

Physics 1B

2023-24

Fall 23

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Textbook: Young & Freedman - Mastering Physics

Topics: Fluids (static fluids, fluid flow), oscillation, mechanical waves & sound, electrostatics (charge, fields, flux, potential), capacitance, circuits (current, resistance)

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Physics 1A Review

9/20/23

Lecture 1

Physics 1A Review

- Newton's Laws:
(1) An object stays at rest, unless acted upon by a force.
(2) $F_{\text{net}} = ma$ [$F_{\text{net}} = \sum \vec{F}$].
(3) For every action, there is an equal and opposite reaction.

$$\text{Force: Newton} = \text{kg m/s}^2$$

- Forces as vectors: have magnitude & direction, can be added.

- Gravity: $\vec{F} = -mg\hat{e}_z$ (where e_z is the vertical distance/height)
 $= -G \frac{m_1 m_2}{r^2} \hat{r}$ [Law of Universal Gravitation] (*) \hat{r} as unit vector

- Hooke's Law: $\vec{F} = -kx$ (where x is the distance from the spring's massless equilibrium position \bar{x}/x_0).

- (The direction of \vec{F} is thus always pointing toward equilibrium).

- Law for ideal springs

- Friction: $F = \mu |N|$ (alt: $|F| = k|V|$ = drag in a fluid)

- Kinetic friction opposes motion [slows object], converting KE to heat

- Conservative forces (e.g. gravity): forces derived from a potential function U [$\vec{F} = -\nabla U$].

- Are path-independent, and conserve mechanical energy $E = K + U$ if isolated

- Energy:

$$\text{Kinetic: } KE = \frac{1}{2} mv^2$$

$$\text{Energy: Joule} = \text{kg m}^2/\text{s}^2$$

$$\text{Potential: } U = mgh \text{ (gravitational)}$$

- Potential energy defined in terms of deviation from an arbitrary baseline;
can be negative, unlike KE

Fluids and Pressure

10/2/23

Lecture 2

Properties of a Fluid

- 3 phases of matter: solid/liquid (assumed non-compressible), vs gas (compressible)
- Density: $\rho = \frac{m}{V}$ [kg/m^3]; assumed constant for liquids per non-compressibility
- (*) Water: 1000; Ice: 917; wood: 700; air: 1.22

Pressure

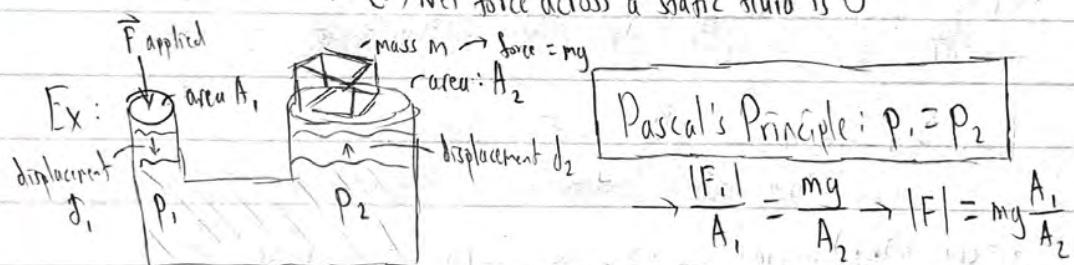
Pressure results from applying force on a liquid:

$$P = \frac{|F_{\perp}|}{A}$$

$[\text{N}/\text{m}^2 - \text{Pascals}]$

Pascal's Principle - pressure is the same everywhere in a liquid (ignoring the weight of the liquid)

→ (*) Net force across a static fluid is 0



→ Force required to lift object $\times \frac{A_1}{A_2}$

Not a force "generator";

simply applies less force

over greater distance

→ equal overall work

since noncompressible

Work: W_1 (work by F) = W_2 (gravity on object)

$$\rightarrow [W_1 = d_1 F_1 = d_1 A_1 P] = [W_2 = d_2 F_2 = d_2 A_2 P]$$

$$d_1 A_1 = d_2 A_2$$

$$\rightarrow d_1 = d_2 \frac{A_2}{A_1}$$

(i.e. $d_1 > d_2$)

(*) Applications: Hydraulic lifts

Drinking straws

Hydrostatic Pressure

10/2/23

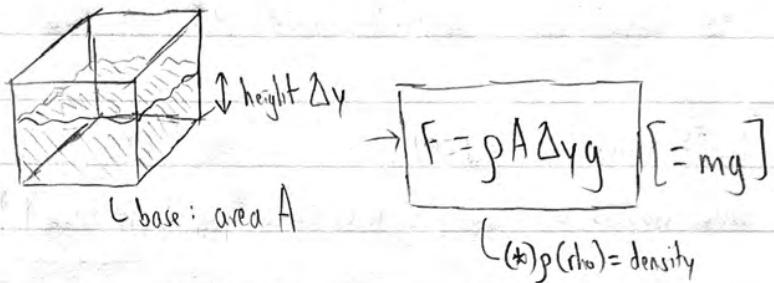
Lecture 2

Hydrostatic Pressure

(cont.)

Def: Hydrostatic pressure is pressure caused by the weight of the fluid

(*) Consequence: Is not constant across the fluid's volume (namely, deeper \rightarrow higher pressure)



Variation in Hydrostatic Pressure

Pressure @ bottom = pressure @ top + hydrostatic pressure

$$\rightarrow p(y_0 - \Delta y)A - p(y)A - \rho A \Delta y g = 0$$

$$\rightarrow \frac{p(y_0 - \Delta y) - p(y)}{\Delta y} = -\rho g \rightarrow \frac{dp}{dy} = -\rho g \quad (\text{Assume } \rho \text{ constant}) \quad \int_{y_0}^{y_0 - \Delta y} dp = -\rho g \int_{y_0}^{y_0 - \Delta y} dy$$

(*) Intuition:

$$p = p(y_0)$$
$$mg \approx ggh$$
$$p = p(y_0) + \rho gh$$

(*) A "column" of fluid isn't 1D? true

$$\rightarrow p(y_0 - \Delta y) = p(y_0) - \rho g((y_0 - \Delta y) - y_0)$$

$$\rightarrow p(y_0 - \Delta y) = p(y_0) + \rho g \Delta y$$

(*) Atmospheric Pressure

Def: Atmospheric pressure is "hydrostatic" pressure from the air [$\approx 10^5$ Pa @ sea level]

$$\hookrightarrow p(y_0) = P_{atm} + P_F \quad \text{pressure from applied force}$$

(*) Fun fact: We don't "feel" P_{atm} in our day-to-day because our bodies operate at P_{atm} themselves

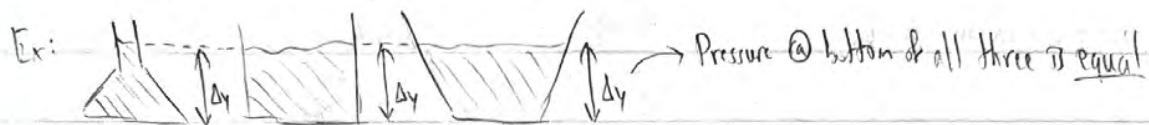
Hydrostatic Pressure (cont.)

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Hydrostatic Pressure (cont.)

Lecture 2

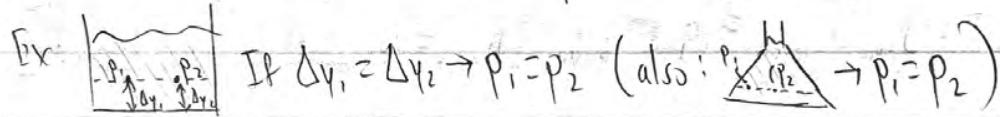
- Hydrostatic pressure is only dependent on height (Δy)



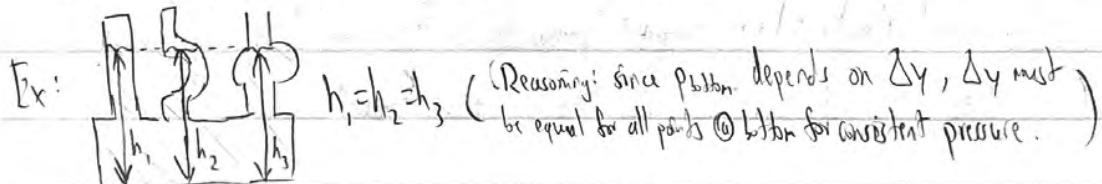
Lecture 3

(*) reasoning: @ A, depth of water directly above is lower, but water pushing up & out on the container creates a normal force in & down (\rightarrow pressure)

- Hydrostatic pressure is constant [equal] at all points at the same height (where Δy is equal)



(*) Consequence: height of water is independent (of) of container shape



(*) Measuring Pressure

- Two different types of pressure measurement: gauge pressure ($p - p_{atm}$) vs absolute pressure (p)
- Gauge pressure [relative pressure] can be negative; absolute pressure cannot (lowest: 0 [vacuum])

The Buoyant Force

10/3/23

Lecture 3

Buoyancy

(cont.)

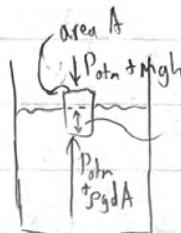
Def: Buoyancy [the buoyant force] is an upward force exerted by a fluid on objects immersed in the fluid.

(*) The net force on a floating object is 0.

$$\rightarrow (P_{atm} + \rho g dA) - P_{atm} - mgh = 0$$

$$\rightarrow F_b = \rho g dA \quad [d = \text{displacement}]$$

$$= \rho g V_{\text{displaced}}$$



→ Archimedes' Principle

$$F_{\text{buoyant}} = g \cdot \rho_{\text{fluid}} \cdot V_{\text{displaced}} = g \cdot M_{\text{displaced}} \quad [F_{\text{buoyant}} = \text{weight of the fluid displaced.}]$$

Notes on Buoyancy

- An object floats when $F_{\text{buoyant}} = F_{\text{gravity}}$ (such that $F_{\text{net}} = 0$).
- Corollary: An object can only float if $\rho_{\text{object}} \leq \rho_{\text{fluid}}$.

Equations of Fluid Flow

10/4/23

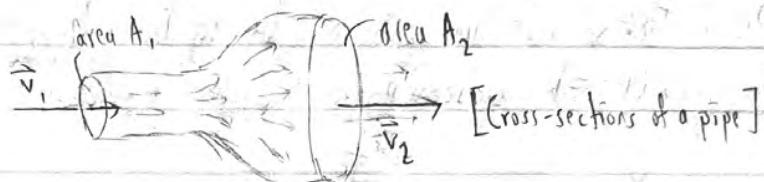
Lecture 4

+ Lecture 5

(*) Fluid assumptions: incompressible, no friction, no turbulence

Fluid Flow

There are two main equations governing fluid flow: the continuity equation and Bernoulli's equation.



(1) Continuity Equation

Given fluid flow through 2 cross-sections of a pipe (pictured above), then:

$$\rho A_1 v_1 = \rho A_2 v_2$$

• Simplifies to $A_1 v_1 = A_2 v_2$ when density is constant

(*) Can be seen as an analogue to conservation of mass: "volume in $[\rho dV_1]$ = volume out $[\rho dV_2]$ "

(*) The value $\frac{dV}{dt} = Av$ is also called the flow rate R_v .

(2) Bernoulli's Equation

Given fluid flow through a pipe, given any two points in the pipe 1 & 2, then:

$$P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \quad [\text{Bernoulli's Equation}]$$

(*) Notes on Bernoulli's Equation

• Can be seen as an analogue to conservation of energy:

• Kinetic energy: $KE = \frac{1}{2} mv^2 \rightarrow \frac{1}{2} \rho v^2$

• Potential energy: $PE = mgh \rightarrow \rho gy$

• Is a generalization of the hydrostatic pressure equation to nonstatic fluids

(*) Can be informally derived from Pascal's Principle

Intro to Oscillations

10/9/23

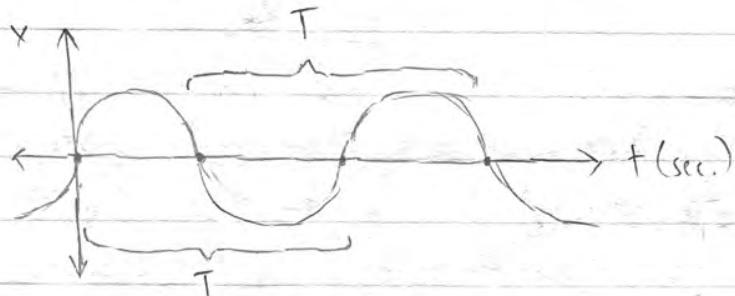
Lecture 6

Periodic Processes

Def: A process that comes back to its initial state and repeats over time (ex: springs, pendulums)

Properties of Periodic Processes

- Period (T): smallest time period of a full repetition [seconds]
- Frequency ($f = \frac{1}{T}$): inverse of period; how often a process repeats [seconds $^{-1}$ = Hertz/Hz]



Simple Harmonic Oscillators (SHO)

Def: Simple harmonic motion (SHM) is the primary physics model for systems involving slight fluctuations around a stable equilibrium

Notes on Simple Harmonic Oscillators

- Model system: ideal [horizontal] spring with a mass
- Example system: vibrations in a solid
- Many specialized physics models are actually variations of simple harmonic motion (e.g. adding an additional term)

(*) SHM: Ideal Springs

- Amplitude (A): maximum distance away from equilibrium in a given repetition [meters]
- Phase constant/shift (ϕ_0): constant encoding the initial position of a spring [radians]

Oscillations of Ideal Springs

10/9/22

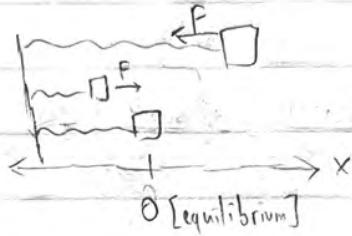
Lecture 6

(cont.)

Recall: Hooke's Law

$$\boxed{\text{Hooke's Law: } F = -kx}$$

(for ideal springs)



[Horizontal spring]

Equations of Simple Harmonic Motion

$$ma = F = -kx$$

$$\rightarrow m \frac{dx}{dt} = -kx$$

$$\rightarrow \frac{d^2x}{dt^2}(t) - \frac{k}{m}x = 0 \Rightarrow$$

$$\boxed{\text{Def: } \omega = \sqrt{\frac{k}{m}} \text{ [angular frequency]}}$$

(*) ω = "omega"

$$\boxed{\frac{d^2x}{dt^2}(t) - \omega^2 x = 0 \text{ [SHM]}}$$

$$\boxed{\text{General solution: } x(t) = A \cos(\omega t + \phi_0)}$$

(Solve 2nd-order diff. eq.)

(*) cosine, ϕ_0 in radians (not degrees)

(*) Characterizing Angular Frequency

$$x(t) = A \cos(\omega t + \phi_0)$$

$$\rightarrow x(t+T) = x(t)$$

$$\rightarrow A \cos(\omega t + \phi_0) = A \cos(\omega(t+T) + \phi_0)$$

$$\boxed{T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}} \text{ [Ideal spring]}$$

$$\boxed{\text{Consequence: Period is independent of amplitude (A)}} \quad \boxed{\text{(*) Also applies to } w, f}$$

(*) Derivatives of Position

$$(i) x(t) = A \cos(\omega t + \phi_0) \Rightarrow x(t)^2 + \frac{v(t)^2}{\omega^2} = A^2$$

$$(ii) v(t) = -A \omega \sin(\omega t + \phi_0)$$

$$(iii) a(t) = -A \omega^2 \cos(\omega t + \phi_0)$$

$$\boxed{A = \sqrt{x(t)^2 + \frac{v(t)^2}{\omega^2}}}$$

Oscillations (cont.)

10/10/23

Lecture 7

Equations of Simple Harmonic Motion (cont.) $\tan(\phi_0)$

$$x(t) = A \cos(\omega t + \phi_0) \quad v(t=0) = -\omega \frac{\sin(\phi_0)}{\cos(\phi_0)} \Rightarrow \phi_0 = \tan^{-1} \left(-\frac{1}{\omega} \cdot \frac{v(t=0)}{x(t=0)} \right)$$

$$v(t) = -A \omega \sin(\omega t + \phi_0)$$

→ Adjust formula to give full range (2π) of values

$$\phi_0 = \begin{cases} x(0) > 0 : \tan^{-1} \left(-\frac{1}{\omega} \cdot \frac{v(t=0)}{x(t=0)} \right) \\ x(0) \leq 0 : \tan^{-1} \left(-\frac{1}{\omega} \cdot \frac{v(t=0)}{x(t=0)} \right) \pm \pi \end{cases}$$

Energy of Simple Harmonic Motion

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t + \phi_0)$$

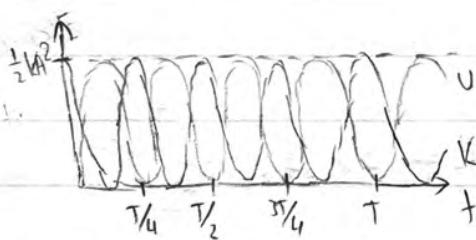
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi_0)$$

$$E_{\text{mech}} = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t + \phi_0) + \frac{1}{2}kA^2 \cos^2(\omega t + \phi_0)$$

$$\downarrow \text{substitute } \omega = \sqrt{\frac{k}{m}}$$

Total energy is independent of time \leftarrow

$$E_{\text{mech}} = \frac{1}{2}kA^2 = \frac{1}{2}mA^2\omega^2$$



- (*) Since K, U have x^2, v^2 terms, sign of $x(t), v(t)$ ignored → period of K, U oscillations = $T/2$.
- (*) Note: U is independent of mass; K is not
- (*) Amplitude determined by potential energy

Applications of Simple Harmonic Motion

1) Torsion pendulum ($x = \theta$)

$m \rightarrow I$ [moment of inertia]

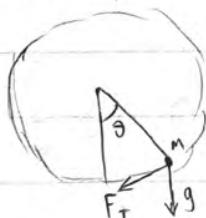
$x \rightarrow \theta$ [angular displacement (rotation)]

$\rightarrow \tau = -\gamma_2 \theta$ [torque is linear]

$(k \rightarrow \gamma_2)$

$$\text{Let } \omega = \sqrt{\frac{\gamma_2}{I}} \Rightarrow \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

2) Simple pendulum



Motion equation

$$ml \frac{d^2\theta}{dt^2} = -mg \sin(\theta)$$

Not an SHO, but becomes one w/ approx. $\sin(\theta) \approx \theta$ at small angles

Oscillations (cont.)

