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On an inverse problem for inhomogeneous thermoelastic rod

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A B S T R A C T

In recent years, different fields of engineering have been increasingly incorporating functionally graded materials with variable physical properties that significantly improve a quality of elements of designs. The efficiency of practical application of thermoelastic inhomogeneous materials depends on knowledge of exact laws of heterogeneity, and to define them it is necessary to solve coefficient inverse problems of thermoelasticity.

In the present research a scheme of solving the inverse problem for an inhomogeneous thermoelastic rod is presented. Two statements of the inverse problem are considered: in the Laplace transform space and in the actual space. The direct problem solving is reduced to a system of the Fredholm integral equations of the 2nd kind in the Laplace transform space and an inversion of the solutions obtained on the basis of the theory of residues. The inverse problem solving is reduced to an iterative procedure, at its each step it is necessary to solve the Fredholm integral equation of the 1st kind; to solve it the Tikhonov method is used. Specific examples of a reconstruction of variable characteristics required are given.

1. Introduction

For many years, layered composites have been widely used as a coating for elements of structures operating in high temperature environment (such as airplanes and spaceships, gas turbine blades, cutting tools, implants in biomedical industry, etc.) as they have been providing the structure's mechanical and thermal properties required. However, the jumps of material properties through the interface between discrete materials may cause large stress concentration and formation of plastic deformation or cracking.

In recent years, functionally graded materials have been used as an alternative to layered composites (Aboudi et al., 1995; Lee et al., 1996; Suresh and Mortensen, 1998; Wetherhold et al., 1996) to avoid material properties jumps through the interface due to their continual change. In this case, the thermomechanical characteristics are not constants but some functions of spatial coordinates, i.e. the material acquires the spatial heterogeneity. Such inhomogeneous structure may be obtained not only within the manufacturing process, but also during an operation under radiation, strong magnetic fields, and heavy temperature drops.

It is almost impossible to predict changes in the structure of materials caused by external actions. The efficiency of practical application of thermoelastic inhomogeneous materials depends on knowing exact laws of heterogeneity. The problem of finding the thermomechanical characteristics of inhomogeneous bodies is the coefficient inverse problem of thermoelasticity.

To date, there is already a considerable experience of investigation inverse problems. The general methods of solving inverse problems are presented in the monographs and papers (Alifanov et al., 1988; Denisov, 1994; Isakov, 2005; Kabanikhin, 2009; Vatulyan, 2007; Gockenbach et al., 2008; Jadamba et al., 2011), etc. But still there is a lack of researches of the coefficient inverse problems of thermoelasticity (Apbasov and Yahno, 1986; Lomazov, 2002; Lukasievicz et al., 1996). However, some specific problems concerned with finding variable coefficients of operators of thermal conductivity (Alifanov et al., 1988; Dimitriau, 2001; Hao, 1998; Isakov and Bindermann, 2000; Pobedrya et al., 2008; Xu et al., 2002) and of the elasticity theory (Alekseev, 1967; Belishev and Blagovecshenskey, 1999; Chen and Gockenbach, 2007; Jadamba et al., 2008; Kabanikhin, 1988; Rakesh, 1993; Vatulyan, 2010; Yakhno, 1990) are separately studied good enough. One of the basic approaches to a solving of inverse problems of heat conduction is its reduction to minimize non-quadratic residual functional in a finite dimensional subspace (Alifanov et al., 1988; Kabanikhin et al., 2008; Pobedrya et al., 2008). It is necessary to use iterative processes requiring the calculation of the functional gradient at each step. There is an extensive theoretical foundation for gradient minimization techniques (Alifanov et al., 1988; Hao, 1998). However, the shortcomings of such techniques are a strong influence of a choice of initial approximation on a convergence of the iteration process, and requirements to the objective function. In addition, with increasing number of unknowns delivering the minimum of

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the objective function, the amount of calculations significantly increases.

When solving inverse problems of the theory of elasticity, the most commonly used method are the method of Volterra operators (Yakhno, 1990), the method of inversion of the difference scheme (Kabanikhin, 1988), the linearization method (Alekseev, 1967), and the boundary control method (Belishev and Blagovecshenskey, 1999). At the same time for a number of modern materials, when solving the inverse problems, it is necessary to take into account the coupling of thermal and mechanical fields. The inverse problems of thermoelasticity for inhomogeneous bodies are scarcely investigated and mainly limited by weakly inhomogeneous materials (Lomazov, 2002) due to the difficulties in the construction of nonlinear operator relations that bind the desired and the measured (during an experiment) functions. However, in some papers (Vatulyan, 2007; Dudarev and Vatulyan, 2011; Nedin and Vatulyan, 2011; Nedin and Vatulyan, 2013a; Nedin and Vatulyan, 2013b) devoted to the inverse problems of the mechanics of related fields, this difficulty was overcome with the help of generalized reciprocal relations. At that the linearized Fredholm integral equations of the 1st kind were obtained to find the corrections of the coefficients recovered.

In the present paper the formulations and solutions of the coefficient inverse problem for an inhomogeneous rod are described in case of an arbitrary coupling parameter. After applying the Laplace transformation, the direct problem was reduced to solving a system of the Fredholm integral equations of the 2nd kind with respect to the transforms of temperature and pressure, and finding the actual space on the basis of the theory of residues. On the basis of the generalized reciprocity relation and the linearization method, the inverse problem was reduced to stepwise solving of the Fredholm integral equation of the 1st kind. A series of computational experiments was conducted for exact and noisy input data. The recommendations for a practical employment of the approach proposed are given.

