ECE 102: Systems & Signals

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Contents

1 Signals

Continuous vs. discrete signals

Operations on Signals:

1. ***Amplitude scaling***: $x(t) \mapsto ax(t)$
   (i) $0 < |a| < 1 \implies$ "*attenuation*", $|a| > 1 \implies$ "*amplification*"
   (ii) $a < 0 \implies$ "inversion"
2. ***Time scaling***: $x(t) \mapsto x(at)$
   (i) $0 < |a| < 1 \implies$ "*expansion*", $|a| > 1 \implies$ "*compression*"
   (ii) $a < 0 \implies$ "*reversal*"
3. ***Time shifting***: $x(t) \mapsto x(t + t_0)$
   (i) $x(t - t_0) \implies$ "*delayed*"; $x(t + t_0) \implies$ "*advanced*"

($*$) Order of operations: reverse PEMDAS

Classes of Signals:

- ***Even signals*** $x(t) = x(-t)\ \forall\, t$, e.g. $\sin(t)$; ***odd signals*** $x(t) = -x(-t)\ \forall\, t$, e.g. $\cos(t)$
  - Else, antisymmetric; can be written as sum of even & odd $x_{e/o}(t) = \frac{1}{2}(x(t) + / - x(-t))$
- ***Periodic signals***: $\exists\, T_0 > 0$ s.t. $x(t + T_0) = x(t)\ \forall\, t$
  - Sum of two periodic signals with periods T_1, T_2 has period $\operatorname{lcm}(T_1, T_2)$ if both rational; if one rational and one irrational, then not periodic
  - *Periodic extension*: can take a time-limited/aperiodic signal and repeat it
- ***Causal signals***: nonzero only for $t \geq 0$; *anticausal*: nonzero only for $t \leq 0$; *noncausal*: nonzero for some $t < 0$

Sinusoids: $x(t) = A\cos(\omega t - \theta)$

- Variables: angular frequency $\omega = \frac{2\pi}{T}$, amplitude A, phase shift θ
- Trigonometry rules:
  - $\sin\theta = \cos(\theta - \pi/2)$
  - $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
  - $\cos(a + b) = \cos(a)\cos(b) - \sin(a) - \sin(b)$
  - $\sin^2(\theta) = (1 - \cos(2\theta))/2$
  - $\cos^2(\theta) = (1 + \cos(2\theta))/2$

Signal energy & power: A signal is either energy, power, or neither

1. If $0 < E < \infty$, $x(t)$ called an ***energy signal***:

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt$$

2. If $0 < P < \infty$, $x(t)$ called a ***power signal***:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

Complex signals: $z(t) = x(t) + jy(t)$ $[x = \Re(z), y\Im(z)$ real signals$]$

- ***Euler's formula***: $e^{j\phi} = \cos\phi + j\sin\phi$
- Phasor representation: $z = x + jy = r \cdot e^{j\phi}$ $[r = \sqrt{x^2+y^2},\ \phi = \arctan(\frac{y}{x})]$
- Complex relations: complex conjugate $z = x + jy \implies z^* = x - jy$
  - Modulus/magnitude: $|z^2| = zz^*$; inverse: $-j = \frac{1}{j}$
- Important relations:
  - $\cos\theta = \frac{1}{2}[e^{j\theta} + e^{-j\theta}]$; $\quad \sin\theta = \frac{1}{2j}[e^{j\theta} - e^{-j\theta}]$

Signal Models

1. Real sinusoid/cosine: $x(t) = A\cos(\omega t - \theta)$
2. Complex sinusoid: $x(t) = Ae^{j(\omega t+\theta)} = A\cos(\omega t + \theta) + jA\sin(\omega t + \theta)$
   - Real part drawn as solid line; imaginary part as dotted line (shifted $\pi/2$ from real)
3. Exponential signal: $x(t) = e^{\sigma t}$ [$\sigma > / < 0 \implies$ exponential growth/decay]
4. Damped/growing sinusoid: $x(t) = e^{\sigma t}\cos(\omega t + \theta)$
5. Complex exponential: $x(t) = e^{(\sigma + j\omega)t} = e^{\sigma t}e^{j\omega t}$
   - Can plot σ, ω on x & y axes; rep. rates of growth/decay, oscillation
6. ***Unit heavyside/step function***: $u(t) = 1_{t\geq 0} + 0 \cdot 1_{t<0}$
7. ***Unit rectangle*** - two definitions:

$$\text{rect}(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}; \qquad \text{rect}_\Delta(t) = \begin{cases} \frac{1}{\Delta} & |t| \leq \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

