Math 120A: Differential Geometry

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Contents

1 Curves and Parametrizations

Definition. A ***parametrized curve*** is a function $\gamma : I \to \mathbb{R}^n$ (where I is a potentially unbounded interval $I \subseteq \mathbb{R}$).

- Say that γ is ***smooth*** if γ is (i) continuous and (ii) infinitely differentiable
- Say that γ is ***regular*** if its 1^{st} derivative is never-vanishing, i.e. $\dot{\gamma} \neq \vec{0} \; \forall \, t \in I$

Definition. A set $C \subseteq \mathbb{R}^n$ is called a ***(smooth) curve*** if it is (locally) the image of a smooth, regular parametrized curve.

- Smoothness is a <u>local</u> property!
- Can still have multiple connected components, discontinuities

1.1 Tangents to Curves

Let $\gamma : (a, b) \to \mathbb{R}^n$ be a smooth parametrized curve:

- The vector $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$ is tangent to the curve $C = \mathrm{im}(\gamma)$ at point $\gamma(t)$
- The ***speed*** at time t is given by $||\dot{\gamma}(t)||$
- The ***arc length*** of γ is given by:

$$\boxed{l(\gamma) = \int_a^b ||\dot{\gamma}(t)|| \, dt}$$

Prop. If a parametrized curve γ is ***constant speed*** (i.e. $||\dot{\gamma}(t)|| = c \; \forall \, t \in I$ for some $c \in \mathbb{R}$), then $\dot{\gamma}(t) \perp \ddot{\gamma}(t) \; \forall \, t \in I$.

1.2 Reparametrization

Prop. Let $\gamma_1 : (a_1, b_1) \to \mathbb{R}^n, \gamma_2 : (a_2, b_2) \to \mathbb{R}^n$ be smooth and regular parametrizations of the same curve C. Then for each $t \in (a_1, b_1)$, $\exists$ intervals $(c_1, d_1) \subseteq (a_1, b_1)$, and $(c_2, d_2) \subseteq (a_2, b_2)$ [with $t \in (c_1, d_1)$] and a smooth bijection [with smooth inverse] $\rho : (c_1, d_1) \to (c_2, d_2)$ s.t.

$$\implies \underline{\gamma_1 = \gamma_2 \circ \rho} \text{ on the interval } (c_1, d_1)$$

1.3 Level Sets

Definition. Let f be a (smooth) function $f : \mathbb{R}^n \to \mathbb{R}$, $c \in \mathbb{R}$; then the ***level set*** of f at height c is defined by:

$$\boxed{f^{-1}(c) = \{x \in \mathbb{R}^n : f(x) = c\}}$$

Definition. Given a function f as described above:

- A point $x \in \mathbb{R}^n$ is called a ***critical point*** of f if $\nabla f(x) = 0$
- A scalar $c \in \mathbb{R}$ is called a ***critical value*** of f if $f(x) = c$ for some critical point $x \in \mathbb{R}^n$; otherwise, c is called a ***regular value***

Thm. Let $f : \mathbb{R}^n \to \mathbb{R}$; then for any regular value $c \in \mathbb{R}$ of f, $f^{-1}(c)$ is a (smooth) curve.

2 Vector Fields

Notation:

- *Base-pointed vectors*: Let $p, v \in \mathbb{R}^n \to$ denote the vector v based at point p as (p, v)
- Also denote the vector space of vectors $\mathbb{R}^n$ based at p as $\mathbb{R}^n_p$
  - Base point p unchanged under addition, scalar multiplication

Definition. Let $U \subseteq \mathbb{R}^n$. A ***(smooth) vector field*** on U is a smooth function $X : U \to U \times \mathbb{R}^n$ such that $\forall\, p \in U, X(p) = (p, v)$ for some $v \in \mathbb{R}^n$.

Note. For any curve C with parametrization γ, can define its ***tangent vector field*** $X : C \to C \times \mathbb{R}^2$ by $X(\gamma(t)) = (\gamma(t), \dot{\gamma}(t))$.

Notation. Given a parametrized curve $\gamma : (a, b) \to \mathbb{R}^n$, a vector field along γ is a function $X : (a, b) \to \mathbb{R}^n \times \mathbb{R}^n$ s.t. $X(t) = (\gamma(t), v(t))$ [for some smooth function $v : (a, b) \to \mathbb{R}^n$].

