

Moth 32B: Calculus of Several Variables (23W)

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Textbook: Jon Rogawski - Multivariable Calculus (4th Edition) - 2019

Topics: Integration of multivariate functions, line integrals, surfaces & surface integrals, vector fields & associated operations, Green's/Stokes'/Divergence Theorems

Table of Contents:

1) Integration of Multiple Variables - 39

i) Double Integrals - 39

(*) Fubini's Theorem - 40

ii) Triple Integrals - 42

iii) Integration in Polar Coordinates - 43

(*) Integration in Cylindrical & Spherical Coordinates - 44

iv) Change of Variables in Multiple Variables - 45

a) Change of Variables in Two Dimensions - 46

b) Change of Variables in Three Dimensions - 47

2) Vector Fields - 48

i) Intro to & Operations on Vector Fields - 48

ii) Properties of Vector Fields - 49

(*) Conservative Vector Fields - 49

3) General Forms of Integration

i) Line Integrals - 50

a) Scalar Line Integrals - 50

b) Vector Line Integrals - 51

(*) Line Integrals of Conservative Vector Fields - 52

ii) Surfaces - 54

a) Surface Integrals - 55

b) Surface Integrals of Vector Fields - 56

4) Major Integration Theorems - 57

i) Green's Theorem - 57

ii) Stokes' Theorem - 58

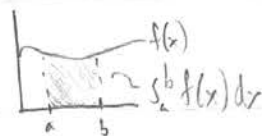
iii) Divergence Theorem - 59

Double and Iterated Integrals

1/9/23
Lecture 1

Double Integrals

$\int_a^b f(x) dx$ = area under $f(x)$ on xy -plane from a to b



Double Integral

Given region D on the xy plane, the double integral of $f(x, y)$ over D :

$\iint_D f(x, y) dA$ represents the volume under the surface $f(x, y)$ in region D



("Negative" volume (volume in regions where $f(x, y) < 0$) is counted as negative)

Iterated Integrals

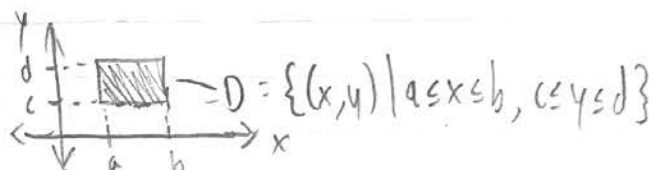
An iterated integral is an expression of the following form:

$$\int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

is equivalent to a double integral over the region $\{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$

$C = R = [a, b] \times [c, d]$ (notation for the rectangle)

- Order of computation does not matter for iterated integrals



1/11/23
Lecture 2

Iterated Integration for General Regions

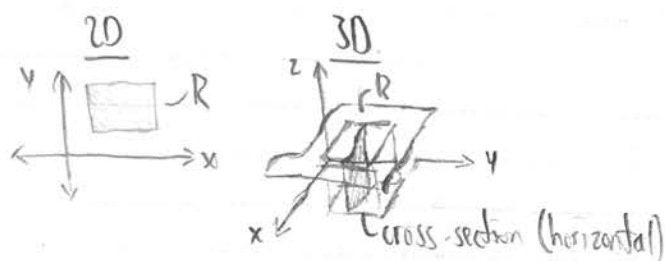
Fubini's Theorem (§16.2)

Given rectangular domain $R = [a, b] \times [c, d]$ and $z = f(x, y)$:

$$SS_R f(x, y) dA = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

$$(*) R = [a, b] \times [c, d] = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$$

Reasoning: Each inner integral generates a cross-section of $f(x, y)$ at a given x -coordinate/ y -coordinate; the integral of these cross-sections gives a volume.



Iterated Integration for General Regions

We can use iterated integrals to calculate double integrals for two types of general region:

General Regions

(1) Horizontally simple (y is bounded $\rightarrow a \leq y \leq b, g_1(y) \leq x \leq g_2(y)$)

$$\iint_D f(x, y) dA = \int_a^b \left(\int_{g_1(y)}^{g_2(y)} f(x, y) dx \right) dy \quad (x \text{ as a function of } y)$$

(2) Vertically simple (x is bounded $\rightarrow a \leq x \leq b, h_1(x) \leq y \leq h_2(x)$)

$$\iint_D f(x, y) dA = \int_a^b \left(\int_{h_1(x)}^{h_2(x)} f(x, y) dy \right) dx \quad (y \text{ as a function of } x)$$

Notes on Double Integrals

1/13/23
Lecture 3

Properties of Double Integration

1) $\iint_D \lambda dA, \lambda \in \mathbb{R} = \lambda \cdot \text{area}(D)$

2) $\iint_D (f(x,y) + g(x,y)) dA = \iint_D f(x,y) dA + \iint_D g(x,y) dA$

3) $\iint_D \lambda f(x,y) dA, \lambda \in \mathbb{R} = \lambda \iint_D f(x,y) dA$

4) Given $D = D_1 \cup D_2, D_1 \cap D_2 = \emptyset$ (D_1 and D_2 disjoint):

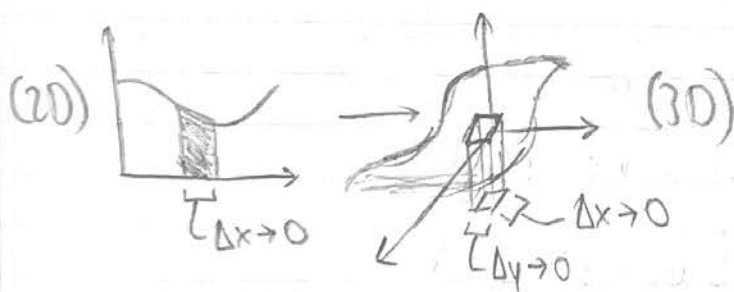
$$\iint_D f(x,y) dA = \iint_{D_1} f(x,y) dA + \iint_{D_2} f(x,y) dA$$

