Physics 1B

2023-24
Fall 23

Instructor: Michael Gutperle (+ Casey Li)
Textbook: Young & Freedman - Mastering Physics
Topics: Fluids (static fluids, fluid flow), oscillation, mechanical waves & sound, electrostatics (charge, fields, flux, potential), capacitance, circuits (current, resistance)

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Physics 1A Review

9/20/23

Lecture 1

Physics 1A Review

- Newton's Laws: (1) An object stays at rest, unless acted upon by a force.
  (2) $F_{net} = ma$ $[F_{net} = \Sigma \vec{F}]$.
  (3) For every action, there is an equal and opposite reaction.
  } Force: Newton = $kg\, m/s^2$
- Forces as vectors: have magnitude & direction, can be added
- Gravity: $F = -mg\hat{e}_z$ (where e_z is the vertical distance/height)
  $= -G\frac{m_1 m_2}{|r|^2}\hat{r}$ [Law of Universal Gravitation] (*) $\hat{r}$ as unit vector
- Hooke's Law: $\vec{F} = -kx$ (where x is the distance from the spring's massless equilibrium position $\bar{x}$/x_0).
  - (The direction of $\vec{F}$ is thus always pointing toward equilibrium).
  (*) Law for ideal springs
- Friction: $F = \mu|N|$ (alt: $|F| = k|v|$ = drag in a fluid)
  - Kinetic friction opposes motion [slows object], converting KE to heat
- Conservative forces (e.g. gravity): forces derived from a potential function U $[\vec{F} = -\nabla \vec{U}]$.
  - Are path-independent, and conserve mechanical energy $E = K + U$ if isolated
- Energy:
  - Kinetic: $KE = \frac{1}{2}mv^2$ } Energy: Joule = $kg\, m^2/s^2$
  - Potential: $U = mgh$ (gravitational)
    (*) Potential energy defined in terms of deviation from an arbitrary baseline; can be negative, unlike KE

Fluids and Pressure

10/2/23
Lecture 2

Properties of a Fluid

- 3 phases of matter: solid/liquid (assumed non-compressible) vs gas (compressible)
- Density: $\rho = \frac{m}{V}$ [kg/m^3]; assumed constant for liquids per non-compressibility
  - (*) Water: 1000; lead: 11,340; wood: 700; air: 122

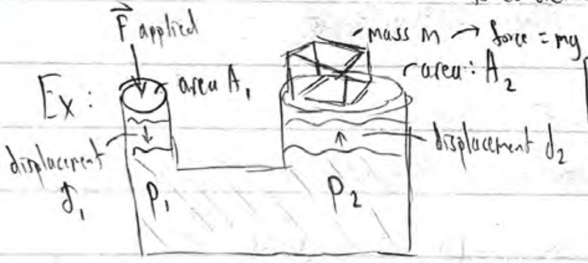
Pressure

Pressure results from applying force on a liquid: $\boxed{p = \frac{|F_\perp|}{A}}$ [N/m^2 - Pascals]

Pascal's Principle - pressure is the same everywhere in a liquid (ignoring the weight of the liquid)

→ (*) Net force across a static fluid is 0

Ex:



$\boxed{\text{Pascal's Principle: } p_1 = p_2}$

$\rightarrow \frac{|F_1|}{A_1} = \frac{mg}{A_2} \rightarrow |F| = mg\frac{A_1}{A_2}$

→ Force required to lift object $\times \frac{A_1}{A_2}$

Not a force "generator"; simply applies less force over greater distance → equal overall work

→ Work: W_1 (work by F) $= W_2$ (gravity on object) (since noncompressible)

$\rightarrow W_1 = d_1 F_1 = d_1 A_1 p$

$W_2 = d_2 F_2 = d_2 A_2 p$

$[p = p] \rightarrow d_1 A_1 = d_2 A_2$

$\rightarrow d_1 = d_2 \frac{A_2}{A_1}$

(i.e. $d_1 > d_2$)

(*) Applications: Hydraulic lifts
Drinking (straws)

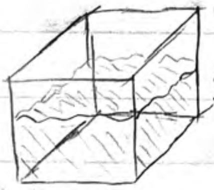
Hydrostatic Pressure

10/2/23
Lecture 2
(cont.)

Hydrostatic Pressure

Def: Hydrostatic pressure is pressure caused by the weight of the fluid

(*) Consequence: Is not constant across the fluid's volume (namely, deeper → higher pressure)



$$\rightarrow \boxed{F = \rho A \Delta y g} \; [= mg]$$

└ (*) ρ (rho) = density

Variation in Hydrostatic Pressure

Pressure @ bottom = pressure @ top + hydrostatic pressure

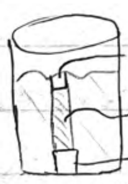
$$\rightarrow p(y_0 - \Delta y)A - p(y)A - \rho A \Delta y g = 0$$

$$\rightarrow \frac{p(y_0 - \Delta y) - p(y)}{\Delta y} = -\rho g \rightarrow \frac{dp}{dy} = -\rho g \rightarrow \text{(Assume } \rho \text{ constant)} \int_{y_0}^{y_0 - \Delta y} dp = -\rho g \int_{y_0}^{y_0 - \Delta y} dy$$

$$\rightarrow p(y_0 - \Delta y) = p(y_0) - \rho g((y_0 - \Delta y) - y_0)$$

$$\boxed{\rightarrow p(y_0 - \Delta y) = p(y_0) + \rho g \Delta y}$$

(*) Intuition:



(*) A "column" of fluid: not 100% true

(*) Atmospheric Pressure

Def: Atmospheric pressure is "hydrostatic" pressure from the air $[\approx 10^5$ Pa @ sea level$]$

$$\hookrightarrow p(y_0) = P_{atm} + P_F$$

pressure from applied force

(*) Fun fact: We don't "feel" P_{atm} in our day-to-day because our bodies operate at P_{atm} themselves

Hydrostatic Pressure (cont.)

10/2/23
Lecture 2
(cont.)
+Lecture 3

Hydrostatic Pressure (cont.)

- Hydrostatic pressure is only dependent on height (Δy)

Ex:  → Pressure @ bottom of all three is equal

(*) reasoning: @ , depth of water directly above is lower, but water pushing up & out on the container creates a normal force in & down (→ pressure)

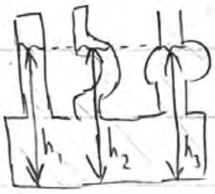
- Hydrostatic pressure is constant [equal] at all points at the same height (where Δy is equal)

Ex: 

If $\Delta y_1 = \Delta y_2 \rightarrow p_1 = p_2$ (also: 

$\rightarrow p_1 = p_2$)

(*) Consequence: height of water is independent (not) of container shape

Ex: 

$h_1 = h_2 = h_3$ (Reasoning: since p_{bottom} depends on Δy, Δy must be equal for all points @ bottom for consistent pressure.)

(*) Measuring Pressure

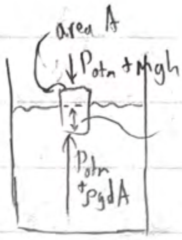
- Two different types of pressure measurement: gauge pressure ($p - p_{atm}$) vs absolute pressure (p)
  - Gauge pressure [relative pressure] can be negative; absolute pressure cannot (lowest: 0 [vacuum])

The Buoyant Force

10/3/23
Lecture 3
(cont.)

Buoyancy

Def: Buoyancy [the buoyant force] is an upward force exerted by a fluid on objects immersed in the fluid.



(*) The net force on a floating object is 0.

$\rightarrow (P_{atm} + \rho g d A) - P_{atm} - mgh = 0$

$\rightarrow F_b = \rho g d A$ [d = displacement]

$= \rho g V_{displaced}$

$\rightarrow$ Archimedes' Principle

$F_{buoyant} = g \cdot \rho_{fluid} \cdot V_{displaced} = g \cdot m_{displaced}$ [$F_{buoyant}$ = weight of the fluid displaced.]

Notes on Buoyancy

- An object floats when $F_{buoyant} = F_{gravity}$ (such that $F_{net} = 0$).
  - Corollary: An object can only float if $\rho_{object} \leq \rho_{fluid}$.

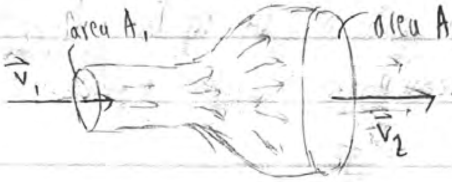
Equations of Fluid Flow

10/4/23
Lecture 4
+Lecture 5

(*) Fluid assumptions: incompressible, no friction, no turbulence

Fluid Flow

There are two main equations governing fluid flow: the continuity equation and Bernoulli's equation.



[Cross-sections of a pipe]

(1) Continuity Equation

Given fluid flow through 2 cross-sections of a pipe (pictured above), then: $\boxed{\rho A_1 v_1 = \rho A_2 v_2}$

- Simplifies to $A_1v_1 = A_2v_2$ when density is constant

(*) Can be seen as an analogue to conservation of mass: "volume in [ρdV_1] = volume out [ρdV_2]"

(*) The value $\frac{\Delta V}{dt} = Av$ is also called the flow rate R_v.

(2) Bernoulli's Equation

Given fluid flow through a pipe, given any two points in the pipe 1 & 2, then:

$$\boxed{P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2} \text{ [Bernoulli's Equation]}$$

(*) Notes on Bernoulli's Equation

- Can be seen as an analogue to conservation of energy:
  - Kinetic energy: $KE = \frac{1}{2}mv^2 \longrightarrow \frac{1}{2}\rho v^2$
  - Potential energy: $PE = mgh \longrightarrow \rho g y$
- Is a generalization of the hydrostatic pressure equation to nonstatic fluids

(*) Can be informally derived from Pascal's Principle

Intro to Oscillations

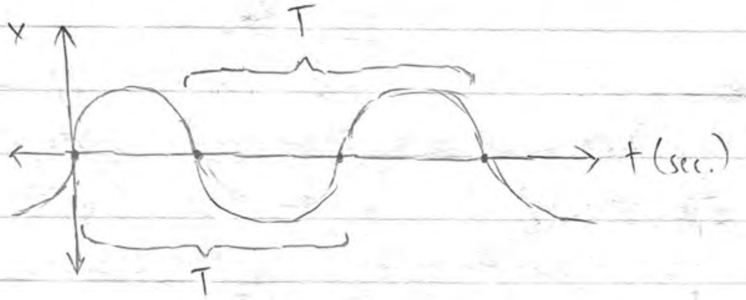
10/9/23
Lecture 6

Periodic Processes

Def: A process that comes back to its initial state and repeats over time (ex: springs, pendulums)

Properties of Periodic Processes

- Period (T): smallest time period of a full repetition [seconds]
- Frequency ($F = \frac{1}{T}$): inverse of period; how often a process repeats [seconds^{-1} = Hertz/Hz]



Simple Harmonic Oscillators (SHO)

Def: Simple harmonic motion (SHM) is the primary physics model for systems involving slight fluctuations around a stable equilibrium

Notes on Simple Harmonic Oscillators

- Model system: ideal [horizontal] spring with a mass
  - (*) Example system: vibrations in a solid
- (*) Many specialized physics models are actually variations of simple harmonic motion (e.g. adding an additional term)

(*) SHM: Ideal Springs

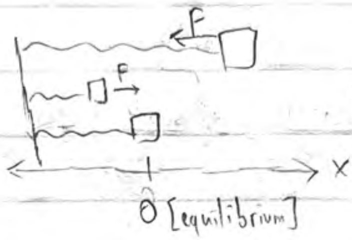
- Amplitude (A): maximum distance away from equilibrium in a given repetition [meters]
- Phase constant/shift (ϕ_0): constant encoding the initial position of a spring [radians]

Oscillations of Ideal Springs

10/9/23
Lecture 6
(cont.)

Recall: Hooke's Law

$$\boxed{\text{Hooke's Law: } F = -kx}$$
(for ideal springs)



[Horizontal spring]

Equations of Simple Harmonic Motion

$ma = F = -kx$

$\rightarrow m\frac{d^2x}{dt} = -kx$

$\rightarrow \frac{d^2x}{dt}(t) - \frac{k}{m}x = 0 \implies$

$$\boxed{\text{Def: } \omega = \sqrt{\frac{k}{m}} \text{ [angular frequency]}}$$
(*) ω = "omega"

$$\boxed{\frac{d^2x}{dt}(t) - \omega^2 x = 0 \text{ [SHO]}}$$

(Solve 2^{nd}-order diff. eq.)

General solution: $\boxed{x(t) = A\cos(\omega t + \phi_0)}$

(*) cosine, ϕ_0 in radians (not degrees)

(*) Characterizing Angular Frequency

$x(t) = A\cos(\omega t + \phi_0)$

$\rightarrow x(t+T) = x(t)$

$\rightarrow A\cos(\omega t + \phi_0) = A\cos(\omega(t+T) \cdot \phi_0)$

$$\boxed{T = \frac{2\pi}{\omega} \implies T = 2\pi\sqrt{\frac{m}{k}}}$$
[Ideal spring]

$\Downarrow$

$$\boxed{\text{Consequence: Period is independent of amplitude (A)}}$$
(*) Also applies to ω, f

(*) Derivatives of Position

(i) $x(t) = A\cos(\omega t + \phi_0)$

(ii) $v(t) = -A\omega\sin(\omega t + \phi_0)$

$\Big\rangle \; x(t)^2 + \frac{v(t)^2}{\omega^2} = A^2$

(iii) $a(t) = -A\omega^2\cos(\omega t + \phi_0)$

$$\boxed{A = \sqrt{x(t)^2 + \frac{v(t)^2}{\omega^2}}}$$

Oscillations (cont.)

10/10/23
Lecture 7

Equations of Simple Harmonic Motion (cont.)

$x(t) = A\cos(\omega t + \phi_0)$

$v(t) = -A\omega\sin(\omega t + \phi_0)$

$$\frac{v(t=0)}{x(t=0)} = -\omega \overbrace{\frac{\sin(\phi_0)}{\cos(\phi_0)}}^{\tan(\phi_0)} \Rightarrow \phi_0 = \tan^{-1}\left(-\frac{1}{\omega}\cdot\frac{v(t=0)}{x(t=0)}\right)$$

→ Adjust formula to give full range (2π) of values

$$\boxed{\phi_0 = \begin{cases} x(0) > 0: \tan^{-1}\left(-\frac{1}{\omega}\cdot\frac{v(t=0)}{x(t=0)}\right) \\ x(0) < 0: \tan^{-1}\left(-\frac{1}{\omega}\cdot\frac{v(t=0)}{x(t=0)}\right) \pm \pi \end{cases}}$$

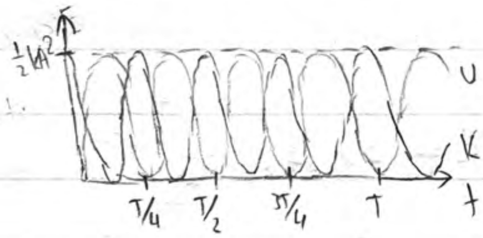
Energy of Simple Harmonic Motion

- $\boxed{K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\sin(\omega t + \phi_0)^2}$
- $\boxed{U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos(\omega t + \phi_0)^2}$

$$E_{mech} = \frac{1}{2}mA^2\omega^2\sin(\omega t + \phi_0)^2 + \frac{1}{2}kA^2\cos(\omega t + \phi_0)^2$$

↓ substitute $\omega = \sqrt{\frac{k}{m}}$

Total energy is independent of time ← $\boxed{E_{mech} = \frac{1}{2}kA^2 = \frac{1}{2}mA^2\omega^2}$



(*) Since K, U have x^2, v^2 terms, sign of $x(t)$, $v(t)$ ignored → period of K, U oscillations = $T/2$.

