Math 33A - Linear Algebra and Applications (23W)

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Textbook: Otto Bretscher - Linear Algebra with Applications (5^{th} Edition - 2012)
Topics: Linear systems, matrices & matrix operations, linear transformations, linear independence, base & subspaces, orthogonality, determinants, eigenvalues & eigenvectors, diagonalization

Table of Contents:

Linear Equations

1/9/23
Lecture 1

Linear Equation
A linear equation is an equation taking the following form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

- $x_1, x_2, \ldots, x_n$ represent distinct variables
- $a_1, a_2, \ldots, a_n, b$ are constants
  - $a_1, a_2, \ldots, a_n$ are the coefficients of a linear equation

A <u>solution</u> for a linear equation represents a set of values for variables $x_1, \ldots, x_n$ such that the linear equation evaluates as true.

A <u>system</u> of linear equations represents a set of linear equations; a solution to a system of linear equations must cause all of the contained linear equations to evaluate as true simultaneously.

A solution may or may not be <u>unique</u> (the only solution).

1/11/23
Lecture 2

Solutions to [systems of] linear equations can be written as ordered tuples, e.g. $(s_1, s_2, \dots, s_n)$ s.t. $x_1 = s_1, \dots, x_n = s_n$ is a solution.

Methods for solving linear equations: (1) Substitution
(2) Elimination

Matrices

1/13/23
Lecture 3

Linear systems are <u>equivalent</u> if they share the same solutions

Linear systems can be rewritten as matrices, e.g.: $\begin{cases} x_1 - x_2 + 2x_3 = 4 \\ 2x_1 - x_2 + x_3 = 4 \\ 3x_1 + x_2 - x_3 = 1 \end{cases} \rightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & \text{constants} \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -1 & 1 & 4 \\ 3 & 1 & -1 & 1 \end{array}\right]$

> **<u>Matrices</u>**
>
> A <u>matrix</u> is a rectangular array of numbers, arranged in rows & columns.
>
> An individual item in a matrix is known as an "entry".

(*) All entries in a matrix must be filled
(*) The size of a matrix is defined by the <u># of rows and columns</u> - a matrix with m rows and n columns is called <u>$m \times n$</u> (has dimensions $m \times n$)

- Entries in a matrix are named by their location - "a_{ij}" refers to an entry in matrix A, row i, col. j
- Matrices A and B are equal if they share the same size, have the same corresponding entries ($a_{ij} = b_{ij} \; \forall i,j$)

<u>Definitions</u>

<u>Square matrix</u> - a matrix with an equal number of rows and columns
<u>Diagonal matrix</u> - all entries above/below the diagonals are 0 ($\{a_{ij} \mid a_{ij} = 0 \text{ if } i \neq j\}$)
- Upper triangular matrix - all entries below the diagonals are 0 (converse is lower-triangular)
- (*) Diagonal and x-triangular matrices are only defined for square matrices

<u>Vector</u> - a matrix with either only one row (row vector), or only one column ([column] vector)
$\mathbb{R}^{m \times n}$ - set of all matrices with dimension $m \times n$

<u>Matrix Algebra (§2.1)</u>

- The sum of two matrices [of the same size only] is defined as the individual addition of corresponding entries, i.e. $A + B = C = \{c_{ij} = a_{ij} + b_{ij}\}$
- The scalar multiple of a matrix is computed by multiplying each element by a scalar, i.e. $B = \lambda A = \{b_{ij} = \lambda \cdot a_{ij}\}$
- The dot product of two vectors is defined as the scalar obtained from multiplying all pairs of corresponding entries, i.e. $\vec{v} \cdot \vec{w}; \vec{v}, \vec{w} \in \mathbb{R}^n = \sum_{j=1}^{n} v_j \cdot w_j$
- A matrix-vector product is the vector obtained from taking the dot product between a vector and each of a matrix's rows/columns (where each row/column is treated as an individual vector)
  - Given matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, it can be written as a collection of row vectors or column vectors:
    $\hookrightarrow \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}, r_i = [r_{i1} \ldots r_{in}]$ $\quad \hookrightarrow [c_1 \ldots c_n], c_i = \begin{bmatrix} c_{1i} \\ \vdots \\ c_{\cdot} \end{bmatrix}$

Solving Linear Systems

1/18/23
Lecture 4

Linear Combinations

Vector $\vec{b} \in \mathbb{R}^n$ is a <u>linear combination</u> of vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ if $\exists\, x_1, \dots, x_n \in \mathbb{R}$ s.t. $\vec{b} = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$

e.g. $\begin{bmatrix}2\\3\end{bmatrix} = \overset{x_1}{2}\overset{v_1}{\begin{bmatrix}1\\0\end{bmatrix}} + \overset{x_2}{3}\overset{v_2}{\begin{bmatrix}0\\1\end{bmatrix}}$

Writing Linear Systems

Given a series of linear equations represented in the form $A\vec{v} = \vec{b}$, where A represents the coefficients of variables in $\vec{v}$ and $\vec{b}$ represents the constants of each equation, we can write $A\vec{v} = \vec{b}$ in another form:

$$A\vec{v} = \vec{b} \rightarrow \begin{bmatrix} A \end{bmatrix}\begin{bmatrix} \vec{v} \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix} \rightarrow \left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \text{ (Augmented matrix)}$$

(*) To solve linear systems, we can express them in the form of an augmented matrix and convert the left-hand side to an identity matrix to determine the values of the variables $x_1, \dots x_n$, i.e.: (*)(convert w/ row operations)

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \rightarrow \left[\begin{array}{c|c} I^n & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \end{array}\right]$$

Solving Linear Systems (cont.)

1/20/23
Lecture 5
+Lecture 6
(1/23/23)

Reduced Row Echelon Form (rref)

A matrix M is said to be in <u>reduced row echelon form</u> (rref) if:

a) For every row with nonzero entries, the first entry is a 1 (called a <u>leading 1</u>/pivot)

b) If a column contains a leading 1, all other entries in the column are 0

c) If a row contains a leading 1, then each row above it has a leading 1 further to the left

Solving Linear Systems

- Solutions for any arbitrary system of equations can be found by reducing the associated [augmented] matrix to RREF via <u>elementary row operations</u>
  - (*) Every matrix can be reduced to a <u>unique</u> RREF matrix

e.g.: $A = \left[\begin{array}{cccc|c} 1 & -3 & 0 & -5 & -7 \\ 3 & -12 & -2 & -27 & -33 \\ 2 & 10 & 2 & 24 & 29 \\ -1 & 6 & 1 & 14 & 17 \end{array}\right] \longrightarrow \mathrm{rref}(A)^{*} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} x_1 + x_4 = 6 \\ x_2 + 2x_4 = 0 \\ x_3 + 3x_4 = 1 \end{cases}$ (solutions)

*(not actually)

- The <u>rank</u> $R(A)$ of a matrix A is defined as the number of leading 1s in rref(A)
- The number of solutions of a linear system depends on its rank and rref form:
  - If rref(A) contains a row $[0\ 0 \ldots 0 \mid 1]$, the system is <u>inconsistent</u> (has no solutions)
  - If the system is <u>consistent</u>, it can either have:
    - <u>Infinitely many solutions</u>, if there is at least one <u>free (i.e. non-pivot) variable</u> $(R(A) < m)$ (*) given $A \in \mathbb{R}^{m \times n}$
    - <u>A unique solution</u>, if all variables are leading $(R(A) = m)$
      - (*) Can only occur if there are at least as many equations as unknowns $(m \geq n)$

Types of Elementary Row Operations

(1) <u>Scalar Multiplication</u>: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\lambda r_1} \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ (*) (for $\lambda \neq 0$)

(2) <u>Row Addition/Subtraction</u>: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_2 + \lambda r_1} \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$

(3) <u>Swapping Rows</u>: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[r_1 = r_2]{r_2 = r_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Misc. Notes: Matrices

2/5/23
Notes

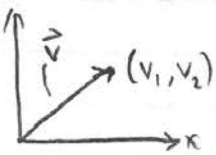
Writing Linear Systems

$$\left.\begin{aligned} c_1x_1 + c_2x_2 + c_3x_3 &= b_1 \\ c_4x_1 + c_5x_2 + c_6x_3 &= b_2 \\ c_7x_1 + c_8x_2 + c_9x_3 &= b_3 \end{aligned}\right\} \rightarrow x_1\begin{bmatrix} c_1 \\ c_4 \\ c_7 \end{bmatrix} + x_2\begin{bmatrix} c_2 \\ c_5 \\ c_8 \end{bmatrix} + x_3\begin{bmatrix} c_3 \\ c_6 \\ c_9 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \overset{A}{\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{\vec{b}}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

$$\rightarrow \boxed{= A\vec{x} = b} \quad (A = \underline{\text{coefficient matrix}})$$

Vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (v_1, v_2) \neq \begin{bmatrix} v_1 & v_2 \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$$



("Standard representation")

Matrix-Vector Multiplication

A matrix-vector can be thought of either in terms of the rows of the matrix, or in terms of the columns.