(*) Disjoint: non-overlapping

Notes

2D integrals can be seen as the limit of 2D Riemann Sums; similarly, in the context of 3D, 3D integrals (double integrals) can be seen as the limit of 3D Riemann Sums,

i.e.: $\iint_R f(x,y) dA = \lim_{\Delta x, \Delta y \rightarrow 0} \sum_i \sum_j f(x_i, y_j) \Delta A_{ij}$



Triple Integrals

1/18/23
Lecture 4

Triple Integrals (§16.3)

Given $f(x, y, z)$, and region w in 3D space (\mathbb{R}^3), we define the triple integral of $f(x, y, z)$ over w as:

$$\rightarrow := \iiint_w f(x, y, z) dV$$

(*) $V = \text{volume}$

Fubini's Theorem for Triple Integrals

Given region $w_0 = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, p \leq z \leq q\}$ for some constants/expressions a, b, c, d, p, q , we can write the expression:

$$\iiint_{w_0} f(x, y, z) dV = \int_a^b \left(\int_c^d \left(\int_p^q f(x, y, z) dz \right) dy \right) dx$$

(*) Order of integration can be changed, per Fubini's

Notes

- Region w is z-simple if: $\{(x, y) \in D \text{ in } \mathbb{R}^2 \mid z_1(x, y) \leq z \leq z_2(x, y) \text{ for some } z_1(x, y), z_2(x, y)\}$
- Given z-simple region w : $\iiint_w f(x, y, z) dV = \iint_D \left(\int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \right) dA$
- Similar corollaries exist for "x-simple" and "y-simple" regions

Just like double integrals, triple integrals can be seen as the limit of [4D] Riemann Sums, i.e.: $\iiint_w f(x, y, z) dV = \lim_{i, j, k \rightarrow 0} \sum_i \sum_j \sum_k f(x, y, z) \Delta V_{i, j, k}$

Polar Integration

1/20/23
Lecture 5

Properties of Triple Integration

1) $\iiint_W \lambda dV, \lambda \in \mathbb{R} = \lambda V$ (Constants)

2) $\iiint_W \lambda f(x, y, z) dV = \lambda \iiint_W f(x, y, z) dV$ (Scalar Multiplication)

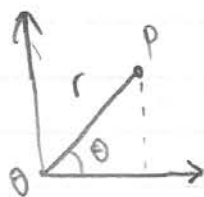
3) $\iiint_W (f(x, y, z) + g(x, y, z)) dV =$

$\rightarrow \iiint_W f(x, y, z) dV + \iiint_W g(x, y, z) dV$ (Integral Sums)

4) Given region $W = W_1 \cup W_2, W_1 \cap W_2 = \emptyset$ (disjoint):

$\iiint_W f(x, y, z) dV = \iiint_{W_1} f(x, y, z) dV + \iiint_{W_2} f(x, y, z) dV$ (Disjoint Regions)

Polar Coordinates

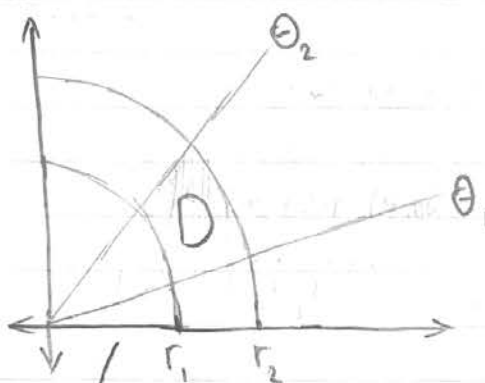


Polar coordinates: $\{(r, \theta) \mid r = |\vec{OP}|, \theta = \text{angle between } x\text{-axis and } \vec{OP}\}$
 $\rightarrow x = r \cos \theta, y = r \sin \theta$ ($r = \sqrt{x^2 + y^2}, \tan(\theta) = \frac{y}{x}$)

Polar Integration

Given region D in polar coordinates ($D = \{(r, \theta) \mid r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\}$ for some expressions $r_1, r_2, \theta_1, \theta_2$, we write:

$$\iint_D f(x, y) dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(x, y) r dr d\theta$$



Polar rectangle

$$D = \{\theta_1 \leq \theta \leq \theta_2 \mid \theta_1, \theta_2 \in \mathbb{R} \text{ (constant)}\} \\ \{r_1 \leq r \leq r_2 \mid r_1, r_2 \in \mathbb{R} \text{ (constant)}\}$$

(*) Polar integrals can also be thought of as the limits of Riemann Sums; however, the shape of the Riemann Sum slices differs in polar vs xy (circle vs square), hence the inclusion of an addl. r in the expression for polar integration

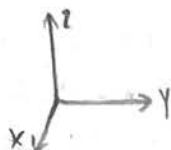
(*) Such a region D is "radially simple"

Cylindrical & Spherical Coordinates

1/23/23
Lecture 6

Types of 3D coordinate spaces:

1) Euclidean - x, y, z

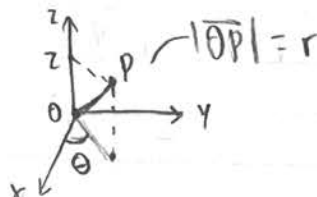


2) Cylindrical - r, θ, z

$$\rightarrow x = r \cos \theta$$

$$y = r \sin \theta$$

$z = z$ (relative to x, y, z)



Given region $w: \begin{cases} \theta_1 \leq \theta \leq \theta_2 \\ r_1(\theta) \leq r \leq r_2(\theta) \\ z_1(r, \theta) \leq z \leq z_2(r, \theta) \end{cases}$ we can compute triple integrals in 3D space via cylindrical coordinates:

$$\rightarrow \iiint_w f(x, y, z) dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r, \theta)}^{z_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

