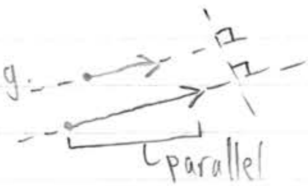
Vector Properties - 9/23/22

Vectors in the plane (XY) defined as: $\vec{v} = \{(x,y) \in \mathbb{R}^2 \mid x, y \in \mathbb{R}\}$

Properties: Initial point ("start" of vector)
Terminal point ("end")
Magnitude (length $= \|\vec{v}\|$)
Direction

Position vector - vector that starts at the origin

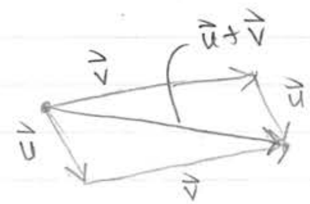
Vectors are parallel if they are on parallel lines, e.g.
Opposite direction vectors can be parallel



Equivalent vectors ($\equiv$) have the same magnitude & direction

$\langle\,\rangle$ for vectors, $(\,)$ for points

$\|-\vec{v}\| = \|-1\| \cdot \|\vec{v}\| = 1 \cdot \|\vec{v}\| = \|\vec{v}\|$ $\|\lambda\vec{v}\| = |\lambda| \cdot \|\vec{v}\|$



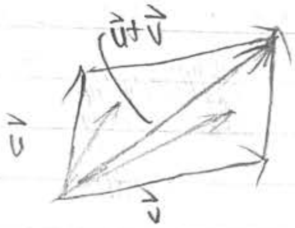
Vector Algebra

9/26/22
Lecture 2

Given $\vec{v}$, $\lambda \in \mathbb{R} \longrightarrow$ if $\lambda > 0$, $\lambda\vec{v}$ same direction as $\vec{v}$
if $\lambda < 0$, $\lambda\vec{v}$ opposite direction as $\vec{v}$

Given $\vec{v} = \langle a_1, b_1 \rangle$ and $\vec{u} = \langle a_2, b_2 \rangle \longrightarrow \vec{v} + \vec{u} = \langle a_1 + a_2, b_1 + b_2 \rangle$
$\vec{v} - \vec{u} = \langle a_1 - a_2, b_1 - b_2 \rangle$
$\lambda\vec{v} = \langle \lambda a_1, \lambda b_1 \rangle$

Given $\vec{w} = \alpha\vec{u} + \beta\vec{v}$, $\vec{w}$ is a <u>linear combination</u> of $\vec{u}$ and $\vec{v}$
$\alpha, \beta \in \mathbb{R}$



linear combination of $\vec{u}$ and $\vec{v}$

Given $\vec{w} = \alpha\vec{u} + \beta\vec{v}$, $0 \le \alpha, \beta \le 1$

every $\vec{w}$ is contained within the parallelogram resulting from $\vec{u}$ and $\vec{v}$

Given $\alpha\vec{v} + \beta\vec{w} = \vec{u}$ and $\vec{v}, \vec{w}, \vec{u}$ are known, α and β can be found via system of equations

$$\alpha\langle v_1, v_2 \rangle + \beta\langle w_1, w_2 \rangle = \langle u_1, u_2 \rangle$$

$$\rightarrow \begin{cases} \alpha v_1 + \beta w_1 = u_1 \\ \alpha v_2 + \beta w_2 = u_2 \end{cases}$$

Unit vector - vector of length 1 $\quad \frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector
$\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ are the standard basis of the XY plane

9/27/22
Disc. 1

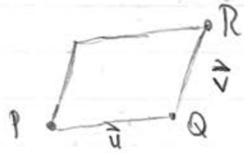
Vector Algebra Review

$P = (p_1, \dots, p_n)$
$Q = (q_1, \dots, q_n)$ $\longrightarrow$ $\boxed{\overrightarrow{PQ} = \vec{v} = \langle q_1 - p_1, \dots, q_n - p_n \rangle}$

Two vector operations defined - addition & scalar multiplication

Addition

$\vec{u} = \langle u_1, \dots, u_n \rangle$
$\vec{v} = \langle v_1, \dots, v_n \rangle$ $\longrightarrow$ $\boxed{\vec{u} + \vec{v} = \langle u_1 + v_1, \dots, u_n + v_n \rangle}$



$$\vec{u} + \vec{v} = \langle q_1 - p_1, \dots, q_n - p_n \rangle + \langle r_1 - q_1, \dots, r_n - q_n \rangle$$
$$= \langle r_1 - p_1, \dots, r_n - p_n \rangle = \overrightarrow{PR}$$

Scalar multiplication

$c \in \mathbb{R}$ $\quad$ $\vec{u} = \langle u_1, \dots, u_n \rangle \longrightarrow \boxed{c\vec{u} = \langle cu_1, \dots, cu_n \rangle}$

Magnitude

$\vec{u} = \langle u_1, \dots, u_n \rangle \longrightarrow \boxed{\|\vec{u}\| = \sqrt{u_1^2 + \dots + u_n^2}}$

Intro to 3D

9/28/22
Lecture 3

(set of points on the
Surface of a sphere w/ center A : $\{P \in \mathbb{R}^3,\ AP = \rho,\ \rho\ \text{cst.}$ (where P is a point on the surface of the sphere)

ρ = rho

$\{(x,y,z) \mid (x-a)^2+(y-b)^2+(z-c)^2 = r\}$

Surface of a cylinder $\{(x,y,z) \mid (x-a)^2+(y-b)^2 = r^2\}$ where (a,b) is the center & r is the radius

Add inequalities to sets of points to "cut" shapes (e.g. $y \geq 0$, $x+y > 1$)

Intro to Parametric Equations

9/30/22
Lecture 4

Vectors $\vec{u}$ and $\vec{v}$ parallel if $\vec{u}, \vec{v} \neq \vec{0}$ and there exists λ, s.t. $\vec{u} = \lambda \vec{v}$ for some $\lambda \in \mathbb{R}$

$\mathbb{R}^* = \mathbb{R}$ (non-zero)

For a line with fixed point P_0, unknown point P, and parallel vector $\vec{u}$, there exists $t \in \mathbb{R}$ such that $\overrightarrow{P_0P} = t\vec{u}$

Parametric equation:
$\vec{R}(t) = \vec{R}_0 + t\vec{u}$
$-\infty \leq t \leq \infty$
$\vec{u}$ = parallel vector
$\vec{R}_0$ = initial point

$\overrightarrow{P_0P} = \overrightarrow{P_0O} + \overrightarrow{OP}$

P = arbitrary point at some t

$\overrightarrow{OP} = \langle x(t), y(t), z(t) \rangle$

$\rightarrow \overrightarrow{OP_0} = \langle x(t_0), y(t_0), z(t_0) \rangle$
constant vector

$\longrightarrow \overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle$

$\longrightarrow \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c} = t$ (symmetric form)

$$\begin{cases} x = x_0 + tu_0 \\ y = y_0 + tu_1 \\ z = z_0 + tu_2 \end{cases}$$

Dot Product $\bullet$

10/3/22
Lecture 5

Given $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, $\boxed{\text{the dot product } \vec{v} \bullet \vec{w} = v_1w_1 + v_2w_2 + v_3w_3}$

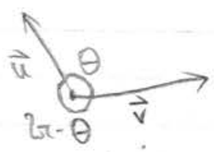
$\vec{a} = \langle a_1, \ldots, a_n \rangle$
$\vec{b} = \langle b_1, \ldots, b_n \rangle \rightarrow \vec{a} \bullet \vec{b} = a_1b_1 + \ldots + a_nb_n$

$\vec{v} \bullet \vec{v} = v_1v_1 + \ldots + v_nv_n$
$= v_1^2 + \ldots + v_n^2 = \|\vec{v}\|^2$