(1) $A\vec{x} = b \rightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\vec{x} = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \end{bmatrix}\vec{x} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\vec{x} = \begin{bmatrix} r_1 \cdot \vec{x} \\ r_2 \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} \langle c_1, c_2 \rangle \cdot \langle x_1, x_2 \rangle \\ \langle c_3, c_4 \rangle \cdot \langle x_1, x_2 \rangle \end{bmatrix} = \vec{b}$ (Rows)

(2) $A\vec{x} = b \rightarrow \begin{bmatrix} \text{col. } 1 & \text{col. } 2 \end{bmatrix}\vec{x} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 = x_1\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} + x_2\begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} x_1c_1 + x_2c_2 \\ x_1c_3 + x_2c_4 \end{bmatrix} = \vec{b}$ (Columns)

(*) The column picture characterizes $\vec{b}$ as a linear combination of the columns of A w/ coeffs. x_1, x_2

Row-Echelon Form

Conditions for Row-Echelon Form (REF):

1) The leading coefficient (leftmost nonzero entry) of any row is to the right of the leading coefficient of the row above it
2) Any rows of all zeros are at the bottom of the matrix

Conditions for RREF:

1) Matrix is in REF
2) All pivots [leading coefficients] are equal to 1
3) All entries above a pivot are 0

Intro to Linear Transformations

1/25/23
Lecture 7

Functions Review

A function $T: X \to Y$ that maps elements of the domain (X) to elements of the target space (Y), such that for all $x \in X$ $\exists$ $T(x) = y$ for some $y \in Y$.

$$T: X \longrightarrow Y$$

$$(\text{input})\ x \longrightarrow y\ (\text{output})$$

Linear Transformations

A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is called a linear transformation if $\exists$ a matrix $A \in \mathbb{R}^{n \times m}$ such that, for all $\vec{x} \in \mathbb{R}^m$, the following expression is true:

$$\longrightarrow \boxed{T(\vec{x}) = A\vec{x}} \quad (T \text{ is represented by a matrix } A)$$

Linear Transformations

- A function $T: \mathbb{R}^m \to \mathbb{R}^n$ takes as input a vector of size m, and outputs a vector of size n, i.e. takes a vector as input and also returns a vector
- The expression $T(\vec{x}) = A\vec{x}$ is also written $\vec{y} = A\vec{x}$ $(\vec{y} = T(\vec{x}))$

(*) Standard Bases of a Vector Space

- Given a vector space $\mathbb{R}^m$, we can define a set of standard bases of $\mathbb{R}^m = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m\}$, where each vector $\vec{e}_i \in \mathbb{R}^m$ takes the form $(0, 0, \dots, 0, 1, 0, \dots, 0)$ (contains all 0s except for a 1 as the i^{th} component)

$$\downarrow$$

Matrices of Linear Transformations

Given a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ for some matrix A, we can write:

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}$$

Linear Transformations (cont.)

1/27/13
Lecture 8

Properties of Linear Transformations

A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear if (and only if):

1) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ for all vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ (Vector Addition)

2) $T(k\vec{v}) = kT(\vec{v})$ for all vectors $\vec{v} \in \mathbb{R}^m$, scalars $k \in \mathbb{R}$ (Scalar Multiplication)

(*) A transformation can only be linear if it can be expressed as a matrix

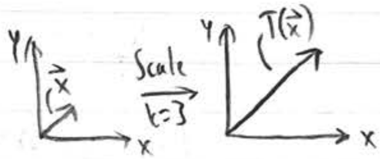
Linear Transformations in Geometry (§2.2)

1) Scaling - a scaling is a transformation that increases/decreases the size of an input by a certain factor k; i.e. $\text{size}(T(X)) = k \cdot \text{size}(X)$

$\longrightarrow$ $\boxed{\text{Scaling: } T(\vec{x}) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}}$ (Scaling by a factor of k)

(*) Notation:

- A scaling is a dilation/enlargement if $k \geq 1$
- A scaling is a contraction/shrinking if $0 < k < 1$



2) Orthogonal Projection - given a line $L \in \mathbb{R}^2$, any vector $\vec{x} \in \mathbb{R}^2$ can be written in the form $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel}$ is parallel to L (the projection of $\vec{x}$ onto L) and $\vec{x}^{\perp}$ is perpendicular to L. An orthogonal projection is the transformation $T(\vec{x}) = \text{proj}_L \vec{x}$ (projection of x onto L) $= \vec{x}^{\parallel}$.

$\rightarrow$ Given a vector $\vec{w}$ parallel to L, we can write that:

$\text{proj}_L(\vec{x}) = k\vec{w}$ (for some scalar k) $= \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \vec{w}$,

which [in the case that $\vec{w} = \vec{u}$ (unit vector parallel to L)] becomes:

$$\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u} \quad \left[= \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right]$$

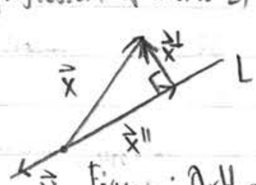


Figure: Orthogonal Projection

(*) This equation only holds for line L that passes through the origin

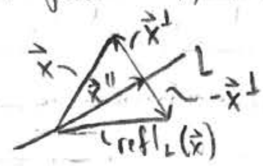
$\longrightarrow$ $\boxed{\text{Orthogonal Projection: } \text{proj}_L(\vec{x}) = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \vec{x}}$ (Orthogonal projection of x onto L, where $\vec{u} \parallel L$, $\|\vec{u}\| = 1$)

Linear Transformations in Geometry

1/30/23
Lecture 9

Linear Transformations in Geometry (cont.)

3) Reflection - Given a line L and vector $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ (in regards to L), the reflection of $\vec{x}$ over L is the following:

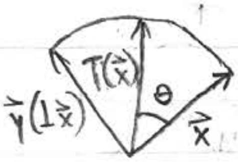


$\longrightarrow ref_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$ $[= 2proj_L(\vec{x}) - \vec{x}]$

$$\longrightarrow \boxed{\text{Reflection: } ref_L(\vec{x}) = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}\vec{x}}$$

(Reflection of $\vec{x}$ over line L, where $\vec{u}$ is a unit vector $\parallel L$)

4) Rotation - a rotation is a transformation that rotates a vector $\vec{x}$ by a certain number of degrees (θ) about the origin.



We can characterize a rotation as a linear combination of $\vec{x}$ and a vector perpendicular to $\vec{x}$, denoted $\vec{y}$ ($\|\vec{y}\| = \|\vec{x}\|$)

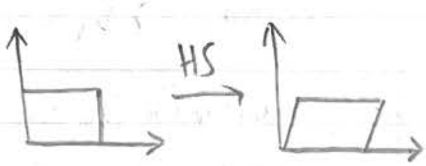
$$\rightarrow T(\vec{x}) = \cos(\theta)\cdot\vec{x} + \sin(\theta)\cdot\vec{y} = \cos(\theta)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sin(\theta)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$\longrightarrow \boxed{\text{Rotation: } T(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{x}}$$

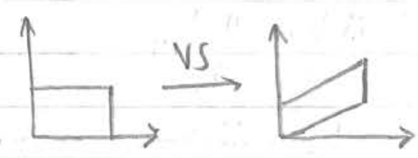
(Rotation of $\vec{x}$ about the origin by θ degrees)

5) Shears - a shear is a transformation that "slants" an input in either the y-direction (horizontal shear) or the x-direction (vertical shear)

$$\longrightarrow \text{Horizontal Shear: } T(\vec{x}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\vec{x}$$



$$\longrightarrow \text{Vertical Shear: } T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\vec{x}$$



Matrix Products

2/1/23 Lecture 10

Matrix Products

Notation: Given two functions $g(x)$ and $f(x)$, the composite of $f(x)$ and $g(x)$ is the function $z = g(f(x))$.

