Analyzing a Circuit with Singular Inverse Inductance Matrix

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$$\hat{H} = \frac{1}{2}\hat{\mathbf{\Phi}}^T \boldsymbol{L}^\star \hat{\mathbf{\Phi}} + \frac{1}{2}\hat{\boldsymbol{Q}}^T \boldsymbol{C}^{-1}\hat{\boldsymbol{Q}} - \sum_j E_{J,j}\cos\left(2\pi\phi_{J,j}/\Phi_0\right)) \tag{1.1}$$

$$[\hat{\phi}_i, \hat{q}_j] = (\hat{\mathbf{\Phi}}\hat{\boldsymbol{Q}}^T - \hat{\boldsymbol{Q}}\hat{\mathbf{\Phi}}^T)_{ij} = i\hbar\delta_{ij} \tag{1.2}$$

A canonical transformation preserves the commutator. Define a linear transformation:

$$\hat{\mathbf{\Phi}}' = \boldsymbol{R}\hat{\mathbf{\Phi}}, \quad \hat{\boldsymbol{Q}}' = \boldsymbol{S}\hat{\boldsymbol{Q}} \tag{1.3}$$

It is clear the commutator is conserved iff $\boldsymbol{S}^T = \boldsymbol{R}^{-1}$.

The transformed Hamiltonian is then:

$$\hat{H} = \frac{1}{2}\hat{\mathbf{\Phi}}'^T \tilde{\boldsymbol{L}}^\star \hat{\mathbf{\Phi}}' + \frac{1}{2}\hat{\boldsymbol{Q}}'^T \tilde{\boldsymbol{C}}^{-1}\hat{\boldsymbol{Q}}' - \sum_j E_{J,j}\cos\left(2\pi\phi'_{J,j}/\Phi_0\right)\big) \tag{1.4}$$

with

$$\tilde{\boldsymbol{C}}^{-1} = (\boldsymbol{S}^{-1})^T \boldsymbol{C}^{-1}(\boldsymbol{S}^{-1}), \quad \tilde{\boldsymbol{L}}^\star = (\boldsymbol{R}^{-1})^T \boldsymbol{L}^\star(\boldsymbol{R}^{-1}) \tag{1.5}$$

and the appropriate transformation for the junction fluxes.

Since the transformation defined above is a linear symplectic transformation (preserved commutator), there is a corresponding unitary transformation such that $\hat{U}^\dagger\hat{\mathbf{\Phi}}\hat{U} = \boldsymbol{R}\hat{\mathbf{\Phi}}$ and $\hat{U}^\dagger\hat{\boldsymbol{Q}}\hat{U} = \boldsymbol{S}\hat{\boldsymbol{Q}}$.

This unitary will preserve the energy spectrum as long as some care is taken to keep within the same Hilbert space (see section on CPB below).

1.3 Applied to Taha's Circuit

For Taha's circuit, we have have two nodes, and

$$\hat{H} = \frac{1}{2}\hat{\mathbf{\Phi}}^T \boldsymbol{L}^\star \hat{\mathbf{\Phi}} + \frac{1}{2}\hat{\boldsymbol{Q}}^T \boldsymbol{C}^{-1}\hat{\boldsymbol{Q}} + \hat{H}_J \tag{1.6}$$

with

$$\boldsymbol{L}^\star = \begin{bmatrix} 1/L_{12} & -1/L_{12} \\ -1/L_{12} & 1/L_{12} \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \tag{1.7}$$

and

$$\hat{H}_J = -E_J\cos\left(\boldsymbol{E}^T\hat{\mathbf{\Phi}}(2\pi/\Phi_0)\right) \tag{1.8}$$

where $\boldsymbol{E}^T = [1, 0]$.

Under a transformation $\boldsymbol{R}$ that makes $\tilde{\boldsymbol{L}}^\star = [1/\tilde{L}_1, 0; 0, 0]$, we find a new Hamiltonian:

$$\hat{H} = \frac{(\hat{\phi}_1')^2}{2\tilde{L}_1} + \frac{(\hat{q}_1')^2}{2\tilde{C}_1} + \frac{(\hat{q}_2')^2}{2\tilde{C}_2} - \frac{\hat{q}_1'\hat{q}_2'}{2\tilde{C}_{12}} + \hat{H}_J \tag{1.9}$$

with

$$\begin{aligned}\hat{H}_J &= -E_J \cos\left(\boldsymbol{E}^T \hat{\boldsymbol{\Phi}}'(2\pi/\Phi_0)\right) \\ &= -E_J \cos\left(\boldsymbol{E}^T \boldsymbol{R}^{-1} \hat{\boldsymbol{\Phi}}(2\pi/\Phi_0)\right) \\ &= -E_J \cos\left((c_1\hat{\phi}_1 + c_2\hat{\phi}_2)(2\pi/\Phi_0)\right)\end{aligned}$$

From here we see that the Hamiltonian is periodic in ϕ_2, with a period of Φ_0/c_2. The Josephson term couples charge eigenstates $|Q_2 + 2e_0c_2 \times m\rangle$ for $m \in \mathbb{Z}$ to one another and so we choose a $\mathcal{H}_{Q_2} = \{|Q_2 + 2ec_2 \times m\rangle | m \in \mathbb{Z}\}$ as our Hilbert space. Using this definition, the Hamiltonian can be written in the basis of $\mathcal{H}_{\text{fock}} \otimes \mathcal{H}_{Q_2}$.