• Properties

1. $\vec{0} \bullet \vec{v} = \vec{v} \bullet \vec{0} = 0$
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
3. $(\lambda\vec{v}) \cdot \vec{u} = \vec{v} \cdot (\lambda\vec{u}) = \lambda(\vec{v} \cdot \vec{u})$
4. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
5. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

dot product results in a number/scalar
- cannot take dot product of multiple vectors (≥ 3)



$(0 \leq \theta \leq \pi)$

$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta \qquad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}}$$

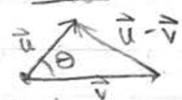
$\downarrow$

$$\boxed{\vec{u} \perp \vec{v} \rightarrow \vec{u} \cdot \vec{v} = 0}$$

(acute) $0 \leq \theta < \pi/2 \rightarrow \cos\theta > 0$
(obtuse) $\pi/2 < \theta \leq \pi \rightarrow \cos\theta < 0$

$\widehat{\vec{u}, \vec{w}}$ = angle between $\vec{u}$ and $\vec{w}$

<u>Proof</u>



via Law of Cosines

$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|u\|\|\vec{v}\| \cos\theta$
$\hookrightarrow = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v}$
$\longrightarrow \cancel{\|\vec{u}\|^2} - 2(\vec{u} \cdot \vec{v}) + \cancel{\|\vec{v}\|^2} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$
$\rightarrow -2(\vec{u} \cdot \vec{v}) = -2\|\vec{u}\|\|\vec{v}\| \cos\theta$

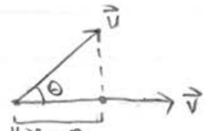
Vector Operations

Dot Product

Given $\vec{u} = \langle u_1, \dots, u_n \rangle$, $\vec{v} = \langle v_1, \dots, v_n \rangle$ $\rightarrow \vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n = \sum_{j=1}^{n} v_j w_j$

$= \|\vec{u}\| \|\vec{v}\| \cos\theta$, θ = angle between $\vec{u}$ and $\vec{v}$

Projection of $\vec{u}$ onto $\vec{v}$ $= \text{proj}_{\vec{v}} \vec{u} = (\|\vec{u}\| \cos\theta) \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \cdot \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{unit vector in direction of } \vec{v}}$



$\|\vec{u}\| \cos\theta$ = component of $\vec{u}$ along $\vec{v}$

$\vec{u}$ and $\vec{v}$ orthogonal if $\vec{u} \cdot \vec{v} = 0$

Cross Product

Cross product strictly 3D (not 2D, 4D, etc.)

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| (\sin\theta) \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$

$\rightarrow \vec{u} \times \vec{v}, \hat{n} \perp \vec{u}, \vec{v}$

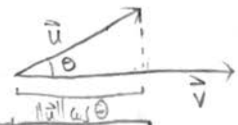
$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| |\sin\theta|$

Projection & Cross Product

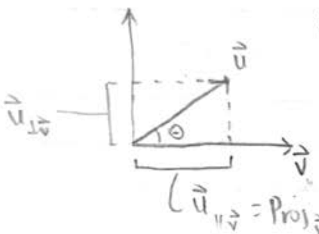
10/5/22
Lecture 6

Projection of $\vec{u}$ onto $\vec{v}$ = $\mathrm{Proj}_{\vec{v}}\vec{u} = (\|\vec{u}\|\cos\Theta)\frac{\vec{v}}{\|\vec{v}\|}$

$$= (\|\vec{u}\|)\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$



$$\boxed{\mathrm{Proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = (\|\vec{u}\|\cos\Theta)\frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}}$$



$\vec{u}_{\perp\vec{v}}$ = orthogonal projection of $\vec{u}$ onto $\vec{v}$

$\vec{u} = \vec{u}_{\parallel\vec{v}} + \vec{u}_{\perp\vec{v}} \rightarrow \vec{u}_{\perp\vec{v}} = \vec{u} - \vec{u}_{\parallel\vec{v}} = \vec{u} - \mathrm{proj}_{\vec{v}}\vec{u}$

Matrices: n rows × n columns
n^{th} order [square] matrix = $M_{n\times n}$

$$\underset{m\times n \text{ matrix}}{M_{m\times n}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix} \quad a_{ij} \rightarrow i = \text{row \#},\ j = \text{col. \#}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$ (for 2nd order square matrices)

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13}$$ (coefficient of $a_{1n} = (-1)^{\overbrace{i+j}^{=\text{row \#} + \text{col \#}}}$)

Cross product $\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k}$

$\hat{i}, \hat{j}, \hat{k}$ = standard basis vectors $\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$

$$\boxed{\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\Theta}$$

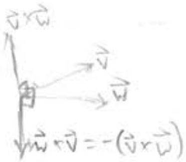
Properties of Cross Product

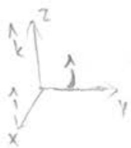
10/7/22
Lecture 7

Given non-zero non-parallel $\vec{v}$ and $\vec{w}$,
$\vec{v}\times\vec{w}$ is the <u>unique</u> vector satisfying:

a. $\vec{v}\times\vec{w} \perp \vec{v}, \vec{w}$
b. $\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$
c. $\{\vec{v}, \vec{w}, \vec{v}\times\vec{w}\}$ form a right handed system

$\vec{v}\times\vec{w} = -(\vec{w}\times\vec{v})$



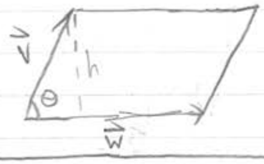


<u>Properties of Cross Product</u>
1. $\vec{v}\times\vec{w} = -(\vec{w}\times\vec{v})$
2. $\vec{v}\times\vec{v} = \vec{0}$
3. $\vec{v}\times\vec{w} = \vec{0} \rightarrow \vec{v} = \lambda\vec{w}$ or $\vec{v}$ or $\vec{w} = \vec{0}$
4. $(\lambda\vec{v})\times\vec{w} = \vec{v}\times(\lambda\vec{w}) = \lambda(\vec{v}\times\vec{w})$
5. $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$

Regarding the standard basis vectors $\hat{i}, \hat{j}, \hat{k}$:

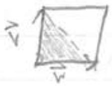
$\hat{i}\times\hat{j} = \hat{k} \quad \hat{k}\times\hat{i} = \hat{j} \quad \hat{j}\times\hat{k} = \hat{i}$

Right hand rule: i (pointer), j (middle), k (thumb)



<u>Area of a parallelogram</u>

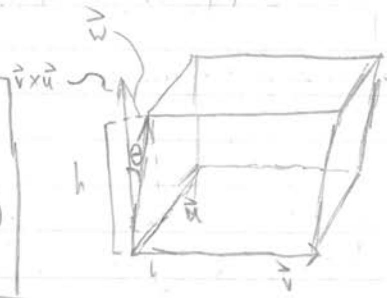
$A = \text{base}\cdot\text{height}$
$= \|\vec{w}\|\cdot h$
$= \|\vec{w}\|\cdot\|\vec{v}\|\sin\theta$
$= \|\vec{w}\times\vec{v}\|$



Area of triangle
$= \frac{1}{2}(\|\vec{v}\times\vec{w}\|)$

<u>Area of a parallelopiped</u>

$A = \text{base}\cdot\text{height}$
$= \|\vec{v}\times\vec{u}\|\cdot h$
$= \|\vec{v}\times\vec{u}\|\cdot\|\vec{w}\|\cos\theta$
$= (\vec{v}\times\vec{u})\cdot\vec{w}$



10/10/22
Lecture 8

Cross Product and 3D Geometry

<u>Scalar triple product</u>: $\vec{w}\cdot(\vec{u}\times\vec{v}) = (\vec{u}\times\vec{v})\cdot\vec{w}$

$\vec{w}\cdot(\vec{u}\times\vec{v}) = \left|\det\begin{pmatrix}\vec{u}\\ \vec{v}\\ \vec{w}\end{pmatrix}\right| \rightarrow$

Volume V of parallelopiped spanned by $\vec{u},\vec{v},\vec{w}$.