2. Statement of the inverse problem

Let us consider a problem of longitudinal oscillations of the inhomogeneous thermoelastic rod of length l rigidly fixed at the end $x = 0$, and distinguish two ways of oscillations excitation: the thermal way and the mechanical one.

In case of the excitation of oscillations under the action of the heat flow $Q = q_0\varphi(t)$ applied to the end $x = l$ the initial-boundary value problem takes the following form (Nowacki, 1970; Vatulyan and Nesterov, 2012):

$$\frac{\partial \sigma_x}{\partial x} = \rho(x)\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$\sigma_x = E(x)\frac{\partial u}{\partial x} - \gamma(x)\theta, \tag{2}$$

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial \theta}{\partial x}\right) = c_\varepsilon(x)\frac{\partial \theta}{\partial t} + T_0\gamma(x)\frac{\partial^2 u}{\partial x \partial t}, \tag{3}$$

$$\theta(0,t) = u(0,t) = 0, \quad -k(l)\frac{\partial \theta}{\partial x}(l,t) = q_0\varphi(t), \quad \sigma_x(l,t) = 0, \tag{4}$$

$$\theta(x,0) = u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0, \tag{5}$$

If the rod is oscillated by the force $F = p_0\lambda(t)$ applied to the end $x = l$ then the boundary conditions (4) in the problem (1)–(5) will take another form:

$$\frac{\partial \theta}{\partial x}(l,t) = 0, \quad \sigma_x(l,t) = p_0\lambda(t). \tag{6}$$

The inverse problem is to determine one of the thermomechanical characteristics of the rod (the specific volumetric heat capacity $c_\varepsilon(x)$, the thermal conductivity $k(x)$, the rod's density $\rho(x)$, the Young modulus $E(x)$, the coefficient of thermal stress $\gamma(x)$) when knowing the rest characteristics of (1)–(5) on the basis of some additional data at the boundary.

In case of the thermal loading of the rod the temperature increment at its end is used as the additional data:

$$\theta(l,t) = f(t), \quad t \in [T_1, T_2] \tag{7}$$

and in case of the mechanical loading the displacement at the rod's end is used:

$$u(l,t) = g(t), \quad t \in [T_3, T_4]. \tag{8}$$

If the temperature and the displacement are known at any given time then the inverse problem may be formulated in the Laplace transform space.

In this case the additional information is

$$\theta(l,p) = \tilde{f}(p), \quad p \in [0,\infty), \tag{9}$$

$$u(l,p) = \tilde{g}(p), \quad p \in [0,\infty). \tag{10}$$

3. Solving the direct problem for an inhomogeneous thermoelastic rod

The direct problem on the vibration of a thermoelastic rod with arbitrary laws of variation of coefficients of differential operators can be solved only numerically.

Let us rewrite (1)–(5) in a dimensionless form. To do this we introduce the following parameters and variables: $z = \frac{x}{l}$, $z \in [0,1]$, $\bar{k}(z) = \frac{k(zl)}{k_0}$, $\bar{c}(z) = \frac{c(zl)}{c_0}$, $\bar{\rho}(z) = \frac{\rho(zl)}{\rho_0}$, $\bar{E}(z) = \frac{E(zl)}{E_0}$, $\bar{\gamma}(z) = \frac{\gamma(zl)}{\gamma_0}$, $t_1 = \frac{l^2 c_0}{k_0}$, $t_2 = l\sqrt{\frac{\rho_0}{E_0}}$, $\tau_1 = \frac{t}{t_1}$, $W_1 = \frac{\gamma_0\theta}{E_0}$, $U_1 = \frac{u}{l}$, $\Omega_1 = \frac{\sigma_x}{E_0}$, $\delta = \frac{\gamma_0^2 T_0}{c_0 E_0}$, $\omega = \frac{q_0\gamma_0 l}{k_0 E_0}$, $\varepsilon = \frac{t_2}{t_1} = \frac{k_0}{c_0 l}\sqrt{\frac{\rho_0}{E_0}}$, $k_0 = \max_{x\in[0,l]} k(x)$, $c_0 = \max_{x\in[0,l]} c(x)$, $E_0 = \max_{x\in[0,l]} E(x)$, $\rho_0 = \max_{x\in[0,l]}\rho(x)$, $\gamma_0 = \max_{x\in[0,l]}\gamma(x)$.

Here δ is the dimensionless coupling parameter, ε is the ratio of the characteristic time of the sound vibration t_2 to the time of the thermal vibration t_1.

Hence, the boundary-value problem (1)–(5) takes the form:

$$\frac{\partial \Omega_1}{\partial z} = \varepsilon^2\bar{\rho}(z)\frac{\partial^2 U_1}{\partial \tau_1^2}, \tag{11}$$

$$\Omega_1 = \bar{E}(z)\frac{\partial U_1}{\partial z} - \bar{\gamma}(z)W_1, \tag{12}$$

$$\frac{\partial}{\partial z}\left(\bar{k}(z)\frac{\partial W_1}{\partial z}\right) = \bar{c}_\varepsilon(z)\frac{\partial W_1}{\partial \tau_1} + \delta\bar{\gamma}(z)\frac{\partial^2 U_1}{\partial z \partial \tau_1}, \tag{13}$$

$$\begin{aligned} U_1(0,\tau_1) = W_1(0,\tau_1) = 0, \quad -\bar{k}(1)\frac{\partial W_1}{\partial z}(1,\tau_1) \\ = \omega\varphi(\tau_1), \quad \Omega_1(1,\tau_1) = 0, \end{aligned} \tag{14}$$

$$W_1(z,0) = U_1(z,0) = \frac{\partial U_1}{\partial \tau_1}(z,0) = 0. \tag{15}$$

In case of the excitation of longitudinal oscillations under the action of mechanical load $p_0\lambda(t)$ we may formulate the dimensionless problem in a same way as previously except the following difference: $\mu = \frac{p_0}{E_0}$, $\tau_2 = \frac{t}{t_2}$, $W_2 = \frac{\gamma_0\theta}{E_0}$, $U_2 = \frac{u}{l}$, $\Omega_2 = \frac{\sigma_x}{E_0}$.

