Intro to Functional Analysis

Lecture 1 test

Problem 1

Let metric space be $\mathbf{X} = \mathbb{R}$ and

$$\rho(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y, \end{cases} \qquad x, y \in \mathbf{X}.$$

1. Prove that ρ is metric.
2. Draw balls $D_{1/2}(0)$, $D_2(0)$.
3. What are the open sets, what are the closed sets in this metric?

Problem 2

Let l_1 be a space of infinite sequences $x = (x_1, x_2, \dots, x_n, \dots)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$. Show that function

$$\rho(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|, \quad x, y \in l_1$$

is a metric.

Problem 3

Consider the following sequence from l_2: $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, 0, \dots), \dots$. Check if there is a Cauchy subsequence.

Problem 4

Prove that if a sequence $\{x_n\}$ converges to a limit x then it is a Cauchy sequence.

Problem 5

Let K be some set in metric space $\mathbf{X}$. Prove that if $\forall \varepsilon > 0$ for K there exists a compact ε-net K_ε in $\mathbf{X}$, then K is compact.

Lecture 2 test

Problem 1

Show that the set of continuous functions on $[a, b]$ satisfying $f(a) = \alpha,\ f(b) = \beta$ is linear iff $\alpha = \beta = 0$

Problem 2

1. Let $x = (x_1, x_2) \in \mathbb{R}^2$. Is $\|x\| = |x_1 + x_2|$ a norm?
2. Prove that $\rho(x, y) = \|x - y\|$ is metric.

Problem 3

Prove Cauchy–Bunyakovsky–Schwarz inequality.

Problem 4

Prove that if a sequence x_n converges to x strongly, then it converges to x weakly as well.

Problem 5

Using contraction mapping principle find condition on λ when the equation

$$\phi(x) = \lambda \int_a^b K(x, y)\phi(y)\,dy + f(x)$$

has unique solution $\phi \in C([a, b])$, where $f(x) \in C([a, b])$ and $K \in C([a, b]^2)$ are given functions.

Lecture 3 test

Problem 1

Let $A : C([0,1]) \to C([0,1])$,

$$(Ax)(t) = \int_0^t x(s)\,ds.$$

- (0.5 pts) Prove that A is linear and bounded;
- (1 pt) Find the norm of A;
- (0.5 pts) Find $R(A)$.

Problem 2

(1 pt) Let $A_n \in \mathcal{L}(X)$, $B_n \in \mathcal{L}(X)$, $A, B \in \mathcal{L}(X)$. Prove that if $A_n \to A$ and $B_n \to B$, then $A_n B_n \to AB$.

Problem 3

(1 pt) Prove that A^{-1} exists iff its null space contains only zero element.

Problem 4

(2 pts) Let H be Hilbert space with two scalar products $(\cdot,\cdot)$ and $[\cdot,\cdot]$. Given

$$(x,x) \leq 2[x,x] \quad \forall x \in H,$$

prove that for a fixed $f \in H$ there exists $F \in H$ such that

$$(f,x) = [F,x] \quad \forall x \in H.$$

Is F unique?