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Math Methods Homework #2

Question 1 : $\Omega = \{z \in \mathbb{C} \mid Im(z) > 0, |z| > 1\}$

$\partial\Omega = \{z \in \mathbb{C} \mid |Re(z)| > 1\} \cup \{z \in \mathbb{C} \mid Im(z) \geq 0, |z| = 1\}$

a) Given $f(z) = z + \frac{1}{z}$, show that $\forall z \in \partial\Omega, f(z) \in \mathbb{R}$.

For points on the real axis: $\forall x \in \mathbb{R}; f(x) = x + \frac{1}{x} \in \mathbb{R}$.

Let $z = e^{i\theta}$ on $\theta \in [0, \pi]$.

$f(e^{i\theta}) = e^{i\theta} + \frac{1}{e^{i\theta}} = e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \in \mathbb{R}$ ∎

b) Given $f(z) = z + \frac{1}{z}$, show that $\forall z \in \mathbb{C}, f(z) \in \Omega$.

Let $z = Re^{i\theta}$ on $\theta \in (0, \pi)$

$f(Re^{i\theta}) = Re^{i\theta} + \frac{1}{Re^{i\theta}} = Re^{i\theta} + \frac{1}{R}e^{-i\theta}$

$= \left(R\cos\theta + \frac{1}{R}\cos\theta\right) + i\left(R\sin\theta - \frac{1}{R}\sin\theta\right)$

$= \left(R + \frac{1}{R}\right)\cos\theta + i\sin\theta\left(R - \frac{1}{R}\right)$

$\sin\theta > 0, \theta \in (0, \pi)$

$-1 \leq \cos\theta \leq 1, \theta \in (0, \pi)$

$\left(R - \frac{1}{R}\right) > 0, R > 1$

$\Rightarrow \left(R - \frac{1}{R}\right)\sin\theta > 0 \quad \forall \theta \in (0, \pi)$

$\boxed{\therefore f : \Omega \rightarrow \mathbb{H}}$

Question 2

Given $f(z) = z + \frac{1}{z} = u(x,y) + iv(x,y)$

a) Let $z = x+iy$, $f(x+iy) = (x+iy) + \frac{1}{(x+iy)} = x+iy + \frac{x-iy}{x^2+y^2}$

$$f(z) = \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right) \Rightarrow \boxed{\begin{aligned} u(x,y) &= x\left(1+\frac{1}{x^2+y^2}\right) \\ v(x,y) &= y\left(1-\frac{1}{x^2+y^2}\right) \end{aligned}}$$

b) See Diagram Attached.

Question 3

a) $$\vec{\nabla} u(x,y) = \frac{\partial}{\partial x}\left(x + \frac{x}{x^2+y^2}\right)\hat{x} + \frac{\partial}{\partial y}\left(x + \frac{x}{x^2+y^2}\right)\hat{y}$$

$$= \left(1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right)\hat{x} + \left(-\frac{2xy}{(x^2+y^2)^2}\right)\hat{y}$$

$$\vec{\nabla} v(x,y) = \frac{\partial}{\partial x}\left(y - \frac{y}{x^2+y^2}\right)\hat{x} + \frac{\partial}{\partial y}\left(y - \frac{y}{x^2+y^2}\right)\hat{y}$$

$$= \left(\frac{2xy}{(x^2+y^2)^2}\right)\hat{x} + \left(1 - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right)\hat{y}$$

b) $\vec{\nabla} u(x,y)$: Fluid flow field (velocity field).

$\vec{\nabla} v(x,y)$: Fluid normal field.

I graphed the gradients and saw $\vec{\nabla} u$ was the direction of fluid traveling left to right.

