Math 115B: Linear Algebra

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Textbook: S. Friedberg, A. Insel, & L. Spence - Linear Algebra (5^{th} ed.)

Topics: Linear transformations & duality; invariant subspaces & the Cayley-Hamilton Theorem; unitary & orthogonal operators; projections & the Spectral Theorem; the Jordan canonical form

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Vector Spaces (Review)

4/1/24

Lecture 1

Vector Spaces (Review)

generally: $\mathbb{R}$ [real nos.], $\mathbb{C}$ [complex nos.]

Def: A vector space V over field F is a set with associated operations of addition & scalar multiplication such that [$\forall x, y \in V;\ a, b \in F$]:

(i) $x + y = y + x$

(ii) $(x+y)+z = x+(y+z)$

(iii) $\exists\, 0_V \in V$ s.t. $x + 0_V = x$

(iv) $\forall x \in V$, $\exists -x \in V$ s.t. $x + (-x) = 0_V$

(v) $1 \cdot x = x$ [$1 \in F$]

(vi) $a(bx) = (ab)x$

(vii) $a(x+y) = ax + ay$

(viii) $(a+b)x = ax + bx$

(*) Notable vector spaces: (i) All fields (e.g. $\mathbb{R}$) over themselves

(ii) $\mathbb{R}^n$: vectors [real-number tuples] of length n

(iii) $P_n(\mathbb{R})$: polynomials w/ degree $\leq n$, real coefficients

(*) $P(\mathbb{R})$: polynomials w/ real coefficients

(iv) $M_{n \times n}(\mathbb{R})$: $n \times n$ real matrices

Def (Subspace): Given V a vector space [v.s.] over field F, a subset $W \subseteq V$ is called a subspace of V if W is itself a vector space over F.

Thm: Let V a v.s. over F, then a subset $W \subseteq V$ is a subspace of V iff W satisfies:

(i) $0_V \in W$

(ii) $x, y \in W \Rightarrow x + y \in W$ [W closed under addition]

(iii) $c \in F$, $x \in W \Rightarrow c \cdot x \in W$ [W closed under scalar multiplication]

→ Corollary: W is a subspace of V iff W satisfies: (i) $0_V \in W$

(ii) $ax + y \in W$ $\forall x, y \in W;\ a \in F$

Linear Independence (Review)

4/1/24

Lecture 1 (cont.)

Thm: Let V a vector space over F with $W_1, \dots, W_n$ subspaces of V; then $\bigcap_{i=1}^{n} W_i$ [intersection; not union] is a subspace of V. $[\bigcap_{i=1}^{n} W_i \subseteq V]$

Linear Combinations (Review)

Def: Let V a v.s. over F, $S \subseteq V$ nonempty:

→ (i) A vector $v \in V$ is called a linear combination of vectors in S if v can be written as $v = a_1 v_1 + \dots + a_n v_n$ for some $v_1, \dots, v_n \in S$; $a_1, \dots, a_n \in F$

(ii) The span of S is defined as the set of all linear combinations of vectors in S:

$$\boxed{\operatorname{span}(S) := \{a_1 v_1 + \dots + a_n v_n : a_1, \dots, a_n \in F;\ v_1, \dots, v_n \in S\}}$$

Span: - Let $S \subseteq V$ nonempty $\Rightarrow$ $\operatorname{span}(S)$ is a subspace of V

- Any subspace of V containing S ($S \subseteq W$) contains $\operatorname{span}(S)$; in particular, $\operatorname{span}(S)$ is the smallest subspace of V that contains S

Def: Let V a v.s., $S \subseteq V$; we say that S generates V if $\operatorname{span}(S) = V$.

[To prove, show $\operatorname{span}(S) \subseteq V$ and $\operatorname{span}(S) \supseteq V$]

Def: Linear Independence

A subset $S \subseteq V$ is called linearly dependent if $\exists$ distinct vectors $v_1, \dots, v_n \in S$ and scalars $a_1, \dots, a_n \in F$ such that:

(i) $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \vec{0}_V$, and

(ii) at least one a_i is nonzero

We say S is linearly independent if S is not linearly dependent.

Bases & Dimension (Review)

4/3/24

Lecture 2 **Prop**: A subset $S \subseteq V$ is linearly independent iff "$v_1, \dots, v_n \in V;\ a_1v_1 + \dots + a_nv_n = 0_V$" $\Rightarrow$ "$a_1 = a_2 = \dots = a_n = 0$".

Bases & Dimension (Review)

Def: Let V a v.s.; a basis for V is a linearly independent subset of V that spans/generates V.

(*) Notable bases: (i) Standard basis for $\mathbb{R}^n$: $\beta = \{e_1, \dots, e_n\}$, where $e_i = \langle 0, \dots, 0, \overset{i^{th}\text{ pos.}}{1}, 0, \dots, 0 \rangle$

(ii) $\beta = \{1, x, \dots, x^n\}$ for $P_n(\mathbb{R})$

Thm: Let V a vector space and $u_1, u_2, \dots, u_n \in V$; then $\{u_1, \dots, u_n\}$ is a basis for V iff every $v \in V$ can be uniquely written as a linear combination of $\{u_1, \dots, u_n\}$.

Prop. Let V be a vector space with a finite basis of size n; then every basis for V will have size n.

$\rightarrow$ **Def**: A vector space is called finite-dimensional if it has a finite basis; we call the size of this basis, the dimension of the vector space.

Dimension: - A vector space is called infinite-dimensional if it is not finite-dimensional [$\nexists$ a finite basis]; denote this as $\dim V = \infty$ (vs finite-dimensional: $\dim V < \infty$)

- We define $\dim(\{0\}) = 0$ [dimension 0]

(*) Notation: "$\dim V = n$" $\Rightarrow$ "V is finite-dimensional with dimension n"

Prop: Let V a vector space w/ $\dim(V) = n$. Then:

(i) Any [finite] generating set for V has at least n elements

(ii) Any generating set for V of size n [exactly n] is necessarily a basis for V

(iii) Any linearly independent subset of V with size n is necessarily a basis for V

(iv) Every linearly independent subset of V can be extended to a basis for V

Linear Transformations (Review)

4/3/24

Lecture 2 (cont.)

Linear Transformations (Review)

Def: Let V, W be vector spaces over a field F; then a linear transformation from $\overset{\text{domain}}{V}$ to $\overset{\text{codomain}}{W}$ is a map $T: V \to W$ that is linear, i.e. that satisfies:

(i) $T(x+y) = T(x) + T(y)$ $\quad [\forall\, x, y \in V;\ c \in F]$

(ii) $T(c \cdot x) = c \cdot T(x)$

Linear transformations: (i) T linear $\Rightarrow$ (a) $T(0_v^{(v)}) = 0_v^{(w)}$ $\quad [0_v^{(v)} \in V,\ 0_v^{(w)} \in W]$

(b) $T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i T(v_i)$ [via induction]

(ii) For proofs: T is linear iff $T(cx+y) = cT(x) + T(y)$ $\forall\, x, y \in V;\ c \in F$

(*) Notation: T may also be called a linear operator/linear map

Def: Let $T: V \to W$ be linear; then we say:

(i) T is injective/one-to-one if $T(u) = T(v) \Rightarrow u = v$ $\forall\, u, v \in V$

(ii) T is surjective/onto if $\forall\, w \in W$, $\exists\, v \in V$ s.t. $T(v) = w$

(iii) T is bijective/invertible if T is both injective and surjective

Def: Let $T: V \to W$ be linear; then we define the (i) null space and (ii) range space of T:

(i) Null space null(T): $\boxed{N(T) = \{v \in V : T(v) = 0_v\}}$

(ii) Range/image im(T): $\boxed{R(T) = \{w \in W : w = T(v) \text{ for some } v \in V\}}$

Prop: (i) $N(T) \subseteq V$ is a subspace of V

(ii) $\text{im}(T) \subseteq W$ is a subspace of V

Linear Transformations (Review)

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Lecture 2

(cont.)

Theorem (Rank-Nullity): Let $\dim V = n$ and $T: V \to W$ linear; then:

$$\Longrightarrow \quad \boxed{\dim[\mathrm{im}(T)] + \dim[\mathrm{null}(T)] = \dim(V)} \; [= n]$$

(*) Notation/Def: (i) Rank: $\mathrm{rank}(T) = \dim(\mathrm{im}(T))$

(ii) Nullity: $\mathrm{nullity}(T) = \dim(\mathrm{null}(T))$

Matrix Representations of Linear Maps

Def: Let $\dim V < \infty$. An ordered basis for V is a basis equipped with an order (e.g. $e_1, e_2, \ldots$).

Recall: Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis for V; then any vector $\vec{v} \in V$ can be expressed uniquely as $\vec{v} = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$ for some scalars $a_1, \ldots, a_n \in F$

$\rightsquigarrow$ Def: Define the coordinates of $\vec{v}$ relative to β as:

$$\boxed{[\vec{v}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}} \qquad \left([\vec{v}]_\beta \in \mathbb{R}^n\right)$$

Def: Let $T: V \to W$ linear and $\beta = \{v_1, v_2, \ldots, v_n\}$, $\gamma = \{w_1, w_2, \ldots, w_n\}$ ordered bases for V and W, respectively. Then the matrix representation of T is defined as:

$$\boxed{[T]_\beta^\gamma = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}} \in M_{m \times n}$$

(*) Notation: If T is $V \to V$ (and β an ordered basis for V), may write $[T]_\beta = [T]_\beta^\beta$.

(*) Infinite-Dimensional Vector Spaces

4/4/24

Discussion I

(*) Linear Independence of Infinite Sets

Def: Let V a vector space, $S \subseteq V$ with $\dim S = \infty$. We say that S is linearly independent if every finite subset $A \subseteq S$ is linearly independent; otherwise, say that S is linearly dependent.

(*) Note: In general, on infinite-dimensional vector spaces, certain theorems (e.g. rank-nullity) may no longer apply

(*) Fields: A field can either have:

(i) $|F| = \infty$, or (ii) $|F| < \infty$ [finite field] $\Rightarrow$ $|F| = p^k$ for some p prime, $k \in \mathbb{Z}^+$

(*) Infinite-Dimensional Vector Spaces

Ex: (i) $P(\mathbb{R})$: vector space of all polynomials with real coefficients

(ii) $\mathbb{R}^\infty = \{(a_i)_{i=0}^{\infty} : a_i \in \mathbb{R}\}$: vector space of all [infinite] real sequences

note: all polynomials $p(x) \in P(\mathbb{R})$ are finite-dimensional [have "final" nonzero coefficient]

$\rightarrow \dim(\mathbb{R}^\infty) > \dim(P(\mathbb{R}))$

Infinite dimensions: (i) The existence of an injective (but not surjective) f'n $f: A \to B$ does not imply $\dim(A) < \dim(B)$; ex: $f: \mathbb{R}^\infty \to \mathbb{R}^\infty$, $(1\ 2\ 3 \ldots) \mapsto (0\ 1\ 0\ 2 \ldots)$

(ii) $|[0,1]| > |\mathbb{Q}|$

For any set S & field F, the set $L(S,F)$ of linear functions $S \to F$ is a vector space

Invertibility & Isomorphism (Review)

4/5/24

Lecture 3

Theorem Let V, W fin. dim. w/ ordered bases β, γ and let $T_1, T_2 : V \to W$ linear; then:

(i) $T_1 = T_2 \iff [T_1]_\beta^\gamma = [T_2]_\beta^\gamma$

(ii) $[T_1 + T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$

(iii) $[cT_1]_\beta^\gamma = c[T_1]_\beta^\gamma \quad \forall\, c \in F$

Composition of Linear Transformations (Review)

Thm: Given vector spaces V, W, Z with ordered bases α, β, and γ, respectively; and $T_1 : V \to W$, $T_2 : W \to Z$ linear, then their composition $T_2 \circ T_1 : V \to Z$ is linear and:

$$\boxed{[T_2 \circ T_1]_\alpha^\gamma = [T_2]_\beta^\gamma [T_1]_\alpha^\beta}$$

Thm: Let V, W fin. dim. v.s.'s with ordered bases β and γ, and $T : V \to W$ linear; then:

$$\boxed{[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta} \quad \forall\, v \in V$$

Invertibility & Isomorphisms (Review)

Def: Let V, W v.s.'s and $T : V \to W$ linear; then a function $U : W \to V$ is called the inverse of T if $T \circ U = id_W$ and $U \circ T = id_V$ [identity function $V \to V$ and $W \to W$].