$$V = \vec{w}\cdot(\vec{u}\times\vec{v}) = \left|\det\begin{pmatrix}\vec{u}\\ \vec{v}\\ \vec{w}\end{pmatrix}\right|$$

Area A of parallelogram spanned by $\vec{u}, \vec{v}$

$$A = \|\vec{u}\times\vec{v}\| = \left|\det\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix}\right|$$

($\vec{u}$ and $\vec{v}$ must only have 2 components)

$\|\vec{v}\times\vec{w}\|^2 = \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2$ (Theorem)

$\hookrightarrow \|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$

Given plane with points P_0, P and vector $\vec{n}$ orthogonal to the plane, $\overrightarrow{P_0P}\cdot\vec{n} = 0$ for all P_0, P (vector $\overrightarrow{P_0P}$ orthogonal to $\vec{n}$)

$P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, $\vec{n} = \langle a, b, c\rangle$

$\rightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$\rightarrow ax + by + cz - d = 0$ (equation of a plane)

$d = ax_0 + by_0 + cz_0$

$\langle a, b, c\rangle$ is vector perpendicular to the plane

Equation of a plane passing through $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is described by:

Vector form: $\vec{n}\cdot\langle x, y, z\rangle = d$ $\quad \langle x,y,z\rangle = \overrightarrow{P_0P}$

Scalar form: $ax + by + cz = d$

$d = ax_0 + by_0 + cz_0$

$\vec{n}\cdot\overrightarrow{P_0P} = 0$ for random point $P = (x, y, z)$ on the plane

$\rightarrow (\overrightarrow{P_0O} + \overrightarrow{OP})\cdot\vec{n} = \vec{0}$

$a\times(b\times c)$ contained on plane spanned by $\vec{b}$ and $\vec{c}$

Planes

10/11/22
Disc 3

Planes

$\vec{u}\cdot\vec{v}=0$

$\vec{n}\cdot\langle x,y,z\rangle=0$

→ Given position vector $\vec{p}$ from the origin to a point P_0 on the plane, we can say: $\vec{n}\cdot(\langle x,y,z\rangle-\vec{p})=0$

Planes and 3D Geometry

10/12/22
Lecture 9

If $\vec{n}$ is normal to a plane, $\lambda\vec{n}$ $(\lambda \neq 0)$ will also be normal

Parallel planes share a common normal vector $\vec{n}$,
e.g. $ax+by+cz=d_1 \parallel ax+by+cz=d_2$ (a b c shared, d different)

Given points P, Q, R on a plane: $\vec{PQ} \times \vec{PR}$ = normal vector of the plane (if $\vec{PQ}, \vec{PR}$ not parallel)
(3 points needed to define a plane)

Given plane $ax+by+cz=d$ and line $(x_0, y_0, z_0) + t\langle x_1, y_1, z_1\rangle = \langle x(t), y(t), z(t)\rangle$:
the point of intersection can be found by substituting (x, y, z) for $(x(t), y(t), z(t))$ & solving for t (t @ point of intersection)

e.g. $ax+by+cz=d \rightarrow ax(t)+by(t)+cz(t)=d$
$\rightarrow a(x_0+x_1t)+b(y_0+y_1t)+c(z_0+z_1t)=d$ (solve for t)

Parametric Equations

10/14/22
Lecture 10
Week 3

Parametric equation of a line:

$\vec{OP} = \vec{OP_0} + t\vec{v}$ $\quad \vec{v} = \lambda\langle 1, m\rangle$

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases} \qquad t\vec{v} = \vec{P_0P}$$

OP_0 akin to a translation vector

Parametric equation: $\vec{OP}(t) = \langle x(t), y(t)\rangle,\ a \le t \le b$

$\vec{OP}(t) = \vec{c}(t) = \langle x(t), y(t)\rangle$

where $\vec{c}(t)$ is the parametricization of the curve C

(t) is simply <u>one possible parameter</u> - the parameter can be anything ("parameter", hence "parametric")

Orientation - "direction" a graphed function is travelling e.g.

$-t \to +t$
(dir. of increasing t)

- Arguably arbitrary

Parametric equations can be converted via expressing $y(t)$ as a function of $x(t)$ e.g. $\langle t, t^2\rangle \to y = x^2$
- Loses orientation information (e.g. no more negative $t \to$ positive t)

Given circle w/ center (x_0, y_0) and radius r, parametric equation of the circle:

$\vec{c}(\theta) = (x_0, y_0) + \langle r\cos\theta, r\sin\theta\rangle$

is equal to $(x-x_0)^2 + (y-y_0)^2 = r^2$

If we define the center of a circle as P_0 and M as a point on the circle, the parametric equation can be derived from the eq.:

$OM = OP_0 + P_0M = \langle x_0, y_0\rangle + \langle r\cos t, r\sin t\rangle$

Ellipses: $\frac{x^2}{A} + \frac{y^2}{B} = 1 \longrightarrow (x(\theta), y(\theta)) = (A\cos\theta, B\sin\theta)$

A single parametric equation will yield a single curve; however, a single curve can have multiple parametrizations

e.g. $\vec{c}(t) = \langle t, t^2\rangle = \langle t^2, t^4\rangle = \langle 2t, 4t^2\rangle$ etc.

Standard functions: $f: \mathbb{R} \mapsto \mathbb{R}$
$x \mapsto f(x)$
scalar $\mapsto$ scalar

Vector-valued functions: $f: \mathbb{R} \mapsto \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ (2D $\mathbb{R}$ space), can be n-D
$t \mapsto \langle x(t), y(t)\rangle$
scalar $\mapsto$ vector

Properties of a function: Domain: values for which f is defined (subset of $\mathbb{R}$) $\to$ intersection of domain of components

Co-domain: set that contains all output values

Calculus of Vector-Valued Functions

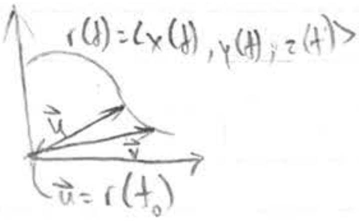
10/17/22
Lecture 12
Week 4

$\boxed{\text{Parametrization of the intersection of two surfaces can be obtained w/ a system of equations by expressing each variable } x, y, z \text{ as functions of } t}$

e.g. $\begin{cases} x^2 - y^2 = z - 1 \\ x^2 + y^2 = 4, \; y \geq 0 \end{cases}$

$\to x = t \to y = \sqrt{4 - x^2} = \sqrt{4 - t^2}$

$\to z = x^2 - y^2 + 1 = t^2 - (\sqrt{4 - t^2})^2 + 1$



A vector-valued function $r(t) \to \vec{u}$ as $t \to t_0$ if $\lim_{t \to t_0} \|\vec{r}(t) - \vec{u}\| = 0$

$$\to \boxed{\lim_{t \to t_0} r(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle}$$

Derivatives of Vector-Valued Functions:

$$\lim_{h \to 0} \left(\frac{r(t+h) - r(t)}{h} \right) = r'(t) = \frac{d}{dt} r(t)$$

$$\boxed{r'(t) = \frac{d}{dt} r(t) = \langle \frac{d}{dt} x(t), \frac{d}{dt} y(t), \frac{d}{dt} z(t) \rangle}$$

Differentiation Rules

1) $(r_1(t) + r_2(t))' = r_1'(t) + r_2'(t)$
2) $(\lambda r(t))' = \lambda r'(t), \; \lambda \in \mathbb{R}$
3) $\frac{d}{dt}(f(t) \cdot r(t)) = f'(t) \cdot r(t) + f(t) \cdot r'(t)$
4) $\frac{d}{dt} r(f(t)) = f'(t) \cdot r'(f(t))$

for vector-valued function $r(t)$; scalar function $f(t)$

Derivative of Dot Product and Cross Product

* $\frac{d}{dt}(r_1(t) \cdot r_2(t)) = r_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t)$
* $\frac{d}{dt}(r_1(t) \times r_2(t)) = r_1'(t) \times r_2(t) + r_1(t) \times r_2'(t)$