(*) Note: U is independent of mass; K is not

(*) Amplitude determined by potential energy

Applications of Simple Harmonic Motion

1) Torsion pendulum

$m \to I$ [moment of inertia]

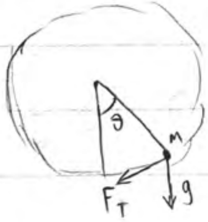
$x \to \phi$ [angular displacement (rotation)]

$\to \tau = -\kappa\phi$ [torque is linear]

($k \to \kappa$)

Let $\omega = \sqrt{\frac{\kappa}{I}} \Rightarrow \boxed{\frac{d^2\phi}{dt^2} + \omega^2\phi = 0}$

2) Simple pendulum



Motion equation

$$\boxed{ml\frac{d^2\theta}{dt^2} = -mg\sin(\theta)}$$

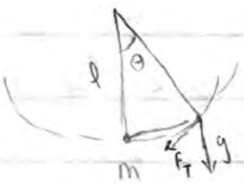
⇓

Not an SHO, but becomes one w/ approx. $\sin(\theta) = \theta$ at small angles

Oscillations (cont.)

10/11/23
Lecture 8

Motion of a Simple Pendulum (cont.)



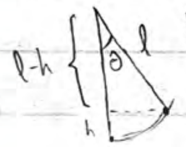
$$m\ell\frac{d^2\theta}{dt^2} = -mg\sin(\theta) \rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin(\theta) = 0 \text{ [not a SHO]}$$

Define stable equilibrium: $\theta = 0$

$$\rightarrow \sin(\theta) \text{ [Taylor exp.]} = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \ldots \rightarrow \sin(\theta) \approx \theta \text{ [small } \theta]$$

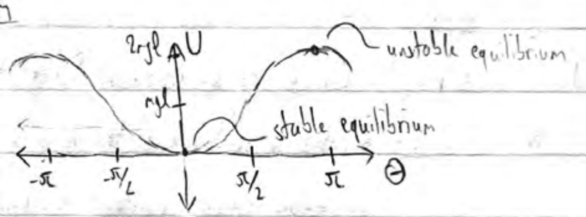
$$\rightarrow \boxed{\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \left[\omega = \sqrt{\frac{g}{\ell}}\right]} \Rightarrow \underline{\text{SHO}} \text{ [for } \theta < 20^\circ\text{, approx.]}$$

(*) Potential Energy of a Simple Pendulum



$h = \ell(1 - \cos\theta)$

$$\rightarrow \boxed{U = mg\ell(1 - \cos\theta)}$$



$$\cos(\theta) \text{ [Taylor exp.]}: 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \ldots \longrightarrow \boxed{U = mg\ell\frac{\theta^2}{2}} \text{ [for small values of } \theta]$$

Damped Oscillators

Damped oscillators introduce a frictional/damping force (viscous friction): $\boxed{F_{fric} = -bv}$

$$\rightarrow F_{net} = m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx \text{ [Ideal spring + friction]}$$

$$\rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Let $\boxed{\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m}}}$; Solutions:

$$\begin{cases} \omega > 0: x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi_0) \text{ [\underline{Underdamped}]} \\ \omega = 0: x(t) = e^{-\frac{b}{2m}t}(C_1 + C_2 t) \text{ [\underline{Critically damped}]} \\ \omega < 0: x(t) = C_{(+)}e^{-\left(\frac{b}{2m} - \omega i\right)t} + C_{(-)}e^{-\left(\frac{b}{2m} + \omega i\right)t} \text{ [\underline{Overdamped}]} \end{cases}$$

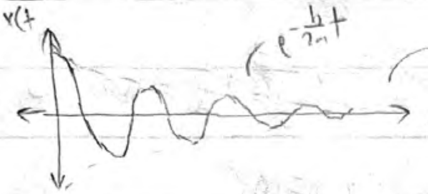
(*) Resonance

10/11/23
Lecture 8
+ Lecture 9

Damped Oscillators (cont)

1) Underdamping:

$E \sim A \sim e^{-\frac{b}{m}t}$



→ Amplitude approaches 0

2) Critical damping:
3) Overdamping:

Overdamping ⟶ no oscillations

Critical damping (fastest rate of decay [exponential])

(*) Resonance

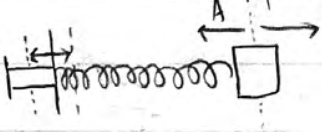
Oscillations may be caused by an external force that is itself oscillatory [oscillates in magnitude]

i) Normal case: oscillation occurs with the driving frequency [frequency of the force's oscillation], amplitude remains constant (steady state) ~ energy from force oscillation "counterbalances" original oscillator

ii) Special case: When the driving frequency approaches the "natural frequency" of the system (of the original oscillator), amplitude becomes very large [resonance]

energy from force oscillation is added to E of original oscillation
→ energy in system grows continuously; potentially dangerous

Ex:



position of block oscillates due to spring

position of spring oscillates

↳ i) Normal case: block oscillation frequency = frequency of spring position oscillation, A constant

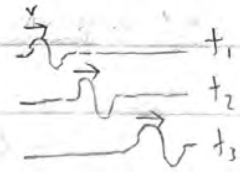
ii) Special case [resonance]: frequency of spring pos. oscillation = frequency of the spring
→ A grows larger & larger

Intro to Mechanical Waves

10/16/23
Lecture 10

Mechanical Waves

- Can be thought of as the transmission of energy/momentum/information between different locations in space



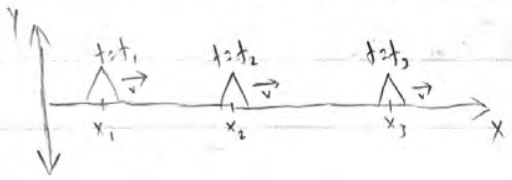
  - Alt: propagation of a disturbance in a medium
  - (*) Ex: waves on strings, waves in a fluid, waves in air (sound waves)
    - (*) Sound waves - changes in air pressure as disturbances
- Similar to oscillators - involve displacement from an equilibrium

Types of Waves

1) Transverse waves: displacement orthogonal to the direction of travel
2) Longitudinal waves: displacement is in the direction of travel (ex: stretching a slinky)

Observations on Mechanical Waves

1) Disturbances move with a constant velocity (wave speed: v)
2) Wave speed does not depend on the "shape" of the disturbance. [$v_1 = v_2$]
3) The shape of a disturbance does not change as it moves.
4) Superposition: if there are multiple disturbances, then they travel past ["through"] each other without influencing each other.



$\Rightarrow v = \frac{x_2 - x_1}{t_2 - t_1}$ [v is constant per (1)]

(*) Describing Waves

Let the shape of the disturbance := $y(x) = f(x)$ at some fixed time

$$\Rightarrow \boxed{y(x,t) = f(x - vt)}$$ [Right-moving/positive direction]

(t increases => x increases to keep y constant

Mechanical Waves (cont.)

10/16/23

Lecture 10 (cont.)

Reminder (Mechanical Waves)

Displacement ["shape"]: $\underline{y(x,t) = f(x-vt)}$

$\Rightarrow$ Change in displacement: $v_y = \frac{\partial y}{\partial t}(x,t) = -vf'(x-vt)$ [change over time]

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\cdot\frac{\partial^2 y}{\partial t^2} = 0} \longrightarrow \text{Wave equation [SHO equation, approx.]}$$

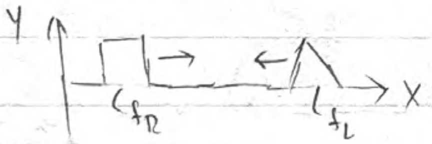
(*) Superposition

Prop: Let $f_R(x-vt)$, $f_L(x+vt)$ be a right-moving & left moving wave, respectively.

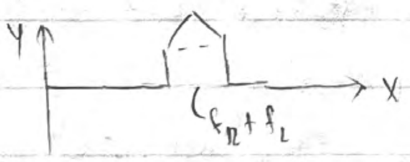
Then if both are traveling in the same medium, then the displacement is:

$$\underline{y(x,t) = f_R(x-vt) + f_L(x-vt)},$$

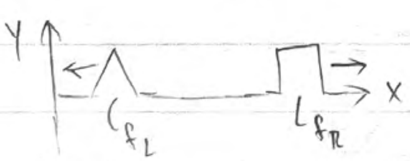
which obeys the wave equation.



[y @ t_1]



[y @ t_2]



[y @ t_3]

Reflections & Sinusoidal Waves

10/17/23

Lecture 11

Reflection of Traveling Waves

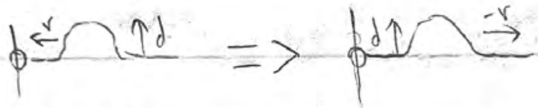
Two cases of wave reflection: fixed end and free end:

1) Fixed end: If the end of the medium is fixed (cannot be displaced/disturbed), then both the directions of travel and displacement are inverted upon reflection.

2) Free end: If the end of the medium is free (i.e. can be displaced/disturbed), then only the direction of travel is inverted upon reflection.

(*) For sinusoidal waves: phase shift of π for fixed-end, no shift for free-end

(1) Fixed end



(2) Free end

Sinusoidal Waves

Def: A wave $y(x,t)$ is sinusoidal if it is periodic across both x and y.

$$\Rightarrow \boxed{y_R(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x-vt)+\phi_0\right)} \quad \text{[Sinusoidal wave]}$$

Variables of Sinusoidal Waves

Def: Wavelength (λ) is the [spatial] distance across which the wave is periodic

$$\Rightarrow \boxed{\lambda = vT} \quad \text{[Wavelength = wave speed} \cdot \text{period]}$$

(*) Corollary: $\omega = \frac{2\pi}{\lambda}v \Rightarrow y(x,t) = A\cos\left(\frac{2\pi}{\lambda}x - \omega t + \phi_0\right)$

(*) $k = \frac{2\pi}{\lambda}$ is sometimes called the "wave number"

Wave Speed & Power

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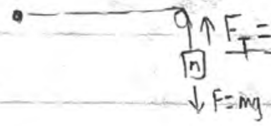
Lecture 11
(cont.)

Wave Speed of a String

We know that "inertia" is present in mechanical waves, and want to quantify its effect on wave speed.

→ Let F_T be the force of tension on the string; let μ [mu] be its linear mass density.

$$\Rightarrow \boxed{v = \sqrt{\frac{F_T}{\mu}} \;;\quad \mu = \frac{m}{l}} = \frac{\text{mass of string}}{\text{length of string}} \quad \text{[Linear mass density]}$$

(*) F_T: 

$\uparrow F_T = (-F) = -mg$ [Force of tension]

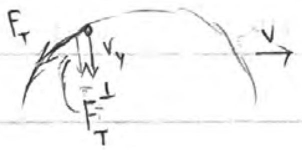
Power

Since waves transport energy, then they are performing work; then power = work/time.

$$1A: \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{x}}{dt} = \vec{F} \cdot \vec{v} \quad \text{[Instantaneous power]}$$

(*) Ex: Transverse Waves

Transverse waves create tension, and this results in points having velocity (i.e. changes in displacement).



$$y(x,t) = A\cos\left(\frac{2\pi}{\lambda}x - \omega t\right)$$

$$\Rightarrow \boxed{P = \mu A^2 \omega^2 v \sin^2\left(\frac{2\pi}{\lambda}x - \omega t\right)} \quad \text{[Instantaneous power]}$$

→ P is oscillating; could we get an average value?

$$\bar{P} = \frac{1}{T}\int_0^T P\,dt = \mu A^2 \omega^2 v \cdot \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi}{x} - \omega t\right) dt$$

$$\Rightarrow \boxed{\bar{P} = \frac{1}{2}\mu A^2 \omega^2 v} \quad \text{[Averaged power; Watts = J/s]}$$

Standing Waves

10/18/23

Lecture 12

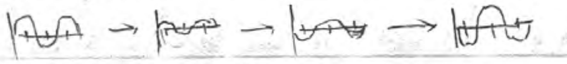
Standing Waves

When a sinusoidal wave $y_R(x,t) = A\cos(\frac{2\pi}{\lambda}x + \omega t)$ is reflected, the superposition of $y(x,t)$ and the reflected wave is:

$$
\begin{aligned}
y_{tot} &= y_R(x,t) + y_L(x,t) \\
&= A\cos\left(\tfrac{2\pi}{\lambda}x + \omega t\right) + A\cos\left(\tfrac{2\pi}{\lambda}x + \omega t + \phi_0\right) \\
&= 2A\cos\left(\tfrac{2\pi}{\lambda}x + \tfrac{\phi_0}{2}\right)\cos\left(\omega t + \tfrac{\phi_0}{2}\right) \qquad \text{[via trig identity]}
\end{aligned}
$$

$$
\Rightarrow \boxed{y_{tot} = \begin{cases} \text{free end}: \phi_0 = 0 \rightarrow 2A\cos\left(\frac{2\pi}{\lambda}x\right)\cos(\omega t) \\ \text{fixed end}: \phi_0 = \pi \rightarrow 2A\sin\left(\frac{2\pi}{\lambda}x\right)\sin(\omega t) \end{cases} > \text{SHO}: \begin{matrix} \cos,\sin(\omega t) \text{ cause oscillation} \\ \cos,\sin(\frac{2\pi}{\lambda}x) \text{ varies amplitude by } x \end{matrix}}
$$

Def: A standing wave is a wave where the locations of peak amplitude are constant.

Ex:  etc.

↳ Wave does not "travel" horizontally - only oscillates up/down

(*) Characteristics of the Standing Wave

- Peak amplitude of 2A [spatially dependent]
- Is a SHO, per the sin, $\cos(\omega t)$ term
- There are special points where displacement is always 0 $\forall t$ - called "nodes".
  - (*) A fixed end is always a node.
- There are points where maximal amplitude is 2A - called "anti-nodes".
  - (*) A free end is always an anti-node.
- Distance between neighboring nodes/anti-nodes is always $\frac{\lambda}{2}$ [half a wavelength].
- Distance between neighboring node, anti-node is always $\frac{\lambda}{4}$.

Standing Waves (cont.)

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Lecture 12 (cont.)

Standing Waves (cont.)

In the case of mediums with two fixed ends, standing waves will only occur at certain wavelengths [alt: certain frequencies].

(*) Only occurs if the value $\lambda/2$ precisely divides the length of the medium, such that the peaks of the reflected waves from both ends align => causes resonance.

(*) Always creates one more anti-node than node (not counting fixed ends)

Standing Waves w/ 2 Ends

Let $x=0$, $x=L$ be the fixed ends. For there to be a standing wave, we need:

$$L \text{ a node} \Rightarrow \sin\left(\frac{2\pi}{\lambda}L\right)=0 \Rightarrow \frac{2\pi L}{\lambda}=n\pi \ [n\in\mathbb{N}] \Rightarrow \boxed{L=n\frac{\lambda}{2}} \ [n\in\mathbb{N}]$$

Def: The fundamental (first harmonic) is the wavelength such that there exists a standing wave with only one anti-node.

$$\Rightarrow \boxed{\lambda=2L;\ f=\frac{v}{2L}}$$

(*) 2^{nd} harmonic: $\lambda=L$ [alt: $L=\lambda_{n=2}$], $f=\frac{v}{L}$

$$\boxed{\text{General formula: } L=\frac{n}{2}\lambda;\ f=\frac{nv}{2L};\ \#\text{anti-nodes}=n;\ \#\text{nodes}=n-1 \text{ (excl. ends)}}$$

└ Called n^{th} harmonic, or $(n-1)$ overtone

Sound Waves

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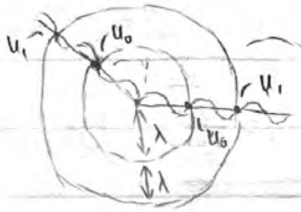
Lecture 14

Sound Waves

Sound waves are longitudinal waves traveling in 3 dimensions, taking air [gas] as a medium.