- Can take derivatives of X w.r.t. t: $\dot{X}(t) = (\gamma(t), \dot{v}(t))$

Definition. Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the ***gradient vector field*** ∇f of f is defined by ($\forall\, p \in \mathbb{R}^n$):

$$\boxed{\nabla f(p) = \left(p, \left(\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p)\right)\right)}$$

- For any parametrization γ of a level set $f^{-1}(c)$ [c regular value] - $\nabla f(\gamma(t)) \perp \dot{\gamma}(t)$

2.1 Integral Curves

Definition. Let $X : U \to U \times \mathbb{R}^2$ be a (smooth) vector field. An ***integral curve*** of X is a parametrization $\gamma : (a, b) \to U$ s.t.

$$\boxed{X(\gamma(t)) = (\gamma(t), \dot{\gamma}(t))\ \forall\, t \in (a, b)}$$

Thm. Given $X : U \to U \times \mathbb{R}^2$ and $p \in U$, $\exists$ a “maximal” integral curve of X through p - $\gamma : (a, b) \to U$ s.t. (assuming WLOG that $a < 0 < b$):

1. $\gamma(0) = p$
2. If $\beta : I \to U$ is another integral curve of X with $\beta(0) = p$, then $I \subseteq (a, b)$ and $\beta(t) = \gamma(t)\ \forall\, t \in I$

2.2 Tangent Spaces

Definition. Let $C \subseteq \mathbb{R}^n$ be a (smooth) curve, and let $p \in C$. A vector (p, v) based at p is said to be ***tangent*** to C if $v = \dot{\gamma}(t)$ for some (smooth, not necessarily regular) parametrization γ of C with $\gamma(t) = p$.

$\to$ The ***tangent space*** of C at p is defined by:

$$\boxed{T_pC = \{(p, v) : (p, v) \text{ is tangent to } C\}}$$

Prop. Let C be a smooth curve and γ a (smooth, regular) parametrization of C with $\gamma(t) = p$; then:

$$\boxed{T_pC = \operatorname{span}\{(p, \dot{\gamma}(t))\}}$$

In particular, T_pC is a 1D subspace of $\mathbb{R}^n_p$.

2.3 Orientation

Definition. Let $\gamma : (a, b) \to \mathbb{R}^n$ be a parametrized curve, and let $X(t) = (\gamma(t), v(t))$ be a vector field along γ.

- X is called a ***unit vector field*** if $||v(t)|| = 1 \; \forall \; t \in (a, b)$
- X is called a ***normal vector field*** if $v(t)$ is orthogonal to $\dot{\gamma}(t) \; \forall \; t \in (a, b)$

Obtaining orientation $[\mathbb{R}^2]$:

1. If $C = f^{-1}(c)$ is a level set of a function f, take $\nabla f / ||\nabla f||$
2. If γ parametrizes C, find $N(t)$ by rotating $\dot{\gamma}(t)$ CCW $\pi/2$ $[(x, y) \mapsto (-y, x)]$

Definition. Given a plane curve $C \subseteq \mathbb{R}^2$, an ***orientation*** of C is a choice of unit normal vector field along C.

Prop. Let $C \subseteq \mathbb{R}^2$ be a plane curve. If C is connected, then C has exactly 2 orientations.

Definition. For a parametrized plane curve $\gamma : (a, b) \to \mathbb{R}^2$, say that an orientation N is ***consistent*** with the parametrization if $\forall \; t \in (a, b)$, $N(t)$ is the vector obtained by rotating the tangent direction $\frac{\dot{\gamma}(t)}{||\dot{\gamma}(t)||}$ $\pi/2$ counterclockwise, i.e.:

$$N(t) = \frac{(-\dot{\gamma}_2(t), \dot{\gamma}_1(t))}{||\dot{\gamma}(t)||}$$

3 Curvature

Definition. Let $C \subseteq \mathbb{R}$ be a simple [i.e. non-self-intersecting] plane curve with orientation N. Then the curvature of C at point $p \in C$ is given by:

$$\boxed{\kappa(p) = \frac{\ddot{\gamma}(t) \cdot N(\gamma(t))}{||\dot{\gamma}(t)||^2}}$$

where γ is any smooth, regular parametrization of C and $t \in \mathbb{R}$ s.t. $\gamma(t) = p$.