10/11/23

Lecture 8

Motion of a Simple Pendulum (cont.)



$$ml \frac{d^2\theta}{dt^2} = -mg \sin(\theta) \rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l} \sin(\theta) = 0 \quad [\text{not a SHO}]$$

Define static equilibrium: $\theta = 0$

$$\rightarrow \sin(\theta) [\text{Taylor exp.}] = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \dots \rightarrow \sin(\theta) \approx \theta \quad [\text{small } \theta]$$

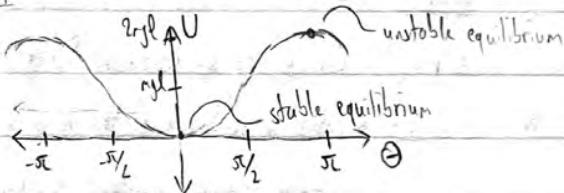
$$\rightarrow \frac{d^2\theta}{dt^2} + \omega^2 \theta = 0 \quad [\omega = \sqrt{\frac{g}{l}}] \Rightarrow \text{SHO} \quad [\text{for } \theta < 20^\circ, \text{ approx.}]$$

(*) Potential Energy of a Simple Pendulum

$$V = mgh \quad h = l(1 - \cos\theta)$$

$$\cos(\theta) [\text{Taylor exp.}] = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \dots$$

$$U = mg l \frac{\theta^2}{2} \quad [\text{for small values of } \theta]$$



Damped Oscillators

Damped oscillators introduce a frictional/damping force (viscous friction): $F_{\text{fric}} = -bv$

$$\rightarrow F_{\text{net}} = m \frac{d^2x}{dt^2} = -b \frac{dx}{dt} - kx \quad [\text{Ideal spring + friction}]$$

$$\rightarrow \frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \quad \left\{ \begin{array}{l} \omega > 0 : x(t) = A e^{-\frac{b}{2m}t} \cos(\omega t + \phi_0) \quad [\text{Underdamped}] \\ \omega = 0 : x(t) = C_1 + C_2 t \quad [\text{Critically damped}] \\ \omega < 0 : x(t) = (C_1 e^{-\frac{(b/2m - \omega)}{2}t} + C_2 e^{-\frac{(b/2m + \omega)}{2}t}) \quad [\text{Overdamped}] \end{array} \right.$$

$$\text{Let } \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m}}$$

$$\text{Solutions: } \left\{ \begin{array}{l} \omega > 0 : x(t) = A e^{-\frac{b}{2m}t} \cos(\omega t + \phi_0) \quad [\text{Underdamped}] \\ \omega = 0 : x(t) = C_1 + C_2 t \quad [\text{Critically damped}] \\ \omega < 0 : x(t) = (C_1 e^{-\frac{(b/2m - \omega)}{2}t} + C_2 e^{-\frac{(b/2m + \omega)}{2}t}) \quad [\text{Overdamped}] \end{array} \right.$$

(*) Resonance

10/11/23

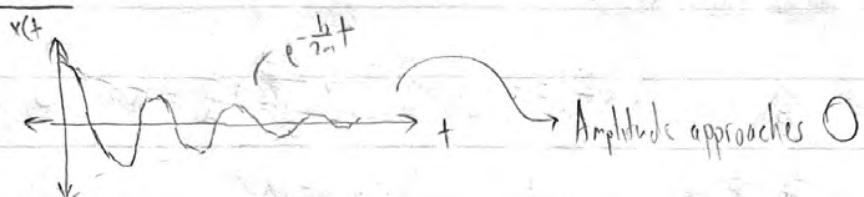
Lecture 8

+ Lecture 9

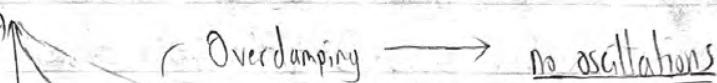
Damped Oscillators (cont.)

1) Underdamping:

$$E_n A \sim e^{-bt}$$



2) Critical damping:



3) Overdamping:

Critical damping (fastest rate of decay [exponential])

(*) Resonance

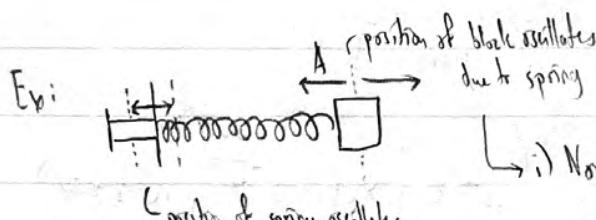
Oscillations may be caused by an external force that is itself oscillatory [oscillates in magnitude]

i) Normal case: oscillation occurs with the driving frequency [frequency of the force's oscillation],

amplitude remains constant (steady state) \rightarrow energy from force oscillation "cancelates" original oscillator.

ii) Special case: When the driving frequency approaches the "natural frequency" of the system (of the original oscillator), amplitude becomes very large [resonance]

\hookrightarrow energy from force oscillation is added to E of original oscillation
 \rightarrow energy in system grows continuously; potentially dangerous



\hookrightarrow i) Normal case: block oscillation frequency = frequency of spring position oscillation, A constant

ii) Special case [resonance]: frequency of spring pos.

oscillation = frequency of the spring

\rightarrow A grows larger & larger

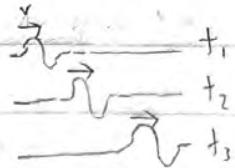
Intro to Mechanical Waves

10/10/27

Lecture 10

Mechanical Waves

- Can be thought of as the transmission of energy/momentum/information between different locations in space
 - Alt: propagation of a disturbance in a medium
 - (*) Ex: waves on strings, waves in a fluid, waves in air (sound waves)
 - (*) Sound waves - changes in air pressure as disturbances
- Similar to oscillators - involve displacement from an equilibrium

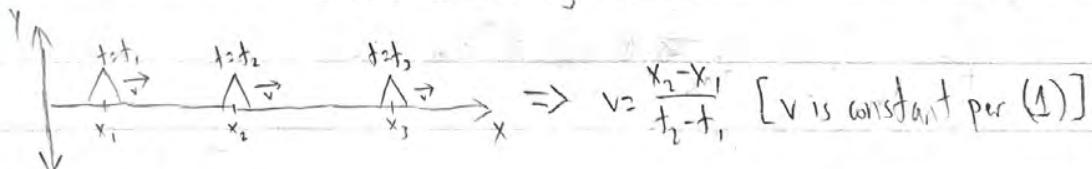


Types of Waves

- 1) Transverse waves: displacement orthogonal to the direction of travel
- 2) Longitudinal waves: displacement is in the direction of travel (ex: stretching a slinky)

Observations on Mechanical Waves

- 1) Disturbances move with a constant velocity (wave speed: v)
- 2) Wave speed does not depend on the "shape" of the disturbance. [$\sqrt{\mu}$ vs. $\sqrt{\rho}$]
- 3) The shape of a disturbance does not change as it moves.
- 4) Superposition: if there are multiple disturbances, then they travel past ["through"] each other without influencing each other.



(*) Describing Waves

Let the shape of the disturbance := $y(x) = f(x)$ at some fixed time

$$\Rightarrow \boxed{y(x,t) = f(x-vt)} \quad \begin{array}{l} [\text{Right-moving/positive direction}] \\ [t \text{ increases} \Rightarrow x \text{ increases} \text{ to keep } y \text{ constant}] \end{array}$$

Mechanical Waves (cont.)

10/16/23

Lecture 10

Reminder (Mechanical Waves)

(cont.)

$$\text{Displacement ["shape"]}: y(x, t) = f(x - vt)$$

$$\Rightarrow \text{Change in displacement: } v_y = \frac{\partial y}{\partial t}(x, t) = -v f'(x - vt) \quad [\text{change over time}]$$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \cdot \frac{\partial^2 y}{\partial t^2} = 0} \rightarrow \text{Wave equation [SHO equation, approx.]}$$

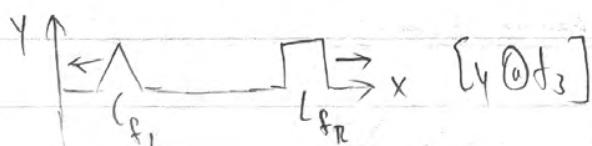
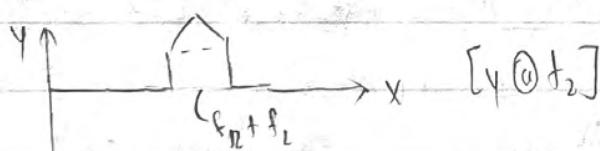
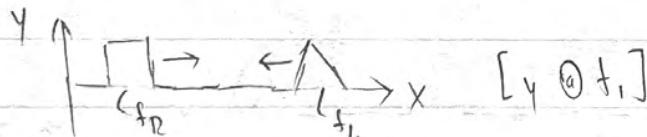
(*) Superposition

Prop: Let $f_R(x-vt)$, $f_L(x+vt)$ be a right-moving & left moving wave, respectively.

Then if both are traveling in the same medium, then the displacement is:

$$y(x, t) = f_R(x-vt) + f_L(x+vt),$$

which obeys the wave equation.



Reflections & Sinusoidal Waves

10/17/23

Lecture 11

Reflection of Traveling Waves

Two cases of wave reflection: fixed end and free end:

- 1) Fixed end: If the end of the medium is fixed (cannot be displaced/disturbed), then
both the directions of travel and displacement are inverted upon reflection.
- 2) Free end: If the end of the medium is free (i.e. can be displaced/disturbed), then
only the direction of travel is inverted upon reflection.

(*) For sinusoidal waves: phase shift of π for fixed-end, no shift for free-end



Sinusoidal Waves

Def: A wave $y(x, t)$ is sinusoidal if it is periodic across both x and y .

$$\Rightarrow \boxed{y_R(x, t) = A \cos\left(\frac{2\pi}{\lambda}(x - vt) + \phi_0\right)} \quad [\text{Sinusoidal wave}]$$

Variables of Sinusoidal Waves

Def: Wavelength (λ) is the [spatial] distance across which the wave is periodic

$$\Rightarrow \boxed{\lambda = vt} \quad [\text{Wavelength} = \text{wave speed} \cdot \text{period}]$$

(*) Corollary: $\omega = \frac{2\pi}{\lambda} v \Rightarrow y(x, t) = A \cos\left(\frac{2\pi}{\lambda} x - \omega t + \phi_0\right)$

(*) $k = \frac{2\pi}{\lambda}$ is sometimes called the "wave number"

Wave Speed & Power

10/17/23

Lecture 11

Wave Speed of a String

(cont.)

We know that "inertia" is present in mechanical waves, and want to quantify its effect on wave speed.

→ Let F_T be the force of tension on the string; let μ [kg/m] be its linear mass density.

$$\Rightarrow v = \sqrt{\frac{F_T}{\mu}} ; \quad \mu = \frac{m}{l} = \frac{\text{mass of string}}{\text{length of string}} \quad [\text{Linear mass density}]$$

$$(*) F_T : \begin{array}{c} \uparrow \\ \text{N} \end{array} \quad \begin{array}{c} \uparrow \\ F_T = (-F) = -mg \end{array} \quad [\text{Force of tension}]$$

$\downarrow F = mg$

Power

Since waves transport energy, then they are performing work; then power = work/time.

$$1A: \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{x}}{dt} = \vec{F} \cdot \vec{v} \quad [\text{Instantaneous power}]$$

(*) Ex: Transverse Waves

Transverse waves create tension, and this results in points having velocity (i.e. changes in displacement).


$$\rightarrow v(x, t) = A \cos\left(\frac{2\pi}{\lambda}x - \omega t\right)$$

$$\Rightarrow P = \mu A^2 \omega^2 v \sin^2\left(\frac{2\pi}{\lambda}x - \omega t\right) \quad [\text{Instantaneous power}]$$

→ P is oscillating; could we get an average value?

$$\bar{P} = \frac{1}{T} \int_0^T P dt = \mu A^2 \omega^2 v \cdot \frac{1}{T} \int_0^T \sin^2\left(\frac{2\pi}{\lambda}x - \omega t\right) dt$$

$$\Rightarrow \bar{P} = \frac{1}{2} \mu A^2 \omega^2 v \quad [\text{Averaged power; Watts} = \text{J/s}]$$

Standing Waves

10/18/23

Lecture 12

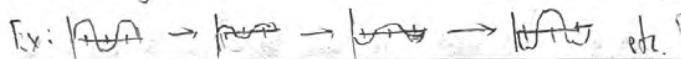
Standing Waves

When a sinusoidal wave $y_1(x, t) = A \cos\left(\frac{2\pi}{\lambda}x + wt\right)$ is reflected, the superposition of $y_1(x, t)$ and the reflected wave is:

$$\begin{aligned}y_{\text{tot}} &= y_1(x, t) + y_2(x, t) \\&= A \cos\left(\frac{2\pi}{\lambda}x + wt\right) + A \cos\left(\frac{2\pi}{\lambda}x + wt + \phi_0\right) \\&= 2A \cos\left(\frac{2\pi}{\lambda}x + \frac{\phi_0}{2}\right) \cos\left(ct + \frac{\phi_0}{2}\right)\end{aligned}\quad [\text{using identity}]$$

$$\Rightarrow y_{\text{tot}} = \begin{cases} \text{free end: } \phi_0 = 0 \rightarrow 2A \cos\left(\frac{2\pi}{\lambda}x\right) \cos(ct) \\ \text{fixed end: } \phi_0 = \pi \rightarrow 2A \sin\left(\frac{2\pi}{\lambda}x\right) \sin(ct) \end{cases} \rightarrow \text{SHO: } \begin{array}{l} \cos, \sin(ct) \text{ cause oscillation} \\ \cos, \sin\left(\frac{2\pi}{\lambda}x\right) \text{ varies amplitude by } x \end{array}$$

Def: A standing wave is a wave where the locations of peak amplitude are constant.

Ex: 

(Wave does not "travel" horizontally - only oscillates up/down)

(*) Characteristics of the Standing Wave



- Peak amplitude of $2A$ [Spatially dependent]
- Is a SHO, per the $\sin, \cos(ct)$ term
- There are special points where displacement is always 0 - called "nodes".

(*) A fixed end is always a node.

- There are points where maximal amplitude is $2A$ - called "anti-nodes".

(*) A free end is always an anti-node.

- Distance between neighboring nodes/anti-nodes is always $\frac{\lambda}{2}$ [half a wavelength].
- Distance between neighboring node, anti-node is always $\frac{\lambda}{4}$.

Standing Waves (cont.)

10/18/27

Lecture 12

Standing Waves (cont.)

(cont.)

In the case of mediums with two fixed ends, standing waves will only occur at certain wavelengths [alt: certain frequencies].

(*) Only occurs if the value $\frac{N}{2}$ precisely divides the length of the medium, such that the peaks of the reflected waves from both ends align \Rightarrow causes resonance.

(*) Always creates one more anti-node than node (not counting fixed ends)

Standing Waves w/ 2 Ends

Let $x=0$, $x=L$ be the fixed ends. For there to be a standing wave, we need:

$$\text{at a node} \Rightarrow \sin\left(\frac{2\pi}{\lambda}L\right) = 0 \Rightarrow \frac{2\pi L}{\lambda} = n\pi \quad [n \in \mathbb{N}] \Rightarrow \boxed{L = \frac{n\lambda}{2}} \quad [n \in \mathbb{N}]$$

Def: The fundamental (first harmonic) is the wavelength such that there exists a standing wave with only one anti-node.

$$\Rightarrow \boxed{\lambda = 2L; f = \frac{V}{2L}} \quad (\rightarrow 2^{\text{nd}} \text{ harmonic: } \lambda = L \quad [\text{alt: } L = \lambda_{n=2}], f = \frac{V}{L})$$

General formula: $L = \frac{n}{2} \lambda$; $f = \frac{nV}{2L}$; # anti-nodes = n ; # nodes = $n - 1$ (excl. ends)

λ (called n^{th} harmonic, or $(n-1)$ overtone)

Sound Waves

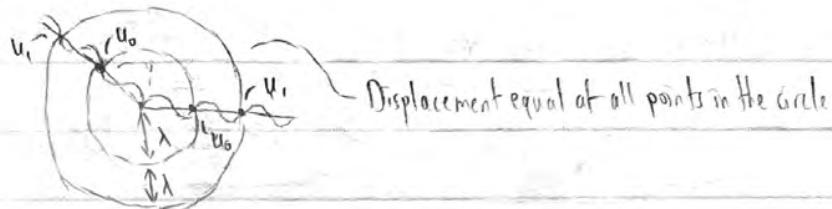
10/23/21

Lecture 14

Sound Waves

Sound waves are longitudinal waves traveling in 3 dimensions, taking air [gas] as a medium.

