Problems for the mini-course "Probability theory"

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Problem 1 (Random walk, 1.5). There is a point on on one-dimensional integer lattice Z. With probability 0.5 the point moves to the right on distance 1 and with probability 0.5 moves to the left on the distance 1. Show that with probability 1 the point will return to the starting point during infinitely time period.

Problem 2 (Hoeffding's Lemma, 1). Show that for any random variable x with $|x| \le 1$ and E[x]=0 the following inequality is true $E\left[\exp(tx)\right] \le \exp\left(t^2/2\right)$.

Problem 3 (Concentration of measure, 1). Consider two unit balls of radius 1 in R^n. Let the center of first ball is at the point $(0,...,0)^T$ and the center of the second ball is at $(1,...,0)^T$. Show that the volume of intersection of these two balls divided by the volume of a unit ball tends to zero with n tending to infinity.

Problem 4 (Concentration of measure, 1). Let a random vector X has a uniform distribution on the unit ball in R^n with center at the point $(0,...,0)^T$. Show that for any a>0 the probability of the event $\|X\|_2 > 1-a$ tends to 1 with n tending to infinity.

Problem 5 (Concentration of measure, 1). Choose two independent random vectors X and Y with the uniform distribution on the unit ball in R^n with center at the point $(0,...,0)^T$. Proof that with high probability $\langle X,Y\rangle \approx 0$ with n tending to infinity.

Literature

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