- Air as a medium - must consider volume/density/pressure/temperature (thermodynamics)
- Displacement as a function of radius [distance from a source]; will be constant at all points across a sphere around the source

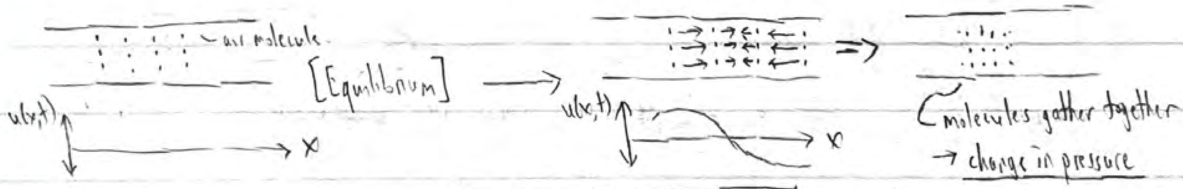


Displacement equal at all points in the circle

Sound Waves in Pipes

Sound waves in narrow pipes ($d < \lambda$) start to mimic 1D motion:

Displacement $u(x,t)$ of sound waves characterized as displacement from equilibrium:



Pressure equation: $\boxed{p(x,t) = -B\dfrac{\partial u}{\partial x}}$ (Gauge pressure)

→ B = bulk modulus [how easy the air is to compress]: $\boxed{\Delta p = -B\dfrac{\Delta V}{V}}$

Wave equation: $u(x,t) = \underbrace{u_0}_{\text{Amplitude of displacement}}\cos\left(\frac{2\pi}{\lambda}x - \omega t\right)$

$\Rightarrow p(x,t) = \underbrace{B\frac{2\pi}{\lambda}u_0}_{}\sin\left(\frac{2\pi}{\lambda}x - \omega t\right)$ > note: phase shift (cos → sin)

p_{max} [amplitude of pressure]: $\boxed{p_{max} = B\frac{2\pi}{\lambda}u_0}$

Properties of Sound Waves

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Lecture 14
(cont.)

(*) Standing Sound Waves in Pipes

Standing wave equation: $u(x,t) = u_0 \sin\left(\frac{2\pi}{\lambda}\right)\sin(\omega t)$

$\Rightarrow$ $p(x,t) = -B\frac{2\pi}{\lambda}u_0\cos\left(\frac{2\pi}{\lambda}x\right)\sin(\omega t)$

> note: phase shift ($\sin \to \cos$)
> $\Rightarrow$ swap nodes, antinodes

Wave Speed of Sound Waves

Similar to strings: sound wave speed dependent only on medium.

Formulas: (1) $v = \sqrt{\frac{\text{restoring force}}{\text{inertia}}} \Rightarrow \boxed{v = \sqrt{B/\rho}}$ [Physical intuition]

(2) $\boxed{v = \sqrt{\frac{\gamma RT}{M}}}$ [Thermodynamics]

$\hookrightarrow$ T = temperature (K); R = gas constant ($\sim 8.314\ \mathrm{J/mol\,K}$)

M = molar mass ($\mathrm{g/mol}$); γ = ratio of heat capacities (monatomic: 1.66 [He]; diatomic: 1.4)

(*) Wave Speed in Air

Air: $\begin{cases} M = 28.97 \\ \gamma = 1.4 \end{cases} \Rightarrow \boxed{v_{air} = 0.4\, v_{He}}$ [v_{He}: wave speed in helium]

Intensity

Power of waves in strings becomes intensity in sound waves.

Formula:

Power: $\bar{P} = \frac{1}{2}\sqrt{T\mu}\,\omega^2 A^2$; can replace $T \to B$, $\mu \to \rho$, $A \to u_0$

$\Rightarrow \boxed{\bar{I} = \frac{1}{2}\sqrt{B\rho}\,\omega^2 u_0^2} \Rightarrow \boxed{\bar{I} = \frac{1}{2}\frac{p_{max}^2}{\sqrt{B\rho}}}$ [Intensity]

Intensity

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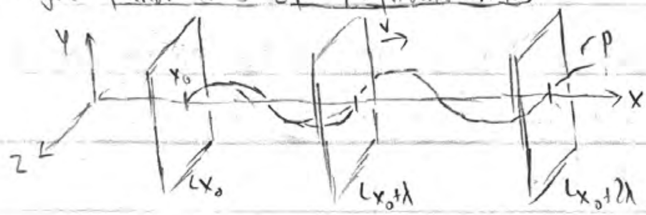
Lecture 15

Intensity & Power

Recall: $\bar{I} = \frac{1}{2}\frac{p_{max}^2}{\sqrt{\Pi \rho}}$ → Units: $\frac{Watts}{m^2}$ [Intensity = $\frac{power}{area}$]

Reasoning: Sound waves 3-dimensional => total power is distributed across large area

(*) Analogue: planar waves [p only dependent on x]



=> $p(x,t)$ constant [power distributed] across the points in each plane

=> Intensity akin to "power" of a single string / set of points $(-, y, z)$,
such that: total power $P = IA$ [intensity · area]

(*) Spherical Waves

Wavefronts in spherical waves are spheres (as opposed to planar waves: planes)

→ $u(\vec{x},t) = \frac{u_0}{r}\sin\left(\frac{2\pi}{\lambda}r - \omega t\right)$

=> $\boxed{I \sim \frac{1}{r^2}}$ [Intensity inversely proportional to r^2] Astrophysics: Standard candles use I to measure distance

=> Power = $IA = \frac{I_0}{r^2} \cdot r^2 = I_0$ = constant [Power is constant]

(*) Decibels

Decibels defined using logarithmic scale, relative to threshold of hearing $I_0 = 10^{-12}\,\frac{W}{m^2}$.

Formula: $\boxed{\beta = 10\,dB \cdot \log\left(\frac{I}{I_0}\right)}$ (*) logarithmic scale used since human ear can hear across many orders of magnitude

Standing Acoustic Waves

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(cont.)

Standing Acoustic Waves in Pipes

Boundary conditions: (i) Closed end: cannot move [displace] past wall → node for displacement; antinode for pressure

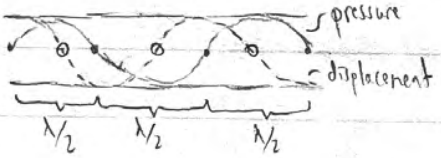
(ii) Open end: air can freely move → node for pressure; antinode for displacement

(*) Intuition: open end pressure = P_{atm} → gauge pressure = 0

(*) Displacement: closed end = fixed end; open end = free end

Pressure: closed end = free end; open end = fixed end

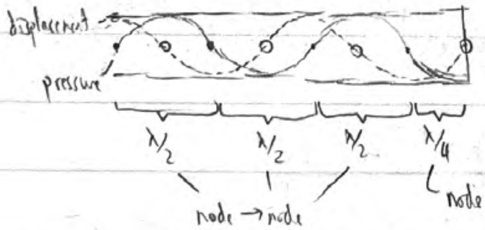
Cases: (i) 2 open ends



2 nodes for pressure (1 at each end)

→ n^{th} harmonic: $L = n\frac{\lambda}{2}$; $\lambda_n = \frac{2}{n}L$

(ii) 1 open end; 1 fixed end



Free end: node for pressure
Fixed end: node for displacement

→ n^{th} harmonic: $L = \frac{2n-1}{4}\lambda_n$; $\lambda_n = \frac{4}{2n-1}L$ [L only comes in odd multiples of λ/4]

Sound Waves (cont.)

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Lecture 16

(*) Beats

<u>Beats</u> – when two sounds of similar frequencies are heard, they may be perceived to be <u>slightly oscillating in volume</u> due to interference.

Ex: $u_1 = A\sin(k_1x + \omega_1 t)$, $u_2 = A\sin(k_2x + \omega_2 t)$ [superimposed]

$$\Rightarrow u_{tot} = u_1 + u_2 = 2A\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)\sin\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)$$

$$\rightarrow \underline{f_{avg} = \frac{f_1+f_2}{2}}\ ;\ \underline{f_{beat} = |f_1 - f_2|}$$

if $f_{beat} \ll f_1, f_2 \Rightarrow u_{tot} = \cos\left(\frac{k_1-k_2}{2}x - \pi(f_1-f_2)t\right)\sin\left(k_{avg}x - \omega_{avg}t\right)$

Treat $\frac{k_1-k_2}{2}x$ as phase constant $\phi \rightarrow \underline{T_{beat} = \frac{1}{f_1-f_2}}$

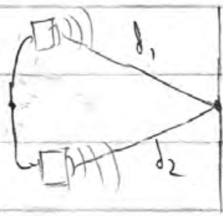
$$\Rightarrow \boxed{f_{beat} = |f_1 - f_2|}$$

(*) Interference

Destructive vs constructive interference:

<u>Destructive</u>: when the superposition of two sound waves results in <u>lower total amplitude</u>

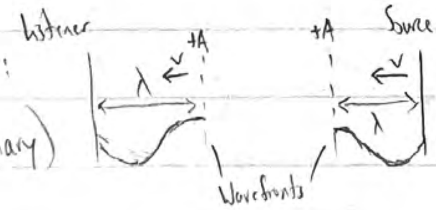
<u>Constructive</u>: when the superposition of two sound waves results in <u>higher total amplitude</u>



(*) Doppler Effect

<u>Doppler Effect</u> – Sounds are heard as being of <u>higher frequency</u> if <u>the source is approaching</u>, and of <u>lower frequency</u> if the <u>source is getting further away</u>

Ex:
(both stationary)



Source: $v = \lambda f_s \Rightarrow f_s = \frac{v}{\lambda}$

Listener: $T_L = \frac{\lambda}{v} \Rightarrow f_L = \frac{v}{\lambda}$

$\Rightarrow \underline{f_s = f_L}$

(time between hearing the maximums of incoming wavefronts

The Doppler Effect

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(cont.)

(*) The Doppler Effect (cont.)

Case 2: Source stationary, Listener moving

→ Listener: time between hearing +A wavefronts

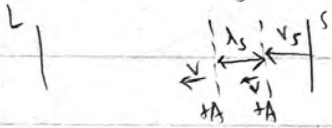
→ $\lambda = (v \pm v_L) T_L$ ⇒ $\boxed{f_L = \frac{v \pm v_L}{\lambda}}$

Source still stationary (frequency unchanged) ⇒ $f_s = \frac{v}{\lambda}$ ⇒ $\boxed{f_L = \frac{v \pm v_L}{v} f_s}$

$v_L > 0$ if listener approaching source ⇒ $f_L > f_s$

$v_L < 0$ if listener approaching source ⇒ $f_L < f_s$ [(*) $f_L = 0$ if $-v_L > v$]

Case 3: Source moving, listener stationary



Source moving → λ_s = distance between +A wavefronts affected

⇒ $\lambda_s = (v - v_s) T_s$ [source] → $f_s = \frac{v - v_s}{\lambda_s}$

Listener: perceived wavelength change → $\lambda_s f_L = v \rightarrow f_L = \frac{v}{\lambda_s}$ ⇒ $\boxed{f_L = \frac{v}{v - v_s} f_s}$

$f_L > f_s$ if $v_s > 0$ [source approaching]; otherwise $f_L < f_s$

→ Sound barrier: $v_s > v$ ⇒ shockwave [$\frac{v}{v - v_s} = \infty$]

Case 4: Source and Listener moving

General formula: $\boxed{f_L = \frac{v + v_L}{v - v_s} f_s}$ (treats v_L, v_s as independent)

v_L, v_s positive if approaching the other; negative if moving away

(*) Assumes head-on motion; take tangential velocities otherwise

Charge

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Lecture 17

Matter & Charge

2 kinds of charge: positive & negative (+ & -)

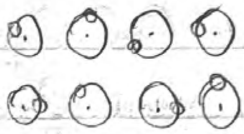
Atoms [matter] made up of protons (p^+), electrons (e^-), and neutrons

[Hydrogen atom]

Electrons much lighter than protons: $\begin{cases} m_p = 1.67 \cdot 10^{-27} \text{ kg} \\ m_e = 9.1 \cdot 10^{-31} \text{ kg} \end{cases}$

$\Rightarrow$ Consequence: Electrons much easier to move

↳ Ex: Protons in a solid lattice are largely fixed, but electrons may or may not be free to move between nuclei.


[Solid lattice]

Charge

Fundamental unit of charge: 1 unit of positive charge = charge of a single proton

1 unit of negative charge = charge of a single electron

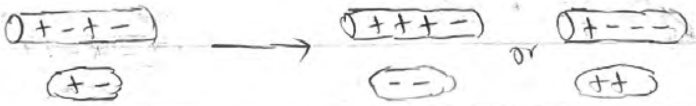
(*) SI unit: Coulombs (1 C $\simeq$ charge of $6.242 \cdot 10^{18}$ protons)

(☆) Important principle: charge is conserved (i.e. no change in a closed system)

Conductors & Insulators

Def: A material (e.g. a solid) is called a conductor if charge can move inside it (i.e. some electrons can freely move inside of it). It is called an insulator if electrons cannot freely move (i.e. are bound to their nuclei).

(*) An insulator can still become charged by having electrons donated/stripped away.



(e.g.)

Properties of Charge

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(cont.)

Properties of Charge

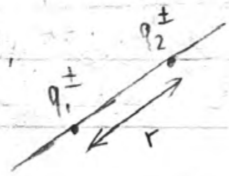
Observation: opposite charges attract; like charges repel. [generate attractive/repulsive forces].

(*) More charge → stronger force

(*) Further distance → weaker force

Coulomb's Law

Given two point charges q_1, q_2 that are r meters apart: the force they exert on each other proportional to charge and inversely proportional to distance.



$$\Rightarrow \boxed{F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{|q_1 q_2|}{r^2} \text{ [Coulomb's Law]}}$$

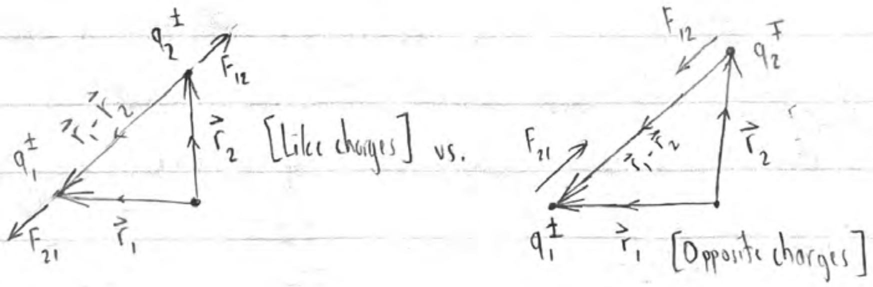
(*) ε_0 is a physical constant: $\boxed{\varepsilon_0 = 8.854 \cdot 10^{-12} \frac{C}{Nm^2} \text{ [SI]}}$

Forces $F_{2\to1}, F_{1\to2}$ [q_2 on q_1, q_1 on q_2] pointing towards each other if q_1, q_2 opposite charges; away from each other if q_1, q_2 are like charges

$$\Rightarrow \boxed{\underline{\text{Coulomb's Law}}: \vec{F}_{2\to1} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} \cdot \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}}$$

(*) For $\vec{F}_{1\to2}$: swap $\vec{r}_1, \vec{r}_2$

(*) $\hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$ [unit vector $1\to2$]



Electric Fields

10/31/23
Lecture 18

Coulomb's Law (cont.)

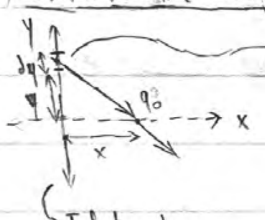
Sum of forces from multiple point charges computed by vector sum of individual forces from Coulomb's law [superposition principle]

Distributed charge. Define linear charge density μ across an object

Ex: Charge of a ring: $Q_{tot} = \mu C = \mu 2\pi r$

(*) Simplifying assumption: constant charge density [not always true; ex: insulators]

(*) Infinite Line Charges



Force of infinitesimal on q_0: $d\vec{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{(q_0)(\mu\, dy)}{x^2+y^2} \cdot \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ -y \end{pmatrix}$

$$\rightarrow \vec{F}_{tot} = \int_{-\infty}^{+\infty} \frac{1}{4\pi\varepsilon_0} \frac{(q_0)(\mu\, dy)}{x^2+y^2} \cdot \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ -y \end{pmatrix} \Rightarrow \vec{F}_{tot} = \frac{\mu q_0}{2\pi\varepsilon_0} \frac{1}{x} \hat{e}_x$$

Inf. line charge ↳ Exerts force $\vec{F}_{tot}$ on q_0

(*) Spherical Coordinates: $\vec{F}_{tot} = \frac{\mu q_0}{2\pi\varepsilon_0} \frac{1}{r} \hat{e}_r$

Electric Field

Given $q_1, \dots, q_n$ at positions $\vec{r}_1, \dots, \vec{r}_n$; force on a charge q_0 at $\vec{r}$

$$\Rightarrow \vec{F}_{tot\, q_0} = \sum_{j=1}^{n} \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_j q_0}{|\vec{r}_j - \vec{r}|} \cdot \hat{r}_{0j} = \Bigg(\underbrace{\sum_{j=1}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{|\vec{r}_j - \vec{r}|} \hat{r}_{0j}}_{\text{Electric field created by } q_1, \dots, q_n \text{ at } \vec{r}} \Bigg) q_0 \longrightarrow \vec{E}(\vec{r}) = \frac{\vec{F}_{tot\, q_0}}{q_0}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \sum_{j=1}^{n} \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_j}{|\vec{r}_j - \vec{r}|} \hat{e}_{\vec{r} \to \vec{r}_j}}$$

[Electric field: property of $q_1, \dots, q_n$ for any point $\vec{r}$]

$\Rightarrow$ Force by $q_1, \dots, q_n$ on a charge q_0 @ $\vec{r}$: $\boxed{\vec{F}_{tot} = \vec{E}(\vec{r}) \cdot q_0}$

(*) $\vec{E}(\vec{r})$ for point charge q: $\vec{E}(\vec{r}) = \frac{q}{2\pi\varepsilon} \frac{1}{\|\vec{r}\|} \hat{r}$ [points radially outward/inward]

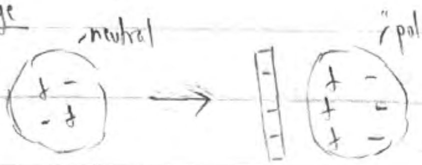
Electric Fields (cont.)