Matrix Products

→ Given two linear transformations $T_1(\vec{x}) = A\vec{x}$, $T_2(\vec{x}) = B\vec{x}$, we define the product of matrices A and B as the matrix of the composite of $T_1(\vec{x})$ and $T_2(\vec{x})$:

$$\longrightarrow \boxed{T(\vec{x}) = T_2(T_1(\vec{x})) = B(A\vec{x}) = (BA)\vec{x}} \quad \text{(Matrix product of A and B)}$$

(*) The composite of two linear transformations is also a linear transformation

Note: Given two matrices $A \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{q \times m}$, the product AB is only defined if $p = q$
⟶ The size of such a matrix AB would be $n \times m$.

Matrix Multiplication

Given matrices $B \in \mathbb{R}^{n \times p}$ and $A \in \mathbb{R}^{p \times m}$, $A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m]$, the product BA is

$$\rightarrow \quad BA = B\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \end{bmatrix} \quad (\text{size } n \times m) \quad \left(\begin{matrix}\text{Product BA of} \\ \text{matrices A and B}\end{matrix}\right)$$

Properties of Matrix Multiplication

1) Matrix multiplication is noncommutative (i.e. $AB \neq BA$ (can be))
   (*) If $AB = BA$ for some matrices A and B, we say A and B commute
2) Given matrix product BA, the (i,j) entry of BA is the dot product between the i^{th} row of B, j^{th} col. of A.
   $\rightarrow$ i.e. $\boxed{BA_{ij} = \sum_{k=1}^{p} b_{ik} a_{kj}} = b_{i1}a_{1j} + b_{i2}a_{2j} + \dots + b_{ip}a_{pj}$ $\quad (B \in \mathbb{R}^{n \times p}, A \in \mathbb{R}^{p \times m})$
3) Given matrix $A \in \mathbb{R}^{n \times n}$, $AI_n = I_nA = A$ (product of a matrix A and the identity matrix, is A)
4) Matrix multiplication is associative, i.e. $(AB)C = A(BC)$
5) Matrix multiplication is distributive, i.e. $A(C+D) = AC + AD$
6) Scalar multiplication is commutative for matrix products, i.e. $(kA)B = A(kB) = k(AB)$
7) Block multiplication: Given a matrix partitioned into blocks, those blocks can be treated like entries when multiplying

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} I_2 & I_2 \\ 0 & I_2 \end{bmatrix} \quad \text{(Block matrices)}$$

Invertibility

2/3/23
Lecture 11

Transition Matrices

Definition: A vector $\vec{x} \in \mathbb{R}^n$ is a <u>distribution vector</u> if its components add up to 1, and all components ≥ 0.

Definition: A matrix A is a <u>transition matrix</u> (/stochastic matrix) if all of its columns are distribution vectors.

(*) A transition matrix is <u>positive</u> if all of its entries are positive (i.e. nonzero)

(*) A transition matrix A is <u>regular</u> (eventually positive) if A^m is positive (for some m)

→ Given regular transition matrix $A \in \mathbb{R}^{n \times n}$:

- There exists a unique <u>equilibrium distribution vector</u> $\vec{x}_{equ}$, s.t. $A\vec{x}_{equ} = \vec{x}_{equ}$.
  - (*) All elements of $\vec{x}_{equ}$ are positive.
- Given any distribution vector $\vec{x} \in \mathbb{R}^n$, $\lim_{m \to \infty} (A^m \vec{x}) = \vec{x}_{equ}$ (the system will approach <u>globally stable equilibrium distribution</u> $\vec{x}_{equ}$ on the long run)
- $\lim_{m \to \infty} A^m = \begin{bmatrix} | & | & & | \\ \vec{x}_{equ} & \vec{x}_{eq} & \cdots & \vec{x}_{eq} \\ | & | & & | \end{bmatrix}$

Matrix Inverses

Background: Given a function $f: X \to Y$, f is <u>invertible</u> if, for all $y \in Y$, $\exists$ a <u>unique</u> solution $x \in X$ such that $f(x) = y$, in which case we can define <u>inverse of f</u> $f^{-1}: Y \to X$, $f^{-1}(y) = x$.

(*) Linear transformations, being functions, can also be invertible.

Invertible Matrices

A square matrix A is <u>invertible</u> if the associated linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible, in which case we define the <u>inverse transformation</u> $\vec{x} = T^{-1}(\vec{y}) = A^{-1}y$ (A^{-1} being the matrix of T^{-1}).

A square matrix $A \in \mathbb{R}^{n \times n}$ is invertible <u>iff $\text{rref}(A) = I_n$</u>, i.e. iff <u>$\text{rank}(A) = n$</u>.

(*) Given matrices $A, A^{-1} \in \mathbb{R}^{n \times n}$, $\boxed{A^{-1}A = AA^{-1} = I_n}$

(*) Invertibility and Linear Systems

Given matrix $A \in \mathbb{R}^{n \times n}$, linear system represented by $A\vec{x} = \vec{b}$:

- If <u>A is invertible</u>, the system has a <u>unique solution</u> $\vec{x} = A^{-1}\vec{b}$
- If <u>A is not invertible</u>, the system has <u>either infinite solutions or no solution</u>.

Invertibility (cont.)

2/6/23
Lecture 12

Finding Matrix Inverses

Given matrix $A \in \mathbb{R}^{n \times n}$, A^{-1} can be found by forming the $n \times 2n$ matrix $[A \mid I_n]$ and computing $\text{rref}[A \mid I_n]$.

1) If $\text{rref}[A \mid I_n] = [I_n \mid B]$ for some matrix B, A is invertible and $A^{-1} = B$.
2) If $\text{rref}[A \mid I_n] = [B \mid C]$ for some matrices B and C $(B \neq I_n)$, A is not invertible.

Properties of Inverse Matrices

- Given invertible matrices $A, B \in \mathbb{R}^{n \times n}$, BA is also invertible $((BA)^{-1} = A^{-1}B^{-1})$
- Given two matrices $A, B \in \mathbb{R}^{n \times n}$ s.t. $BA = I_n$, A and B are both invertible $(A^{-1} = B \,;\, B^{-1} = A)$

Inverse of a 2×2 Matrix

Given 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A is invertible iff $ad - bc \neq 0$.

If A is invertible: $\rightarrow \boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

(*) Determinant of a 2×2 Matrix

Determinant of a 2×2 Matrix

The quantity $ad-bc$ is called the determinant of 2×2 matrix $A = [a\ b\ ;\ c\ d]$, i.e.:

$\longrightarrow \det(A) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (Determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$)

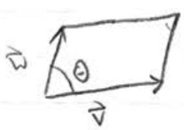
Notes

If we write $A = [\vec{v}\ \vec{w}]$ (2×2 matrix with columns $\vec{v}, \vec{w}$):

$\longrightarrow \boxed{\det(A) = \|\vec{v}\|\,\|\vec{w}\| \sin(\theta)}$ (where θ is the angle from $\vec{v}$ to $\vec{w}$ $(-\pi < \theta \leq \pi)$)

(*) $\det(A) = 0$ if $\vec{v} \parallel \vec{w}$ ($\vec{v}$ and $\vec{w}$ parallel)

Geometry: $|\det(A)|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$



(Parallelogram spanned by $\vec{v}$ and $\vec{w}$)

Image, Span, and Kernel

2/8/23
Lecture 13

Definition: Image
Given a function $f: X \to Y$, the <u>image of f</u> is defined as <u>all the values taken by the function in its target space</u>:

$$\rightarrow \underline{\text{image}(f)} = \{f(x) \mid x \in X\}$$
$$= \{y \mid y \in Y, \exists x \in X \text{ s.t. } f(x) = y\}$$

(*) $\text{im}(f) \subseteq \text{codomain}(f)$
(*) The image of a linear transformation $T(\vec{x}) = A\vec{x}$ (denoted $\text{im}(T)$ or $\text{im}(A)$) is the <u>span of the column vectors of A</u>

Definition: Span
Given a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$, the <u>span of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$</u> is defined as the <u>set of all linear combinations</u> $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m$ of the vectors $v_1, \ldots, v_m$:

$$\underline{\text{span}(\vec{v}_1, \ldots, \vec{v}_m)} = \{c_1\vec{v}_1 + \ldots + c_m\vec{v}_m \mid c_1, \ldots, c_m \in \mathbb{R}\}$$

Images of Linear Transformations

Given linear transformation $T(\vec{x}) = A\vec{x}$, $A = [\vec{v}_1 \ldots \vec{v}_m]$ –
$\rightarrow \text{im}(A) = \text{span}(\vec{v}_1, \ldots, \vec{v}_m)$ has the following properties:

- $(\vec{0} \in \mathbb{R}^n) \in \text{im}(\vec{A})$
- $\text{im}(\vec{A})$ is <u>closed under addition</u>, i.e.:
  (*) $\vec{v}_1, \vec{v}_2 \in \text{im}(\vec{A}) \rightarrow (\vec{v}_1 + \vec{v}_2) \in \text{im}(A)$
- $\text{im}(A)$ is <u>closed under scalar multiplication</u>, i.e.:
  (*) $\vec{v}_1 \in \text{im}(\vec{A}) \rightarrow k\vec{v}_1 \in \text{im}(\vec{A})$ $(k \in \mathbb{R})$

Definition: Kernel
Given a linear transformation $T(\vec{x}) = A\vec{x}$, the <u>kernel of $T(\vec{x})$</u> is the set of all <u>zeros of $T(\vec{x})$</u>, i.e. of all vectors $\vec{x}$ such that $T(\vec{x}) = A\vec{x} = 0$ (set of solutions).

$$\rightarrow \text{kernel of } T = \underline{\ker(A)} = \underline{\ker(T)}$$
$$= \{\vec{x} \mid T(\vec{x}) = 0\}$$

(*) $\ker(T) \subseteq \text{domain}(T)$
(*) The kernel of $T(\vec{x}) = A\vec{x}$ can be seen as the set of solutions of the linear system $A\vec{x} = 0$

(*) The notion of the zeros of a function is not limited to linear algebra

Subspaces

2/10/23
Lecture 14

Properties of Kernel
Given a linear transformation T:
1) $\vec{0} \in \ker(T)$
2) The kernel is closed under addition
3) The kernel is closed under scalar multiplication.

Case: $\ker(A) = \{\vec{0}\}$
a) iff $\text{rank}(A) = m$ $(A \in \mathbb{R}^{n \times m})$
b) $\ker(A) = 0 \rightarrow m \leq n$
(*) $n < m \rightarrow \ker(A) \neq \{\vec{0}\}$
c) A square $\rightarrow \ker(A) = 0$ iff A invertible

Properties of Invertible Matrices
Given matrix $A \in \mathbb{R}^{n \times n}$, the following statement are equivalent:
i. A is invertible
ii. The system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ $\forall \vec{b} \in \mathbb{R}^n$
iii. $\text{rref}(A) = I_n$
iv. $\text{rank}(A) = n$
v. $\text{im}(A) = \mathbb{R}^n$
vi. $\ker(A) = \{\vec{0}\}$

(*) equivalent - all true or all false

Subspaces
A subset W of the vector space $\mathbb{R}^n$ is called a [linear] subspace of $\mathbb{R}^n$ if it satisfies:

1) $\vec{0} \in W$
2) W is closed under addition (i.e. $\vec{w}_1, \vec{v}_2 \in W \rightarrow \vec{w}_1 + \vec{v}_2 \in W$)
3) W is closed under scalar multiplication (i.e. $\vec{w}_1 \in W \rightarrow kw_1\ (k \in \mathbb{R}) \in W$)

Theorem: Given linear transformation $T(\vec{x}) = A\vec{x} : \mathbb{R}^m \rightarrow \mathbb{R}^n$:
- $\ker(A) = \ker(T) \leq \mathbb{R}^m$ (kernel is a subspace of the domain)
- $\text{im}(A) = \text{im}(T) \leq \mathbb{R}^n$ (image is a subspace of the codomain/target space)

Subspace Dimensions:

	Subspaces of $\mathbb{R}^2$	Subspaces of $\mathbb{R}^3$
3D	N/A	$\mathbb{R}^3$
2D	$\mathbb{R}^2$	Planes through $\vec{0}$
1D	Lines through $\vec{0}$	Lines through $\vec{0}$
0D	$\{\vec{0}\}$	$\{\vec{0}\}$

Bases and Linear Independence

2/13/23
Lecture 15

Bases and Linear Independence

Definition: A vector $\vec{v}_i$ in the list $\vec{v}_1, \ldots, \vec{v}_n$ is considered <u>redundant</u> if $\vec{v}_i$ can be expressed as a linear combination of the vectors $\{\vec{v}_j \mid 1 \le j \le n, j \ne i\}$ [other vectors in the list]

(* "Redundant" is not an established linear algebra term)

Linear Independence

A set of vectors $\vec{v}_1, \ldots, \vec{v}_n$ are called <u>linearly independent</u> if no vector v_i in the set can be expressed as a linear combination of the other vectors in the set.

i.e. $\nexists\ \vec{v}_i$ s.t. $\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$ (for any constants $c_1, \ldots, c_k$)

A set of vectors that does not fulfill the above (contains a vector that is expressible as a linear combination of the other vectors in the set) is called <u>linearly dependent</u>.

Bases

A set S of vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ in a subspace V of $\mathbb{R}^n$ is a <u>basis</u> of V if they span V (i.e. $V \subseteq \text{span}(S)$) and are linearly independent.

i.e. $\forall\ \vec{v} \in V \rightarrow \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$ (for some constants $c_1, \ldots, c_k$)

Basis of an Image

Given a matrix A with column vectors $\vec{v}_1, \ldots, \vec{v}_n$, a basis of the image $\text{im}(A)$ of A can be formed from the set of column vectors of A, omitting any redundant vectors.

Linear Relations

Given a set of vectors $\vec{v}_1, \ldots, \vec{v}_n$, a <u>[linear] relation</u> among the vectors $\vec{v}_1, \ldots, \vec{v}_n$ is an equation of the form:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0} \quad (c_1, \ldots, c_k \in \mathbb{R})$$

Types of Linear Relations

- The <u>trivial relation</u> is the relation $c_1 = c_2 = \ldots = c_n = 0$ (always exists)
- A <u>nontrivial relation</u> (if one exists) is a relation with at least one $c_i \neq 0$

(*) $\exists$ nontrivial $\rightarrow$ $v_1, \ldots, v_n$ linearly dependent

Bases and Dimension

2/17/23
Lecture 16

Kernel and Relations

The vectors $\in$ ker(A) correspond to linear relations among column vectors $v_1, \dots, v_m$ of A, i.e.:

$$\vec{x} \in \ker(A)\ (A\vec{x} = \vec{0}) \rightarrow x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m = \vec{0}$$

Properties of Linear Independence

Given vectors $v_1, \dots, v_m \in \mathbb{R}^n$, the following statements are equivalent:

i. Vectors $v_1, \dots, v_m$ are linearly independent

ii. No vector v_i is a linear combination of the other vectors in the list $\{v_j\}_{j \neq i}$

iii. The only linear relation among the vectors $v_1, \dots, v_m$ is the trivial relation

iv. $\ker\begin{bmatrix}\vec{v}_1 \dots \vec{v}_m\end{bmatrix} = \{\vec{0}\}$

v. $\text{rank}\begin{bmatrix}\vec{v}_1 \dots \vec{v}_m\end{bmatrix} = m$

(*) In a vector space $\mathbb{R}^n$, we can find at most n linearly independent vectors

Bases

Given a subspace V of $\mathbb{R}^n$, a set of vectors $\vec{v}_1, \dots, \vec{v}_m$ form a basis of V iff every vector $\vec{v} \in V$ can be expressed as a linear combination:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$$

($c_1, \dots c_m$ being the coordinates of $\vec{v}$ with respect to basis $v_1, \dots, v_m$)

Bases and Dimension

- Given a subspace $V \subseteq \mathbb{R}^n$, all bases of V will contain the same number of vectors
  - This number is called the dimension of V (dim(V))
- Given a subspace $V \subseteq \mathbb{R}^n$, $\dim(V) = m$:
  - We can find at most m linearly independent vectors in V
  - We need at least m vectors to span V
  - If m vectors in V are linearly independent, they form a basis of V (span V)

Rank and Dimension: $\dim(\text{im}(A)) = \text{rank}(A)$

Rank-Nullity, B-Coordinates, & Similarity

2/22/23
Lecture 17

Rank-Nullity Theorem

Given a matrix A, the <u>nullity of A</u> is the dimension of the kernel of A <u>dim(ker(A))</u>.
For any matrix $A \in \mathbb{R}^{n\times m}$:

$$\boxed{\text{nullity}(A) + \text{rank}(A) = \dim(\ker(A)) + \dim(\text{im}(A)) = m}$$

(also written <u>$N(A)+R(A)=m$</u>)

Properties of Invertible Matrices (cont.)

vii. The column vectors of A are linearly independent.
viii. The column vectors of A span $\mathbb{R}^n$.
ix. The column vectors of A form a basis of $\mathbb{R}^n$.

