Math 120A: Differential Geometry

2024-25
Fall '24

Instructor: Konstantinos Varvorezos
Textbook: J. A. Thorpe - Elementary Topics in Differential Geometry
Topics: Curves & vector fields, surfaces; orientation & curvature of curves & surfaces; intrinsic properties of surfaces

Table of Contents

Curves

9/27/24

Lecture 1

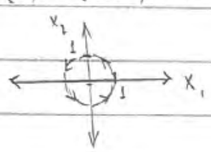
Def: Parametrized Curves

A parametrized curve is a function $\gamma: I \to \mathbb{R}^n$, where I is an interval $\subseteq \mathbb{R}$ (that is potentially unbounded).

(i) Say that γ is smooth if all derivatives of γ exist (γ is infinitely differentiable; $\frac{d^k\gamma}{dt^k}(t)$ exists $\forall\, t \in I,\ k \in \mathbb{R}$).

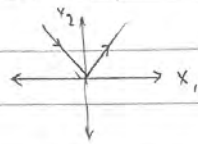
(ii) Say that γ is regular if the 1st derivative $\dot{\gamma} = \frac{d\gamma}{dt} \neq 0\ [\vec{0}]\ \forall\, t \in I$ (the 1st derivative is never-vanishing).

(*) Ex: $\gamma(t) = (\cos t, \sin t)$



[Smooth + regular curve]

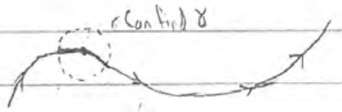
$\gamma(t) = (t, |t|)$



[Neither smooth nor regular]

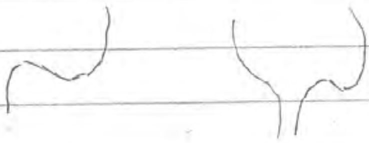
Def: Smooth Curves

A set $C \subseteq \mathbb{R}^n$ is a (smooth) curve if it is (locally) the image of a smooth regular parametrized curve $\gamma: (a,b) \to \mathbb{R}^n$, i.e.: $\forall\, x \in C\ \exists\ \gamma: I \to \mathbb{R}^n$ smooth & regular, $\varepsilon > 0$ s.t. $B_\varepsilon(x) \cap C \subseteq Im(\gamma)$.



(*) Ex:

Curves:



[Multiple components]

Smooth [2 γ's]

NOT Curves:

Not smooth

Can't be parametrized by 1 γ

Level Sets

9/29/24

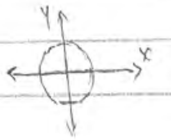
Lecture 2

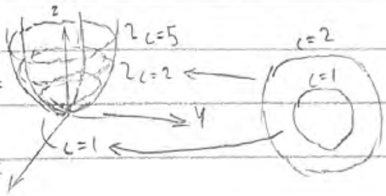
Def: Level Sets

Let f be a (smooth) function $f: \mathbb{R}^n \to \mathbb{R}$, $c \in \mathbb{R}$ $\to$ define the level set of f at height c as:

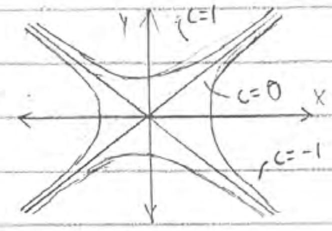
$$\boxed{f^{-1}(c) = \{x \in \mathbb{R}^n : f(x) = c\}} \subseteq \mathbb{R}^n$$

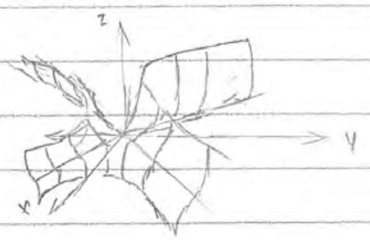
(*) Ex: $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = x^2 + y^2 \to f^{-1}(1) =$ 

$\operatorname{Im}(f) =$ 

(*) Ex: $g: \mathbb{R}^2 \to \mathbb{R}$

$g(x,y) = y^2 - x^2 \to$ Level sets: 



$\to$ (*) Notice: Many level sets are curves (but not all).

Critical Points

Def: Let $f: \mathbb{R}^n \to \mathbb{R}$ smooth:

$\to$ • $x_0 \in \mathbb{R}^n$ is called a critical point of f if $\nabla f(x_0) = 0$

• $c \in \mathbb{R}$ is called a critical value of f if $f(x_0) = c$ for some critical point $x_0 \in \mathbb{R}^n$.

Otherwise, call c a regular value.

$\to$ Thm: Let $f: \mathbb{R}^n \to \mathbb{R}$ smooth, and let $c \in \mathbb{R}$ be a regular value of f. Then $f^{-1}(c)$ [the level set of f at height c] is a (smooth) curve.

Note: γ is called a global parametrization of a curve C if $C = \operatorname{Im}(\gamma)$ (*) Equality: show $\supseteq$ both ways!