11/1/23

Lecture 19

(*) Induced Charge

Polarization:

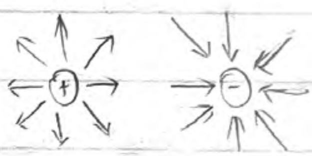


Ground: infinite reservoir of +/- charge

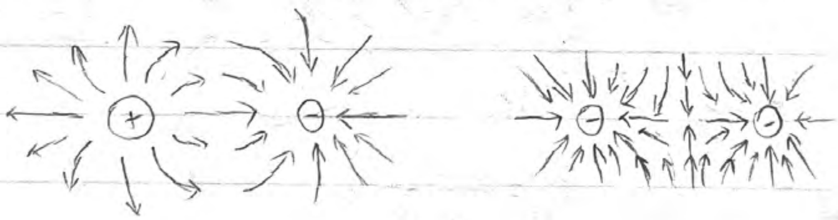
⇒ Induced charge

Electric Field

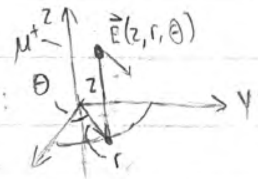
Is a vector field (as opposed to temperature - scalar field) of every point in space



- Electric field points away from positive charges; toward negative charges
- Electric field from multiple charges equivalent to vector sum (superposition) of individual electric fields

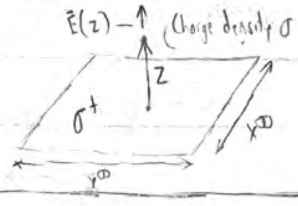


Infinite line charge:



$$\boxed{\vec{E}(z,r,\theta) = \frac{\mu}{2\pi\varepsilon_0}\cdot\frac{1}{r}\cdot\hat{e}_r}$$

Infinite "sheet charge":



$$\boxed{\vec{E} = \begin{cases} +\frac{\sigma}{2\varepsilon_0}\hat{e}_z & z>0 \\ -\frac{\sigma}{2\varepsilon_0}\hat{e}_z & z<0 \end{cases}}$$

(*) $\sigma = \frac{Q}{\text{Area}}$

(*) Electric Field Lines

Visualizing electric fields: Electric field lines (drawn tangent to electric fields)

(1) Magnitude of $\vec{E}$ proportional to density of field lines

(2) Field lines: ⊕ → ⊖

(3) Field lines do not cross

(4) Conductors' field lines orthogonal to surface

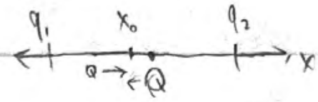
(*) Notes on Coulomb's Law

11/3/23

Lecture 20

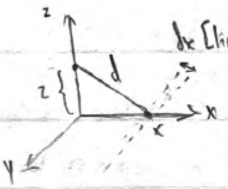
<u>Notes on Coulomb's Law</u>

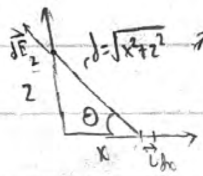
(*) Small variations in x around an equilibrium → can model/approximate <u>Coulomb's Law as restoring force in a SHO</u>



Ex: $F_x = \frac{Qq}{4\pi\varepsilon_0} \cdot \frac{-4dx}{(d^2-x^2)} \Rightarrow$ Linear approx: $F = \frac{-Qq}{4\pi\varepsilon_0} \cdot \frac{d}{d^4} x \Rightarrow \underline{F = -kx}$ $\left[k = \frac{Qq}{4\pi\varepsilon_0} \cdot \frac{d}{d^4}\right]$

(*) Deriving electric field on infinite sheet - <u>treat infinite sheet as union of infinite line charges</u>





$\rightarrow dE_z = \frac{\sigma}{2\pi\varepsilon_0} \cdot \frac{z}{z^2+x^2} dx$

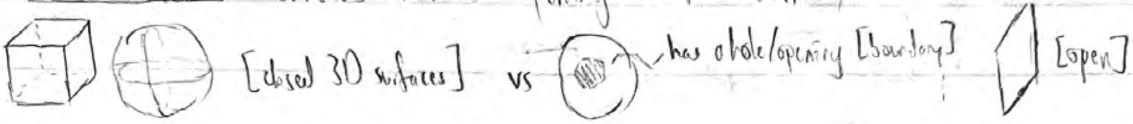
$\Rightarrow E_z = \int_{-\infty}^{+\infty} \frac{\sigma}{2\pi\varepsilon_0} \cdot \frac{z}{z^2+x^2} dx \overset{x=zu}{=} \frac{\sigma}{2\pi\varepsilon_0} z \int_{-\infty}^{+\infty} \frac{z\,du}{z^2+z^2u^2} = \underline{\frac{\sigma}{2\varepsilon_0}}$

Electric Flux

11/6/23

Lecture 21 Electric Flux & Gauss's Law

Define "closed surfaces" - surfaces with no "openings" [informal defn.],

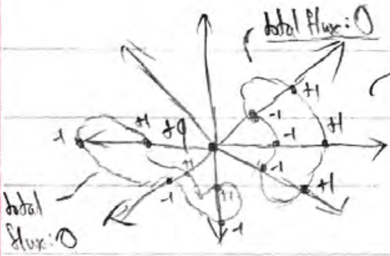
Ex: 

→ For closed surfaces, can unambiguously define an "inside" and an "outside"

Recall: Field lines can intersect surfaces. 

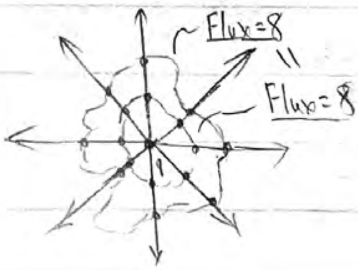
→ Define flux units relative to # of intersections.

- Field line enters surface => +1 units. } Defn. only "makes sense" for closed surfaces.
- Field line leaves surface => -1 units. } Open surfs.: what does "leave" mean? ex: fig. above



~ +q not in the surface => flux through surface is 0 for any closed surface [of any shape]

+q in the surface => flux through surface = # field lines from q for any closed surface [of any shape]

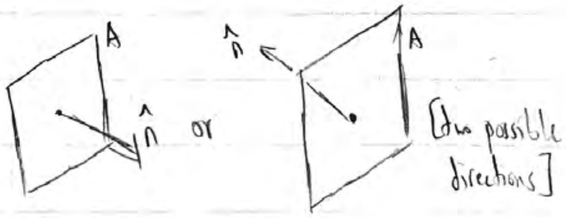


Recall: # of field lines proportional to magnitude of charge

$\boxed{\text{Gauss's Law: The electric flux through a closed surface is proportional to the enclosed charge.}}$

Surfaces

For any surface: can describe direction of surface via normal vector $\hat{n}$ orthogonal to surface.



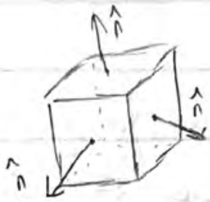
Gauss's Law

11/6/23

Lecture 21 (cont.)

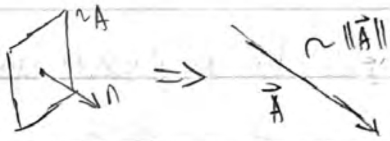
Electric Flux

For any closed surface: set normals to all be <u>pointing outward</u>.



[<u>outward normals</u>]

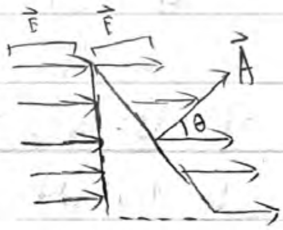
For a surface w/ area A, normal $\hat{n}$; define area vector $\underline{\vec{A} = A\hat{n}}$. [units: m^2]



$\Rightarrow$ $\sim \|\vec{A}\| = A,\ \vec{A} \parallel \hat{n}$

(*) Electric flux (simple case): uniform electric field $\vec{E}$ across surface (e.g. square) w/ area vector $\vec{A}$.

$$\Rightarrow \boxed{\Phi = \vec{E}\cdot\vec{A}}$$

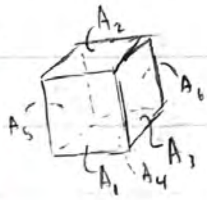


$\Rightarrow \underline{\Phi = EA\cos\theta}$

(*) Special cases:

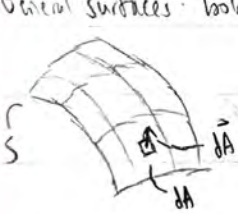
$\Rightarrow \Phi < 0$;

$\Rightarrow \Phi = 0\ [\vec{A} \perp \vec{E}]$



$\rightarrow \Phi = \sum_{i=1}^{6} \Phi_i$ [flux is sum of flux through subsurfaces]

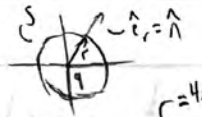
General surfaces: look at <u>flux through infinitesimal squares & sum</u>. [surface integral]



$\Rightarrow d\Phi = d\vec{A}\cdot\vec{E} \Rightarrow \Phi = \oint_S \vec{E}\cdot d\vec{A}$

$$\Rightarrow \boxed{\text{Gauss's Law: } \Phi_S = \frac{q_{\text{enclosed}}}{\varepsilon_0}}$$

(*) Ex: flux of a point charge.



Know: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{e}_r$; $d\vec{A} = dA\,\hat{e}_r$

$\Rightarrow d\Phi = \vec{E}\cdot d\vec{A} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}dA(\hat{e}_r\cdot\hat{e}_r) \Rightarrow \Phi = \oint_{S_{R^2}} d\Phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\overbrace{\oint_{S_{R^2}} dA}^{=4\pi R^2 \text{ [surface area]}} = \frac{q}{\varepsilon_0}$.

11/7/23
Lecture 22

Gauss's Law (cont.)

Applications of Gauss's Law

(*) If there are multiple charges $q_1, q_2, \ldots$ in closed surface $S \Rightarrow q_{encl} = \sum_{i=1}^{n} q_i$.

(*) Note: differential form of Gauss's Law: $\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$ [equivalent to integrative form]

☆ Given a charge distribution, can calculate flux integral relatively simply to find $\vec{E}$ using Gauss's Law.

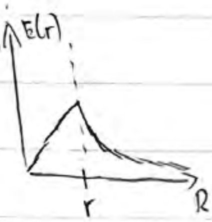
Ex: Ball w/ uniform charge density around origin

Know $\vec{E}(r) = E(r)\hat{e}_r$ [points radially outward]

→ Choose ball enclosing original ball as Gaussian surface S.

Then: $d\vec{A} = dA\,\vec{e}_r \Rightarrow \Phi = \oint_S \vec{E} \cdot d\vec{A} = E(r) \int dA = E(r) \cdot 4\pi r^2$

Know: $q_{encl} = \int \rho\, dV = \rho \frac{4}{3}\pi R^3 \Rightarrow E_r 4\pi r^2 = \frac{q_{encl}}{\varepsilon_0} \Rightarrow E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_{encl}}{r^2}$



(*) Electric field inside: take sphere inside (note: adjust q_{encl} for smaller volume)

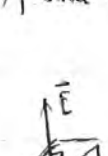
$\rightarrow E(r) 4\pi r^2 = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho \rightarrow E(r) = \frac{1}{3\varepsilon_0} r\rho$

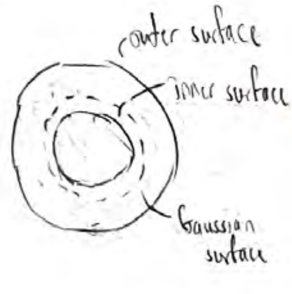
$$\boxed{E(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{q_{encl}}{r^2} & r > R \\ \frac{1}{3\varepsilon_0} r\rho & r < R \end{cases}}$$

(*) Gauss's Law & Conductors

Charges can move freely inside conductors, but the locations of charges inside a conductor eventually stabilize (conductor becomes electrostatic) → where do the charges go? [excess charges]

Conductor electrostatic ⇒ no electric field in conductor ($E=0$), since otherwise the charges would move. Then $E \neq 0$ only on the surface, but only if E is orthogonal to the surface (such that charges don't move, since they can't escape into air). Since $E=0$ inside, then for any S inside, $\Phi_S = 0 \Rightarrow q_{encl} = 0$. Then all charges are on the surface only.





If conductor has outer, inner surfaces: no excess charge on inner surface.

Pf: ∃ excess charge inside ⇒ surface in conductor enclosing inner surface has $q_{encl} \neq 0 \Rightarrow \Phi \neq 0$.

Gauss's Law (cont.)

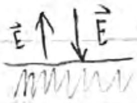
11/7/23

Lecture 22
+Lecture 23

Gauss's Law: Consequences

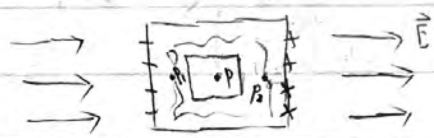
(i) Any excess charge in an electrostatic conductor must be on the surface of the conductor.
(ii) If the conductor has distinct inner & outer surfaces, there is no excess charge on the inner surface.