We can perform this procedure for two different transformations,

$$\boldsymbol{R} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \quad \text{and} \quad \boldsymbol{R} = \begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}. \tag{1.10}$$

Due to the way that Q_2 is transformed, setting $Q_2 = 0$ for both gives two different representations of the same physical system. The Hamiltonians do not look similar on paper. However, numerical diagonalization verifies that they have the same spectrum, as expected. Key parts of the code are shown below.

```
function get_rotated_matrices(R,C1,C2,L12)


    Cmat = [C1 0; 0 C2];
    Lstar = [1/L12 -1/L12; -1/L12 1/L12]

    S = (R')^(-1);
    Rinv = R^(-1);
    Sinv = R';

    Lstar_t = Rinv'*Lstar*Rinv;
    println("rotated Lstar matrix: ")
    println(Lstar_t)

    Cmatinv = Cmat^(-1);

    Cmatinv_t = Sinv'*Cmatinv*Sinv;

    Evec = [1 0];

    C1t = 1/Cmatinv_t[1,1];
    C2t = 1/Cmatinv_t[2,2];
    C12t = -1/Cmatinv_t[1,2];

    L1t = 1/Lstar_t[1,1]


    c1 = (Evec*Rinv)[1]
    c2 = (Evec*Rinv)[2]

```

```
31
32      return [C1t, C2t, C12t, L1t, c1, c2]
33
34  end
35
36
37  function get_generic_circ_params(R,C1, C2, L12, EJ, Q2)
38
39
40      (C1t, C2t, C12t, L1t, c1, c2) = get_rotated_matrices(R,C1,C2,L12)
41
42
43      w1 = 1/sqrt(L1t*C1t);
44      qzp = sqrt(ħ*w1*C1t/2);
45       zp = sqrt(ħ*w1*L1t/2);
46
47      EC = (2*e0*c2)^2/(2*C2t);
48      E  = (2*e0*c2)*qzp/(C12t);
49
50      expfactor = 1im* zp*2 /Φ0*c1;
51
52
53      return [w1, qzp,  zp, C2t, C12t, EC, E , expfactor, Q2/(2*e0*c2)]
54  end
55
56
57  function get_circ1_params_v3(C1, C2, L12, EJ, Q2)
58      R = [0 1; 1 1]^(-1);
59      #R = [1 -1; 1 1]
60      return get_generic_circ_params(R,C1, C2, L12, EJ, Q2)
61  end
62  function get_circ2_params_v3(C1, C2, L12, EJ, Q2)
63      R = [1 -1; 1 1]/sqrt(2)
64      return get_generic_circ_params(R,C1, C2, L12, EJ, Q2)
65  end
66
67  function get_H(circ_param_method,C1,C2,L12,EJ,Q2)
68
69      #[w1, qzp,  zp, C2t, C12t, EC, E , expfactor, Q2] =
70      (w1, qzp,  zp, C2t, C12t, EC, E , expfactor, Q2)=circ_param_method(C1,C2,L12,EJ,Q2)
71      N = solver_params["N"]
72      M = solver_params["M"]
73      dim = N*M;
74
75      a = get_a_op(solver_params);
76      n = get_n_op(solver_params);
77      D  = get_D_alpha(solver_params,expfactor)
78      nn1 = get_nn1_op(solver_params)
79      id = get_id(solver_params)
80
81
82      H = zeros((Complex{Float64}),dim,dim);
83
84
```

```
    H = (ħ*w1*a'a + EC*(id.*Q2+n)^2 + 1.0im*E*(a-a')*(id.*Q2+n) - EJ/2*(D*nn1 +
    ↪ D'*nn1'))/(ħ*w1)


end
```

---

2 Transforming coordinates and its effect on the spectrum

2.1 Generalities

Can the spectrum of a Hamiltonian change under a unitary transformation? Naively, this seems impossible. Let

$$\hat{S}\hat{H}\hat{S}^\dagger = \hat{\tilde{H}} \tag{2.1}$$

Then, if $|n\rangle$ is an eigenvector of $\hat{H}$,

$$\begin{aligned} \hat{H}|n\rangle &= E_n|n\rangle \\ \hat{S}\hat{H}\hat{S}^\dagger\hat{S}|n\rangle &= E_n\hat{S}|n\rangle, \end{aligned} \tag{2.2}$$

and so $\hat{S}|n\rangle$ is an eigenvector of $\hat{\tilde{H}}$ with the same eigenvalue E_n. Therefore the spectrum hasn't changed.

There is however a caveat. The above argument holds so long as $\hat{S}|n\rangle$ is a valid vector in the Hilbert space. This may not be necessarily the case. But what is our Hilbert space? It's useful to look at an example to understand what this means.