If T has an inverse, say T is invertible.

Facts: (i) If T is invertible, then its inverse is unique [denoted T^{-1}].

(ii) T is invertible $\iff$ T is bijective $\iff$ T surjective and injective

(*) $T : V \to W$ linear $\to$ T surjective iff $im(T) = W$; T injective iff $ker(T) = \{0\}$

(*) $T : V \to W$, $\dim V = \dim W \to$ T injective $\iff$ T surjective

Dual Spaces

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Lecture 3

+ Lecture 4

Facts (cont.): If T_1, T_2 composable, linear, and invertible, then:

(a) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$ exists [$T_2 \circ T_1$ invertible]

(b) (T_1^{-1}) is invertible with $(T_1^{-1})^{-1} = T_1$

Thm: Let V, W fin. dim. v.s.'s with ordered bases β, γ and $T: V \to W$ linear; then T is invertible $\iff [T]_\beta^\gamma$ invertible; furthermore, $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$.

Def: Let V, W be vector spaces; we say that V is isomorphic to W if $\exists$ an isomorphism [invertible linear transformation] from V to W.

Dual Spaces

Def: Let V a v.s. over F. A linear functional on V is a linear transformation from V to F.

(*) Ex: $A \mapsto \text{tr} A$ is a linear functional $M_{n \times n}(F) \to F$

Def: Let V a v.s. w/ ordered basis $\beta = \{v_1, \ldots, v_n\}$. The i^{th} coordinate function w.r.t. β is the function $f_i(v) = a_i$, where $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$ $([v]_\beta = \langle a_1, a_2, \ldots, a_n \rangle)$.

(*) Notation: Also denoted as $f_i(v_j) = \delta_{ij}$ [$=1$ if $i=j$; 0 otherwise]

Def: Given a vector space V over field F, define the dual space of V as $V^* = L(V, F)$ [the set of linear transformations $V \to F$].

Note: Let V a v.s. over F; then V^* is itself a vector space over F.

Dual Spaces (cont.)

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Lecture 4 – Lecture 5

Thm: Let V a v.s. with ordered basis $\beta = \{v_1, \dots, v_n\}$. We define the dual basis of β to be the set $\beta^* = \{f_1, \dots, f_n\}$, where f_i is the i^{th} coordinate function w.r.t. β.

Prop: Let V a v.s. with ordered basis $\beta = \{v_1, \dots, v_n\}$; then the dual basis β^* is a basis for V^*. In particular, for any $f \in V^*$:

$$\boxed{f = \sum_{i=1}^{n} f(v_i) f_i} \quad \forall\, f: V \to F \in V^*$$

(*) **Proof**: Let $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ $[v \in V]$.

Then
$$\left[\sum_{i=1}^{n} f(v_i) f_i\right](v) = \sum_{i=1}^{n} \left[f(v_i) f_i\right](v)$$
$$= \sum_{i=1}^{n} f(v_i) \cdot f_i(v)$$
$$= \sum_{i=1}^{n} f(v_i) \cdot a_i$$
$$= \sum_{i=1}^{n} a_i f(v_i) = f\left(\sum_{i=1}^{n} a_i v_i\right) = f(v).$$ ▨

Def: Let V, W f.d.v.s.'s with ordered bases β and γ, respectively. Let $T: V \to W$ linear; we define the transpose of T to be the function $T^t: W^* \to V^*$ defined by:

$$\boxed{T^t(g) = g \circ T} \quad \forall\, g \in W^*$$

Theorem: Let V, W, β, γ as above; let $T: V \to W$ linear. Then T^t is a linear transformation from V to W; in particular:

$$\boxed{[T^t]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t}$$

Dual Spaces (cont.)

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Lecture 5

(cont.)

(*) Proof: Need to check 3 claims:

(i) $T^t(g) \in V^*$ [is linear] $\forall g \in W^*$: $(T^t(g))(0) = g \circ T(0) = g(0) = 0$

$(T^t(g))(cx+y) = g(T(cx+y)) = g(cT(x)+T(y)) = cg(T(x)) + g(T(y)) = c(T^t(g))(x) + (T^t(g))(y)$.

(ii) T^t is linear: $(T^t(0))(v) = 0$ [trivial]

$(T^t(cg+h))(v) = (cg+h)(T(v)) = cg(T(v)) + h(T(v)) = (cT^t(g) + T^t(h))(v)$

(iii) $[T^t]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$: Let $\beta = \{x_1, \dots, x_n\}$ & $\gamma = \{y_1, \dots, y_m\}$; $\beta^* = \{f_1, \dots, f_n\}$, $\gamma^* = \{g_1, \dots, g_m\}$.

Let $A = [T]_\beta^\gamma$, i.e.:

$$A = \big([T(x_1)]_\gamma \; \dots \; [T(x_n)]_\gamma \big) = (A_{ij})$$

Looking at $[T^t]_{\gamma^*}^{\beta^*} = B$:

$$[T^t]_{\gamma^*}^{\beta^*} = \big([T^t(g_1)]_{\beta^*} \; \dots \; [T^t(g_m)]_{\beta^*} \big) = B$$

Recall that $T^t(g_j) = g_j \circ T = \sum_{s=1}^{n} (g_j \circ T)(x_s) \cdot f_s$

$$\to B_{ij} = (g_j \circ T)(x_i) = g_j\Big(\overbrace{\sum_{k=1}^{m} A_{ki} y_k}^{T(x_i) = Ax_i}\Big) = \sum_{k=1}^{m} A_{ki} \overbrace{g_j(y_k)}^{=1 \text{ if } j=k,\ 0 \text{ otherwise}} = A_{ji} \quad \blacksquare$$

Rec: The vector spaces $L(V,W)$ [$\dim V = n$, $\dim W = m$] and $M_{m \times n}(F)$ are isomorphic through the isomorphism $\Phi_\beta^\gamma : L(V,W) \to M_{m \times n}(F)$ defined by $T \mapsto [T]_\beta^\gamma$.

→ Cor: The linear transformation T^t is the unique linear map with $[T^t]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$.

(*) $\dim(L(V,W)) = \dim(M_{m \times n}(F)) = m \cdot n$.

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Discussion 2

(*) Dual Spaces

Let V a fin.-dim. v.s. over F w/ basis $\beta = \{v_1, \dots, v_n\}$

→ $V^* = L(V, F) := \{\text{linear transformations } V \to F\}$ is a vector space; called dual space of V

with dual basis $\{f_i : v_j \mapsto \delta_{ij}\}$ [→ $f \in V^* \Rightarrow f = \sum_{i=1}^{n} f(v_i) f_i$]

Can take double dual V^{**} of V, dual space of V^*: $V^{**} = L(V^*, F)$

Define $e_j(f_i) = f_i(v_j) \rightsquigarrow$ form basis for V^{**}: $e(f \in V^*) = \left(\sum_{i=1}^{n} c_i e_i\right)(f) = f\left(\sum_{i=1}^{n} c_i v_i\right)$

└ evaluating f_i at v_j

Observe: $e = c_1 e_1 + \dots + c_n e_n \Rightarrow e(f) = f(c_1 v_1 + \dots + c_n v_n)$, evaluates f at $v = c_1 v_1 + \dots + c_n v_n$

In particular can find an isomorphism $V \to V^{**}$, $v \mapsto e_v$ [$e_v : V^* \to F$, $f \mapsto f(v)$]

+ isomorphism is basis-agnostic: does not depend on choice of basis (or having a basis)

└ alt.: is a canonical isomorphism

→ Consequence: Taking infinite duals of V gives just two vector spaces: V [odd duals], V^* [even duals]

(*) Ex: V^{***} isomorphic to V^*

Let V, W vector spaces; then any $T : V \to W$ linear induces transpose $T^* : W^* \to V^*$ defined by $T^*(f \in W^*) = f \circ T$

- For fixed bases β, γ for V, W + duals β^*, γ^* for V^*, W^*: $[T^*]_{\gamma^*}^{\beta^*} = \left([T]_{\beta}^{\gamma}\right)^T$

Dual Spaces (cont)

4/12/24

Lecture 6

Isomorphisms

Thm: Let V, W v.s.'s over field F; then V isomorphic to W $\iff$ $\dim(V) = \dim(W)$.

Dual Spaces (cont)

Let $\dim V < \infty$; may define $V^{**} := (V^*)^*$.

→ Observe: $\dim(V) = \dim(V^*) = \dim(V^{**}) \Rightarrow V, V^{**}$ are isomorphic

Def: For $x \in V$, define $\hat{x} : V^* \to F$ by $\hat{x}(f) = f(x)$ $\forall f \in V^*$. $[\hat{x} \in V^{**}]$

(*) Proof $[\hat{x} \in V^{**};\ \hat{x}$ linear$]$: $\hat{x}(0) = 0$, trivial

$\hat{x}(cf+g) = (cf+g)(x) = cf(x) + g(x) = c\hat{x}(f) + \hat{x}(g)$. ∎

Thm: Let $\dim V < \infty$; then the function $\psi : x \mapsto \hat{x}$ is an isomorphism $V \to V^{**}$.

$$\begin{tikzcd} x \in V \arrow[d, shift right, "\psi"'] \\ \hat{x} \in V^{**} \arrow[u, shift right, "\psi^{-1}"'] \end{tikzcd}$$

(*) Proof

Lemma: If $\hat{x}(f) = 0$ $\forall f \in V^*$, then $x = 0_V$.

(*) Proof: $x = a_1v_1 + \dots + a_nv_n$; $\hat{x}(f_i) = 0$ $\forall i \Rightarrow a_1 = a_2 = \dots = a_n = 0$.

(i) ψ is linear: $x, y \in V$; $c \in F \to (\psi(cx+y))(f) = \widehat{(cx+y)}(f) = f(cx+y) = cf(x) + f(y) = (c\psi(x) + \psi(y))(f)$.

(ii) ψ is bijective: Know $N(\psi) = \{0\} \iff \psi$ injective.

By lemma: since "$\hat{x} = 0$" $\Rightarrow x = 0_V$, then $N(\psi) = \{0_V\} \Rightarrow \psi$ injective.

Since $\dim V = \dim V^{**}$ & ψ linear, then ψ injective $\iff \psi$ bijective. ∎

Eigenvalues & Eigenvectors

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Lecture 6

Eigenvalues & Eigenvectors

Def. Let T a linear operator on V [i.e. $T: V \to V$]; then a vector $v \in V$, $v \neq 0_V$ is called an eigenvector of T if $T(v) = \lambda v$ for some scalar $\lambda \in F$ (called an eigenvalue of T).

Thm: Let $\dim V < \infty$, T a linear operator on V. Then for any two bases β, γ for V, have that $\det([T]_\beta) = \det([T]_\gamma)$, i.e. determinant does not depend on basis. Can denote this unique value $\det(T)$.

Prop: Let T a linear operator on V; then:

(i) T is bijective $\iff$ $\det(T) \neq 0$

(ii) If T is bijective, then $\det(T^{-1}) = \frac{1}{\det(T)}$.

(iii) Let U another linear operator on V; then $\det(T \circ U) = \det(T) \cdot \det(U)$.

Thm: Let $A \in M_{n \times n}(F)$; then a scalar $\lambda \in F$ is an eigenvalue of A iff:

$$\boxed{\det(A - \lambda I_n) = 0}$$

→ Def: Define the characteristic polynomial of A as the polynomial $f(t) = \det(A - tI_n)$.

→ (*) Note: (i) The characteristic polynomial of $A \in M_{n \times n}(F)$ has degree n & leading coeff. $(-1)^n$

(ii) A matrix $A \in M_{n \times n}(F)$ has at most n distinct eigenvalues.

Invariant Subspaces

4/15/24

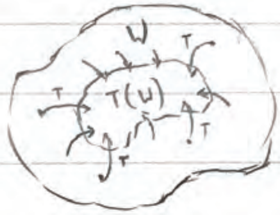
Lecture 7

Invariant Subspaces

<u>Def</u>: Let T a linear operator on v.s. V. We call a subspace $W \subseteq V$ <u>T-invariant</u> if <u>$T(W) \subseteq W$</u>, i.e. <u>$T(w) \in W \ \forall \ w \in W$</u>.