(*) Theorem Proof (Outline)

Via the implicit function theorem & smoothness of f:

- $\forall$ points $(x,y) \in f^{-1}(c)$: locally, $f^{-1}(c)$ "looks like" $y = g(x)$ or $x = g(y)$ for some smooth function $g: \mathbb{R}^{n-1} \to \mathbb{R}$
- In particular, $f^{-1}(c)$ is locally the graph of g (which is a smooth curve)

  $\Rightarrow f^{-1}(c)$ is a smooth curve [smoothness is a local property]

Smooth Vector Fields

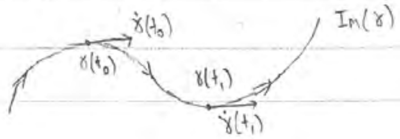
10/2/24

Lecture 3

Recall (Tangents to Curves)

Let $\gamma : (a,b) \to \mathbb{R}^n$ be a smooth parametrized curve:

- The vector $\dot{\gamma}(t)$ [called the velocity vector] is tangent to the curve $C = \mathrm{Im}(\gamma)$ at point $\gamma(t)$
- The speed at time t is given by $\|\dot{\gamma}(t)\|$



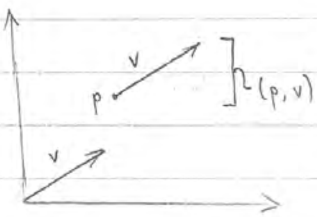
→ **Def:** Given a smooth parametrized curve $\gamma : (a,b) \to \mathbb{R}^n$, its arc length is given by:

$$\boxed{\ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\, dt}$$

Vector Fields

Notation:

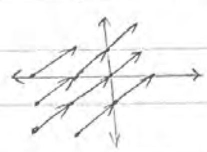
- Base-pointed vectors: Let $p, v \in \mathbb{R}^n$ → denote the vector v based at point p as (p,v); ex. $((1,2),(3,4))$
- Denote the vector space of vectors $\in \mathbb{R}^n$ based at p as $\mathbb{R}^n_p$
  - Base point unchanged under addition, scalar multiplication



Def: Let $U \subseteq \mathbb{R}^n$. A (smooth) vector field on U is a smooth function $X : U \to U \times \mathbb{R}^n$ such that $\forall\, p \in U$, $X(p) = (p,v)$ for some $v \in \mathbb{R}^n$.

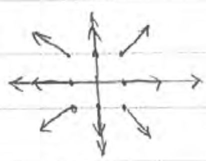
(*) Ex: $U = \mathbb{R}^2$, $X : U \to \mathbb{R}^2 \times \mathbb{R}^2$

(i)



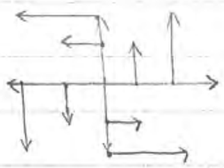
$X(x,y) = ((x,y),(1,1))$

(ii)



$X((x,y)) = ((x,y),(x,y))$

(iii)



$X(x,y) = ((x,y),(-y,x))$

Notice: The vectors in (iii) are tangent to the unit circle C; in fact, parametrizing C by $\gamma(t) = (\cos t, \sin t) \Rightarrow \dot{\gamma}(t) = (-\sin t, \cos t)$, $X(\gamma(t)) = (\gamma(t), (-\sin t, \cos t)) = (\gamma(t), \dot{\gamma}(t))$.

⟶ More generally: for any smooth parametrized curve γ (parametrizing a curve C), can get a tangent vector field $X : C \to C \times \mathbb{R}^2$ defined by $X(\gamma(t)) = (\gamma(t), \dot{\gamma}(t))$.

Integral Curves

10/2/24

Lecture 3

Integral Curves

Tangent vector fields associated a vector field with any parametrized curve $\gamma:(a,b)\to\mathbb{R}^2$; can do "in reverse" with integral curves:

Def: Let $X: U \to U\times\mathbb{R}^2$ be a (smooth) vector field. An integral curve of X is a parametrization $\gamma:(a,b)\to U$ s.t. $X(\gamma(t)) = (\gamma(t), \gamma'(t))\ \forall\ t\in(a,b)$.

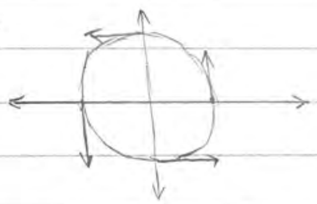
Thm: Given $X: U\to U\times\mathbb{R}^2$ and $p\in U$, $\exists$ a "maximal" integral curve $\gamma:(a,b)\to U$ [(a,b) potentially unbounded] s.t. (assuming WLOG that $a<0<b$):

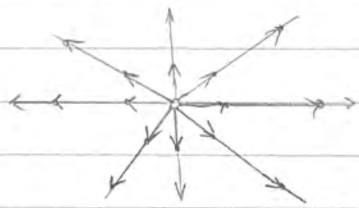
(i) $\gamma(0)=p$

(ii) If $\beta: I\to U$ is another integral curve of X with $\beta(0)=p$, then $I\subseteq(a,b)$ and $\beta(t)=\gamma(t)\ \forall\ t\in I$.

$\to$ γ is called the maximal integral curve of X through p.

(*) Ex:





(i) $X: (x,y)\mapsto((x,y),(-y,x))$

[All max. ICs are circles about the origin]

(*) Pf: $X(\gamma(t)) = (\gamma(t), (-\gamma_2(t), \gamma_1(t)))$

$\to \frac{d}{dt}\gamma_2 = \gamma_1$; $\frac{d}{dt}\gamma_1 = -\gamma_2$

$\to \gamma_1 = \sin t$, $\gamma_2 = \cos t$

(ii) $X: (x,y)\to((x,y),(x,y))$

[All max ICs are lines from the origin]

(*) Pf: $X(\gamma(t)) = (\gamma(t), \gamma(t))$

$\to \frac{d}{dt}\gamma_1 = \gamma_1$; $\frac{d}{dt}\gamma_2 = \gamma_2$

$\to \gamma_1 = v_1 e^t$, $\gamma_2 = v_2 e^t$, $\gamma(t) = (v_1, v_2)e^t$

(*) Proof Outline of Theorem

γ integral curve through $p \Rightarrow X(x_1,\dots,x_n) = ((x_1,\dots,x_n),(X_1(x_1,\dots,x_n),\dots,X_n(x_1,\dots,x_n)))$

$\Rightarrow \frac{d\gamma_1}{dt}(t) = X_1(\gamma_1(t),\dots,\gamma_n(t))$, $\frac{d\gamma_2}{dt}(t) =$ etc.