- Air as a medium - must consider volume/density/pressure/temperature (thermodynamics)
- Displacement as a function of radius [distance from a source] will be constant at all points across a sphere around the source

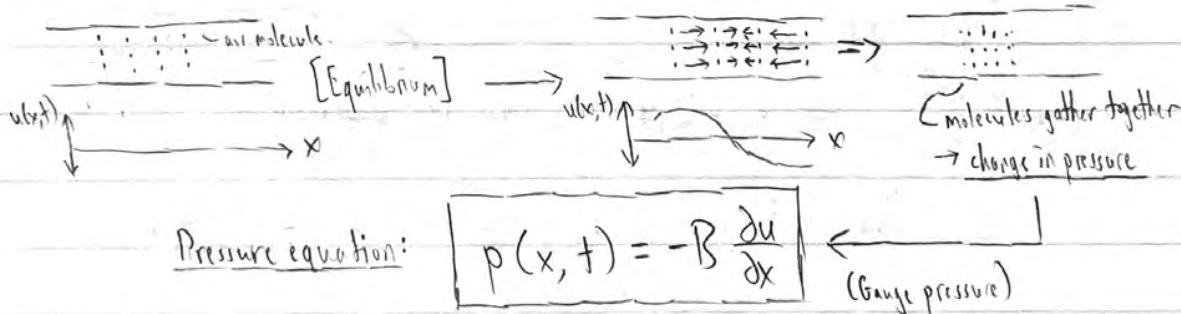


Sound Waves in Pipes

Sound waves in narrow pipes ($\delta \ll \lambda$) start to mimic 1D motion:



Displacement $u(r, t)$ of sound waves characterized as displacement from equilibrium:



$$\rightarrow B = \text{bulk modulus} \quad [\text{how easy the air is to compress}] : \Delta p = -B \frac{\Delta V}{V}$$

Wave equation: $u(x, t) = u_0 \cos\left(\frac{2\pi}{\lambda} x - \omega t\right)$ note: phase shift ($\omega t \rightarrow \sin$)

$$\Rightarrow p(x, t) = B \frac{2\pi}{\lambda} u_0 \sin\left(\frac{2\pi}{\lambda} x - \omega t\right)$$

P_{max} [Amplitude of pressure]: $p_{\max} = B \frac{2\pi}{\lambda} u_0$

Properties of Sound Waves

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Lecture 14

(*) Standing Sound Waves in Pipes

(cont.)

$$\text{Standing wave equation: } u(x, t) = U_0 \sin\left(\frac{2\pi}{\lambda}x\right) \sin(\omega t) \quad \Rightarrow \text{note: phase shift (sin} \rightarrow \text{us)}$$
$$\Rightarrow p(x, t) = -B \frac{2\pi}{\lambda} U_0 \cos\left(\frac{2\pi}{\lambda}x\right) \sin(\omega t) \quad \Rightarrow \text{swap nodes, antinodes}$$

Wave Speed of Sound Waves

Similar to strings: sound wave speed dependent only on medium.

Formulas:

(1) $v = \sqrt{\frac{\text{restoring force}}{\text{inertia}}} \Rightarrow v = \sqrt{B/\rho}$ [Physical intuition]

(2) $v = \sqrt{\frac{\gamma RT}{M}}$ [Thermodynamics]

$\hookrightarrow T = \text{temperature (K)}$; $R = \text{gas constant} (\sim 8314 \text{ J/mol K})$

$M = \text{molar mass (g/mol)}$; $\gamma = \text{ratio of heat capacities}$ (monatomic: 1.66 [H₂]; diatomic: 1.4)

(*) Wave Speed in Air

Air: $\begin{cases} M = 28.97 \\ \gamma = 1.4 \end{cases} \Rightarrow v_{\text{air}} = 0.4 v_{\text{He}}$ [v_{He} : wave speed in helium]

Intensity

Power of waves in strings becomes intensity in sound waves.

Formula:

Power: $\bar{P} = \frac{1}{2} \int T \mu' \omega^2 A^2$; can replace $T \rightarrow B$, $\mu \rightarrow \rho$, $A \rightarrow u_0$.

$$\Rightarrow \bar{I} = \frac{1}{2} \int B \rho' \omega^2 u_0^2 \Rightarrow \bar{I} = \frac{1}{2} \frac{P_{\max}^2}{\int B \rho'} \quad [\text{Intensity}]$$

Intensity

10/25/23

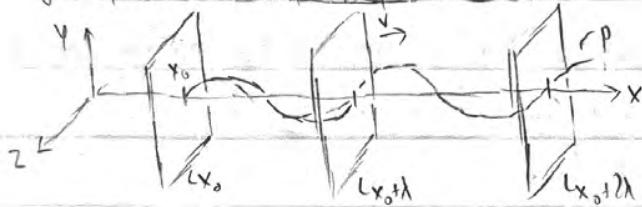
Lecture 15

Intensity & Power

$$\text{Recall: } \bar{I} = \frac{1}{2} \frac{P_{\text{max}}}{\pi r^2} \rightarrow \text{Units: } \frac{\text{Watts}}{\text{m}^2} \quad [\text{Intensity} = \frac{\text{Power}}{\text{Area}}]$$

Reasoning: Sound waves 3-dimensional \Rightarrow total power is distributed across large area

(*) Analogue: planar waves [power dependent on x]



$\Rightarrow p(x, t)$ constant [power distributed] across the points in each plane

\Rightarrow Intensity contains "power" of a single string/set of points (x, y, z) ,

such that: total power $P = IA$ [intensity · area]

(*) Spherical Waves

Wavefronts in spherical waves are spheres (as opposed to planar waves: planes)

$$\rightarrow u(\vec{r}, t) = \frac{U_0}{r} \sin\left(\frac{2\pi}{\lambda} r - wt\right)$$

$$\Rightarrow \boxed{I \sim \frac{1}{r^2}} \quad [\text{Intensity inversely proportional to } r^2] \sim \begin{matrix} \text{Astrophysics: Standard} \\ \text{candles use I to measure distance} \end{matrix}$$

$$\Rightarrow \text{Power} = IA = \frac{I_0}{r^2} \cdot r^2 = I_0 = \text{constant} \quad [\text{Power is constant}]$$

(*) Decibels

Decibels defined using logarithmic scale, relative to threshold of hearing $I_0 = 10^{-12} \frac{\text{W}}{\text{m}^2}$.

$$\text{Formula: } \boxed{B = 10 \text{ dB} \cdot \log\left(\frac{I}{I_0}\right)} \quad (*) \text{ logarithmic scale used since human ear can hear across many orders of magnitude}$$

Standing Acoustic Waves

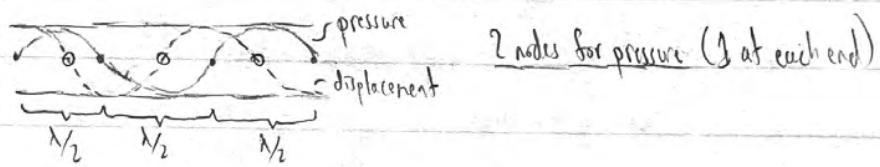
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Lecture 15 Standing Acoustic Waves in Pipes

(cont.)

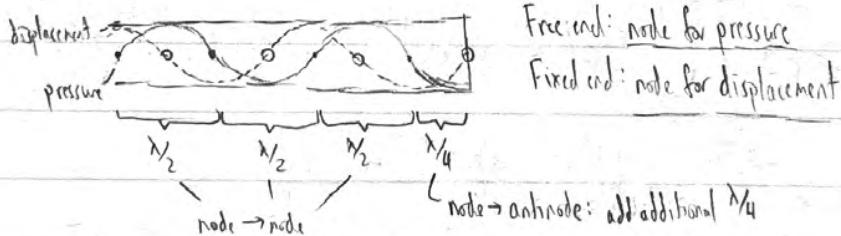
- Boundary conditions:
- (i) Closed end: cannot move [displace] past wall \rightarrow node for displacement; antinode for pressure
 - (ii) Open end: air can freely move \rightarrow node for pressure; antinode for displacement
 - (*) Intuition: open end pressure = p_{atm} \rightarrow gauge pressure = 0
 - (*) Displacement: closed end = fixed end; open end = free end
 - Pressure: closed end = free end; open end = fixed end

Cases: (i) 2 open ends



$$\rightarrow n^{\text{th}} \text{ harmonic: } L = n \frac{\lambda}{2}; \lambda_n = \frac{2}{n} L$$

(ii) 1 open end; 1 fixed end



$$\rightarrow n^{\text{th}} \text{ harmonic: } L = \frac{2n-1}{4} \lambda_n; \lambda_n = \frac{4}{2n-1} L \quad [L \text{ only comes in odd multiples of } \lambda/4]$$

Sound Waves (cont.)

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Lecture 16

(*) Beats

Beats - when two sounds of similar frequencies are heard, they may be perceived to be slightly oscillating in volume due to interference.

$$\text{Ex: } u_1 = A \sin(k_1 x + \omega_1 t), u_2 = A \sin(k_2 x + \omega_2 t) \quad [\text{superimposed}]$$

$$\Rightarrow u_{\text{tot}} = u_1 + u_2 = 2A \cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right) \sin\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)$$

$$\rightarrow f_{\text{avg}} = \frac{f_1 + f_2}{2}; f_{\text{beat}} = |f_1 - f_2|$$

$$\text{if } f_{\text{beat}} \ll f_1, f_2 \Rightarrow u_{\text{tot}} = \cos\left(\frac{k_1 - k_2}{2}x - \pi(f_1 - f_2)t\right) \sin(k_{\text{avg}}x - \omega_{\text{avg}}t)$$

$$\text{Treat } \frac{k_1 - k_2}{2}x \text{ as phase constant } \phi \rightarrow T_{\text{beat}} = \frac{1}{f_1 - f_2}$$

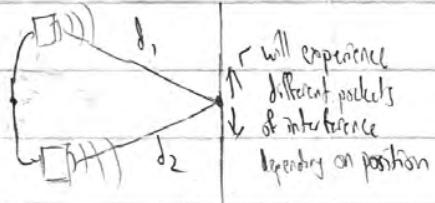
$$\Rightarrow f_{\text{beat}} = |f_1 - f_2|$$

(*) Interference

Destructive vs constructive interference:

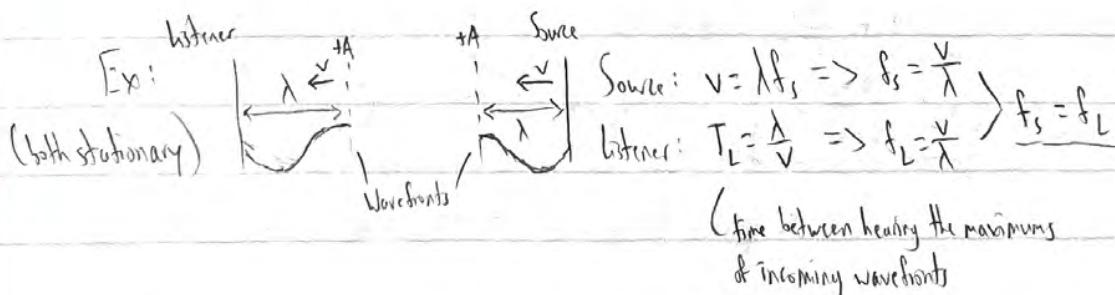
Destructive: when the superposition of two sound waves results in lower total amplitude

Constructive: when the superposition of two sound waves results in higher total amplitude



(*) Doppler Effect

Doppler Effect - Sounds are heard as being of higher frequency if the source is approaching, and of lower frequency if the source is getting further away



The Doppler Effect

10/27/23

Lecture 18

(*) The Doppler Effect (cont.)

(cont.)

Case 2: Source stationary, Listener moving

$$\text{Listener: time between hearing } +\Delta \text{ wavefronts} \rightarrow \lambda = (v \pm v_L) T_L \Rightarrow f_L = \frac{v \pm v_L}{\lambda}$$

Source still stationary (frequency unchanged) $\Rightarrow f_s = \frac{v}{\lambda}$ $\Rightarrow f_L = \frac{v \pm v_L}{v} f_s$

$v_L > 0$ if listener approaching source $\Rightarrow f_L > f_s$

$v_L < 0$ if listener approaching source $\Rightarrow f_L < f_s$ [(*) $f_L = 0$ if $-v_L > v$]

Case 3: Source moving, Listener stationary

$$\text{Source moving} \rightarrow \lambda_s = \text{distance between } +\Delta \text{ wavefronts affected} \Rightarrow \lambda_s = (v - v_s) T_s [\text{source}] \rightarrow f_s = \frac{v - v_s}{\lambda_s}$$

Listener: perceived wavelength change $\rightarrow \lambda_s f_L = v \rightarrow f_L = \frac{v}{\lambda_s} \Rightarrow f_L = \frac{v}{v - v_s} f_s$

$f_L > f_s$ if $v_s > 0$ [source approaching]; otherwise $f_L < f_s$

\rightarrow Sound barrier: $v_s > v \Rightarrow$ shockwave [$\frac{v}{v - v_s} = \infty$]

Case 4: Source and Listener moving

General formula:
$$f_L = \frac{v + v_L}{v - v_s} f_s$$
 (treats v_L, v_s as independent)

v_L, v_s positive if approaching the other; negative if moving away

(*) Assumes head-on motion; take tangential velocities otherwise

Charge

10/30/23

Lecture 17

Matter & Charge

2 kinds of charge: positive & negative (+ & -)

Atoms [matter] made up of protons (p^+), electrons (e^-), and neutrons

Electrons much lighter than protons: $m_p = 1.67 \cdot 10^{-27} \text{ kg}$

\Rightarrow Consequence: Electrons much easier to move $m_e = 9.1 \cdot 10^{-31} \text{ kg}$



[Hydrogen atom]

Ex: Protons in a solid lattice are largely fixed,



but electrons may or may not be free to



move between nuclei.

[Solid lattice]

Charge

Fundamental unit of charge: 1 unit of positive charge = charge of a single proton

1 unit of negative charge = charge of a single electron

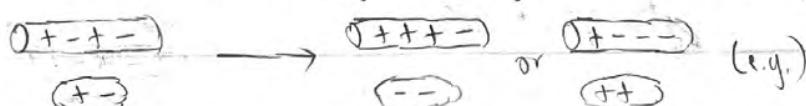
(*) SI unit: Coulombs (1 C \cong charge of $6.242 \cdot 10^{18}$ protons)

(*) Important principle: charge is conserved (i.e. no change in a closed system)

Conductors & Insulators

Def: A material (e.g. a solid) is called a conductor if charge can move inside it (i.e. some electrons can freely move inside of it). It is called an insulator if electrons cannot freely move (i.e. are bound to their nuclei).

(*) An insulator can still become charged by having electrons donated/stripped away.



Properties of Charge

18/30/23

Lecture 17

(cont.)

Properties of Charge

Observation: opposite charges attract; like charges repel. [Generate attractive/repulsive forces].

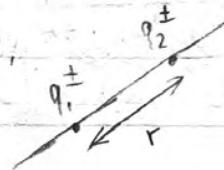
(*) More charge \rightarrow stronger force

(*) Further distance \rightarrow weaker force

Coulomb's Law

Given two point charges q_1, q_2 that are r meters apart: the force they exert on each other proportional to charge, and inversely proportional to distance,

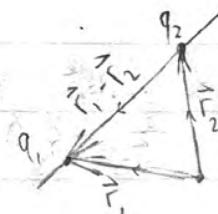
$$\Rightarrow F = \frac{1}{4\pi\epsilon_0} \cdot \frac{|q_1 q_2|}{r^2} \quad [\text{Coulomb's Law}]$$



(*) ϵ_0 is a physical constant: $\epsilon_0 = 8.854 \cdot 10^{-12} \frac{\text{C}}{\text{Nm}^2}$ [SI]

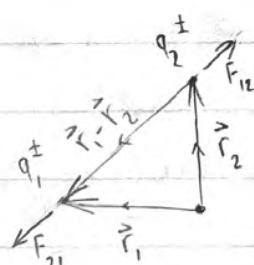
Forces $F_{2 \rightarrow 1}, F_{1 \rightarrow 2}$ [q_2 on q_1, q_1 on q_2] pointing towards each other if q_1, q_2 opposite charges; away from each other if q_1, q_2 are like charges

$$\Rightarrow \text{Coulomb's Law: } \vec{F}_{2 \rightarrow 1} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} \cdot \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

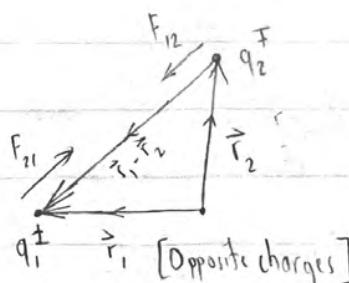


(*) For $\vec{F}_{1 \rightarrow 2}$: swap \vec{r}_1, \vec{r}_2

$$(\#) \hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} \quad [\text{unit vector } 1 \rightarrow 2]$$



[like charges]



[Opposite charges]

Electric Fields

10/31/21

Lecture 18

Coulomb's Law (cont.)