<u>Def</u>: Let T a lin. op. on V, $W \subseteq V$ T-invariant. We define the <u>restriction of T to W</u> as the function <u>$T_W : W \to W$</u> defined by <u>$T_W(w) = T(w)$</u> $\forall \ w \in W$.

└ linear operator



(*) <u>Notable T-invariant subspaces</u>: $\{0_V\}$, V, $\mathrm{im}(T)$, $\mathrm{null}(T)$

<u>Thm</u>: Let T a linear operator on V ($\dim V < \infty$), and let $W \subseteq V$ T-invariant. Then the characteristic polynomial of T_W divides the characteristic polynomial of T.

(*) <u>Proof</u>: Let $\dim W = k \le \dim V = n$. Let $\gamma = \{v_1, \dots, v_k\}$ be an ordered basis for W, and let $\beta = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ be its extension to a basis for V. Let $A = [T]_\beta$, $B = [T_W]_\gamma$; have that:

$$A = \left([T(v_1)]_\beta \ \dots \ [T(v_n)]_\beta \right); \quad B = \left([T_W(v_1)]_\gamma \ \dots \ [T(v_k)]_\gamma \right)$$

Observe that since W is T-invariant, then $T(v_i)$ $[1 \le i \le k]$ only maps to $v_1, \dots, v_k$; namely:

$$[T_W(v_i)]_\gamma = \langle a_1, \dots, a_k \rangle \Rightarrow [T(v_i)]_\beta = \langle a_1, \dots, a_k, 0, \dots, 0 \rangle = \langle [T_W(v_i)]_\gamma, 0, \dots, 0 \rangle$$

Thus:

$$A = \left([T(v_1)]_\beta \dots [T(v_k)]_\beta \dots [T(v_n)]_\beta \right) = \left(\begin{array}{c|c} B & C \\ \hline 0 & D \end{array} \right)$$

$$\to \det(A - tI_n) = \det \left(\begin{array}{c|c} B - tI_k & C \\ \hline 0 & D - tI_{n-k} \end{array} \right) = \underbrace{\det(B - tI_k)}_{\text{char. poly. of } T_W} \det(D - tI_{n-k}). \quad \blacksquare$$

The Cayley-Hamilton Theorem

4/17/24

Lecture 8

The Cayley-Hamilton Theorem

Def: Let T be a linear operator on V, and let $v \in V$, $v \neq 0_V$. Then we call the subspace $W = \text{span}\{v, T(v), T^2(v), \ldots\}$, the T-cyclic subspace generated by v.

From last time: saw that the characteristic polynomial of T_W provides info about T; want to find W with easy-to-find char. poly. [Observe: W T-cyclic $\Rightarrow$ W T-invariant]

Thm. Let T linear operator on V, and let W be the T-cyclic subspace generated by $v \in V$ [$v \neq 0_V$]; let $\dim(W) = k$. Then:

(i) The set $\{v, T(v), \ldots, T^{k-1}(v)\}$ is a basis for W

(ii) Let $a_1, \ldots, a_{k-1} \in F$ s.t. $a_0 v + a_1 T(v) + \ldots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$; then the characteristic polynomial of T_W is:

$$\boxed{f(t) = (-1)^k (a_0 + a_1 t + \ldots + a_{k-1} t^{k-1} + t^k)}$$

→ Corollary (Cayley-Hamilton): Let T be a linear operator on V [$\dim V < \infty$], and let $f(t)$ be the characteristic polynomial of T. Then $f(T)$ is the zero transformation. [$(f(T))(v) = 0 \ \forall v \in V$]

Alternatively, let $A \in M_{n \times n}(\mathbb{R})$ with characteristic polynomial $f(t)$; then:

$$\boxed{f(A) = 0_{n \times n}}$$ [$n \times n$ matrix of all zeroes]

(*) Proof: Let $v \in V$ nonzero; want to show that $(f(T))(v) = 0$. Let W be the T-cyclic subspace generated by v, with char. poly $g(t)$; by the (above) theorem, have that $(g(T))(v) = (-1)^k (a_0 + \ldots + T^{k-1})(v) = 0$. Know that W T-invariant $\Rightarrow$ $\underbrace{g(t) \text{ divides } f(t)}_{\text{see Lec. 7}}$; then $(f(T))(v) = ((g \cdot h)(T))(v) = 0$. ∎

for some polynomial h

Inner Products

4/19/24

Lecture 9

Def: Let V v.s. over field $\mathbb{C}$. A complex inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies $\forall\, x, y, z \in V$ and $c \in \mathbb{C}$:

(i) Linearity: $\langle cx + z, y \rangle = c\langle x, y \rangle + \langle z, y \rangle$

(ii) Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(iii) Positive-definite: If $x \neq 0_V$, then $\langle \cdot, \cdot \rangle > 0$ and is real.

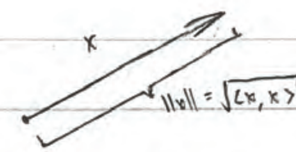
Properties: (i) $\langle 0_V, x \rangle = \langle x, 0_V \rangle = 0$

(ii) $\langle x, x \rangle$ is real and non-negative

(iii) $\langle x, x \rangle = 0$ iff $x = 0_V$

(iv) $\langle x, ay + bz \rangle = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle$ [conjugate linearity]

(v) $\langle x + y, x + y \rangle = \langle x, x \rangle + 2\,\mathrm{Re}\langle x, y \rangle + \langle y, y \rangle$



(*) Ex: Define the standard inner product on $\mathbb{C}^n$ as $\boxed{\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i}$, $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

Def: Let V v.s. over field $\mathbb{R}$. A real inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies $\forall\, x, y, z \in V$ and $c \in \mathbb{R}$:

(i) Linearity: $\langle cx + z, y \rangle = c\langle x, y \rangle + \langle z, y \rangle$

(ii) Symmetry: $\langle x, y \rangle = \langle y, x \rangle$

(iii) Positive-definite: If $x \neq 0_V$, then $\langle x, x \rangle$ is positive [and real].

Properties: Same as above, but applied to real numbers: (iv) $\bar{a}, \bar{b}$ [$\mathbb{C}$] $\to a, b$ [$\mathbb{R}$]

(v) $\mathrm{Re}\langle x, y \rangle$ [$\mathbb{C}$] $\to \langle x, y \rangle$ [$\mathbb{R}$]

(*) Ex: Define the standard inner product on $\mathbb{R}^n$ as $\boxed{\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i}$, $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

Inner Product Spaces

4/19/24
Lecture 9
(cont.)

(*) **Note**: Define an inner product space as a vector space equipped with an inner product

Inner Product Spaces

Def: Let $A \in M_{n\times n}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$. We define the conjugate transpose of A to be the matrix $A^* \in M_{n\times n}(F)$ such that $(A^*)_{ij} = \overline{A}_{ji}$.

Def: Let V a real or complex inner product space. For $x \in V$, define the norm of X by:

$$\boxed{\|x\| = \sqrt{\langle x, x\rangle}}$$

Prop. Let V real or complex IPS; if $\langle x, y\rangle = \langle x, z\rangle \ \forall x \in V$, then $y = z$.

Theorem: Let V be an IPS over F, where $F = \mathbb{R}$ or $\mathbb{C}$. Then $\forall x, y \in V$ and $c \in F$:

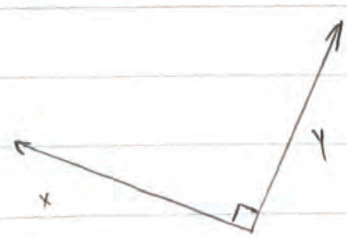
(i) $\|cx\| = |c| \cdot \|x\|$

(ii) $\|x\| \geq 0 \ \forall x \in V$. In particular, $\|x\| = 0$ iff $x = 0_V$.

(iii) Cauchy-Schwarz inequality: $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

(iv) Triangle inequality: $\|x+y\| \leq \|x\| + \|y\|$

Def: Let V real/complex IPS. Say that two vectors $x, y \in V$ are orthogonal if $\langle x, y\rangle = 0$.
A subset $S \subseteq V$ is called orthogonal if any 2 distinct vectors in S are orthogonal, and called orthonormal if it is orthogonal and consists only of vectors with norm 1.



[Orthogonal vectors]

Orthonormal Bases

4/19/24
Lecture 9
+ Lecture 10

Orthonormal Bases

Def: Let V an IPS. A subset $\beta \subseteq V$ is an orthonormal basis for V if it is an ordered basis for V and orthonormal; i.e. for any $v, w \in \beta$:

$$\boxed{\langle v, w \rangle = \begin{cases} 1 & v = w \\ 0 & v \neq w \end{cases}}$$

Theorem: Let V be a nonzero finite-dimensional inner product space; then V admits an orthonormal basis β. Furthermore, if $\beta = \{v_1, \dots, v_n\}$ and $x \in V$, then:

$$\boxed{x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i}$$

Adjoints of Linear Operators

Def: Let V fin-dim. IPS, and let T be a linear operator on V. Then we define the adjoint of T to be the [unique] function $T^*: V \to V$ that satisfies $\forall x, y \in V$:

$$\boxed{\langle T(x), y \rangle = \langle x, T^*(y) \rangle}$$

Prop. The adjoint T^* always exists for any lin. op. T. Furthermore, T^* is linear.

(*) Gram-Schmidt (Review)

Theorem: Let $S = \{v_1, v_2, \dots, v_k\}$ orthonormal set in a vector space V ($\dim V = n$). Then $k \leq n$ and S can be extended to an orthonormal basis for V. In particular, if $k = n$, then S is an orthonormal basis for V.

Adjoints of Linear Operators

4/24/24
Lecture 10
(cont.)

Adjoints of Linear Operators

Theorem: Let V IPS and let T, U linear operators on V. Then:

(i) T+U has adjoint $(T+U)^* = T^* + U^*$

(ii) For any $c \in F$, cT has adjoint $(cT)^* = \bar{c}T^*$.

(iii) $T \circ U$ has adjoint $(T \circ U)^* = U^* \circ T^*$.

(iv) T^* has adjoint $T^{**} = T$.

(v) The identity mapping has adjoint $I^* = I$.

Theorem: Let V be a fin.-dim. IPS, and let β be an orthonormal basis for V. Let T be a linear operator on V; then:

$$\boxed{[T^*]_\beta = [T]_\beta^*}$$ [Conjugate transpose]

Self-Adjoint Operators

Def: Let T linear operator on IPS V. We say that T is self-adjoint if $T = T^*$. Similarly, say that an $n \times n$ [real or complex] matrix A is self-adjoint if $A = A^*$.

Unitary & Orthogonal Operators

Def: Let T linear operator on fin.-dim. IPS V (over field $F = \mathbb{R}$ or $\mathbb{C}$). If $\|T(x)\| = \|x\| \ \forall x \in V$, we say that:

(a) T is a unitary operator if $F = \mathbb{C}$

(b) T is an orthogonal operator if $F = \mathbb{R}$

(*) Ex: Rotation [rotation matrices], reflection are unitary operators

Unitary & Orthogonal Operators

4/24/24
Lecture 10
+Lecture 11

Lemma: Let U a self-adjoint operator on an IPS V, and suppose that $\langle x, U(x)\rangle = 0 \ \forall x \in V$. Then U is the zero operator.

(*) Proof: For any $x \in V$, have:

$$\begin{aligned} 0 &= \langle x + U(x), U(x + U(x))\rangle \\ &= \langle x + U(x), U(x) + U^2(x)\rangle \\ &= 0 + \langle x, U^2(x)\rangle + \langle U(x), U(x)\rangle + 0 \\ &= \langle U^*(x), U(x)\rangle + \langle U(x), U(x)\rangle = 2\|U(x)\| = 0 \ \forall x \in V. \ \blacksquare \end{aligned}$$

Theorem: Let T linear operator on fin-dim. IPS V over $\mathbb{R}$ or $\mathbb{C}$; then TFAE:

(i) $T \circ T^* = I$

(ii) $T^* \circ T = I$

(iii) $\langle T(x), T(y)\rangle = \langle x, y\rangle \ \forall x, y \in V$ [T preserves the inner product]

(iv) If β is an orthonormal basis for V, then $T(\beta)$ is an orthonormal basis for V

(v) $\exists$ orthonormal basis β for V s.t. $T(\beta)$ an orthonormal basis for V

(vi) $\|T(x)\| = \|x\| \ \forall x \in V$, i.e. T unitary/orthogonal [T preserves the norm]

(*) Proof: (i)$\Rightarrow$(ii) / (ii)$\Rightarrow$(i): Follows from the fact that $[T]_\beta [T^*]_\beta = I \Rightarrow [T^*]_\beta = [T]_\beta^{-1}$.