$\to$ Form system of ordinary diff. eqs [ODE], use initial condition $\gamma(0)=p$ to solve for $\gamma(p)$

- X smooth $\Rightarrow$ solution exists & is unique

10/4/24

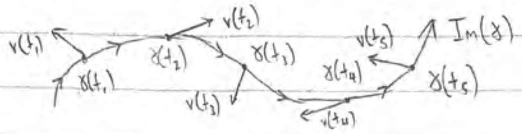
Lecture 4

+Lecture 5

Gradient Vector Fields

Vector Fields Along Curves

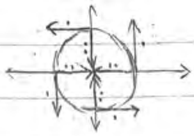
Def: Given a parametrized curve $\gamma:(a,b)\to\mathbb{R}^n$, a vector field along γ is a function $X:(a,b)\to\mathbb{R}^n\times\mathbb{R}^n$ s.t. $X(t)=(\gamma(t),v(t))$ [for some smooth function $v:(a,b)\to\mathbb{R}^n$].



$\to$ Can take derivatives of X w.r.t. t: $\dot{X}(t)=(\gamma(t),\dot{v}(t))$ ~ [note: base point unchanged]

(*) Ex: Velocity vector field $X(t)=(\gamma(t),\dot\gamma(t))$

$\xrightarrow{d/dt}$ Acceleration VF $\dot{X}(t)=(\gamma(t),\ddot\gamma(t))$

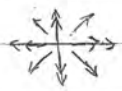


Gradient Vector Fields

Def: Given a smooth function $f:\mathbb{R}^n\to\mathbb{R}$, the gradient vector field ∇f of f is defined by ($\forall\, p\in\mathbb{R}^n$):

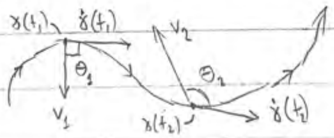
$$\boxed{\nabla f(p)=\left(p,\left(\tfrac{\partial f}{\partial x_1}(p),\dots,\tfrac{\partial f}{\partial x_n}(p)\right)\right)}$$

(*) Ex: $f(x,y)=x^2+y^2 \Rightarrow \nabla f(x,y)=((x,y),(2x,2y))$



Recall: $v,w\in\mathbb{R}^n$ nonzero vectors $\Rightarrow v\cdot w=\|v\|\cdot\|w\|\cdot\cos(\theta)$ $\Rightarrow \theta=\cos^{-1}\left(\frac{v\cdot w}{\|v\|\cdot\|w\|}\right)$

$\to$ Def: Let $\gamma:(a,b)\to\mathbb{R}$ be a smooth & regular parametrized curve, and let $(\gamma(t),v\neq 0)$ be a vector based at $\gamma(t)$ $\Rightarrow$ define the angle between $(\gamma(t),v)$ and γ as the angle θ between $\dot\gamma(t)$ and v. In particular, say that $(\gamma(t),v)$ is orthogonal to γ if $v\cdot\dot\gamma(t)=0$.



Prop: Let $f:\mathbb{R}^2\to\mathbb{R}$ smooth, $c\in\mathbb{R}$ a regular value of f, and γ a local parametrization of the level set $f^{-1}(c)$ [near some point p]. Then $\forall\, t\in(a,b)$, $\nabla f(\gamma(t))$ is orthogonal to $\dot\gamma(t)$.

(*) Proof of Prop: Let $\gamma(t)=(\gamma_1(t),\gamma_2(t))$. Since γ parametrizes $f^{-1}(c)$, then $f(\gamma(t))=c\ \forall\, t\in(a,b)$.

$\rightsquigarrow 0=\frac{d}{dt}f(\gamma(t))=\frac{\partial f}{\partial x}f(\gamma(t))\frac{d\gamma_1}{dt}(t)+\frac{\partial f}{\partial y}(\gamma(t))\frac{d\gamma_2}{dt}(t)=\nabla f(\gamma(t))\cdot\dot\gamma(t).$

Orientation & Reparametrization

10/7/24
Lecture 5 (cont.)

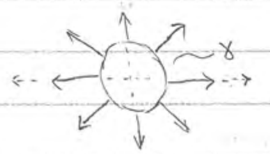
Orientation

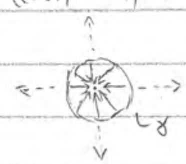
Def: Let $\gamma:(a,b)\to\mathbb{R}^n$ be a parametrized curve, and let X be a vector field along γ. $[X(t)=(\gamma(t),v(t))]$
(i) X is called a unit vector field if $\|v(t)\|=1\ \forall\ t\in(a,b)$.
(ii) X is called a normal vector field if $v(t)$ is orthogonal to $\dot\gamma(t)\ \forall\ t\in(a,b)$.

(*) Ex: Let $f:\mathbb{R}^2\to\mathbb{R}$, $c\in\mathbb{R}$ regular value $\Rightarrow$ for any parametrization γ of level set $f^{-1}(c)$, $\nabla f(\gamma(t))$ is a normal vector field to γ.

→ **Def:** Given a plane curve C [i.e. $C\subseteq\mathbb{R}^2$], an orientation of C is a choice of unit normal vector field along C.

(*) Ex: $\gamma(t)=(\cos t,\sin t)$
→ (i) $N_1=((\cos t,\sin t),(\cos t,\sin t))$ (ii) $N_2=((\cos t,\sin t),(-\cos t,-\sin t))$



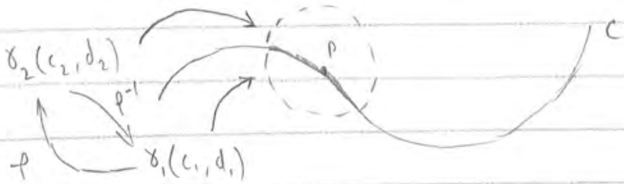


(*) Notice: Any plane curve has at least 2 orientations $[N\ \&\ -N]$.