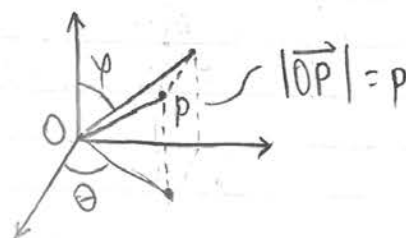
1/25/23
Lecture 7

3) Spherical - ρ, θ, φ

$$\rightarrow x = \rho \sin(\varphi) \cos(\theta)$$

$$y = \rho \sin(\varphi) \sin(\theta)$$

$$z = \rho \cos(\varphi)$$



Given region $w:$

$$w: \begin{cases} \theta_1 \leq \theta \leq \theta_2 \\ \varphi_1 \leq \varphi \leq \varphi_2 \\ \rho_1(\theta, \varphi) \leq \rho \leq \rho_2(\theta, \varphi) \end{cases}$$

We can compute triple integrals in 3D space via spherical coordinates with the formula:

$$\rightarrow \iiint_w f(x, y, z) dV = \int_{\theta_1}^{\theta_2} \int_{\varphi_1}^{\varphi_2} \int_{\rho_1(\theta, \varphi)}^{\rho_2(\theta, \varphi)} f(\rho \sin(\varphi) \cos(\theta), \rho \sin(\varphi) \sin(\theta), \rho \cos(\varphi)) \rho^2 \sin(\varphi) d\rho d\varphi d\theta$$

Intro to Change of Variables

1/27/23
Lecture 8

Applications of Double & Triple Integrals (§16.5)

- Averages over regions (volumes): e.g. average of $f(x, y)$ over $D = \iint_D f(x, y) dA$
- Centroid of a region $D = (\bar{x}, \bar{y}) = \text{average of } x\text{-coords } / y\text{-coords in } D \rightarrow \bar{x} = \frac{1}{\text{area}} \iint_D x dA, \bar{y} = \frac{1}{\text{area}} \iint_D y dA$
- Mass (from density): Given density function $p(x, y)$, total mass (over region D) = $\iint_D p(x, y) dA$
 - Center of mass: $(x_{cm}, y_{cm}) := x_{cm} = \frac{1}{\text{mass}} \iint_D x p(x, y) dA, y_{cm} = \frac{1}{\text{mass}} \iint_D y p(x, y) dA$
- Probability: Given probability density function $p(x, y)$ (i.e. $P(x=x_0, y=y_0) = p(x_0, y_0)$, region J s.t. $\iint_J p(x, y) dA = 1$,
 - $\rightarrow P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d p(x, y) dy dx = \text{probability that } a \leq x \leq b, c \leq y \leq d \text{ for random } x, y$

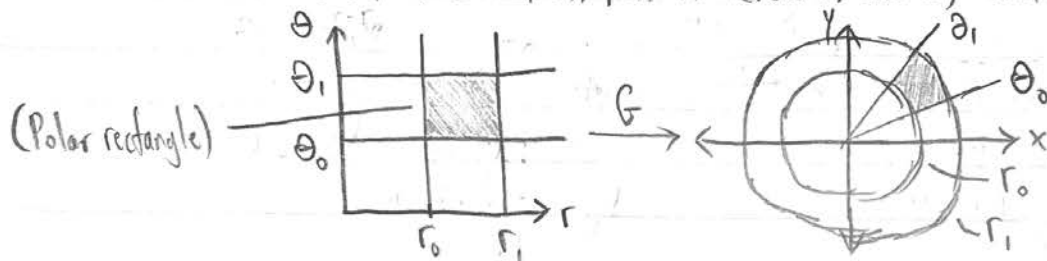
Change of Variables (§16.6)

Definition: A function $G: X \rightarrow Y$ is also called a map or a mapping (from X to Y)

- Given $x \in X$, $G(x) \in Y$ is called the image of x
- The set of all images $\{G(x) | x \in X\}$ is called the range or image of X , $G(X)$

(*) Notation: A function $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is written as $G(u, v) = (x(u, v), y(u, v))$

- Ex: $G(r, \theta) = (x(r, \theta), y(r, \theta)) = (r \cos \theta, r \sin \theta)$ (Polar $\rightarrow xy$)

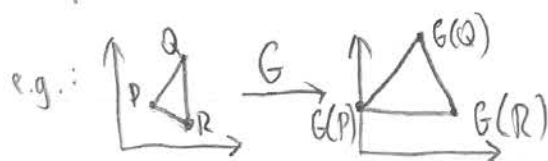


(*) Linear Maps

A map $G(u, v) \rightarrow (x(u, v), y(u, v))$ is called linear if $(x(u, v), y(u, v))$ takes the form $(Au + Bv, Cu + Dv)$ for some constants $A, B, C, D \in \mathbb{R}$

Properties of a Linear Map

- 1) $G(u_1 + u_2, v_1 + v_2) = G(u_1, v_1) + G(u_2, v_2)$
- 2) $G(cu, cv) = cG(u, v) \forall \text{ constants } c$
- 3) A line between any two points $P = (u_1, v_1), Q = (u_2, v_2)$ is mapped as a line between the points $G(P) = G(u_1, v_1)$ and $G(Q) = G(u_2, v_2)$



Change of Variables

1/30/23
Lecture 9

The Jacobian

Given a map $G(u, v) \rightarrow (x(u, v), y(u, v))$, we define the Jacobian determinant (the Jacobian) of map G :

$$\text{Jac}(G) = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

(*) The Jacobian of a linear map $G(u, v) = (Au + Bv, (u + Dv))$ is $AD - BC \forall u, v$ (is constant)



Area and the Jacobian

Given a map G that maps a domain D to a codomain $G(D)$, we can state that:

$$\text{area}(G(D)) \approx |\text{Jac}(G)(P)| \text{area}(D)$$

(.) The approximation grows more precise as $\text{area}(D) \rightarrow 0$,

$$\text{i.e.: } \lim_{|D| \rightarrow 0} |\text{Jac}(G)(P)| \text{area}(D) = \text{area}(G(D))$$

