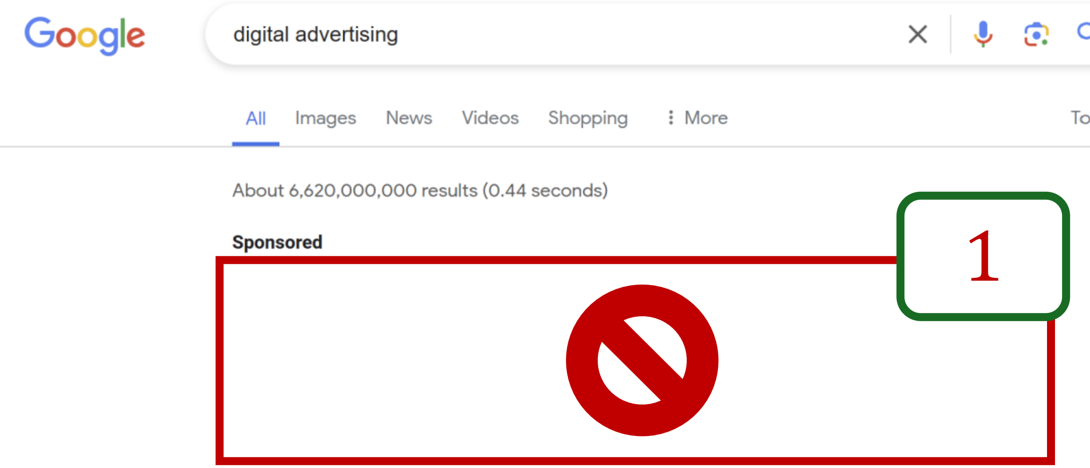


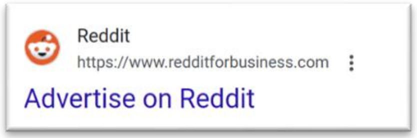
$v_2 \sim U[0,100]$ → $v_2 = 51$ → $\phi_2 = 2$

Non-Identically Distributed Bidders

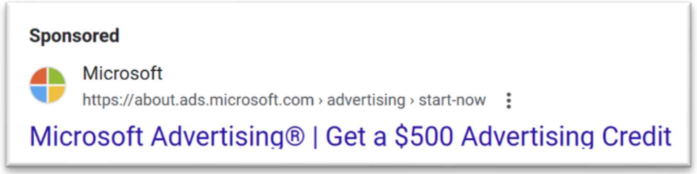
- Allocate to highest virtual value (if positive)!

$$\operatorname{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$





$\boxed{v_1 \sim U[0,1]} \rightarrow \boxed{v_1 = .49} \rightarrow \boxed{\phi_1 = -.02}$



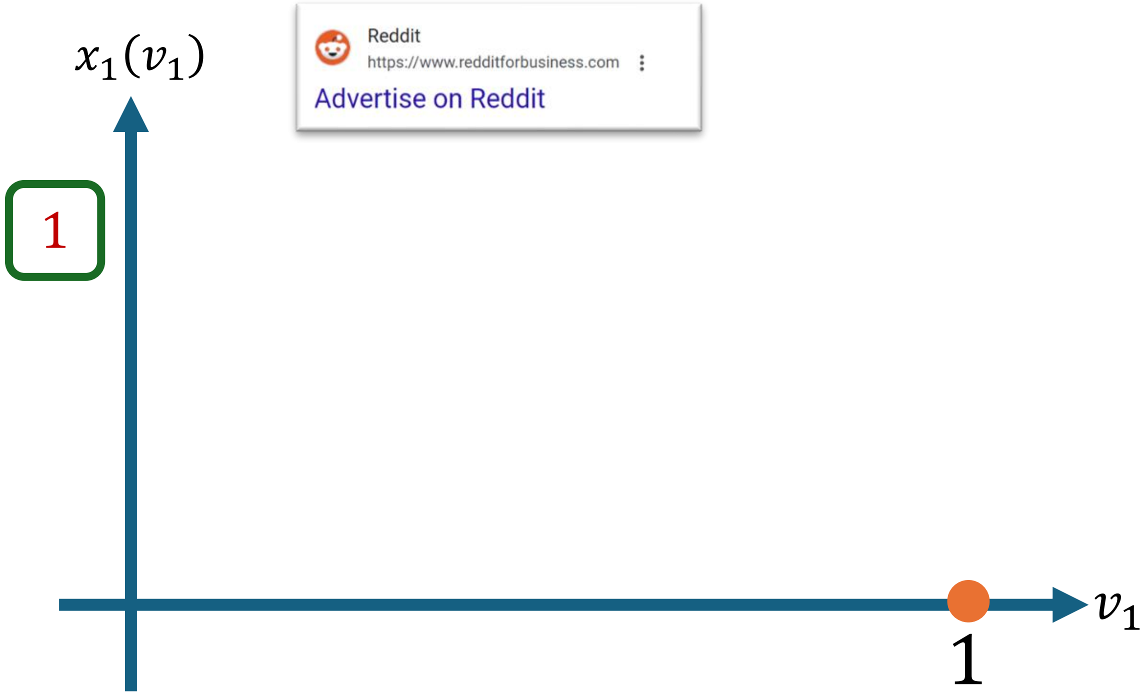
$\boxed{v_2 \sim U[0,100]} \rightarrow \boxed{v_2 = 49} \rightarrow \boxed{\phi_2 = -2}$