Reparametrizations

Q: What can we say about 2 different parametrizations γ_1,γ_2 of the same underlying curve?

⇒ **Prop:** Let $\gamma_1:(a_1,b_1)\to\mathbb{R}^n$, $\gamma_2:(a_2,b_2)\to\mathbb{R}^n$ be smooth + regular parametrizations of the same curve. Then for each $t\in(a_1,b_1)$, $\exists$ intervals $(c_1,d_1)\subseteq(a_1,b_1)$ and $(c_2,d_2)\subseteq(a_2,b_2)$ [with $t\in(c_1,d_1)$] and smooth bijection (with smooth inverse) $\rho:(c_1,d_1)\to(c_2,d_2)$ s.t.:
$$\boxed{\gamma_1=\gamma_2\circ\rho}\quad \text{on } (c_1,d_1).$$



(*) Note: A smooth bijection w/ smooth inverse also called a diffeomorphism.

Tangent Spaces

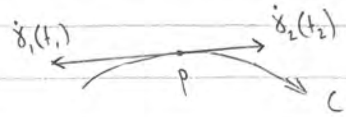
10/9/24
Lecture 6

Reparametrizations (cont.)

Corollaries of Prop:

(1) If γ_1, γ_2 are smooth + regular parametrizations of the same curve near $p = \gamma_1(t_1) = \gamma_2(t_2)$, then $\exists\ \lambda \in \mathbb{R}, \lambda \neq 0$ s.t.:

$$\boxed{\dot{\gamma}_1(t_1) = \lambda \dot{\gamma}_2(t_2)}$$



(2) Let γ_1, γ_2 as in (1) near $p \in \mathbb{R}$ → a vector (p, v) based at p is orthogonal to $\dot{\gamma}_1(t_1)$ iff (p, v) is orthogonal $\dot{\gamma}_2(t_2)$ [i.e. orthogonality is independent of parametrization].

(*) Proofs: (1) $\gamma_1 = \gamma_2 \circ \rho$ (by Prop) → $\dot{\gamma}_1(t_1) = \dot{\gamma}_2(\overbrace{\rho(t_1)}^{t_2}) \cdot \underbrace{\rho'(t_1)}_{\lambda}$;

ρ^{-1} smooth $\Rightarrow \rho'(t_1)\ [= \lambda] \neq 0$

(2) Via (1)

Tangent Spaces

Def: Let $C \subseteq \mathbb{R}^n$ be a (smooth) curve, and let $p \in C$. A vector (p, v) based at p is said to be tangent to C if $v = \dot{\gamma}(t)$ for some (smooth, not necessarily regular) parametrization γ of C with $\gamma(t) = p$.

→ The tangent space of C at p is defined by:

$$\boxed{T_p C = \{(p, v) : (p, v) \text{ is tangent to } C\}}$$

Prop: Let C be a smooth curve and γ a (smooth, regular) parametrization of C with $\gamma(t) = p$

$$\Rightarrow \boxed{T_p C = \operatorname{span}\{(p, \dot{\gamma}(t))\}}$$

In particular, $T_p C$ is a 1D subspace of $\mathbb{R}^n_p$.

(*) Proof:

(i) $\supseteq$ is trivial

(ii) $\subseteq$ via Prop (Reparametrization) - reparametrize by $\beta(t) = \gamma(\lambda(t - t_0) + t_0)$ for any $\lambda \in \mathbb{R}$

More on Orientation

10/11/24

Lecture 7

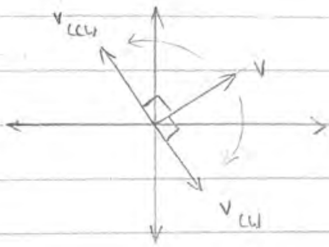
More on Orientation

Notice: If γ is a (local) parametrization of $f^{-1}(c)$ for some $f: \mathbb{R}^2 \to \mathbb{R}$ and regular value $c \in \mathbb{R}$ [$f^{-1}(c)$ a level set], then $\nabla f(\gamma(t)) \perp \gamma'(t)$ $\forall t \in (a,b)$

$\to$ • $\nabla f / \|\nabla f\|$ is a unit normal vector field (i.e. an orientation)

• For any $t \in (a,b)$: $T_{\gamma(t)} C = \{(\gamma(t), v) : v \cdot \nabla f(\gamma(t)) = 0\}$

+ Recall: If $v = (v_1, v_2) \in \mathbb{R}^2$, can either rotate v CCW 90° ($v \mapsto (-v_2, v_1)$) or CW 90° ($v \mapsto (v_2, v_1)$) to obtain vectors orthogonal to v



$\to$ • For a parametrized curve γ, can find orientation:

$$\boxed{N(t) = \left(\gamma(t), \frac{\text{rot}(\dot\gamma(t))}{\|\dot\gamma(t)\|}\right)}$$

$\Rightarrow$ 2 ways to obtain orientation:

(1) If $C = f^{-1}(c)$ a level set, take $\nabla f / \|\nabla f\|$

(2) If γ parametrizes C, can find orientation $N(t) = \left(\gamma(t), \text{rot}(\dot\gamma(t)) / \|\dot\gamma(t)\|\right)$

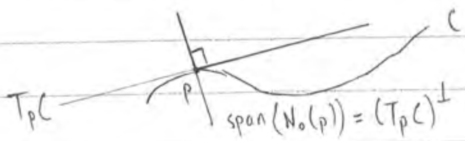
> Prop: Let $C \subseteq \mathbb{R}^2$ be a plane curve. If C is connected [i.e. has only 1 connected component], then C has exactly 2 orientations.