(*) Bases of $\mathbb{R}^n$

A set of vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ form a basis of $\mathbb{R}^n$ iff the matrix $A = [\vec{v}_1 \dots \vec{v}_n]$ is invertible.

B-Coordinates

Given a basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)$ of a subspace $V \subseteq \mathbb{R}^n$, any vector $\vec{x} \in V$ can be written uniquely as $\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$. The scalars $c_1, \dots, c_m$ are called the <u>B-coordinates</u> of $\vec{x}$; we then define the <u>B-coordinate vector of $\vec{x}$</u>, $[\vec{x}]_{\mathfrak{B}} = \langle c_1, \dots, c_m \rangle$. Thus, we write:

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \quad (S \in \mathbb{R}^{n\times m})$$

Linearity of Coordinates

Given basis $\mathfrak{B}$ of subspace $V \subseteq \mathbb{R}^n$:

i. $[\vec{x}+\vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}} \;\forall \vec{x}, \vec{y} \in V$
ii. $[k\vec{x}]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}} \;\forall \vec{x} \in V, k \in \mathbb{R}$

Matrices of Linear Transformations

Given $T: \mathbb{R}^n \to \mathbb{R}^n$, basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$ of $\mathbb{R}^n$, $\exists$ a unique <u>B-matrix of T</u> transforming $[\vec{x}]_{\mathfrak{B}}$ into $[T(\vec{x})]_{\mathfrak{B}}$:

$$T(\vec{x}) = B[\vec{x}]_{\mathfrak{B}} \to B = \left[[T(\vec{v}_1)]_{\mathfrak{B}} \dots [T(\vec{v}_n)]_{\mathfrak{B}} \right]$$

Similar Matrices

Given two $n\times n$ matrices A and B, we say <u>A is similar to B</u> if $\exists$ invertible matrix S such that:

$$AS = SB \longrightarrow \underline{B = S^{-1}AS}$$

(*) Notes on Similarity

- Similarity is an <u>equivalence relation</u>
- Given $T: \mathbb{R}^n \to \mathbb{R}^n$, basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$ of $\mathbb{R}^n$, standard matrix A and B-matrix B of T:

$$A = SBS^{-1} \quad (S = [\vec{v}_1 \dots \vec{v}_n])$$

Orthogonality & Projection

2/22/23
Lecture 17
(cont.)

Orthogonality + Vectors Review

- Vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are perpendicular or orthogonal if $\vec{v} \cdot \vec{w} = 0$
- The length (or magnitude/norm) of a vector $\vec{v}$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$
- A vector is a unit vector if its length is 1

Orthonormal Vectors

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ are called orthonormal if they are all unit vectors and orthogonal to each other.

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$

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Properties of Orthonormal Vectors

i. Orthonormal vectors are linearly independent
ii. Orthonormal vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ form an [orthonormal] basis of $\mathbb{R}^n$

(*) Cauchy-Schwarz Inequality

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

A vector $\vec{x} \in \mathbb{R}^n$ is called orthogonal to a subspace $V \subseteq \mathbb{R}^n$ if $\vec{x}$ is orthogonal to all vectors $\in V$.
→ Any vector $\vec{w} \in \mathbb{R}^n$ can be uniquely represented $\vec{w} = \vec{w}^{\parallel} + \vec{w}^{\perp}$, where $\vec{w}^{\parallel} \in V$ and $\vec{w}^{\perp} \perp V$.

Orthogonal Projection

Given vector $\vec{x} \in \mathbb{R}^n$, subspace $V \subseteq \mathbb{R}^n$, the orthogonal projection of $\vec{x}$ onto V is:

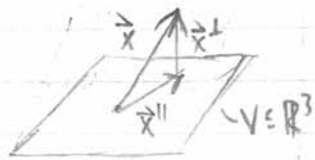
$$\text{proj}_V \vec{x} = \vec{x}^{\parallel}$$

(*) If $V = \mathbb{R}^n$ $\rightarrow \text{proj}_V \vec{x} = \vec{x}$

(*) Orthogonal Projection

Orthogonal projection onto subspaces can be represented as a linear transformation, i.e. $T(\vec{x}) = \text{proj}_V \vec{x}$.

(*) $\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|$



Orthogonal Projection

Given vector $\vec{x} \in \mathbb{R}^n$, $V \subseteq \mathbb{R}^n$ with orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$, then:

$$\boxed{\text{proj}_V \vec{x} = \vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \ldots + (\vec{u}_m \cdot \vec{x})\vec{u}_m} \quad [= \vec{x} \text{ if } V = \mathbb{R}^n]$$

(*) Angle Between Vectors

$\vec{x}, \vec{y} \rightarrow \theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}\right)$

$(0 \leq \theta \leq \pi)$

Orthogonal Complement

Given subspace $V \subseteq \mathbb{R}^n$, the orthogonal complement $V^{\perp}$ of V is the set of vectors $\vec{x} \in \mathbb{R}^n$ orthogonal to all vectors in V:

$$V^{\perp} = \{\vec{x} \mid \vec{x} \in \mathbb{R}^n, \vec{x} \cdot \vec{v} = 0 \ \forall \vec{v} \in V\}$$

(*) $V^{\perp} = \ker(\text{proj}_V \vec{x})$

Properties of Orthogonal Complement

i. $V^{\perp} \subseteq \mathbb{R}^n$
ii. $V \cap V^{\perp} = \{\vec{0}\}$
iii. $\dim(V) + \dim(V^{\perp}) = n$
iv. $(V^{\perp})^{\perp} = V$

Gram-Schmidt & Orthogonal Transformations

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Lecture 19

Gram-Schmidt (§5.2)

Given a basis $\vec{v}_1, \dots, \vec{v}_m$ of subspace $V \subseteq \mathbb{R}^n$, we can construct an orthonormal basis $u_1, \dots, u_m$ of V by expressing each vector $\vec{v}_j$, $j \geq 2$ as the sum of components parallel and perpendicular to the span of the preceding vectors $\vec{v}_1, \dots, \vec{v}_{j-1}$:

$$\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp} \quad (\text{relative to } \operatorname{span}(\vec{v}_1, \dots, \vec{v}_{j-1}))$$

$$\rightarrow \vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1,\ u_2 = \frac{1}{\|v_2^{\perp}\|}\vec{v}_2^{\perp},\ \dots,\ u_m = \frac{1}{\|v_m^{\perp}\|}\vec{v}_m^{\perp} \quad (u_1, \dots, u_m \text{ \underline{orthonormal basis} of } V)$$

THM: Gram-Schmidt

$$\vec{v}_j^{\perp} = \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j)\vec{u}_1 - \dots - (\vec{u}_{j-1} \cdot \vec{v}_j)\vec{u}_{j-1} \quad \left[= \vec{v}_j - \vec{v}_j^{\parallel}\right] \quad (j = 2, \dots, m)$$

(*) The Gram-Schmidt process represents a <u>change of basis</u> from $B = (\vec{v}_1, \dots, \vec{v}_m)$ to orthonormal $U = (\vec{u}_1, \dots, \vec{u}_m)$

$$\rightarrow \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}}_{Q} R \quad (\text{\underline{QR factorization of } M})$$

QR Factorization

Given an $n \times m$ matrix M with linearly independent columns $\vec{v}_1, \dots, \vec{v}_m$, $\exists$ a unique representation <u>$M = QR$</u>, where Q is an $n \times m$ matrix with orthonormal columns $\vec{u}_1, \dots, \vec{u}_m$ and R is an upper triangular matrix with positive diagonal entries.

$\therefore r_{11} = \|\vec{v}_1\|,\ r_{jj} = \|v_j^{\perp}\|,\ r_{ij} = \vec{u}_i \cdot \vec{v}_j \ (\text{for } i < j)$ [Gram-Schmidt coefficients]

Orthogonal Transformations & Matrices (§5.3)

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is called <u>orthogonal</u> if it preserves the length of vectors,

i.e. $\|T(\vec{x})\| = \|\vec{x}\|,\ \forall\, \vec{x} \in \mathbb{R}^n$

If $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation, A is an <u>orthogonal matrix</u>.

Orthogonal Matrices

2/27/23
Lecture 19
(cont.)