Sum of forces from multiple point charges computed by vector sum of individual forces from Coulomb's law [superposition principle]

Distributed charge. Define linear charge density μ across an object

$$\text{Ex: Charge of a ring: } Q_{\text{tot}} = \mu C = \mu 2\pi r$$

(*) Simplifying assumption: constant charge density [not always true; ex: insulators]

(*) Infinite Line Charges

Force of infinitesimal on q_0 : $d\vec{F} = \frac{1}{4\pi\epsilon_0} \cdot \frac{(q_0)(\mu dy)}{x^2+y^2} \cdot \frac{1}{\sqrt{x^2+y^2}} (-\hat{x})$

$$F_{\text{tot}} = \int_{-\infty}^{+\infty} \frac{1}{4\pi\epsilon_0} \frac{(q_0)(\mu dy)}{x^2+y^2} \cdot \frac{1}{\sqrt{x^2+y^2}} (-\hat{x}) \Rightarrow \vec{F}_{\text{tot}} = \frac{\mu q_0}{2\pi\epsilon_0} \frac{1}{x} \hat{e}_x$$

Infinite line charge \hookrightarrow Exerts force \vec{F}_{tot} on q_0 .

$$(*) \text{ Spherical Coordinates: } \vec{F}_{\text{tot}} = \frac{\mu q_0}{2\pi\epsilon_0} \frac{1}{r} \hat{e}_r$$

Electric Field

Given q_1, \dots, q_n at positions $\vec{r}_1, \dots, \vec{r}_n$; force on a charge q_0 at \vec{r}

$$\Rightarrow \vec{F}_{\text{tot}, q_0} = \sum_{j=1}^n \frac{1}{4\pi\epsilon_0} \cdot \frac{q_j q_0}{|\vec{r}_j - \vec{r}|} \cdot \hat{r}_{0j} = \underbrace{\left(\sum_{j=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\vec{r}_j - \vec{r}|} \hat{r}_{0j} \right) q_0}_{\text{Electric field created by } q_1, \dots, q_n \text{ at } \vec{r}} \rightarrow \vec{E}(\vec{r}) = \frac{\vec{F}_{\text{tot}, q_0}}{q_0}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \sum_{j=1}^n \frac{1}{4\pi\epsilon_0} \cdot \frac{q_j}{|\vec{r}_j - \vec{r}|} \hat{e}_{r \rightarrow r_j}} \quad [\text{Electric field: property of } q_1, \dots, q_n \text{ for any point } \vec{r}]$$

$$\Rightarrow \text{Force by } q_1, \dots, q_n \text{ on a charge } q_0 @ \vec{r}: \boxed{\vec{F}_{\text{tot}} = \vec{E}(\vec{r}) \cdot q_0}$$

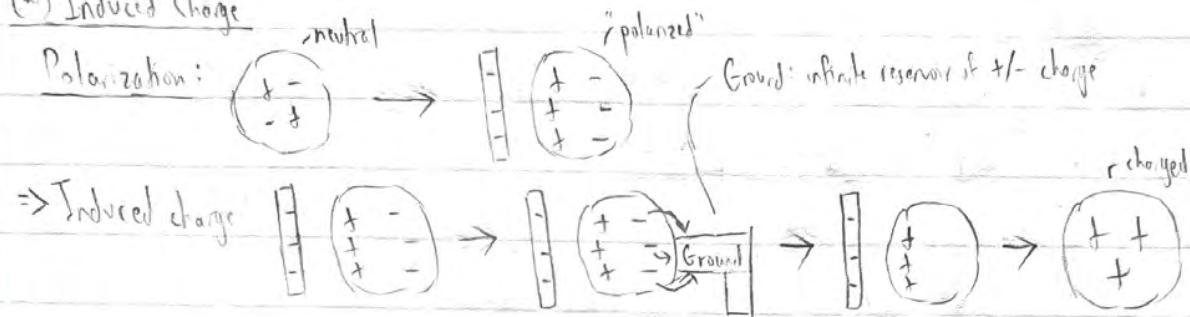
$$(*) \vec{E}(\vec{r}) \text{ for point charge } q: \vec{E}(\vec{r}) = \frac{q}{2\pi\epsilon_0 r^2} \hat{r} \quad [\text{points radially outward/inward}]$$

Electric Fields (cont.)

11/1/23

Lecture 19

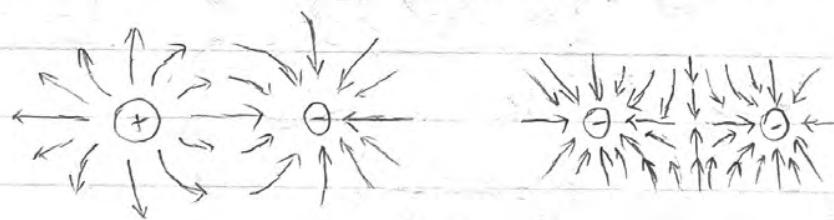
(*) Induced Charge



Electric Field

Is a vector field (as opposed to temperature - scalar field) of every point in space

- Electric field points away from positive charges; toward negative charges
- Electric field from multiple charges equivalent to vector sum (superposition) of individual electric fields



Infinite line charge:

$$\vec{E}(z, r, \theta) = \frac{\mu}{2\pi\epsilon_0} \cdot \frac{1}{r} \cdot \hat{e}_r$$

Infinite "sheet charge":

$$\vec{E} = \begin{cases} +\frac{\sigma}{2\epsilon_0} \hat{e}_z & z > 0 \\ -\frac{\sigma}{2\epsilon_0} \hat{e}_z & z < 0 \end{cases}$$

(*) $\sigma = \frac{Q}{\text{Area}}$

(*) Electric Field Lines

Visualizing electric fields: Electric field lines (drawn tangent to electric fields)

(1) Magnitude of \vec{E} proportional to density of field lines

(3) Field lines do not cross

(2) Field lines: $\oplus \rightarrow \ominus$

(4) Conductors' field lines orthogonal to surface



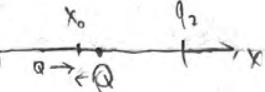
(*) Notes on Coulomb's Law

11/3/23

Lecture 20

Notes on Coulomb's Law

(*) Small variations in x around an equilibrium \rightarrow can model/approximate



Coulomb's Law as restoring force in a SHO

$$\text{Ex: } F_x = \frac{Qq}{4\pi\epsilon_0} \cdot \frac{-4dx}{(x^2 - x_0^2)} \Rightarrow \text{linear approx: } F = -\frac{Qq}{4\pi\epsilon_0} \cdot \frac{1}{x^2} x \Rightarrow F = -kx \quad [k = \frac{Qq}{4\pi\epsilon_0} \cdot \frac{1}{x_0^2}]$$

(*) Deriving electric field on infinite sheet - treat infinite sheet as union of infinite line charges

$$dE_z = \frac{\sigma}{2\pi\epsilon_0} \cdot \frac{z}{z^2 + x^2} dx$$

$$\Rightarrow E_z = \int_{-\infty}^{+\infty} \frac{\sigma}{2\pi\epsilon_0} \cdot \frac{z}{z^2 + x^2} dx = \frac{\sigma}{2\pi\epsilon_0} z \int_{0}^{+\infty} \frac{2dx}{z^2 + u^2} = \frac{\sigma}{2\pi\epsilon_0}$$

Electric Flux

11/6/23

Lecture 21 Electric Flux & Gauss's Law

Define "closed surfaces" - surfaces with no "openings" [informal defn.]

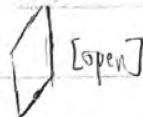
Ex:



[closed 3D surfaces]



vs [has hole/opening [boundary]]



[open]

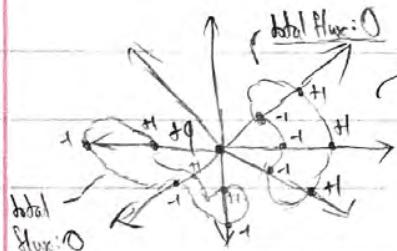
→ For closed surfaces, can unambiguously define on "inside" and on "outside"

Recall: Field lines can intersect surfaces.



→ Define flux units relative to # of intersections.

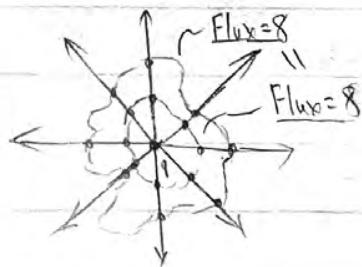
- Field line enters surface $\Rightarrow +1$ units.
 - Field line leaves surface $\Rightarrow -1$ units.
- } Defn. only "makes sense" for closed surfaces.
Open surf.: what does "leave" mean? ex: fig. above



$\sim +q$ not in the surface \Rightarrow flux through surface is 0
for any closed surface [of any shape]

$+q$ in the surface \Rightarrow flux through surface = # field lines

from q for any closed surface [of any shape]

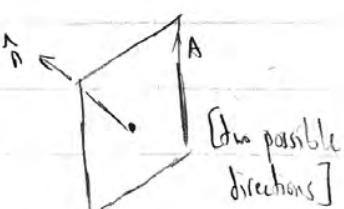
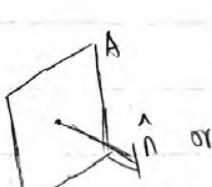


Recall: # of field lines proportional to magnitude of charge

Gauss's Law: The electric flux through a closed surface is proportional to the enclosed charge.

Surfaces

For any surface: can describe direction of surface via normal vector \hat{n} orthogonal to surface.



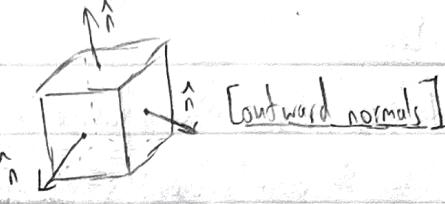
[two possible directions]

Gauss's Law

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Electric Flux

For any closed surface: set normals to all be pointing outward.



Lecture 2

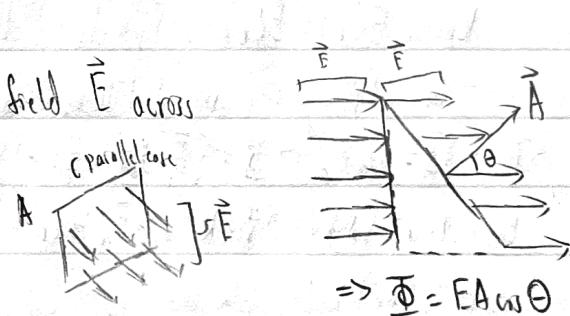
(cont.)

For a surface w/ area A , normal \hat{n} ; define area vector $\vec{A} = A\hat{n}$. [units: m^2]

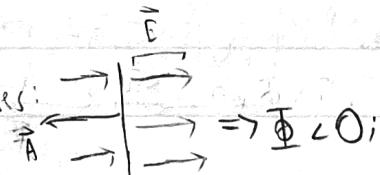
$$\Rightarrow \|\vec{A}\| = A, \vec{A} \parallel \hat{n}$$

(*) Electric flux (simple case): uniform electric field \vec{E} across surface (e.g. square) w/ area vector \vec{A} :

$$\Rightarrow \Phi = \vec{E} \cdot \vec{A}$$



(*) Special cases:



$$\Rightarrow \Phi = 0 \quad [\vec{A} \perp \vec{E}]$$

$$\begin{array}{c} A_1 \\ | \\ A_2 \\ | \\ A_3 \\ | \\ A_4 \\ | \\ A_5 \end{array} \rightarrow \Phi = \sum_{i=1}^6 \Phi_i \quad [\text{flux is sum of flux through subsurfaces}]$$

General surfaces: look at flux through infinitesimal squares & sum. [surface integral]

$$S \int dA = \int d\vec{A} \cdot \vec{E} \Rightarrow \Phi = \int_S \vec{E} \cdot d\vec{A}$$

$$\Rightarrow \boxed{\text{Gauss's Law: } \Phi_S = \frac{q_{\text{enclosed}}}{\epsilon_0}}$$

(*) Ex: flux of a point charge.

$$V_{\text{point}}: \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}, \quad d\vec{A} = dA \hat{n}$$

$$\Rightarrow d\Phi = \vec{E} \cdot d\vec{A} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dA (\hat{r} \cdot \hat{n}) \Rightarrow \Phi = \int_S d\Phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \int_S dA = \frac{q}{\epsilon_0}$$

$$\int_S dA = 4\pi r^2 \quad [\text{Surface area}]$$

Gauss's Law (cont.)

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Lecture 22

Applications of Gauss's Law

(*) If there are multiple charges q_1, q_2, \dots in closed surface $S \Rightarrow q_{\text{enc}} = \sum_{i=1}^n q_i$.

(*) Note: differential form of Gauss's Law: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ [equivalent to integrative form]

* Given a charge distribution, can calculate flux integral relatively simply to find \vec{E} using Gauss's Law.

Ex: Ball w/ uniform charge density around origin

Know $\vec{E}(r) = E(r) \hat{r}$, [points radially outward]

→ Choose ball enclosing original ball as Gaussian surface S .

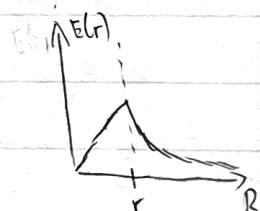
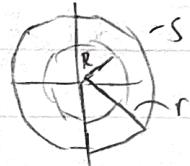
$$\text{Then: } d\vec{A} = dA \hat{e}_r \Rightarrow \oint_S \vec{E} \cdot d\vec{A} = E(r) S dA = E(r) \cdot 4\pi r^2$$

$$\text{Know: } q_{\text{enc}} = \int dV = \frac{4}{3}\pi r^3 \rho \Rightarrow F_r 4\pi r^2 = \frac{q_{\text{enc}}}{\epsilon_0} \Rightarrow E(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{enc}}}{r^2}$$

(*) Electric field inside: take sphere inside (note: adjust q_{enc} for smaller volume)

$$\rightarrow E(r) 4\pi r^2 = \frac{1}{\epsilon_0} \frac{4}{3}\pi r^3 \rho \rightarrow E(r) = \frac{1}{3\epsilon_0} r \rho$$

$$E(r) = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q_{\text{enc}}}{r^2} & r > R \\ \frac{1}{3\epsilon_0} r \rho & r \leq R \end{cases}$$



(*) Gauss's Law & Conductors

Charges can move freely inside conductors, but the locations of charges inside a conductor

eventually stabilize (conductor becomes electrostatic) → where do the charges go? [excess charges]

Conductor electrostatic \Rightarrow no electric field in conductor ($E=0$), since otherwise the

charges would move. Then $E \neq 0$ only on the surface, but only

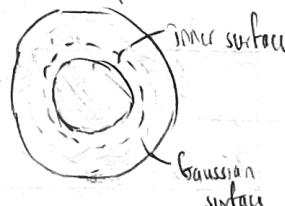
& E is orthogonal to the surface (such that charges don't move, since

they won't move into air). Since $E=0$ inside, then for any S inside,

$$\oint_S \vec{E} \cdot d\vec{A} = 0 \Rightarrow q_{\text{enc}} = 0. \text{ Then all charges are on the surface only.}$$



outer surface



If conductor has outer, inner surfaces: no excess charge on inner surface.

Pf: If outer charge inside \Rightarrow surface in conductor enclosing inner surface has $q_{\text{enc}} \neq 0 \Rightarrow \oint \vec{E} \cdot d\vec{A} \neq 0$.

Gauss's Law (cont.)

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Lecture 22

+ Lecture 23

Gauss's Law: Consequences

- (i) Any excess charge in an electrostatic conductor must be on the surface of the conductor.
- (ii) If the conductor has distinct inner & outer surfaces, there is no excess charge on the inner surface.