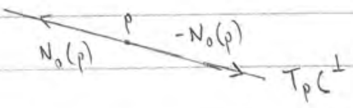
(*) Proof:

(i) Existence: From above, have that C has at least 1 orientation N

$\Rightarrow$ C has at least 2 orientations (N & $-N$).

(ii) "At most 2": Suppose N_1 is a 3rd orientation (besides $N_0, -N_0$). Then $\forall p \in C$, $\lambda \in \mathbb{R}$, $N_1(p) \perp T_pC$. In particular, $\dim T_pC = 1 \Rightarrow \dim (T_pC)^\perp = 1 \Rightarrow N_1(p) = \lambda(p) N_0(p)$.





Know $|\lambda(p)| = 1$ $\forall p \in C$ [$\lambda(p) = \pm 1$] + by smoothness of N_1, λ can't jump (within a connected component)

$\Rightarrow \lambda = \pm 1$ constant $\Rightarrow$ by connectedness of C, $N_1(p) = N_0(p)$ or $-N_0(p)$ $\forall p \in \mathbb{R}$. ∎

Orientation & Direction

10/11/24

Lecture 7

+Lecture 8

Orientation & Direction

For any parametrized curve γ, can find an associated tangent direction: $\forall\ t \in \mathbb{R}$, assign unit vector

$$\boxed{\frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}} \quad \text{[Tangent direction]}$$

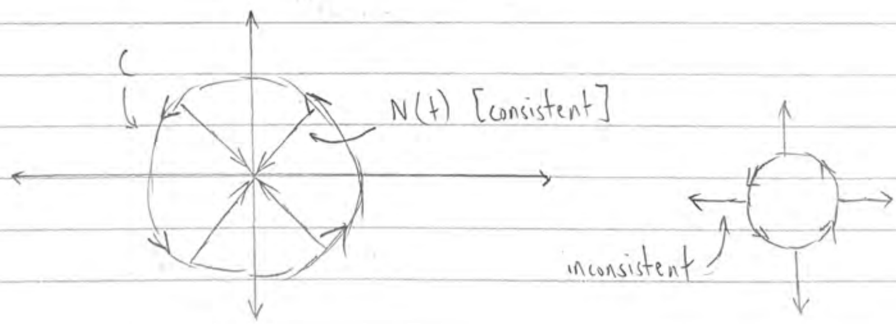
→ Def: For a parametrized plane curve $\gamma : (a,b) \to \mathbb{R}^2$, say that an orientation N is consistent with the parametrization if $\forall\ t \in (a,b)$, $N(t)$ is the vector obtained by rotating the tangent dir. $\dot{\gamma}(t)/\|\dot{\gamma}(t)\|$ $\pi/2$ counterclockwise, i.e.:

$$\boxed{N(t) = \frac{(-\dot{\gamma}_2(t), \dot{\gamma}_1(t))}{\|\dot{\gamma}(t)\|}}$$

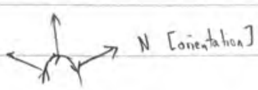
(*) Equivalently: $N(t) = (\gamma(t), n(t))$ is consistent with $\gamma(t)$ if:

$$\boxed{\det\begin{pmatrix} \dot{\gamma}_1(t) & \dot{\gamma}_2(t) \\ n_1(t) & n_2(t) \end{pmatrix} > 0}$$

(*) Ex: Circle parametrized by $\gamma(t) = (\cos t, \sin t)$



(*) Curvature (Intuition)



- Shallow curve → low curvature $|\kappa|$
- Orientation in dir. of curve → $\kappa > 0$ ("inward")

- Sharp curve → high $|\kappa|$
- Orientation in opposite dir. of turn → $\kappa < 0$ ("outward")

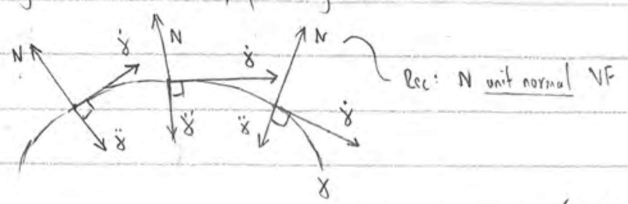
Curvature

10/14/24
Lecture 8
(cont.)

Curvature

Let $\gamma(t)$ be a unit-speed parametrization of a curve $C \subseteq \mathbb{R}^2 \to$ have that:

- $\dot{\gamma}(t)$ is tangent to C w/ magnitude 1
- $\|\dot{\gamma}\|$ constant $\Rightarrow \dot{\gamma}(t) \perp \ddot{\gamma}(t) \; \forall \, t \in I \Rightarrow \ddot{\gamma} \parallel N \; \forall \, t \in I$
  - $\ddot{\gamma}$ is a vector orthogonal to the curve, pointing "inward"



$\to$ Can define curvature for unit-speed γ, orientation N: $\underline{\kappa = \ddot{\gamma}(t) \cdot N(\gamma(t))} = \begin{cases} +\|\ddot{\gamma}\| & \ddot{\gamma}, N \text{ same dir} \\ -\|\ddot{\gamma}\| & \ddot{\gamma}, N \text{ opp. dirs.} \end{cases}$

More generally:

> **Curvature**
>
> Def: Let $C \subseteq \mathbb{R}$ be a simple [i.e. non-self-intersecting] plane curve with orientation N.
> Then the curvature of C at point $p \in C$ is given by:
>
> $$\boxed{\kappa(p) = \frac{\ddot{\gamma}(t_0) \cdot N(\gamma(t_0))}{\|\dot{\gamma}(t_0)\|^2}}$$
>
> where γ is *any* smooth, regular parametrization of C with $\gamma(t_0) = p$.

