Problems from lecture 1

1. Solve the problem mentioned in the part **Complete metric spaces**.
2. Let $\mathbf{X}$ be a metric space with some metric ρ. Prove that all subsets of $\mathbf{X}$ are open iff every subset of $\mathbf{X}$ which consists of a single point is open.

Problem mentioned in part Complete metric spaces

A: $\tilde{L}_2(a, b)$ is incomplete. Indeed, consider $\tilde{L}_2(-1, 1)$ and prove that the following sequence

$$f_n(x) = \begin{cases} 1, & x \in [1/n, 1], \\ nx, & x \in [-1/n, 1/n], \\ -1, & x \in [-1, -1/n]. \end{cases}$$

is fundumental, but the limit is not in $\tilde{L}_2(-1, 1)$ (**home assignment task**).

Problems from lecture 2

1. Write Newton's method for the equation $f(x) = x^2 - a = 0$, where $a > 0$ is a given constant. Find a region where the starting point x_0 should be such that the method converges. **Note:** Use fixed-point theorem.
2. Let x_n and y_n be sequences such that $\|x_n\| \leq 1$ and $\|y_n\| \leq 1$, $(x_n, y_n) \to 1$. Prove that $\|x_n - y_n\| \to 0$.
3. Let ϕ_k be orthogonal system in H, $x \in H$ and C_k be Fourier coefficients. Prove that if $n > m$, then

$$\|x - \sum_{k=1}^{n} C_k \phi_k\| \leq \|x - \sum_{k=1}^{m} C_k \phi_k\|$$

4. Let $D_1 \supset D_2 \supset \dots$ be a sequence of nested closed balls in Banach space. Prove that their intersection is not empty.

Problems from lecture 3

Problem 1

Consider $\mathbb{R}^2$ with standard scalar product and linear subset L defined as

$$L = \{(x, y) : 2x - y = 0\}$$

with inherited scalar product. Define following functional on L

$$\phi(x, y) = x$$

- Find the norm of $\phi : L \to \mathbb{R}$
- Prove that it can be extended uniquely to $\mathbb{R}^2$ with the same norm
- Find explicit form of the extended functional

Problem 2

Consider following normed space:

$$B = (C^\infty[0, 1], \|f(x)\| = \max_{[0,1]} |f(x)|)$$

and linear map

$$\frac{d}{dx} : B \to B$$

Find its norm or show that it is unbounded.