Recall: (i) $\vec{E} = 0$ inside the conductor
(ii) $\vec{E} \perp$ to the surface of the conductor

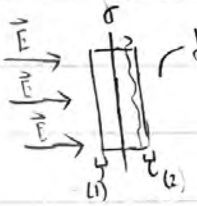


(*) Faraday's Cage

Conductor inside external $\vec{E}$:



[$\vec{E}=0$ @ $P_1, P_2 \Rightarrow \vec{E}=0$ @ P [by continuity]



take ϕ_1, ϕ_2 @ orthogonal sides

$\vec{E}$ @ $2 = 0 \Rightarrow \phi_2 = 0$
$\phi_1 = EA$

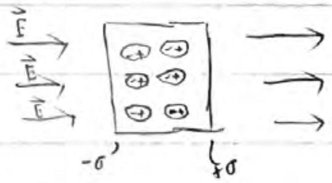
$\Rightarrow \phi = -EA = \frac{\sigma A}{\varepsilon_0} \Rightarrow \boxed{\sigma = -E\varepsilon_0}$

can be applied to infinitesimals

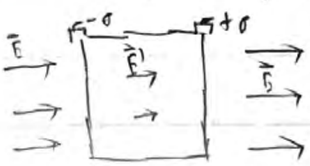
(*) Insulators in an Electric Field

Insulators - charge cannot move, but individual atoms may still orient dipoles toward electric field

$\Rightarrow$ net surface charges still occur (just smaller)



[Dielectric]



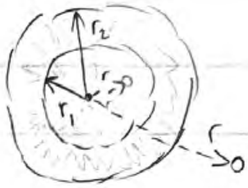
Electric field inside insulator: $|\vec{E}'| < |\vec{E}|$

Gauss's Law (cont)

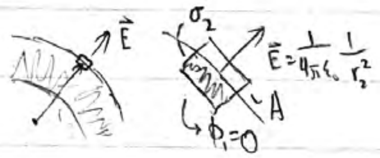
11/8/23
Lecture 23

Charge in a Conducting Shell

Know: $E(r)\ [r<r_1,\ r>r_2]$: $\underline{\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$



$\underline{\vec{E} \text{ at } r=r_2}$:



$\Rightarrow \phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_2^2}A = \frac{\sigma_2 A}{\varepsilon_0}$

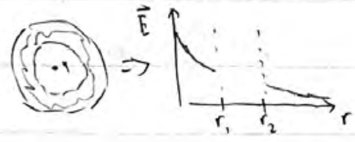
$\Rightarrow \boxed{\sigma_2 = \frac{q}{4\pi r_2^2}}$ [σ @ r_2]

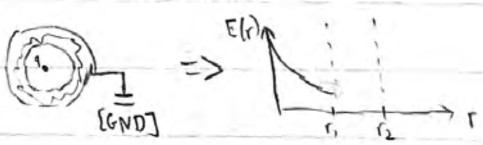
$\underline{\vec{E} \text{ at } r=r_1}$:

$\phi = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r_1^2}A = \frac{\sigma_1 A}{\varepsilon_0}$ [E only entering => flux negative]

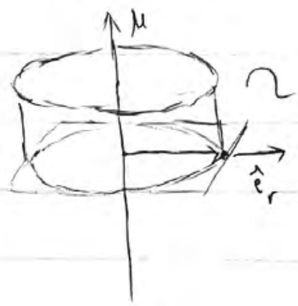
$\Rightarrow \boxed{\sigma_1 = \frac{q}{4\pi r_1^2}}$ [σ @ r_1]

(*) Electric Field:

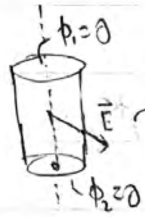




(*) Electric Field on an Infinite Line Charge



Gaussian surface:



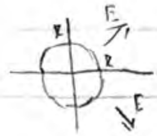
$\phi_3 = \Phi = \oint dA\, E(r)$
$= E(r)(2\pi r h)$

$\Rightarrow q_{encl} = \mu h \Rightarrow \Phi = \frac{\mu h}{\varepsilon_0} \Rightarrow \boxed{E(r) = \frac{1}{2\pi\varepsilon_0}\frac{\mu}{r}}$

(*) Electric Field Examples

11/13/23
Lecture 24

Ex 1: Spherical charge distribution with $\rho = \begin{cases} \rho_0 \frac{R}{r} & r<R \\ 0 & r>R \end{cases}$, radius R



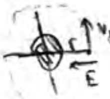
→ find E for $r<R$, $r>R$.

$r>R$: $E = \frac{1}{4\pi\varepsilon_0}\frac{q_{enc}}{r^2}$ $\quad q_{enc} = \int_0^R dV\,\rho(r) = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \int dr\, r^2 \rho(r) = 4\pi \int_0^R \rho_0 \frac{R}{r} r^2\, dr = 2\pi\rho_0 R^3$

→ can find E

$r<R$: $q_{enc} = \int_0^r dV\,\rho(r') = 4\pi \int_0^r \rho_0 \frac{R}{r'}\, dr' = 2\pi\rho_0 R r^2$ $\quad \Phi(r) = 4\pi r^2 E(r) = \frac{q_{enc}}{\varepsilon_0} \to E(r) = \frac{\rho_0}{2\varepsilon_0} R$

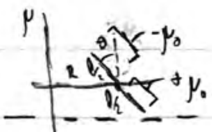
~ Q: Given particle charge q @ r, what v_{tan} is needed for the particle to orbit?



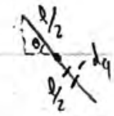
$F_E = F_{centripetal} \to qE = -m\frac{v_{tan}^2}{r} = -|q|\frac{q_{enc}}{4\pi\varepsilon_0}\frac{1}{r^2}$ ← assuming ρ_0, q opposite signs

→ $v_{tan} = \sqrt{\frac{|\rho_0 q|}{2m\varepsilon_0}\frac{R^3}{r}}$

Ex 2: Infinite line charge μ, tilted "dipole" $(-\mu_0, +\mu_0)$ ← length l R away → find F_{net} on dipole.



$E(r) = \frac{1}{2\pi\varepsilon_0}\frac{1}{r}\hat{e}_r$ $\quad$ Divide dipole into infinitesimals:

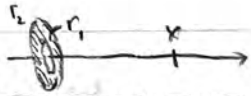


→ $dq = \pm\mu_0\, dl \to dF = (\pm\mu_0\, dl)\cdot\frac{1}{2\pi\varepsilon_0}\frac{1}{r}$

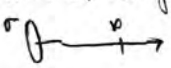
$r = R \pm l\sin\theta \to dF = (\pm\mu_0\, dl)\cdot\frac{1}{2\pi\varepsilon_0}\frac{1}{R\pm l\sin\theta}$; $F^{(-)}_{tot} = \int_0^{l/2} \frac{\mu\mu_0}{2\pi\varepsilon_0}\frac{dl}{R-l\sin\theta} = \frac{\mu\mu_0}{2\pi\varepsilon_0}\frac{1}{\sin\theta}\left(\ln(R-\frac{l}{2}\sin\theta) - \ln(R)\right)$

→ $= \frac{1}{2\pi\varepsilon_0}\frac{1}{\sin\theta}\ln\left(1-\frac{l}{2R}\sin\theta\right)$; similar for $F^{(+)}_{tot}$; $F_{tot} = F^{(+)}_{tot} + F^{(-)}_{tot}$

Ex 3: Finite ring of charge w/ inner radius r_1, outer radius r_2 → find $E(x)$



Recall: Disc of charge → $E(x) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{1}{\sqrt{\frac{R^2}{x^2}+1}}\right)$ [from HW: integrate rings of charge]



↳ For ring: superimpose 2 discs

→ $E(x) = E^+ - E^- = \frac{1}{2\varepsilon}\left(\frac{1}{\sqrt{\frac{r_1^2}{x^2}+1}} - \frac{1}{\sqrt{\frac{r_2^2}{x^2}+1}}\right)$

↳ if $x \ll r_1, r_2$: $E \approx \frac{1}{4\pi\varepsilon_0}\frac{\sigma\pi(r_1^2 - r_2^2)}{x^2}$

Electrostatic Potential

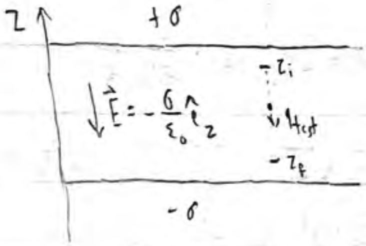
11/14/23
Lecture 25

Electrostatic Potential Energy

Observation: Electrostatic force equation [Coulomb's Law] similar to that of gravity (a conservative force)
→ electrostatic force is conservative (can derive an electrostatic potential energy U)

Recall (1A: Work): (i) $W_{i\to f} = \int_{C_{i\to f}} \vec{F}\cdot d\vec{x}$
(ii) $W_{i\to f} = K_f - K_i$
(iii) Conservative forces: $W_{i\to f} = -\Delta U = U_i - U_f$ (W independent of $C_{i\to f}$)
⇒ conservation of energy: $K_i + U_i = K_f + U_f$

Ex: Constant Electric Field

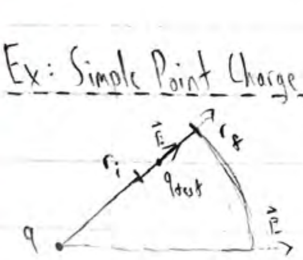


$F = q_{test}\vec{E} = -q_{test}\frac{\sigma}{\varepsilon_0}\hat{e}_z \rightarrow W_{z_i\to z_f} = \int_{z_i}^{z_f}\left(-q_{test}\frac{\sigma}{\varepsilon_0}\hat{e}_z\right)\cdot d\vec{z}$
$\rightarrow -\Delta U = -W = \left(-q_{test}\frac{\sigma}{\varepsilon_0}\right)(z_f - z_i) = U_f - U_i$

$$\boxed{\rightarrow U(z) = q_{test}\frac{\sigma}{\varepsilon_0} z}$$
(*) analogous to $U = mgh$ [gravitation]

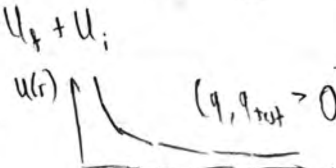
(*) Reminder: potential energy is relative to any arbitrary "0" (ex: set $0 = U(\infty)$)
Observation: For a positive charge, U decreases in direction of electric field (K increases)
↳ Similar to gravity (U decreases in direction of gravitational field - "falling")

Ex: Simple Point Charge



$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_{test}q}{r^2}\hat{e}_r \longrightarrow W_{r_i\to r_f} = \int_{r_i}^{r_f}\vec{F}\cdot d\vec{r} = \int_{r_i}^{r_f}\frac{1}{4\pi\varepsilon_0}\frac{q_{test}q}{r^2}(\hat{e}_r\cdot\hat{e}_r)\,dr$
$\rightarrow W_{r_i\to r_f} = \frac{1}{4\pi\varepsilon_0}q_{test}q\left(-\frac{1}{r_f} + \frac{1}{r_i}\right) = -U_f + U_i$

$$\boxed{\rightarrow U(r) = \frac{q_{test}q}{4\pi\varepsilon_0}\frac{1}{r}}$$



$(q, q_{test} > 0)$

Electric Potential

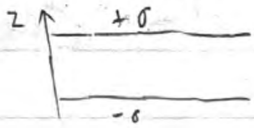
11/14/23
Lecture 25
(cont.)

Electric Potential

Deriving electric potential V from U akin to deriving electric field $\vec{E}$ from $\vec{F}$ (i.e. $\vec{E} = \frac{\vec{F}}{q_{test}}$).

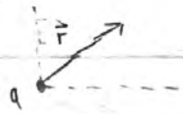
$\rightarrow \boxed{\Delta V = \frac{\Delta U}{q_{test}}}$; units: $\text{Volts} = \frac{\text{Joule}}{\text{Coulomb}}$

Ex: Infinite Charged Sheets



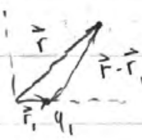
$U(z) = q_{test}\frac{\sigma}{\varepsilon_0}z \Rightarrow \boxed{V(z) = \frac{\sigma}{\varepsilon_0}z}$ [note: is a scalar]

Ex: Simple Point Charge



$U(z) = \frac{1}{4\pi\varepsilon_0}\frac{q_{test}q}{r} \Rightarrow \boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$ $[r = |\vec{r}|]$

Alt.:



$\Rightarrow \boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{r}-\vec{r}_1|}}$

(*) Superposition

Superposition holds for electric potential.

(*) Ex: Point charges $q_1, \dots, q_n$ at $\vec{r}_1, \dots, \vec{r}_n$

$$\boxed{\rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r}_i|}}$$

Electric Potential (cont)

11/15/23
Lecture 26

Methods for Finding Electric Potential

1) Integrate electric field $\vec{E}$ along a path ($\vec{E} \to V$)
2) Superposition principle

1) Integrating Electric Field

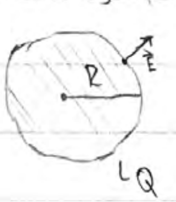
$$\Delta V = V_f - V_i = \frac{1}{q_{test}}\left(\int_{C_{i\to f}} \vec{F}\cdot d\vec{x}\right) \Rightarrow \boxed{\Delta V = -\int_{C_{i\to f}} \vec{E}\cdot d\vec{x}}$$

(*) ΔV independent of path $C_{i\to f}$ taken

Observations:

i) V decreases when traveling along field lines (electric field
ii) If a path is completely orthogonal to the electric field $\to \Delta V = 0$
Consequence: $V_f = V_i$ if $\vec{r}_f, \vec{r}_i$ connected by a path always orthogonal to $\vec{E}$
iii) In a region with no electric field, all points have the same potential (inside electrostatic conductor, e.g.)
(*) Corollary [(ii) & (iii)]: All points inside, or on the surface of, an electrostatic conductor have the same potential

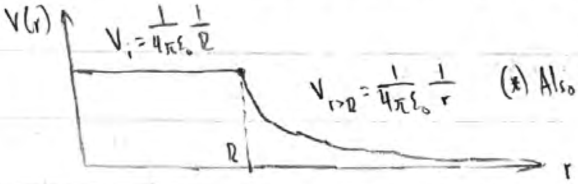
Ex: Conducting Sphere



$\vec{E}_{inside} = 0$
$\vec{E}_{outside} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{e}_r$

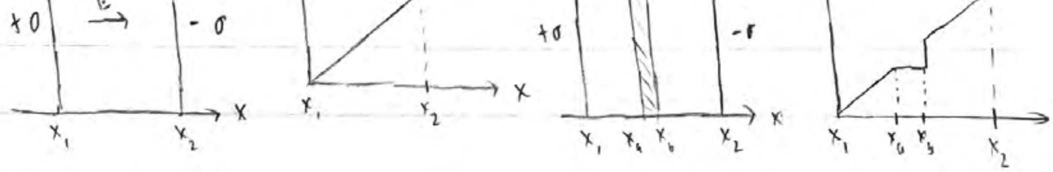
true for $r \geq R$; but electric potential is continuous, so we can find that $V_{r=R} = V_{r<R}$

$\Rightarrow$



(*) Also true for conducting shells

Ex: Charged Sheets



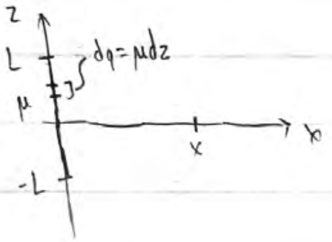
Electric Potential (cont.)

11/15/23
Lecture 26
(cont.)

2) Superposition Principle

Electric potential from multiple charges is the superposition of the electric potential from each individual charge.

Ex: Finite Line Charge

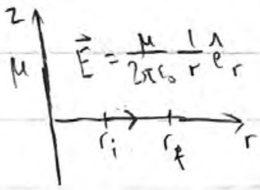


$$\rightarrow V(x, z=0) = \frac{\mu}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{dz}{\sqrt{r^2+z^2}^{\,3/2}} = \frac{\mu}{4\pi\varepsilon_0}\ln\left(z+\sqrt{z^2+r^2}\right)\Bigg|_{-L}^{+L}$$

$$= \frac{\mu}{4\pi\varepsilon_0}\ln\left(\frac{1+\sqrt{1+\frac{x^2}{L^2}}}{-1+\sqrt{1+\frac{x^2}{L^2}}}\right)$$

Take $\lim_{L\to\infty}$

Ex: Infinite Line Charge

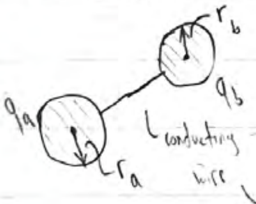


$\vec{E} = \frac{\mu}{2\pi\varepsilon_0}\frac{1}{r}\hat{e}_r$

$$V(r_f) - V(r_i) = -\int \vec{E}\cdot d\vec{x} = \frac{\mu}{2\pi\varepsilon_0}\int_{r_i}^{r_f}\frac{1}{r}\,dr$$

$$= -\frac{\mu}{2\pi\varepsilon_0}\left(\ln(r_f) - \ln(r_i)\right) \rightarrow \boxed{V(r) = -\frac{\mu}{2\pi\varepsilon_0}\ln(r)}$$

(*) Tesla Coils: Twin Conducting Spheres



Know: $V(r_a) = \frac{q_a}{4\pi\varepsilon_0}\frac{1}{r_a}$, $V(r_b) = \frac{q_b}{4\pi\varepsilon_0}\frac{1}{r_b}$

$\rightarrow V(r_a) = V(r_b)$

$$\rightarrow \boxed{\frac{q_a}{r_a} = \frac{q_b}{r_b}}$$

Recall: $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

$$\rightarrow \frac{E_a}{E_b} = \frac{q_a}{r_a^2}\frac{r_b^2}{q_b} = \frac{r_a}{r_b} \rightarrow \boxed{E_a r_a = E_b r_b}$$

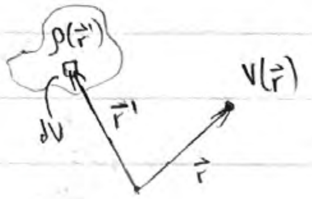
Equipotential Surfaces

11/20/23

Lecture 27

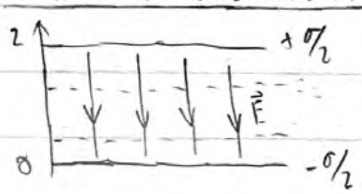
Superposition (cont.)

