Solutions to the problems of Project Euler

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Problem 101

Solution:

Answer: —

Problem 102

Three distinct points are plotted at random on a Cartesian plane, for which $-1000 \leq x, y \leq 1000$, such that a triangle is formed.

Consider the following two triangles:

$$\begin{array}{rll} A(-340, 495), & B(-153, -910), & C(835, -947) \\ X(-175, 41), & Y(-421, -714), & Z(574, -645) \end{array}$$

It can be verified that triangle ABC contains the origin, whereas triangle XYZ does not.

Using `triangles.txt`, a 27k text file containing the co-ordinates of one thousand "random" triangles, find the number of triangles for which the interior contains the origin.

NOTE: The first two examples in the file represent the triangles in the example given above.

Solution:

Given a triangle ABC and the origin O, consider the section of the 'area' corresponding to the quantity $|ABO| + |BCO| + |CAO|$. If the point O lies inside the triangle, then this quantity will be equal to the area $|ABC|$, whereas if it is outside, this quantity will be appropriately larger. Applying this test for each triangle will tell us whether it contains the origin or not.

To find the areas, I used Heron's formula:

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \frac{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}}{4}.$$

Answer: 228

Problem 103

Solution:

Answer: —

Problem 104

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad \text{where } F_1 = 1 \text{ and } F_2 = 1.$$

It turns out that F_{541}, which contains 113 digits, is the first Fibonacci number for which the last nine digits are 1-9 pandigital (contain all the digits 1 to 9, but not necessarily in order). And F_{2749}, which contains 575 digits, is the first Fibonacci number for which the first nine digits are 1-9 pandigital.

Given that F_k is the first Fibonacci number for which the first nine digits AND the last nine digits are 1-9 pandigital, find k.

Solution:

This can be solved with an 'intelligent' brute force solution. We can generate a list of Fibonacci numbers, keeping only the last 9 digits with ease, checking if the end digits are 1-9 pandigital. Once we find one, we need to check the first 9 digits as well, and for this we use a trick. It is well known that $F_n = (\varphi^n + \varphi^{-n})\sqrt{5}$. Now the second term is always less than 1, so when looking at the first nine digits, we can neglect it. Observe that $\log_{10} F_n = n \log_{10} \varphi - \log_{10} \sqrt{5}$, so by reducing $\log_{10} F_n$ by an integer quantity until it lies between 8 and 9, we get a number with the same leading digits, but less than 10^9. Exponentiating it, and dropping the fractional part, we have the first 9 digits to check for pandigitality.

Answer: 329468

Problem 105

Solution:

Answer: —

Problem 106

Let $S(A)$ represent the sum of elements in set A of size n. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

1. $S(B) \neq S(C)$; that is, sums of subsets cannot be equal.
2. If B contains more elements than C then $S(B) > S(C)$.

For this problem we shall assume that a given set contains n strictly increasing elements and it already satisfies the second rule.

Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which $n = 4$, only 1 of these pairs need to be tested for equality (first rule). Similarly, when $n = 7$, only 70 out of the 966 subset pairs need to be tested.

For $n = 12$, how many of the 261625 subset pairs that can be obtained need to be tested for equality?

NOTE: This problem is related to problems 103 and 105.

Solution:

Ok, step one will be to generate all the 12 element subsets, and classify them by length (as the second condition need not be checked). Then we'll need to run the sets of the same length against each other to see if they need be compared. If they have a common element, we can reduce it to a smaller case (without that element), so we don't check it. And, if the elements of one sequence can be matched up to be all bigger than the elements of the second sequence, we again don't need to compare. As the sequences are given as ordered, we just need to check the indices, keeping track of two boolean variables – one for sequence one being bigger, one the other way round. If they are both true, we need to count this one (eg $a_1 + a_4$ vs $a_2 + a_3$).

Answer: 21384

Problem 107

Solution:

Answer: —

Problem 108

In the following equation x, y, and n are positive integers.

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$$

For $n=4$ there are exactly three distinct solutions:

$$\frac{1}{5}+\frac{1}{20}=\frac{1}{6}+\frac{1}{12}=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

What is the least value of n for which the number of distinct solutions exceeds one-thousand?

NOTE: This problem is an easier version of problem 110; it is strongly advised that you solve this one first.

