

MS&E 125: Intro to Applied Statistics

Linear regression

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Announcements

- ▶ section today
- ▶ hw5 out today, due next Tuesday
- ▶ not sure if you're on the right track on hw? come to OH!

Outline

Linear models

Prediction

Fitting linear regression

Maximum likelihood

Multiple regression

Motivation: linear models

Linear models can be used for

- ▶ **prediction:** given a set of input variables, predict a value for the output variable
- ▶ **understanding:** how are the input variables related to the output variable, and to each other?
- ▶ **inference:** how much do the input variables affect the output variable?
- ▶ **counterfactuals:** what would happen if we changed the input variables?
- ▶ **control:** how can we change the input variables to achieve a desired output?

Prediction examples

- ▶ **House Price Prediction:**

- ▶ Input variables (x): square footage, number of bedrooms, age of the house, location, etc.
- ▶ Output variable (y): price of the house

- ▶ **Sales Forecasting:**

- ▶ Input variables (x):

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- ▶ Output variable (y): student's final exam score or GPA

Prediction examples

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▶ **Energy Consumption Forecasting:**

- ▶ Input variables (x): temperature, humidity, time of day, day of the week, etc.
- ▶ Output variable (y): energy consumption of a building or household

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Regression setup

we want to predict output given inputs

- ▶ input variables $x \in \mathbf{R}^p$
 - ▶ also called “predictors”, “independent variables”, “covariates”
 - ▶ a row of a data table
- ▶ output variable $y \in \mathbf{R}$
 - ▶ also called “outcome”, “response”, “dependent variable”, “label”, “target” ...

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example: to predict the cost of an insurance claim,

- ▶ y is the cost of an insurance claim.
- ▶ entries of x are the properties of the insured and his/her vehicle, e.g., credit score, age of the vehicle, ...

Demo: simple linear regression

`https://colab.research.google.com/github/
stanford-mse-125/demos/blob/main/regression.ipynb`

Simple linear regression

simple linear regression: $p = 1$

- ▶ predict

$$\hat{y} = \beta_0 + \beta_1 x$$

- ▶ $\beta_0, \beta_1 \in \mathbf{R}$ are called **regression coefficients**
- ▶ \hat{y} is called the **prediction** for input x

Predictions: example

In the fathers and sons dataset, we found

$$\hat{y} = 34 + 0.5x$$

where x is the height of the father in inches.

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Q: What do the numbers 34 and .5 mean?

A: A father with height 0 inches has a son with height 34 inches. For each inch of height, the son is expected to be 0.5 inches taller.

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Residuals

look at **residual** r to understand how well the model fits the data

$$r = y - \hat{y} = y - \beta_0 - \beta_1 x_1$$

pick β so the residuals are small

Dataset

to find the best line, we need a dataset! suppose we have

- ▶ n data points $(x_1, y_1), \dots, (x_n, y_n)$
 - ▶ also called **dataset**, **examples**, **observations**, **samples** or **measurements**
- ▶ each $x_i \in \mathbf{R}^p$ is a vector of p input variables
 - ▶ a row from the data table
- ▶ each $y_i \in \mathbf{R}$ is a scalar output variable

Linear regression: two perspectives

how to choose β ?

- ▶ **optimization perspective:** find β to minimize the sum of squared errors

$$\text{minimize} \quad \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_1)^2$$

- ▶ **statistical perspective:** find the line that maximizes the likelihood of the data

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- ▶ **statistical perspective:** find the line that maximizes the likelihood of the data

theorem: for appropriate assumptions, the two perspectives give the same answer

(coming in a few slides, or see All of Statistics ch. 14)

Least squares fitting

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Least squares fitting

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Q: Given $x_i \in \mathbf{R}$, what is β_1 ?

A: Set derivative to zero; solution is slope of the line of best fit.

Solve for β_0

$$\text{minimize} \quad \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

take derivative wrt β_0 and set to zero:

$$\sum_{i=1}^n -2(y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^n y_i = \beta_0 n - \beta_1 \sum_{i=1}^n x_i$$

$$\frac{1}{n} \sum_{i=1}^n y_i = \beta_0 - \beta_1 \frac{1}{n} \sum_{i=1}^n x_i$$

\implies the model goes through the point of averages

Solve for β_1

$$\text{minimize} \quad \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

take derivative wrt β_1 and set to zero:

$$\sum_{i=1}^n -2(y_i - \beta_0 - \beta_1 x_i)x_i = 0$$

$$\beta_0 \sum_{i=1}^n x_i + \beta_1 \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i y_i$$

$$\beta_1 = \frac{\sum_{i=1}^n x_i y_i - \beta_0 \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2}$$

interpretation:

- ▶ suppose x and y have been standardized so that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0$ and $\frac{1}{n} \sum_{i=1}^n x_i^2 = \frac{1}{n} \sum_{i=1}^n y_i^2 = 1$.
- ▶ then $\beta_1 = \frac{1}{n} \sum_{i=1}^n x_i y_i$ is the **correlation** between x and y

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Linear regression model

probabilistic model for linear regression:

suppose the x s are fixed, and y s are generated by

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2)$$

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under the model, the likelihood of observing residual $r = y - \hat{y}$ is

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

Demo: are errors iid normal?