Recall: (i) $\vec{E} = 0$ inside the conductor



(ii) $\vec{E} \perp$ to the surface of the conductor

(*) Faraday's Cage

Conductor inside external \vec{E} :



$$(\vec{E} = 0 \text{ at } p_1, p_2 \Rightarrow \vec{E} = 0 \text{ at } p) \text{ [by continuity]}$$

$$\begin{array}{c} \vec{E} \\ \vec{E} \\ \vec{E} \end{array} \quad \begin{array}{c} \sigma \\ | \\ \text{Cage} \\ | \\ \sigma_{(1)} \quad \sigma_{(2)} \end{array} \quad \text{Take } \phi_1, \phi_2 \text{ at orthogonal sides}$$

$$\vec{E} @ 2 = 0 \Rightarrow \phi_2 = 0$$

$$\phi_1 = EA$$

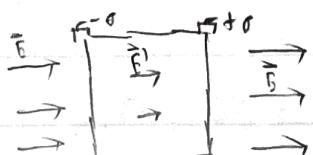
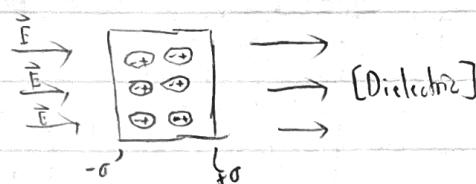
$$\Rightarrow \phi = -EA = \frac{\sigma A}{\epsilon_0} \Rightarrow \boxed{\sigma = -E \epsilon_0}$$

(can be applied to infinitesimals)

(*) Insulators in an Electric Field

Insulators - charge cannot move, but individual atoms may still orient dipoles toward electric field

\Rightarrow net surface charges still occur (just smaller)



$$\text{Electric field inside insulator: } |\vec{E}'| < |\vec{E}|$$

Gauss's Law (cont)

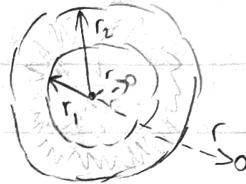
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Lecture 23

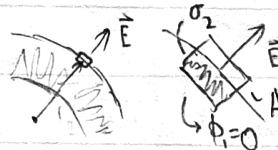
Charge in a Conducting Shell



Know: $E(r) [r \in r_1, r > r_2]$: $\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$



\vec{E} at $r = r_2$:



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \Rightarrow \phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} A = \frac{\sigma_2 A}{\epsilon_0}$$

$$\Rightarrow \sigma_2 = \frac{q}{4\pi r_2^2} \quad [\sigma @ r_2]$$

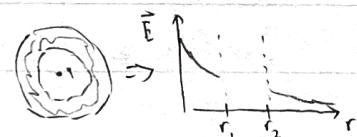
\vec{E} at $r = r_1$:



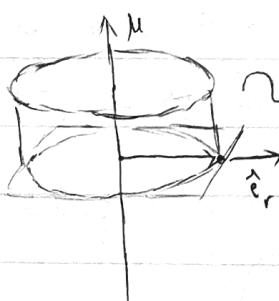
$$\phi = -\frac{1}{4\pi\epsilon_0} \frac{q}{r_1^2} A = \frac{\sigma_1 A}{\epsilon_0} \quad [E \text{ only entering} \Rightarrow \text{flux negative}]$$

$$\Rightarrow \sigma_1 = \frac{q}{4\pi r_1^2} \quad [\sigma @ r_1]$$

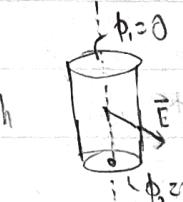
(*) Electric Field:



(*) Electric Field on an Infinite Line Charge



Gaussian surface:



$$\oint \vec{E} \cdot d\vec{A}$$

$$\begin{aligned} \phi_1 &= \Phi = \oint \vec{E} \cdot d\vec{A} \\ &= E(r) (2\pi h) \end{aligned}$$



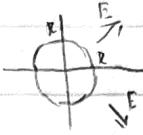
$$h \Rightarrow q_{\text{enc}} = \mu h \Rightarrow \Phi = \frac{\mu h}{\epsilon_0} \Rightarrow E(r) = \frac{1}{2\pi\epsilon_0} \frac{\mu}{r}$$

(*) Electric Field Examples

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Electric 24

Ex 1: Spherical charge distribution with $\rho = \begin{cases} \rho_0 & r \leq R \\ 0 & r > R \end{cases}$, radius R



\rightarrow find E for $r < R$, $r > R$.

$$r > R: E = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{ext}}}{r^2} \quad q_{\text{ext}} = \int_0^R dV \rho(r) = \int_0^R \theta \sin\theta \int_0^\pi d\phi \int_0^R r^2 \rho(r) dr = 4\pi \int_0^R \frac{\rho_0}{r} r^2 dr = 2\pi\rho_0 R^3$$

\rightarrow can find $E(r)$:

$$r < R: q_{\text{ext}} = \int_0^r dV \rho(r) = 4\pi \int_0^r \frac{\rho_0}{r} dr = 2\pi\rho_0 R r^2 \quad \Phi(r) = 4\pi r^2 E(r) = \frac{q_{\text{ext}}}{\epsilon_0} \rightarrow E(r) = \frac{\rho_0}{2\epsilon_0} R$$

~ Q: Given particle charge q @ r , what v_{tan} is needed for the particle to orbit?

$$\text{Diagram: } \begin{array}{c} \text{charge } q \\ \text{at } r \\ \text{with velocity } v_{\text{tan}} \\ \text{and force } F_E = F_{\text{centrifugal}} \end{array} \rightarrow qE = -mv_{\text{tan}}^2/r = -mv_{\text{tan}}^2 \frac{q_{\text{ext}}}{4\pi\epsilon_0 r^2} \quad \text{assuming } \rho_0, q \text{ opposite signs}$$

$$v_{\text{tan}} = \sqrt{\frac{q_{\text{ext}}}{2\pi\epsilon_0} \frac{R^3}{r}}$$

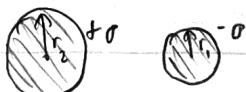
Ex 2: Infinite line charge μ_0 , titled "dipole" ($-\mu_0$, $+\mu_0$) R away \rightarrow find F on dipole.

$$E(r) = \frac{1}{2\pi\epsilon_0} \frac{1}{r} \hat{r} \quad \text{Dipole: } \begin{array}{c} \text{length } l \\ \text{into infinitesimals} \end{array} \rightarrow dF = (\pm \mu_0 dl) \cdot \frac{1}{2\pi\epsilon_0} \frac{1}{r}$$

$$r = R + l \sin\theta \rightarrow dF = (\pm \mu_0 dl) \cdot \frac{1}{2\pi\epsilon_0} \frac{1}{R \sin\theta} ; F_{\text{tot}}^{(+)} = \int_0^l \frac{\mu_0}{2\pi\epsilon_0} \frac{dl}{R \sin\theta} = \frac{\mu_0}{2\pi\epsilon_0} \frac{1}{\sin\theta} \left(\ln(R + \frac{l}{2} \sin\theta) - \ln(R) \right) \\ \rightarrow = \frac{1}{2\pi\epsilon_0 \sin\theta} \ln\left(1 + \frac{l}{2R} \sin\theta\right) ; \text{ similar for } F_{\text{tot}}^{(-)} ; F_{\text{tot}}^{(+)} = F_{\text{tot}}^{(+)}/F_{\text{tot}}^{(-)}$$

Ex 3: Find ring of charge w/ inner radius r_1 , outer radius $r_2 \rightarrow$ find $E(x)$

$$\text{Recall: Disc of charge } \rightarrow E(x) = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{1}{\sqrt{\frac{x^2}{R^2} + 1}} \right) \quad [\text{from HW: integrate rings of charge}]$$

\rightarrow For ring: superpose 2 discs 

$$\rightarrow E(x) = E^+ - E^- = \frac{1}{2\epsilon_0} \left(\frac{1}{\sqrt{\frac{x^2}{r_2^2} + 1}} - \frac{1}{\sqrt{\frac{x^2}{r_1^2} + 1}} \right)$$

$$\text{Let } x \ll r_1, r_2: E \approx \frac{1}{4\pi\epsilon_0} \frac{\sigma\pi(r_1^2 - r_2^2)}{x^2}$$

Electrostatic Potential

11/14/23

Lecture 25: Electrostatic Potential Energy

Observation: Electrostatic force equation [Coulomb's Law] similar to that of gravity (a conservative force)

⇒ electrostatic force is conservative (can derive an electrostatic potential energy U)

Recall (1A: Work): (i) $W_{i \rightarrow f} = \int_{i \rightarrow f} \vec{F} \cdot d\vec{x}$

(ii) $W_{i \rightarrow f} = K_f - K_i$

(iii) Conservative forces: $W_{i \rightarrow f} = -\Delta U = U_i - U_f$ (W independent of $i \rightarrow f$)

⇒ conservation of energy: $K_i + U_i = K_f + U_f$

Ex: Constant Electric Field

$$\vec{E} = -\frac{\sigma}{\epsilon_0} \hat{z}_2$$

$$F = q_{\text{test}} \vec{E} = -q_{\text{test}} \frac{\sigma}{\epsilon_0} \hat{z}_2 \rightarrow W = \int_{z_i \rightarrow z_f} \left(-q_{\text{test}} \frac{\sigma}{\epsilon_0} \hat{z}_2 \right) \cdot d\vec{z}$$

$$\rightarrow -\Delta U = -W = \left(-q_{\text{test}} \frac{\sigma}{\epsilon_0} \right) (z_f - z_i) = U_f - U_i$$

$$\rightarrow U(z) = q_{\text{test}} \frac{\sigma}{\epsilon_0} z \quad (*) \text{ analogous to } U = mgh \text{ [gravitation]}$$

(*) Reminder: potential energy is relative to any arbitrary "0" (ex: set $0 = U(\infty)$)

Observation: For a positive charge, U decreases in direction of electric field (E increases)

↳ Similar to gravity (U decreases in direction of gravitational field - "falling")

Ex: Simple Point Charge

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{test}} q}{r^2} \hat{r}_r \rightarrow W_{r_i \rightarrow r_f} = \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r} = \int_{r_i}^{r_f} \frac{1}{4\pi\epsilon_0} \frac{q_{\text{test}} q}{r^2} (\hat{r}_r \cdot \hat{r}_r) dr$$

$$\rightarrow W_{r_i \rightarrow r_f} = \frac{1}{4\pi\epsilon_0} q_{\text{test}} q \left(-\frac{1}{r_f} + \frac{1}{r_i} \right) = -U_f + U_i$$

$$\rightarrow U(r) = \frac{q_{\text{test}} q}{4\pi\epsilon_0} \frac{1}{r} \quad U(r) \quad (q, q_{\text{test}} > 0)$$

Electric Potential

11/14/23

Lecture 25

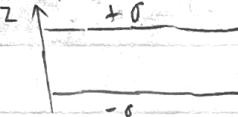
Electric Potential

Deriving electric potential V from U akin to deriving electric field \vec{E} from \vec{F} (i.e. $\vec{E} = \frac{\vec{F}}{q_{test}}$)

(cont.)

$$\rightarrow \Delta V = \frac{\Delta U}{q_{test}} ; \text{ units: Volts} = \frac{\text{Joule}}{\text{Coulomb}}$$

Ex: Infinite Charged Sheets

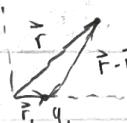


$$U(z) = \frac{q_{test} \sigma}{\epsilon_0} z \Rightarrow V(z) = \frac{\sigma}{\epsilon_0} z \quad [\text{Note: is a scalar}]$$

Ex: Simple Point Charge



$$U(z) = \frac{1}{4\pi\epsilon_0} \frac{q_{test} q}{r} \Rightarrow V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad [r = |\vec{r}|]$$



$$\text{Alt.: } V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

(*) Superposition

Superposition holds for electric potential.

(*) Ex: Point charges q_1, \dots, q_n at $\vec{r}_1, \dots, \vec{r}_n$

$$\rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

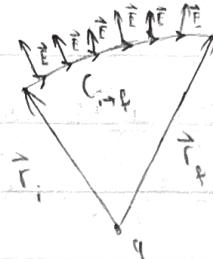
Electric Potential (cont.)

11/15/23

Lecture 26 Methods for Finding Electric Potential

1) Integrate electric field \vec{E} along a path: ($\vec{E} \rightarrow V$)

2) Superposition principle



1) Integrating Electric Field

$$\Delta V = V_f - V_i = \frac{1}{q_{\text{test}}} \left(\int_{C_{\text{test}}} \vec{E} \cdot d\vec{r} \right) \Rightarrow \boxed{\Delta V = - \int_{C_{\text{test}}} \vec{E} \cdot d\vec{r}}$$

(*) ΔV independent of path C_{test} taken

Observations:

i) V decreases when traveling along field lines (electric field)

ii) If a path is completely orthogonal to the electric field $\rightarrow \Delta V = 0$

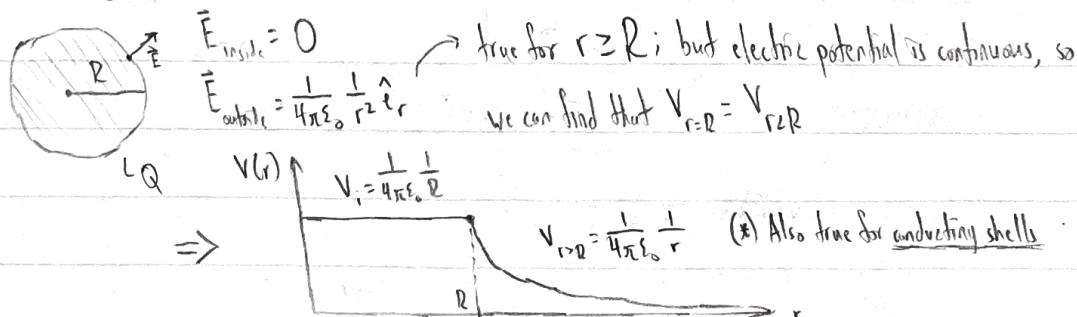
Consequence: $V_f = V_i$ if \vec{r}_f, \vec{r}_i connected by a path always orthogonal to \vec{E}

iii) In a region with no electric field, all points have the same potential (inside electrostatic conductor, e.g.)

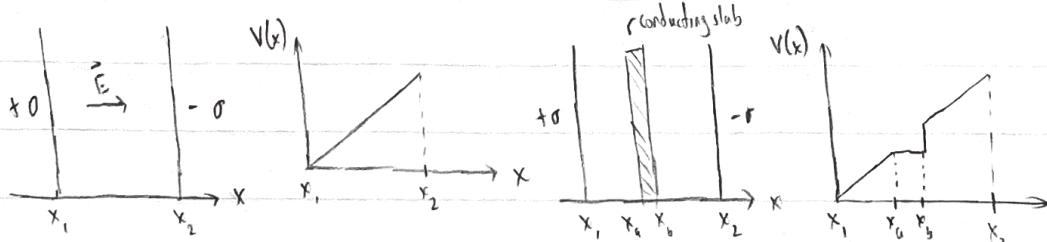
(*) Corollary [ii) & iii): All points inside, or on the surface of, an electrostatic conductor

have the same potential

Ex: Conducting Sphere



Ex: Charged Sheets



Electric Potential (cont.)

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2) Superposition Principle

Electric potential from multiple charges is the superposition of the electric potential from each individual charge.

Ex: Finite Line Charge

$$V(x, z=0) = \frac{\mu}{4\pi\epsilon_0} \int_{-L}^{+L} \frac{dz}{\sqrt{x^2 + z^2}} = \frac{\mu}{4\pi\epsilon_0} \ln(z + \sqrt{z^2 + r^2}) \Big|_{-L}^{+L}$$

$$= \frac{\mu}{4\pi\epsilon_0} \ln \left(\frac{1 + \sqrt{1 + \frac{x^2}{L^2}}}{-1 + \sqrt{1 + \frac{x^2}{L^2}}} \right)$$

Take $\lim_{L \rightarrow \infty}$

Ex: Infinite Line Charge

$$\vec{E} = \frac{\mu}{2\pi\epsilon_0} \frac{1}{r} \hat{r}$$

$$V(r_f) - V(r_i) = - \int \vec{E} \cdot d\vec{r} = \frac{\mu}{2\pi\epsilon_0} \int_{r_i}^{r_f} \frac{1}{r} dr$$

$$= - \frac{\mu}{2\pi\epsilon_0} (\ln(r_f) - \ln(r_i)) \rightarrow V(r) = - \frac{\mu}{2\pi\epsilon_0} \ln(r)$$

(*) Tesla Coils: Twin Conducting Spheres

Know:

$$V(r_a) = \frac{q_a}{4\pi\epsilon_0 r_a}$$

$$V(r_b) = \frac{q_b}{4\pi\epsilon_0 r_b}$$

$$V(r_a) = V(r_b) \rightarrow \frac{q_a}{r_a} = \frac{q_b}{r_b}$$

Recall: $E = \frac{1}{4\pi\epsilon_0 r} \frac{q}{r}$

$$\frac{E_a}{E_b} = \frac{q_a}{r_a^2} \frac{r_b^2}{q_b} = \frac{r_a}{r_b} \rightarrow E_a r_a = E_b r_b$$

Equipotential Surfaces

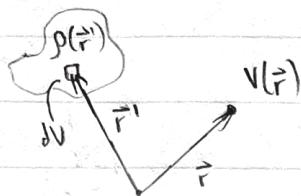
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Lecture 27 Superposition (cont.)