2.2 The Cooper Pair Box

The most clear example of where the above argument breaks down is the Cooper pair box (CPB) Hamiltonian.

In the cooper pair box, we have the Hamiltonian

$$\hat{H} = \frac{\hat{q}^2}{2C} - E_J\cos(2\pi\hat{\phi}/\Phi_0), \quad [\hat{\phi},\hat{q}] = i\hbar. \tag{2.3}$$

The Hamiltonian is Φ_0-periodic in the flux basis with $\hat{H}(\phi) = \hat{H}(\phi+\Phi_0)$.

Charge displacement operator

Given a charge eigenvector $|Q_0\rangle$ with $\hat{q}|Q_0\rangle = Q_0|Q_0\rangle$, the charge displacement operator is

$$\exp\left(-\frac{i\hat{\phi}p}{\hbar}\right)|Q_0\rangle = |Q_0 - p\rangle. \tag{2.4}$$

Assume, our Hilbert space contains $|Q_0\rangle$. What other vectors must it contain for the charge displacement operator to make sense? Since the operator displaces charge by $\pm p$, the states $|Q_0 + mp\rangle$ for all integer m should be in the Hilbert space. So a minimal Hilbert space is defined by

$$\mathcal{H}_{Q_0} = \{|Q_0 + mp\rangle | m \in \mathbb{Z}\} \tag{2.5}$$

We can then write the operator as

$$\exp\left(-\frac{i\hat{\phi}p}{\hbar}\right) = \sum_m |Q_0 + p(m-1)\rangle\langle Q_0 + pm|. \tag{2.6}$$

to describe its action in $\mathcal{H}_{Q_0}$.
In the above notation, the label of the ket is the actual charge eigenvalue. As a convenient shorthand, we can write $\exp\left(-\frac{i\hat{\phi}p}{\hbar}\right) = \sum_m |m-1\rangle\langle m|$, keeping in mind that $\hat{q}|m\rangle = (Q_0 + pm)|m\rangle$.

We can write the cosine term in the Hamiltonian in terms of the charge displacement operator:

$$\hat{H} = \frac{\hat{q}^2}{2C} - \frac{E_J}{2}\sum_m \left(|Q_0 + 2e_0(m-1)\rangle\langle Q_0 + 2e_0 m| + \text{h.c.}\right) \tag{2.7}$$

Note that we've chosen the Hilbert space $\mathcal{H}_{Q_0} = \{|Q_0 + mp\rangle | m \in \mathbb{Z}\}$ (see the discussion above). The Hamiltonian couples between states within this Hilbert space. The flux displacement operator operates on the flux basis as

$$\exp\left(\frac{i\hat{q}f}{\hbar}\right)|\phi_0\rangle = |\phi_0 - f\rangle. \tag{2.8}$$

Expressing the charge eigenstates in the flux basis, we see that

$$\begin{aligned}\langle\phi|\exp\left(-\frac{i\hat{q}f}{\hbar}\right)|Q_0 + 2e_0 m\rangle &= \langle\phi_0 - f|Q_0 + 2e_0 m\rangle \\ &= \exp\left(-\frac{i(Q_0 + 2e_0 m)f}{\hbar}\right)\langle\phi|Q_0 + 2e_0 m\rangle.\end{aligned} \tag{2.9}$$

Setting $f = \Phi_0$, we find

$$\langle\phi_0 - \Phi_0|Q_0 + 2e_0 m\rangle = \exp\left(-\frac{i2\pi Q_0}{2e_0}\right)\langle\phi|Q_0 + 2e_0 m\rangle. \tag{2.10}$$

This means that all of the basis vectors in our Hilbert space $\mathcal{H}_{Q_0}$ are periodic by Φ_0 modulo a phase $e^{i2\pi n_0}$ with $n_0 = Q_0/2e_0$.

Now we are ready to see where the argument above regarding the unitary transformation of the Hamiltonian can break. More specifically, if transformation $\hat{S}$ operates on $\mathcal{H}_{Q_0}$, it transforms a vector $|n\rangle \in \mathcal{H}_{Q_0}$ such that $S|n\rangle \in \mathcal{H}_{Q_0}$. Then we have:

$$\begin{aligned} \hat{H}|n\rangle &= E_n|n\rangle \\ \hat{S}\hat{H}\hat{S}^\dagger\hat{S}|n\rangle &= E_n\hat{S}|n\rangle, \end{aligned} \tag{2.11}$$

and the spectrum remains unchanged. However, if we have a transformation like $\hat{S} = \exp(-i\hat{\phi}p/\hbar)$ for some arbitrary p, it's pretty easy to see that $\hat{S}$ no longer keeps us in the same Hilbert space. It effectively connects $\mathcal{H}_{Q_0}$ and $\mathcal{H}_{Q_1}$.

Finally, note that we should think of Q_0 (modulo $2e_0$) as a real physical property describing the state of the system. It is related to the initial state of the system, the charge that is present on some isolated island at an initial time. Since the system dynamics do not affect this charge (modulo $2e_0$), we are always stuck in this Hilbert space and the energy levels are dependent this initial state.