(ii)$\Rightarrow$(iii): $T^* \circ T = I \Rightarrow \langle x, y\rangle = \langle T^* \circ T(x), y\rangle = \langle T(x), T(y)\rangle$

(iii)$\Rightarrow$(iv): $T(\beta)$ orthonormal per (iii) $[\langle x,y\rangle = 0 \Rightarrow \langle T(x), T(y)\rangle = 0]$, has $\dim V$-many elements

(iv)$\Rightarrow$(v): Know $\exists$ orthonormal basis β for V; then $T(\beta)$ also an orthonormal basis for V

(v)$\Rightarrow$(vi): Let $\beta = \{v_1, \dots, v_n\}$, $x = a_1 v_1 + \dots + a_n v_n \in V$; then:

$$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle \sum_{i=1}^n a_i T(v_i), \sum_{i=1}^n a_i T(v_i)\rangle = \sum_{i=1}^n a_i \bar{a}_i = \sum_{i=1}^n |a_i|^2 = \langle \sum_{i=1}^n a_i v_i, \sum_{i=1}^n a_i v_i\rangle = \|x\|^2.$$

(vi)$\Rightarrow$(ii): $\langle x, x\rangle = \langle T(x), T(x)\rangle = \langle x, T^* \circ T(x)\rangle \Rightarrow \langle x, (I - T^* \circ T)(x)\rangle = 0$

$U := I - T^* \circ T$ self-adjoint $[\langle (I - T^* \circ T)(x), y\rangle = \langle x, (I - T^* \circ T)(y)\rangle] \Rightarrow I - T^* \circ T = 0$ transformation. $\blacksquare$

Unitary & Orthogonal Matrices

4/26/24

Lecture 11 + Lecture 12

Def: A square matrix $A \in M_{n\times n}(F)$ is called: (i) orthogonal if $A^tA = AA^t = I_n$.
(ii) unitary if $A^*A = AA^* = I_n$.

Prop. The following are equivalent:
(i) $A \in M_{n\times n}(\mathbb{R})$ is an orthogonal matrix
(ii) The rows of A form an orthonormal basis for $\mathbb{R}^n$
(iii) The columns of A form an orthonormal basis for $\mathbb{R}^n$

(*) Can write similar for $A \in M_{n\times n}(\mathbb{C})$ unitary, rows/cols orthonormal basis for $\mathbb{C}^n$

(*) **Proof:** (i) => (ii): Let A orthogonal:

$$\to \delta_{ij} = (AA^t)_{ij} = \sum_{k=1}^{n} A_{ik}(A^t)_{kj} = \underbrace{\sum_{k=1}^{n} A_{ik}A_{jk}}_{= \langle i^{th}\text{ row}, j^{th}\text{ row of } A\rangle = \delta_{ij}} \Rightarrow \text{rows of } A \text{ are orthonormal}$$

A has $n = \dim(\mathbb{R}^n)$ rows, therefore rows of A comprise an orthonormal basis.

(ii) => (i): Via opposite calc. as above, rows of A orthogonal => $(AA^t)_{ij} = \delta_{ij}$ => $AA^t = I_n$.
(*) (i) => (iii), (iii) => (i) via similar arguments [using A^tA]. ∎

Def: A real or complex matrix $A \in M_{n\times n}(F)$ is called normal if A commutes with A^*, i.e.:

$$\boxed{A^*A = AA^*}$$

(*) **Note:** If A real symmetric $[A = A^t]$ or Hermitian $[A = A^*]$, then A is normal

(*) **Theorem:** A matrix $A \in M_{n\times n}(F)$ is normal iff $\exists$ an orthonormal basis for F^n consisting of eigenvectors of A.

Unitary Equivalence

4/29/24

Lecture 12

Def: Let $A \in M_{n\times n}(F)$ real or complex. Then A is called <u>orthogonally [unitarily] equivalent</u> to a matrix $D \in M_{n\times n}(F)$ if $\exists$ an orthogonal [unitary] matrix P such that:

$$\boxed{D = P^* A P}$$

Theorem: (i) Let $A \in M_{n\times n}(\mathbb{C})$; then <u>$A$ is normal</u> iff <u>A is unitarily equivalent to a diagonal matrix</u>.

(ii) Let $A \in M_{n\times n}(\mathbb{R})$; then <u>$A$ is symmetric</u> iff <u>A is orthogonally equivalent to a real diagonal matrix</u>.

(*) <u>Proof [just (i)]:</u>

($\Rightarrow$) Let A normal $[A^*A = AA^*]$. Per prev. theorem, A normal $\Rightarrow \exists$ orthonormal basis for $\mathbb{C}^n$ comprised of eigenvectors of A. Let $\beta = \{v_1, \dots, v_n\}$ orthonormal basis, where $v_1, \dots, v_n$ are eigenvectors w/ eigenvalues $\lambda_1, \dots, \lambda_n$

$$\Rightarrow A \text{ unitarily equivalent to } \boxed{D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}} \text{ w/ } \boxed{P = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}$$

($\Leftarrow$) Let $A = P^* D P$ [D diagonal, P unitary]; then:

$$A^*A = (P^*DP)^*(P^*DP) = P^*D^*\underbrace{PP^*}_{P \text{ unitary} \Rightarrow PP^* = I}DP = P^*D^*DP = P^*\overbrace{DD^*}^{D \text{ diagonal} \Rightarrow D = D^*}P = AA^*. \quad \blacksquare$$

(*) <u>Note:</u> To orthogonally/unitarily diagonalize a matrix:

(i) Find the eigenvalues & corresponding eigenvectors [forming a basis for V]

(ii) Orthonormalize the set of eigenvectors:

(a) Within each eigenspace, orthogonalize its basis of eigenvectors (via Gram-Schmidt, e.g.)

(b) Normalize all eigenvectors, i.e. divide each eigenvector by its norm

(iii) Set D, P as given in above proof

Rigid Motions

5/3/24

Lecture 13

Recall: Let $V = \mathbb{R}^2$ and $\theta \in \mathbb{R}$. Have the following matrices:

(i) Rotation by θ about the origin: [counterclockwise] $\boxed{[T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$ [Rotation matrix] $\det(T) = 1$

(ii) Reflection about a line θ away from the positive x-axis: $\boxed{[T]_\beta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}}$ [Reflection matrix] $\det(T) = -1$

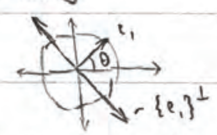
Theorem: Let T an orthogonal operator on $\mathbb{R}^2$ and $A = [T]_\beta$, w/ β standard basis for $\mathbb{R}^2$. Then exactly one of the following is true:

(i) Either T is a rotation and $\det(A) = 1$, or

(ii) T is a reflection [about a line through the origin] and $\det(A) = -1$.

(*) **Proof**: Since T orthogonal, $\exists$ orthonormal basis $\{T(e_1), T(e_2)\}$ for $\mathbb{R}^2$ [by past thm.]

Since $\|T(e_1)\| = 1$, then $T(e_1) = (\cos\theta, \sin\theta)$ for some $\theta \in [0, 2\pi)$.



Since $T(e_1) \perp T(e_2)$, have 2 possibilities for $T(e_2)$:

(*) $T(e_2) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2))$

(i) $T(e_2) = (-\sin\theta, \cos\theta) \Rightarrow T$ a rotation, or

(ii) $T(e_2) = (\sin\theta, -\cos\theta) \Rightarrow T$ a reflection. ⊠

Def: Let V real IPS. A function $f: V \to V$ is called a rigid motion if, $\forall\, x, y \in V$:

$$\boxed{\|f(x) - f(y)\| = \|x - y\|}$$

(*) **Note**: Intuitively, the norm of a difference of two vectors represents the distance between them; then a rigid motion is a function that preserves distances between elements

(*) **Note**: Any orthogonal operator on V is a rigid motion.

Rigid Motions (cont.)

5/3/24

Lecture 13 (cont.)

Theorem: Let $f: V \to V$ be a rigid motion on a fin. dim. real IPS V. Then $\exists$ unique orthogonal operator T on V and unique translation g on V s.t. $f = g \circ T$.

Recall: A translation on V is a function $g: x \mapsto x + v_0$ for some $v_0 \in V$ fixed. Any translation is a rigid motion.

(*) Proof (Theorem: Existence)

Let $T: V \to V$ defined by $T(x) = f(x) - f(0)$ $\forall x \in V$, and let $g: V \to V$ defined by $g(x) = x + f(0)$. Clearly, $f = g \circ T$ and g is a translation; remains to show that T is orthogonal. [Note: no assumption of T linear.]

For any $x \in V$, have: $\|T(x)\| = \|f(x) - f(0)\| = \|x\|$, f a rigid motion

$$\Rightarrow \|T(x) - T(y)\|^2 = \langle T(x) - T(y), T(x) - T(y) \rangle$$
$$= \langle T(x), T(x) \rangle - 2\langle T(x), T(y) \rangle + \langle T(y), T(y) \rangle$$
$$= \|T(x)\|^2 - 2\langle T(x), T(y) \rangle + \|T(y)\|^2.$$

Know: (i) $\|x-y\|^2 = \langle x-y, x-y \rangle = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$, (ii) $\|x-y\|^2 = \|T(x-y)\|^2 = \|T(x) - T(y)\|^2$

$\Rightarrow \langle x, y \rangle = \langle T(x), T(y) \rangle$ [T preserves inner product]

Remains to show T linear: $\|T(x+cy) - T(x) - cT(y)\|^2 = \|\{T(x+cy) - T(x)\} - cT(y)\|^2$

$\to = \|T(cx+y) - T(x)\|^2 - 2c\langle T(x+cy) - T(x), T(y) \rangle + c^2\|T(y)\|^2$

$=$

Direct Sums

5/6/24
Lecture 14

Def: A vector space V is called the direct sum of $W_1, W_2, \dots, W_k$ [denoted $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$] if $W_1, W_2, \dots, W_k$ are subspaces of V s.t.:

(i) $W_1 + W_2 + \dots + W_k = V$

(ii) $W_1, W_2, \dots, W_k$ are disjoint, i.e. $W_i \cap \left[\sum_{j \neq i} W_j\right] = \{0_V\}$ $\forall$ $1 \leq i \leq k$.

(*) Ex: Let $W_1 = \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and $W_2 = \text{span}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\}$; then $V = \mathbb{R}^2 = W_1 \oplus W_2$

Prop. Let V a v.s. and $W_1, W_2, \dots, W_k$ subspaces of V. Then $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ iff for every $v \in V$, $\exists$ unique $w_i \in W_i$ for $1 \leq i \leq k$ s.t. $v = w_1 + w_2 + \dots + w_k$.

(*) Corollary: If V fin. dim. and $V = W_1 \oplus \dots \oplus W_k$, then $\dim(V) = \sum_{i=1}^{k} \dim(W_i)$.

(*) Proof (Prop): Proof for $k=2$:

($\Rightarrow$) Let $V = W_1 \oplus W_2$, then $V = W_1 + W_2$; hence, for any $v \in V$, $v = w_1 + w_2$ for some $w_1 \in W_1$, $w_2 \in W_2$.

For uniqueness: suppose $v = \tilde{w}_1 + \tilde{w}_2$ for some $\tilde{w}_1 \in W_1$, $\tilde{w}_2 \in W_2$.

Then $w_1 + w_2 = \tilde{w}_1 + \tilde{w}_2 \Rightarrow w_1 - \tilde{w}_1 = \tilde{w}_2 - w_2$.

$w_1, \tilde{w}_1 \in W_1 \Rightarrow w_1 - \tilde{w}_1 \in W_1$, and similar for $\tilde{w}_2 - w_2 \in W_2$; then have that $w_1 - \tilde{w}_1 = \tilde{w}_2 - w_2 \in W_1 \cap W_2$.

