Test 1, Oct 2nd

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Oct 2nd

Note: NO CALCULATORS ARE ALLOWED. SHOW YOUR WORK.

Problem	Points	
1	10	10
2	10	0
3	10	10
4	20	17
5	15	13
6	15	10
7	20	16
Total	100	

76

1.(10 points) Find the norm of $f(x) = x$ on $[0, \pi]$.

$$\|f(x)\|^2 = \int_0^{\pi} f(x)f(x)\,dx = \int_0^{\pi} x^2\,dx = \frac{x^3}{3}\Big|_0^{\pi} = \frac{\pi^3}{3}$$

$$\boxed{\|f(x)\| = \sqrt{\frac{\pi^3}{3}}}$$

2.(10 points) Classify the equation $xyu_{xx} - u_{xy} + u_{yy} = 0$.

Second order, linear, homogeneous partial differential equation.

-10

$b^2 - 4ac = 1 - 4xy > 0$ H

< 0 E

$= 0$ P

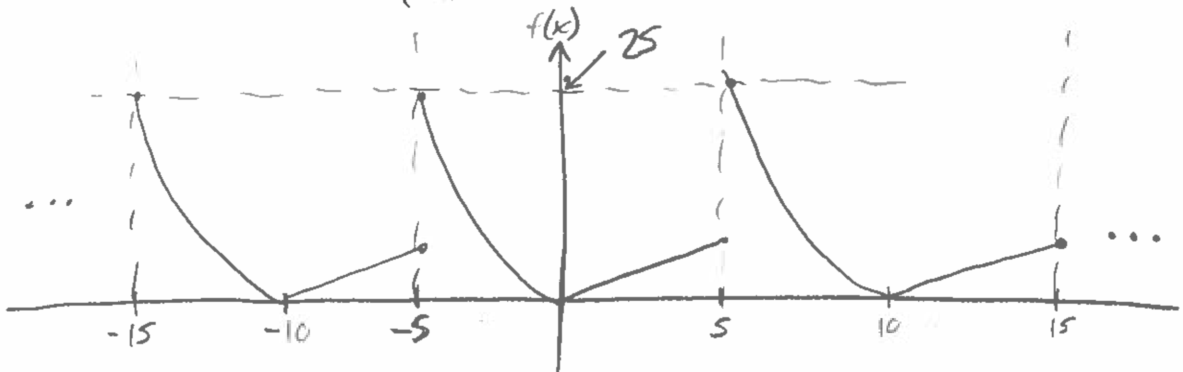
3.a)(3 points) Write the definition of an even function.

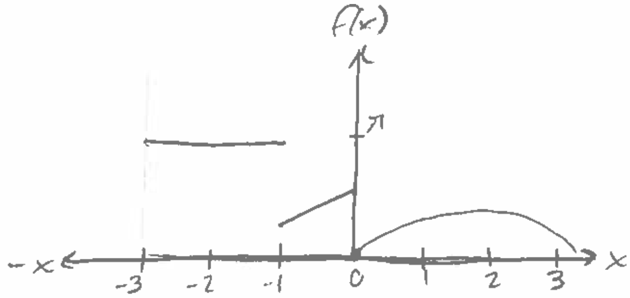
$f(-x) = f(x) \ \forall x \in \Omega$

b)(3 points) Write the definition of a periodic function with period 7.

$f(x+7n) = f(x) \ , \ n \in \mathbb{Z}$

c)(4 points) Draw the graph of the 10-periodic extension on R of the function $f(x) = \begin{cases} x^2, & -5 < x \leq 0 \\ x, & 0 < x < 5 \end{cases}$





4.(20 points) Use the convergence theorem to find the convergence of the Fourier series of $f(x) = \begin{cases} \pi, & -3 < x < -1 \\ x+2, & -1 < x < 0 \\ \sin x, & 0 < x < 3 \end{cases}$ at

$\pi/2 \sim \frac{3.14}{2} = 1.57$

$0 < 1.57 < 3$

$\Rightarrow 0 < \pi/2 < 3$

a) $x = \frac{\pi}{2}$ $\quad \sin(\frac{\pi}{2}) = \boxed{1}$

b) $x = -1$ $\quad \frac{(\pi)+(-1+2)}{2} = \boxed{\frac{\pi+1}{2}}$

c) $x = 3$

$= 2$

Assuming the 1-sided limit:

$f(3) = \lim_{x \to 3^-} f(x) = \boxed{\sin(3)}$ $+ \pi$

-3 $\quad \frac{f(L^-)+f(-L^+)}{2}$ $\quad 2$

d)on $(1, 2)$ $\quad$ Open set 1 and 2 means

$\boxed{f(x) = \sin(x), \quad 1 < x < 2}$

5.(15 points) Use the separation of variables method to solve the equation

$$yu_{xy} = u_{xx}$$

Let $u(x,y) = X(x)Y(y)$, then

$$y \cdot X'(x)Y'(y) = X''(x)Y(y)$$

$$\Rightarrow \frac{X''(x)}{X'(x)} = y\frac{Y'(y)}{Y(y)} = k$$

$$X'' - kX' = 0$$

$$r^2 - kr = 0$$

$$r = 0$$

$$r = k$$

$$\Rightarrow \int X''(x)\,dx = \int kX'(x)\,dx \qquad \Rightarrow \int \frac{Y'(y)}{Y(y)}dy = \int \frac{k}{y}dy$$

$$\Rightarrow X(x) = \cancel{Ae^{kx} + B} \qquad \ln|Y(y)| = k\ln|y| + c$$

$$= 0 \qquad Y(y) = C|y|^k$$

$$k = 0$$

$$X = Ax + B$$

?