Derivative (geometric interpretation) - akin to a tangent vector to a curve at a given point

$\to$ eq. of tangent line @ $t = t_0$:
$c(t) = r(t_0) + t r'(t_0)$

Line tangent to $r(t)$ at $r(t_0)$:

$$\boxed{c(t) = \underbrace{r(t_0)}_{\text{point}} + t \underbrace{r'(t_0)}_{\text{orthogonal direction vector}}}$$

Calculus of Vector-Valued Functions

10/19/2
Lecture 1
Week 4

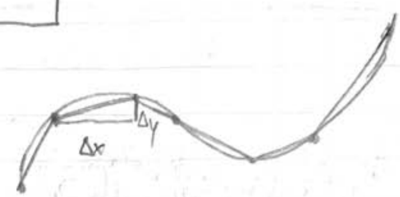
$$\boxed{\begin{array}{l}\text{Line tangent to } r(t) \text{ at } r(t_0): \\ c(t) = \underbrace{r(t_0)}_{\text{initial point}} + t\,\underbrace{r'(t_0)}_{\text{tangent direction vector}}\end{array}}$$

$r(t) \perp \frac{d}{dt} r(t)$ for spheres

$$\boxed{\begin{array}{c}\text{Integration of Vector-Valued Functions} \\ \int_a^b r(t)\,dt = \left\langle \int_a^b x(t)\,dt,\ \int_a^b y(t)\,dt,\ \int_a^b z(t)\,dt \right\rangle \\ \text{where } r(t) = \langle x(t), y(t), z(t) \rangle\end{array}}$$

<u>Integration Properties</u>

- $\int_a^b r(t) = R(b) - R(a)$
- $\frac{d}{dt}\left(\int_a^b r(t)\right) = r(t)$



$$\boxed{\begin{array}{l}\text{Given arc } r(t) \text{ with length } \lambda \text{ from } t_1 \text{ to } t_2: \\ \lambda = \int_{t_1}^{t_2} \|r'(t)\|\,dt = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt\end{array}}$$

Arc Length Parameterization

10/24/22
Lecture 13
Week 5

$s(t)$ = arc length from t_0 to variable t = <u>curvilinear absisce</u>

$\rightarrow \frac{ds(t)}{dt} = \boxed{\|r'(t)\| = \text{speed at time } t}$

$s = s(t) = \int_a^t \|r'(u)\|\,du = g(t)$ $\rightarrow$ e.g. $s(t) = 2\sqrt{2}\,t$ $\rightarrow t = \frac{s(t)}{2\sqrt{2}} = g^{-1}(s)$

Arc Length Parametrization:
alt. formulation of a curve as a function of the arc length

$\rightarrow$ if g is invertible, $g^{-1}(g(t)) = g^{-1}(s) = t$

$\rightarrow \boxed{r(t) = r(g^{-1}(s)) = \langle x(g^{-1}(s)), y(g^{-1}(s)), z(g^{-1}(s)) \rangle \text{ = Arc Length Parametrization of } r(t)}$

$\boxed{T = \frac{r'(t)}{\|r'(t)\|} = \text{unit tangent vector of } r(t)}$

$\frac{r'(t_0)}{\|r'(t_0)\|}$ = unit tangent vector @ t_0

Given arc length parametrization $r(s)$ and unit tangent T, the curvature of the underlying curve is:

$\boxed{K(s) = \left\|\frac{dT}{ds}\right\| \text{ (Curvature)}}$ $\longrightarrow$ $T'(t) = \frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt}$

= rate of change of the direction of the unit tangent vector

$= \frac{dT}{ds} \cdot v(t)$ $\quad v(t) = \text{speed} = \|r'(t)\| = \frac{ds}{dt}$

$\rightarrow \frac{dT}{ds} = \frac{T'(t)}{v(t)}$

<u>Curvature</u>

$$\boxed{K(t) = \left\|\frac{dT}{ds}\right\| = \frac{1}{v(t)}\left\|\frac{dT}{dt}\right\|}$$

$T = \frac{r'(t)}{v(t)}$

Arc Length Parameterization Review

10/25/22
Disc 5

Arc length parametrization: a parametrization for which speed is always 1 $r_1(s)$ s.t. $\|r_1'(s)\| = 1$

Given $r(t)$ w/ nonzero $r'(t)$, we can find the arc length parametrization:

1. Find $s = g(t) = \int_a^t \|r'(u)\|\,du$ (s = arc length from time a to time t)
2. Solve for $t = g^{-1}(s) \rightarrow r_1(t) = r_1(g^{-1}(s)) = r_1(s)$

For arc len. param.: $\boxed{\kappa(s) = \left\|\frac{d\vec{T}}{ds}\right\|}$, where $\vec{T}$ = unit tangent vector $\vec{T}(s) = r'(s)$ $\left(\vec{T}(t) = \frac{r'(t)}{\|r'(t)\|}\right)$

ds represents speed: arc. len. param. holds speed at 1, providing intrinsic curvature of curve (independent of speed)

General parameterization: $\boxed{\kappa(t) = \frac{\|d\vec{T}/dt\|}{v(t)} = \frac{\|r'(t) \times r''(t)\|}{v(t)^3}}$ $(v(t) = \text{velocity} = \|r'(t)\|)$

$v(t) = 1$ if using arc length param.

κ = kappa

For plane curve $r(t) = \langle x(t), y(t) \rangle$: $\kappa(t) = \frac{|x'y'' - x''y'|}{\left(\sqrt{x'^2 + y'^2}\right)^3}$

$$\boxed{\kappa(t) = \frac{\left\|\frac{d\vec{T}}{dt}\right\|}{v(t)}}$$

Given curve r, we can find velocity $= v(t) = r'(t)$ and acceleration $= a(t) = r''(t)$

$$\boxed{\begin{aligned}\vec{a} &= a_{\vec{T}}\vec{T} + a_{\vec{N}}\vec{N} \\ &= v'\vec{T} + \kappa v^2\vec{N}\end{aligned}}$$

(T, N are tangential & normal components, respectively)
($a_{\vec{T}}, a_{\vec{N}}$ coefficients)

Plane of curvature: plane defined by $\vec{T}$ and $\vec{N}$

Curvature

10/26/22
Lecture 14
Week 5

$$\kappa(s) = \kappa(t) = \left\| \frac{1}{v(t)} \cdot \frac{d\vec{T}}{dt} \right\| = \frac{1}{v(t)} \left\| \frac{d\vec{T}}{dt} \right\|$$

Curvature of a line $= 0$
Curvature of circle (radius $= r$) $= \frac{1}{r}$

$$\kappa(t) = \frac{1}{v(t)} \left\| \frac{d\vec{T}}{dt} \right\| = \frac{\|r'(t) \times r''(t)\|}{v(t)^3} = \left\| \frac{d\vec{T}}{ds} \right\|$$

$$\hookrightarrow = \frac{1}{v(t)} \|T'(t)\|$$

$$\boxed{\begin{array}{c}\text{Principal normal} = \text{unit vector tangent to } \vec{T} \\ N = \frac{1}{\kappa} \cdot \frac{dT}{ds} = \frac{\frac{dT}{ds}}{\left\|\frac{dT}{ds}\right\|} = \frac{T'(t)}{\|T'(t)\|}\end{array}}$$

Principal normals are unique

Given $r(t) = \langle x(t), y(t) \rangle$:

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}}$$

Given graph of $y = f(x)$, curvature of $(x, f(x))$:

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

Curvature κ of ellipse $r(t) = \langle a\cos t, b\sin t \rangle$ or $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$:

$$\kappa(t) = \frac{ab}{(b^2\cos^2 t + a^2\sin^2 t)^{3/2}}$$

Frenet Frame

10/28/22
Lecture 15
Week 5

$\boxed{\vec{T} \perp \frac{d\vec{T}}{ds}}$ Proof: $\|\vec{T}\| = 1 \to \vec{T}\cdot\vec{T} = 1 \to \frac{d}{ds}(\vec{T}\cdot\vec{T}) = \frac{d}{ds}(1) = 0$
$\to \frac{d\vec{T}}{ds}\cdot\vec{T} = 0 \to \vec{T} \perp \frac{d\vec{T}}{ds}$

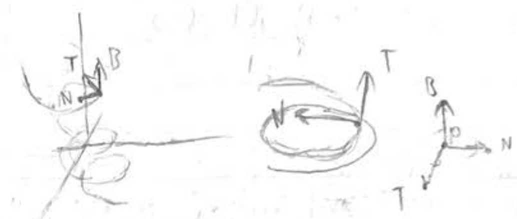
$$N = \frac{\frac{d\vec{T}}{ds}}{\|\frac{d\vec{T}}{ds}\|} = \frac{1}{K}\cdot\frac{d\vec{T}}{ds} = \boxed{\frac{dT}{dt}\cdot\frac{1}{\|\frac{dT}{dt}\|} = \vec{N}}$$

Proof: $\frac{\frac{dT}{ds}}{\|\frac{dT}{ds}\|} = \frac{\frac{dT}{dt}\cdot\frac{dt}{ds}}{\|\frac{dT}{dt}\cdot\frac{dt}{ds}\|} = \frac{\frac{dT}{dt}}{\|\frac{dT}{dt}\|}$

Frenet Frame

Given unit tangent T and principal normal N, we define binormal vector $\vec{B}$:

$$\boxed{\vec{B} = \vec{T} \times \vec{N}}$$



* $\vec{B} \perp \vec{T}, \vec{N}$
* $\|\vec{B}\| = 1 \ (= \|\vec{T}\| = \|\vec{N}\|)$
* T, N, B form a right-handed system

Osculating plane: plane spanned by $\vec{T}, \vec{N}$

Circle of curvature (osculating circle): At point P, C of C is the circle on the osculating plane, tangent to the curve at P w/ the same curvature as the curve at P with its center on the inside of the curve with radius $\rho(M)$ $\left(\rho(M) = \frac{1}{K(M)}\right)$

* Center of curvature: At point P, center of curvature is the center of the osculating circle at P

Given Frenet frame (T, N, B): $\frac{dB}{ds} = \frac{d}{ds}(\vec{T}\times\vec{N}) = \vec{T}\times\frac{d}{ds}\vec{N}$ $\quad \frac{dB}{ds} \perp B$

$\frac{dT}{ds} \parallel \vec{N} \to \frac{dT}{ds}\times N = 0$

$\to \frac{dB}{ds} \perp T,\ B \to \frac{dB}{ds} \parallel \vec{N}$

$\to \frac{dB}{ds} = -\tau\vec{N} \to \tau = -\left(\frac{dB}{ds}\cdot\vec{N}\right)$

> Given $\vec{B} = \vec{T}\times\vec{N}$, we define torsion function τ:
>
> $$\tau = -\frac{dB}{ds}\cdot\vec{N}$$

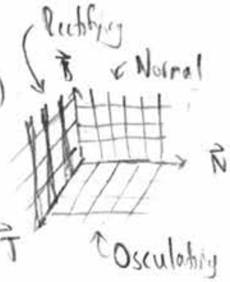
Torsion & Acceleration

Stanley Li
10/31/22
Lecture 26
Week 6

Given $\vec{B} = \vec{T} \times \vec{N}$, we define torsion function:

$$\boxed{\tau = -\frac{dB}{ds} \cdot \vec{N}}$$

Frenet Frame Planes: Osculating plane (T, N)
Rectifying plane (T, B)
Normal plane (B, N)



Torsion (τ): twist (change in direction) of rectifying plane at each point on the curve

Given $r(t) = \langle x(t), y(t), z(t) \rangle$:
$$\boxed{\tau(t) = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{\|v \times a\|^2} \quad (v \times a \neq 0)}$$

Given $r(t) = \langle x(t), y(t), z(t) \rangle$: $v(t) = \frac{dr}{dt}(t) = \frac{dr}{ds}(t) \cdot \frac{ds}{dt}$

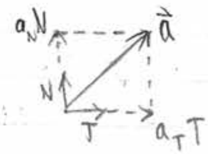
Velocity: $v(t) = r'(t)$
Acceleration: $a(t) = v'(t) = r''(t)$
Speed: $v(t) = \|v(t)\| = \|r'(t)\|$

$$\rightarrow a(t) = \frac{dv}{dt}(t) = \frac{d}{dt}\left(\frac{dr}{ds}(t) \cdot \frac{ds}{dt}\right) = \frac{d}{dt}\left(T \cdot \frac{ds}{dt}\right) = \frac{d^2s}{dt^2} \cdot T + \frac{ds}{dt}\frac{dT}{dt}$$

$$= \frac{d^2s}{dt^2} T + \frac{ds}{dt} \cdot \frac{dT}{ds} \cdot \frac{ds}{dt} = \underbrace{\frac{d^2s}{dt^2}}_{a_T \text{ (tangential component)}} \vec{T} + \underbrace{K \cdot \left(\frac{ds}{dt}\right)^2}_{a_N \text{ (normal component)}} \vec{N}$$

Acceleration

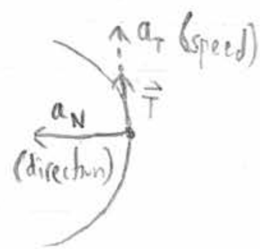
$$\boxed{a(t) = a_T \vec{T} + a_N \vec{N} = \frac{d^2s}{dt^2}\vec{T} + K\left(\frac{ds}{dt}\right)^2 \vec{N}}$$



a_T (tangential component): $a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\|v\| = a \cdot \vec{T} = \frac{a \cdot v}{\|v\|}$

a_N (normal component): $a_N = K\left(\frac{ds}{dt}\right)^2 = K\|v\|^2 = a \cdot \vec{N} = \sqrt{\|a\|^2 - a_T^2}$

$$a_T \vec{T} = (a \cdot T)\frac{v}{\|v\|} = \left(a \cdot \frac{v}{\|v\|}\right)\frac{v}{\|v\|} = \left(\frac{a \cdot v}{v \cdot v}\right) v$$



Tangential acceleration: acceleration as a result of change in speed
Normal acceleration: acceleration as a result of change in direction

Acceleration in a Frenet frame: $a = \langle a_T, a_N, 0 \rangle_{(T, N, B)}$

2D Limits Review

11/1/22
Disc 6

1D Limits: $\lim_{x\to a} f(x)$

2D Limits: $\lim_{(x,y)\to(a,b)} f(x,y) = A$ if, for all $\varepsilon>0$, there exists $\delta>0$ such that $0<\|(x,y)-(a,b)\|<\delta$
$\to |f(x,y)-A|<\varepsilon$

Showing a Limit Exists	Showing a Limit Doesn't Exist
- Continuity	- Check along different paths of same type
- Squeeze theorem	- Polar coordinates (prove limit depends on $\Theta \to$ varies based on diff angles)
- Polar coords. + squeeze theorem (eliminate Θ)	

Proving a Limit Doesn't Exist via Diff. Paths: $\lim_{(x,y)\to(0,0)}\left(\frac{x^2}{x^2+y^2}\right) \longrightarrow y=mx: \lim_{x\to 0} f(x,mx) = \frac{1}{1+m^2}$ (limits not equal)

$y=x^2: \lim_{x\to 0} f(x,mx) = \frac{2}{2+12x^2} \neq \frac{1}{1+m^2}$

$\to \lim_{(x,y)\to(a,b)} f(x,y)$ DNE

(only two paths needed to disprove)

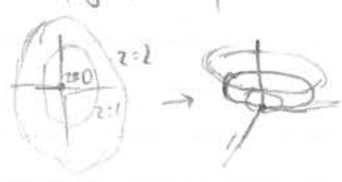
Multidimensional (Scalar) functions: $f(x)$, $f(x,y)$, $f(x,y,z)$, etc.