By defn. of direct sum, $W_1 \cap W_2 = \{0_V\}$; then $w_1 - \tilde{w}_1 = \tilde{w}_2 - w_2 = 0_V$
$\Rightarrow w_1 = \tilde{w}_1$, $w_2 = \tilde{w}_2$.

($\Leftarrow$) Obviously, $V \subseteq W_1 + W_2$. Since $W_1, W_2 \subseteq V$, then $W_1 + W_2 \subseteq V \Rightarrow W_1 + W_2 = V$.

Assume, for contradiction, that $W_1 \cap W_2 \supsetneq \{0\}$. Then $\exists$ $w \neq 0_V$ s.t. $w \in W_1$, $w \in W_2$.

Then: $0_V^{\in W_1} + 0_V^{\in W_2} = w^{\in W_1} + (-w)^{\in W_2} = 0_V \in V$, violating the uniqueness condition. $\blacksquare$

Projections & Orthogonal Complement

5/6/24
Lecture 14
+Lecture 15

Def: Let V a v.s., and W_1, W_2 subspaces of V s.t. $V = W_1 \oplus W_2$. Then the function $T: V \to V$ defined by $T(v) = w_1$, where $w_1 \in W_1$ is the [unique] vector s.t. $v = w_1 + w_2$ for some $w_2 \in W_2$, is called the projection of V on W_1 along W_2.

(*) **Note:** The projection of V on W_1 along W_2 is linear.

Def: Let V an IPS, and let $S \subseteq V$ be a nonempty subset of V. Then we define the orthogonal complement of S to be the set $S^\perp$ defined by:

$$\boxed{S^\perp = \{x \in V : \langle x, y \rangle = 0 \ \forall\, y \in S\}}$$

Prop: Properties of the Orthogonal Complement

(i) If S is a subset of V, then its orthogonal complement $S^\perp$ is a subspace of V

(ii) $\{0_V\}^\perp = V$; $V^\perp = \{0_V\}$

(iii) If W is a subspace of V, then $W \cap W^\perp = \{0_V\}$

(iv) If W is a fin. dim. subspace of V, then $(W^\perp)^\perp = W$.

(*) **Proof:** (i) Trivially, $0_V \in S^\perp$. Let $u, v \in S^\perp$ and $c \in F$; then $\forall\, y \in V$:

$\langle cu + v, y \rangle = \langle cu, y \rangle + \langle v, y \rangle = c\langle u, y \rangle + \langle v, y \rangle = 0 + 0 = 0 \Rightarrow cu + v \in S^\perp$.

(ii) Clearly, $\langle 0_V, y \rangle = 0 \ \forall\, y \in V$. $[\{0_V\}^\perp = V]$ From prev. lectures: $\langle x, y \rangle = 0 \ \forall\, y \in V$ iff $x = 0_V$. $[V^\perp = \{0_V\}]$

(iii) Clearly, $0_V \in W \cap W^\perp$. Let $x \in W \cap W^\perp$; then $\|x\|^2 = \langle x, x \rangle = 0 \Rightarrow x = 0_V \Rightarrow W \cap W^\perp = \{0_V\}$.

(iv) Let $w \in W$: then $\langle w, y \rangle = 0 \ \forall\, y \in W^\perp \Rightarrow w \in (W^\perp)^\perp \Rightarrow W \subseteq (W^\perp)^\perp$.

Orthogonal Projections

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Lecture 15
(cont.)

Prop. Let V an IPS, W a finite-dimensional subspace of V. Then we have that:

$$\boxed{V = W \oplus W^\perp}$$

(*) Proof: Want to show $V = W + W^\perp$. Let $\{v_1, .., v_k\}$ orthonormal basis for W, and let $v \in V$.

Then have: $v = \underbrace{\langle v, v_1\rangle v_1 + ... + \langle v, v_k\rangle v_k}_{:= w_1 \in W} + \underbrace{v - \langle v, v_1\rangle v_1 - ... - \langle v, v_k\rangle v_k}_{:= w_2 \in V}$,

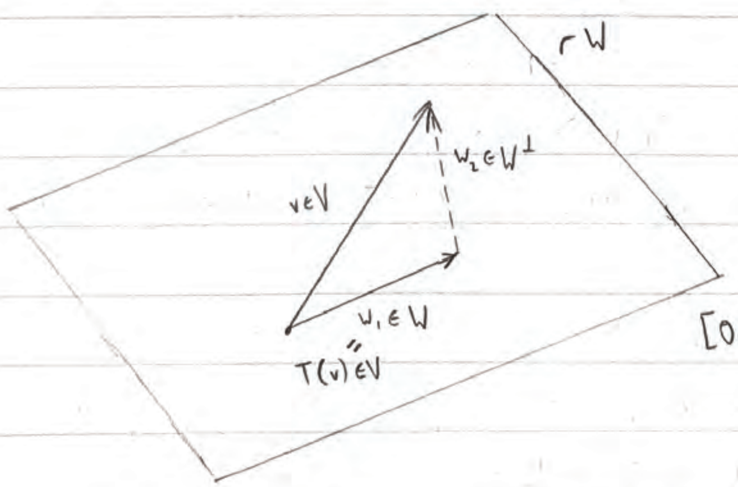
Then for each v_i, $1 \leq i \leq k$: $\langle w_2, v_i\rangle = \langle v - \sum_{j=1}^{k} \langle v, v_j\rangle v_j, v_i\rangle = \langle v, v_i\rangle - \sum_{j=1}^{k} \langle v, v_j\rangle \langle v_j, v_i\rangle$

$\rightarrow = \langle v, v_i\rangle - \langle v, v_i\rangle = 0 \Rightarrow w_2 \in W^\perp \Rightarrow V = W + W^\perp$. Since $W \cap W^\perp = \{0_V\}$, then $V = W \oplus W^\perp$. ∎

Def: Let V an IPS, W fin-dim subspace of V. Then we define the <u>orthogonal projection of V onto W [along $W^\perp$]</u> to be the function $T: V \to V$ defined by $T(v) = w_1$, where $w_1 \in W$ is the unique vector $\in W$ s.t. $v = w_1 + w_2$ for some $w_2 \in W^\perp$.

(*) Note: Since T is a projection, then T is linear.

(*) Ex: Let $V = \mathbb{R}^2$, $W = \text{span}\{\begin{pmatrix}1\\0\end{pmatrix}\}$ $[\Rightarrow W^\perp = \text{span}\{\begin{pmatrix}0\\1\end{pmatrix}\}]$; then $T(\begin{pmatrix}a\\b\end{pmatrix}) = \begin{pmatrix}a\\0\end{pmatrix}$ for any $\begin{pmatrix}a\\b\end{pmatrix} \in V$.



$V = \mathbb{R}^3$, $\dim(W) = 2$

[Orthogonal projection onto W]

(*) Ex: Let V IPS, and let $w \in V$, $w \neq 0$; define $W = \text{span}\{w\}$ [1D subspace]

$\rightarrow$ for any $v \in V$: $v = \frac{\langle v, w\rangle}{\|w\|^2} w + (v - \frac{\langle v, w\rangle}{\|w\|^2} w) \rightarrow T(v) = \frac{\langle v, w\rangle}{\|w\|^2} w$ orthogonal projection on W

Orthogonal Projections (cont.)

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Lecture 16

Prop. Let V an IPS, W a fin.-dim. subspace of V. Let T be the orthogonal projection of V onto W along $W^\perp$; then:

(i) $v - T(v) \in W^\perp \ \forall \ v \in V$

(ii) $T(w) = w$ iff $w \in W$

(iii) $T(u) = 0_V$ iff $u \in W^\perp$

(iv) $\underline{R(T) = W}$ and $\underline{N(T) = W^\perp}$

(v) $\underline{T^2 = T}$

(vi) Let $w_1, w_2, \dots, w_k$ orthonormal basis for W; then:

$$\boxed{T(v) = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 + \dots + \langle v, w_k \rangle w_k}$$

(*) Proof: (i) Let $v \in V$; then v has unique decomposition $v = w_1 + w_2$ w/ $w_1 \in W$, $w_2 \in W^\perp$. Then $T(v) = w_1$ [by defn.] $\Rightarrow v - T(v) = v - w_1 = w_2 \in W^\perp$.

(ii) For any $w \in W$, have $w = \overset{\in W}{w} + \overset{\in W^\perp}{0_V} \Rightarrow T(w) = w$. Conversely, if $w \in V$ has $T(w) = w$, then $T(w) \in W \Rightarrow w \in W$.

(iii) For any $u \in W^\perp$, have $u = \overset{\in W}{0_V} + \overset{\in W^\perp}{u} \Rightarrow T(u) = 0_V$. Conversely, if $u \in V$ has $T(u) = 0_V$, then per (i): $u - T(u) = u - 0_V \in W^\perp \Rightarrow u \in W^\perp$.

(iv) Clearly, $R(T) \subseteq W$. Let $w \in W$; then $T(w + 0_V) = w \Rightarrow W \subseteq R(T) \Rightarrow R(T) = W$. Per (iii), for any $u \in V$, $T(u) = 0_V$ iff $u \in W^\perp$; hence $W^\perp = N(T)$.

(v) Let $v \in V$ w/ $v = \overset{W}{w_1} + \overset{W^\perp}{w_2}$; then $T^2(v) = T(w)$; $w \in W \overset{(ii)}{\Rightarrow} T^2(v) = w = T(v) \Rightarrow T^2 = T$.

(vi) Proven previously ▨

Prop. Let V fin.-dim. IPS, and let T be a projection. Then T is an orthogonal projection on some subspace W iff $\underline{R(T)^\perp = N(T)}$.

(*) Proof [($\Rightarrow$)]: T orthogonal projection $\Rightarrow R(T) = W$, $N(T) = W^\perp \Rightarrow R(T)^\perp = N(T)$. ▨

Orthogonal Projections (cont.)

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Lecture 16

Thm: Let V fin.-dim. IPS and T linear operator on V. Then T is an orthogonal projection on some subspace W iff $\underline{T^2 = T = T^*}$.

(*) Proof: ($\Rightarrow$) Let T an orthogonal projection. From before, have that $T^2 = T$; remains to show that $T = T^*$.

Let $x, y \in V$; can write $x = x_1 + x_2$, $y = y_1 + y_2$ for some $x_1, y_1 \in W$; $x_2, y_2 \in W^\perp$. Hence:

(i) $\langle x, T(y)\rangle = \langle x_1 + x_2, y_1\rangle = \langle x_1, y_1\rangle + \langle x_2, y_1\rangle = \langle x_1, y_1\rangle$

(ii) $\langle T(x), y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle + \langle x_2, y_1\rangle = \langle x_1, y_1\rangle$

$\rightarrow \langle x, T(y)\rangle = \langle T(x), y\rangle \ \forall x, y \in V \Rightarrow T^* = T$.

($\Leftarrow$) Lemma: Let T linear $V \to V$; then $T^2 = T$ iff T is a projection on $W_1 = \{y : T(y) = y\}$ along $N(T)$. [proof in HW]

Assume $T^2 = T = T^*$; by the lemma, T is a projection on $W_1 = \{y : T(y) = y\}$ along $N(T)$. To show T an orthogonal projection, want to show that $R(T)^\perp = N(T)$.

First, want to show that $R(T) = W_1$. Obviously, $W_1 \subseteq R(T)$. Let $y \in R(T)$; then $\exists\, x \in V$ s.t. $T(x) = y$. Since $T^2 = T$: $T(y) = T(T(x)) = T(x) = y \Rightarrow y \in W_1$, therefore $R(T) \supseteq W_1$.

Since $W_1 = R(T)$, then T is a projection on $R(T)$ along $N(T)$. Let $x \in R(T)$, $y \in N(T)$; using that $T = T^*$, $x = T(x)$: $\langle x, y\rangle = \langle T(x), y\rangle = \langle x, T(y)\rangle = \langle x, 0\rangle = 0 \Rightarrow y \in R(T)^\perp \Rightarrow N(T) \subseteq R(T)^\perp$.