In this case the dimensionless boundary conditions take the form:

$$W_2(0,\tau_2) = U_2(0,\tau_2) = 0, \quad \frac{\partial W_2}{\partial z}(1,\tau_2) = 0, \quad \Omega_2(1,\tau_2) = \mu\lambda(\tau_2). \tag{16}$$

After applying the Laplace transformation to the dimensionless boundary value problem (11)–(15) is simply reduced to a system of the Fredholm integral equations of the 2nd kind in the transform space (Vatulyan and Nesterov, 2011):

$$\tilde{W}_1(z,p) = \int_0^1 K_1(z,\xi,p)\tilde{W}_1(\xi,p)d\xi + \int_0^1 K_2(z,\xi,p)\tilde{\Omega}_1(\xi,p)d\xi + f_1(z,p),$$

$$\tilde{\Omega}_1(z,p) = \int_0^1 K_3(z,\xi,p)\tilde{W}_1(\xi,p)d\xi + \int_0^1 K_4(z,\xi,p)\tilde{\Omega}_1(\xi,p)d\xi. \qquad (17)$$

Here the kernels $K_1(z,\xi,p),\ K_2(z,\xi,p),\ K_3(z,\xi,p),\ K_4(z,\xi,p)$ and the right part $f_1(z,p)$ has the form:

$$K_1(z,\xi,p) = -p\left(\bar{c}_\varepsilon(\xi) + \delta\frac{\bar{\gamma}^2(\xi)}{\bar{E}(\xi)}\right)\int_0^{\min\{z,\eta\}}\frac{d\eta}{\bar{k}(\eta)},$$

$$K_2(z,\xi,p) = -p\delta\frac{\bar{\gamma}(\xi)}{\bar{E}(\xi)}\int_0^{\min\{z,\eta\}}\frac{d\eta}{\bar{k}(\eta)},$$

$$K_3(z,\xi,p) = -\varepsilon^2p^2\frac{\bar{\gamma}(\xi)}{\bar{E}(\xi)}\int_{\min}^{\{z,\eta\}1}\bar{\rho}(\eta)d\eta,$$

$$K_4(z,\xi,p) = -\varepsilon^2p^2\frac{1}{\bar{E}(\eta)}\int_{\min}^{\{z,\eta\}1}\bar{\rho}(\eta)d\eta,$$

$$f_1(z,p) = -\omega\tilde{\varphi}(p)\int_0^z\frac{d\xi}{\bar{k}(\xi)}.$$

The system of Eq. (17) is solved numerically on the basis of the collocation method by using the quadrature trapezium rule.

Similarly, a system of the Fredholm integral equations of the 2nd kind is obtained when solving the problem on the oscillations of a thermoelastic rod under a mechanical excitation.

Numerically analytical solutions of the system of linear equations obtained after the discretization of the system (17) showed that transformants of the temperature and the stress at the nodal points are fractionally rational functions of the Laplace transformation parameter p that do not have singular points except for the complex poles determined by the kernels of the system (17) and the type of load. Therefore, to find the actual space of the temperature and the stress we used the theory of residues: the functions in the actual space were determined in a form of a finite sum of exponential functions, and their power exponents correspond to the roots of the denominator. When $\delta = 0$, these roots are divided into two subsets. The first one includes pairs of pure imaginary numbers corresponding to the problem of the theory of elasticity, and the second one contains the negative real numbers corresponding to the heat conduction problem. If $\delta \neq 0$ then pairs of complex conjugate roots are obtained with small negative real parts. The accuracy of the solution of the direct problem was checked by comparing with analytical solution obtained for the homogeneous rod.

The Table 1 presents the values of the temperature at the rod's end for some time points under the load $\varphi(\tau_1) = H(\tau_1)$ with dimensionless parameters $\delta = 0.05,\ \varepsilon = 10^{-6}$ derived analytically and by the theory of residues.

Table 2 represents the values of the temperature at the end of the homogeneous rod $W_1(1,\tau_1)$ calculated for different thermal loads for some time points.

Within the experiment it was assumed $n = 20,\ \delta = 0.05$ and $\varepsilon = 10^{-6}$.

Table 1
The values of the temperature for several time points calculated analytically and by means of the theory of residue.

Time points τ_1	Analytical solution	Solution on the basis of the theory of residue $n = 20$	Solution on the basis of the theory of residue $n = 30$
0.001	0.033074	0.030741	0.032796
0.01	0.112838	0.112329	0.112541
0.1	0.356823	0.356264	0.356572
1	0.931260	0.931167	0.931205
5	0.999996	0.999996	0.999996

Table 2
The values of the temperature calculated in several time points for different thermal loads (H is the Heaviside function, δ is the Dirac delta function).