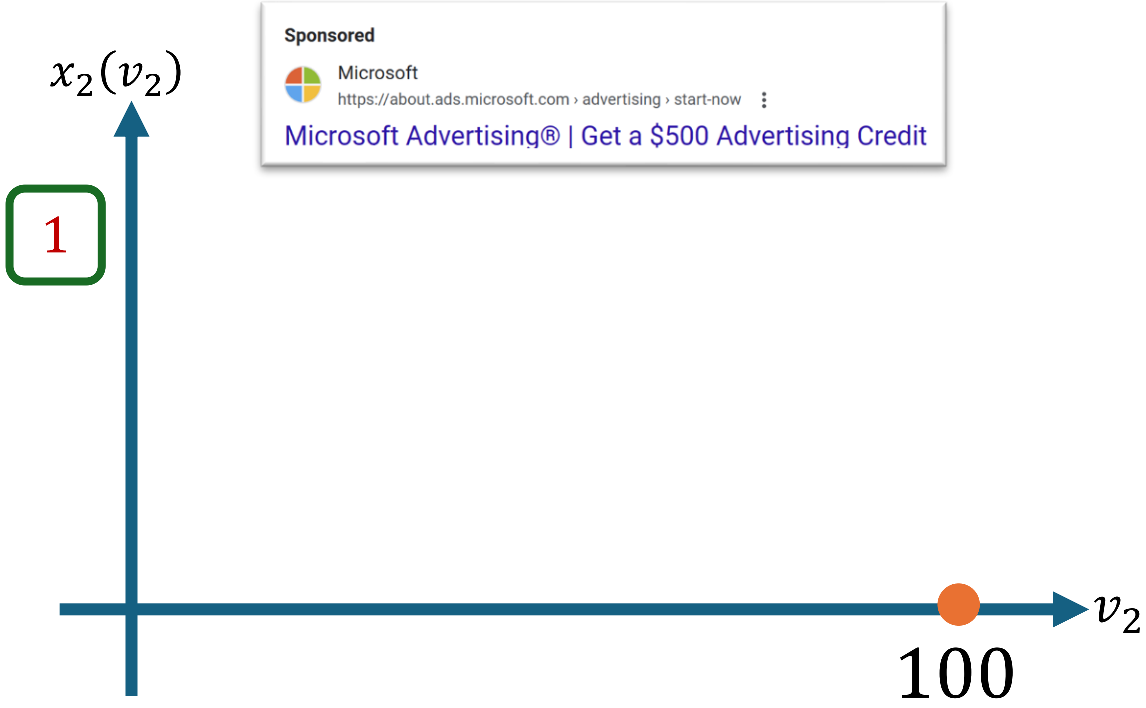
Non-Identically Distributed Bidders

- Allocate to highest virtual value (if positive)!