Functions: Domain $\longrightarrow$ Codomain (mapping remains 1:1 for higher dimensions)
(Domain can take higher-dimensional values, e.g. $\mathbb{R}^3$)

Traces: "slices" of a function at certain values (e.g. $f(x,y)|_{y=2}$ is a vertical trace of $f(x,y)=z$, equiv. to intersection of $f(x,y)$ and plane $y=2$)
- Can be used to approximate the shape of a function w/o drawing it out entirely

Vertical trace: e.g. $x=2$, $y=4$
Horizontal trace: e.g. $z=3$ ($f(x,y)=3$)
Level curve: Projecting horizontal traces onto the xy plane, a la a topographic map

Level curve = indiv. horiz. trace



Motion

10/2/22
Lecture 17
Week 6

Motion Equations

Velocity: $v(t) = r'(t)$
Acceleration: $a(t) = v'(t) = r''(t)$
Speed: $v(t) = \|v(t)\| = \|r'(t)\|$

$\rightarrow$ $v(t) = \int a(t)\,dt + v_0$, v_0 = unknown constant
$r(t) = \int v(t)\,dt + r_0$; r_0 = unknown constant

Uniform Circular Motion

Given circular curve $r(\theta) = R\langle\cos\theta, \sin\theta\rangle$:

Angular Velocity

$w = \frac{\theta}{t}$ $(\theta = wt)$

(w = Omega)

$\rightarrow r(t) = R\langle\cos wt, \sin wt\rangle$

$\rightarrow \frac{dr(t)}{dt} = R\langle -w\sin wt, w\cos wt\rangle$

$= Rw\langle -\sin wt, \cos wt\rangle = v(t)$

$\rightarrow$ speed $= \|\frac{dr}{dt}\| = \|Rw\langle -\sin wt, \cos wt\rangle\| = Rw$ (constant)

$\rightarrow a(t) = \frac{dv(t)}{dt} = Rw^2\langle -\cos wt, -\sin wt\rangle$ (not constant)

Functions of Multiple Variables

11/4/2
Lecture 1
Week 6

Functions of Multiple Variables

Definition: $f: \mathbb{R}^n \longmapsto \mathbb{R}$

$(x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)$, where f is a scalar function

Domain of $f(x,y)$: (x,y), s.t. $f(x,y)$ is defined $\left({}^{D}f = \{(x,y);\ f(x,y) \text{ is defined}\} \right)$

$h(x,y) = \frac{1}{xy} \rightarrow {}^{D}h = \mathbb{R}^2 \setminus^{\text{(exclude)}} \{\{(x,0)\} \cup \{0,y\}\}$ ($\mathbb{R}^2$ excluding the x-axis, y-axis)

2 #s, 1 tuple [of 2 #s]

$f(x,y) = f((x,y))$ Domain of $f(x_1, \ldots, x_n)$: ${}^{D}f = \{(x_1, \ldots, x_n;$ [conditions s.t. $f(x_1, \ldots, x_n)$ is defined]$\}$

$$\boxed{\begin{array}{c} \text{(Scalar) Functions of Multiple Variables} \\ f: \mathbb{R}^n \longmapsto \mathbb{R}^1 \\ (x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n) = f((x_1, \ldots, x_n)) \\ \text{(Domain)} \rightarrow {}^{D}f = \{(x_1, \ldots, x_n);\ [\text{conditions s.t. } f(x_1, \ldots, x_n) \text{ is defined}]\} \end{array}}$$

$f(x,y)$ still only takes one input: (x,y), a single tuple (hence $f(x,y) = f((x,y))$)
not two

3D Surfaces

11/7/22
Lecture 19
Week 7

Level curves can also be taken for 2D plots, e.g.

Average rate of change between 2 points on a level curve contour plot: $\frac{\Delta \text{altitude}}{\Delta(\text{horiz. dist})}$ ← Δz, ← $\|\overrightarrow{P_0P_1}\|$

Level surface (3D) / level curve (general) of 4D surfaces

<u>Quadric Surfaces</u>

A <u>quadric surface</u> has general equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + ax + by + cz + d = 0$$

(all constants $\in \mathbb{R}$)

Ellipsoid: $\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$

Paraboloid (elliptic): $\frac{x^2}{A^2} + \frac{y^2}{B^2} = z$

Hyperboloid: $\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1$ (one sheet), $\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = -1$ (two sheets)
↳ def. when $|z| \geq |c|$

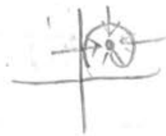
Cone (elliptic): $\frac{x^2}{A^2} + \frac{y^2}{B^2} = \frac{z^2}{C^2}$ (x or y const. → eq. of lines)

Hyperbolic Paraboloid: $\frac{y^2}{B^2} - \frac{x^2}{A^2} = \frac{z}{c}$

2D Limits Review

11/8/22
Disc 7

1D Limits: only approachable from 2 directions (+ or −)
2D Limits: approachable from ∞ paths
(incl. curves - not just lines)



Prove limit E w/ continuity: if a function is continuous @ (a,b), $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Prove limit !E w/ diff. paths: $\lim_{(x,g_1(x))\to(a,b)} f(x,y) \neq \lim_{(x,g_2(x))\to(a,b)} f(x,y)$ for diff. funcs, $g_1(x)$, $g_2(x)$ → no limit @ (a,b)

Prove Limit E w/ squeeze: $g(x,y) \le f(x,y) \le h(x,y)$, $\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = A \to \lim_{(x,y)\to(a,b)} f(x,y) = A$

(In a limit where $x \to 0$, x can still be factored out b/c it $!= 0$ for the purposes of the limit)

Checking along any specific paths is <u>not sufficient to prove a limit exists</u> - it is only sufficient to prove a limit does <u>not</u> exist
∟ (Even checking all polynomial paths is insufficient).

Triangle inequality: $|x+y| \le |x| + |y|$ (can be used for squeeze thm.), $x^2+y^2 \ge 2|xy|$

Larger polynomial exp. degree in denom → lim. prob DNE; larger exp. in num. → limit may E

θ must be eliminated to prove limit E w/ polar b/c θ is itself a variable (a function of r)

2D Limits

11/9/22
Lecture 20
Week 7

Limits of Functions of Several Variables

Given $f(x,y) \to L$, as $(x,y) \to (x_0, y_0)$:

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L} \longleftrightarrow \boxed{\lim_{(x_0,y_0)} f(x,y) = L}$$

Definition:

If $\forall\, \varepsilon > 0\ \exists\ \delta > 0$ s.t.:

$\forall M(x,y) \in \mathbb{R}^2,\ d(MA) < \delta \to |f(x,y) - L| < \varepsilon$ (can be extended for higher dimensions)

A = fixed point $(x_0, y_0) \to \exists$ (there exists) disc (radius δ) around A where given point (x,y) within the disc (neighborhood)

any point $(x,y) \to f(x,y) - L$ will not exceed ε

↳ therefore $\lim_{A} f(x,y) = L$

($d(MA)$ = distance from M to A)

Given a point $A = (x_0, y_0)$, if, for any real number $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any point (x,y) within a disc of radius δ w/ center A, $|f(x,y) - \lim_{(x_0,y_0)} f(x,y)| < \varepsilon$, then there exists limit L @ A

2D Limits and Continuity

Lecture 21

Given: $\lim\limits_{(x_0,y_0)} f(x,y) = L$, $\lim\limits_{(x_0,y_0)} g(x,y) = M$, $L, M, k \in \mathbb{R}$ $\longrightarrow$

Properties of 2D Limits

a) $\lim\limits_{(x_0,y_0)} \left(f(x,y) + g(x,y)\right) = \lim\limits_{(x_0,y_0)} f(x,y) + \lim\limits_{(x_0,y_0)} g(x,y)$

b) $\lim\limits_{(x_0,y_0)} k f(x,y) = k \cdot \lim\limits_{(x_0,y_0)} f(x,y)$

c) $\lim\limits_{(x_0,y_0)} \left(f(x,y) \cdot g(x,y)\right) = \lim\limits_{(x_0,y_0)} f(x,y) \cdot \lim\limits_{(x_0,y_0)} g(x,y)$

d) $\lim\limits_{(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim\limits_{(x_0,y_0)} f(x,y)}{\lim\limits_{(x_0,y_0)} g(x,y)}$, $\lim\limits_{(x_0,y_0)} g(x,y) \neq 0$

e) $\lim\limits_{(x_0,y_0)} \left(f(x,y)\right)^n = \left(\lim\limits_{(x_0,y_0)} f(x,y)\right)^n$

f) $\lim\limits_{(x_0,y_0)} k = k$

2-Path Non-existence of a Limit:
If a function $f(x,y)$ has two diff. limits along two diff. paths in the domain of $f(x,y)$ approaching (x_0, y_0), then $\lim\limits_{(x_0,y_0)} f(x,y)$ <u>does not exist</u>.