Let $y \in R(T)^\perp$. Let $y_1 \in R(T)$; $y_2 \in N(T)$ s.t. $y = y_1 + y_2$. Then $\forall\, x \in R(T)$:

$\rightarrow \langle x, y\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle = \langle x, T(y_1)\rangle + \langle x, T(y_2)\rangle = \langle x, y_1\rangle$. Since $y \in R^\perp$, then $\langle x, y\rangle = \langle x, y_1\rangle = 0 \ \forall\, x \in R(T)$; hence $y_1 = 0_V \Rightarrow y = y_2 \Rightarrow y \in N(T)$, hence $R(T)^\perp \subseteq N(T)$. ▨

The Spectral Theorem

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Lecture 17

The Spectral Theorem / Spectral Decomposition

Theorem: Let T be a linear operator on a fin-dim. IPS V over F with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k \in F$. Additionally, assume that T is normal $[F=\mathbb{C}]$ / self-adjoint $[F=\mathbb{R}]$. For each $1 \le i \le k$, let W_i be the eigenspace corresponding to eigenvalue λ_i, and let T_i be the orthogonal projection of V onto W_i. Then:

(i) $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$

(ii) For each W_i, $W_i^\perp = \bigoplus_{j \ne i} W_j$ [the direct sum of the subspaces $W_{j \ne i}$]

(iii) $T_i T_j = \delta_{ij} T_i$ for $1 \le i,j \le k$

(iv) $I = T_1 + T_2 + \dots + T_k$

(v) $\boxed{T = \lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k}$ [Spectral decomposition]

Recall: Given a linear operator T with eigenvalue λ, the eigenspace corresponding to λ is the subspace $E_\lambda = \{v \in V : T(v) = \lambda v\}$ of all eigenvectors with associated eigenvalue λ

(*) Proof (The Spectral Theorem)

(i): Trivially, $W_i \cap [\bigcup_{j \ne i} W_j] = \{0_V\}$ and $W_1 + W_2 + \dots + W_k \subseteq V$. Since T is normal, then $\exists$ an orthonormal basis for V consisting of eigenvalues of T, hence $W_1 + W_2 + \dots + W_k \subseteq V$.

(ii): Fix $1 \le i,j \le k$, $i \ne j$. Let $x_i \in W_i$, $x_j \in W_j$. $[T(x_i) = \lambda_i x_i ; T(x_j) = \lambda_j x_j]$. Then:

$$\lambda_i \langle x_i, x_j \rangle = \langle T(x_i), x_j \rangle = \langle x_i, T^*(x_j) \rangle$$

$$\to = \langle x_i, \overline{T(x_j)} \rangle \text{ [T self-adjoint]} = \lambda_j \langle x_i, x_j \rangle$$

Since $\lambda_i \ne \lambda_j$, then $\langle x_i, x_j \rangle = 0$; holds for any $j \ne i$

$\rightsquigarrow$ By linearity: $\langle x_i, y \rangle = 0 \;\; \forall y \in \bigoplus_{j \ne i} W_j$.

(iii) For any $1 \le i,j \le k$: if $i \ne j$, then $W_i \perp W_j \Rightarrow T_i \circ T_j = T_0$. Otherwise, $T_i^2 = T_i$ [orthogonal projection].

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Lecture 17

(cont.)

The Spectral Theorem (cont.)

(*) Proof (Spectral Theorem, cont.)

(iv) Per (ii): since T_i orthogonal projection, have that $N(T_i) = W_i^{\perp} = \bigoplus_{j \neq i} W_j$ for each $1 \leq i \leq k$.

Per (i): since $W_1 \oplus W_2 \oplus \ldots \oplus W_k = V$, then for every $v \in V$ $\exists$ unique vectors $v_1 \in W_1$, $v_2 \in W_2, \ldots, v_k \in W_k$ s.t. $v = v_1 + v_2 + \ldots + v_k$.

For each $1 \leq i \leq k$:

$$v = \underbrace{v_i}_{\in W_i} + \overbrace{v_1 + v_2 + \ldots + v_{i-1} + v_{i+1} + \ldots + v_k}^{\in \bigoplus_{j \neq i} W_j = W^{\perp}} \Rightarrow T_i(v) = v_i$$

Hence:

$$\begin{aligned}(T_1 + T_2 + \ldots + T_k)(v) &= (T_1 + T_2 + \ldots + T_k)(v_1 + v_2 + \ldots + v_k) \\ &= v_1 + v_2 + \ldots + v_k = v \ [= I(v)]\end{aligned}$$

(v) Let $v \in V$. As in (iv), have that $\exists$ unique $v_1 \in W_1, v_2 \in W_2, \ldots, v_k \in W_k$ such that $v = v_1 + v_2 + \ldots + v_k$. [Recall: $v_i \in W_i \Rightarrow T(v_i) = \lambda v_i$]

Then:

$$\begin{aligned}T(v) &= T(v_1 + v_2 + \ldots + v_k) \\ &= T(v_1) + T(v_2) + \ldots + T(v_k) \\ &= \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k \\ &= \lambda_1 T_1(v) + \lambda_2 T_2(v) + \ldots + \lambda_k T_k(v) \\ &= (\lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_k T_k)(v)\end{aligned}$$

$\rightarrow T = \lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_k T_k$. ▨

(*) Note: The spectral decomposition allows for decomposing any normal/self-adjoint operator into [much simpler] orthogonal projections $\Rightarrow$ many uses & applications.

The Spectral Theorem, Corollaries

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Lecture 18

Corollary: Let T be a normal $[F=\mathbb{C}]$/self-adjoint operator, and let g be a polynomial. Let $T=\lambda_1T_1+\lambda_2T_2+\ldots+\lambda_kT_k$ be the spectral decomposition of T; then:

$$\boxed{g(T)=g(\lambda_1)T_1+g(\lambda_2)T_2+\ldots+g(\lambda_k)T_k}$$

(*) Ex: Let $A\in M_{n\times n}(\mathbb{R})$ symmetric [=> orthogonally equivalent to a real diagonal matrix]. Can compute its powers A^p $[p\in\mathbb{N}]$ using its spectral decomposition.

Since A symmetric, then A self-adjoint => $\exists$ orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A. Let $W_1,\ldots,W_k$ eigenspaces of A corresponding to [distinct] eigenvalues $\lambda_1,\ldots,\lambda_k$, and let $m_i=\dim(W_i)$ for $1\le i\le k$. Let β be an orthonormal basis for $\mathbb{R}^n$ of eigenvectors.

$$\rightarrow [T]_\beta=\left[\sum_{i=1}^k\lambda_iT_i\right]_\beta=\sum_{i=1}^k\lambda_i[T_i]_\beta$$

For each T_i, have: $$[T_i]_\beta=\begin{pmatrix}0 & & & & 0\\ & \ddots & & & \\ & & I_{m_i} & & \\ & & & \ddots & \\ 0 & & & & 0\end{pmatrix}$$

$$\Rightarrow \sum_{i=1}^k\lambda_i[T_i]_\beta = \boxed{[T]_\beta=\begin{pmatrix}\lambda_1I_{m_1} & & & \\ & \lambda_2I_{m_2} & & \\ & & \ddots & \\ & & & \lambda_kI_{m_k}\end{pmatrix}} \Rightarrow [T^p]_\beta=\begin{pmatrix}\lambda_1^pI_{m_1} & & & \\ & \lambda_2^pI_{m_2} & & \\ & & \ddots & \\ & & & \lambda_k^pI_{m_k}\end{pmatrix}$$

Corollary: If $F=\mathbb{C}$, then T is normal iff, for some polynomial g:

$$\boxed{T^*=g(T)}$$

The Spectral Theorem, Corollaries (cont.)

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Lecture 18

Corollary (restated): If $F = \mathbb{C}$, then T is normal iff $T^* = g(T)$ for some polynomial g.

(*) Proof:

($\Rightarrow$) Suppose T is normal, i.e. $T^*T = TT^*$. Let $T = \sum_{i=1}^{k} \lambda_i T_i$ spectral decomposition for T; then we have that:

$$T^* = \sum_{i=1}^{k} (\lambda_i T_i)^* = \sum_{i=1}^{k} \overline{\lambda_i} T_i \quad \text{(*) } T_i \text{ orthogonal proj.} \Rightarrow T_i \text{ self-adjoint}$$

Can find polynomial g s.t. $g(\lambda_i) = \overline{\lambda_i}$ $\forall\ 1 \le i \le k$; e.g. define:

$$g_i(x) = \frac{(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_{i-1})(x-\lambda_{i+1})\dots(x-\lambda_k)}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\dots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\dots(\lambda_i-\lambda_k)} \quad [g_i(\lambda_j) = \delta_{ij}]$$

$$g(x) = \sum_{i=1}^{k} \overline{\lambda_i}\, g_i(x)$$

$$\Rightarrow g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i = \sum_{i=1}^{k} \overline{\lambda_i} T_i = T^*$$

($\Leftarrow$) Suppose $T^* = g(T)$ for some polynomial g; then:

$$TT^* = Tg(T) = g(T)T = T^*T \Rightarrow T \text{ normal.} \quad \blacksquare$$

Corollary: If $F = \mathbb{C}$, then T is self-adjoint iff T is normal and all eigenvalues λ_i of T are real-valued $[\lambda_i \in \mathbb{R}]$.

Corollary: Let T be a linear operator with spectral decomposition $T = \sum_{i=1}^{k} \lambda_i T_i$. Then $\forall\ 1 \le i \le k$, have that $T_i = g_i(T)$ for some polynomial g_i.

(*) Proof ($T_i = g_i(T)$): Take g_i as above, s.t. $g_i(\lambda_j) = \delta_{ij}$. Then $\forall\ 1 \le i \le k$:

$$g_i(T) = \sum_{j=1}^{k} g_i(\lambda_j) T_j = T_i \quad \blacksquare$$

The Geometry of Orthogonal Operators

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Lecture 19

Can show that any orthogonal operator on a fin.-dim. real IPS is describable in terms of rotations and reflections, starting with $\dim V = 1$ & $\dim V = 2$:

One-Dimensional Vector Spaces

Def: A linear operator T on a 1D real IPS is called:

(i) A rotation if $T(x) = x \ \forall \ x \in V$, and

(ii) A reflection if $T(x) = -x \ \forall \ x \in V$.

Prop: Any orthogonal operator on a real IPS V is either a reflection or a rotation.

Two-Dimensional Vector Spaces

Recall: Let T be a linear operator on a 2D real IPS. Then we say that T:

(i)... is a rotation if $\exists$ orthonormal basis β for V and $\theta \in \mathbb{R}$ such that:

$$\boxed{[T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \qquad [\det(T) = 1]$$

(ii)... is a reflection if $\exists$ orthonormal basis β for V and $\theta \in \mathbb{R}$ such that:

$$\boxed{[T]_\beta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}} \qquad [\det(T) = -1]$$

Prop: A linear operator T on 2D real IPS is a reflection iff $\exists$ a one-dimensional subspace W of V such that: (i) $T(x) = -x \ \forall \ x \in W$

(ii) $T(x) = x \ \forall \ x \notin W$

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Lecture 19 (cont.)

The Geometry of Orthogonal Operators (cont.)

Theorem: Let T be an orthogonal operator on a 2D real IPS V. Then either (i) T is a rotation and $\det(T) = 1$, or (ii) T is a reflection and $\det(T) = -1$.

Looking at vector spaces of arbitrary dimension:

Lemma: Let T be a linear operator on nonzero fin.-dim. real IPS V. Then $\exists$ a T-invariant subspace W of V s.t. $1 \leq \dim W \leq 2$.

$\downarrow$

Theorem: Let T be a linear operator on a nonzero fin.-dim. real IPS V. Then $\exists$ pairwise orthogonal T-invariant subspaces $W_1, W_2, \ldots, W_m$ of V such that:

(i) $1 \leq \dim W_i \leq 2$ for each $i = 1, 2, \ldots, m$

(ii) $V = W_1 \oplus W_2 \oplus \ldots \oplus W_m$

(*) Proof of Thm.: Via induction on $\dim(V)$. Base case: $\dim(V) = 1 \Rightarrow$ trivial.