Orthogonal Transformations

- Orthogonal transformations preserve orthogonality, i.e. $\vec{v} \perp \vec{w} \rightarrow T(\vec{v}) \perp T(\vec{w})$
- A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal iff the vectors $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$ (i.e. column vectors of A, $T(\vec{x}) = A\vec{x}$)
- An $n \times n$ matrix A is orthogonal iff its columns form an orthonormal basis of $\mathbb{R}^n$ ☆

(*) Properties: 1) The product of two orthogonal matrices is orthogonal
2) The inverse of an orthogonal matrix is orthogonal

Definition: Transpose
Given an $m \times n$ matrix A, the transpose A^T of A is the $n \times m$ matrix such that $A^T_{ij} = A_{ji}$ (and vice versa), i.e. the "reflection" of A over its diagonal.

(*) Definitions: 1) A square matrix A is symmetric if $A^T = A$
2) A square matrix A is skew-symmetric if $A^T = -A$

(!) A square matrix A is orthogonal iff $A^T A = I$ (i.e. $A^{-1} = A^T$)

Properties of Transpose
1) $(A+B)^T = A^T + B^T$
2) $(kA)^T = k(A^T)$
3) $(AB)^T = B^T A^T$
4) $\text{rank}(A^T) = \text{rank}(A)$
5) $(A^T)^{-1} = (A^{-1})^T$

Properties of Orthogonal Matrices
Given $A \in \mathbb{R}^{n \times n}$, the following statements are equivalent:
i. A is an orthogonal matrix.
ii. $T(\vec{x}) = A\vec{x}$ preserves length, i.e. $\|A\vec{x}\| = \|\vec{x}\| \ \forall \vec{x}$.
iii. The columns of A form an orthonormal basis of $\mathbb{R}^n$.
iv. $A^{-1} = A^T$.
vi. A preserves dot product, i.e. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y} \ \forall \vec{x}, \vec{y}$.

Matrices of Orthogonal Projections
Given a subspace $V \subseteq \mathbb{R}^n$ with orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$, the matrix P of the orthogonal projection of a vector $\in \mathbb{R}^n$ onto V is:

$$P = QQ^T \quad \left(Q = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{bmatrix}\right) \quad [\text{Orthogonal projection onto } V]$$

(*) The matrix P is symmetric ($P = P^T$)

Intro to Determinants

3/1/23
Lecture 20

Determinants (§6.1)

The determinant of a 3×3 matrix $A = [\vec{u}\ \vec{v}\ \vec{w}]$ is:

$$\det(A) = \det([\vec{u}\ \vec{v}\ \vec{w}]) = \vec{u} \cdot (\vec{v} \times \vec{w}) \quad \text{[Determinant of a 3x3]}$$

$\boxed{\text{THM A square matrix } A \text{ is invertible iff } \det(A) \neq 0.}$

(*) $\det(A) = 0 \leftrightarrow A$ not invertible

(*) Linearity of Determinant

Given $L(\vec{x}) = \det(\langle \vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{x}, \vec{v}_{i+1}, \ldots, \vec{v}_n \rangle)$, $L(\vec{x})$ is a linear transformation [linearity of determinant in the i^{th} row]. The determinant is also linear in all columns.

(*) Reminder: $L(\vec{x})$ linear $\rightarrow L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$; $L(k\vec{x}) = kL(\vec{x})$

Determinant of a Square Matrix

Given $n \times n$ matrix A, we define a pattern P as a selection of n items $\in A$, such that every row & column of A contains exactly one element $\in P$.

Definitions:
- prod P denotes the product of all entries in a given pattern P
- An inversion in a given pattern P is a pair of elements $\in P$, such that one is located above & to the right of the other in A.
- The sign/signature/signum of pattern P is defined as $\operatorname{sgn} P = (-1)^{\#\text{ of inversions in } P}$

$\boxed{\text{THM Determinant} \quad \det(A) = \sum (\operatorname{sgn} P)(\operatorname{prod} P) \quad \forall \text{ patterns } P \text{ in } A.}$

(*) Determinants for Special Matrices

- The determinant of an upper triangular, lower triangular, or diagonal matrix A is the product of its diagonal entries.
- Given square matrices A, C [of potentially different size]:

$$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = (\det A)(\det C) \quad \text{[Determinant of Block Matrices]}$$

(*) "$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (\det A)(\det D) - (\det B)(\det D)$" does not always hold

Determinant (cont.)

3/1/23 Lecture 20 (cont.)

$\boxed{\begin{array}{l} \underline{\text{Properties of Determinant (§6.2)}} \\ \text{i. } \det(A^T) = \det(A) \text{ (A square) [Transpose]} \\ \text{ii. } \det(AB) = \det(A)\det(B) \text{ [Multiplication]} \\ \text{iii. } \det(A^m) = \det(A)^m \text{ [Exponentiation]} \\ (*)\ \det(A^{-1}) = \det(A)^{-1} \text{ [Inverse]} \end{array}}$

(*) Row Operations & Determinant

i. $B = A$, multiplying a row by scalar k:

$$\det(B) = k\det(A)$$

ii. $B = A$, swapping two rows of A:

$$\det(B) = -\det(A)$$

iii. $B = A$, adding a [multiple of a] row to another row:

$$\det(B) = \det(A)$$

3/3/23 Lecture 21

(*) Determinant can be calculated by reducing to rref via Gauss-Jordan, then multiplying the determinant of the rref matrix by scaling factors (as appropriate)

(*) Laplace Expansion

Definition: Given $n \times m$ matrix A, we denote the $(n-1)\times(m-1)$ matrix obtained from removing the i^{th} row and j^{th} column of A, $\underline{A_{ij}}$. The determinant of A_{ij} is called a minor of A.

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nm} \end{bmatrix} = A \longrightarrow A_{ij} = \begin{bmatrix} a_{11} & \cdots & \cancel{a_{1j}} & \cdots & a_{1m} \\ \vdots & & & & \vdots \\ \cancel{a_{i1}} & \cdots & \cancel{a_{ij}} & \cdots & \cancel{a_{im}} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & \cancel{a_{nj}} & \cdots & a_{nm} \end{bmatrix} \quad (A \in \mathbb{R}^{n-1 \times m-1}) \quad \text{[Minors of A]}$$

$\longrightarrow$ $\boxed{\begin{array}{l} \underline{\text{Determinant (Laplace)}} \\ \text{Given } n\times n \text{ matrix A:} \\ \longrightarrow \det(A) = \sum_{i=1}^{n} (-1)^{i+j}(A_{ij}), \text{ for some } j \text{ [Expansion by Column]} \\ \longrightarrow \det(A) = \sum_{j=1}^{n} (-1)^{i+j}(A_{ij}), \text{ for some } i \text{ [Expansion by Row]} \end{array}}$

Misc Properties

- If two matrices A and B similar, $\underline{\det(A) = \det(B)}$.
- Given linear transformation w/ β-matrix B (for any basis β, incl. standard): $\det(T) = \det(B)$

  (*) $T(\vec{x}) = A\vec{x} \rightarrow \det(T) = \det(A)$

Determinants and Geometry

3/3/23
Lecture 21 (cont.)

Determinants of Orthogonal Matrices

- A determinant of an orthogonal matrix is either 1 or -1 ($|\det A| = 1$)
  - (*) $\det A = 1 \rightarrow A$ is a rotation matrix

Determinant and Vectors

Given matrix A w/ columns $\vec{v}_1, \dots, \vec{v}_n$:

$$\boxed{|\det A| = \|\vec{v}_1\| \|\vec{v}_2^{\perp}\| \dots \|\vec{v}_n^{\perp}\| \quad \text{THM}}$$

Determinants and Geometry (§6.3)

Given a set of vectors $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$, we define the m-parallelopiped defined by $\vec{v}_1, \dots, \vec{v}_n$ as the set of all points $(x_1, \dots, x_n) \in \mathbb{R}^n = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ $(0 \le c_i \le 1)$. The m-volume of the m-parallelopiped is defined as:

$$V(v_1, \dots, v_n) = V(v_1, \dots, v_{n-1}) \|v_n^{\perp}\| \; ; \; V(\vec{v}_1) = \|\vec{v}_1\| \quad \text{[M-Volume]}$$

(*) $A = [\vec{v}_1 \dots \vec{v}_m] \rightarrow V(v_1, \dots, v_m) = \sqrt{\det(A^T A)}$

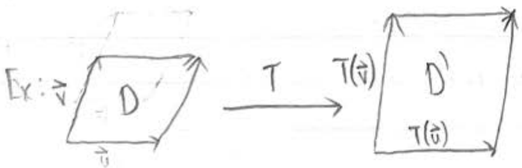
(*) $m = n \rightarrow V(v_1, \dots, v_n) = |\det(A)|$

Ex: $V\Big($  $\Big) = \det([\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3])$

(*) Determinants and Transformations

Given a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$, $|\det A|$ is the <u>expansion factor</u> of T on n-parallelopipeds:

$$V(A\vec{v}_1, \dots, A\vec{v}_n) = |\det A| V(\vec{v}_1, \dots, \vec{v}_n) \quad (\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n)$$

Ex: 

$\rightarrow$ area(D') = $|\det A| \cdot$ area(D)

(*) Cramer's Rule

Given a linear system $A\vec{x} = \vec{b}$ ($A \in \mathbb{R}^{n \times n}$, invertible), the components x_i of the solution vector $\vec{x}$ are:

$$x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

where $A_{\vec{b},i}$ is the matrix obtained by replacing the i^{th} column of A with $\vec{b}$.