Solution:

Well, observe that $y=\frac{nx}{x-n}$, and the question can be brute forced from here. But I'd rather do it properly, so check out the solution to 110.

Answer: 180180

Problem 109

In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty.

The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero. The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively.

At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points.

There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system, which means that the player must land a double (including the double bulls-eye at the centre of the board) on their final dart to win; any other dart that would reduce their running total to one or lower means the score for that set of three darts is "bust".

When a player is able to finish on their current score it is called a "checkout" and the highest checkout is 170: T20 T20 D25 (two treble 20s and double bull).

There are exactly eleven distinct ways to checkout on a score of 6:
D3; D1 D2; S2 D2; D2 D1; S4 D1; S1 S1 D2; S1 T1 D1; S1 S3 D1; D1 D1 D1; D1 S2 D1; S2 S2 D1

Note that D1 D2 is considered different to D2 D1 as they finish on different doubles. However, the combination S1 T1 D1 is considered the same as T1 S1 D1.

In addition we shall not include misses in considering combinations; for example, D3 is the same as 0 D3 and 0 0 D3.

Incredibly there are 42336 distinct ways of checking out in total.

How many distinct ways can a player checkout with a score less than 100?

Solution:

Ok, this is a bit like baby's first dynamic programming. Can even just do it in a triple loop over the possible scores for each dart and see if it comes out below 100.

Answer: 38182

Problem 110

In the following equation x, y, and n are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

It can be verified that when $n = 1260$ there are 113 distinct solutions and this is the least value of n for which the total number of distinct solutions exceeds one hundred.

What is the least value of n for which the number of distinct solutions exceeds four million?

NOTE: This problem is a much more difficult version of problem 108 and as it is well beyond the limitations of a brute force approach it requires a clever implementation.

Solution:

If $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$, then rearranging gives $y = \frac{nx}{x-n}$. The inspirational bit is to notice that $y = \frac{nz}{x-n} = n + \frac{n^2}{x-n}$, so $(x-n)$ must be a factor of n^2. Due to the symmetric nature of (x, y), for a given n there will be $\lceil d(n^2) \rceil = \frac{1}{2}(d(n^2) + 1)$.

Now if $n = \prod p_i^{\alpha_i}$, then $n^2 = \prod p_i^{2\alpha_i}$ and $d(n^2) = \prod(2\alpha_i + 1)$. So we need to find a set of exponents $\{\alpha_i\}$, which when arranged in decreasing order to the first primes will give the lowest total. Next observe that $3^{15} > 8 \times 10^{10}$, so the product of the first 15 primes will have over 4 million solutions, so we need not go higher than that.

Also observe that in a problem such as this, repeating a small prime will not be that beneficial, as we're considering a product of $(2\alpha+1)$'s, so a new prime will give a factor of 3, so eliminating a prime will require increasing an α to 4 (or spreading it out somehow), and after the first couple primes it will become very costly.

So we 'guess' that only the first 5 prime will involve repeats. Run a loop over the first α_i's (keeping them decreasing), evaluating the product so far, and the number of solutions to the equation. From that, we can calculate how many other primes need to be included (once each), and include them in the product. Compare to best solution so far. And that is it.

Answer: 9350130049860600

Problem 111

Solution:

Answer: —

Problem 112

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

Clearly there cannot be any bouncy numbers below one-hundred, but just over half of the numbers below one-thousand (525) are bouncy. In fact, the least number for which the proportion of bouncy numbers first reaches 50

Surprisingly, bouncy numbers become more and more common and by the time we reach 21780 the proportion of bouncy numbers is equal to 90%.

Find the least number for which the proportion of bouncy numbers is exactly 99%.

Solution:

Given that there are 2178 non-bouncy numbers up to 21780, to get this down to 1% we'll need to go to a couple of million. As we expect to find more non-bouncy numbers, a fair guess for the answer would be 10 to 20 million. This is sufficiently small as to be brute forced. Just run through all the numbers from 101 upwards, counting the bouncy ones, waiting for it to reach 99% of the total.

Answer: 1587000

Problem 113

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

As n increases, the proportion of bouncy numbers below n increases such that there are only 12951 numbers below one-million that are not bouncy and only 277032 non-bouncy numbers below 10^{10}.