(size (diameter) of D)

(*) For some sample point $P \in D$

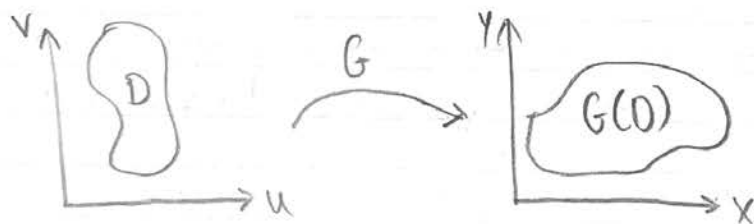
Change of Variables Formula (2D)

Given map $G(u, v) \rightarrow (x(u, v), y(u, v))$ that maps a given region D to $G(D)$ one-to-one, we can say:

$$\iint_D f(u, v) du dv = \iint_{G(D)} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Notes

- (*) One-to-one: " $G(P) = G(Q) \rightarrow P = Q$ "
- (*) Assumes $f(x, y)$ is continuous
- (*) The act of integration sends the error of $\text{Jac}(G)$ to zero (hence the use of " \approx ")



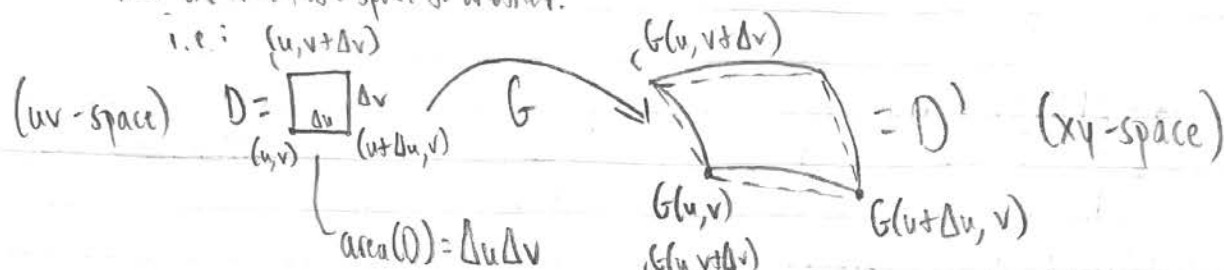
Change of Variables (cont.)

2/1/23
Lecture 10

The Jacobian

Justification for the inclusion of the term $|\text{Jac}(G)|$ when changing variables:

The Jacobian accounts for the change of the area of regions (i.e. of dA) when mapped from one coordinate space to another.



- (*) Changing variables allows us to map difficult-to-integrate regions into more integration-friendly ones (e.g. an ellipse \rightarrow a unit circle)
- (*) Given $G: D' \rightarrow D$, $G^{-1}: D \rightarrow D'$:
- $$\text{Jac}(G) = (\text{Jac}(G^{-1}))^{-1} \text{ (assuming there exists a function } G^{-1})$$
- $$\text{area}(E) = \left(\frac{\partial x}{\partial u} \right) \Delta u + \left(\frac{\partial y}{\partial v} \right) \Delta v = \text{Jac}(G) \Delta u \Delta v$$

Change of Variables for Triple Integrals

Change of variables in 3D is virtually the same as for 2D, i.e.:

2/3/23
Lecture 11

Change of Variables Formula (3D)

Given map $G(u, v, w) \rightarrow (x(u, v, w), y(u, v, w), z(u, v, w))$ that maps a 3D region D to $G(D)$ one-to-one, we can say:

$$\iiint_{G(D)} f(x, y, z) dV = \iiint_D f(G(u, v, w)) |\text{Jac}(G)| dV$$

The Jacobian (3D)

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad (3D)$$

- (*) The Jacobian for maps to polar, cylindrical, and spherical coordinates are $\text{Jac}(G) = r$, $\text{Jac}(G) = r$, and $\text{Jac}(G) = r^2 \sin \theta$, respectively (this can be proven via the determinant definition)

Intro to Vector Fields

2/3/23
Lecture 11

Vector Fields

Vector Fields

A vector field is a type of function that assigns to each point $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ a vector output

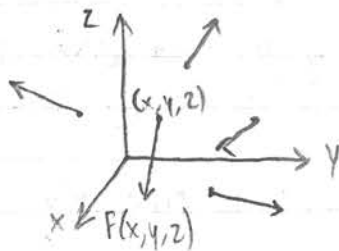
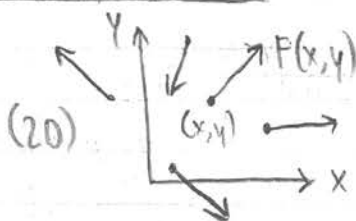
$$\vec{F}(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \longrightarrow \\ \longrightarrow = \langle F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n) \rangle$$

Vector Fields in \mathbb{R}^2 and \mathbb{R}^3

$$\mathbb{R}^2: \vec{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle \\ = F_1 \hat{i} + F_2 \hat{j}$$

$$\mathbb{R}^3: \vec{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle \\ = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

Visualizing Vector Fields



(*) Vector fields akin to "vectors living in a plane"

Types of Vector Fields

- Constant vector field: $\vec{F}(x, y, z)$ constant for all (x, y, z)
- Unit vector field: $\vec{F}(P)$ such that $\|\vec{F}(P)\| = 1$ for all points P
- Radial vector field: $\vec{F}(P)$ such that $\vec{F}(P)$ is pointing away from the origin ($\vec{OP} \parallel \vec{F}(P)$) and $\|\vec{F}(P)\|$ depends only on the distance from the origin for all points P
- Unit radial vector field (\mathbb{R}^2): $\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$

Operations on Vector Fields

∇ (del operator) = $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ ("Quasi-vector of partial derivatives" (used for notation))

• ∇f (scalar function f) = $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle =$ gradient of f

• Divergence of vector field F : $\text{div}(F) = \nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

• Curl of vector field $F = \text{curl}(F) = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$

Vector Fields (cont.)