$U = [-10, -9, \ldots, 10]$

U = 21 element list

$$x\left(1 + \frac{1}{x^2 + y^2}\right) = k \cdot U\{x^2 + y^2 > 1\}$$

$V = [-10, -9, \ldots, 10]$

V = 21 element list

$$y\left(1 - \frac{1}{x^2 + y^2}\right) = k \cdot V\{x^2 + y^2 > 1\}$$

$k = 0.44$

−1 1

$$x^2 + y^2 = 1$$

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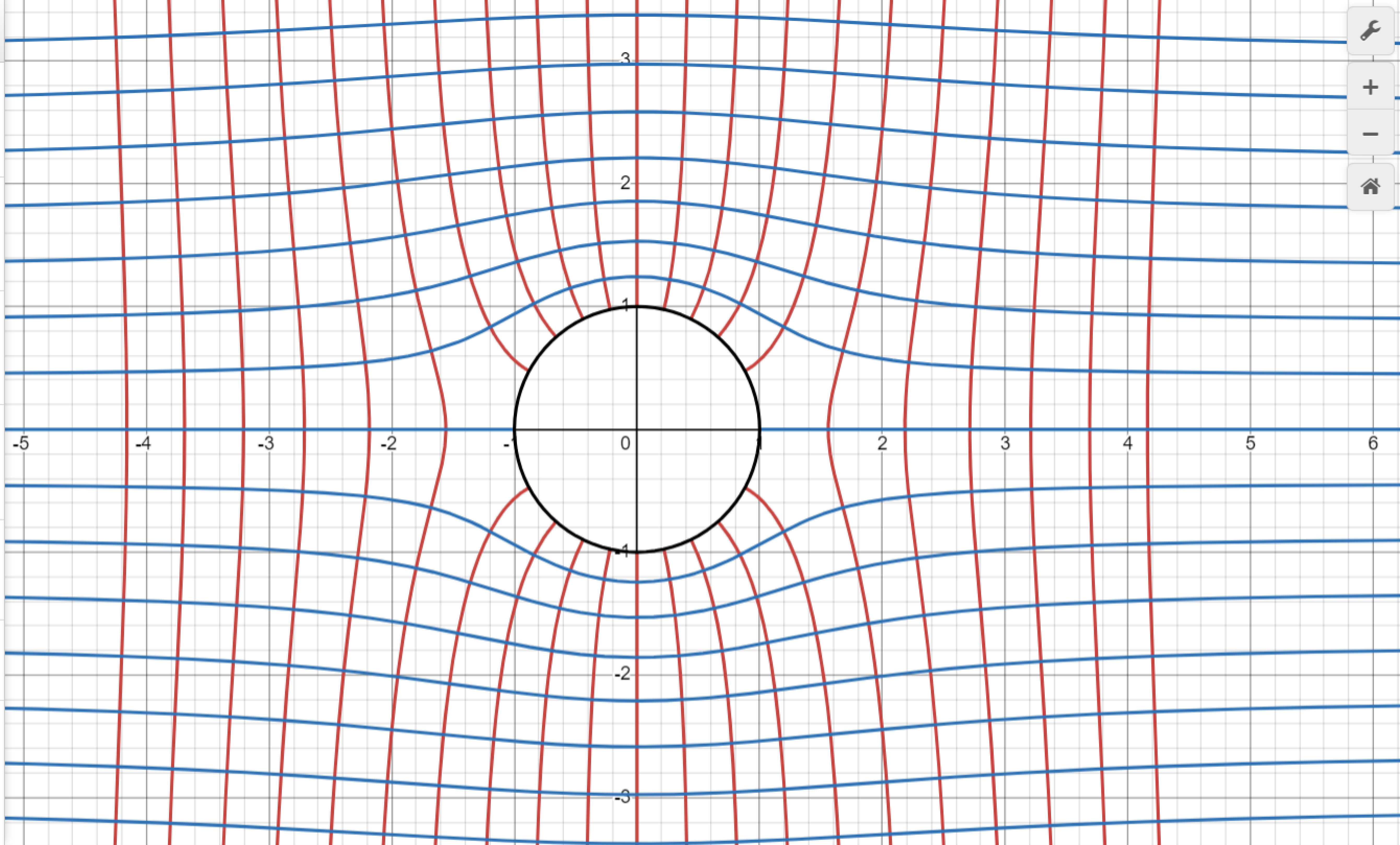

$U = $ 21 element list

2 $V = [-10, -9, \ldots, 10]$

$V = $ 21 element list

3 $x\left(1 + \frac{1}{x^2 + y^2}\right) = k \cdot U\{x^2 + y^2 > 1\}\{y >$

4 $y\left(1 - \frac{1}{x^2 + y^2}\right) = k \cdot V\{x^2 + y^2 > 1\}\{|y|$

5 $y \le \sqrt{1 - x^2}\ \{y > 0\}$

6 $k = 0.44$

-1 1

7 $y \le 0$

8