Given charge distribution $\rho(r)$, point $\vec{r}$: $\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3 r'}$ (integrate over charge distribution)



(*) Integral guaranteed to converge for finite charge distributions ; may or may not converge for infinite charge distributions

Electric Potential & Field Lines



$\vec{E}(z) = -\frac{\sigma}{\varepsilon_0}\hat{e}_z$ [constant $\forall$ z inside]

$V(z) = \frac{\sigma}{\varepsilon_0} z$ [not constant] (0 @ z = 0)

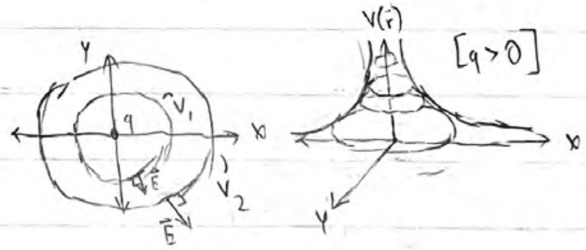
=> at any z : all points at that z have equal potential V

→ plane at any z is an equipotential surface

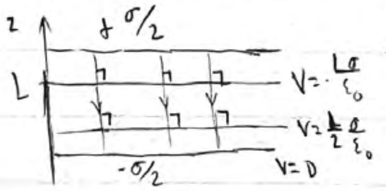
(*) Equipotential Surfaces

Simple point charge:

$V = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$



Infinite Charged Sheets:



Observations:

(i) Electric field lines are orthogonal to equipotential surfaces at every point.

(ii) Electric potential decreases in the direction of the electric field. [Electric field lines point toward lower potential].

Electric Potential & Electric Field

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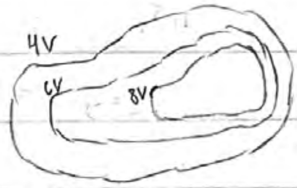
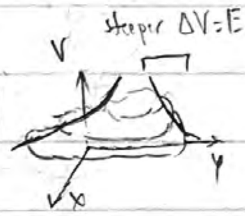
Lecture 27
(cont.)

Equipotential Surfaces (cont.)

(*) Equipotential surfaces have V everywhere the same; but not necessarily $\vec{E}$ everywhere the same.

Can read equipotential lines like a topographic map:
<u>density of lines ~ rate of change in potential [E]</u>
(<u>closer lines → larger electric field</u>)





Electric Potential → Electric Field ($V \to E$)

<u>Recall</u>: $V = -\int \vec{E}\cdot d\vec{x} \longrightarrow V(\vec{x}) = V(\vec{x}_1) - \int_{\vec{x}_1}^{\vec{x}} \vec{E}\cdot d\vec{x} \Rightarrow \boxed{\vec{E} = -\vec{\nabla} V(\vec{x})}$

[32A: $-\nabla f(\vec{v})$ = direction of greatest descent]

<u>(*) Ex</u>:

<u>Charged Sheet</u>: $V(\vec{x}) = \frac{\sigma}{\varepsilon_0} z \longrightarrow -\nabla V(\vec{x}) = -\langle 0, 0, -\frac{\sigma}{\varepsilon_0}\rangle = -\frac{\sigma}{\varepsilon_0}\hat{e}_z$ ✓

<u>Point Charge</u>: $V(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} \to -\nabla V(\vec{x}) = -\nabla\left(\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$\to = \frac{-q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(\frac{x}{r}\hat{e}_x, \frac{y}{r}\hat{e}_y, \frac{z}{r}\hat{e}_z\right) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{e}_r \ \checkmark$$

<u>Line Charge</u>: (Use cylindrical coordinates)

(*) Potential Energy of a Charge Distribution

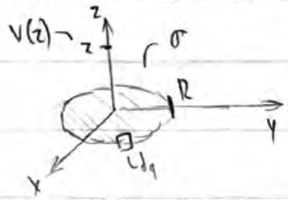
$\vec{r}_1$, q_1 $\xrightarrow{\text{(bring } q_2 \text{ from } \infty)}$ $\vec{r}_1$, q_1; $\vec{r}_2$, q_2 $\xrightarrow{(q_3 \text{ from } \infty)}$ $\vec{r}_1$, q_1; $\vec{r}_2$, q_2; $\vec{r}_3$, q_3

$U = 0 \longrightarrow U_{tot} = U_{1\,2} = \frac{q_1 q_2}{4\pi\varepsilon_0|\vec{r}_2 - \vec{r}_1|} \longrightarrow \boxed{U_{tot} = \frac{q_1 q_2}{4\pi\varepsilon_0|\vec{r}_1-\vec{r}_2|} + \frac{q_1 q_3}{4\pi\varepsilon_0|\vec{r}_1-\vec{r}_3|} + \frac{q_2 q_3}{4\pi\varepsilon_0|\vec{r}_2-\vec{r}_3|}}$

Capacitance

11/21/23

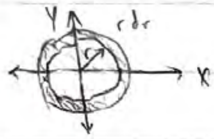
Lecture 28 (*) Potential of a Solid Disk



1) Take potential of ring w/ radius r, linear charge density λ:

$$dq = \lambda r\, d\phi \rightarrow V_\lambda = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{\lambda r}{\sqrt{z^2+r^2}}\, d\phi = \frac{2\pi}{4\pi\varepsilon_0}\frac{\lambda r}{\sqrt{z^2+r^2}} = \frac{1}{2\varepsilon_0}\frac{\lambda r}{\sqrt{z^2+r^2}}.$$

2) Integrate whole disk via rings:



$$\lambda = \sigma\, dr \rightarrow V_R = \int_0^R \frac{r\sigma\, dr}{2\varepsilon_0\sqrt{z^2+r^2}} = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r}{\sqrt{r^2+z^2}}\, dr = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)$$

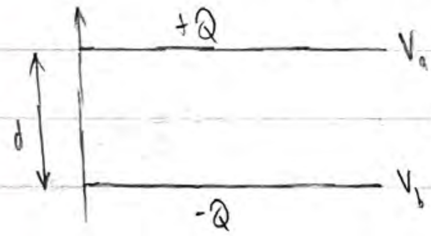
$$= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2} - z\right) \quad [z > 0]$$

(*) 3) Find electric field: $\vec{E} = -\vec{\nabla} V = -\frac{\partial}{\partial z} V \hat{e}_z$ [since x, y not in expression for V on z-axis]

$$\rightarrow \vec{E} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+R^2}} - 1\right)\hat{e}_z \; [z>0]; \quad \vec{E} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+R^2}} + 1\right) \; [z<0]$$

Capacitance

When two charges (initially together) are separated, then a potential difference will be created.



(inf. charged sheets)

$\rightarrow V_a - V_b = \frac{\sigma}{\varepsilon_0} d$ => more charge results in a larger potential diff.

(*) also denoted V_{ab}

→ Def: Capacitance is the "proportionality factor" between charge and potential difference.

$$\Rightarrow V_{ab} C = Q \rightarrow \boxed{C = \frac{V}{Q}}; \text{ units: F } \left[\text{Farad}: \frac{\text{Coulomb}}{\text{Volt}}\right]$$

→ A capacitor is a device that can store charge when a potential difference is applied.

(*) Creating a Potential Difference

Typically use a battery/power supply to create & maintain a potential difference.

(non-electrical)

Can be done via chemical processes (e.g. a battery), or mechanically (Van der Graaff generator).

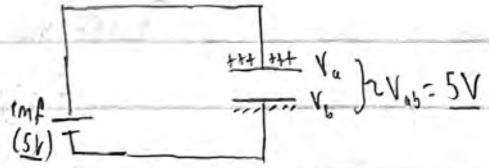
Capacitance (cont.)

11/21/23
Lecture 28
+ Lecture 29

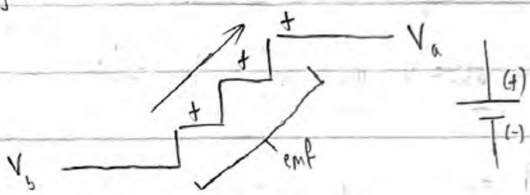
Electromotive Force (emf)

Potential difference supplied by emf [electromotive force], e.g. from a battery. (Units: Volts)

↳ "emf" - use as catch-all for black-box processes supplying ΔV

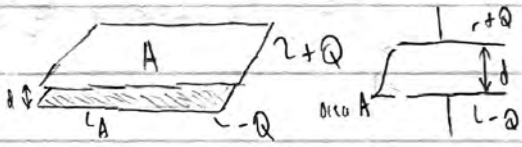


(*) Analogy - emf as a "charge elevator"



(*) Capacitance of a Plate Capacitor

Approximate a plate capacitor [2 parallel plates] by assuming inf. area ($\Rightarrow \vec{E} = 0$ outside, uniform inside)



↳ $V_a - V_b = \frac{\sigma}{\varepsilon_0} d$, $\sigma = \frac{Q}{A}$ $\Rightarrow$ $\boxed{C = \frac{A\varepsilon_0}{d}}$

(*) Approximation works best when plates are close together [d is small]

(*) Capacitance of a Spherical Capacitor

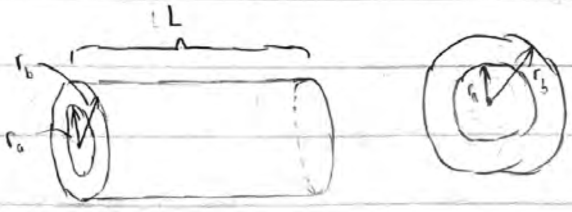
Recall: Spherical shell w/ charge q, radius R ↳ $V = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{1}{R} & r \le R \\ \frac{1}{4\pi\varepsilon_0}\frac{1}{r} & r > R \end{cases}$

$V_{tot} = V(+Q, r_a) + V(-Q, r_b)$

$\Rightarrow V(r_a) - V(r_b) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{r_b - r_a}{r_a r_b}\right)$ ↳ $\boxed{C = 4\pi\varepsilon_0 \frac{r_a r_b}{r_b - r_a}}$

(*) Capacitance of a Cylindrical Capacitor

Approximate a cylindrical capacitor by assuming infinite length ($L = \infty$)



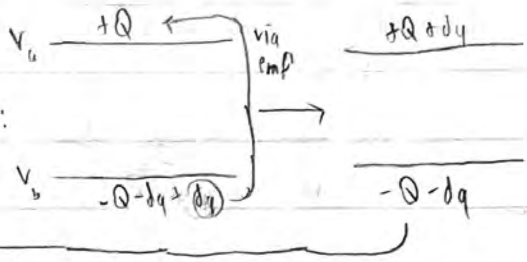
↳ $\boxed{C = 2\pi\varepsilon_0 \frac{L}{\ln(r_b/r_a)}}$

Capacitors

11/22/23
Lecture 29
(cont.)

Capacitors

Capacitors store energy. Ex (Plate capacitors):



Moving charge = work performed.

$\hookrightarrow dU = V_{ab}\,dq$. Recall: $V_{ab} = \frac{Q}{C} \longrightarrow dU = \frac{1}{C} q\,dq \rightarrow U = \int_0^Q dU \Rightarrow \boxed{U = \frac{Q^2}{2C}}$

(*) Stored Energy Equation (Variants)

$$\boxed{U = \frac{1}{2C}Q^2 \iff U = \frac{1}{2}QV \iff U = \frac{1}{2}CV_{ab}^2}$$

(☆) $U = \frac{1}{2}CV^2$

(*) Capacitors as Batteries

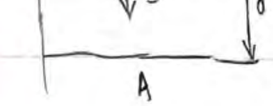
Capacitors are good at storing & releasing energy quickly (useful in circuits, e.g.); are not good for storing large quantities of energy (i.e. are not good batteries).

(*) Ex: EV battery: 60 kWh → $2.16\cdot 10^8$ J [standard for EVs]

if using plate capacitor w/ $V = 1000$ V [substantial], $d = 0.5$ mm $\Rightarrow A = \frac{2Ud}{\varepsilon_0 V^2} = 3.15\cdot 10^{10}$ m [too large]

└ preferred: store energy as chemicals (e.g. lithium ion batteries)

Energy Density of an Electric Field [1C]



→ $U = \frac{1}{2}CV_{ab}^2$. Recall: $V_{ab} = Ed$ [inf. charged sheets]

$\hookrightarrow U = \frac{1}{2}\frac{\varepsilon_0 A}{d}V_{ab}^2 = \frac{1}{2}\frac{\varepsilon_0 A}{\cancel{d}}d^{\cancel{2}}E^2 = \frac{1}{2}\varepsilon_0 A d E$

→ Observe V [volume] $= A\cdot d \Rightarrow U = \frac{1}{2}\varepsilon_0 E^2 V \Rightarrow \boxed{\frac{U}{V} = u = \frac{1}{2}\varepsilon_0 \vec{E}^2}$ [energy density of $\vec{E}$]

(used for electric waves, e.g.)

Dielectrics

11/22/23
Lecture 29
(cont.)

Relative Permittivity

Previously: Assumed potential difference has been across a vacuum. [$V_0 = V_{vacuum}$]

Insert material between capacitors ↳ $\underline{V' = \frac{V_0}{K}}$, where $K > 1$ [material constant]

→ Capacitance: $C_0 = \frac{Q}{V_0} \Rightarrow C_{matter} = K\frac{Q}{V_0} \Rightarrow \boxed{C_{matter} = K \cdot C_0}$ [capacitance increases]

Up to now: $C_0 = \varepsilon_0 \times$ something → $C_{matter} = K \cdot \varepsilon_0 \cdot$ something

"permeability of space"

relative permittivity / dielectric constant

↳ Can define $\varepsilon = K \cdot \varepsilon_0 \Rightarrow \boxed{K = \frac{\varepsilon}{\varepsilon_0}}$

(*) Effect on Stored Energy

Case 1: Q fixed

due to plate doing work on inserted material

$$U_0 = \frac{Q^2}{2C_0} \rightarrow U_{matter} = \frac{Q^2}{2KC_0} = \frac{1}{K}U_0 \quad [U \text{ decreases}]$$

Case 2: V_{ab} fixed (connected to emf)

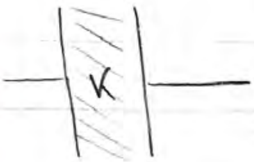
inserting material requires doing work

$$U_0 = \frac{1}{2}C_0V^2 \rightarrow U_{matter} = \frac{1}{2}KC_0V^2 = KU_0 \quad [U \text{ increases}]$$

Dielectrics (cont.)

11/27/23
Lecture 38

Dielectrics



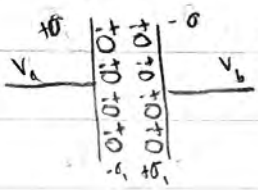
Analogy: ε_0 denotes material constant for "vacuum" [air] ← i.e. "ε"

→ $\varepsilon = K\varepsilon_0$ (material constant for vacuum)

$$E_0 = \frac{1}{\varepsilon_0}\cdot \text{"something"} \; ; \; E = \frac{1}{K\varepsilon_0}\cdot \text{"something"} \quad (\underline{E < E_0})$$

Intuition

In a vacuum:



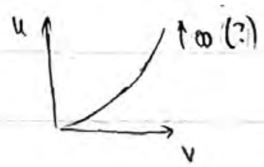
~ Capacitance creates induced charge [dipole] between conductors: magnitude σ_1 on surface

$$\longrightarrow \boxed{E = \frac{\sigma - |\sigma_1|}{\varepsilon_0}} \text{ (between plates)}$$

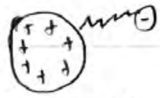
Know: $E_{matter} = \dfrac{\sigma}{K\varepsilon} \longrightarrow \boxed{|\sigma_1| = \left(1 - \frac{1}{K}\right)\sigma}$ ~ In medium, dipoles generated stronger than in air

(*) Limitations of Capacitors

Recall: $U = \frac{1}{2}CV^2$; why can't we keep increasing V to store more energy?