2/6/23
Lecture 12

Properties of Divergence

Given vector field \vec{F} at a point P :

- 1) If $\text{div}(\vec{F}) > 0$ at P , we say P is a source (vectors out > vectors in)
- 2) If $\text{div}(\vec{F}) < 0$ at P , we say P is a sink (vectors out < vectors in)
- 3) If $\text{div}(\vec{F}) = 0$ at P , P is neither a source nor a sink
- (*) If $\text{div}(\vec{F}) = 0$ for all points P , we say \vec{F} is incompressible

Source ($\text{div}(\vec{F}) > 0$)



Sink ($\text{div}(\vec{F}) < 0$)



Properties of Curl

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

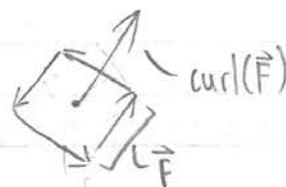
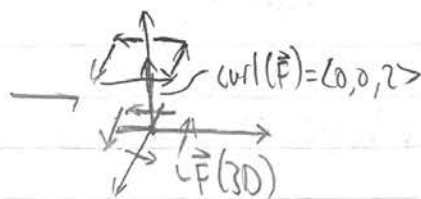
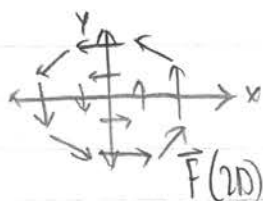
(*) Cross product (and thus curl) is only defined for three dimensions

(*) Curl gives a vector [field] as output

Theorem: The direction of $\text{curl}(\vec{F})$ is the direction of the axis of rotation of \vec{F} ;
the magnitude of $\text{curl}(\vec{F})$ is proportional to the rate of rotation of \vec{F} .

(*) as given by the right-hand rule

Ex: $\vec{F} = \langle -y, x, 0 \rangle \rightarrow \text{curl}(\vec{F}) = \langle 0, 0, 2 \rangle$:



Conservative Vector Fields

(Math 32A: Given scalar function $f(x, y, z)$, $\text{grad}(f) = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$ [vector field])

Conservative Vector Fields

A vector field \vec{F} is conservative [a conservative vector field] if $\vec{F} = \nabla f$ for some scalar function f (called the "potential of F " / "potential function for \vec{F} ").

→ i.e. $\vec{F} = \nabla f = \langle F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z} \rangle$

A vector field \vec{F} can only be conservative if it does not curl (i.e. $\text{curl}(\vec{F}) = 0 \forall \text{ points } P \in \mathbb{R}^3$)

→ 2D: $\text{curl} \langle F_1, F_2, 0 \rangle = 0$

→ 3D: $\text{curl} \langle F_1, F_2, F_3 \rangle = 0$

(*) 0 curl does not necessarily imply a vector field is conservative

Scalar Line Integrals

2/7/23

Lecture B

Conservative Vector Fields

The potential function f of a vector field F can be found through integration, i.e.:

$$F_1 = \frac{\partial f}{\partial x} \rightarrow \int F_1 dx = h(x, y) + g(y) \quad (g(y) = \text{some function not dependent on } x)$$

$$F_2 = \frac{\partial f}{\partial y} = h(x, y) + g'(y) \rightarrow \int F_2 dy = f(x, y) + C \quad (\text{e.g.})$$

Line Integrals (§17.2)

A scalar line integral represents the integral of a scalar function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ over a curve $C \in \mathbb{R}^n$ (line area of the sheet between C and the surface of f).

Line Integral of $f(x, y, z)$ over C

$$\rightarrow \int_C f(x, y, z) ds$$

(*) "ds" represents arc length,

i.e.:

$$C \quad ds \rightarrow \lim_{\Delta s \rightarrow 0} \sum_{i=1}^n f(P_i) \Delta s$$



Curves for line integrals can most easily be expressed via parametrization, i.e. $C = \vec{r}(t)$ [function of time]

Computing Scalar Line Integrals

Given a curve C with parametrization $\vec{r}(t)$, $a \leq t \leq b$, and continuous function $f(x, y, z)$ and $\vec{r}'(t)$, the line integral of C over f is:

$$\int_C f(x, y, z) ds = \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| dt$$

$$(*) ds \rightarrow \|\vec{r}'(t)\| dt$$

Vector Line Integrals

2/10/20
Lecture 14

Properties of Scalar Line Integrals

- 1) $\int_C 1 ds$ = arc length of C
- 2) Given $C = C_1 \cup C_2$ ($C_1 \cup C_2 = \emptyset$) $\rightarrow \int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds$
- 3) The value of $\int_C f ds$ is independent of the choice of parametrization $\vec{r}(t)$ of C

Vector Line Integrals

Definition: Given vector field \vec{F} and oriented curve C (i.e. curve w/ associated direction), the line integral of \vec{F} along C is:

$$\int_C \vec{F} d\vec{r} = \int_C (\vec{F} \cdot \vec{T}) ds$$

(*) \vec{T} = unit vector tangent to C , at point P
 \rightarrow (*) $\vec{F} \cdot \vec{T}$ = tangential component of \vec{F} (at P)

- Vector line integrals, unlike scalar line integrals, depend on the direction along the curve

\rightarrow (*) Intuition: Travelling in a direction opposing the wind is more work (\rightarrow larger integral) than travelling in the same direction as the wind

\rightarrow (*) The "positive" direction is defined as the direction [orientation] along C ; negative = direction against

An oriented curve can be parametrized (expressed in the form $C = \vec{r}(t)$) for easier computation.