Types of load	Time points τ_1	Analytical solution	Solution on the basis of the theory of residue
$\varphi(\tau_1) = H(\tau_1)$	0	0.00001	0. 00031
	0.1	0.356820	0.356264
	0.2	0.504087	0.503680
	0.3	0.613236	0.612886
	0.4	0.697881	0.697572
	0.5	0.763950	0.763679
	0.6	0.815565	0.815330
$\varphi(\tau_1) = \delta(\tau_1)$	0	$+\infty$	2400.95
	0.01	5.541896	5.579683
	0.02	3.989422	3.991751
	0.03	3.257350	3.274732
	0.04	2.280948	2.292167
	0.05	2.523132	2.531131
	0.06	2.303294	2.309363
$\varphi(\tau_1) = \tau_1e^{-\tau_1}$	0	0.000027	0.000043
	0.1	0.021873	0.021577
	0.2	0.057498	0.056893
	0.3	0.097321	0.096701
	0.4	0.137695	0.137115
	0.5	0.178739	0.175736
	0.6	0.210088	0.211058

The analysis of the Tables 1 and 2 show that when in the quadrature trapezium formula the partition is $n = 20$, the relative error of the numerical solution is less than 1%.

4. Formulation of the operator relations on a basis of the generalized reciprocity theorem in transformants for thermoelastic bodies

The inverse problems set previously are nonlinear ill-posed problems. The main difficulty of studying nonlinear inverse problems is the formulation of an operator relation between the unknown coefficients of differential operators and the boundary physical fields.

In Vatulyan and Nesterov (2009) a method of obtaining operator relations linking the desired and the given functions was proposed by using the generalized relation of reciprocity in transform space.

We shall distinguish two states (1) and (2): the first one with the Young modulus $E^{(1)}(x)$, the thermal conductivity $k^{(1)}(x)$, the coefficient of thermal stress $\gamma^{(1)}(x)$, the density $\rho^{(1)}(x)$, the specific volumetric heat capacity $c_\varepsilon^{(1)}(x)$, the transformant of the temperature increment $\tilde{\theta}^{(1)}(x,t)$, the transformant of the displacement $\tilde{u}^{(1)}(x,t)$, and the second one with, respectively, $E^{(2)}(x),\ k^{(2)}(x),\ \gamma^{(2)}(x),\ \rho^{(2)}(x),\ c_\varepsilon^{(2)}(x),\ \tilde{\theta}^{(2)}(x,t), \tilde{u}^{(2)}(x,t)$.

For each state, the equations of motion, the equations of heat conductivity, the constitutive equations, and the boundary conditions in transformants are satisfied; they are the same for both states.

Using a standard technique, the two nonlinear integral relations were obtained:

$$-\int_0^l (E^{(2)}-E^{(1)})\frac{d\tilde{u}^{(1)}}{dx}\frac{d\tilde{u}^{(2)}}{dx}dx-\int_0^l(\rho^{(2)}-\rho^{(1)})\tilde{u}^{(1)}\tilde{u}^{(2)}dx$$
$$-\int_0^l\left(\gamma^{(1)}\frac{d\tilde{u}^{(2)}}{dx}\tilde{\theta}^{(1)}-\gamma^{(2)}\frac{d\tilde{u}^{(1)}}{dx}\tilde{\theta}^{(2)}\right)dx$$
$$=\tilde{F}(\tilde{u}^{(2)}-\tilde{u}^{(1)})|_{x=l}, \tag{18}$$

$$\int_0^l (k^{(2)}-k^{(1)})\frac{d\tilde{\theta}^{(1)}}{dx}\frac{d\tilde{\theta}^{(2)}}{dx}dx+p\int_0^l(c_\varepsilon^{(2)}-c_\varepsilon^{(1)})\tilde{\theta}^{(1)}\tilde{\theta}^{(2)}dx-pT_0$$
$$\times\int_0^l\left(\gamma^{(1)}\frac{d\tilde{u}^{(1)}}{dx}\tilde{\theta}^{(2)}-\gamma^{(2)}\frac{d\tilde{u}^{(2)}}{dx}\tilde{\theta}^{(1)}\right)dx$$
$$=\tilde{Q}(\tilde{\theta}^{(2)}-\tilde{\theta}^{(1)})|_{x=l}. \tag{19}$$

The relations (18) and (19) obtained may be interpreted as nonlinear relations for the inverse problem of thermoelasticity in transform space. On the basis of these relations an iterative process can be constructed. To do this, let us use a conventional technique for a linearization of (18), (19) by setting $\tilde{\theta}^{(1)}=\tilde{\theta}^{(n-1)}$, $\tilde{\theta}^{(2)}=\tilde{\theta}^{(n-1)}+\delta\tilde{\theta}^{(n-1)}$, $\tilde{u}^{(1)}=\tilde{u}^{(n-1)}$, $\tilde{u}^{(2)}=\tilde{u}^{(n-1)}+\delta\tilde{u}^{(n-1)}$, $k^{(1)}=k^{(n-1)}$, $k^{(2)}=k^{(n-1)}+\delta k^{(n-1)}$, $E^{(1)}=E^{(n-1)}$, $E^{(1)}=E^{(n-1)}$, $\rho^{(1)}=\rho^{(n-1)}$, $\rho^{(2)}=\rho^{(n-1)}+\delta\rho^{(n-1)}$, $c_\varepsilon^{(1)}=c_\varepsilon^{(n-1)}$, $c_\varepsilon^{(2)}=c_\varepsilon^{(n-1)}+\delta c_\varepsilon^{(n-1)}$, $\gamma^{(1)}=\gamma^{(n-1)}$, $\gamma^{(2)}=\gamma^{(n-1)}+\delta\gamma^{(n-1)}$; then, maintaining the linear terms in the relationships, and taking into account the additional data, we have:

$$-\int_0^l \delta E^{(n-1)}\left(\frac{d\tilde{u}^{(n-1)}}{dx}\right)^2dx-p^2\int_0^l\delta\rho^{(n-1)}(\tilde{u}^{(n-1)})^2dx$$
$$+\int_0^l\delta\gamma^{(n-1)}\frac{d\tilde{u}^{(n-1)}}{dx}\tilde{\theta}^{(n-1)}dx$$
$$=p_0\tilde{\lambda}(p)(\tilde{g}-\tilde{u}^{(n-1)})|_{x=l}, \tag{20}$$

$$\int_0^l \delta k^{(n-1)}\left(\frac{d\tilde{\theta}^{(n-1)}}{dx}\right)^2dx+p\int_0^l\delta c^{(n-1)}(\tilde{\theta}^{(n-1)})^2dx+pT_0$$
$$\times\int_0^l\delta\gamma^{(n-1)}\frac{d\tilde{u}^{(n-1)}}{dx}\tilde{\theta}^{(n-1)}dx$$
$$=q_0\tilde{\varphi}(p)(\tilde{f}-\tilde{\theta}^{(n-1)})|_{x=l}. \tag{21}$$

5. Iterative scheme of solving the inverse problem of thermoelasticity for the inhomogeneous rod

The relations (20) and (21) are the integral equations for the components $\delta k^{(n-1)}(x)$, $\delta c_\varepsilon^{(n-1)}(x)$, $\delta\gamma^{(n-1)}(x)$, $\delta\rho^{(n-1)}(x)$, $\delta E^{(n-1)}(x)$; they allow to find an approximation of coefficients if the direct problem is previously solved with the thermomechanical characteristics $k^{(n-1)}(x)$, $c_\varepsilon^{(n-1)}(x)$, $\gamma^{(n-1)}(x)$, $\rho^{(n-1)}(x)$, $E^{(n-1)}(x)$.

The dimensionless operator Eqs. (20) and (21) take the form:

$$-\int_0^1 \delta\bar{E}^{(n-1)}\left(\frac{d\tilde{U}_2^{(n-1)}}{dz}\right)^2dz-p^2\int_0^1\delta\bar{\rho}^{(n-1)}\left(\tilde{U}_2^{(n-1)}\right)^2dz$$
$$+\int_0^1\delta\bar{\gamma}^{(n-1)}\frac{d\tilde{U}_2^{(n-1)}}{dz}\tilde{W}_2^{(n-1)}dz$$
$$=\mu\tilde{\lambda}(p)\left(\tilde{g}(p)-\tilde{U}_2^{(n-1)}(1,p)\right), \tag{22}$$

$$\int_0^1 \delta\bar{k}^{(n-1)}\left(\frac{dW_1^{(n-1)}}{dz}\right)^2+p\int_0^1\delta\bar{c}_\varepsilon^{(n-1)}\left(W_1^{(n-1)}\right)^2dz+2\delta p$$
$$\times\int_0^1\delta\bar{\gamma}^{(n-1)}\frac{d\tilde{U}_1^{(n-1)}}{dz}\tilde{W}_1^{(n-1)}dz$$
$$=\omega\tilde{\varphi}(p)\left(\tilde{f}(p)-\tilde{W}_1^{(n-1)}(1,p)\right). \tag{23}$$

If it is required to restore only one thermomechanical characteristic then the Eqs. (22) and (23) are divided into the independent integral equations.

Thereby, to find the corrections while reconstructing the dimensionless specific volumetric heat capacity unit volume it is necessary to solve the integral equation:

$$p\int_0^1\delta\bar{c}_\varepsilon^{(n-1)}\left(\tilde{W}_1^{(n-1)}\right)^2dz=\omega\tilde{\varphi}(p)\left(\tilde{f}(p)-\tilde{W}_1^{(n-1)}(1,p)\right). \tag{24}$$

To solve the inverse problem in the actual space let us apply to the Eq. (24) the convolution theorem and the theorem on the differentiation of the original. Then, when exposing by the heat load $\varphi(\tau_1)=\delta(\tau_1)$, to find the corrections $\delta\bar{c}_\varepsilon^{(n-1)}(z)$ at the finite dimensionless time interval $[a,b]$ we deal with the integral equation:

$$\int_0^1\delta\bar{c}_\varepsilon^{(n-1)}K(z,\tau_1)dz=\omega\left(f(\tau_1)-W_1^{(n-1)}(1,\tau_1)\right),\quad \tau_1\in[a,b], \tag{25}$$

where the kernel of the integral equation (25) has the form:

$$K(z,\tau_1)=\int_0^{\tau_1}\frac{\partial W_1^{(n-1)}(z,\tau)}{\partial\tau}W_1^{(n-1)}(z,\tau_1-\tau)d\tau.$$

In the present paper a computational experiment was conducted instead of a natural one.

The dimensionless thermomechanical characteristics $\bar{a}(z)$ of the rod were restored in two stages. At the first stage the initial approximation was determined in a class of positive bounded linear functions $\bar{a}^{(0)}(z)=kz+b$ on a basis of minimizing the residual functional on a compact set. In case of the thermal loading the residual functional has the form:

$$J_1=\int_0^T\left(f(\tau_1)-W_1^{(n-1)}(1,\tau_1)\right)^2d\tau_1, \tag{26}$$

where $[0,T]$ is the dimensionless time period at which the residual J_1 is calculated. In case of the mechanical loading the residual functional has the form:

$$J_2=\int_0^H\left(g(\tau_2)-U_2^{(n-1)}(1,\tau_2)\right)^2d\tau_2, \tag{27}$$

where $[0,H]$ is the dimensionless time period at which the residual J_2 is calculated.