$$\mathrm{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Bidder 1 wins if:



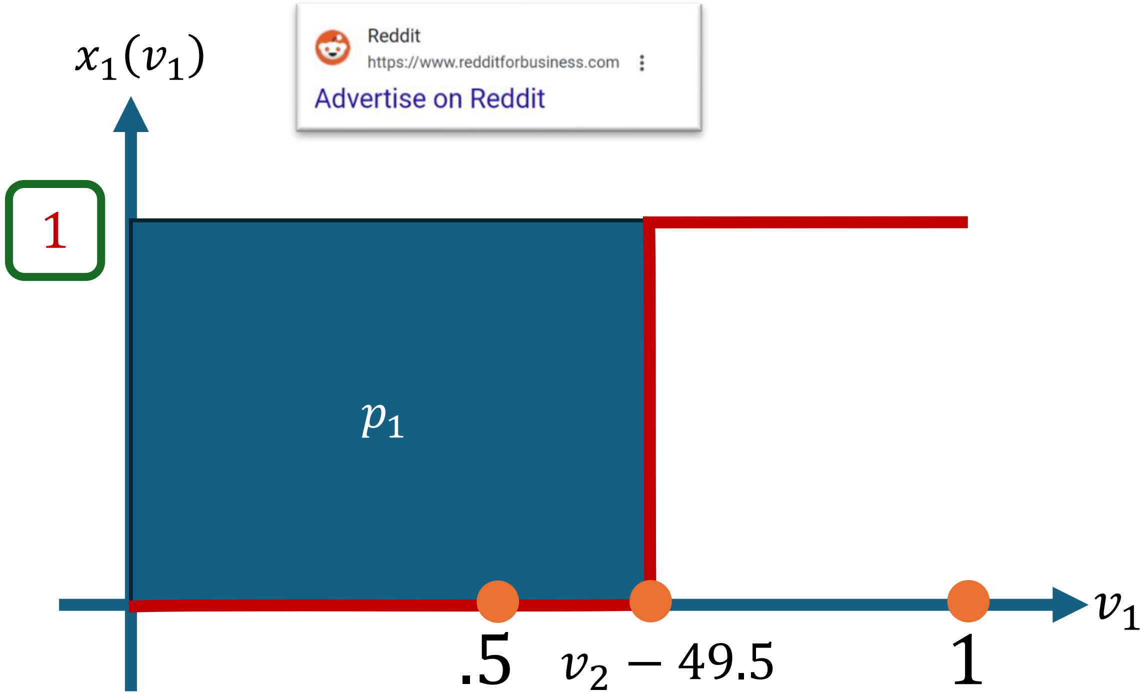


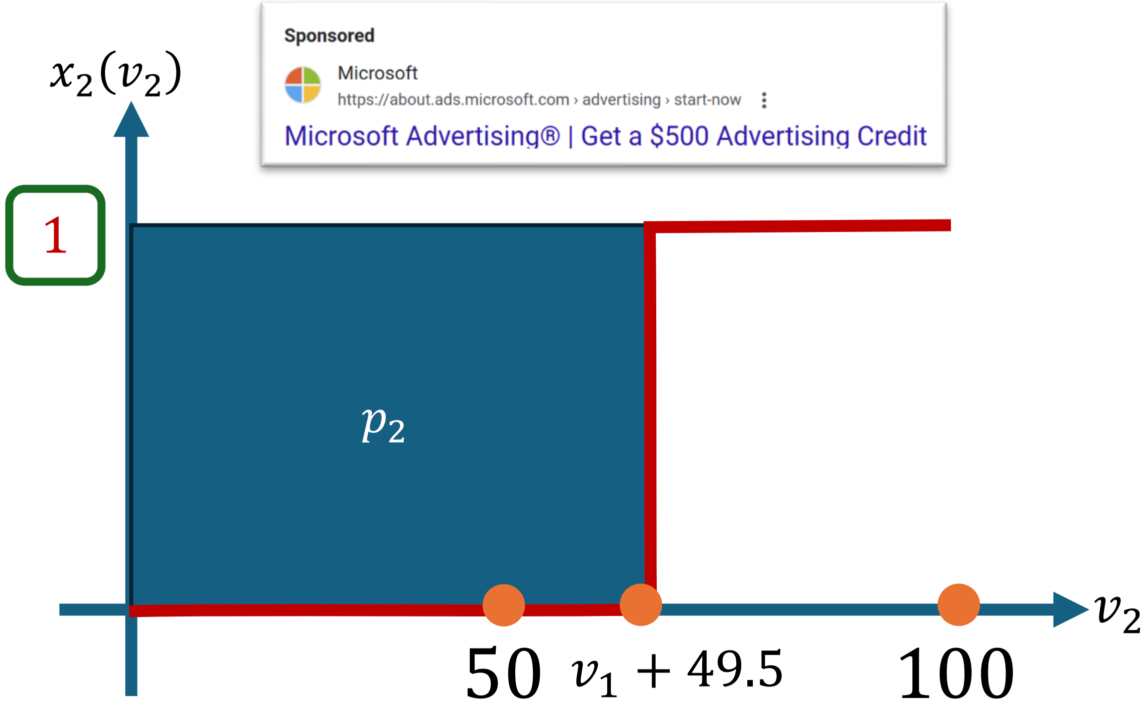
Non-Identically Distributed Bidders

- Allocate to highest virtual value (if positive)!

$$\text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Bidder 1 wins if: $2v_1 - 1 \geq 2v_2 - 100 \Rightarrow v_1 \geq v_2 - \frac{99}{2}$



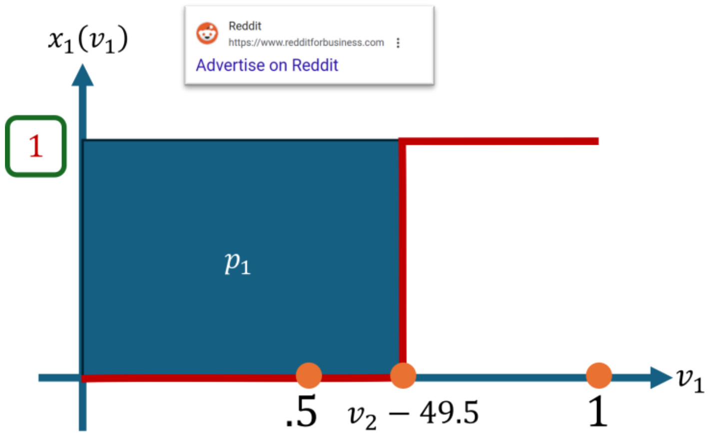


Non-Identically Distributed Bidders

- Allocate to highest virtual value (if positive)!

$$\text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Optimal auction rules



Non-Identically Distributed Bidders

- Allocate to highest virtual value (if positive)!

$$\text{argmax}\{0, 2v_1 - 1, 2v_2 - 100\}$$

Optimal auction rules

- If $v_1 > .5$, $v_2 < 50$, allocate to 1, charge .5
- If $v_1 < .5$, $v_2 > 50$, allocate to 2, charge 50
- If $.5 \le v_1 < v_2 - 49.5$, allocate to 2, charge $v_1 + 49.5$
- If $50 \le v_2 < v_1 + 49.5$, allocate to 1, charge $v_2 - 49.5$

At the optimal auction, we are giving a huge advantage to the weaker bidder! We roughly add 49.5$ to their bid!

We expect more from stronger bidders and make it harder for them to win, to incentivize them to pay more.

Optimal auction is
1) cumbersome, 2) hard to understand, 3) hard to explain, 4) does not always allocate to the highest value player, 5) discriminates a lot, 6) is many times counter-intuitive, 7) can seem unfair!

Are there simpler auctions that always achieve almost as good revenue?