(!) The parametrization $\vec{r}(t)$ must travel in the same direction as the original curve.



Computing Vector Line Integrals

$$\int_C \vec{F} d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt^*$$

$$(C = \vec{r}(t) \quad (t_0 \leq t \leq t_1))$$

(*) Alt. Notation: $= \int_{t_0}^{t_1} (F_1 dx + F_2 dy + F_3 dz) dt$

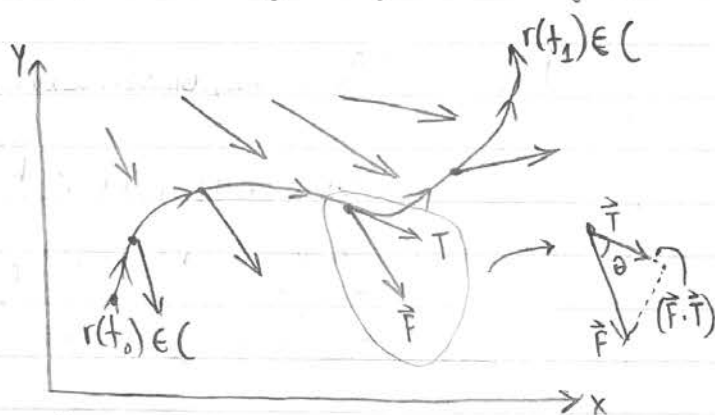


Figure: Visualization of an Oriented Curve $C = \vec{r}(t)$ in a Vector Field \vec{F} (2D)

Proof: $C = \vec{r}(t) \rightarrow \vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \quad ds = \|\vec{r}'(t)\| dt$

$$\rightarrow \int_C (\vec{F} \cdot \vec{T}) ds = \int_{t_0}^{t_1} \left(\vec{F} \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \right) \|\vec{r}'(t)\| dt = \int_{t_0}^{t_1} \vec{F} \cdot \vec{r}'(t) dt$$

Conservative Vector Fields

2/13/23
Lecture 15

Vector Line Integrals

Applications: A vector line integral can be used to calculate the work done by a particle moving along a curve C in vector field \vec{F} , i.e. $W = \int_C \vec{F} \cdot d\vec{r}$

Notations: Given an oriented curve C , we write $-C$ to denote the curve C in the opposite direction

$$(*) \quad \int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$$

Conservative Vector Fields (§17.3)

Definition: A vector field is path-independent if $\int_C \vec{F} \cdot d\vec{r}$ is constant for all curves C with the same start and endpoints, i.e. $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ (C_1 curve $P \rightarrow Q$, C_2 curve $P \rightarrow Q$)

Fundamental Theorem of Line Integrals

Given conservative vector field $\vec{F} = \nabla f$ for some scalar function f , any curve C [path r] from point P to point Q in the domain of \vec{F} , we can say:

$$\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P)$$

(Conservative vector fields are path-independent)

(*) Conversely: On an open, connected domain, all path-independent vector fields are conservative

Definition: A curve C is closed if its start and end points are the same (i.e. $P = Q$)

Notation (Line Integrals for Closed Curves): $\int_C \vec{F} \cdot d\vec{r}$ (C closed) = $\oint_C \vec{F} \cdot d\vec{r}$

Integrals of Closed Curves

Given a conservative vector field \vec{F} , the following expression is true for all closed curves C :

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

Conservative Vector Fields (cont.)

2/15/23
Lecture 11

Conservative Vector Fields

(!) Note: $(\text{curl}(\vec{F}) = \vec{0})$ does not necessarily imply conservativity in all cases

(*) \vec{F} cons. $\rightarrow \text{curl}(\vec{F}) = \vec{0} \forall \vec{F}$, but $\text{curl}(\vec{F}) = \vec{0} \rightarrow \vec{F}$ cons. only on simply connected domains





(*) $\oint_C \vec{F} \cdot d\vec{r} = 0$ also does not imply conservativity, unless proven for all closed curves

(*) Finding a potential function works regardless of domain

Domains

Simply Connected Regions

A region D is considered simply connected if it is connected (i.e. one single region, not multiple parts) and if every loop in D [any loop] can be contracted continuously within D to a single point (i.e. D has no holes).

Ex:   \mathbb{R}^2 ;   $(\mathbb{R}^2 - (0,0))$

• $\text{curl}(\vec{F}) = \vec{0}$ does imply \vec{F} is conservative if the domain of \vec{F} is simply-connected

(*) $\text{curl}(\vec{F}) = \vec{0}$ is true of all conservative vector fields, even in non-simply-connected domains; it is only true of only conservative v.f.s on simply-connected domains

Ex: $\text{curl}(\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle) = \vec{0}$, but it is not conservative

Intro to Surfaces

2/17/23
Lecture 17

Domains

- The domain $(\mathbb{R}^2 - (0,0))$ is not simply connected [\mathbb{R}^2 - a 1D figure]
- The domain $(\mathbb{R}^3 - (0,0,0))$ [\mathbb{R}^3 - a 1D figure] is simply connected (any loop around $(0,0,0)$ can just be pulled "up" the z -axis and then contracted, e.g.)
- The domain $(\mathbb{R}^3 - \{(x,y,z) | x,y=0\})$ [\mathbb{R}^3 - a line] is not simply connected
- Generally, any domain \mathbb{R}^n would require an $(n-2)$ D removal to stop being simply connected
- A disconnected domain of simply connected domains can be treated like a piecewise domain
 - Ex: $D = D_1 \cup D_2 \rightarrow \vec{F} = \nabla f, f = \begin{cases} f_1, & (p \in D_1) \\ f_2, & (p \in D_2) \end{cases}$
- A vector field can be conservative on some parts of its domain, but not others
 - Ex: If f ($\vec{F} = \nabla f$) is not defined on parts of the domain of F