When having a priori information on the coefficients magnitudes in the form $0<a_-\leqslant\bar{a}^{(0)}(z)\leqslant a_+$, one may build the restrictions for the unknown constants k and b in the form of a system of inequalities $a_-\leqslant k\leqslant a_+$, $a_-\leqslant k+b\leqslant a_+$ that describe the range of the coefficients – a compact set. Then, by splitting this area into a grid and minimizing the corresponding functional on a built compact set we select a suitable pair (k,b).

At the second stage on a basis of the solution of the integral equations (24), (25) the iterative process of the reconstructed-function refinement was built under the scheme $\bar{a}^{(n)}(z)=\bar{a}^{(n-1)}(z)+\delta\bar{a}^{(n-1)}(z)$. At each step of the process, by solving a system of the Fredholm integral equations of the 2nd kind (17) the new values $\tilde{W}_1^{(n-1)}(z,p)$ were found, and then $W_1^{(n-1)}(z,\tau_1)$ was; with the help of the latter the right parts of the Fredholm integral equations of the 1st kind (24) and (25) and their kernels were calculated.

The exit from the iterative process was carried out when reaching the maximum number of iterations or when the corresponding residual functional (26) or (27) reached the threshold value 10^{-6}.

6. Implementation of the Tikhonov method when solving the inverse problem in transform space

A process of finding the corrections from the solutions of the integral equation (24) is an ill-posed problem and it requires a regularization. In the present paper the Tikhonov regularization method is employed (Tikhonov and Arsenin, 1979).

Let us carry out a discretization of the integral Eq. (24) based on the collocation method with using the trapezium quadrature formula. For that, we introduce a uniform partition of the segment $[0,1]$ into n segments by the points $z_i = \Delta z(i-1),\ i = 1,\dots,n+1$, where $\Delta z = \frac{1}{n}$ is the subinterval.

Then a set of parameters $\{p_j\}_{j=1}^{m+1}$ is constructed in the following way:

(1) mapping of the set $p \in [0,+\infty)$ into the segment $s \in [0,1]$ according to the rule $s^2 = p^2(1+p^2)^{-1}$;
(2) construction of a uniform partition of the segment $s \in [\kappa, 1-\kappa]$ by the segments points $\{s_j\}_{j=1}^{m+1}$;
(3) calculation of $\{p_j\}_{j=1}^{m+1}$ using the inverse mapping set $\{s_j\}_{j=1}^{m+1}$.

Satisfying (24) in a set of points p_j, we obtain the linear algebraic system with respect to the nodal unknowns.

According to the method of Tikhonov (Tikhonov and Arsenin, 1979), the solving of the inverse problem is reduced to the solving of the regularized system of equations with searching for the regularization parameter by the generalized residual.

7. Results of computational experiments

The numerical experiments were conducted with the following parameters: $n = 20,\ \varepsilon = 10^{-6},\ \lambda = 0.1,\ \omega = 0.05$.

At first, a series of experiments on the reconstruction of the thermo-mechanical characteristics in transform space was conducted. It was revealed that the change in the coupling parameter scarcely effect on the reconstruction results, but the coefficient of thermal stress was restored better with increase of coupling parameter. Monotonic functions were restored with a smaller error than the nonmonotonic. As a result of a reconstruction of piecewise-constant functions, smooth functions were obtained which moments of the zeroth, the first and the second orders almost coincided.

During the reconstruction of the specific volumetric heat capacity, the coefficient of thermal stress and the density, the largest reconstruction error occurred in the vicinity of the end $z = 0$. This is due to the fact that the kernels of the corresponding integral equations vanish for $z = 0$.

Table 3 shows the values of the residuals calculated at the dimensionless segment $[0,1]$, and the relative error of the reconstruction depending on the number of iterations for various laws of the recovery $\bar{k}(z)$: 1st law $\bar{k}(z) = e^{-z}$, 2nd law $\bar{k}(z) = 1 + z^2$, and the 3rd law $\bar{k}(z) = 1.1 + \sin(\pi z) + z^2$.

Table 3
Here N is a number of iterations.

N	Residual J_n			Relative error, %		
	1	2	3	1	2	3
1	0.000315	0.001147	0.002925	7	5	15
2	0.000014	0.000083	0.000242	5	3.5	12
3	0.000006	0.000005	0.000015	3.5	2.5	10
4	0.0000008	0.0000003	0.000002	3	2	9
5	–	–	0.0000004	–	–	8

From the Table 3 it may be seen that with a growth of a number of iterations the residue and the relative error of the reconstruction decrease rapidly for all the laws of heterogeneity.

At all the figures below the solid line shows the exact law, points – the restored one, dotted line – the initial approximation. Fig. 1 represents the result of the recovery of the decreasing function $\bar{c}_\varepsilon(z) = 1 - \frac{z^2}{2}$ when the heat load is $\varphi(\tau_1) = \tau_1 e^{-\tau_1}$. The coupling parameter is $\delta = 0.1$, the initial approximation is $\bar{c}_0(z) = 1.05 - 0.45z$. To achieve the threshold value of the functional (26) it was required 3 iterations. The recovery error at the last iteration did not exceed 5%.