Simple vs. Optimal Auctions

Second-Price with Player-Specific Reserves

- What if we simply run a second price auction but have different reserves for each bidder

- Each bidder i has a reserve price r_i
- Reject all bidders with bid below the reserve
- Among all bidders with value $v_i \geq r_i$, allocate to highest bidder
- Charge winner max of their reserve and the next highest surviving bid

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- Revenue of the optimal auction is the maximum virtual welfare

$$\mathrm{OPT} = E\left[\max_i \phi_i^+(v_i)\right], \qquad \phi_i^+(v_i) = \max\{0, \phi_i(v_i)\}$$

- Assume that reserve prices are at least the monopoly reserves
- Revenue of the second-price with player specific reserves (SP-r)

$$\mathrm{Rev} = E\left[\sum_i x_i(v)\phi_i^+(v_i)\right]$$

- Can we guarantee that the auction collects a $\phi_i^+(v_i)$ that, in expectation, is at least half of the maximum $\phi_i^+(v_i)$?

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- Can we guarantee that the auction collects a $\phi_i^+(v_i)$ that, in expectation, is at least half of the maximum $\phi_i^+(v_i)$?
- Since the auction allocates to some player with $v_i \geq r_i$
- Since ϕ_i^+ are monotone: to some player with $\phi_i^+(v_i) > \theta_i$
- We can think of $\phi_i^+(v_i)$ as non-negative prizes Π_i

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- We can think of $\phi_i^+(v_i)$ as non-negative prizes Π_i
- The optimal auction gets revenue that corresponds to the expected maximum prize $E[\max_i \Pi_i]$
- The SP-r auction gets revenue that corresponds to some price Π_τ that satisfies that it is above some threshold θ_τ
- Is there a threshold rule for collecting prizes that guarantees at least half of the expected maximum prize?

Parenthesis: (Optimal Stopping Problems)

- There are n stages
- In each stage i, we are offered a prize $\Pi_i \sim G_i$
- Distributions G_i are known ahead of time
- Realized prize Π_i only revealed at stage i
- At each stage, we can choose to accept Π_i and end the game or discard the prize and continue opening prizes

Question. Is there a strategy to play the game that guarantees at least half of what an oracle who knows all the prizes ahead of time would achieve?

Parenthesis: (Optimal Stopping Problems)

Question. Is there a strategy to play the game that guarantees at least half of what an "prophet" who knows all the prizes ahead of time would achieve?

Theorem (Prophet Inequality). There exists a threshold strategy APX that accepts the first prize that passes a threshold θ, such that:

$$E[\Pi_\tau] \geq \frac{1}{2} E\left[\max_i \Pi_i\right]$$

τ is the random stopping time induced by the threshold policy.

Parenthesis: (Proof of Prophet Inequality)

- Let's be generous with the optimal benchmark

$$E[\Pi_*] = E\left[\max_i \Pi_i\right] \le E[\theta + [\Pi_* - \theta]_+] \le \theta + \sum_i \boxed{E[[\Pi_i - \theta]_+]}^{A_i}$$

- APX gets θ if there exists some prize above, i.e., $\Pi_* \geq \theta$
- On top of that, we also collect some ***excess*** $[\Pi_\tau - \theta]_+$
- ***Excess*** is A_i, when all rewards other than i is $\le \theta$

$$\text{Excess} \geq \sum_i A_i \Pr(\forall j \neq i : \Pi_j < \theta) \geq \sum_i A_i \Pr(\Pi_* < \theta)$$

Overall: $\text{APX} \geq \theta \Pr(\Pi_* \geq \theta) + \Pr(\Pi_* < \theta) \sum_i A_i$

Choosing $\Pr(\Pi_* \geq \theta) = 1/2$: $\text{APX} \geq \frac{1}{2}\left(\theta + \sum_i A_i\right) \geq \frac{1}{2} E[\Pi_*]$

Parenthesis: (Optimal Stopping Problems)

Question. Is there a strategy to play the game that guarantees at least half of what an "prophet" who knows all the prizes ahead of time would achieve?

Theorem (Prophet Inequality). There exists a threshold strategy APX that accepts the first prize that passes a threshold θ, such that:

$$E[\Pi_\tau] \geq \frac{1}{2} E\left[\max_i \Pi_i\right]$$

τ is the random stopping time induced by the threshold policy.