Summary

$$\begin{array}{l} \vec{F} \text{ conservative} \iff \begin{cases} \vec{F} = \nabla f \\ \oint_C \vec{F} \cdot d\vec{r} = 0 \quad \forall C \\ \text{curl}(\vec{F}) = 0 \end{cases} \\ \text{(for simply connected domains)} \end{array}$$

Surfaces (§17.4)

Parametrized Surfaces

A parametrized surface is a set of points $S \subseteq \mathbb{R}^3$, which can be expressed in the form $G(u,v) = (x(u,v), y(u,v), z(u,v))$ for some region D in the uv -plane.

$$\begin{array}{c} \text{Diagram: } D \text{ in } uv\text{-plane} \rightarrow \text{Surface } S \text{ in } xyz\text{-space} \\ G(D) = S \quad \left(\begin{array}{l} \text{Parametrization of } S \\ \text{by } G \text{ in parameter domain } D \end{array} \right) \end{array}$$

Ex: Cylinder $(x^2 + y^2 = R^2) \rightarrow (x,y,z) = G(\theta, z)$

$$= (R \cos \theta, R \sin \theta, z), \quad D = \begin{cases} 0 \leq \theta \leq 2\pi \\ -\infty \leq z \leq \infty \end{cases}$$

Surface Integrals

2/2/23
Lecture 18

Example Surfaces

- Sphere $(x^2 + y^2 + z^2 = R^2) \rightarrow G(\theta, \phi) = (R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi)$ ($0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$)
- Graph of $f(x, y) \rightarrow G(x, y) = (x, y, f(x, y))$ ($-\infty \leq x, y \leq \infty$)

Tangent & Normal Vectors

Given surface S parametrized by $G(u, v) = (x(u, v), y(u, v), z(u, v))$, we define the vectors tangent to S at a point $P = (u_0, v_0)$:

$$\vec{T}_u(P) = \frac{\partial G}{\partial u}(u_0, v_0) = \left\langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right\rangle$$

$$\vec{T}_v(P) = \frac{\partial G}{\partial v}(u_0, v_0) = \left\langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right\rangle$$

(1) Significance: \vec{T}_u and \vec{T}_v span the tangent plane of S at P ; from then, we can define:

Normal Vector of a Surface

Given tangent vectors $\vec{T}_u(P)$ and $\vec{T}_v(P)$ of a surface S at a point P , we can find the normal vector to the tangent plane of S at P , normal to the surface S :

$$\vec{N} = \vec{N}(P) = \vec{N}(u_0, v_0) = \vec{T}_u(P) \times \vec{T}_v(P) \quad (\text{Normal to surface } S)$$

(*) T_u, T_v , and N do not necessarily need to be unit vectors (and T_u and T_v need not be orthogonal)

Surface Integrals

A parametrization G is regular at a point P if $\vec{N}(P) \neq \vec{0}$.

* $\iint_S 1 dS$ = surface area (S)

Surface Integrals

Given a surface S parametrized by $G(u, v)$ over some parameter domain, where G is continuously differentiable, 1:1, and regular. Then we can express the surface integral of a scalar function $f(x, y, z)$ over S as follows:

$$\iint_S f(x, y, z) dS = \iint_D f(G(u, v)) \|\vec{N}\| dA \quad (\text{Integral of } f \text{ over } S)$$

Vector Surface Integrals

2/24/23
Lecture 19

Surface Integrals for Vector Fields (§17.5)

Given a surface S , an orientation on S is a continuous choice of [direction of] unit normal vector $\hat{n}(P)$ for all points $P \in S$

(*) \exists two normal directions at all points

(*) Given an orientation $\hat{n}(P)$, the opposite orientation is denoted $-\hat{n}(P)$



Vector Surface Integrals

Given an orientable and connected surface S with orientation $\hat{n}(P)$ and parametrization $G(u, v)$ on domain $D \subset \mathbb{R}^2$ a vector field \vec{F} , the vector surface integral of F over S is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \hat{n}) dS = \iint_D (\vec{F}(G(u, v)) \cdot \vec{N}(u, v)) dA \quad [\vec{N} = \vec{T}_u \times \vec{T}_v]$$

Vector Surface Integrals

- The vector surface integral $\iint_S \vec{F} \cdot d\vec{S}$ represents the flux of \vec{F} through S , or the integral of the normal components of \vec{F} on S
- The vector surface integral is only defined for surfaces that are orientable, i.e. on which a continuous orientation $\hat{n}(P)$ can be defined
 - (*) The Möbius strip is an example of an unorientable surface

(*) Notes

$$\hat{n}(P) = \frac{\vec{N}}{\|\vec{N}\|} \quad [\text{normal vectors}]$$

$$\iint_S (\vec{F} \cdot (-\hat{n})) dS = -\iint_S (\vec{F} \cdot \hat{n}) dS \quad [\text{opposite orientation}]$$

Green's Theorem

2/27/23
Lecture 20

Green's Theorem (§18.1)

Definition: A curve C is simple if it has no self-intersections.

Notation: Given a region D in the xy -plane, ∂D denotes the boundary of D .