Given charge distribution $\rho(\vec{r})$, point \vec{r} :

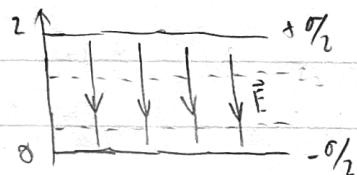
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r'$$

(integrate over charge distribution)



(*) Integral guaranteed to converge for finite charge distributions; may or may not converge for infinite charge distributions

Electric Potential & Field Lines



$$\vec{E}(z) = -\frac{q}{\epsilon_0} \hat{z} \quad [\text{constant } V \text{ z inside}]$$

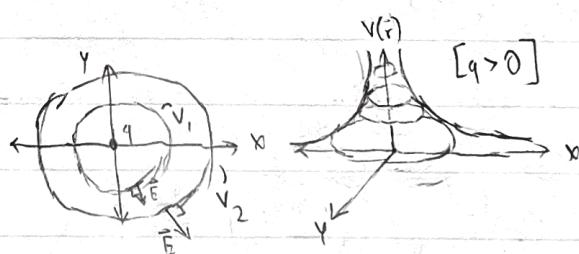
$$V(z) = \frac{q}{\epsilon_0} z \quad [\text{not constant}] \quad (\partial V @ z=0)$$

\Rightarrow at any z : all points at that z have equal potential V
 → plane at any z is an equipotential surface

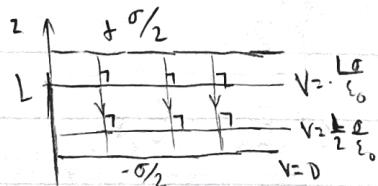
(*) Equipotential Surfaces

Simple point charge:

$$V = \frac{q}{4\pi\epsilon_0 r}$$



Infinite Charged Sheets:



Observations:

(i) Electric field lines are orthogonal to equipotential surfaces at every point.

(ii) Electric potential decreases in the direction of the electric field. [Electric field lines point toward lower potential].

Electric Potential & Electric Field

11/20/23

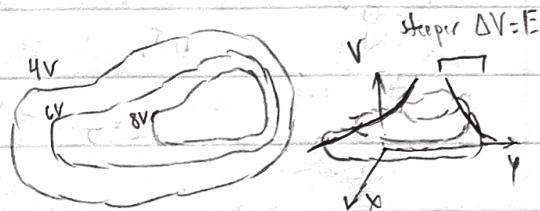
Equipotential Surfaces (cont.)

- (*) Equipotential surfaces have V everywhere the same, but not necessarily \vec{E} everywhere the same. (cont.)

Can read equipotential lines like a topographic map:

Density of lines \approx rate of change in potential [E]

(closer lines \rightarrow larger electric field)



Electric Potential \rightarrow Electric Field ($V \rightarrow E$)

$$\text{Recall: } V = -\int \vec{E} \cdot d\vec{x} \rightarrow V(\vec{x}) = V(\vec{x}_1) - \int_{\vec{x}_1}^{\vec{x}} \vec{E} \cdot d\vec{x} \Rightarrow \vec{E} = -\nabla V(\vec{x})$$

[32A: $-\nabla f(\vec{v})$ = direction of greatest descent]

(*) E_p :

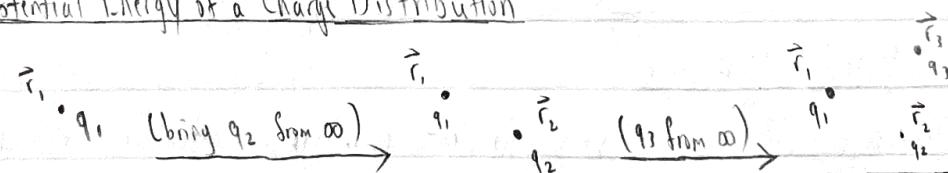
$$\text{Charged Sheet: } V(\vec{x}) = \frac{\sigma}{\epsilon_0} z \rightarrow \nabla V(\vec{x}) = -20, 0, -\frac{\sigma}{\epsilon_0} \hat{e}_z = -\frac{\sigma}{\epsilon_0} \hat{e}_z \quad \checkmark$$

$$\text{Point Charge: } V(\vec{x}) = \frac{q}{4\pi\epsilon_0 r} \rightarrow -\nabla V(\vec{x}) = -\nabla \left(\frac{q}{4\pi\epsilon_0 \sqrt{x_x^2 + x_y^2 + x_z^2}} \right)$$

$$\rightarrow -\frac{q}{4\pi\epsilon_0} \nabla \left(\frac{1}{r} \right) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \left(\frac{x_x}{r} \hat{e}_x + \frac{x_y}{r} \hat{e}_y + \frac{x_z}{r} \hat{e}_z \right) = \frac{q}{4\pi\epsilon_0 r^2} \hat{e}_r \quad \checkmark$$

Line Charge: (Use cylindrical coordinates)

(*) Potential Energy of a Charge Distribution



$$U = 0 \longrightarrow U_{tot} = U_{1,2} = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|}$$

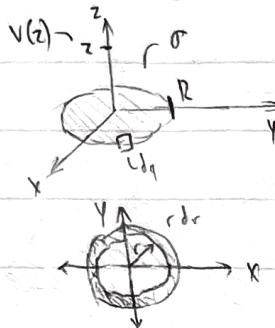
$$U_{tot} = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|} + \frac{q_1 q_3}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_3|} + \frac{q_2 q_3}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_3|}$$

Capacitance

11/21/23

Lecture 28

(*) Potential of a Solid Disk



1) Take potential of ring w/ radius r , linear charge density λ :

$$dq = \lambda r dr \rightarrow V_\lambda = \frac{1}{4\pi\epsilon_0} \int_{0}^{2\pi} \frac{\lambda r}{\sqrt{z^2 + r^2}} d\phi = \frac{2\pi}{4\pi\epsilon_0} \frac{\lambda r}{\sqrt{z^2 + r^2}} = \frac{1}{2\epsilon_0} \frac{\lambda r}{\sqrt{z^2 + r^2}}$$

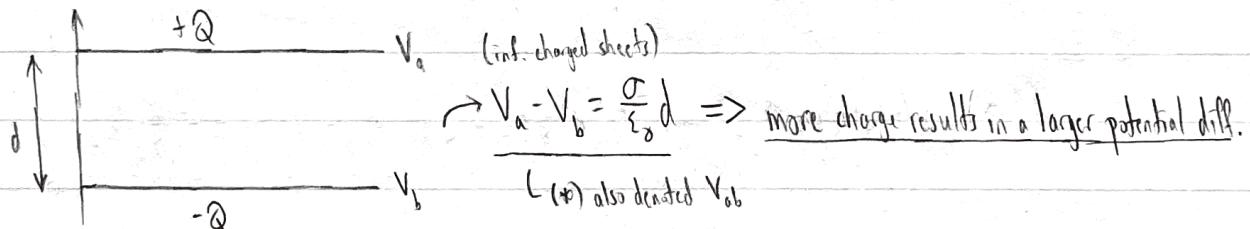
2) Integrate whole disk via rings:

$$\lambda = \sigma dr \rightarrow V_R = \int_{0}^{R} \frac{r \sigma dr}{2\epsilon_0 \sqrt{z^2 + r^2}} = \frac{\sigma}{2\epsilon_0} \int_{0}^{R} \frac{r}{\sqrt{z^2 + r^2}} dr = \frac{\sigma}{2\epsilon_0} (\sqrt{R^2 + z^2} - \sqrt{z^2}) \\ = \frac{\sigma}{2\epsilon_0} (\sqrt{R^2 + z^2} - z) \quad [z > 0]$$

(*) 3) Find electric field: $\vec{E} = -\nabla V = -\frac{\partial}{\partial z} V \hat{e}_z$ [since x, y not in expression for V on x -axis] $\rightarrow \vec{E} = -\frac{\sigma}{2\epsilon_0} \left(\frac{z}{\sqrt{z^2 + R^2}} - 1 \right) \hat{e}_z \quad [z > 0]; \vec{E} = -\frac{\sigma}{2\epsilon_0} \left(\frac{z}{\sqrt{z^2 + R^2}} + 1 \right) \quad [z < 0]$

Capacitance

When two charges (initially together) are separated, then a potential difference will be created.



→ Def: Capacitance is the "proportionality factor" between charge and potential difference.

$$\Rightarrow V_{ab} (= Q) \rightarrow C = \frac{Q}{V} \quad ; \text{ units: F [Farad: } \frac{\text{Coulomb}}{\text{Volt}} \text{]}$$

→ A capacitor is a device that can store charge when a potential difference is applied.

(*) Creating a Potential Difference

Typically use a battery/power supply to create & maintain a potential difference.

(non-electrical)

Can be done via chemical processes (e.g. a battery), or mechanically (Van der Graaff generator).

(Capacitance (cont.))

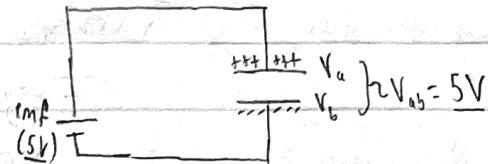
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Lecture 28

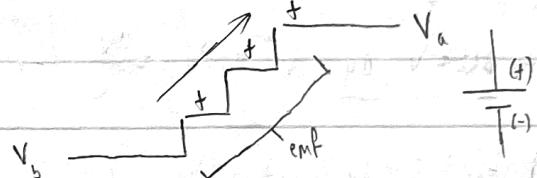
Electromotive Force (emf)

Potential difference supplied by emf [Electromotive force], e.g. from a battery. (Units: Volts)

L "emf" - use as catch-all for black-box processes supplying ΔV



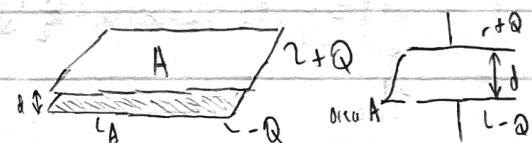
(*) Analogy - emf as a "charge elevator"



(*) Capacitance of a Plate Capacitor

Approximate a plate capacitor [2 parallel plates]

by assuming inf. area ($\Rightarrow \vec{E} = 0$ outside; uniform inside)



$$\hookrightarrow V_a - V_b = \frac{\sigma}{\epsilon_0} d, \sigma = \frac{Q}{A} \Rightarrow C = \frac{A \epsilon_0}{d}$$

(*) Approximation works best when plates are close together [d is small]

(*) Capacitance of a Spherical Capacitor



Recall: Spherical shell w/ charge Q , radius R $\hookrightarrow V = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{1}{R} & r \leq R \\ \frac{1}{4\pi\epsilon_0} \frac{1}{r} & r > R \end{cases}$

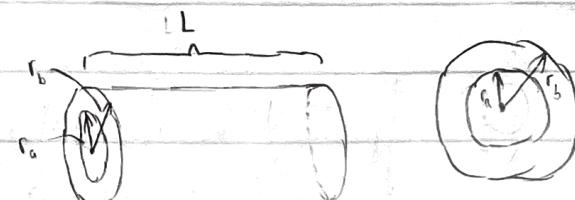
$$V_{\text{tot}} = V(+Q, r_a) + V(-Q, r_b) \leftarrow$$

$$\Rightarrow V(r_a) - V(r_b) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right) \hookrightarrow C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$

(*) Capacitance of a Cylindrical Capacitor

Approximate a cylindrical capacitor by

assuming infinite length ($L = \infty$)



$$\hookrightarrow C = 2\pi\epsilon_0 \frac{L}{\ln(r_b/r_a)}$$

Capacitors

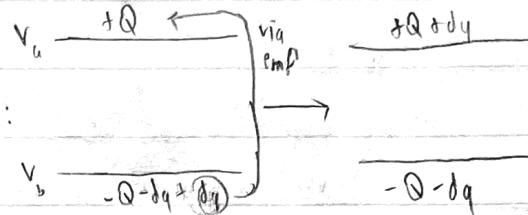
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Lecture 29

Capacitors

(cont.)

Capacitors store energy. For (Plate) capacitors:



Moving charge = work performed.

$$\hookrightarrow \delta U = V_{ab} dq. \text{ Recall: } V_{ab} = \frac{Q}{C} \rightarrow \delta U = \frac{1}{C} q dq \rightarrow U = \int_0^Q \delta U \Rightarrow U = \frac{Q^2}{2C}$$

(*) Stored Energy Equation (Variants)

$$U = \frac{1}{2C} Q^2 \Leftrightarrow U = \frac{1}{2} QV \Leftrightarrow U = \frac{1}{2} (V_{ab})^2$$

$$(*) U = \frac{1}{2} (V)^2$$

(*) Capacitors as Batteries

useful in circuits, e.g.

Capacitors are good at storing & releasing energy quickly; are not good for storing large quantities of energy (i.e. are not good batteries).

(*) Ex: EV battery: 60 kWh $\rightarrow 2.16 \cdot 10^8$ J [standard for EVs]

If using plate capacitor w/ $V=1000$ V [substantial], $d=0.5$ mm $\Rightarrow A = \frac{2Vd}{\epsilon_0 V^2} = 3.15 \cdot 10^{10}$ m [too large]

\hookrightarrow preferred: store energy as chemicals (e.g. lithium ion batteries)

Energy Density of an Electric Field [1C]

$$\hookrightarrow U = \frac{1}{2} (V_{ab})^2, \text{ Recall: } V_{ab} = Ed \text{ [inf. charged sheets]}$$

$$\hookrightarrow U = \frac{1}{2} \frac{\epsilon_0 A}{d} V_{ab}^2 = \frac{1}{2} \frac{\epsilon_0 A}{d} d^2 E^2 = \frac{1}{2} \epsilon_0 A d E^2$$

$$\hookrightarrow \text{Observe } V [\text{volume}] = A \cdot d \Rightarrow U = \frac{1}{2} \epsilon_0 E^2 V \Rightarrow \boxed{\frac{U}{V} = u = \frac{1}{2} \epsilon_0 \bar{E}^2}$$

Energy density
at \bar{E}

[used for
electric waves, e.g.]

Dielectrics

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Lecture 29

Relative Permittivity

Previously: Assumed potential difference has been across a vacuum. $[V_0 = V_{\text{vacuum}}]$

(cont.)

Insert material between capacitors $\rightarrow V' = \frac{V_0}{K}$, where $K > 1$ [material constant]

$$\rightarrow \text{Capacitance: } C_0 = \frac{Q}{V_0} \Rightarrow C_{\text{matter}} = K \frac{Q}{V_0} \Rightarrow \boxed{C_{\text{matter}} = K \cdot C_0} \quad [\text{capacitance increases}]$$

Up to now: $C_0 = \epsilon_0 \times \text{something} \rightarrow C_{\text{matter}} = K \cdot \epsilon_0 \cdot \text{something}$

("permeability of space")

relative permittivity/dielectric constant

$$\rightarrow \text{Can define } \epsilon = K \cdot \epsilon_0 \Rightarrow$$

$$\boxed{K = \frac{\epsilon}{\epsilon_0}}$$

(*) Effect on Stored Energy

Case 1: Q fixed

$$U_0 = \frac{Q^2}{2C_0} \rightarrow U_{\text{matter}} = \frac{Q^2}{2KC_0} = \frac{1}{K} U_0 \quad [U \text{ decreases}]$$

due to plate doing work on

inserted material

Case 2: V_{ab} fixed (connected to emf)

$$U_0 = \frac{1}{2} C_0 V^2 \rightarrow U_{\text{matter}} = \frac{1}{2} K C_0 V^2 = K U_0 \quad [U \text{ increases}]$$

inserting material requires doing work

Dielectrics (cont.)

11/27/23

Lecture 30

Dielectrics

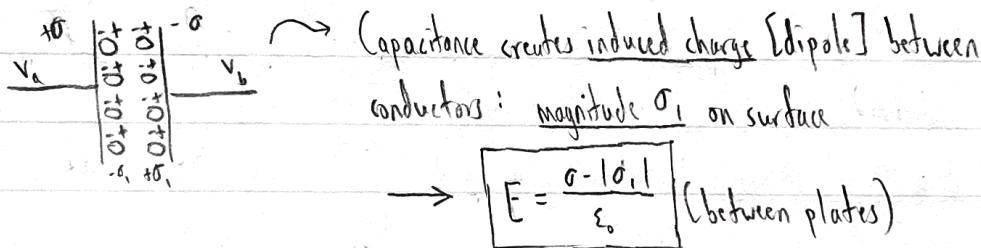


Analogy: ϵ_0 denotes material constant for "vacuum" [air]
 $\rightarrow \epsilon = \kappa \epsilon_0$ (material constant for vacuum)

$$E_0 = \frac{1}{\epsilon_0} \cdot \text{something}^* ; E = \frac{1}{\kappa \epsilon_0} \cdot \text{something}^* \quad (E \propto E_0)$$

Intuition

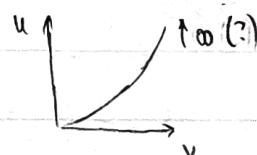
In a vacuum:



Know: $E_{\text{matter}} = \frac{\sigma}{\kappa \epsilon}$ → $|\sigma_1| = \left(1 - \frac{1}{\kappa}\right) \sigma$ ~ In medium, dipoles generated stronger than in air

(*) Limitations of Capacitors

Recall: $U = \frac{1}{2} C V^2$; why can't we keep increasing V to store more energy?
 ↴ or whatever medium



→ Answer: Air acts as an insulator between charged surfaces in a capacitor;

but if V [E] grows too large → causes dielectric breakdown:

the insulator itself becomes conducting [capacitor shorts out].



ex: Van de Graaff generator

Consequence: For any capacitor, \exists a limit on V . Namely: $\frac{V}{d} > \text{dielectric strength} \Rightarrow \text{short}$

(*) Dielectric strength is a material constant (ex: $3.6 \cdot 10^6$ V/m in air)

(*) Observe: maximum V is lower for smaller value of d

Capacitors in Series & in Parallel

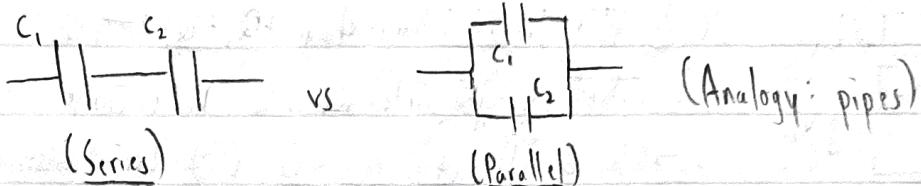
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(cont)

Capacitors in Series & in Parallel

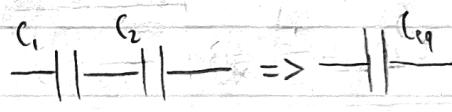
Given two capacitors C_1, C_2 ; have two configurations (in series & in parallel)



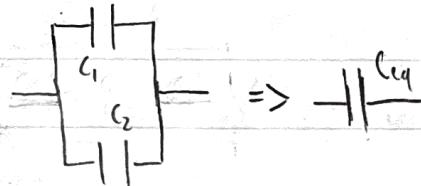
→ Capacitors combined in series/in parallel, still behave as a capacitor when combined.