   - Has area 1 for all Δ
8. ***Unit ramp/relu***: $r(t) = t \cdot 1_{t\geq 0} + 0 \cdot 1_{t<0} = t \cdot u(t)$
9. ***Unit triangle*** [area 1]:

$$\Delta(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{else} \end{cases}$$

10. ***Dirac function/delta***: $\delta(t) = \infty \cdot 1_{t=0} + 0 \cdot 1_{t\neq 0}$

Dirac Delta

- Has area 1; intuitively, $\delta(t) = \lim_{\Delta\to 0} \text{rect}_\Delta(t)$
- Properties:
  1. ***Impulse sampling property***: $x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$ [area $x(0)$]
     - Shifting: $x(t)\delta(t - T) = x(T)\delta(t - T)$
  2. ***Impulse sifting property***: $\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$
     - Can integrate from $-\infty$ to 0^- [doesn't include delta] or 0^+ [includes delta]
  3. Impulse & unit step: $\int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$; $\frac{du(t)}{dt} = \delta(t)$
  4. Scaling: $\delta(bt) = \frac{1}{b}\delta(t)$
- Graphing: Draw as an arrow at $t = 0$ [or $t = T$], write area if area is $x(0)$

2 Systems

Systems transform input signals $x(t)$ into output signals $y(t)$

1. Scaling system: $x(t) \mapsto y(t) = ax(t)$; draw as triangle with an a
2. Differentiator: $x(t) \mapsto x'(t)$; rectangle with $\frac{d}{dt}$
3. Integrator: $x(t) \mapsto \int_a^t x(\tau)d\tau$ $[a = 0 \text{ or } \infty]$; rectangle with $\int$
4. Squarer: $x(t) \mapsto x(t)^2$; rectangle with $(\cdot)^2$
5. Systems with multiple inputs:
    - Summing system: $x_1, x_2 \mapsto x_1 + x_2$; circle with +
    - Difference: Draw + next to x_1, - next to x_2
    - Multiplier: circle with $\times$

System Properties

- System called ***BIBO stable*** if a bounded input results in a bounded output
    - Bounded signal: $\exists$ M constant s.t. $|x(t)| \leq M_x < \infty$ $\forall$ t
- System called ***causal*** if it only uses values of input signal $x(t)$ for $t \leq 0$
- System called ***time-invariant*** if $[\mathcal{S}(x(t-\alpha))](t) = [\mathcal{S}(x(t))](t-\alpha)$
- System is called ***linear*** if the following hold:
    1. ***Homogeneity/scaling***: For any signal x and scalar a, $S(ax) = aS(x)$
    2. ***Superposition/additivity***: For any signals x_1 and x_2, $S(x_1+x_2) = S(x_1)+S(x_2)$
- System called ***LTI*** if it is both linear and time-invariant
    - Can freely swap order of LTI systems
    - Note: differentiation is an LTI system
- System has ***memory*** if its output depends on past or future values of the input; otherwise, called ***memoryless***
- System called ***invertible*** if $\exists$ $\mathcal{S}^{inv}$ s.t. $x(t) = \mathcal{S}^{inv}(\mathcal{S}(x(t)))$

3 Impulse Response

Can define responses of a system H to several different signals:

1. ***Zero response***: $H(0)$
2. ***Impulse response***: $h(t) = H(\delta(t))$
3. ***Step response***: $H(u(t))$
   - Note: $\frac{d}{dt}H(u(t)) = H(\delta(t))$

Define the ***convolution integral***:

$$\boxed{f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau}$$

$\implies$ for any LTI system H with impulse response $h(t) = H(\delta(t))$, input signal $x(t)$:

$$\boxed{y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau)}$$

<ins>**Properties of Convolution**</ins>:

1. **Commutativity**: $(x * h)(t) = (h * x)(t)$ [via change of vars $\gamma = t - \tau$]
2. **Associativity**: $(f * (g * h))(t) = ((f * g) * h)(t)$ [via swapping order of integration]
3. **Distributivity**: $f * (g + h) = f * g + f * h$
4. **LTI**: Convolution systems are <ins>linear & time-invariant</ins>
   - Cascade/composition: $y = (x * f) * g = x * h$ $[h = f * g]$
   - $h(t) = \frac{d}{dt}s(t)$