Policy. Simply choose θ such that $\Pr\left(\max_i \Pi_i \geq \theta\right) = 1/2$

Second-Price with Player-Specific Reserves

Theorem. There exist personalized reserve prices such that the above auction achieves at least ½ of the optimal auction revenue!

- Choose θ such that:

$$\Pr\left(\max_i \phi_i^+(v_i) \geq \theta\right) = 1/2$$

- Then set personalized reserve prices implied by:

$$\phi_i^+(v_i) \geq \theta \Leftrightarrow v_i \geq r_i$$

All these designs required knowledge of distributions of values F_i!

What can we do if we only have data from F_i?

Learning Auctions from Samples

Learning from Samples

- We are given a set S of m samples of value profiles

$$S = \left\{ v^j = \left(v_1^j, \dots, v_n^j \right) \right\}_{j=1}^{m}$$

- Each sample is drawn i.i.d. from the distribution of values

$$v_i^j \sim F_i, \qquad v^j \sim \boldsymbol{F} \stackrel{\text{def}}{=} F_1 \times \cdots \times F_n$$

- Samples can be collected from historical runs of truthful auction
- Bids of each bidder in each of the m historical runs of the auction

Desiderata

- Without knowledge of distributions F_i, we want to produce a mechanism M_S, that achieves good revenue on these distributions
- For some $\epsilon(m) \to 0$ as the number of samples grows:

$$\mathrm{Rev}(M_S) \stackrel{\text{def}}{=} E_{v\sim F}\left[\sum_i p_i^{M_S}(v)\right] \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon(m)$$

- Either in expectation over the draw of the samples, i.e.

$$E_S[\mathrm{Rev}(M_S)] \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon(m)$$

- Or with high-probability over the draw of the samples, i.e.

$$\mathrm{w.p.}\ 1-\delta: \quad \mathrm{Rev}(M_S) \geq \mathrm{OPT}(\boldsymbol{F}) - \epsilon_\delta(m)$$

Easy Start: Pricing from Samples

Pricing from Samples

- Suppose we have only one bidder with $v \sim F$, for simplicity in $[0, 1]$
- Optimal mechanism is to post the monopoly reserve price
- The optimal price r is the one that maximizes

$$\mathrm{Rev}(r) = E_{v \sim F}[r \cdot 1\{v \geq r\}] = r \Pr(v \geq r) = r\left(1 - F(r)\right)$$

which is the monopoly reserve price η that solves:

$$\eta - \frac{1 - F(\eta)}{f(\eta)} = 0$$

- Choosing η requires knowledge of the CDF F and the pdf f
- Can we optimize r if we have m samples of v?

The Obvious Algorithm

- We want to choose r that maximizes

$$\max_{r\in[0,1]} \mathrm{Rev}(r) \stackrel{\text{def}}{=} E_{v\sim F}[r \cdot 1\{v \geq r\}], \qquad \text{(population objective)}$$

- With m samples S, we can optimize average revenue on samples!

$$\max_{r\in[0,1]} \mathrm{Rev}_S(r) \stackrel{\text{def}}{=} \frac{1}{m}\sum_{j=1}^{m} r \cdot 1\{v^j \geq r\}, \qquad \text{(empirical objective)}$$

- This approach is called Empirical Reward Maximization (ERM)
- **Intuition.** Since each value is drawn from distribution F the empirical average over i.i.d. draws from F, by law of large numbers, should be very close to expected value

Same as Empirical Risk Minimization (ERM) in Machine Learning (loss vs reward)

A Potential Problem with ERM

- The Law of Large Numbers applies if we wanted to evaluate the revenue of a fixed reserve price, we had in mind using the samples
- If we optimize over a very large set of reserve prices, then by random chance, it could be that we find a reserve price that has a large revenue on the samples, but small on the distribution

- This behavior is called overfitting to the samples
- We need to argue that overfitting cannot arise when we optimize over the reserve price!