Thm: This is well-defined, i.e. $\kappa(p)$ is independent of choice of parametrization.

(*) Pf: Let β, γ be parametrizations of C with $p = \gamma(t_0) = \beta(t_1)$.
$\to$ Rec: $\exists$ a reparametrization $\phi: I \to J$ s.t. $\beta = \gamma \circ \phi$ and $\phi(t_1) = t_0$.

Computing curvature for β at point p: , rec: $\dot{\gamma}(t_0) \cdot N(\gamma(t_0)) = 0$

$$\frac{\ddot{\beta}(t_1) \cdot N(\beta(t_1))}{\|\dot{\beta}(t_1)\|^2} = \frac{[\ddot{\gamma}(\phi(t_1))\dot{\phi}^2(t_1) + \dot{\gamma}(\phi(t_1))\ddot{\phi}(t_1)] \cdot N(\gamma(\phi(t_1)))}{\|\dot{\gamma}(\phi(t_1))\dot{\phi}(t_1)\|^2} = \frac{\ddot{\gamma}(t_0) \cdot N(\gamma(t_0))}{\|\dot{\gamma}(t_0)\|^2} \quad \blacksquare$$

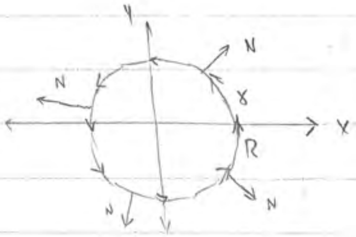
10/14/24
Lecture 8
+Lecture 9

Circles of Curvature

(*) Ex: Curvature

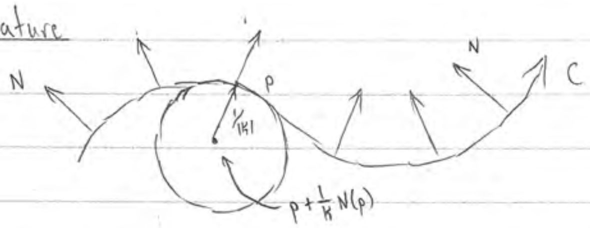
$\gamma = (R\cos t, R\sin t)$,

$N(\gamma(t)) = \dfrac{(x,y)}{\sqrt{x^2+y^2}} \longrightarrow \kappa = \dfrac{\ddot{\gamma}(t)\cdot N(\gamma(t))}{\|\dot{\gamma}(t)\|^2} = \dfrac{(-R\cos t, -R\sin t)\cdot(\cos t, \sin t)}{R^2} = \boxed{-1/R}$



(*) $N(\gamma(t)) = \dfrac{(-x,-y)}{\sim} \rightarrow \kappa = 1/R$

Circle of Curvature



Informally: the circle of curvature at point $p \in C$ is the circle best approximating C in some neighborhood of p

- Is unique for any point $p \in C$
- Say that it has contact [with C] of order 2 [vs. tangent lines - order 1]

> **Circle of Curvature**
>
> Def: Let $C \subseteq \mathbb{R}^2$ be a curve with orientation N, and let $p \in C$ s.t. $\kappa(p) \neq 0$; then the circle of curvature/osculating circle of C at p is the circle $C_c \subseteq \mathbb{R}^2$ with:
>
> (i) Center: $p + \frac{1}{\kappa(p)} N(p)$ [center of curvature]
>
> (ii) Radius: $1/|\kappa(p)|$ [radius of curvature]
>
> (iii) Orientation: N_c s.t. $N_c(p) = N(p)$

→ Properties

(i) C_c is tangent to C at point p

(ii) C_c, C have the same curvature at p

(iii) If C is a circle, then $C_c = C$

(*) C_c not generally defined where $\kappa = 0$; in some cases, may be defined as a straight line $[R = \infty]$

Frenet-Serret Formulas [2D]

10/18/24

Lecture 10

Frenet-Serret Formulas for $\mathbb{R}^2$

Def: Let $C \subseteq \mathbb{R}^2$ be a curve with orientation N, $\gamma(t)$ a unit-speed parametrization of C

$\longrightarrow$ define the unit tangent vector to γ at time t by:

$$\boxed{T(t) = \big(\gamma(t), \dot{\gamma}(t)\big)}$$

(*) Notation: Denote $N(t)$ as $N(t) = \big(\gamma(t), n(t)\big)$ for some function $n(t)$

By definition: (i) $\kappa(\gamma(t)) = \ddot{\gamma}(t) \cdot N(\gamma(t))$

(ii) $\dot{T}(t) = (\gamma(t), \ddot{\gamma}(t))$, and $\ddot{\gamma} \perp \dot{\gamma}$

$\Rightarrow \dot{T} \perp T$; $\dot{T} \parallel N(t)$, i.e. $\dot{T}(t) = \lambda(t) N(t)$ [$\lambda : I \to \mathbb{R}$ scalar]

$\Rightarrow \dot{T}(t) \cdot N(t) = \lambda(t) \underbrace{N(t) \cdot N(t)}_{=1} \Rightarrow \kappa(\gamma(t)) = \lambda(t) \Rightarrow \boxed{\dot{T} = \kappa N} \ \forall t \in I$ (1)

Note: $T \cdot N = 0 \xrightarrow{d/dt} \dot{T} \cdot N + T \cdot \dot{N} = 0 \Rightarrow \dot{N} \cdot T = -\dot{T} \cdot N = -\kappa$

$\|N\| = 1$ [const.] $\Rightarrow \dot{N} \perp N \Rightarrow \dot{N} \parallel T$; $\dot{N}(t) = c(t) T(t)$ [$c : I \to \mathbb{R}$ scalar]