(*) Adjoints

Given invertible matrix $A \in \mathbb{R}^{n \times n}$, the <u>classical adjoint</u> $\text{adj}(A)$ of A is the $n \times n$ matrix, $\{\text{adj}(A)_{ij} = (-1)^{i+j} \det(A_{ji})\}$.

$\rightarrow$ (*) $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

Eigenvectors & Eigenvalues

3/6/23
Lecture 22

Diagonalization

Def.: A linear transformation $T(\vec{x}) = A\vec{x}$ is said to be diagonalizable if the matrix B of T with respect to some basis is diagonal, i.e. if A is similar to some diagonal matrix B [$\exists$ invertible matrix S s.t. $S^{-1}AS = B$ for some diagonal matrix B].

Eigenvectors & Eigenvalues (§6.1)
Given a linear transformation $T(\vec{x}) = A\vec{x}$ $(T: \mathbb{R}^n \to \mathbb{R}^n)$, a nonzero vector $\vec{v} \in \mathbb{R}^n$ is called an eigenvector of A (or T) if $A\vec{v} = \lambda\vec{v}$ for some scalar λ [eigenvalue associated w/ $\vec{v}$].

Notes on Eigenvectors

- A basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ is called an eigenbasis of A if the vectors $v_1, \dots, v_n$ are eigenvectors of A
  - A matrix A is diagonalizable iff $\exists$ an eigenbasis of A, in which case:

$$v_1, \dots, v_n \text{ (eigenbasis of } A) \longrightarrow \boxed{S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}, B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}} \text{ s.t. } S^{-1}AS = B$$

  (*) If matrices S and B diagonalize A, the columns of S form an eigenbasis of A
- The only possible eigenvalues for an orthogonal matrix are 1 and -1
- A matrix A is invertible iff 0 is not an eigenvalue of A

3/8/23
Lecture 23

THM Characteristic Eq.	Given an $n \times n$ matrix A and a scalar λ, λ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$. (Characteristic/Secular Equation)

↳ (*) The eigenvalues of a triangular matrix are its diagonal entries

(*) Trace

Definition: Given a square matrix A, the trace $tr(A)$ of A is the sum of its diagonal entries.

Notes on Eigenvalues

3/8/23
Lecture 23
(cont.)

<u>Reminder</u>: λ eigenvalue of $A \rightarrow \det(A-\lambda I_n)=0$ [Characteristic Equation]

Definition: Characteristic Polynomial (§7.2)

Given $n \times n$ matrix A, $\det(A-\lambda I_n)$ is a polynomial of degree n; this polynomial is called the <u>characteristic polynomial</u> $f_A(\lambda)$ of A.

$$\boxed{\det(A-\lambda I_n) = (-1)^n \lambda^n + (-1)^{n-1}(\operatorname{tr}A)\lambda^{n-1} + \ldots + \det A}$$ [Characteristic Polynomial]

↳ (*) $A \in \mathbb{R}^{2\times 2} \rightarrow f_A(\lambda) = \lambda^2 - (\operatorname{tr}A)\lambda + \det A$

↳ The zeros of the characteristic are the eigenvalues of A

<u>Notes on Eigenvalues</u>

Definition: The <u>algebraic multiplicity</u> of an eigenvalue λ_0 of matrix A is defined as the degree of that root in the characteristic polynomial, i.e. $f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$ [for some polynomial $g(\lambda)$] $\rightarrow$ <u>$\operatorname{almu}(\lambda_0) = k$</u>
[λ_0 is a root of multiplicity k of $f_A(\lambda)$] $\quad$ ↳ $g(\lambda_0) \neq 0$

- An $n \times n$ has <u>at most</u> n real eigenvalues
  (*) n odd $\rightarrow$ has at least 1 real eigenvalue

(*) <u>Properties of Eigenvalues</u>

Given an $n \times n$ matrix A with n eigenvalues $\lambda_1, \ldots, \lambda_n$ $(f_A(\lambda) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda))$:

<u>$\det(A)$</u> $= \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$; <u>$\operatorname{tr}(A)$</u> $= \lambda_1 + \lambda_2 + \ldots + \lambda_n$ ((*) $\lambda_0, \ldots, \lambda_1$ need not be distinct)

Definition: Eigenspaces (§7.3)

Given an eigenvalue λ of an $n \times n$ matrix A, the <u>eigenspace</u> E_λ associated with λ is defined as the kernel of the matrix $(A - \lambda I_n)$, i.e.:

$$\boxed{E_\lambda = \ker(A - \lambda I_n) = \{\vec{v} \mid A\vec{v} = \lambda\vec{v}\}}$$ [Eigenspace ass. w/ λ]

↳ <u>Definition</u>: The <u>geometric multiplicity</u> <u>$\operatorname{gemu}(\lambda)$</u> of λ is defined as the dimension of the eigenspace E_λ, i.e. $\operatorname{gemu}(\lambda) = \operatorname{nullity}(A - \lambda I_n) = n - \operatorname{rank}(A - I_n)$ $\quad$ ((*) $\operatorname{gemu}(\lambda) \leq \operatorname{almu}(\lambda)$)

Diagonalization

3/10/23
Lecture 24

Notes on Eigenspaces

- An eigenspace E_λ is the set of all eigenvectors w/ associated eigenvalue λ
- The set of basis vectors $\vec{v}_1, \ldots, \vec{v}_s$ across all eigenspaces of a matrix $A \in \mathbb{R}^{n \times n}$ is linearly independent
  - $s = \sum \text{gemu}(\lambda)$ for all eigenvalues λ [sum of dimensions of eigenspaces] $(s \leq n)$
  - (*) An $n \times n$ matrix A is diagonalizable iff $s = \sum \text{gemu}(\lambda) = n$ [A $n \times n$]
    - (*) If $A \in \mathbb{R}^{n \times n}$ has n eigenvalues, A is diagonalizable

(*) Eigenvalues and Similarity

Given matrix A similar to B:
1. A and B have the same characteristic polynomial $(f_A(\lambda) = f_B(\lambda))$
2. $\text{rank}\, A = \text{rank}\, B$; $\text{nullity}\, A = \text{nullity}\, B$
3. A and B have the same eigenvalues, with the same algebraic & geometric multiplicities ((*) Eigenvectors may not be shared)
4. $\det(A) = \det(B)$; $\text{tr}(A) = \text{tr}(B)$

Procedure for Diagonalization (☆)

In order to diagonalize $n \times n$ matrix A (i.e. find invertible matrix S s.t. $S^{-1}AS = B$ [B diagonal]):

1) Find eigenvalues of A by solving the characteristic $f_A(\lambda) = \det(A - \lambda I_n) = 0$

2) For each eigenvalue λ, find a basis of the eigenspace $E_\lambda = \ker(A - \lambda I_n)$

3) A is diagonalizable iff the dimensions of the eigenspaces add up to n

→ 4) Combining the basis vectors $\vec{v}_1, \ldots, \vec{v}_n$ of the eigenspaces (from 2))

→ $S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$, $B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ (λ_i being the eigenvalue associated with v_i)

(*) Complex Numbers Review

- Expressed as $z = a + ib$ (a real; ib imaginary), $z = r(\cos\theta + i\sin\theta)$
  $\rightarrow z(\alpha) \cdot z(\beta) = z(\alpha + \beta)$
- De Moivre's Formula: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Orthogonal Diagonalization

3/10/23 Lecture 24 (cont.)