How many numbers below a googol (10^{100}) are not bouncy?

Solution:

Let us count the number of increasing numbers, up to k-digits long. It can be written with leading zeros as a set of zeros, ones, twos, etc. up to nines. Each combination will give a unique number. This can be calculated as taking $k+9$ places and setting up the digit partitions, giving $\binom{k+9}{9}$ numbers. However, of these $9k$ are constant, and 1 is identically zero.

Similarly, we can work with decreasing numbers. However, to include leading zeros we now have zeros, nines, eights, down to ones and zeros. So this time we insert 10 partitions among $k+10$ places, giving $\binom{k+10}{10}$ numbers. Of these, $10k$ will be constant, and 1 is identically zero.

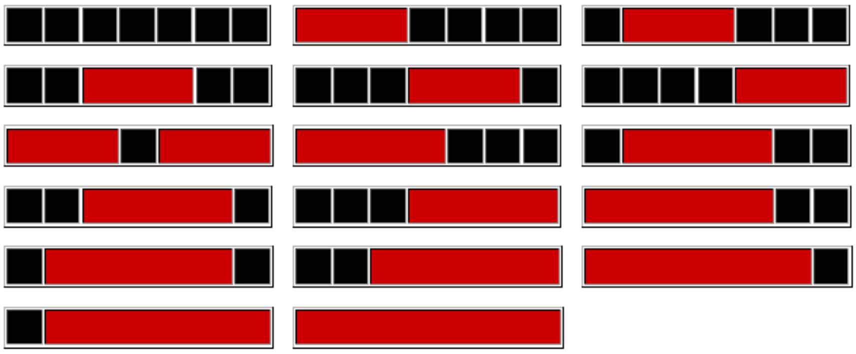
So total number of non-bouncy numbers will be (including constant ones):

$$\mathcal{N} = \binom{109}{9} - 901 + \binom{110}{10} - 1001 + 900 = \binom{109}{9} + \binom{110}{10} - 1002.$$

Answer: 51161058134250

Problem 114

A row measuring seven units in length has red blocks with a minimum length of three units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square. There are exactly seventeen ways of doing this.



How many ways can a row measuring fifty units in length be filled?

NOTE: Although the example above does not lend itself to the possibility, in general it is permitted to mix block sizes. For example, on a row measuring eight units in length you could use red (3), black (1), and red (4).

Solution:

This is a fairly straight forward exercise in DP (well, it's fairly linear), with one trick. Since the red blocks cannot touch, convert the red blocks into a 'red block+black square', and solve for a board of length 51.

So we evaluate the number of combinations up to length n. Then for a length $n+1$, we can add a black square (so same as combinations of length n), or add a block of length 4, 5, etc. Very fast code.

Answer: 16475640049

Problem 115

NOTE: This is a more difficult version of problem 114.

A row measuring n units in length has red blocks with a minimum length of m units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square.

Let the fill-count function, $F(m, n)$, represent the number of ways that a row can be filled.

For example, $F(3, 29) = 673135$ and $F(3, 30) = 1089155$.

That is, for $m = 3$, it can be seen that $n = 30$ is the smallest value for which the fill-count function first exceeds one million.

In the same way, for $m = 10$, it can be verified that $F(10, 56) = 880711$ and $F(10, 57) = 1148904$, so $n = 57$ is the least value for which the fill-count function first exceeds one million.

For $m = 50$, find the least value of n for which the fill-count function first exceeds one million.

Solution:

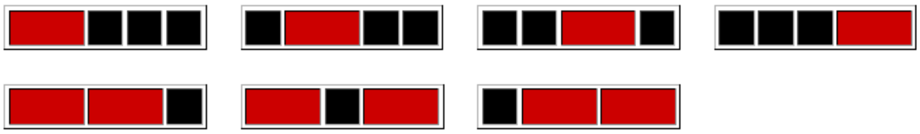
Meh, this question might have been tricky, had it not been for #114. Just repackage the code from problem 114 into a function to evaluate $F(m, n)$, set $m = 50$, start with $n > m$, and work upwards until $F(m, n) > 10^6$.

Answer: 168

Problem 116

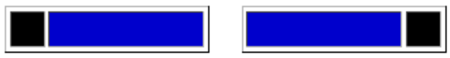
A row of five black square tiles is to have a number of its tiles replaced with coloured oblong tiles chosen from red (length two), green (length three), or blue (length four).