Note.

- The curvature of a curve is independent of parametrization γ.
- Inward/outward $\to$ +/-; sharp/shallow $\to$ high/low

Definition. Let $C \subseteq \mathbb{R}^2$ be a curve with orientation N, and let $p \in C$ s.t. $\kappa(p) \neq 0$; then the ***circle of curvature/osculating circle*** of C at p is the circle $C_O \subseteq \mathbb{R}^2$ with:

1. Center of curvature $p + \frac{1}{\kappa(p)} N(p)$
2. Radius of curvature $\frac{1}{|\kappa(p)|}$
3. Orientation N_O s.t. $N_O(p) = N(p)$

Properties

1. C_O is tangent to C at point p
2. C_O, C have the same curvature at p

3.1 Frenet-Serret [2D]

Definition. Let $C \subseteq \mathbb{R}^2$ be a curve with orientation N, $\gamma(t)$ a unit-speed parametriztion of C
$\to$ define the ***unit tangent vector*** to γ at time t by:

$$\boxed{T(t) = (\gamma(t), \dot{\gamma}(t))}$$

Frenet-Serret Formulas [2D]

$$\boxed{\begin{aligned} &(1)\ \dot{T} = \kappa N \\ &(2)\ \dot{N} = -\kappa T \end{aligned}} \iff \boxed{\begin{bmatrix} \dot{T} \\ \dot{N} \end{bmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{bmatrix} T \\ N \end{bmatrix}}$$

3.2 Frenet-Serret [3D]

Let γ be a unit-speed parametrization of a curve $C \subseteq \mathbb{R}^3$:

1. Define the ***unit tangent vector*** at $\gamma(t)$ by:

$$\boxed{T(t) = (\gamma(t), \dot{\gamma}(t))}$$

2. Define the ***principal unit normal vector*** of $\gamma(t)$ by:

$$\boxed{N(\gamma(t)) = \left(\gamma(t), \frac{\ddot{\gamma}(t)}{||\ddot{\gamma}(t)||}\right)}$$

3. Define the ***binormal unit vector*** of $\gamma(t)$ by:

$$\boxed{B = T \times N}$$

Frenet-Serret Formulas [3D]

$$\boxed{\begin{aligned} &(1)\ \dot{T} = \kappa N \\ &(2)\ \dot{N} = -\kappa T + \tau B \\ &(3)\ \dot{B} = -\tau N \end{aligned}} \iff \boxed{\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}}$$

Consequence: Any curve C completely determined by its initial values + curvature & torsion.

4 Surfaces

Definition. A ***surface/2-manifold*** is a set $S \subseteq \mathbb{R}^n$ s.t. $\forall\ p \in S$, S is locally parametrized in some neighborhood U of p by a smooth function $\phi : D_r^2 \to U$ satisfying $\operatorname{rank}(\operatorname{Jac}(\phi)) = 2$.

3 ways to obtain a surface:

1. As the image of a local parametrization $S = \operatorname{im}(\phi)$
2. As the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$
3. As the level set of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$

2 notable classes of surfaces:

1. Given plane curve $C \subseteq \mathbb{R}^2$, the ***cylinder over*** C is the set $A \subseteq \mathbb{R}^3$ given by

$$\boxed{A = \{(x, y, z) : (x, y) \in C\}}$$

   - C smooth curve $\implies$ cylinder over C is a smooth surface

2. Given $C \subseteq \mathbb{R}^2$ curve lying above the x-axis, its ***surface of revolution*** is:

$$\boxed{S = \left\{(x, y, z) : \left(x, \sqrt{y^2 + z^2}\right) \in C\right\}}$$