(*) Complex Eigenvalues (§7.5)

- Any polynomial $p(\lambda)$ with complex coefficients splits, i.e. $p(\lambda) = k(\lambda-\lambda_1)\ldots(\lambda-\lambda_n)$ [k, λ_i complex]
  - → Has exactly n complex roots ((*) Counted with algebraic multiplicities - may not be distinct)
  - ⟶ A complex $n\times n$ matrix A has n complex eigenvalues (from the characteristic)
- Given real 2×2 matrix A with eigenvalues $a \pm ib$ ($b \neq 0$), eigenvector $\vec{v} + i\vec{w}$ associated with eigenvalue $a+ib$

$$\longrightarrow S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}; \quad S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

(*) The relationship between eigenvalues and determinant/trace hold for complex eigenvalues

Orthogonal Diagonalization (§8.1)

Definition: A matrix A is orthogonally diagonalizable if there exists an orthogonal matrix S that diagonalizes A (i.e. $S^{-1}AS$ [$= S^TAS$] is diagonal)

$$S^{-1}AS = B \text{ [B diagonal]}; \ S \text{ orthogonal} \quad \text{[Orthogonal Diagonalization]}$$

Spectral Theorem	A matrix A is orthogonally diagonizable iff A is symmetric (i.e. $A^T = A$).

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- Given a symmetric matrix A: Let $\vec{v}_1$ and $\vec{v}_2$ be eigenvectors of A associated with eigenvalues λ_1 and λ_2. If $\lambda_1 \neq \lambda_2$ (eigenvalues distinct) → $\vec{v}_1 \cdot \vec{v}_2 = 0$ [$\vec{v}_1, \vec{v}_2$ orthogonal]
- A symmetric $n\times n$ matrix A has n real eigenvalues (counted w/ algebraic multiplicities)

Procedure for Orthogonal Diagonalization (☆)

Given a symmetric $n\times n$ matrix A:

1) Find the eigenvalues of A, and find a basis of each eigenspace

2) Use Gram-Schmidt to find an orthonormal basis of each eigenspace.

3) Form an orthonormal eigenbasis for A $\vec{v}_1, \ldots, \vec{v}_n$ by concatenating the bases from 2):

$$S = [\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n] \quad (S \text{ orthogonal}; \ S^{-1}AS \text{ diagonal})$$

(*) Notes: Linear Spaces & Isomorphisms

4/2/23
Notes

Definition: Linear Spaces (§4.1)

A linear space is a set with an associated rule for addition ($f, g \in$ linear space $V \rightarrow (f+g) \in V$) and scalar multiplication ($f \in V, k \in \mathbb{R} \rightarrow kf \in V$), s.t. $\forall f, g, h \in V$ and $c, k \in \mathbb{R}$:

i) $(f+g)+h = f+(g+h)$
ii) $f+g = g+f$
iii) $\exists$ neutral element n "=" $0 \in V$ ($n+f = f \ \forall f$)
iv) for each $f \in V$, $\exists g$ "=" $-f \in V$ s.t. $f+g = 0$
v) $k(f+g) = kf + kg$
vi) $(c+k)f = cf + kf$
vii) $c(kf) = (ck)f$
vi) $1f = f$

↳ Def: A vector space is a linear space containing vectors (with the associated operations of vector addition and scalar multiplication).

↳ (*) A linear transformation can be considered a function that takes vector spaces as domain and codomain (+ other conditions: $T(f+g) = T(f) + T(g)$, $T(kf) = kT(f)$).

Definition: Isomorphism (§4.2)

An isomorphism is an invertible linear transformation. Given two linear spaces V and W, we say V is isomorphic to W (and vice versa) if $\exists$ an isomorphism $T: V \rightarrow W$.

(*) Isomorphisms

If $\mathfrak{B} = (f_1, f_2, \dots, f_n)$ is a basis of a linear space V, then the coordinate transformation $L_{\mathfrak{B}}(f) = [f]_{\mathfrak{B}}$ from V to $\mathbb{R}^n$ is an isomorphism; in other words, any n-dimensional linear space V is isomorphic to, and can be transformed into, $\mathbb{R}^{n*}$.

↳ (*) (if $n \neq \infty$)

(*) Notes: Least Squares

Notes

Given $A = [\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_m]$ w/ $V = \text{im}(A) \subset \mathbb{R}^n$, can look at $V^\perp$ [orthogonal complement of V].

$V^\perp = \{x \in \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0\ \forall\ v \in V\} = \{x \in \mathbb{R}^n : \vec{v}_i \cdot \vec{x} = 0\ \forall\ i = 1, \dots, n\} = \{\vec{x} \in \mathbb{R}^n : \vec{v}_i^T \vec{x} = 0\ \forall\ i = 1, \dots, m\}$

↳ Observation: $(\text{im } A)^\perp = \ker(A^T)$ => $\ker(A) = \ker(A^T A)$

Least Squares

Recall: Given $\vec{x} \in \mathbb{R}^n$, subspace $V \subset \mathbb{R}^n$, the orthogonal projection of $\vec{x}$ onto V [$\text{proj}_V \vec{x}$] is the vector in V s.t.

$$\|\vec{x} - \text{proj}_V \vec{x}\| < \|\vec{x} - \vec{v}\|\ \forall\ v \in V \text{ s.t. } v \neq \text{proj}_V \vec{x}$$

$\text{proj}_V \vec{x}$ is the vector in V "closest" to $\vec{x}$

→ Given a linear system of eqs. $A\vec{x} = \vec{b}$ w/ no solution, can find "closest" solution $\vec{x}^*$ minimizing $\|\vec{b} - A\vec{x}^*\|$

Least Squares	Given a linear system $A\vec{x} = \vec{b}$, a least-squares solution is a vector $\vec{x}^* \in \mathbb{R}^n$ s.t. $\|\vec{b} - A\vec{x}^*\| \le \|\vec{b} - A\vec{v}\|\ \forall\ v \in \mathbb{R}^n$

→ Prop. The least-squares solutions of $A\vec{x} = \vec{b}$ is an exact solution to normal equation $A^T A \vec{x} = A^T \vec{b}$.

(*) Reasoning: Let $V = \text{im}(A) \to A\vec{x}^* = \text{proj}_V \vec{b} \to (\vec{b} - A\vec{x}^*) \in V^\perp = (\text{im } A)^\perp = \ker(A^T) \to A^T(\vec{b} - A\vec{x}^*) = 0$. □

→ Lemma. The matrix of orthogonal projection onto $\text{im}(A)$ is the matrix $A(A^T A)^{-1} A^T$.

$A(A^TA)^{-1}A^T\vec{b} = A(A^TA)^{-1}A^TA\vec{x}^* = A\vec{x}^*$

(*) Ex: Data Fitting

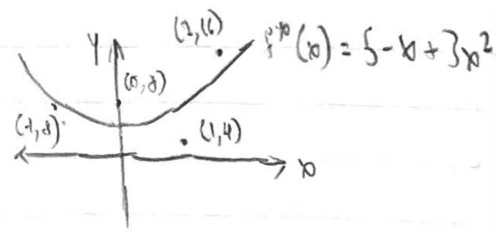
Problem: Fit a quadratic to the points $(-1, 8)$, $(0, 8)$, $(1, 4)$, $(2, 16)$.

$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

↳ Want $f(t) = c_0 + c_1 t + c_2 t^2$ s.t. $\begin{vmatrix} f(x_1) = 8 \\ f(x_2) = 8 \\ f(x_3) = 4 \\ f(x_4) = 16 \end{vmatrix} \to \begin{vmatrix} c_0 + c_1(-1) + c_2(-1)^2 = 8 \\ c_0 = 8 \\ c_0 + c_1(1) + c_2(1)^2 = 4 \\ c_0 + c_1(2) + c_2(2)^2 = 16 \end{vmatrix} \to A \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \vec{b} = \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix}$

Issue: no exact soln. exists (# equations > # unknowns)

↳ find least-squares soln: $\vec{c}^* = \begin{bmatrix} c_0^* \\ c_1^* \\ c_2^* \end{bmatrix} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$



→ $f^*(t) = 5 - t + 3t^2$ is eqn. minimizing $\left\| \begin{bmatrix} f^*(x_1) \\ \vdots \\ f^*(x_4) \end{bmatrix} - \vec{b} \right\| = \|A\vec{c}^* - \vec{b}\|$