Other properties:

(∗) BIBO stability: H is BIBO-stable $\iff$ $h(t)$ is absolutely integrable ($\int_{-\infty}^{\infty} |h(t)|\, dt \in \mathbb{R}$ finite)

(∗) Delay element of convolution: $x(t) * \delta(t) = x(t)$; $x(t) * \delta(t - t_d) = x(t - t_d)$

(∗) Causal systems: $h(t) = 0$ for $t < 0$

(∗) Integrating via convolution: $x(t) * u(t) = \int_{-\infty}^{t} x(t)dt$

4 Fourier Series

Let $f(t)$ cts. periodic signal with fundamental period $T_0 \implies$ can write as:

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

where $\omega_0 = \frac{2\pi}{T_0}$, and c_k are Fourier coefficients of $f(t)$ given by:

$$\boxed{c_k = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) e^{-jk\omega_0 t} dt}$$

- For non-continuous signals, equality for $f(t)$ fails at discontinuities
- c_0 is the time-averaged mean of signal

A signal $x(t)$ is an ***eigenfunction*** of system $\mathcal{S}$ if $\mathcal{S}(x(t)) = ax(t)$ for some constant eigenval. $a \in \mathbb{C}$.

- Complex exponentials $e^{j\omega t}$ are eigenfunctions of LTI systems: $a = re^{j\theta} \implies ax(t) = re^{j(\omega t + \theta)}$
- Disprove: via example, or looking at $y(t)$

Amplitude & phase: can plot A/ϕ vs. ω, corresponding to signal's Fourier series at that ω

- Signal perfectly determined by amplitude, pphase spectrums

Note: $\text{sinc}(t) = \frac{\sin(\pi t)}{t}$

Fourier Symmetries:

- $f(t)$ real $\implies c_k^* = c_{-k}: \quad \Re(c_k) = \Re(c_{-k}),\ \Im(c_k) = -\Im(c_{-k})$
- $|c_k| = |c_{-k}|$
- $\angle c_k = -\angle c_{-k} \left[\angle c_k = \arctan(\frac{\Im(c_k)}{\Re(c_k)})\right]$
- Also:
  - $x(t)$ even & real $\implies c_k$ are real
  - $x(t)$ odd & real $\implies c_k$ are imaginary

Parseval's theorem: Let $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, then power $P = \sum_{k=-\infty}^{\infty} |c_k^2|$

5 Fourier Transforms

Fourier Transform & Inverse FT of a signal $f(t)$ $[t \to \omega \Leftrightarrow \omega \to t]$:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad \Longleftrightarrow \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega}$$

Existence of the FT: Sufficient (but not necessary) condition:

$$\boxed{\int_{-\infty}^{\infty} |f(t)|\, dt < \infty}$$

Fourier Properties:

1. **LTI**: The Fourier transform system $\mathcal{F}[\cdot]$ is LTI
2. **Complex conjugate**: $f^*(t) \Longleftrightarrow F^*(-j\omega)$
3. **Time scaling**: $\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(j\frac{\omega}{a}\right)$
4. **Time shifting**: $\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}\mathcal{F}[f(t)]$
5. **Duality**: $\mathcal{F}[F(t)] = 2\pi f(-j\omega)$
6. **Convolution theorem**: For two arbitrary signals $f_1(t), f_2(t)$:

$$\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$$

7. **Parseval's Theorem**: $\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega|^2\, d\omega$
8. **Derivative**: $\mathcal{F}[f'(t)] = j\omega F(j\omega)$
   - Dual: $(-jt)f(t) = F'(j\omega)$
9. **Modulation**: $\mathcal{F}[f(t)e^{j\omega_0 t}] = F(j(\omega-\omega_0))$
   - $\mathcal{F}[f(t)\cos(\omega_0 t)] = \frac{1}{2}(F(j(\omega-\omega_0)) + F(j(\omega+\omega_0)))$
   - $\mathcal{F}[f(t)\sin(\omega_0 t)] = \frac{1}{2j}(F(j(\omega-\omega_0)) - F(j(\omega+\omega_0)))$

FT Symmetries/Properties:

- For any $f(t)$ real/imag/complex: $f(t)$ even $\implies$ $F(j\omega)$ even [same for odd]
  - Even $\implies$ mag., phase even;
- Real signals have Hermitian FT (Hermitian symmetry): $f(t)$ real $\implies$ $F(-j\omega) = F^*(j\omega)$
  - $|X(j\omega)|$ even; $\angle X(j\omega) = -\angle X(-j\omega)$

- Imaginary signals have anti-Hermitian FT: $f(t)$ imaginary $\implies F(-j\omega) = -F^*(j\omega)$
- Complex signals have neither Hermitian nor anti-Hermitian FT

6 Frequency Response

Frequency response: Fourier transform of impulse response $\underline{h(t) \Longleftrightarrow H(j\omega)}$

$$\boxed{y(t) = h(t) * x(t) \Longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)}$$

- $H(j\omega)$ also called ***transfer function***: characterizes how input changed at every frequency

Filters

1. **Low-pass filters**: $= 0 \ \forall \ \omega \notin [-\omega_c, \omega_c]$ [=1]
   Ideal low-pass: [not causal, infinitely long $h(t)$]
   $$H(j\omega) = \text{rect}\left(\omega/(2\omega_c)\right) \Longleftrightarrow h(t) = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right)$$
2. **High-pass filters**: $= 0 \ \forall \ \omega \in [-\omega_c, \omega_c]$ [=1]
   Ideal high-pass:
   $$H(j\omega) = 1 - \text{rect}\left(\omega/(2\omega_c)\right) \Longleftrightarrow h(t) = \delta(t) - \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right)$$
3. **Band-pass filters**: $= 0 \ \forall \ \omega \notin [\pm\omega_0 - \omega_c, \pm\omega_0 + \omega_c]$ [=1]
   Ideal band-pass:
   $$H(j\omega) = \text{rect}\left((\omega+\omega_0)/(2\omega_c)\right) + \text{rect}\left((\omega-\omega_0)/(2\omega_c)\right) \Longleftrightarrow h(t) = 2\cos(\omega_0 t) \cdot \left[\frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right)\right]$$

Causal filters: can truncate & shift [disortionless]

Distortionless (LTI) system: Output is only a shifted/scaled version of the input $y(t) = Kx(t - t_d)$

- *Frequency response*: $Y(j\omega) = Ke^{-j\omega t_d} \cdot X(j\omega) \implies H(j\omega) = Ke^{-j\omega t_d}$
  (Namely: $|H(j\omega)| = K$, $\angle H(j\omega) = -\omega t_d$)

Group delay - indicates how much a signal will be delayed (constant for LTI)

$$t_d(\omega) = -\frac{d}{d\omega}\angle H(j\omega)$$

7 Sampling

Impulse train (& FT) $[\omega_0 = 2\pi/T]$:

$$\boxed{\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \quad \Longleftrightarrow \quad F(j\omega) = \omega_0 \sum_{k=-\infty}^{\infty} \delta\left(x - k\omega_0\right)}$$

Impulse train sampling:

$$\boxed{f(t)\delta_T(t) = \sum_{k=-\infty}^{\infty} f(kT) \cdot \delta(t - kT) \quad \Longleftrightarrow \quad \tilde{F}(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))}$$

Nyquist Sampling Theorem

Find $B = \frac{1}{2\pi} |\omega_{\max}|$ [maximum frequency in Hz] $\implies$ to perfectly recover a signal, require:

$$\boxed{\omega_0 \geq 4\pi B \; [= 2\omega_{\max}] \Longleftrightarrow \frac{1}{T} = f_{sample} \geq 2B \quad [\textit{Nyquist rate}]}$$

- $\omega_{\max}$: largest $|\omega|$ with $F(\pm j\omega) \neq 0$
- *Nyquist interval*: $T = 1/(2B)$
- To recover signal: find $\tilde{F}(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))$ & use LPF to find central copy
- $\omega_0 < 4\pi B \implies$ **aliasing**: modulated copies start to overlap
  - Taking LPF will reconstruct a different, lower-frequency signal
  - *Anti-aliasing filter*: Use a LPF on signal beore sampling to reduce aliasing (causes distortion)

Interpolation

1. Zero-order hold: At each t, take the value of the last-measured signal/sample
2. Linear interpolation: Connect sampled points with a line
3. Perfect interpolation: Returns the function exactly

   Under Nyquist rate $\implies$ use **Whittaker-Shannon interpolation formula** [IFT of $\tilde{F}(j\omega)H_{\text{LPF}}$]:

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} f(kT) \operatorname{sinc}(2Bt - k)}$$

8 The Laplace Transform

Unilateral Laplace Transform for a causal signal $f(t)u(t)$:

$$\boxed{F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt}$$

- **Region of convergence/ROC**: Range of values (σ, ω) $[s = \sigma + j\omega]$ for which $F(s)$ converges

Laplace Transform Properties

1. The Laplace transform $\mathcal{L}[\cdot]$ is **linear**
2. **Time scaling** $[a > 0]$: $\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$
3. **Time shift** $[T > 0]$: $\mathcal{L}[f(t-T)] = e^{-sT}F(s)$
4. **Frequency shift**: $\mathcal{L}[f(t)e^{s_0 t}] = F(s - s_0)$
5. **Convolution Theorem**: $\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$
6. **Integration**: $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$
7. **Derivative**: $\mathcal{L}[f'(t)] = sF(s) - f(0)$
8. **Multiplication by** t: $\mathcal{L}[tf(t)] = -F'(s)$

Fourier & Laplace transforms: For cases where ROC includes $s = j\omega$ axis:

$$\boxed{F(j\omega) = F(s)|_{s=j\omega}}$$

Laplace transform & Diff. Eqs.: Differentiating a signal multiplies LT by s; integrating multiplies by $1/s$

- Differential equations in time domain $\Leftrightarrow$ algebraic equations in Laplace domain

Partial Fraction Expansion

Can find poles $\lambda_1, \ldots, \lambda_n$ of $F(s)$ & residues $r_1, \ldots, r_n$ s.t.:

$$\boxed{F(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a} \implies F(s) = \frac{r_1}{s - \lambda_1} + \ldots + \frac{r_n}{s - \lambda_n}}$$

- Inverse Laplace transform [for $t \geq 0$]:

$$\boxed{f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{r_1}{s - \lambda_1} + \ldots + \frac{r_n}{s - \lambda_n}\right] = r_1 e^{-\lambda_1 t} + \ldots + r_n e^{-\lambda_n t}}$$

- **Repeated roots**:

$$(s-\lambda^k) \text{ a root} \implies \text{Include a term } \frac{r_{1,i}}{(s-\lambda)^i} \text{ for each } i = 1, \ldots, k$$

$$\longrightarrow \quad \mathcal{L}^{-1}\left[\frac{r}{(s-\lambda)^k}\right] = \frac{r}{(k-1)!}t^{k-1}e^{\lambda t}$$

Finding Partial Fraction Expansions

(i) **Cover-up procedure** [primary method]:

$$\boxed{r_k = (s-\lambda_k)F(s)\Big|_{s=\lambda_k}}$$

- **Repeated roots**: $r_{1,k}$ via cover-up (multiplying by $(s-\lambda)^k$); for $r_{1,k-j}$ $[j \neq 0]$:

$$\boxed{r_{1,k-j} = \frac{1}{j!}\frac{d^j}{ds^j}(F(s)(s-\lambda)^k)\Big|_{s=\lambda}}$$

(ii) **L'Hopital's Rule**: Can use that

$$\boxed{F(s) = \frac{b(s)}{a(s)} \implies r_k = \frac{b(\lambda_k)}{a'(\lambda_k}}$$

(iii) **Quadratic Factors**: Take partial fraction expansion directly

$$\boxed{F(s) = \frac{r_1 s + r_2}{as^2+bs+c} \implies \text{ILT via } e^{-at}\cos(\omega t) \text{ Laplace pair}}$$

(∗) **Nonproper Rational Functions** [Degree m of numerator $\geq$ degree n of denominator]

Split into polynomial + proper rational function:

$$\boxed{F(s) = \underbrace{\frac{b(s)}{a(s)}}_{\text{nonproper}} \longrightarrow F(s) = \underbrace{c(s)}_{\text{polynom.}} + \underbrace{\frac{d(s)}{a(s)}}_{\text{proper}}}$$

- $d(s)/a(s)$ proper $\rightarrow$ use partial fraction expansion
- Obtain $c(s)$ via polynomial long division [stop when order of subtracted < order of denominator]

ILT of $c(s)$: $\underline{c(s) = c_0 + c_1 s + \ldots + c_{m-n}s^{m-n} \iff c_0\delta(t) + c_1\delta^{(1)}(t) + \ldots + c_{m-n}\delta^{(m-n)}(t)}$