If red tiles are chosen there are exactly seven ways this can be done.



If green tiles are chosen there are three ways.

And if blue tiles are chosen there are two ways.



Assuming that colours cannot be mixed there are $7 + 3 + 2 = 12$ ways of replacing the black tiles in a row measuring five units in length.

How many different ways can the black tiles in a row measuring fifty units in length be replaced if colours cannot be mixed and at least one coloured tile must be used?

NOTE: This is related to problem 117.

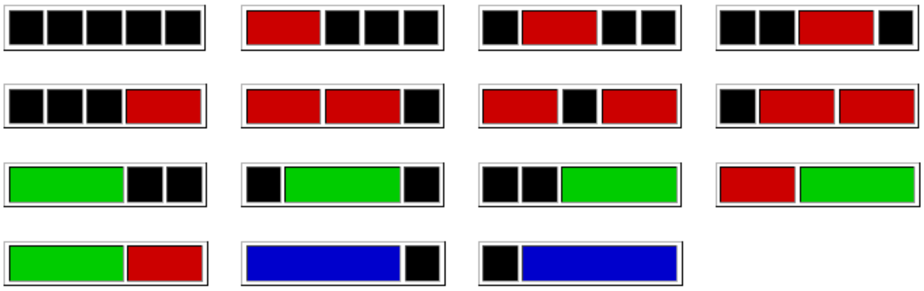
Solution:

Meh, this seems like a three times a simpler version of problem 114. There is no limit on the length of the coloured blocks and they need not be separated by a black block, so $N(l) = N(l) + N(l - c)$, where c is the length of coloured block. So just run for c equal to 2, 3 and 4, and sum the result (remembering to remove 1 from each calculation for the all-black solution).

Answer: 20492570929

Problem 117

Using a combination of black square tiles and oblong tiles chosen from: red tiles measuring two units, green tiles measuring three units, and blue tiles measuring four units, it is possible to tile a row measuring five units in length in exactly fifteen different ways.



How many ways can a row measuring fifty units in length be tiled?

NOTE: This is related to problem 116.

Solution:

Ok seriously, this **is** just a simpler version of problem 114. Simple linear DP.

Answer: 100808458960497

Problem 118

Solution:

Answer: —

Problem 119

The number 512 is interesting because it is equal to the sum of its digits raised to some power: $5 + 1 + 2 = 8$, and $8^3 = 512$. Another example of a number with this property is $614656 = 28^4$.

We shall define a_n to be the n^{th} term of this sequence and insist that a number must contain at least two digits to have a sum.

You are given that $a_2 = 512$ and $a_{10} = 614656$.

Find a_{30}.

Solution:

Well, if we set a guess-limit that the number will have at most 20 digits long, then $10^{20} > N = a^b$, the sum of digits of N will be at most 180, so we have a limit on a. So we loop over a from 2 to 180, evaluating all the exponents, checking of the digit sum corresponds to a. The answer comes out to be 63^8.

This problem is easily scalable to higher indices. For instance $a_{100} = 256^{22}$, has 53 digits.

Answer: 248155780267521

Problem 120

Let r be the remainder when $(a-1)^n + (a+1)^n$ is divided by a^2.

For example, if $a = 7$ and $n = 3$, then $r = 42$: $6^3 + 8^3 = 728 \equiv 42 \bmod 49$. And as n varies, so too will r, but for $a = 7$ it turns out that $r_{\max} = 42$.

For $3 \leq a \leq 1000$, find $\sum r_{\max}$.

Solution:

Notice that if n is even, $(a-1)^n + (a+1)^n \equiv_{a^2} (\frac{a^2}{2} - na + 1) + (\frac{a^2}{2} + na + 1) \equiv_{a^2} 2$, whereas if n is odd, it reduces to $2na$.

So for odd a, we can take $n = (a-1)/2$ to obtain $r = a(a-1)$, which is the largest multiple of a less than a^2, so it is unquestionably the maximum. For even a we can take $n = (a-2)/2$, to obtain $r = a(a-2)$, which is less obviously the maximum, but a brute force check shows that this too is the value of $r_{\max}$.