$\Rightarrow \underbrace{\dot{N} \cdot T}_{-\kappa} = c \underbrace{T \cdot T}_{1} = c \Rightarrow \boxed{\dot{N} = -\kappa T} \ \forall t \in I$ (2)

Frenet-Serret Formulas [2D]

$$\boxed{\begin{aligned} \dot{T} &= \kappa N \quad (1) \\ \dot{N} &= -\kappa T \quad (2) \end{aligned}} \iff \boxed{\begin{bmatrix} \dot{T} \\ \dot{N} \end{bmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{bmatrix} T \\ N \end{bmatrix}}$$

(*) Also called the Frenet-Serret apparatus; $\underbrace{T, N}_{\text{orthonormal basis of } \mathbb{R}^2}$ as the F-S frame

$\hookrightarrow$ Uniqueness of ODEs $\Rightarrow$ diff. eq. has a unique solution

(*) Can solve matrix equation sometimes [but not always]

Frenet-Serret Formulas [3D]

10/18/24

Lecture 10 + Lecture 11

Frenet-Serret Formulas for $\mathbb{R}^3$

Let γ be a unit-speed parametrization [smooth] of a curve $C \subseteq \mathbb{R}^3$ [$\gamma : I \to \mathbb{R}^3$]

→ similar to $\mathbb{R}^2$: define unit tangent vector at $\gamma(t)$ by $T(t) = (\gamma(t), \dot\gamma(t))$. [$T \perp \dot T$]

In $\mathbb{R}^3$: C has infinitely many possible unit normals at any $p \in C$ [∞-ly many orientations]

⟶ define the principal unit normal vector of $\gamma(t)$ at time t by:

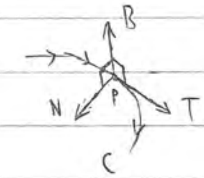
$$\boxed{N(\gamma(t)) = \left(\gamma(t), \frac{\ddot\gamma(t)}{\|\ddot\gamma(t)\|}\right)}$$ ~ (*) Note: Points in the dir. the curve is turning

+ define curvature of $\gamma(t)$ by:

$$\boxed{\kappa(t) = \|\ddot\gamma(t)\|} \longrightarrow \overset{(1)}{\boxed{\dot T = \kappa N}}$$ [same as $\mathbb{R}^2$]

(*) Locally, around $p \in C$, C (+ osculating circle C_0) approximately lies in the plane spanned by T & N [called the osculating plane]

In $\mathbb{R}^3$ → define a 3^{rd} vector [perpendicular to T, N]: $\boxed{B = T \times N}$
(orthogonal to osculating plane)



→ obtain Frenet frame [$\mathbb{R}^3$]: T, N, B [ON basis for $\mathbb{R}^3$]

Regarding B: (i) B unit VF $\Rightarrow \dot B \perp B$

(ii) $B \cdot T = 0 \xrightarrow{d/dt} \dot B \cdot T + \underbrace{B \cdot \dot T}_{B \cdot \kappa N = 0} = 0 \Rightarrow \dot B \cdot T = 0$ ($\dot B \perp T$)

$\Rightarrow \dot B \parallel N$, $\dot B(t) = c(t) N(t)$

→ Define torsion of $\gamma(t)$ as $\tau(t)$ s.t. $\overset{(2)}{\boxed{\dot B = -\tau N}} \Leftrightarrow \boxed{\tau(t) = -\dot B \cdot N}$ ~ also called "2^{nd} generalized curvature"

Can express $\dot N$ as $\overset{(3)}{\boxed{\dot N = -\kappa T + \tau B}}$

Frenet-Serret Formulas [3D]

10/21/24

Lecture 11

Frenet-Serret Formulas [3D]

(i)
$$\boxed{\begin{aligned}&(1)\ \dot{T} = \kappa N\\ &(2)\ \dot{N} = -\kappa T + \tau B\\ &(3)\ \dot{B} = -\tau N\end{aligned}}$$

$\iff$

(ii)
$$\boxed{\begin{bmatrix}\dot{T}\\ \dot{N}\\ \dot{B}\end{bmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{bmatrix}T\\ N\\ B\end{bmatrix}} \in \mathbb{R}^{3\times 3}$$

(ii) is an ODE $\Rightarrow$ a solution exists and is unique

In particular: given $\kappa(t)$, $\tau(t)$, and initial values $T(0)$, $B(0)$, $N(0)$:
- $\rightarrow$ by existence & uniqueness: can solve ODE to find $T(t)$, $B(t)$, $N(t)$ $\forall\, t \in I$
- $\rightarrow$ if $\gamma(0)$ is known, can integrate T from $\gamma(0)$ to find $\gamma(t)$ $\forall\, t \in I$

$\Rightarrow$ Consequence: Any curve C is completely determined by its initial values & curvature + torsion.
- For any 2 curves C_1, C_2 with same κ, τ: C_1, C_2 are identical under rigid motion [rotation + translation]

(*) Can generalize results to $\mathbb{R}^n$:
- $\mathbb{R}^2$: Curve determined by just curvature κ
- $\mathbb{R}^3$: Curve determined by just curvature κ, torsion τ
- $\mathbb{R}^n$: Curve determined by $(n-1)$-many generalized curvatures

(*) Observe: Frenet-Serret matrix is skew-symmetric $(M = -M^T)$ $\Rightarrow$ gives orthogonality of T, N, B

10/28/24

Misc. Notes

(*) Misc. Notes

(*) <u>Note</u>: The graph of any function $f: \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F: \mathbb{R}^{n+1} \to \mathbb{R}$.

