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# Math Methods Homework #1

Let $\omega = f(z)dz = f(z)dx + if(z)dy$

## Section 1: Let $f(z) = z$

1) $\omega = f(z)dz = z\,dx + i\,z\,dy = (x+iy)dx + i(x+iy)dy$
$$= \boxed{(x\,dx - y\,dy) + i(y\,dx + x\,dy)}$$

2) Using $\int_\gamma \omega = \int_\gamma \omega_1 + i\int_\gamma \omega_2$, calculate $\int_\gamma \omega$.

$\gamma : \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases}$ $\quad dx = -R\sin\theta\, d\theta$ $\quad dy = R\cos\theta$

$$\int_\gamma \omega = \int_\gamma (x\,dx - y\,dy) + i\int_\gamma (y\,dx + x\,dy)$$
$$= \int_0^{2\pi} \left(R\cos\theta \cdot (-R)\sin\theta - R\sin\theta \cdot R\cos\theta\right)d\theta$$
$$+ i\int_0^{2\pi} \left[(R\sin\theta)(-R\sin\theta) + (R\cos\theta)(R\cos\theta)\right]d\theta$$
$$= -R^2\int_0^{2\pi} 2\sin\theta\cos\theta\, d\theta + iR^2\int_0^{2\pi}(\cos^2\theta - \sin^2\theta)d\theta = 0 + 0i$$
$$\rightarrow \boxed{\int_\gamma \omega = 0}$$

3) $\vec{F}(x,y) = f(x,y)\hat{x} + f(x,y)\hat{y} = (x+y)\hat{x} + (x+y)\hat{y}$ ?

$$\int(\vec{\nabla}\times\vec{F})d\tau = \oint \vec{F}\cdot d\vec{A} = 0$$

Because there is no singularity on the domain of the function, it does not matter your value of R; all closed contours integrate to 0 with no singularities.

See better explanation on next page.

Section 2: Let $f(z) = \frac{1}{z} = \frac{z^*}{z^* z}$

1) $\omega = f(z)dz = \frac{x - yi}{x^2+y^2} dx + i \frac{x-yi}{x^2+y^2} dy = \left(\frac{xdx + ydy}{x^2+y^2}\right) + i\left(\frac{-ydx + xdy}{x^2+y^2}\right)$

2) $\int_\gamma \omega$, when $\gamma$: $\left[\begin{array}{ll} x = r\cos\theta & dx = -r\sin\theta d\theta \\ y = r\sin\theta & dy = r\cos\theta d\theta \end{array}\right.$

$$\int_\gamma \omega = \int\left(\frac{xdx+ydy}{x^2+y^2}\right) + i\int \frac{-ydx+xdy}{x^2+y^2} =$$

$$= \int_0^{2\pi} \cancel{\frac{-r^2\cos\theta\sin\theta + r^2\sin\theta\cos\theta}{r^2}} d\theta + i\int_0^{2\pi} \frac{r^2\sin\theta\sin\theta + r^2\cos\theta\cos\theta}{r^2} d\theta$$

$$= 0 + i\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)d\theta = 0 + 2\pi i \qquad \boxed{\int_\gamma \omega = 2\pi i}$$

3) Green's theorem tells us it is radius-independent since the singularity is the origin of the circle.
All area in the circle that isn't the singularity contributes nothing to the integral, so therefore is independent of R.

3) Given $f(z) = z \Rightarrow \omega = f(z)dz$

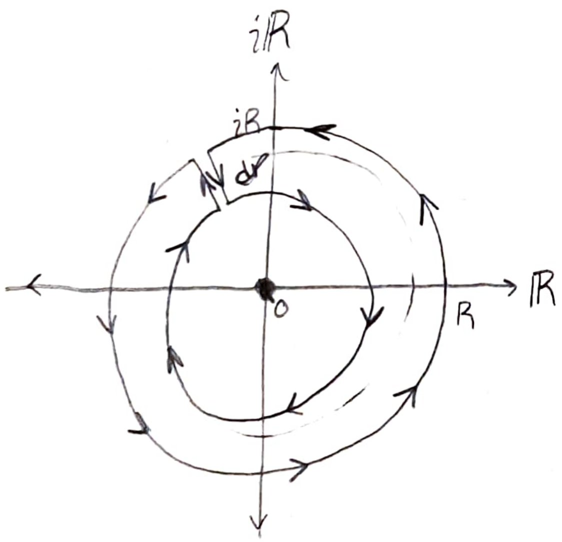
In part 1, we found that $\omega = \overbrace{(xdx - ydy)}^{\omega_1} + i\overbrace{(ydx + xdy)}^{\omega_2}$

In part 2, we found that a path parametrized by $Re^{i\theta}$ had a closed loop integral of $0$.

Calculating $d\omega$, we find: $\boxed{d\omega = d\omega_1 + d\omega_2 = 0}$

$$d\omega_1 = \frac{\partial(-y)}{\partial x} - \frac{\partial(x)}{\partial y} = 0 \quad \& \quad d\omega_2 = i\frac{\partial(x)}{\partial x} - i\frac{\partial(y)}{\partial y} = 0i$$

The integral from part 2 is independent of radius R for the follow reason:




As a path travels this washer of outer radius R and inner radius R−dr, the rotation goes back on itself, and so the net rotation is $0$.

This applies to any choice of R, and so is independent of R. ☒

3) Given $f(z) = \frac{1}{z}$ $\Rightarrow$ $\omega = f(z)\,dz$

Part 1: $\omega = \left(\frac{x\,dx + y\,dy}{x^2+y^2}\right) + i\left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right)$

Part 2: On traversal of $\gamma$ on $\omega$, $\int_\gamma \omega = 2\pi i$

Calculating $d\omega$, we get:

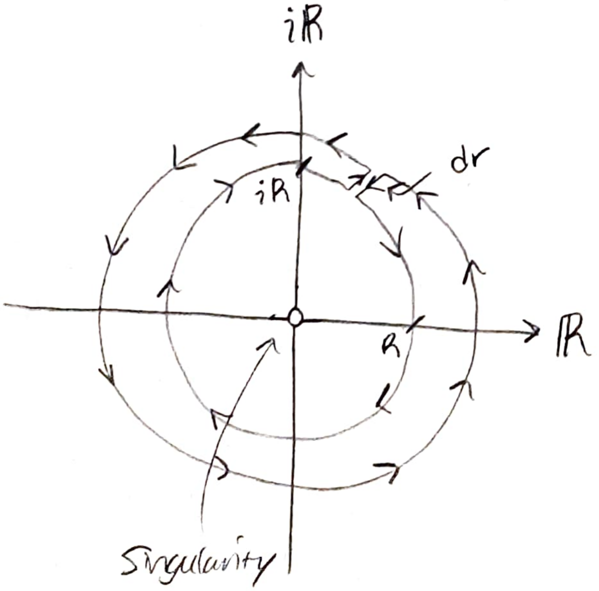
$$d\omega_1 = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{-2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2} = 0$$

$$d\omega_2 = i\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - i\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= i\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} - i\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}\,i$$

Similar to $f(z) = z$, if we draw a washer:




As you traverse this washer, the rotation is equal and opposite and this applies for all $R > 0$.

Since there is no contained curl in this path other than at $(0,0)$, the closed contour integral is independent of radius $R > 0$.