Answer: 333082500

Problem 121

A bag contains one red disc and one blue disc. In a game of chance a player takes a disc at random and its colour is noted. After each turn the disc is returned to the bag, an extra red disc is added, and another disc is taken at random.

The player pays £1 to play and wins if they have taken more blue discs than red discs at the end of the game.

If the game is played for four turns, the probability of a player winning is exactly 11/120, and so the maximum prize fund the banker should allocate for winning in this game would be £10 before they would expect to incur a loss. Note that any payout will be a whole number of pounds and also includes the original £1 paid to play the game, so in the example given the player actually wins £9.

Find the maximum prize fund that should be allocated to a single game in which fifteen turns are played.

Solution:

It would seem tempting to run a simulation of the game, but is actually fairly easy to do it 'properly'. The chance of drawing a blue ball in draw k is $\frac{1}{k+1}$. For a 15 ball game, there are $2^{15} = 32768$ ways of setting up a blue-red pattern, and each one has an appropriate probability (as we know the probability of each colour at each draw). The sum of these can be checked to come to zero, but we only sum those with 8 or more blue balls. The array operations are useful in this question. Once we have the probability of success P, the maximal payout is $\lfloor \frac{1}{P} \rfloor$.

Answer: 2269

Problem 122

The most naive way of computing n^{15} requires fourteen multiplications:

$$n \times n ... \times n = n^{15}$$

But using a "binary" method you can compute it in six multiplications:

$$\begin{aligned} n \times n &= n^2 \\ n^2 \times n^2 &= n^4 \\ n^4 \times n^4 &= n^8 \\ n^8 \times n^4 &= n^12 \\ n^{12} \times n^2 &= n^{14} \\ n^{14} \times n &= n^{15} \end{aligned}$$

However it is yet possible to compute it in only five multiplications:

$$\begin{aligned} n \times n &= n^2 \\ n^2 \times n &= n^3 \\ n^3 \times n^3 &= n^6 \\ n^6 \times n^6 &= n^{12} \\ n^{12} \times n^3 &= n^{15} \end{aligned}$$

We shall define $m(k)$ to be the minimum number of multiplications to compute n^k; for example $m(15) = 5$.

For $1 \leq k \leq 200$, find $\sum m(k)$.

Solution:

The idea is to find the optimal solution for each n, keeping track of which powers were found in order to get there, then going up in n, checking for which smaller n_0 one can obtain n in one more multiplication. The messy part is that one ought to keep *all* the best combinations, otherwise some numbers might not get an optimal solution (as happened for $n = 77$). It can be noted that $m(2n) = m(n) + 1$ for all n, and that makes the calculations a bit quicker.

Answer: 1582

Problem 123

Let p_n be the n^{th} prime: 2, 3, 5, 7, 11, ..., and let r be the remainder when $(p_n - 1)^n + (p_n + 1)^n$ is divided by p_n^2.

For example, when $n = 3$, $p_3 = 5$, and $4^3 + 6^3 = 280 \equiv 5 \bmod 25$.

The least value of n for which the remainder first exceeds 10^9 is 7037.

Find the least value of n for which the remainder first exceeds 10^{10}.

Solution:

There are two cases to deal with here. If n is even, then

$$(p_n - 1)^n + (p_n + 1)^n \equiv_{p_n^2} np_n(-1)^{n-1} + (-1)^n + np_n\ 1^{n-1} + 1^n \equiv_{p_n^2} 2.$$

In the case where n is odd however,

$$(p_n - 1)^n + (p_n + 1)^n \equiv_{p_n^2} np_n(-1)^{n-1} + (-1)^n + np_n\ 1^{n-1} + 1^n \equiv_{p_n^2} 2np_n.$$

So we just need to run through odd n, checking if $2np_n > 10^{10}$. This occurs for $(n, p_n) = (21035, 237737)$.

Answer: 21035

Problem 124

The radical of n, $\text{rad}(n)$, is the product of distinct prime factors of n. For example, $504 = 2^3 \times 3^2 \times 7$, so $\text{rad}(504) = 2 \times 3 \times 7 = 42$.