Basic Elements of Statistical Learning Theory

Uniform Convergence

- Uniform Convergence. Suppose that we show that, w.p. $1-\delta$

$$\forall r \in [0,1]:\ |\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \le \epsilon_\delta(m)$$

- Alert. Note that this is different than: $\forall r \in [0,1]$, w.p. $1-\delta$

$$|\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \le \epsilon_\delta(m)$$

- The first asks that with probability $1-\delta$, the empirical revenue of all reserve prices is close to their population revenue
- The second asks that for a given reserve price, with probability $1-\delta$ its empirical revenue is close to its population
- The second claims nothing about the probability of the *joint event* that this is satisfied for all prices simultaneously

Uniform Converges Suffices for No-Overfitting

- Uniform Convergence. Suppose that we show that, w.p. $1-\delta$

$$\forall r \in [0,1]: \ |\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_\delta(m)$$

- Empirical Risk Maximization reserve:

$$r_S = \operatorname*{argmax}_{r\in[0,1]} \mathrm{Rev}_S(r)$$

Theorem. If uniform convergence holds then, w.p. $1-\delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m) = \mathrm{OPT}(F) - 2\epsilon_\delta(m)$$

Uniform Converges Suffices for No-Overfitting

Theorem. If uniform convergence holds then, w.p. $1-\delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m) = \mathrm{OPT}(F) - 2\epsilon_\delta(m)$$

- By uniform convergence, with probability $1-\delta$:

$$\mathrm{Rev}(\mathrm{r}_S) \geq \mathrm{Rev}_S(r_S) - \epsilon_\delta(m)$$

- Since, r_S optimizes the empirical objective

$$\mathrm{Rev}_S(r_S) \geq \mathrm{Rev}_S(\eta)$$

- By uniform convergence:

$$\mathrm{Rev}_S(\eta) \geq \mathrm{Rev}(\eta) - \epsilon_\delta(m)$$

- Putting it all together:

$$\mathrm{Rev}(r_s) \geq \mathrm{Rev}(\eta) - 2\epsilon_\delta(m)$$

This is the no-overfitting property: It **cannot be** that we found a reserve price that has *large empirical revenue* but very *small population revenue*

The *monopoly reserve* is a **feasible** reserve price but **was not chosen** by ERM. So, it must have had smaller empirical average revenue.

LLN vs Uniform Convergence

Crucial Argument: with probability $1-\delta$: $\mathrm{Rev}(\mathrm{r}_S) \geq \mathrm{Rev}_S(r_S) - \epsilon_\delta(m)$

- Cannot be argued solely using Law of Large Numbers: if we have i.i.d. X^j with mean $E[X]$

$$\left|\frac{1}{m}\sum_{j=1}^{m} X^j - E[X]\right| \to 0$$

- For reserve price r that is chosen before looking at the samples, define $X^j(r) = r \cdot 1\{v^j \geq r\}$

$$|\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| = \left|\frac{1}{m}\sum_j r \cdot 1\{v^j \geq r\} - E[r \cdot 1\{v \geq r\}]\right| \to 0$$

- Problem. The reserve price r_S was chosen by looking at all the samples in S
  - If I tell you r_S you learn something about the samples
  - Conditional on r_S the samples are no-longer i.i.d.

- Uniform convergence, essentially means "*what I learn about S from r_S is not that much…*"

Concentration Inequalities and Uniform Convergence

- Concentration inequalities give us a stronger version of LLN

- Chernoff-Hoeffding Bound. If we have i.i.d. $X^j \in [0,1]$ with mean $E[X]$, w.p. $1-\delta$:

$$\left|\frac{1}{m}\sum_{j=1}^{m} X^j - E[X]\right| \leq \epsilon_\delta(m) \stackrel{\text{def}}{=} \sqrt{\frac{\log(2/\delta)}{2m}}$$

- Crucial. The bound grows only logarithmically with $1/\delta$

Union Bound

- Suppose we had only K possible reserve prices $\left\{\frac{1}{K}, \frac{2}{K}, \frac{3}{K} \dots, 1\right\}$
- For each reserve price r on the grid, for any probability δ', by Chernoff bound

$$\Pr\left((\text{Bad Event})_r\right) = \Pr\left(\left|\frac{1}{m}\sum_{j=1}^{m} X^j(r) - E[X(r)]\right| > \epsilon_{\delta'}(m)\right) \leq \delta'$$