→ Answer: Air ← or whatever medium acts as an insulator between charged surfaces in a capacitor; but if V [E] grows too large → causes dielectric breakdown: the insulator itself becomes conducting [capacitor shorts out]. ~ ex: Van de Graaff generator



Consequence: For any capacitor, ∃ a limit on V. Namely: $\boxed{\frac{V}{d} > \text{dielectric strength} \Rightarrow \text{short}}$

(*) Dielectric strength is a material constant (ex: $3.6\cdot 10^6$ V/m in air)

(*) Observe: maximum V is lower for smaller values of d

Capacitors in Series & in Parallel

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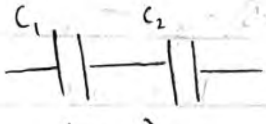
Lecture 30
(cont)

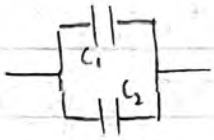
Capacitors in Series & in Parallel

Given two capacitors C_1, C_2: have two configurations (in series & in parallel)



(Series)

vs



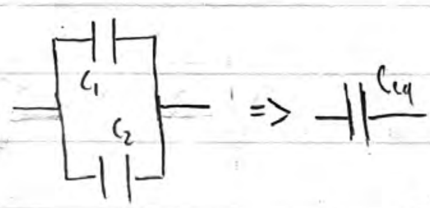
(Parallel)

(Analogy: pipes)

→ Capacitors combined in series/in parallel, still behave as a capacitor when combined.
In particular: can treat as a single capacitor w/ capacitance C_{equiv}.

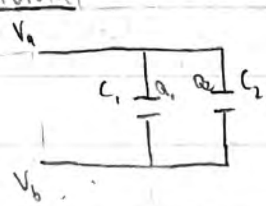
Series: $\boxed{C_{eq} = \frac{C_1 C_2}{C_1 + C_2}}$ Alt: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$

Parallel: $\boxed{C_{eq} = C_1 + C_2}$ [Capacitances sum]



Capacitors in Parallel

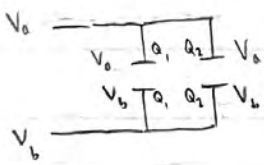
Illustration:



conductor => equipotential

Observe: V_a must be the same for both C_1, C_2 [since they are connected by a conducting wire]; same for V_b

→ $C_1 = \frac{Q_1}{V_{ab}}$, $C_2 = \frac{Q_2}{V_{ab}}$



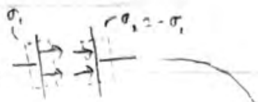
Looking at totals: $C_{tot} = \frac{Q_{tot}}{V_{ab}} = \frac{Q_1 + Q_2}{V_{ab}} = C_1 + C_2$ ✓

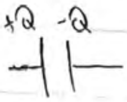
Capacitors in Series

11/27/23
Lecture 30
(cont.)

Capacitors in Series

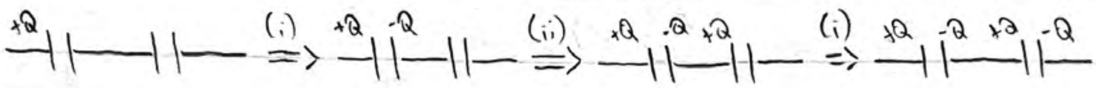
Illustration:

Gauss's Law: 

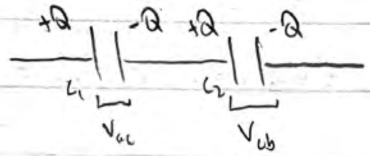
Recall: plate capacitors have charges $+Q, -Q$:  (i)

In initial uncharged state: middle wire has no charge
→ when charged: net charge = 0 (ii) [charge conservation]

Then:



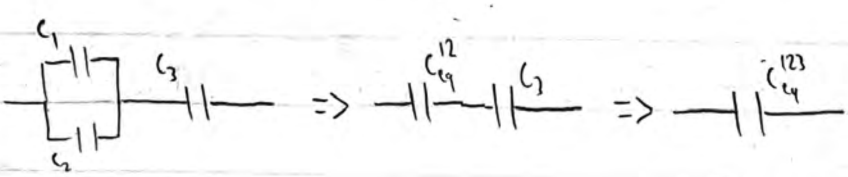
→ Consequence: Charges are the same across capacitors in series.

Know: $V_{ab} = V_{ac} + V_{cb}$; 

$$\rightarrow V_{ab} = \frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C_{eq}} \Rightarrow \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C_{eq}} \quad \left[C_{eq} = \frac{C_1 C_2}{C_1 + C_2}\right] \checkmark$$

(*) Capacitor Configurations

Can reduce complex configurations of charges to simpler ones.

(*) Ex: 

Current

11/28/23
Lecture 31

Current

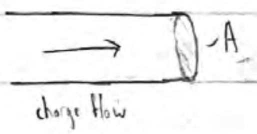
Def: Current is the motion of charge due to a potential difference [flow of charge/unit time]

$$\rightarrow \boxed{I = \frac{\Delta Q}{\Delta t} \overset{\text{inst.}}{=} \frac{dQ}{dt}}$$ unit: Amperes $\left[\frac{\text{Coulomb}}{\text{second}}\right]$; A, amps

(*) Note: 1 Coulomb is a substantial charge → 1 A is a fairly sizable current

(*) Idea of current analogous to flow of fluid; conducting wires as pipes, emf as a "pump"

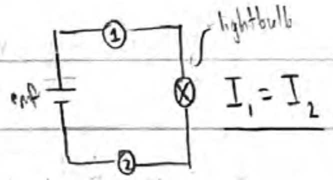
Think of wires as pipes → can define current density: $\boxed{J = \frac{I}{A}}$



Circuits are a closed system [charge conserved]

→ no charge is ever gained or lost at any point ($Q_{in} = Q_{out}$)

⟶ Consequence: Current is equal at all points in a circuit.



Formulation for charge density: $\boxed{J = q_e n v}$ => Charge through A in time dt: $dQ = q_e n A v dt$

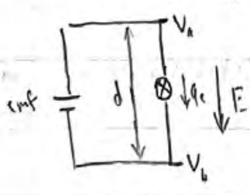
└ q_e = "charge carriers" [electrons]; n = number density $\left(\frac{\#\text{ carriers}}{V}\right)$; v = velocity

(*) Note: v describes speed of individual charge carriers; speed of signal propagation is much faster (electrons push each other, e.g.; closer to speed of light)

(*) Flow of "Free-Charge" Model

Observation: Current resulting from a constant potential difference, is constant (does not change over time).

Problem: Assume free charge carriers.



assume constant $\vec{E} = \frac{V_{ab}}{d}$

$\rightarrow m_e a = F = q_e E$

$\rightarrow a = \frac{q_e V_{ab}}{m_e d}$ => current is changing [contradiction]

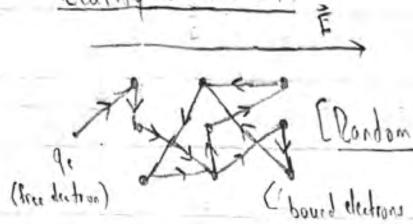
Ohm's Law

11/28/23
Lecture 31
+ Lecture 32

Ohm's Law (Intuition)

Free-Carrier Model:

"Reality" (Drude model)



[Random walk]
- Due to collisions between carriers (à la bouncing pinballs)
- Affected by temperature (random motion)
- Carriers still drift in dir. of E

Properties

$\langle v \rangle$ - average velocity across all particles over time

$\langle \tau \rangle$ - average time between collisions for free electrons [small timescale: 10^{-30}, e.g.]

- During time between collisions, free electrons act as point particles [are accelerated by electric field]; is "reset" on collision → v does not increase over time
- Drift velocity - velocity obtained by charge carriers during time between collisions

Equation: $\boxed{v_D = a\langle \tau \rangle = \frac{q_e}{m_e} E \langle \tau \rangle}$

Ohm's Law (Derivation)

Plugging v_D into charge density eq: $J = q_e n v_D = \underbrace{\frac{q_e^2 \langle \tau \rangle}{m_e} n}_{\text{material constants}} E$ ← not dependent on geometry!

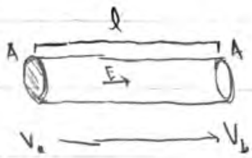
→ define conductivity: $\boxed{\sigma = \frac{q_e^2 \langle \tau \rangle}{m_e} n}$ → $\boxed{J = \sigma E}$

Define resistivity: $\boxed{\rho = \frac{1}{\sigma}}$ → $J = E/\rho$. Recall: $I = JA$

→ $I = \frac{1}{\rho} A E$

unit: Ohm = $\frac{\text{Volt}}{\text{Ampere}}$ $[\Omega]$

Recall:



$E = \frac{V_{ab}}{l}$ ⟶ $I = \frac{1}{\rho} \frac{A}{l} V_{ab}$; Define resistance: $\boxed{R = \rho \frac{l}{A}}$

⟶ Ohm's Law: $\boxed{I = \frac{1}{R} V_{ab} \iff V_{ab} = IR}$

note: Ohm's Law is in direction of current (w.r.t. V_{ab})

Resistance

11/29/23

Lecture 32 (cont.)

Resistance

(analogy: wider pipe → more flow)

<u>Observation</u>: resistance proportional to length, inversely proportional to cross-sectional area

<u>Extrema</u>: (i) <u>High resistance</u> → no meaningful charge can flow [<u>insulator</u>]

(ii) <u>Low resistance</u> → V_{ab} (potential diff.) minimized [<u>conductors</u>]

→ $I = \frac{V_{ab}}{R}$ can go to infinity; current dependent only on power supply

↳ <u>Consequence</u>: [Nearly] constant potential across a conductor

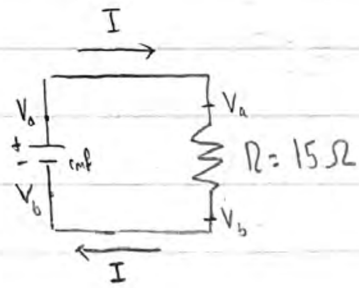
(*) Convention - "Direction of charge flow" is dir. of positive charge flow [high V → low V]

Resistance dependent on <u>material</u>, <u>temperature</u> [<u>lower temperature → lower resistance</u>]

(*) <u>Superconductors</u> - materials where resistance → 0 at sufficiently low temperatures

(*) Ex: With V fixed: lower temperature → lower R → higher I → more light from lightbulb, e.g.

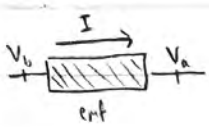
(*) <u>Ohm's Law</u>



Approximate conducting wires as ideal conductors ($V_{ab} = 0$)

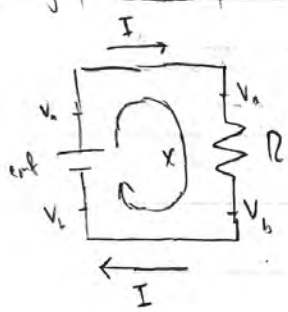
→ Voltage drops occur only at resistors

$V_a > V_b$

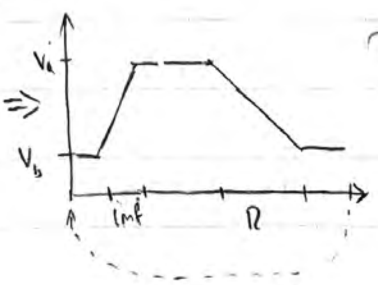


$V_a > V_b$

Can graph <u>electric potential</u> across circuit:



⇒



↳ inside emf: current goes from V_b to V_a [<u>against electric field</u>]

↳ battery <u>does work to move charges against</u> $\vec{E}$ [maintain potential diff.]

Ohm's Law (cont.)

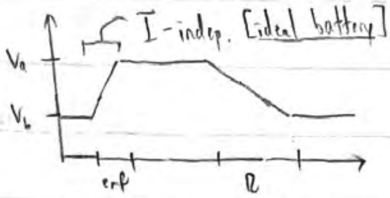
12/1/23

Lecture 33

Ohm's Law Approximations

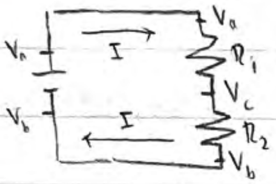
i) Neglect resistance of conducting wire

ii) Assume emf is independent of current I



(*) Intuition: Resistors take energy out of system; emf inserts energy back in

(*) Ex: Resistors in Series



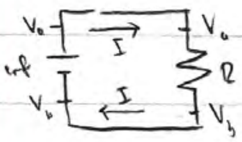
$V_a - V_c = IR_1$

$V_c - V_b = IR_2$

same I → $\boxed{V_a - V_b = I(R_1 + R_2)}$

Non-Ideal Batteries

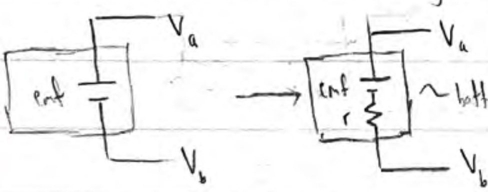
Ideal battery: emf = V_{ab} [Ideal emf]



Ohm's Law: $I = V_{ab}/R$; what if R small?

$R \to 0 \Rightarrow I \to \infty \Rightarrow$ battery shorts out

Solution: Treat batteries as having internal resistors



r = internal resistance

→ $\boxed{V_a - V_b = \text{emf} - Ir}$ Ir has little effect when I low; decrease max V_{ab} when I is high

→ Ohm's Law [Modified]: $V_a - V_b = IR = \text{emf} - Ir$ → $\boxed{I = \frac{\text{emf}}{R + r}}$ ~ Used batteries: r ↑

Current & Energy

12/1/23

Lecture 33

(cont.)

Current & Energy

Observation: Current to a lightbulb [resistor] causes it to heat up, eventually produce light (more current → more light)

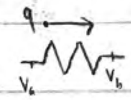
→ Current is a way of transporting energy.

Resistors, motors take energy out of the system (to produce kinetic energy, e.g.)

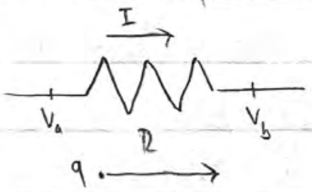
Ex: Resistors convert energy to heat [analogue: friction]

Conversely: batteries, power supplies put energy back in

Think of current as potential energy (from moving charges)



$$\boxed{\Delta U = q(V_b - V_a)}$$



(*) Resistors:

→ $\Delta U = U_b - U_a = q(V_b - V_a)$, $V_a > V_b \Rightarrow \Delta U < 0$

→ $\Delta K = -\Delta U > 0$

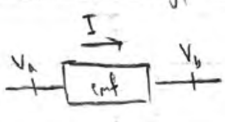
vs Batteries: $\Delta U > 0$, $\Delta K < 0$

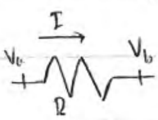
Look at infinitesimal charges: $dK = dq(V_a - V_b)$.

Recall: $dq = I\,dt \rightarrow dK = (V_a - V_b) I\,dt$ → Power: $\boxed{P = \frac{dK}{dt} = (V_a - V_b) I}$

Ohm's Law: $V_{ab} = RI$ → Power (Equivalent formulas): $\boxed{P = RI^2,\ P = \frac{(V_a - V_b)^2}{R}}$

$P > 0 \Rightarrow$ energy leaving; $P < 0 \Rightarrow$ energy being put in





$V_a < V_b \rightarrow P_{battery} < 0$ $\qquad$ $V_a > V_b \rightarrow P_{resistor} > 0$

Power

12/4/13
Lecture 34

(*) Power

In complex circuits: may have $P_{battery} > 0$ [may occur if there are multiple batteries, e.g.]