4.1 Tangent Spaces

Definition. $S \subseteq \mathbb{R}^3$ a surface, $p \in S$ a point $\implies$ (p, v) is ***tangent to*** S if $\exists$ smooth PC $\gamma : (a, b) \to S$ s.t. for some $t_0 \in \mathbb{R}$:

$$\boxed{\gamma(t_0) = p \text{ and } \dot{\gamma}(t_0) = v}$$

$\implies$ The ***tangent space*** of S at p is the set $T_pS \subseteq \mathbb{R}^3_p$ $[\dim T_pS = 2]$ defined by:

$$\boxed{T_pS = \{(p, v) : (p, v) \text{ is tangent to } S\}}$$

4.2 Orientation for Surfaces

Definition. Let $S \subseteq \mathbb{R}^n$ a surface and $p \in S$ $\implies$ the ***tangent plane*** of S at p is:

$$\boxed{P = \{p + v : (p, v) \in T_pS\}}$$

Definition. A vector (p, v) is ***orthogonal*** to a surface S if it is orthogonal to all vectors $(p, v_0) \in T_pS$

$\implies$ Let $S \subseteq \mathbb{R}^3$ a surface; then an ***orientation*** of S is a choice of unit normal vector field $N : S \to S : \times\mathbb{R}^3$ s.t. $\forall\, p \in S$: $N(p) \perp S, ||N(p)|| = 1$.

- Note: S a connected surface $\implies$ S has either 0 or 2 orientations

4.3 The Gauss Map

Definition. Given an oriented curve/surface $A \in \mathbb{R}^n$ with orientation N, the associated ***Gauss map*** is the fuction $G : A \to S^{n-1}$ defined by:

$$\boxed{N(p) = (p, G(p))\ \forall\ p \in A}$$

- Notation: S^n is unit sphere in $\mathbb{R}^{n+1}$

Definition. Define the ***spherical image*** of A as the image of the Gauss map of A

- **Thm**: If $A = f^{-1}(c)$ compact, then its Gauss map under orientation $N = \pm\frac{\nabla f}{||\nabla f||}$ is surjective
- *Rec*: $A = f^{-1}(c)$ for f smooth $\implies$ A closed; $A \subseteq \mathbb{R}^n$ compact $\iff$ A closed & bounded [Heine-Borel]

4.4 Curves on Surfaces

Definition. Let $S \subseteq \mathbb{R}^3$ a surface, $P \subseteq \mathbb{R}^3$, and $p \in S \cap P \implies$ say P is ***orthogonal*** to S at p if $\exists\ (p, v) \neq 0$ orthogonal to S and tangent to P

5 Curvatures of Surfaces

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ smooth, and let $(p, v) \in \mathbb{R}^n_p \implies$ define ***derivative of f w.r.t.*** (p, v) [linear] by (for $\gamma : (a, b) \to \mathbb{R}^n$ PC w/ $\gamma(t_0) = p, \dot{\gamma}(t_0) = v$):

$$\boxed{\nabla_{(p,v)} f = \frac{d}{dt}(f \circ \gamma)(t_0) \; [= \nabla f(p) \cdot v]}$$

& f or a vector field $X : U \to U \times \mathbb{R}^n$:

$$\boxed{\nabla_{(p,v)} X = (X \dot{\circ} \gamma)(t_0) \; [= (p, \; (\nabla X_1(p) \cdot v, \dots, \nabla X_n(p) \cdot v))]}$$

Definition. Let $S \subseteq \mathbb{R}^3$ be smooth surface oriented by N, and let $p \in S \implies$ define the ***Weingarten map*** of f at p by:

$$\boxed{L_p(v) = -\nabla_{(p,v)} N(p)}$$

- The Weingarten map is linear & self-adjoint: $L_p(v) \cdot w = L_p(w) \cdot v$
- For a curve $\gamma : (a, b) \to S$ on S with $\gamma(t_0) = p, \dot{\gamma}(t_0) = v$: $\ddot{\gamma} \cdot N(p) = L_p(v) \cdot v$
- Note: $\nabla_{(p,v)} N$ only depends on values of N on S