If we calculate $\text{rad}(n)$ for $1 \leq n \leq 10$, then sort them on $\text{rad}(n)$, and sorting on n if the radical values are equal, we get:

n	$\text{rad}(n)$
1	1
2	1
3	3
4	2
5	4
6	6
7	7
8	2
9	3
10	10

n	$\text{rad}(n)$	k
1	1	1
2	2	2
4	2	3
8	2	4
3	3	5
9	3	6
5	5	7
6	6	8
7	7	9
10	10	10

Let $E(k)$ be the k^{th} element in the sorted n column; for example, $E(4) = 8$ and $E(6) = 9$.

If $\text{rad}(n)$ is sorted for $1 \leq n \leq 100000$, find $E(10000)$.

Solution:

Well, we need to calculate $\text{rad}(n)$ for each n in the range, and as it consists of the product of the constituent primes, we can generated this table while looking for the primes (modified Sieve algorithm). We can then sort the radical values to find the 10000^{th} one. However, there will be several numbers that map to that radical. To fix this, I chose to encode the number into the 'radical', by storing $\text{rad}(n) + \frac{n}{10^6}$. The sort works just as well, and we can then reconstruct the original number, as is required.

Answer: 21417

Problem 125

The palindromic number 595 is interesting because it can be written as the sum of consecutive squares: $6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2$.

There are exactly eleven palindromes below one-thousand that can be written as consecutive square sums, and the sum of these palindromes is 4164. Note that $1 = 0^2 + 1^2$ has not been included as this problem is concerned with the squares of positive integers.

Find the sum of all the numbers less than 10^8 that are both palindromic and can be written as the sum of consecutive squares.

Solution:

While the problem seems daunting at first, there is a great simplification we can do. First observe that since a sum of squares includes at least two terms, the largest square to consider is 7071^2. Next, let $S_k = \sum_1^k i^2$. Then any sum of consecutive squares is a a difference of two S_k's. And while there are 25 million such terms, less than 400000 of them are below 10^8. Those just need to be checked if they are palindromes, and added to the list (need to watch for repeats, of which there are two).

Answer: 2906969179

Problem 126

Solution:

Answer: —

Problem 127

Solution:

Answer: —

Problem 128

Solution:

Answer: —

Problem 129

A number consisting entirely of ones is called a repunit. We shall define $R(k)$ to be a repunit of length k; for example, $R(6) = 111111$.

Given that n is a positive integer and $\mathrm{GCD}(n, 10) = 1$, it can be shown that there always exists a value, k, for which $R(k)$ is divisible by n, and let $A(n)$ be the least such value of k; for example, $A(7) = 6$ and $A(41) = 5$.

The least value of n for which $A(n)$ first exceeds ten is 17.

Find the least value of n for which $A(n)$ first exceeds one-million.

Solution:

Well, since $R(k) = \frac{1}{9}(10^k - 1)$, observe that if $k|10^n - 1$ then $A(n) \leq k$. And since, by Euler's theorem, $10^{\varphi(n)} \equiv_n 1$, we know that $A(n)|\varphi(n)$. To find the smallest n for which $A(n)$ is greater than a million, it should be easiest to start looking for above a million, and evaluating their $A(n)$ directly.

For numbers not divisible by three, can just check when $10^k \equiv_n 1$. For multiples of three, can evaluate the repunit (mod p) directly. Slow, but gets the job done.

Answer: 1000023

Problem 130

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, $R(6) = 111111$.

Given that n is a positive integer and $\text{GCD}(n, 10) = 1$, it can be shown that there always exists a value, k, for which $R(k)$ is divisible by n, and let $A(n)$ be the least such value of k; for example, $A(7) = 6$ and $A(41) = 5$.

You are given that for all primes, $p > 5$, that $p - 1$ is divisible by $A(p)$. For example, when $p = 41$, $A(41) = 5$, and 40 is divisible by 5.

However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703.

Find the sum of the first 25 composite values of n for which $\text{GCD}(n, 10) = 1$ and $n - 1$ is divisible by $A(n)$.

Solution:

If $R(A(n))$ is divisible by n, and $n - 1$ is divisible by $A(n)$, then $R(n - 1)$ willa also divisible by n. And as $R(n) = \frac{1}{9}(10^n - 1)$, for n not divisible by 3, we need to check if $10^n \equiv_n 1$. We use the `pow` function, and the code runs nice and fast.

Answer: 149253