In particular: can treat as a single capacitor w/ capacitance C_{equiv} .

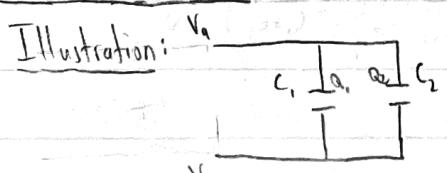
$$\text{Series: } C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2} \quad \text{Alt: } \frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$



$$\text{Parallel: } C_{\text{eq}} = C_1 + C_2 \quad [\text{Capacitance sum}]$$

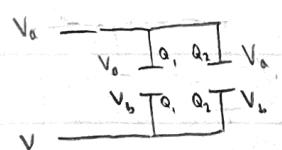


Capacitors in Parallel



Observe: V_a must be the same for both C_1, C_2 [since they are connected by a conducting wire] i same for V_b

$$\rightarrow C_1 = \frac{Q_1}{V_{ab}}, \quad C_2 = \frac{Q_2}{V_{ab}}$$



$$\text{Looking at totals: } C_{\text{tot}} = \frac{Q_{\text{tot}}}{V_{ab}} = \frac{Q_1 + Q_2}{V_{ab}} = C_1 + C_2 \quad \checkmark$$

Capacitors in Series

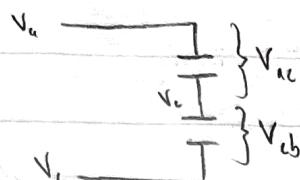
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Lecture 30

(cont.)

Capacitors in Series

Illustration:



Gauss's law:

Recall: plate capacitors have charges $+Q, -Q$: (i)

In initial uncharged state: middle wire has no charge

\rightarrow when charged: net charge = 0 (ii) [charge conservation]

Then:



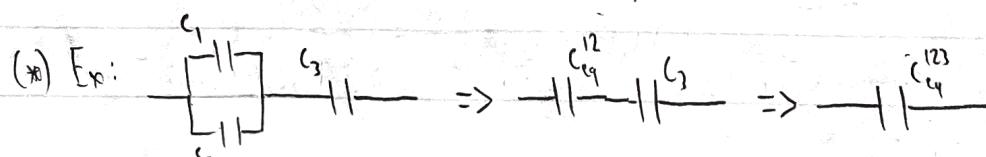
\rightarrow Consequence: Charges are the same across capacitors in series.

Know: $V_{ab} = V_{ac} + V_{cb}$; $\frac{+Q}{C_1} \parallel \frac{-Q}{C_2} \parallel \frac{+Q}{C_3} \parallel \frac{-Q}{C_4}$

$$\rightarrow V_{ab} = \frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C_{eq}} \Rightarrow \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C_{eq}} \quad \left[C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \right] \quad \checkmark$$

(*) Capacitor Configurations

Can reduce complex configurations of charges to simpler ones.



Current

11/28/23

Lecture 31

Current

Def: Current is the motion of charge due to a potential difference [flow of charge/unit time]

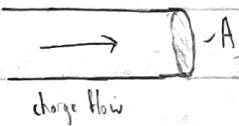
$$\rightarrow I = \frac{\Delta Q}{\Delta t} \stackrel{\text{inst.}}{=} \frac{dQ}{dt} \quad \text{unit: Amperes} \left[\frac{\text{Coulombs}}{\text{second}} \right]; A, \text{amps}$$

(*) Note: 1 Coulomb is a substantial charge $\rightarrow 1 \text{ A}$ is a fairly sizable current

(*) Idea of current analogous to flow of fluid; conducting wires as pipes, emf as a "pump"

Think of wires as pipes \rightarrow can define current density:

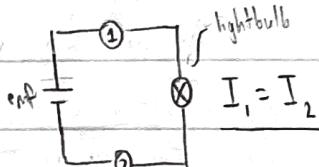
$$J = \frac{I}{A}$$



Circuits are a closed system [charge conserved]

\rightarrow no charge is ever gained or lost at any point ($Q_{in} = Q_{out}$)

\rightarrow Consequence: Current is equal at all points in a circuit.



Formulation for charge density: $J = q_e n V \Rightarrow$ Charge through A in time Δt : $dQ = q_e n A V dt$

q_e = "charge carrier" [electrons]; n = number density ($\frac{\# \text{ carriers}}{V}$); V = velocity

(*) Note: v describes speed of individual charge carriers; speed of signal propagation is much faster (electrons push each other, e.g., closer to speed of light)

(*) Flaws of "Free-Charge" Model

Observation: Current resulting from a constant potential difference, is constant (does not change over time).

Problem: Assume free charge carriers.

$$\begin{array}{l}
 \text{assume constant } \vec{E} = \frac{V_{ab}}{d} \\
 \text{inf } \boxed{d} \quad V_a \quad V_b \\
 \downarrow \quad \downarrow \quad \downarrow \\
 \text{assume } q_e V_{ab} = F = m_e a \\
 \rightarrow m_e a = F = q_e E \\
 \rightarrow a = \frac{q_e V_{ab}}{m_e d} \Rightarrow \text{current is changing [contradiction]}
 \end{array}$$

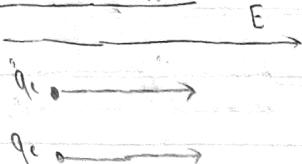
Ohm's Law

11/28/23

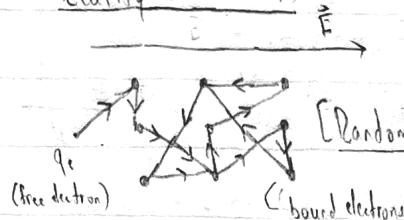
Lecture 31 Ohm's Law (Intuition)

+ Lecture 32

Fermi-Carrier Model:



"Reality" (Drude model)



- Due to collisions between carriers

- Affected by temperature (random motion)

- Carriers still drift in dir. of E

Properties

$\langle v \rangle$ - average velocity across all particles over time

$\langle \tau \rangle$ - average time between collisions for free electrons [small timescale: 10^{-10} s, e.g.]

- During time between collisions, free electrons act as point particles [are accelerated by electric field]; is "reset" on collision $\rightarrow v$ does not increase over time

- Drift velocity - velocity obtained by charge carriers during time between collisions

Equation:

$$v_d = a \langle \tau \rangle = \frac{q_e}{m_e} E \langle \tau \rangle$$

Ohm's Law (Derivation)

Plugging v_d into charge density eq: $J = q_e n v_d = \frac{q_e^2 \langle \tau \rangle}{m_e} n E$ not dependent on geometry!

material constants

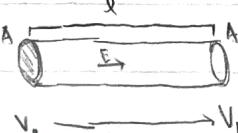
\rightarrow define conductivity:

$$\sigma = \frac{q_e^2 \langle \tau \rangle}{m_e} n$$

Define resistivity: $\rho = \frac{1}{\sigma}$ $\rightarrow J = E/\rho$ Recall: $I = JA$

$$\rightarrow I = \frac{1}{\rho} AE$$

(unit: Ohm = $\frac{\text{V}\cdot\text{A}}{\text{m}^2}$ [Ω])

Recall:  $E = \frac{V_{ab}}{l} \rightarrow I = \frac{1}{\rho} \frac{A}{l} V_{ab}$; Define resistance: $R = \rho \frac{l}{A}$

$$\rightarrow \text{Ohm's Law: } I = \frac{1}{R} V_{ab} \Leftrightarrow V_{ab} = IR$$

note: Ohm's law is in direction of current (w.r.t. V_{ab})

Resistance

11/29/23

Resistance

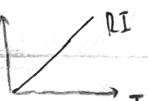
analog: water pipe \rightarrow more flow

Lecture 32

Observation: resistance proportional to length, inversely proportional to cross-sectional area

(cont.)

Extrema: (i) High resistance \rightarrow no meaningful charge can flow [insulator]



(ii) Low resistance $\rightarrow V_{ab}$ (potential diff.) minimized [conductors]

$\rightarrow I = \frac{V_{ab}}{R}$ can go to infinity; current dependent only on power supply

\hookrightarrow Consequence: [Nearly] constant potential across a conductor

(*) Convention - "Direction of charge flow" is dir. of positive charge flow [high V \rightarrow low V]

Resistance dependent on material, temperature [lower temperature \rightarrow lower resistance]

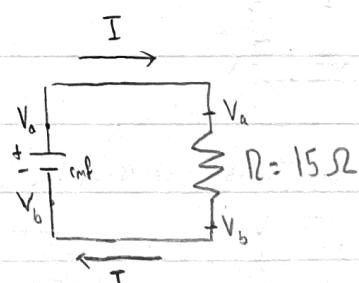
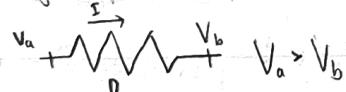
(*) Superconductors - materials where resistance $\rightarrow 0$ at sufficiently low temperatures

(*) Ex: With V fixed: lower temperature \rightarrow lower R \rightarrow higher I \rightarrow more light from lightbulb, e.g.

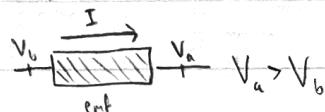
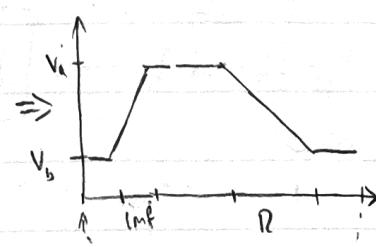
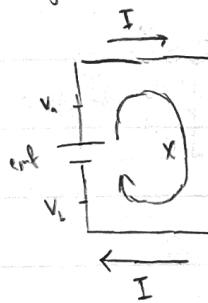
(*) Ohm's Law

Approximate conducting wires as ideal conductors ($V_{ab} = 0$)

\rightarrow Voltage drops occur only at resistors



Can graph electric potential across circuit:



\hookrightarrow inside emf: current goes from V_b to V_a [against electric field]

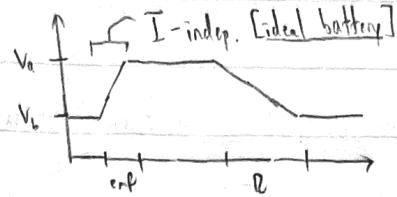
\hookrightarrow battery does work to move charges against \vec{E} [maintain potential diff.]

Ohm's Law (cont.)

12/1/23

Lecture 33 Ohm's Law Approximations

- i) Neglect resistance of conducting wire
- ii) Assume emf is independent of current I
- (*) Intuition: Resistors take energy out of system; emf inserts energy back in

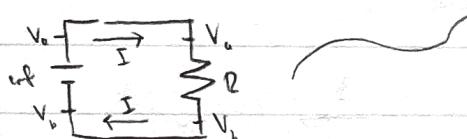


(+) Ex: Resistors in Series

$$\begin{array}{c} \text{Circuit diagram: Two resistors } R_1 \text{ and } R_2 \text{ in series with a battery of emf } V_a \text{ and terminal voltage } V_o. \\ \text{Current } I \text{ flows through both resistors.} \\ \text{Voltage drop across } R_1: V_a - V_c = IR_1 \\ \text{Voltage drop across } R_2: V_c - V_b = IR_2 \\ \text{Total voltage drop: } V_a - V_b = I(R_1 + R_2) \end{array}$$

Non-Ideal Batteries

Ideal battery: $\text{emf} = V_{ab}$ [Ideal emf]



Ohm's Law: $I = \frac{V_{ab}}{R}$; what if R small?

$R \rightarrow 0 \Rightarrow I \rightarrow \infty \Rightarrow$ battery shorts out

Solution: Treat batteries as having internal resistors

$r = \text{internal resistance}$

$$\begin{array}{c} \text{Circuit diagram: A battery with emf } V_a \text{ and terminal voltage } V_b. \\ \text{Internal resistance } r \text{ is shown in parallel with the battery.} \\ \text{Voltage drop across } r: V_a - V_b = \text{emf} - Ir \end{array}$$

\downarrow Ir has little effect when I low;
decreases more Vab when I is high

$$\rightarrow \text{Ohm's Law [Modified]: } V_a - V_b = IR = \text{emf} - Ir \rightarrow I = \frac{\text{emf}}{R+r}$$

\sim Used batteries: $r \uparrow$

Current & Energy

12/1/23

Lecture 33

Current & Energy

more current \rightarrow more light

Observation: Current to a lightbulb [resistor] causes it to heat up, eventually produce light

(cont.)

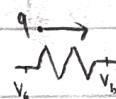
→ Current is a way of transporting energy.

Resistors, motors take energy out of the system (to produce kinetic energy, e.g.)

Ex: Resistors convert energy to heat [analogue: friction]

Conversely: batteries, power supplies put energy back in

Think of current as potential energy (from moving charges)



$$\Delta U = q(V_b - V_a)$$

(*) Resistors:

$$\begin{aligned} \text{---} & \xrightarrow{\quad I \quad} \text{---} \\ V_a & \xrightarrow{\quad R \quad} V_b \end{aligned} \rightarrow \Delta U = U_b - U_a = q(V_b - V_a), V_a > V_b \Rightarrow \Delta U < 0$$

$$\rightarrow \Delta K = -\Delta U > 0$$

In Batteries: $\Delta U > 0, \Delta K < 0$

Look at infinitesimal charges: $dK = dq(V_a - V_b)$.

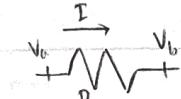
Recall: $dq = I dt \rightarrow dK = (V_a - V_b) I dt$

$$\text{Power: } P = \frac{dK}{dt} = (V_a - V_b) I$$

Ohm's Law: $V_{ab} = RI \rightarrow \text{Power (Equivalent formulas):}$

$$P = RI^2, P = \frac{(V_a - V_b)^2}{R}$$

$P > 0 \Rightarrow$ energy leaving; $P < 0 \Rightarrow$ energy being put in



$$V_a < V_b \rightarrow P_{\text{battery}} < 0$$

$$V_a > V_b \rightarrow P_{\text{resistor}} > 0$$

Power

12/4/23

Lecture 34

(*) Power

In complex circuits: may have $P_{\text{battery}} > 0$ [may occur if there are multiple batteries, e.g.]

Interpretation: Battery being charged, or discharged as heat

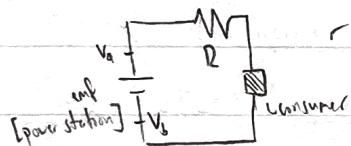
(*) Ex: High-Voltage Transmission Line

Constants: $\lambda = 100 \text{ km}$; $A = 500 \text{ mm}^2$; $V = 765 \text{ kV}$; $P = 1000 \text{ MW}$

→ Q: How much energy is lost due to Joule heating? [Due to resistance of copper wire]

$$\text{i) Find } R: \rho_{\text{cu}} = 177 \cdot 10^8 \Omega \text{ m} \rightarrow R = \frac{\rho}{A} = 3.54 \text{ [= } 3.54 \text{]}$$

ii) Draw circuit model:



Interpretation: P is power put into circuit by emf (power station);

$$\rightarrow V = V_a - V_b \text{ [= emf]}$$

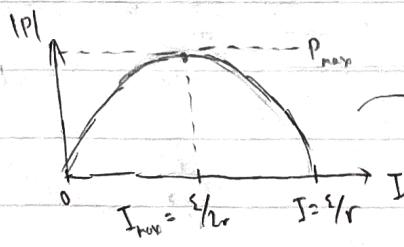
$$\rightarrow P = VI \rightarrow I = \frac{P}{V} = 1307 \text{ A}$$

$$\text{iii) Know } I, R \rightarrow P_{\text{Joule}} = RI^2 = 6.84 \cdot 10^6 \text{ W [less than 1% of 1000 MW]}$$

(*) Power in a Battery

Ideal battery: $|P| = \text{emf} \cdot I$. Recall (non-ideal batteries): $V = \epsilon - Ir$

$$\rightarrow \text{"Real" battery: } P = \epsilon I - I^2 r \quad \text{Interpretation: } I^2 r = \text{power dissipated by internal resistance}$$



Power maximized at

$$I_{\text{max}} = \frac{\epsilon}{2r}$$

$$\rightarrow P_{\text{max}} = \frac{1}{4} \frac{\epsilon^2}{r}$$

Resistors in Series & Parallel

12/4/23

Resistors in Series & Parallel

Lecture 34

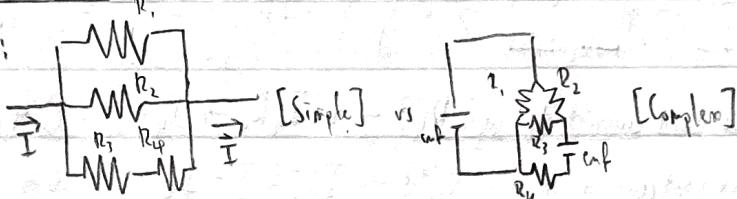
Observation: Given fixed V , resistors in parallel $\rightarrow I \uparrow$; in series $\rightarrow I \downarrow$

+ Lecture 35

Similar to capacitors - given R_1, R_2 , want to find equivalent resistance R_{eq} . Can use to simplify circuits.