Definition. Let $S \subseteq \mathbb{R}^3$ surface oriented by N, and let $p \in S$, $(p, v) \in T_p S$ unit vector. Then the ***normal curvature*** of S at p in the direction v is defined by:

$$\boxed{k(p, v) = L_p(v) \cdot v}$$

Corollary: Let $S \subseteq \mathbb{R}^3$ oriented surface, $p \in S$, P plane through p orthogonal to S at p; then $P \cap S$ is (near p) a smooth curve satisfying:

$$\boxed{\kappa(p) = k(p, v)}$$

Definition. Let $S \subseteq \mathbb{R}^3$ surface oriented by N, and let $p \in S$. Let $\{v_1, v_2\}$ be an orthonormal basis (for $T_p S$) of eigenvectors of L_p with eigenvalues λ_1, λ_2; then the principal curvatures of S at p are defined by:

$$\boxed{k_1(p) = k(p, v_1) = \lambda_1; \quad k_2(p) = k(p, v_2) = \lambda_2}$$

- The principal curvatures $k_1(p) \leq k_2(p)$ are the min & max of the normal curvatures $k(p, v)$ at p:

$$\underline{k_1(p) = \min_v k(p, v); \quad k_2(p) = \max_v k(p, v)}$$

Definition. Let $S \subseteq \mathbb{R}^3$ an oriented surface, and let $p \in S$. Then the ***Gauss curvature*** of S at p is given by:

$$\boxed{K(p) = k_1(p) \cdot k_2(p)[= \det L_p]}$$

Similarly, the ***mean curvature*** of S at p is defined by:

$$\boxed{H(p) = \frac{1}{2}\left(k_1(p) + k_2(p)\right)}$$

Theorem. Let $S \subseteq \mathbb{R}^3$ a compact oriented surface; then $\exists\ p \in S$ such that either (i) $k(p,v) > 0 \forall\ v$, or (ii) $k(p,v) < 0\ \forall\ v$.

[Equivalently: $K(p) > 0$]

Lemma. Let $S \subseteq \mathbb{R}^3$ oriented by N, and let $X : \mathbb{R}^3 \to \mathbb{R}^3$ a VF s.t. $X/\left|\left|X\right|\right| = N$. Let $p \in S$ & $\{v_1, v_2\}$ any basis for T_pS; then:

$$K(p) = \det\begin{pmatrix} \nabla_{(p,v_1)}X \\ \nabla_{(p,v_2)}X \\ X(p) \end{pmatrix} \cdot \frac{1}{\left|\left|X(p)\right|\right|^2 \det\begin{pmatrix} v_1 \\ v_2 \\ X(p) \end{pmatrix}}$$

5.1 Fundamental Forms

Definition. Llet $V \subseteq \mathbb{R}^n$ a subspace of $\mathbb{R}^n$, and let $T : V \to V$ a self-adjoint linear map. Then the quadratic form associated with T is the map $Q : V \to \mathbb{R}$ given by:

$$Q(v) = T(v) \cdot v$$

Definition. Let $S \subseteq \mathbb{R}^3$, and let $p \in S$. Then the ***fundamental forms*** of S at p are:

1. 1^{st} fundamental form: $\ell_p : T_pS \to \mathbb{R}$ defined by

$$\boxed{\ell_p(v) = v \cdot v = \left|\left|v\right|\right|^2}$$

2. 2^{nd} fundamental form: $\rho_p : T_pS \to \mathbb{R}$ defined by:

$$\boxed{\rho_p(v) = L_p(v) \cdot v}$$

Notice: Let $(p,v) \in T_pS, v \neq 0$; then:

$$\underline{\rho_p(v) = \left|\left|v\right|\right|^2 k\left(p, \frac{v}{\left|\left|v\right|\right|}\right)}$$

In particular, have the following relationships:

1. ρ_p definite $\iff K(p) > 0$
2. ρ_p indefinite $\iff K(p) < 0$
3. ρ_p semi-definite $\iff K(p) = 0$

[**Corollary**: Let $S \subseteq \mathbb{R}^3$ a compact oriented surface; then $\exists\ \rho_p \in S$ with ρ_p definite.]

5.2 Isometries

Definition. Let $X_1, X_2 \subseteq \mathbb{R}^n$, and let $\phi : X_1 \to X_2$ a diffeomorphism. We call ϕ an ***isometry*** if $\forall\ \gamma : (a, b) \to X_1$, the arc length of γ is equivalent to the arc length of $\phi \circ \gamma$, i.e.:

$$\underline{\int_a^b ||\dot{\gamma}(t)||\, dt = \int_a^b ||\phi \dot{\circ} \gamma(t)||\, dt}$$