Fig. 2 depicts an example of the reconstruction of an increasing function $\bar{E}(z) = \frac{1}{\sqrt{2-z^2}}$ under mechanical load $\lambda(\tau_2) = H(\tau_2)$. The initial approximation is $\bar{E}_0(z) = 0.7 + 0.25z$, the coupling parameter is $\delta = 0.05$. To achieve the threshold value of the functional (27) it took 4 iterations. The recovery error in the last iteration did not exceeded 4%.

Fig. 3 shows an example of the reconstruction of increasing function $\bar{\rho}(z) = z + e^{-z} - 0.5$ under mechanical load $\lambda(\tau_2) = H(\tau_2)$.

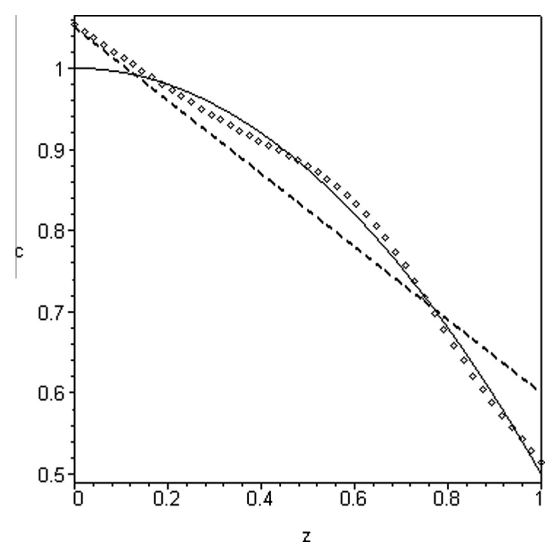

Fig. 1. Reconstruction of $\bar{c}(z)$.

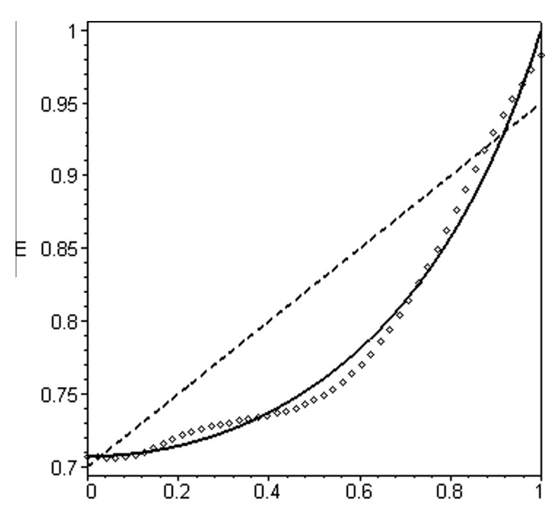

Fig. 2. Reconstruction of $\bar{E}(z)$.

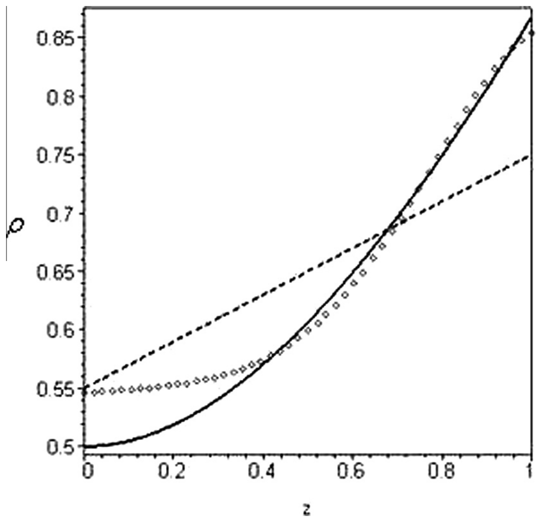

Fig. 3. Reconstruction of $\bar{\rho}(z)$.

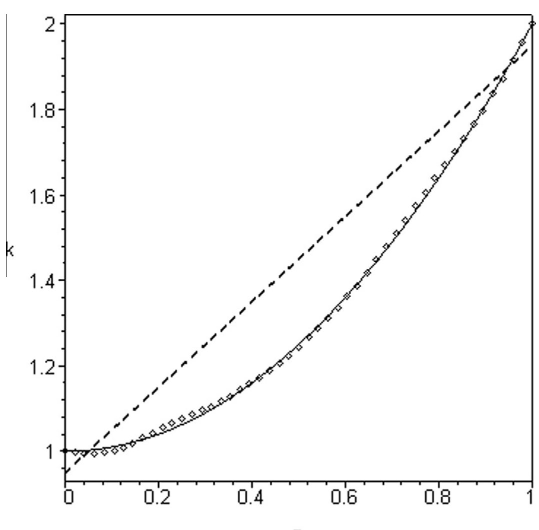

Fig. 4. Reconstruction of $\bar{k}(z)$.

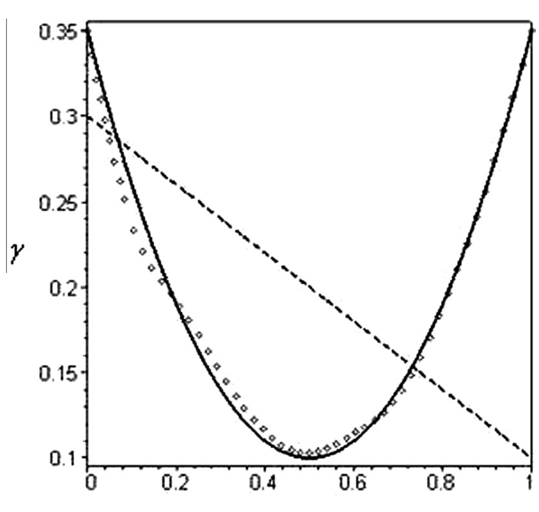

Fig. 5. Reconstruction of $\bar{\gamma}(z)$.