Continuity

A surface (x,y) is continuous at (x_0, y_0) if:

a) $\lim\limits_{(x_0,y_0)} f(x,y)$ exists

b) $f(x,y)$ is defined at (x_0, y_0)

c) $\lim\limits_{(x_0,y_0)} f(x,y) = f(x_0, y_0)$

* All rational functions are continuous on their domain

Polar-Notated Limits

11/14/22
Lecture 22
Week 8

Polar-notated limits (i.e. r, θ instead of x, y) cannot be evaluated if a θ term remains within the expression
(θ term can be eliminated w/ squeeze theorem)

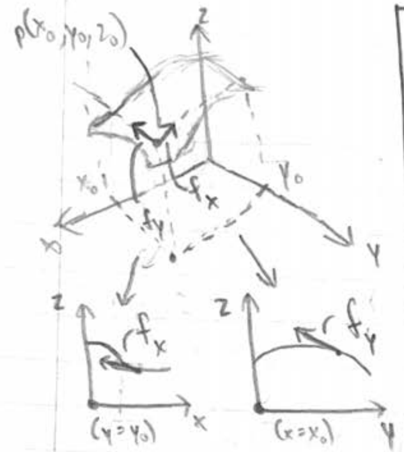
e.g. $\lim_{r \to 0} r\sin\theta \rightarrow$ cannot be evaluated

$\lim_{r \to 0} r\sin\theta = \lim_{r \to 0} r$ (via squeeze) $\rightarrow$ can be evaluated ($L = 0$)

$\lim_{r \to 0} \sin\theta \rightarrow$ D.N.E (limit varies based on $\theta \rightarrow$ does not converge, DNE)

Partial Derivatives

11/16/22
Lecture 23
Week 8



Partial Derivatives

$$\frac{\partial}{\partial x} f \Big|_{(x_0, y_0)} = \lim_{h \to 0} \left(\frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \right)$$

$$= f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \partial_x f(x_0, y_0)$$

(Partial derivative of $f(x,y)$ with respect to x at (x_0, y_0))

$$\frac{\partial}{\partial y} f \Big|_{(x_0, y_0)} = \lim_{h \to 0} \left(\frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \right)$$

$$= f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = \partial_y f(x_0, y_0)$$

(Partial derivative of $f(x,y)$ w/ respect to y at (x_0, y_0))

Partial derivative of $f(x,y)$ w/ respect to x/y at (x_0, y_0, z_0): derivative of the curve created by vertical trace $y = y_0 / x = x_0$ at (x_0, y_0, z_0)

- Derivative of $f(x,y)$ @ (x_0, y_0) s.t. the derivative is parallel to the x-axis/y-axis

To take the partial derivative of $f(x,y)$ w/ respect to x/y: take the derivative of $f(x,y)$ with respect to x/y, treating y/x as a constant

Higher-dimension partial derivatives: hold all but 1 variable constant (same process)

Clairaut's Theorem & Tangent Planes

11/21/22
Lecture 24
Week 9

Higher order partial derivatives exist, e.g. $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = f_{xy}$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial^2 f}{\partial x^2}(x,y) = f_{xx}$$

Notation: $\underbrace{\frac{\partial^n f}{\partial v_1 \dots \partial v_n}}_{\text{computed right to left, i.e. } \partial v_n \text{ then } \partial v_{n-1} \dots}(x,y) = f_{v_n \dots v_1}$] computed left to right, i.e. f_{v_n} then $f_{v_n v_{n-1}} \dots$

> **Clairaut's Theorem**
>
> If f_{xy} and f_{yx} both exist and are continuous on a disk D, then it follows that:
>
> $$f_{xy}(a,b) = f_{yx}(a,b),\ (a,b) \in D$$

(Disk D = domain where the theorem applies)

→ $f_{xy} = f_{yx}$ more generally within D

↳ entails partial derivatives can be taken in any order (as long as the conditions are true), i.e. $f_{xyxy} = f_{xxyy} = \dots$ as long as Clairaut's Thm. applies

Tangent Planes

Given $f(x,y)$ where f_x, f_y exist at (a,b):

> Tangent plane of f: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
> (Plane tangent to f @ (a,b)) $= L(x,y)$ (Linear approximation of f @ (a,b))

- Tangent plane spanned by $\langle 1, 0, f_x \rangle, \langle 0, 1, f_y \rangle$
  → Normal vector (of tangent plane) $:= \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle$

Differentiability & Gradients

11/23/22
Lecture 25
Week 9

Differentiability
$f(x,y)$ is differentiable @ (a,b) if:

$$\boxed{\lim_{(a,b)} \left(\frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} \right) = 0}$$

- We say $f(x,y)$ is differentiable on domain D if $f(x,y)$ is differentiable at all points on D

- We say $f(x,y)$ is differentiable @ (a,b) if f_x, f_y are continuous at (a,b)
  - If a function is differentiable at a point, it is also continuous at that point; however, if a function is continuous, it is not necessarily differentiable (at that point)
    - If a function is not continuous at a point, it is also not differentiable (at that point)

Linear Approximation of $f(x,y)$

If $f(x,y)$ is differentiable at (a,b) and (x,y) is close to (a,b):

$$f(x,y) \simeq L(x,y)$$
$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$\rightarrow f(x,y) - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$\rightarrow \boxed{\Delta f = f_x(a,b)\Delta x + f_y(a,b)\Delta y}$ $\quad \Delta f$: Linear approximation of $f(x,y) - f(a,b)$ at point (x,y) close to (a,b)

$\rightarrow f(a+\Delta x, b+\Delta y) \simeq f(a,b) + \Delta f$

Gradient and Directional Derivative

Given $f(x,y)$, gradient of $f(x,y)$ at point $P=(a,b)$ is:

$$\boxed{\begin{aligned}\nabla f_P &= \langle f_x(a,b), f_y(a,b) \rangle \\ &= \nabla f(P) = \nabla f(a,b)\end{aligned}}$$

- Gradient of $f(x,y)$ confined to plane (x,y) (2D vector)

$\nabla f_{P(a,b,c)} = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle$ (Higher-dimensional gradients)

Gradient & Directional Derivatives

11/28/22
Lecture 26
Week 10

Properties of Gradient

Given $f(x,y,z)$ and $g(x,y,z)$:

i) $\nabla(f+g) = \nabla f + \nabla g$ (Addition)
ii) $\nabla(cf) = c\nabla f$, $c \in \mathbb{R}$ (Scalar Multi.)
iii) $\nabla(fg) = g\nabla f + f\nabla g$ (Product Rule)
iv) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ (Quotient Rule)
v) $\nabla F(f(x,y,z)) = \nabla f \cdot F'(f(x,y,z))$ (Chain Rule)
(Scalar func. $F(x)$)