Suppose claim holds $\forall \dim(V) \leq n-1$. Let $\dim(V) = n$. By the lemma, $\exists$ a T-invariant subspace W_1 of V s.t. $1 \leq \dim W_1 \leq V$. Since W_1 is a T-invariant subspace, then $W_1 \oplus W_1^\perp = V$ and $W_1^\perp$ is T-invariant [using that T is orthogonal]; furthermore, the restriction $T_{W_1^\perp}$ of T to $W_1^\perp$ is an orthogonal operator on $W_1^\perp$. Then per the inductive assumption, $\exists$ pairwise orthogonal T-invariant subspaces $W_2, W_3, \ldots, W_m$ of $W_1^\perp \subseteq V$ s.t. $W_2 \oplus \ldots \oplus W_m = W_1^\perp$ and $\dim(W_i) = 1$ or 2 for each $i = 2, 3, \ldots, m$. Since $W_2, \ldots, W_m \subseteq W_1^\perp$, then W_1 and W_i are pairwise orthogonal for each $i = 2, 3, \ldots, m$; hence $W_1, W_2, \ldots, W_m$ are pairwise orthogonal T-invariant subspaces satisfying (i) [$W_1 \oplus W_1^\perp = W_1 \oplus W_2 \oplus \ldots \oplus W_m = V$] and (ii). ∎

Remark: We have that (i) any orthogonal operator on a 1D/2D real IPS is either a rotation or a reflection, and (ii) that any fin.-dim. real IPS can be decomposed into 1D/2D pairwise orthogonal T-invariant subspaces W_i. Since the restriction of T to W_i is either a rotation or a reflection [by (i)], we find that <u>T itself is composed of rotations & reflections</u> [on the subspaces W_i].

The Jordan Canonical Form

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Lecture 20

Recall: Let T be a linear operator over a fin-dim V.S. V with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. Then T is diagonalizable iff, for every $i = 1, 2, \dots, k$, the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$.

$\rightsquigarrow$ Q: What if T is not diagonalizable?

(i) Currently, have Schur's lemma: for any $A \in M_{n\times n}(\mathbb{C})$, $\exists$ Q unitary, U upper-triangular [$Q, U \in M_{n\times n}(\mathbb{C})$] s.t. $A = QUQ^{-1}$.

(ii) Even better: Jordan canonical form:

$$A \sim \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ 0 & & & 0 & \lambda_i \end{pmatrix} \text{ called a Jordan block}$$

Formally:

Theorem: Let $A \in M_{n\times n}(F)$ s.t. the characteristic polynomial of A splits [into linear factors].

Then A is similar to the block diagonal matrix

$$\boxed{J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}} \in M_{n\times n}(F)$$

where each block J_i is of the form

$$\boxed{J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & & & \lambda_i & 1 \\ & & & 0 & \lambda_i \end{pmatrix}} \text{ [upper-triangular]}$$

and $\lambda_1, \lambda_2, \dots, \lambda_k$ are eigenvalues of A [not necessarily distinct].

Recall: Say that a characteristic polynomial $f(t)$ splits over a field F if $\exists$ scalars $c, \lambda_1, \lambda_2, \dots, \lambda_n \in F$ [not necessarily distinct] s.t. $f(t) = c(t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_n)$. In particular, every char. poly. for $A \in M_{n\times n}(\mathbb{C})$ splits.

The Jordan Canonical Form (cont)

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Lecture 20

> **Prop**: Let $B \in M_{k\times k}(\mathbb{C})$ be a Jordan block with diagonal element λ. Then:
>
> (i) B has only one eigenvalue λ with multiplicity k.
>
> (ii) $\dim(E_\lambda) = 1$. [note: implies B not diagonalizable unless $k=1$]
>
> (iii) $(B-\lambda I_k)^p \neq O_{k\times k}$ if $1 \le p < k$, but $(B-\lambda I_k)^p = O_{k\times k}$ $\forall$ $p \ge k$.

(*) Proof of Prop:

(i) Since B is of the form

$$B = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & 0 & & \lambda \end{pmatrix} \quad \text{[upper-triangular]}$$

then B has characteristic polynomial $\underline{f(t) = \det(B - tI_k) = (-1)^k (t-\lambda)^k}$ $\Rightarrow$ λ is the only eigenvalue.

(ii) Notice that:

$$\underline{B - \lambda I_k} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

By definition: $E_\lambda = N(B - \lambda I_k) \Rightarrow \underline{E_\lambda = \text{span}\{(1,0,0,\dots,0)\}}$, $\underline{\dim(E_\lambda) = 1}$.

(iii) Looking at the form of $(B-\lambda I_k)$ from (ii), see that each instance of $(B-\lambda I_k)$ "shifts" each 1 up one row [alt: right one column]

$\rightsquigarrow$ before $p=k$, at least one 1 will remain.

after $p=k$, all 1s will be gone $\Rightarrow$ zero matrix.

$\blacksquare$

The Jordan Canonical Form (cont.)

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Lecture 21

Previously, had that some matrices were not diagonalizable because their eigenspaces were too small. To get around this, can "extend" the notion of an eigenspace - generalized eigenspaces.

Def. Let $A \in M_{n\times n}(\mathbb{C})$, and let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then the generalized eigenspace of A corresponding to eigenvalue λ is defined as:

$$\boxed{K_\lambda = \{x \in \mathbb{C}^n : (A-\lambda I_n)^p(x) = 0 \text{ for some } p \in \mathbb{N}\}}$$

A vector $x \in \mathbb{C}^n$ is called a generalized eigenvector of A corresponding to eigenvalue λ if $x \neq 0_V$ and $x \in K_\lambda$. [x is contained in the generalized eigenspace]

Observation: For every eigenvalue λ of A, K_λ is a subspace of $\mathbb{C}^n$ with $E_\lambda \subseteq K_\lambda$.

Def: Let $A \in M_{n\times n}(\mathbb{C})$. Let $x \in \mathbb{C}^n$ be a generalized eigenvector of A corresponding to λ $[x \in K_\lambda]$. Suppose $p \geq 1$ is the smallest integer > 0 s.t. $(A-\lambda I_n)^p(x) = 0$. Then the ordered set

$$\boxed{\gamma = \{(A-\lambda I_n)^{p-1}(x), (A-\lambda I_n)^{p-2}(x), \dots, (A-\lambda I_n)(x), x\}}$$

is called a cycle of generalized eigenvectors [corresponding to eigenvalue λ] of length p.

Prop: For any such cycle $\{(A-\lambda I_n)^{p-1}(x), \dots, (A-\lambda I_n)(x), x\}$, the set is linearly independent.

(*) Proof: By definition, all vectors $v \in \gamma$ are nonzero. Suppose $\exists\, c_0, c_1, c_2, \dots, c_{p-1} \in \mathbb{C}$ such that $\sum_{i=0}^{p-1} c_i (A-\lambda I_n)^i(x) = 0$.

Applying $(A-\lambda I_n)^{p-1}$ to both sides: $\sum_{i=0}^{p-1} c_i (A-\lambda I_n)^{i+p-1}(x) = c_0 \overbrace{(A-\lambda I_n)^{p-1}(x)}^{\neq 0} = 0 \rightarrow c_0 = 0$.

By a similar argument, can find that $c_1, c_2, c_3, \dots$ are all 0 [induct up to p-1].

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Lecture 21
+ Discussion 8

The Jordan Decomposition Theorem

Theorem (Jordan decomposition): Let $A \in M_{n\times n}(\mathbb{C})$ with distinct eigenvalues $\lambda_1, \dots, \lambda_k$ with corresponding multiplicities $m_1, \dots, m_k$. Then:

(i) $\dim(K_{\lambda_i}) = m_i$ for each $i = 1, 2, \dots, k$

(ii) $\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \dots \oplus K_{\lambda_k}$.

(iii) Each K_{λ_i} has an ordered basis consisting of a union of cycles of generalized eigenvectors corresponding to λ_i, for each $i = 1, 2, \dots, k$.

(*) Note: To find the union of cycles for some eigenvalue λ, do the following:

(i) Let $\dim(K_\lambda) = r$. Can compute $\text{null}((A-\lambda I)^i) = r_i$ for each $i = 1, 2, \dots$ until eventually $r_{N+1} = r_N \Rightarrow r_N = r$. [To show that N exists: $r_1 < r_2 < \dots < r_N \leq r$]

(ii) The value N [from (i)] corresponds to the length of the longest cycle. Define $s_1 = r_1$, $s_2 = r_2 - r_1$, $s_3 = r_3 - r_2, \dots, s_N = r_N - r_{N-1}$. Then for each $k = 1, 2, \dots, N$, the no. of cycles of length at least k is given by s_k.

(iii) Once the no. of cycles + corresponding lengths have been determined, can solve for the generalized eigenvectors:

$$v \text{ a [gen.] eigenvector} \rightarrow \text{solve for } v' \in V \text{ s.t. } (A - \lambda I_n)v' = v$$

(*) Note: Let $\beta = \{v_1, \dots, v_m\}$ cycle of generalized eigenvectors

$$\rightarrow \beta = \{v_1, v_2, \dots, v_{m-1}, v_m\}, \quad T(v_i) = \begin{cases} \lambda v_i + v_{i-1} & i \neq 1 \\ \lambda v_i & i = 1 \end{cases}$$

The Jordan Decomposition Theorem (cont.)

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The Jordan Canonical Form

Lecture 22

+Lecture 23

Let $A \in M_{n \times n}(\mathbb{C})$. Let $\beta = \bigcup_{i=1}^{k} \beta_i$, where each $\beta_i = \{v_1^{(i)}, \dots, v_{m_i}^{(i)}\}$ is the union of cycles for K_{λ_i}; call β the Jordan basis. Then we have that:

$$\boxed{A = QJQ^{-1}} \quad \text{[Jordan form]}$$

w/ Q, J defined by:

$$\boxed{Q = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1^{(1)} & v_2^{(1)} & \cdots & v_{m_k}^{(k)} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}} \quad \text{and} \quad \boxed{J = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_k} \end{pmatrix}} \quad \left[J_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix} \in M_{m_i \times m_i}(F) \right]$$

To prove Jordan decomposition theorem, first want to prove 2 lemmas:

> **Lemma (1):** Let $A \in M_{n \times n}(\mathbb{C})$, λ eigenvalue of A w/ corresponding gen. eigenspace K_λ. Then:
>
> (i) K_λ is T-invariant under the operator $T = L_A : \mathbb{C}^n \to \mathbb{C}^n$.
>
> (ii) For any scalar $\mu \neq \lambda$, the operator $L_{A-\mu I_n} : K_\lambda \to K_\lambda$ is injective.
>
> (iii) For any eigenvalue μ of A such that $\mu \neq \lambda$, have that:
>
> $$\boxed{K_\mu \cap K_\lambda = \{0_v\}}$$

(*) Proof of Lemma:

(i) Suffices to show that $L_A(K_\lambda) \subseteq K_\lambda$. Let $x \in K_\lambda$; then by definition of K_λ, $\exists\ p \geq 1$, $p \in \mathbb{N}$ s.t. $(A - \lambda I)^p(x) = 0$. Then:

$$(A - \lambda I_n)^p (L_A(x)) = (A - \lambda I_n)(A(x)) = A(A - \lambda I_n)^p(x) = A(0_v) = 0_v$$

Hence $x \in K_\lambda$ by definition of K_λ.

The Jordan Decomposition Theorem (cont)

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Lecture 23

+ Lecture 24

(*) Proof of Lemma (cont.)

(ii) Assume, for the sake of contradiction, that $L_{A-\mu I_n}$ is not injective. Then $\exists$ $x \in K_\lambda$, $x \neq 0_v$ s.t. $(A-\mu I_n)(x) = 0$. Since $x \in K_\lambda$, $\exists$ smallest integer $p \geq 1$ s.t. $(A-\lambda I_n)^p(x) = 0$. Let $y = (A-\lambda I_n)^{p-1}(x)$; then $y \neq 0$, but $y \in N(A-\lambda I_n)$. We also find that $y \in N(A-\mu I_n)$:

$$(A-\mu I_n)(y) = (A-\mu I_n)(A-\lambda I_n)^{p-1}(x) = (A-\lambda I_n)^{p-1}(A-\mu I_n)(x) = 0$$

Then $Ay = \lambda y = \mu y$, but $y \neq 0$ and we assumed $\lambda \neq \mu$. [contradiction]

(iii) Via similar argument as in (ii). ∎

Lemma (2): Let $A \in M_{n\times n}(\mathbb{C})$, λ eigenvalue of A w/ multiplicity m. Then:

(i) $\dim(K_\lambda) \leq m$

(ii) $K_\lambda = N((A-\lambda I_n)^m)$.

