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Math Methods Homework #1

Let $\omega = f(z)dz = f(z)dx + if(z)dy$

Section 1: Let $f(z) = z$

1) $\omega = f(z)dz = z\,dx + i\,z\,dy = (x+iy)dx + i(x+iy)dy$
$$= \boxed{(x\,dx - y\,dy) + i(y\,dx + x\,dy)}$$

2) Using $\int_\gamma \omega = \int_\gamma \omega_1 + i\int_\gamma \omega_2$, calculate $\int_\gamma \omega$.

$$\int_\gamma \omega = \int_\gamma (x\,dx - y\,dy) + i\int_\gamma (y\,dx + x\,dy) \qquad \gamma: \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \quad \begin{matrix} dx = -R\sin\theta\, d\theta \\ dy = R\cos\theta \end{matrix}$$

$$= \int_0^{2\pi} \left(R\cos\theta \cdot (-R)\sin\theta - R\sin\theta \cdot R\cos\theta\right)d\theta$$

$$+ i\int_0^{2\pi} \left[(R\sin\theta)(-R\sin\theta) + (R\cos\theta)(R\cos\theta)\right]d\theta$$

$$= -R^2\int_0^{2\pi} 2\sin\theta\cos\theta\, d\theta + iR^2\int_0^{2\pi}(\cos^2\theta - \sin^2\theta)\,d\theta = 0 + 0i$$

$$\rightarrow \boxed{\int_\gamma \omega = 0}$$

3) $\vec{F}(x,y) = f(x,y)\hat{x} + f(x,y)\hat{y} = (x+y)\hat{x} + (x+y)\hat{y}$?

$$\int(\vec{\nabla}\times\vec{F})d\tau = \oint \vec{F}\cdot d\vec{A} = 0$$

Because there is no singularity on the domain of the function, it does not matter your value of R; all closed contours integrate to 0 with no singularities.

See better explanation on next page.

Section 2: Let $f(z) = \frac{1}{z} = \frac{z^*}{z^* z}$

1) $\omega = f(z)dz = \frac{x - yi}{x^2+y^2} dx + i \frac{x-yi}{x^2+y^2} dy = \left(\frac{xdx + ydy}{x^2+y^2}\right) + i\left(\frac{-ydx + xdy}{x^2+y^2}\right)$

2) $\int_\gamma \omega$, when γ: $\left[\begin{array}{ll} x = r\cos\theta & dx = -r\sin\theta d\theta \\ y = r\sin\theta & dy = r\cos\theta d\theta \end{array}\right.$

$$\int_\gamma \omega = \int \left(\frac{xdx + ydy}{x^2+y^2}\right) + i\int \frac{-ydx + xdy}{x^2+y^2} =$$

$$= \int_0^{2\pi} \cancel{\frac{-r^2\cos\theta\sin\theta + r^2\sin\theta\cos\theta}{r^2}} d\theta + i\int_0^{2\pi} \frac{r^2\sin\theta\sin\theta + r^2\cos\theta\cos\theta}{r^2} d\theta$$

$$= 0 + i\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)d\theta = 0 + 2\pi i \qquad \boxed{\int_\gamma \omega = 2\pi i}$$

3) Green's theorem tells us it is radius-independent since the singularity is the origin of the circle.
All area in the circle that isn't the singularity contributes nothing to the integral, so therefore is independent of R.