Given surface $z = f(x,y)$, 2D path $\vec{r}(t) = \langle x(t), y(t)\rangle$:

$$\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

$$= \langle f_x, f_y \rangle\Big|_{(x,y)=r(t)} \cdot \langle x'(t), y'(t)\rangle$$

Directional Derivative

Given surface $z = f(x,y)$, 2D line $r(h) = (a,b) + h\vec{u}$ (where h = scalar, $\vec{u} = \langle u_1, u_2\rangle$ is a vector indicating direction), the directional derivative of f at (a,b) in direction $\vec{u}$ is:

$$\frac{d}{dh} f(r(h))\Big|_{h=0}$$

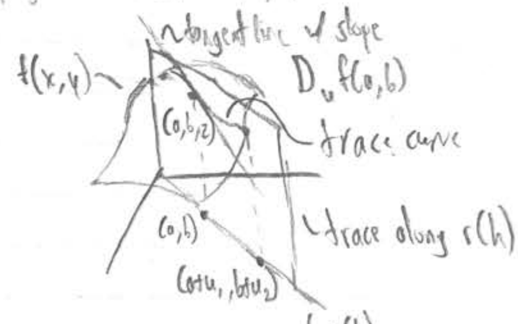
Directional Derivative

$$D_u f(a,b) = \lim_{h\to 0}\left(\frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}\right)$$

* Directional derivative of $f(x,y)$ in direction $\vec{u} = \langle u_1, u_2\rangle$ at (a,b)

- Represents slope of tangent line of $f(x,y)$ along trace w/ 2D projection $= r(h)$, at (a,b)

• $\vec{u}$ must be a unit vector ($\|\vec{u}\| = 1$) to be a directional derivative



Directional Derivative

$$D_u f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle \cdot \vec{u}$$
$$= \nabla f(a,b) \cdot \vec{u}$$

$\rightarrow D_u f(a,b) = \|\nabla f(a,b)\| \cos\Theta$

Gradient & Directional Derivatives

11/30/22
Lecture 27
Week 10

$\boxed{\text{Directional Derivative} \quad D_u f(P) = \nabla f_p \cdot \vec{u}}$ ↙ at point P (directional derivative @ P)

$\rightarrow D_u f(P) = \|\nabla f_p\| \cos\theta$ (θ = angle between ∇f_p, $\vec{u}$)

$\rightarrow -\|\nabla f_p\| \le D_u f(P) \le \|\nabla f_p\|$

Theorem: Gradient Properties

Given function f and point P:

1) ∇f_p points in the direction of fastest increase of f at point P
   - The rate of fastest increase is equal to $\|\nabla f_p\|$
2) $-\nabla f_p$ points in the direction of fastest decrease of f at point P
   - The rate of fastest decrease is equal to $-\|\nabla f_p\|$
3) ∇f_p is normal to the level curve of f at P

Surfaces can be explicitly defined in the form $F(x,y,z)=0$

e.g. $x^2+y^2+z^2=r^2 \rightarrow F(x,y,z)=x^2+y^2+z^2-r^2=0$ (sphere)

$F(x,y,z)=0 \rightarrow$ Tangent plane to the surface at $P=(a,b,c)$:

$$F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0$$

$F(x,y,z)=0 \rightarrow f(x,y)-z=F(x,y,z)=0 \rightarrow z=f(x,y)$

$\rightarrow F_x((a,b,c=f(a,b))=f_x(a,b)$

$F_y((a,b,c=f(a,b))=f_y(a,b)$

$F_z((a,b,c=f(a,b))=-1$

(Converting $F(x,y,z)=0$ surface definition back into the form $z=f(x,y)$)

Chain Rule for Paths

Given surface $z=f(x,y)$, 2D path $r(t)=\langle x(t),y(t)\rangle$:

$\rightarrow f(r(t))=f(x(t),y(t))$

$\rightarrow \frac{d}{dt}f(r(t))=\nabla f_{r(t)}\cdot r'(t)$

Chain Rule for Paths

12/2/22
Lecture 28
Week 10

Chain Rule for Paths

Given surface $f(x,y)$, 2D path $r(t) = \langle x(t), y(t) \rangle$: — x, y are auxiliary/intermediate variables; t is an independent variable

$f(r(t)) = f(x(t), y(t))$

$$\boxed{\frac{d}{dt} f(r(t)) = \nabla f_{r(t)} \cdot r'(t)}$$ Dot product (produces a scalar)

$$= f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

N-Dimensional Case: given $f(x_1, \dots, x_n)$, $r(t) = \langle x_1(t), \dots, x_n(t) \rangle$:

$$\frac{d}{dt} f(r(t)) = \nabla f_{r(t)} \cdot r'(t) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(r(t)) \left(\frac{dx_k}{t} \right)$$ (also works for variables not $= t$)

Chain Rule for Independent Variables

Given multiple independent variables (e.g., $r(s,t) = \langle x(s,t), y(s,t) \rangle$), consider only partial derivatives, i.e. $\frac{\partial}{\partial t} f(r(s,t)), \frac{\partial}{\partial s} f(r(s,t))$

$$\frac{\partial}{\partial s} f(r(s,t)) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$, e.g.

Optimization

12/5/22
Lecture 29
Finals Week

Implicit Differentiation

Given $F(x,y,z)=0$, where z is implicitly defined by independent variables x, y:

$$F_x = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\left(\frac{\partial y}{\partial x}=0\right) + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \rightarrow F_x + F_z\frac{\partial z}{\partial x} = 0 \rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}}$$

(Can be used for writing linear approximations, e.g.)

$$F_y = \frac{\partial F}{\partial x}\cdot\left(\frac{\partial x}{\partial y}=0\right) + \frac{\partial F}{\partial y}\cdot\left(\frac{\partial y}{\partial y}=1\right) + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \rightarrow F_y + F_z\frac{\partial z}{\partial y} = 0 \rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

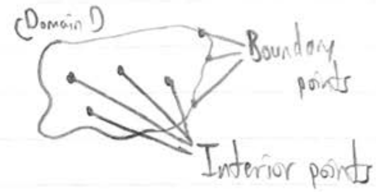
Optimization

A function $f(x,y)$ has local extremum at $P=(a,b)$ if $\exists$ open disk $D(P,r)$ s.t.: $\longrightarrow$

Local Extremum
- Local minimum: $f(P) \le f(x,y)\ \forall (x,y)\in D$
- Local maximum: $f(P) \ge f(x,y)\ \forall (x,y)\in D$

Critical point: An interior point $P(a,b)$ of domain f is a critical point of f if:

Conditions for Critical Points
1) $f_x(P)=0$ or $f_x(P)$ D.N.E., and:
2) $f_y(P)=0$ or $f_y(P)$ D.N.E.



Fermat's Theorem: If $f(x,y)$ has a local extremum at $P=(a,b)$, (a,b) is a critical point of $f(x,y)$

alt.: Fermat's Theorem: All local extrema are critical points.

Saddle point: Critical point of $f(x,y)$ at $P=(a,b)$ s.t. f is differentiable @ P but P does not represent a local extremum. $\rightarrow$ $f_x(a,b)=f_y(a,b)=0$, $\forall r>0\ \exists D(P,r)$ s.t. $\exists (x,y)\in D$ s.t. $f(x,y)<f(P)$ && $\exists (x,y)\in D$ s.t. $f(x,y)>f(P)$

Discriminant: We can define discriminant D at $P=(a,b)$ of $f(x,y)$ as follows:

$$\boxed{D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f^2_{xy}(a,b)} \longrightarrow$$

$$= \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

($f_{xy}=f_{yx}$ per Clairaut's)

Second Derivative Test
1) If $D>0$, and $f_{xx}(a,b)>0$, $f(a,b)$ is a local minimum.
2) If $D>0$ and $f_{xx}(a,b)<0$, $f(a,b)$ is a local maximum.
3) If $D<0$, f has a saddle point at (a,b)
4) If $D=0$, the test is inconclusive.