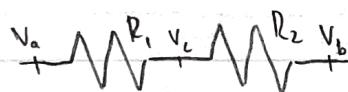
(*) Note: R_{eq} reduction only guaranteed to work for simple circuits ($1 I_{in}, 1 I_{out}$); may fail on more complex circuits

(*) Ex:



(Simplify to one R_{eq} incrementally (similar to capacitors))

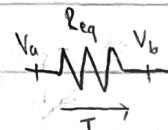
Resistors in Series



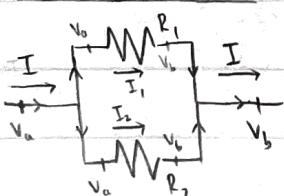
Observe: Current is the same across all resistors (conservation of charge)

$$\begin{aligned} V_a - V_c &= IR_1 \\ V_c - V_b &= IR_2 \end{aligned} \quad \rightarrow V_a - V_b = I(R_1 + R_2) = IR_{eq}$$

$$R_{eq} = R_1 + R_2$$



Resistors in Parallel



Observe: $I = I_1 + I_2$; potential difference is the same across all resistors

$$\begin{aligned} V_a - V_b &= I_1 R_1 \\ V_a - V_b &= I_2 R_2 \end{aligned} \quad \rightarrow V_a - V_b = I R_{eq} = I_1 R_1 + I_2 R_2$$

$$I_1 = \frac{V_a - V_b}{R_1}, \quad I_2 = \frac{V_a - V_b}{R_2}, \quad I = \frac{V_a - V_b}{R_{eq}}$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

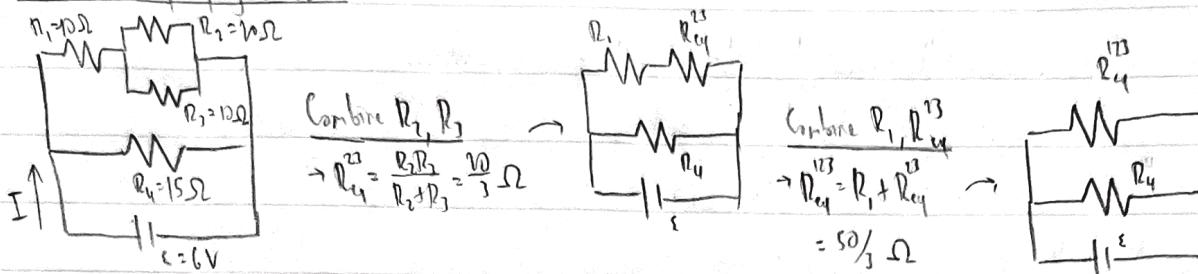
$$(\star) \text{Alt: } R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

Kirchhoff's Laws

12/5/23

Lecture 35
(cont.)

(*) Ex: Simplifying Circuits



$$\text{Combine } R_2, R_3 \rightarrow R_{23} = \frac{R_2 R_3}{R_2 + R_3} = \frac{10}{3} \Omega$$

$$\text{Combine } R_1, R_{23} \rightarrow R_{eq123} = R_1 + R_{23} = 50/3 \Omega$$

$$I = \frac{6}{50/3} = 0.72 \text{ A} \quad [\text{Ohm's Law}]$$

$$\rightarrow \text{What if we want current through } R_4? \rightarrow I_4 = \frac{6}{R_4} = 0.4 \text{ A} \quad [\text{Ohm's Law}]$$

$$\rightarrow \text{What if we want current through } R_{eq123}? \rightarrow I_{eq123} = I - I_4 = 0.36 \text{ A} \quad [\text{Conservation of charge}]$$

$$\rightarrow \text{What if we want } \Delta V \text{ across } R_1? \rightarrow \Delta V = I_{eq123} R_1 = 3.6 \text{ V} \quad [\text{Ohm's Law}]$$

Kirchhoff's Laws

Kirchhoff's circuit laws - rules governing current, potential differences in circuits

1) Junction rule (charge conservation): Current into a junction is equivalent to current out of it

$$\text{Ex: } I \rightarrow I_1 \downarrow I_2 \quad I - I_1 - I_2 = 0 \quad I \rightarrow I_1 \uparrow I_2 \quad I + I_2 - I_1 = 0$$

2) Loop rule: The sum of potential differences around any closed loop is 0.

$$\text{Ex: } \begin{array}{c} V_1 \\ | \\ \text{---} \\ | \\ V_2 \\ | \\ \text{---} \\ | \\ V_3 \\ | \\ \text{---} \\ | \\ V_4 \end{array} \rightarrow (V_1 - V_2) + (V_2 - V_3) + (V_3 - V_4) = IR_1 + IR_2 - \epsilon = 0$$

↳ (*) Loops also called "Jouls"

(*) Loop Rule - Derivation

Recall: electric field is conservative $\Rightarrow \oint \mathbf{E} \cdot d\mathbf{r}$ around any closed loop is 0

C $\oint \mathbf{E} \cdot d\mathbf{r}$ is eq. for potential \rightarrow potential does not change going around a loop [Loop Rule]

Kirchhoff's Laws (cont.)

12/6/23

Lecture 36

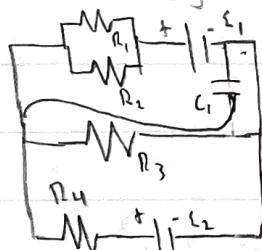
(*) Rules for Solving Circuits

- i) Simplify circuit with equivalent resistors (i.e. by "merging" resistors)
- ii) Eliminate capacitors [once charged]
- iii) Choose currents I_1, I_2, \dots and label their directions (using arrows, e.g.) directions are arbitrary
(math handles it)
- iv) Use Junction rule to eliminate as many I 's as possible
- v) Choose loops with direction around faces for Loop Rule
- vi) Use Loop Rule w/ Ohm's Law (resistors), emf (batteries) to relate variables & solve

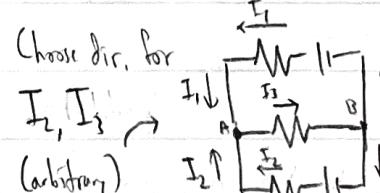
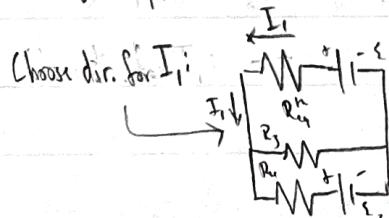
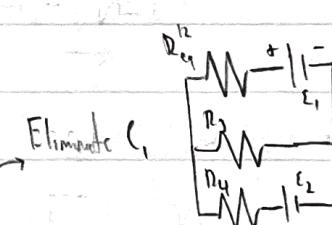
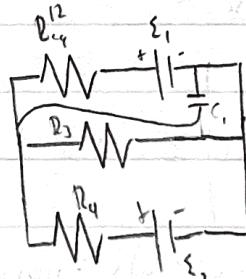
Kirchhoff's Rules (summarized)

- i) Junction Rule: The sum of incoming, outgoing currents are equal at all junctions.
- ii) Loop Rule: The sum of potential differences when going around any loop, is 0.

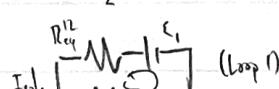
(*) Ex: Solving Circuits



Combine R_1, R_2
 $\rightarrow R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$



Junction Rule:
A: $I_1 + I_2 - I_3 = 0$
B: $I_3 - I_1 - I_2 = 0$

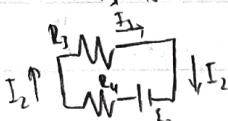


$$V_{R_{eq}} + V_{R_3} + V_{\epsilon_1} = 0$$

$$\rightarrow I_1 R_{eq} + I_3 R_3 - \epsilon_1 = 0$$

$$\therefore I_1 R_{eq} + (I_1 + I_2) R_3 - \epsilon_1 = 0$$

← (choose loops) ← Set $I_3 = I_1 + I_2$ (arbitrary)



←

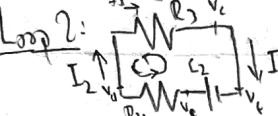
Lenz's Law (cont.)

12/6/23

Lecture 36

(*) Ex: Solving Circuits (cont.)

(cont.)

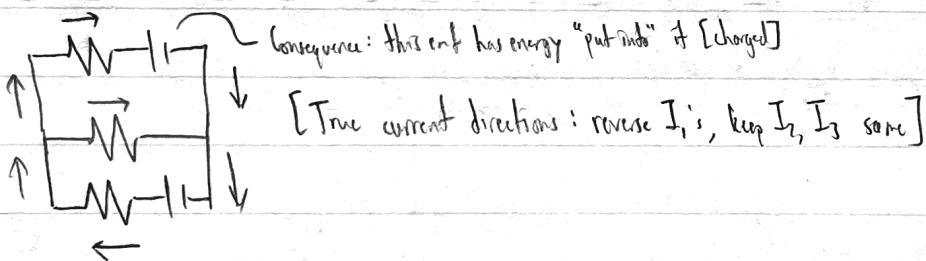
Loop 2:  Loop Rule: $(V_b - V_c) + (V_e - V_f) + (V_f - V_d) = 0$
 $\rightarrow -I_2 R_4 + \epsilon_2 - I_3 R_3 = 0 \rightarrow -I_2 R_4 + \epsilon_2 - (I_1 + I_2) R_3 = 0$

Known Eqs.: $I_1 R_{eq} + (I_1 + I_2) R_3 - \epsilon_1 = 0$
 $-I_2 R_4 - (I_1 + I_2) R_3 + \epsilon_2 = 0$ [Know 5 variables → can solve]

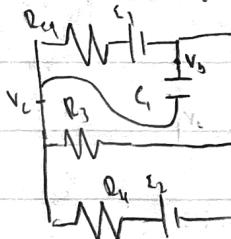
Ex: $\epsilon_1 = 12V$, $\epsilon_2 = 24V$, $R_{eq} = 0.5\Omega$, $R_4 = 1\Omega$, $R_3 = 100\Omega$

$\hookrightarrow 0.5I_1 + 100(I_1 + I_2) - 12 = 0$ [Solve: $I_1 = \frac{(\epsilon_1 - \epsilon_2)R_3 + \epsilon_1 R_4}{0.5R_4 + R_3(R_4 + R_3)}$, $I_2 = \frac{(\epsilon_2 - \epsilon_1)R_3 + \epsilon_2 R_4}{R_4 R_3 + R_3(R_4 + R_3)}$]
 $-I_2 - 100(I_1 + I_2) + 24 = 0$

$\rightarrow I_1 = -7.89A$, $I_2 = +8.05A$, $I_3 = 0.16A$



Q: What was the charge on the capacitor? Use formula for capacitance $[C = \frac{Q}{V}]$



$$Q = (V_b - V_c) C_1 = (I_3 R_3) C_1; C_1 = 3 \cdot 10^{-6} \rightarrow Q = 4.8 \cdot 10^{-5} C$$

(*) Once capacitor fully charged - no more current through that specific wire

(*) RC Circuits

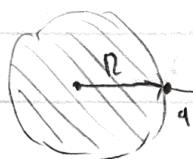
Def: An RC circuit is a circuit composed of resistors & capacitors.

(#) E&M Examples

12/8/23

Lecture 37

Problem: Given sphere w/ radius R , constant charge density $\rho_0 > 0$ and point charge $q > 0$ with mass m (initially at R), what is the velocity of q at distance $4R$?



Use conservation of energy. Know $V(r) =$

$$\begin{cases} \frac{q_{\text{ext}}}{4\pi\epsilon_0} \frac{1}{r} & r > R \\ \frac{q_{\text{ext}}}{4\pi\epsilon_0} \frac{R^2}{R^3} & r < R \end{cases}$$

$$\begin{cases} \frac{q_{\text{ext}}}{4\pi\epsilon_0} \frac{1}{r^2} & r > R \\ \frac{q_{\text{ext}}}{4\pi\epsilon_0} \frac{R^2}{R^3} & r < R \end{cases}$$

$$q_{\text{ext}} = \rho_0 \frac{4}{3}\pi R^3 [q \cdot V] \rightarrow U_i^{(q)} = qV(R) ; U_f^{(q)} = qV(4R)$$

$$U_i + (K_i = 0) = U_f + K_f \rightarrow K_f = \frac{q_{\text{ext}}}{4\pi\epsilon_0} \frac{3}{4R} \rightarrow v^2 = \frac{2}{m} K_f \rightarrow v = \sqrt{\frac{K_f}{m}}$$

b) If the point charge had charge $-q$ $[-q < 0]$, we observe that it will oscillate inside the sphere; what is the frequency?

Find $E(z)$: Use Gauss's law



$$EA = 4\pi z^2 E(z) = \phi = \frac{1}{\epsilon_0} \frac{4}{3}\pi z^3 \rho_0 \rightarrow E(z) = \frac{\rho_0}{3\epsilon_0} z$$

$$\rightarrow F = -qE(z) = -\frac{\rho_0 q}{3\epsilon_0} z \quad [\text{linear restoring force} \rightarrow \text{SHM}]$$

$$ma = \left(\frac{d^2}{dt^2} z\right)_m = F = -\frac{\rho_0 q}{3\epsilon_0} z \rightarrow \frac{d^2}{dt^2} z = -\frac{\rho_0 q}{3\epsilon_0 m} z$$

$$\text{Recall: } \frac{d^2}{dt^2} z = -\omega^2 z \rightarrow \omega = \sqrt{\frac{\rho_0 q}{3\epsilon_0 m}} \rightarrow f = \frac{1}{2\pi} \omega$$

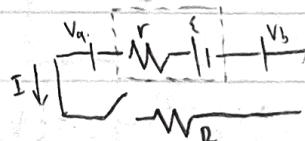
Problem: Battery with emf ϵ , internal resistance r

a) With the switch open, we measure $V_a - V_b = 12V$. What

$\rightarrow I$ and what is ϵ ?

b) Switch open \rightarrow no current flows $\rightarrow I=0$

Ohm's Law: $V_a - V_b = \epsilon - Ir = \epsilon \rightarrow \epsilon = 12V$



(*) E&M Examples (cont.)

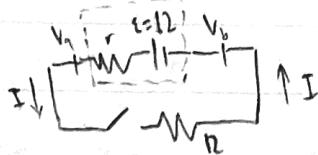
12/8/23

Lecture 37
(cont.)

FINAL

- b) Circuit as before; with switch open, measure

$$V_a - V_b = 9 \text{ V}, I = 3 \text{ A}. \text{ What are } r, R?$$



$$V_a - V_b = 9 = \epsilon - Ir = 12 - 3r \rightarrow r = 1 \Omega$$

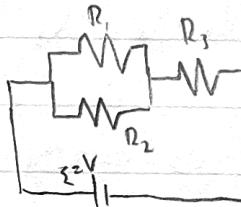
Know: potential diff. across R is $V_a - V_b$: [assuming ideal conductor]

$$\rightarrow I = \frac{V}{R} = \frac{9}{3} \rightarrow R = 3 \Omega$$

Problem: Given 3 lightbulbs w/ emf $\epsilon = V$:

- a) What relation do R_1, R_2 satisfy so that

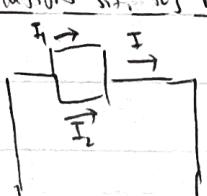
they are equally bright? [i.e. same power]



\downarrow resistors in parallel $\rightarrow P_1 = \frac{(V_a - V_b)^2}{R_1}$; sum for $P_2 \rightarrow P_1 = P_2 \Rightarrow R_1 = R_2$

- b) What are the relations s.t. R_3 half as, or twice, as bright as R_1, R_2 ?

Draw currents:



$$\text{look at power: } P_3 = R_3 I^2 = 4R_3 I_1^2$$

$$\text{Half: } 4R_3 I_1^2 = \frac{1}{2}(R_1 I_1^2) \rightarrow R_3 = \frac{1}{8} R_1, \quad P_1 = R_1 I_1^2$$

$$\text{Twice: } 4R_3 I_1^2 = 2R_1 I_1^2 \rightarrow R_3 = \frac{1}{2} R_1$$