`https://colab.research.google.com/github/
stanford-mse-125/demos/blob/main/regression.ipynb`

Maximum likelihood

likelihood function: probability of data given parameters

$$\ell(\beta_0, \beta_1) = \prod_{i=1}^n (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right)$$

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maximum likelihood estimation (MLE):

choose β_0 and β_1 to maximize the likelihood function

$$\hat{\beta}_0, \hat{\beta}_1 = \underset{\beta_0, \beta_1}{\operatorname{argmax}} \ell(\beta_0, \beta_1) = \underset{\beta_0, \beta_1}{\operatorname{argmax}} \log \ell(\beta_0, \beta_1)$$

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maximum likelihood estimation (MLE):

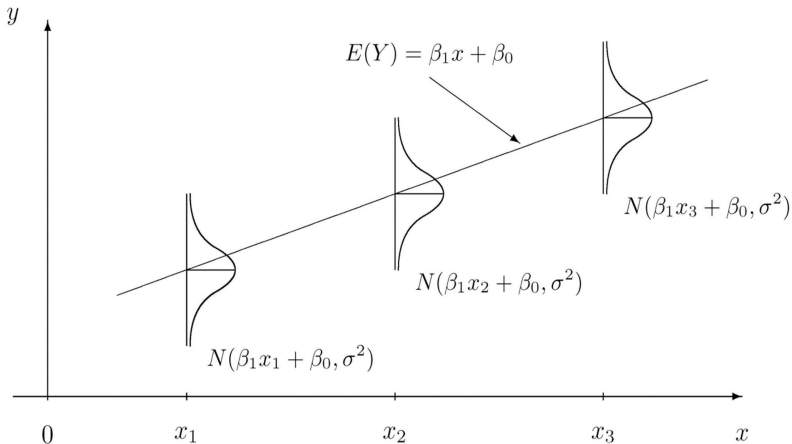
choose β_0 and β_1 to maximize the likelihood function

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\implies least squares finds the maximum likelihood estimate!

Probabilistic interpretation

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$



Estimation puts a hat on it

statisticians use hats to denote estimates:

- ▶ $\hat{\beta}_0$ is the estimate of β_0
- ▶ \hat{y} is the estimate of y

these estimates are **random quantities** that depend on the data

```
https://colab.research.google.com/github/  
stanford-mse-125/demos/blob/main/  
regression-uncertainty.ipynb
```

Properties of the estimator

putting it together, we have found:

$$\hat{\beta}_1 = \rho(x, y) \hat{\sigma}_y / \hat{\sigma}_x, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

where

- ▶ $\rho(x, y)$ is the correlation between x and y
- ▶ $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are the sample standard deviations of x and y
- ▶ \bar{x} and \bar{y} are the sample means of x and y

under the normal model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

these estimates are unbiased:

$$\mathbb{E}[\hat{\beta}_1] = \beta_1, \quad \mathbb{E}[\hat{\beta}_0] = \beta_0$$

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All of Statistics ch 14 derives the variance of the estimates

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Matrix notation

$$\hat{y}_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

rewrite using linear algebra:

- ▶ form **response vector** $y \in \mathbf{R}^n$: each y_i is an entry of y
 - ▶ also called **target vector**
- ▶ form **design matrix** $X \in \mathbf{R}^{n \times p}$: each $x^{(i)}$ is a row of X
 - ▶ also called **feature matrix**
 - ▶ if the model includes a constant term, the 0th column of $X \in \mathbf{R}^{n \times p+1}$ is all ones

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} X_{11} & \cdots & X_{1p} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ X_{n1} & \cdots & X_{np} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}$$
$$y = X\beta + \epsilon$$

Least squares in matrix notation

rewrite error:

$$\sum_{i=1}^n (y_i - \hat{y}_i)^2 = \|y - X\beta\|^2$$

interpretation:

- ▶ $X\beta$ is a linear combination of the columns of X
- ▶ we seek the linear combination that best matches y

Linear regression: model

we can rewrite the model as

$$\hat{y}_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \cdots + \beta_p X_{i,p} + \varepsilon_i$$

- ▶ notice that $\beta_0, \beta_1, \dots, \beta_p$ do not depend on i
- ▶ the columns of the data table are $Y_i, X_{i,1}, \dots, X_{i,p}$

i	y_i	$X_{i,1}$	$X_{i,2}$	\dots	$X_{i,p}$
1	2.3	1.1	6.2	\dots	5.9
2	12.7	2.4	5.4	\dots	9.6
3	6.3	0.9	6.9	\dots	1.5

Example: electricity usage

- ▶ We are managing a large complex of apartments in the Northeast.
- ▶ We pay for the electricity used by our residents.
- ▶ We would like to predict electricity usage so that we can estimate how much money should be set aside.

Demo: multiple linear regression

`https://colab.research.google.com/github/
stanford-mse-125/demos/blob/main/electricity.ipynb`