The initial approximation is $\bar{\rho}_0(z) = 0.55 + 0.2z$, the coupling parameter is $\delta = 0.01$. To achieve the threshold value of the functional (27) it took 6 iterations. The error recovery at the last iteration did not exceed 8%.

Next, the results of the second series of experiments on the reconstruction of thermomechanical characteristics in actual space are described. We will discuss the influence of the noise of input data on the accuracy of the reconstruction, the selection of the loading type, the time interval, and the number of measurement points from it.

The noise masking of the input data was simulated using the relationship $f_\beta(\tau) = f(\tau)(1 + \beta\psi)$, where β is the noise magnitude, ψ is a random variable with a uniform distribution law. The procedure of a reconstruction of the coefficients was stable up to 3%-noise ratio. At that the error of the reconstruction did not exceed 15%.

Fig. 4 shows the result of a restoration of the increasing function $\bar{k}(z) = 1 + z^2$ with a load $\varphi(\tau_1) = H(\tau_1)$, assuming that end values are given. The initial approximation is $\bar{k}_0(z) = 0.95 + z$. The calculations were made at the dimensionless time interval $[0, 1]$ and 4 observation points from it. To achieve the threshold value of the functional (26) 3 iterations were required.

Fig. 5 shows the result of the reconstruction of a nonmonotonic function $\bar{\gamma}(z) = 0.1 + (z - 0.5)^2$ under load $\varphi(\tau_1) = \delta(\tau_1)$ and given end values. The initial approximation is $\bar{\gamma}_0(z) = 0.3 - 0.2z$. The calculations were made at the dimensionless time interval, at 5 points of observation. The iterative process was completed when the maximum number of iterations equaling eight was achieved. The maximum error of reconstruction at the last iteration reached 9%.

Fig. 6 represents a result of a reconstruction of the function $\bar{c}_\varepsilon(z) = -z^2 + z + 1$ for the heat load $\varphi(\tau_1) = \tau_1 e^{-\tau_1}$. The coupling parameter is $\delta = 0.01$, the initial approximation is $\bar{c}_0(z) = 1.05 + 0.25z$. The calculations were made at the dimensionless time interval $[0, 1.5]$ with 5 observation points taken from this segment. To achieve the threshold value of the functional (26) it took six iterations.

Fig. 7 shows plots of the relative error of a reconstruction of the specific heat capacity with the law of variation $\bar{c}_\varepsilon(z) = -z^2 + z + 1$ for different values of noise magnitude ($\beta = 0$, $\beta = 0.01$, $\beta = 0.03$).

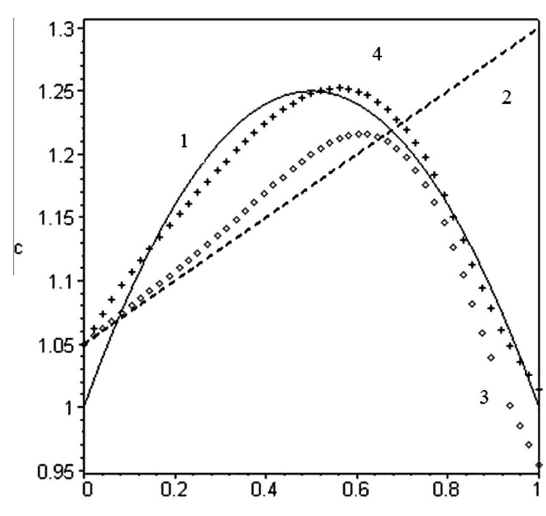

Fig. 6. The process of reconstruction of the function $\bar{c}_\varepsilon(z) = -z^2 + z + 1$. Here: 1 – exact law, 2 – initial approximation, 3 – result of reconstruction on the 1st iteration, 4 – final result of the reconstruction.

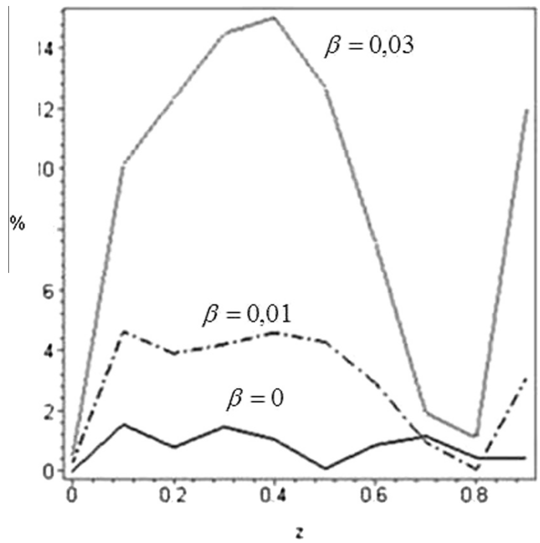

Fig. 7. Relative errors of reconstruction of $\bar{c}_{\varepsilon}(z) = -z^2 + z + 1$ for different β.

8. Conclusions

1. The statements of the inverse coefficient problems of thermoelasticity are given for an inhomogeneous rod both in the Laplace transform space and in the actual space.
2. The techniques of solving the direct problems on longitudinal oscillations of an inhomogeneous thermoelastic rod under mechanical and thermal loading are developed.
3. The methods of constructing operator relations and iterative processes in inverse problems on an identification of heterogeneous thermoelastic characteristics of a rod are proposed.
4. The computational experiments on a reconstruction of thermomechanical characteristics both in the transform space and in the actual space are conducted.

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