Green's Theorem (2D)

Given a region D with a simple closed curve boundary ∂D (oriented counterclockwise) and vector field \vec{F} with continuously differentiable components on D :

$$\text{Then: } \boxed{\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \int_{\partial D} \vec{F} \cdot d\vec{r}} \quad [\partial D: \vec{F} = \langle F_1, F_2 \rangle]$$

3/1/23
Lecture 21

↳ Can be used to convert hard-to-compute double integrals into line integrals (or vice versa)

Ex: $D = \text{unit circle} \rightarrow \partial D: \vec{r}(t) = \langle \cos(t), \sin(t) \rangle, 0 \leq t \leq 2\pi$

$$\rightarrow \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \int_0^{2\pi} (\vec{F} \cdot \vec{r}'(t)) dt$$

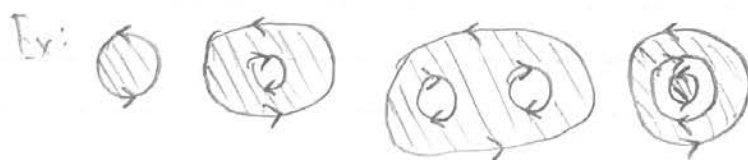


(*) Computing Area

$$\text{Area}(D) = \iint_D 1 dA \rightarrow [\vec{F} = \langle 0, x \rangle] = \int_{\partial D} x dy \text{ (e.g.)}$$

Domains w/ Multiple Boundaries

Given a domain D with boundary ∂D consisting of multiple simple closed curves C_1, \dots, C_n ($\partial D = C_1 + \dots + C_n$), the boundary orientation on each curve is such that the region D is always on the left when following the orientation direction.



(!) Green's Theorem still holds for such domains (can be expressed by decomposing/splitting D , e.g.)

Stokes' Theorem

3/3/23

Lecture 22

Green's Theorem (cont.)

Given the integral of vector field \vec{F} over closed curved boundary $\partial D = \partial D_1 + \partial D_2 + \dots + \partial D_n$ for non-overlapping regions D_1, \dots, D_n intersecting only on their boundaries [or not intersecting]:

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D_1} \vec{F} \cdot d\vec{r} + \dots + \oint_{\partial D_n} \vec{F} \cdot d\vec{r} \quad [\text{Additivity of Closed Curves}]$$

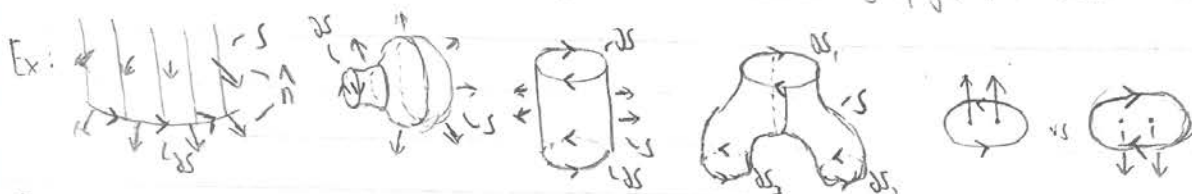
(*) ∂D should be composed of only simple closed curves (no subtracting single points, e.g.)

31/6/23

Lecture 23

Stokes' Theorem (§18.2)

Definition: Given a surface S oriented by \hat{n} with boundary ∂S , we define a boundary orientation of ∂S such that the surface S is on the left when following the orientation and "standing upright" in the direction of \hat{n} .



(*) A surface S with no boundary is called a closed surface $\rightarrow \partial S = \emptyset$. Ex: $\partial(\odot) = \emptyset$
(finite)



Stokes' Theorem

Given a surface S with oriented boundary ∂S , and vector field \vec{F} with continuously differentiable components ($\vec{F}(x, y, z)$) on S :

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

(*) Notes on Stokes' Theorem

- Green's Theorem represents a 2D instance of Stokes' Theorem
- If S is closed, then $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$
- Given two surfaces S_1, S_2 s.t. $\partial S_1 = \partial S_2$
 $\rightarrow \iint_{S_1} \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_{S_2} \text{curl}(\vec{F}) \cdot d\vec{S}$ [Only depends on the boundary]
- Can be used to convert hard surface integrals to line integrals (or vice versa)
- (*) Can also "chain" Stokes' Theorem (e.g. hard surface \rightarrow line \rightarrow easier surface)

Divergence Theorem

3/8/23
Lecture 24

(*) Notes on Stokes' Theorem (cont.)

- A boundary is defined (loosely) as the edge of a surface
- Alternate formulation of Stokes' Theorem: If a vector field \vec{F} can be expressed as the curl of another vector field G (i.e. $F = \text{curl}(G)$), then:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{G} \cdot d\vec{r}} \quad [F = \text{curl}(G)] \quad (\text{Stokes' Theorem})$$

Reminder: given an oriented curve C , all surfaces S with $\partial S = C$ will share the same integral of curl under Stokes' Theorem

* In the case that $\text{curl}(\vec{F}) = 0$ [e.g. for conservative vector fields], $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all curves $C = \partial S$ for some surface S .


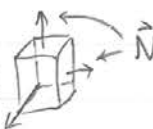
* " $C = \partial S$ for some surface S " only holds true for all curves C on simply-connected regions

Divergence Theorem (§18.3)

3/10/23
Lecture 25

Definition: Given a 3D region W , the boundary ∂W of W is the surface that encloses W

Ex: $W =$  $\rightarrow \partial W =$  $W: x^2 + y^2 + z^2 \leq 4$
 \downarrow
 $\partial W: x^2 + y^2 + z^2 = 4$

$W =$  $\rightarrow \partial W =$ 

Divergence Theorem

Given 3D region W in \mathbb{R}^3 enclosed by surface ∂W , where ∂W is piecewise smooth and oriented with normal vectors pointing outside W , then for any vector field $F(x, y, z)$ with partial derivatives continuous in W :

$$\boxed{\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \text{div}(\vec{F}) dV} \quad [\text{Divergence Theorem}]$$

(*) The symbol \oiint can be used to denote integration over a closed surface, i.e. $\iint_S \vec{F} \cdot d\vec{S} \rightarrow \oiint \vec{F} \cdot d\vec{S}$

3/13/22

lecture 26

(*) 3D Regions w/ Multiple Boundaries

Given a 3D region W , if ∂W is composed of multiple surfaces (i.e. $\partial W = S_1 + S_2 + \dots + S_n$), then we orient each surface S_i such that the normal vector \hat{n} points away from W .

Ex: $W = \{1 \leq x^2 + y^2 + z^2 \leq 2\} \rightarrow W =$    [Orientation]