(*) Proof of Lemma:

(i) Let $W = K_\lambda$; by lemma (1), W is T-invariant for $T = L_A$. Define $d = \dim(W)$. We notice that the operator $T_W: W \to W$ has eigenvalue λ: $T_W(x) = T(x) = \lambda x$ $\forall$ eigenvectors $x \in \mathbb{C}^n$ w/ eigenvalue λ. In particular, by lemma (1), this eigenvalue is the only eigenvalue for T_W; then T_W has char. poly.:

$$h(t) = (-1)^d (t-\lambda)^d \quad \text{[characteristic polynomial for } T_W]$$

Know that $h(t)$ divides the characteristic polynomial $f(t)$ of T; since λ has multiplicity m in $f(t)$, therefore $\dim(W) = d \leq m$.

(ii) Let $x \in N((A-\lambda I)^m)$; clearly, $x \in K_\lambda$ $[N((A-\lambda I_n)^m) \subseteq K_\lambda]$. Conversely, let $x \in K_\lambda = W$; by Cayley-Hamilton, $h(T_W) = (-1)^d (T_W - \lambda I)^d = 0$. In particular, $(T_W - \lambda I_n)^d(x) = 0$; since $m \geq d$, therefore $(T_W - \lambda I_n)^m(x) = 0$ $[K_\lambda \subseteq N((A-\lambda I_n)^m)]$. ∎

The Jordan Decomposition Theorem (cont.)

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Lecture 24 (cont.)

Theorem (Jordan decomposition, restated): Let $A \in M_{n\times n}(\mathbb{C})$ and $\lambda_1, \dots, \lambda_k$ distinct eigenvalues of A with multiplicities $m_1, \dots, m_k$. Then:

(i) $\dim(K_{\lambda_i}) = m_i$ for each $i = 1, \dots, k$

(ii) $\mathbb{C}^n = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \dots \oplus K_{\lambda_m}$

(iii) Each K_{λ_i} has an ordered basis consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ_i, for each $i = 1, \dots, k$.

(*) Proof (Jordan decomposition)

(ii) Via induction on # eigenvalues k.

(a) Base case: $k = 1 \Rightarrow A$ has characteristic polynomial $(-1)^n(t-\lambda_1)^n$. By Cayley-Hamilton, $(A-\lambda_1)^n = O_{n\times n} \Rightarrow$ by Lemma 2: $K_{\lambda_1} = N((A-\lambda_1 I_n)^n) = N(O_{n\times n}) = \mathbb{C}^n$.

(b) Inductive step: Assume result holds for matrices with $k-1$ distinct eigenvalues, $k \geq 2$. Let $A \in M_{n\times n}(\mathbb{C})$ with characteristic polynomial $f(t) = (-1)^n (t-\lambda_1)^{m_1} \cdot \ldots \cdot (t-\lambda_k)^{m_k}$.

Let $W = R(A-\lambda_k I_n)^{m_k}$; claim that W invariant under $T = L_A$. Let $x \in \mathbb{C}^n$, $y = (A-\lambda_k I_n)^{m_k} x \in W$; then:

$$Ay = A(A-\lambda_k I_n)^{m_k} x = (A-\lambda_k I_n)^{m_k}(Ax) \in W$$

Then by Lemma 1, for each $1 \leq i \leq k$, $L_{A-\lambda_k I}: K_{\lambda_i} \to K_{\lambda_i}$ is injective; by rank-nullity, $L_{A-\lambda_k I}$ is bijective.

Let $W = R(A-\lambda_k I_n)^{m_k}$; see that $K_{\lambda_i} \subseteq W \ \forall\ 1 \leq i \leq k-1$. Additionally, notice: $T_W: W \to W$ has eigenvalues $\lambda_1, \dots, \lambda_{k-1}$, but not λ_k. By the inductive hypothesis, $\forall\ x \in \mathbb{C}^n$, $\exists$ unique $w_i \in K_{\lambda_i}\ \forall\ 1 \leq i \leq k-1$ s.t. $(A-\lambda_k I)^{m_k} x = w_1 + w_2 + \dots + w_{k-1}$.

Since $L_{(A-\lambda_k I)^{m_k}}: K_{\lambda_k} \to K_{\lambda_k}$ bijective, $\exists$ unique $v_i \in K_{\lambda_i}$ s.t. $(A-\lambda_k I)^{m_k} v_i = w_i \ \forall\ 1 \leq i \leq k-1$

$\rightsquigarrow (A-\lambda_k I_n)^{m_k}\underbrace{(x - v_1 - \dots - v_{k-1})}_{= v_k \in K_{\lambda_k}} = 0 \Rightarrow x = v_1 + v_2 + \dots + v_{k-1} + v_k$.

The Jordan Decomposition Theorem (cont.)

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Lecture 24 (cont.)

(*) Proof (Jordan decomposition, cont.)

(i) Let β_i an ordered basis for K_{λ_i} for $i=1,\dots,k$. By Lemma 1, have that $\beta_i \cap \beta_j = \emptyset \; \forall \; i \neq j$; by part (ii), have that $\beta = \beta_1 \cup \dots \cup \beta_k$ spans $\mathbb{C}^n$.

Let $q = |\beta|$; then $q \geq n$. However, by Lemma 2, $\dim K_{\lambda_i} \leq m_i$ for each $i=1,\dots,k$

$$\rightarrow q = \sum_{i=1}^{k} \dim(K_{\lambda_i}) \leq \sum_{i=1}^{k} m_i = n$$

Hence $q = n$ [β basis for $\mathbb{C}^n$] and $\dim(K_{\lambda_i}) = m_i$ for each $i=1,\dots,k$.

(iii) Observe: for any cycle γ of generalized eigenvectors of T, $W = \text{span}(\gamma)$ is T-invariant.

Let $\gamma_1,\dots,\gamma_q$ cycles of generalized eigenvectors of T corresponding to an eigenvalue λ s.t. the initial vectors of each γ_i form a linearly-independent set. Clearly, the γ_i's are disjoint; can find that $\gamma = \bigcup_{i=1}^{q} \gamma_i$ is linearly independent via induction on $|\gamma|$.

To show that we can find such a basis γ for every gen. eigenspace K_λ of T, can prove via induction on $\dim(K_\lambda)$ [using that $R([T-\lambda I]_{K_\lambda}) \subseteq K_\lambda$ is a gen. eigenspace of the restriction of T to K_λ] that can find γ lin. indep. [from above] with $\dim(\gamma) = \dim(K_\lambda)$.

Clearly, the set $\beta = \bigcup_\lambda \gamma_\lambda$ consisting of the union of bases of generalized eigenspaces forms a linearly independent set. Importantly, can find that β is a basis for F^n iff the characteristic polynomial of T splits into linear factors $(t - \lambda_i I_n)$; guaranteed for $\mathbb{C}$ [not $\mathbb{R}$].

Assume the characteristic polynomial $f(t)$ of T splits, i.e. that $f(t)$ has form $f(t) = (t-\lambda_1 I)^{m_1} \dots (t-\lambda_k I)^{m_k}$ for some eigenvalues $\lambda_1,\dots,\lambda_k$. By Cayley-Hamilton, $f(A) = (A-\lambda_1 I)^{m_1} \dots (A-\lambda_k I)^{m_k} = 0 \Rightarrow f(A)v = (A-\lambda_1 I)^{m_1} \dots (A-\lambda_k I)^{m_k} v = 0_v \;\; \forall \; v \in \mathbb{C}^n$, therefore the generalized eigenspaces $K_{\lambda_1},\dots,K_{\lambda_k}$ span $\mathbb{C}^n \Rightarrow \beta = \bigcup_{i=1}^{k} \gamma_{\lambda_i}$ is a basis for $\mathbb{C}^n$. ▨

The Minimal Polynomial

6/5/24

Lecture 25

<u>Recall</u>: A monic polynomial is a polynomial with highest-degree term having coefficient 1.

<u>Def</u>: Let $A \in M_{n\times n}(F)$. The <u>minimal polynomial</u> $p(t)$ of A is defined as the <u>monic polynomial of least positive degree for which $p(A) = O_{n\times n}$</u>.

<u>Theorem (Existence & Uniqueness)</u>: Let $A \in M_{n\times n}(F)$. Then:

(i) There is a monic polynomial $p(t) \in P(F)$ of least degree s.t. $p(A) = O_{n\times n}$.

(ii) For any polynomial $g(t)$: if $g(A) = O_{n\times n}$, then $p(t)$ divides $g(t)$. In particular, $p(t)$ divides the characteristic polynomial of A.

(iii) The minimal polynomial is unique.

<u>(*) Proof of Theorem</u>

(i) By Cayley-Hamilton, the characteristic polynomial $f(t)$ of A is a monic polynomial s.t. $f(A) = O_{n\times n}$, hence $\{p(t) \in P(F) : p(t) \text{ monic}, p(A) = O_{n\times n}\}$ nonempty $\Rightarrow$ has a member w/ least degree.

(ii) Let $p(t)$ be a [the] minimal polynomial, and let $g(t) \in P(F)$ monic s.t. $g(A) = O_{n\times n}$. By definition, $\deg(g(t)) \geq \deg(p(t)) \Rightarrow$ can perform polynomial long division to find:

$$g(t) = q(t)p(t) + r(t)$$

for some $q(t), r(t)$; $\deg(r(t)) < \deg(p(t))$. Then $g(A) = q(A)\underbrace{p(A)}_{=O_{n\times n}} + r(A) = O_{n\times n}$, thus $r(A) = O_{n\times n}$. Since $p(t)$ was a minimal polynomial of A and $\deg(p(t)) > \deg(r(t))$, thus $r(t) \equiv 0$.

(iii) Let p_1, p_2 be two minimal polynomials of A, $\deg(p_1) = \deg(p_2)$. Per (ii), p_1 divides p_2; then $p_2 = cp_1$ for some $c \in F$. Since p_1, p_2 are both monic, then $c = 1 \Rightarrow p_1 = p_2$. ▨

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Lecture 25 + Lecture 26

The Minimal Polynomial (cont.)

Theorem: Let $A \in M_{n\times n}(F)$, $p(t)$ minimal polynomial of A. Then a scalar $\lambda \in F$ is an eigenvalue of A iff $p(\lambda) = 0$.

Corollary: The minimal and characteristic polynomials share the same 0s.

(*) Proof of Theorem

($\Leftarrow$) Since $p(t)$ divides the characteristic polynomial, then for any $\lambda \in F$:

$$p(\lambda) = 0 \Rightarrow f(\lambda) = q(\lambda)p(\lambda) = 0 \Rightarrow \lambda \text{ is an eigenvalue of } A.$$

($\Rightarrow$) Let $\lambda \in F$ eigenvalue of A, $x \in V$ eigenvector of A corresponding to λ. Then $Ax = \lambda x \Rightarrow p(A)x = p(\lambda)x$. Since $p(A)x = 0_{n\times n} \cdot x = 0_V$ but $x \neq 0_V$ [by definition], then it must be the case that $p(\lambda) = 0$. ▨

Applications of Minimal Polynomials

Prop (1). Let $A \in M_{n\times n}(F)$ and let $\lambda_1, \dots, \lambda_k$ be the eigenvalues [distinct] of A. Then A is diagonalizable iff it has minimal polynomial $p(t) = (t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_k)$.

Prop (2). Let $A \in M_{n\times n}(F)$ and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the [distinct] eigenvalues of A. Then for each $i = 1, \dots, k$, the size of the largest Jordan block associated with λ_i is equal to the exponent of $(t-\lambda_i)$ in the minimal polynomial.

Consequently: can use characteristic/minimal polynomials to verify Jordan decomposition; in some cases (not all), may be sufficient to determine Jordan canonical form of A outright

The Minimal Polynomial (cont.)

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Lecture 26
(cont.)
FINAL

Def: Let T be a linear operator on a fin.-dim. v.s.. Then a polynomial $p(t)$ is called the minimal polynomial of T if $p(t)$ is the monic polynomial of least positive degree s.t. $p(T) = \theta_T$.

Theorem: The minimal polynomial of T on n-dimensional vector space V is equal to the minimal polynomial of each matrix representation of T.

(*) **Note:** Existence & uniqueness of the minimal polynomial of T follows from same argument as previously.