Interpretation: Battery being charged, or discharged as heat

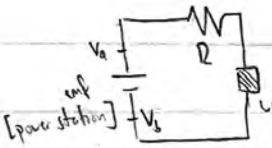
(*) Ex: High-Voltage Transmission Line

Constants: $\ell = 100\,km$; $A = 500\,mm^2$; $V = 765\,kV$; $P = 1000\,MV$

→ Q: How much energy is lost due to Joule heating? [Due to resistance of copper wire]

i) Find R: $\rho_{Cu} = 1.77 \cdot 10^{-8}\,\Omega m \rightarrow R = \rho \frac{\ell}{A}$ [$= 3.54$]

ii) Draw circuit model:



Interpretation: P is power put into circuit by emf (power station);

→ $V = V_a - V_b$ [= emf]

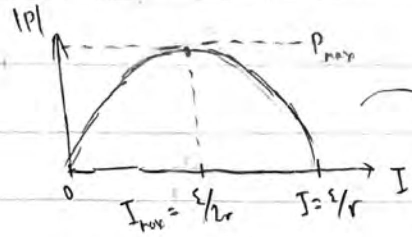
→ $P = VI \rightarrow I = P/V = 1307\,A$

iii) Know $I, R \rightarrow P_{Joule} = RI^2 = 6.04 \cdot 10^6\,W$ [less than 1% of 1000 MW]

(*) Power in a Battery

Ideal battery: $|P| = \text{emf} \cdot I$. Recall (non-ideal batteries): $V = \varepsilon - Ir$

→ "Real" battery: $\boxed{P = \varepsilon I - I^2 r}$ interpretation: $I^2 r$ = power dissipated by internal resistance



Power maximized at $\boxed{I_{max} = \frac{\varepsilon}{2r}}$

→ $\boxed{P_{max} = \frac{1}{4}\frac{\varepsilon^2}{r}}$

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Resistors in Series & Parallel

12/4/23

Lecture 34

+Lecture 35

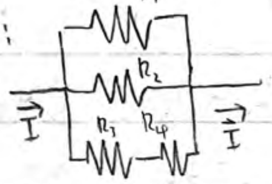
Resistors in Series & Parallel

Observation: Given fixed V, resistors in parallel $\rightarrow I\uparrow$; in series $\rightarrow I\downarrow$

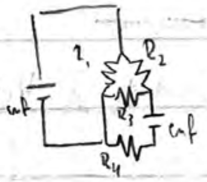
Similar to capacitors - given R_1, R_2, want to find equivalent resistance R_{eq}. Can use to simplify circuits.

(*) Note: R_{eq} reduction only guaranteed to work for simple circuits (1 I in, 1 I out); may fail on more complex circuits

(*) Ex:



[Simple] vs



[Complex]

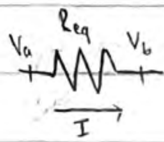
↳ Simplify to one R_{eq} incrementally (similar to capacitors)

Resistors in Series

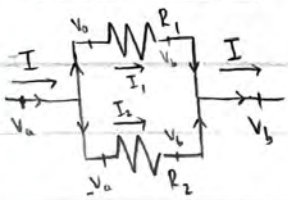
Observe: Current is the same across all resistors (conservation of charge)

$$\left.\begin{aligned} V_a - V_c &= IR_1 \\ V_c - V_b &= IR_2 \end{aligned}\right\} \rightarrow V_a - V_b = I(R_1+R_2) = IR_{eq}$$

$$\rightarrow \boxed{R_{eq} = R_1 + R_2}$$



Resistors in Parallel



Observe: $I = I_1 + I_2$; potential difference is the same across all resistors

$$\left.\begin{aligned} V_a - V_b &= I_1R_1 \\ V_a - V_b &= I_2R_2 \end{aligned}\right\} V_a - V_b = IR_{eq} = I_1R_1 + I_2R_2$$

$$\rightarrow I_1 = \frac{V_a - V_b}{R_1}, \quad I_2 = \frac{V_a - V_b}{R_2}, \quad I = \frac{V_a - V_b}{R_{eq}}$$

$$\rightarrow \boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}}$$ (☆) Alt.: $R_{eq} = \frac{R_1R_2}{R_1+R_2}$

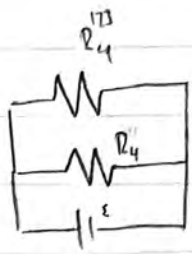
Kirchhoff's Laws

12/5/23
Lecture 35
(cont.)

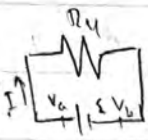
(*) Ex: Simplifying Circuits

Combine R_2, R_3
$\rightarrow R_{eq}^{23}=\frac{R_2R_3}{R_2+R_3}=\frac{20}{3}\Omega$ $\rightarrow$

Combine R_1, R_{eq}^{23}
$\rightarrow R_{eq}^{123}=R_1+R_{eq}^{23}$
$=50/3\ \Omega$ $\rightarrow$



Combine R_{eq}^{123}, R_4
$\rightarrow R_{eq}=7.89\Omega$ $\rightarrow$



$\rightarrow \varepsilon=V_a-V_b=IR_{eq}\rightarrow I=\frac{\varepsilon}{R_{eq}}=0.76A$ [Ohm's Law]

→ What if we want current through R_4? $\rightarrow \varepsilon=I_4R_4\rightarrow I_4=\frac{\varepsilon}{R_4}=0.4A$ [Ohm's Law]

→ What if we want current through R_{eq}^{123}? $\rightarrow I_{123}=I-I_4=0.36A$ [Conservation of charge]

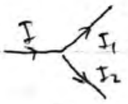
→ What if we want ΔV across R_1? $\rightarrow \Delta V=I_{123}R_1=3.6V$ [Ohm's Law]

Kirchhoff's Laws

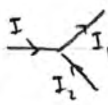
Kirchhoff's circuit laws - rules governing current, potential differences in circuits

1) Junction rule (Charge conservation): Current into a junction is equivalent to current out of it

Ex:



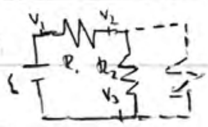
$I-I_1-I_2=0$



$I+I_2-I_1=0$

2) Loop rule: The sum of potential differences around any closed loop is 0.

Ex:



$\rightarrow (V_1-V_2)+(V_2-V_3)+(V_3-V_1)=IR_1+IR_2-\varepsilon=0$

└ (*) Loops also called "loci"

(*) Loop Rule - Derivation

Recall: electric field is conservative $\Rightarrow \oint E\cdot d\vec{s}$ around any closed loop is 0

└ $\int E\cdot d\vec{s}$ is eq. for potential → potential does not change going around a loop [Loop Rule]

Kirchhoff's Laws (cont.)

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Lecture 36

(*) Rules for Solving Circuits

i) Simplify circuit with equivalent resistors (i.e. by "merging" resistors)

ii) Eliminate capacitors [once charged]

iii) Choose currents I_1, I_2, etc. and label their directions (using arrows, e.g.) — directions are arbitrary (math handles it)

iv) Use Junction rule to eliminate as many I's as possible

v) Choose loops with direction around faces for Loop Rule

vi) Use Loop Rule w/ Ohm's Law (resistors), emf (batteries) to relate variables & solve

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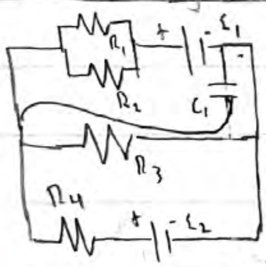
Kirchhoff's Rules (summarized)

i) Junction Rule: The sums of incoming, outgoing currents are equal at all junctions.

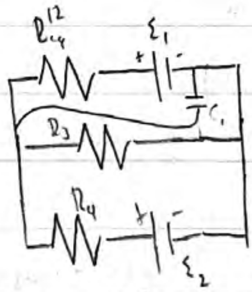
ii) Loop Rule: The sum of potential differences when going around any loop, is 0.

(*) Ex: Solving Circuits



Combine R_1, R_2

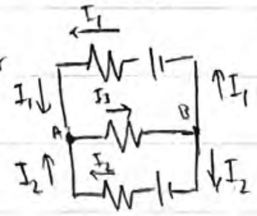
$\rightarrow R_{eq}^{12} = \frac{R_1 R_2}{R_1 + R_2}$



Eliminate C_1

Choose dir. for I_1:

Choose dir. for I_2, I_3 (arbitrary)



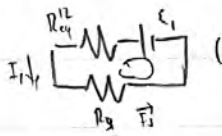
Junction Rule:

A: $I_1 + I_2 - I_3 = 0$

B: $I_3 - I_1 - I_2 = 0$

Set $I_3 = I_1 + I_2$

Choose loops (arbitrary)



Loop 1:

$V_{R_{eq}^{12}} + V_{R_3} + V_{\varepsilon_1} = 0$

$\rightarrow I_1 R_{eq}^{12} + I_3 R_3 - \varepsilon_1 = 0$

$\hookrightarrow I_1 R_{eq}^{12} + (I_1 + I_2) R_3 - \varepsilon_1 = 0$

Kirchhoff's Laws (cont.)

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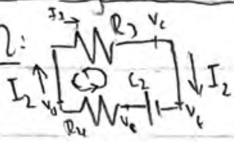
Lecture 36 (cont.)

(*) Ex: Solving Circuits (cont.)

Loop 2:



Loop Rule: $(V_d - V_e) + (V_e - V_f) + (V_f - V_d) = 0$

$\to -I_2R_4 + \varepsilon_2 - I_3R_3 = 0 \to -I_2R_4 + \varepsilon_2 - (I_1+I_2)R_3 = 0$

→ Known Eqs.: $I_1R_{eq} + (I_1+I_2)R_3 - \varepsilon_1 = 0$

$-I_2R_4 - (I_1+I_2)R_3 + \varepsilon_2 = 0$ [Know 5 variables → can solve]

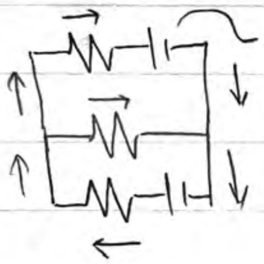
Ex: $\varepsilon_1 = 12V$, $\varepsilon_2 = 24V$, $R_{eq} = 0.5\Omega$, $R_4 = 1\Omega$, $R_3 = 100\Omega$

↳ $0.5I_1 + 100(I_1+I_2) - 12 = 0$

$-I_2 - 100(I_1+I_2) + 24 = 0$

[Solve: $I_1 = \frac{(\varepsilon_1-\varepsilon_2)R_3 + \varepsilon_1R_4}{R_1R_4 + R_3(R_1+R_4)}$, $I_2 = \frac{(\varepsilon_2-\varepsilon_1)R_3 + \varepsilon_2R_1}{R_1R_4 + R_3(R_1+R_4)}$]

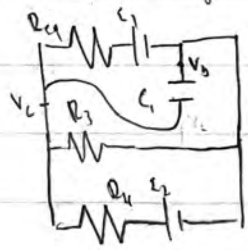
→ $I_1 = -7.89A$, $I_2 = +8.05A$, $I_3 = 0.16A$



Consequence: this emf has energy "put into" it [charged]

[True current directions: reverse I_1's, keep I_2, I_3 same]

Q: What was the charge on the capacitor? Use formula for capacitance $[C = \frac{Q}{V}]$



$Q = (V_b - V_c)C_1 = (I_3R_3)C_1$; $C_1 = 3\cdot10^{-6} \to Q = 4.8\cdot10^{-5}$ C

↳ (*) Once capacitor fully charged - no more current through that specific wire

(*) RC Circuits

Def: An RC circuit is a circuit composed of resistors & capacitors.

(*) E&M Examples

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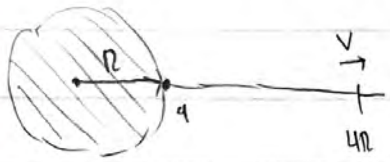
Lecture 37

<u>Problem</u>: Given sphere w/ radius R, constant charge density $\rho_0 > 0$ and point charge $q > 0$ with mass m (initially at R), what is the velocity of q at distance $4R$?



Use conservation of energy. Know $V(r) = \begin{cases} \frac{q_{encl}}{4\pi\varepsilon_0}\frac{1}{r} & r > R \\ \frac{q_{encl}}{4\pi\varepsilon_0}\frac{r^2}{R^3} & r < R \end{cases}$

$q_{encl} = \rho_0 \frac{4}{3}\pi R^3$ $[\rho \cdot V] \rightarrow U_0^{(q)} = qV(R)$; $U_1^{(q)} = qV(4R)$

$U_i + (K_i = 0) = U_f + K_f \rightarrow K_f = \frac{qq_{encl}}{4\pi\varepsilon_0}\frac{3}{4R} \rightarrow v^2 = \frac{2}{m}K_f \rightarrow v = \sqrt{K_f \frac{2}{m}}$

b) If the point charge had charge $-q$ $[-q < 0]$, we observe that it will oscillate inside the sphere; what is the frequency?

Find $E(z)$: Use Gauss's law

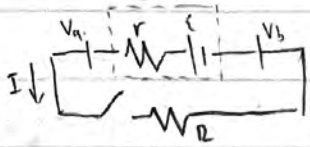
$EA = 4\pi z^2 E(z) = \Phi = \frac{1}{\varepsilon_0}\frac{4}{3}\pi z^3 \rho_0 \rightarrow E(z) = \frac{\rho_0}{3\varepsilon_0} z$

$\rightarrow F = -qE(z) = -\frac{\rho_0 q}{3\varepsilon_0} z$ [linear restoring force $\rightarrow$ SHM]

$ma = \left(\frac{d^2}{dt^2} z\right) m = F = -\frac{\rho_0 q}{3\varepsilon_0} z \rightarrow \frac{d^2}{dt^2} z = -\frac{\rho_0 q}{3\varepsilon_0 m} z$

Recall: $\frac{d^2}{dt^2} z = -\omega^2 z \rightarrow \omega = \sqrt{\frac{\rho_0 q}{3\varepsilon_0 m}} \rightarrow f = \frac{1}{2\pi}\omega$

<u>Problem</u>: Battery with emf ε, internal resistance r



a) With the switch open, we measure $V_a - V_b = 12V$. What is I and what is ε?

Switch open $\rightarrow$ no current flows $\rightarrow I = 0$

Ohm's Law: $V_a - V_b = \varepsilon - Ir = \varepsilon \rightarrow \varepsilon = 12V$

(*) E&M Examples (cont.)

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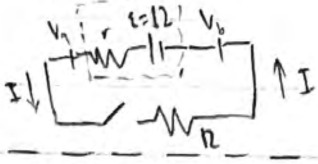
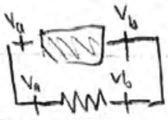
Lecture 37
(cont.)
FINAL

b) Circuit as before; with switch open, measure $V_a - V_b = 9\,V$, $I = 3A$. What are r, R?



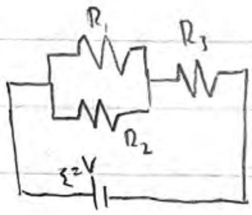
$V_a - V_b = 9 = \varepsilon - Ir = 12 - 3r \rightarrow r = 1\,\Omega$



Know: potential diff. across R is $V_a - V_b$: [assuming ideal conductor]

$\rightarrow I = 3 = \frac{V}{R} = 9/R \rightarrow R = 3\,\Omega$

Problem: Given 3 lightbulbs w/ emf $\varepsilon = V$:

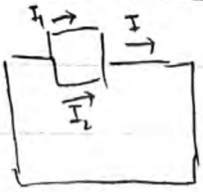


a) What relation do R_1, R_2 satisfy so that they are equally bright? [i.e. same power]

resistors in parallel $\rightarrow P_1 = \frac{(V_a - V_b)^2}{R_1}$; same for $P_2 \rightarrow P_1 = P_2 \Rightarrow R_1 = R_2$

b) What are the relations s.t. R_3 half as, or twice, as bright as R_1, R_2?

Draw currents:



$R_1 = R_2 \Rightarrow I_1 = I_2$, know: $I = I_1 + I_2 = 2I_1$

look at power: $P_3 = R_3 I^2 = 4R_3 I_1^2$

$P_1 = R_1 I_1^2$

Half: $4R_3 I_1^2 = \frac{1}{2}(R_1 I_1^2) \rightarrow R_3 = \frac{1}{8} R_1$

2x: $4R_3 I_1^2 = 2R_1 I_1^2 \longrightarrow R_3 = \frac{1}{2} R_1$