- Union Bound. The probability of the union of events is at most the sum of the probabilities

$$\Pr\left(\cup_{r=1}^{K} (\text{Bad Event})_r\right) \leq \sum_{r=1}^{K} \Pr\left((\text{Bad Event})_r\right) \leq K \cdot \delta'$$

- Apply Chernoff bound with $\delta' = \delta/K$

$$\Pr\left(\cup_{r=1}^{K} (\text{Bad Event})_r\right) \leq \delta$$

- Probability(*exists reserve price whose empirical revenue is far from its population*) at most δ

Uniform Convergence via Union Bound

Theorem. Suppose we had K possible reserve prices $\mathrm{Grid}_{\mathrm{K}} \stackrel{\text{def}}{=} \left\{\frac{1}{K}, \frac{2}{K}, \frac{3}{K} \dots, 1\right\}$

Then with probability at least $1-\delta$

$$\forall r \in \mathrm{Grid}_{\mathrm{K}}: |\mathrm{Rev}_S(r) - \mathrm{Rev}(r)| \leq \epsilon_{\delta/K}(m) \stackrel{\text{def}}{=} \sqrt{\frac{\log(2K/\delta)}{2m}}$$

Problem. The optimal reserve η can potentially not be among these K reserves

Intuition. For a sufficiently large K, for any reserve price, we can find a reserve price on this discretized grid that achieves almost as good revenue

We don’t lose much by optimizing over the grid!

Discretization

- For a reserve price r, pick largest reserve price below r on the grid
- Denote this discretization of r as r_K

- By doing so, you allocate to any value you used to allocate before
- For any such value you receive revenue at least $r - 1/K$

- Overall, you lose revenue at most $1/K$

$$\mathrm{Rev}(r_K) \geq \mathrm{Rev}(r) - 1/K$$

Discretized ERM

- Let’s modify ERM to optimize only over the grid

$$r_S = \max_{r \in \mathrm{Grid}_K} \mathrm{Rev}_S(r)$$

- We can apply the uniform convergence over the grid

$$\boxed{\mathrm{Rev}(r_S) \geq \mathrm{Rev}_S(r_S) - \epsilon_{\delta/K}(m)}$$

We cannot overfit, when optimizing over the grid of reserves

- Since, r_S optimizes the empirical objective over the grid

$$\boxed{\mathrm{Rev}_S(r_S) \geq \mathrm{Rev}_S(\eta_K)}$$

The *discretized monopoly reserve* is a **feasible** reserve in the grid but **was not chosen** by ERM.

- By uniform convergence over the grid:

$$\mathrm{Rev}_S(\eta_K) \geq \mathrm{Rev}(\eta_K) - \epsilon_{\delta/K}(m)$$

- By the discretization error argument:

$$\mathrm{Rev}(\eta_K) \geq \mathrm{Rev}(\eta) - 1/K$$

Theorem. The revenue of the reserve price output by discretized ERM over the K-grid satisfies, with probability $1-\delta$

$$\mathrm{Rev}(r_S) \geq \mathrm{OPT}(F) - 2\sqrt{\frac{\log(2K/\delta)}{2m}} - \frac{1}{K}$$

Choosing $K = 1/m$

$$\mathrm{Rev}(r_S) \geq \mathrm{OPT}(F) - 3\sqrt{\frac{\log(2m/\delta)}{2m}}$$

Desideratum satisfied!
$\epsilon_\delta(m) \to 0$ as m grows

The Limits of Discretization

- Do we really need to optimize over the discrete grid?
- What if we insist on optimizing over $[0,1]$. Can we still overfit?

- Now that we have infinite possible reserves, we cannot apply the union bound argument ($K = \infty$)!
- How do we argue about optima over continuous, infinite cardinality spaces?

Sneak Peek

- Would have been ideal if we only have to argue about behavior of our optimization space, *on the given set of samples*
- As opposed to the unknown distribution of values

- What if we can find a small set of reserves and argue that for all reserves there is an approximately equivalent one in the small set, in terms of revenue on the samples

- Maybe then it suffices to invoke the union bound over the smaller space, even though we optimize over the bigger space