If $k=0$: $\boxed{u(x,y) = Ax+B}$

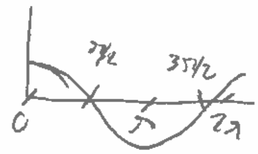
$$k \neq 0$$

If $k \neq 0$: $u(x,y) = (Ae^{kx}+B)(C|y|^k) = \boxed{(A'e^{kx}+B')|y|^k}$

$$X = C_1 + C_2e^{kx}$$

$$u(x,y) = \begin{cases} A_0x + B_0, & k = 0 \\ (A_1e^{kx}+B_1)|y|^k, & k \neq 0 \end{cases}$$

-2



6.(15 points) Solve the Sturm-Liouville problem: $y''(x) + \lambda y(x) = 0$, $0 < x < 6$ with Robin conditions $y'(0) = 0 = y(6)$.

$y''(x) = -\lambda y(x) \Rightarrow y(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

-2 $\quad y'(0) = \sqrt{-\lambda}A - \sqrt{-\lambda}B = 0 \Rightarrow A - B = 0 \Rightarrow A = B$

$y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

If $\lambda > 0$: $\cancel{y(x) = A\left(e^{i\sqrt{|\lambda|}x} + e^{-i\sqrt{|\lambda|}x}\right)} = A' \cos(\sqrt{|\lambda|}x)$, $y'(0) = 0$

$\lambda = \alpha^2$ $\quad y(6) = A'\cos(\sqrt{\lambda}\cdot 6) = 0 \Rightarrow 6\sqrt{\lambda} = \left(n + \frac{1}{2}\right)\pi$

$\sqrt{+\lambda} = \frac{(n+\frac{1}{2})}{6}\pi \Rightarrow \frac{(n+\frac{1}{2})^2}{36}\pi^2 = +\lambda \Rightarrow \boxed{\lambda_n = +\frac{(n+\frac{1}{2})^2}{36}\pi^2, \ n \in \mathbb{Z}}$

If $\lambda < 0$: $y(x) = A\left(e^{\sqrt{|\lambda|}x} + e^{-\sqrt{|\lambda|}x}\right) = \cancel{A\cosh(\sqrt{|\lambda|}x)}$ $\quad 1 \quad 4 \quad 2$

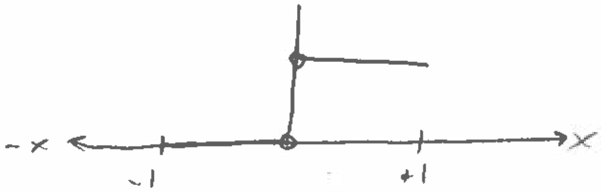
$\lambda = -\alpha^2$ $\quad y(6) = A\cosh(\sqrt{|\lambda|}\,6) = 0$ $\quad$ Not possible, so $\boxed{\lambda \not< 0}$ $\quad -3$

If $\lambda = 0$: $y(x) = Ax + B$, $y'(x) = A$

$y'(0) = A = 0 \Rightarrow A = 0 \quad y(6) = B = 0 \Rightarrow B = 0$

$\cancel{y = 0}$ Not valid.

$$\boxed{\begin{aligned} y_n(x) &= A\cos\left(\left(\frac{n+\frac{1}{2}}{6}\right)\pi x\right), \quad n \in \mathbb{N} = \mathbb{Z}^+/\{0\} \\ \lambda_n &= \left(\left(\frac{n+\frac{1}{2}}{6}\right)\pi\right)^2, \quad n \in \mathbb{N} = \mathbb{Z}^+/\{0\} \end{aligned}}$$



7.(20 points) a) Find the Fourier series of $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 5, & 0 \le x < 1 \end{cases}$ on $(-1, 1)$

$$a_0 = \frac{1}{\cancel{2}}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_{-1}^{0} 0\,dx + \frac{1}{2}\int_{0}^{1} 5\,dx = \frac{5}{\cancel{2}}$$

$$a_K = \int_{-1}^{1} f(x)\sin(Kx)\,dx = \int_{-1}^{1} 0\sin(Kx)\,dx + \int_{0}^{1} 5\sin(Kx)\,dx$$

$$= 5\int_{0}^{1}\sin(Kx)\,dx = -\frac{5}{K}\cos(Kx)\Big|_0^1 = -\frac{5}{K\pi}(1-\cos(K\pi))$$

$$= -(-1)^n$$

-2

$$b_K = \int_{-1}^{1} f(x)\cos(Kx)\,dx = \int_{-1}^{0} 0\cos(Kx)\,dx + \int_{0}^{1} 5\cos(Kx)\,dx$$

$$= 5\int_{0}^{1}\cos(Kx)\,dx = \frac{5}{K}\sin(Kx)\Big|_0^1 = \frac{5}{K}\sin(K\pi) = 0$$

-2

series: $$\frac{5}{2} + \sum_{K=1}^{\infty}\left[\frac{5}{K}(1-\cos(K))\sin(Kx) + \frac{5}{K}\sin(K)\cos(Kx)\right]$$

For the π basis: $$\frac{5}{2} + \frac{1}{\pi}\sum_{K=1}^{\infty}\left[\frac{5}{K}\left(1-(-1)^K\right)\sin(K\pi x)\right]$$

Either works!