(*) <u>Circle of Curvature (Alt. Defn.)</u>

Let $C \subseteq \mathbb{R}^n$ a curve, and let $p \in C$ s.t. $\kappa(p) \neq 0$. Then the <u>circle of curvature of C at p</u> is the unique oriented circle O satisfying:

(i) O is tangent to C at p $[T_pC = T_pO]$

(ii) O is oriented consistently with C (i.e. $N_C(p) = N_O(p)$)

(iii) $\nabla_v N_O = \nabla_v N_C \ \forall \ v \in T_pC = T_pO$ $(\nabla_v N = \nabla N \cdot v)$

Intro to Surfaces

10/23/24

Lecture 12

Informally: a parametrized surface in $\mathbb{R}^n$ is given by a [smooth] function $\Phi: U \to \mathbb{R}^n$, where U is a (potentially unbounded) open set $U \subseteq \mathbb{R}^2$.

(*) At every point $p \in Im(\Phi)$, want to be able to define a tangent plane to $Im(\Phi)$ at p

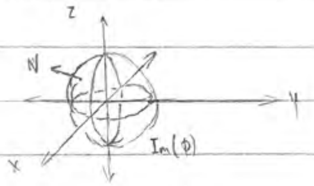
$\to$ require $\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}$ to be nonzero and linearly independent at all points $(u,v) \in U$

[Alt: Require Jacobian to be rank 2]

(*) Ex: Sphere $\Phi(u,v) = (R\cos(u)\cos(v), R\cos(u)\sin(v), R\sin(u))$

$\rightsquigarrow$ not a PS: $\frac{\partial \Phi}{\partial v}$ vanishes at $v = \pm \pi/2$

- Rmk: No sphere can be covered in a single parametrization



Def: A set $S \subseteq \mathbb{R}^n$ $[n \geq 3]$ is a surface in $\mathbb{R}^n$ if $\forall\, p \in S$, $\exists$ open neighborhood U of p in S (i.e. $U = B_\varepsilon(p) \cap S$ for some $\varepsilon > 0$) and function $\Phi: D_r^2 \to U$ smooth satisfying:

(i) Φ is a local parametrization of "patch" U of S at point p

(ii) The Jacobian of Φ has full rank $[=2]$ at every point $x \in D_r^2$

(*) Notation: Write "D_r^2" to indicate the open disk of radius r in $\mathbb{R}^2$: $D_r^2 = \{(x,y) : \sqrt{x^2+y^2} \leq r\}$

3 common ways to obtain surfaces:

1. As the image of a local parametrization:

   Let $r > 0$ and $\Phi: D_r^2 \to \mathbb{R}^n$ smooth 1:1 w/ $Jac(\Phi)$ full rank $\to$ $S = im(\Phi)$ is a surface in $\mathbb{R}^n$.

2. As the graph of a function $f: \mathbb{R}^2 \to \mathbb{R}$:

   Let $f: D_r^2 \to \mathbb{R}$ smooth & define $Graph(f) = \{(x,y,z) : (x,y) \in D_r^2,\ z = f(x,y)\}$

   $\to$ $Graph(f)$ is a surface with a single local parametrization $\Phi(u,v) = (u, v, f(u,v))$, $\Phi: D_r^2 \to \mathbb{R}^3$

3. As the level set of a smooth function $f: \mathbb{R}^3 \to \mathbb{R}$:

   Let $f: \mathbb{R}^3 \to \mathbb{R}$ smooth, $c \in \mathbb{R}$ a regular value of f — (*) Proof via implicit function theorem

   $\to$ $S = f^{-1}(c)$ is a surface (or a union of disjoint surfaces)

(*) Note: Surfaces are, more generally, 2-manifolds in $\mathbb{R}^n$

- Can generalize: via local parametrizations $\Phi: D_n^r \to \mathbb{R}^n$ for n-d manifolds

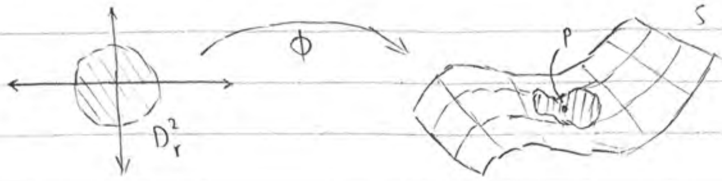
Intro to Surfaces (cont.)

10/28/24
Lecture 13
+ Lecture 14

(*) Surfaces (Note)

(i) Full-rank req. for $\mathrm{Jac}(\Phi)$ analogous to regularity requirement for curves

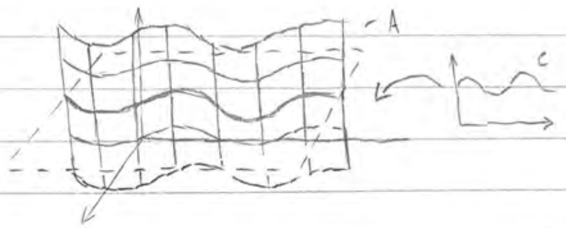
(ii) Locally, $S \subseteq \mathbb{R}^n$ smooth surface has:



Generalized Cylinders

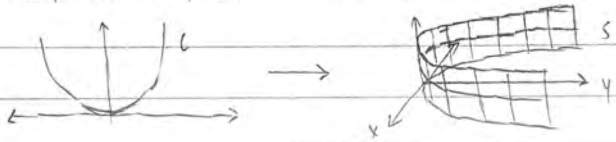
Def: Let $C \subseteq \mathbb{R}^2$ be a plane curve. Then the cylinder over C is the set $A \subseteq \mathbb{R}^3$ given by:

$$\boxed{A = \{(x,y,z) : (x,y) \in C\}}$$



→ Fact: If C is a smooth curve, then the cylinder over C is a smooth surface.

(*) Ex: Cylinder over $y = x^2$:



Surfaces of Revolution

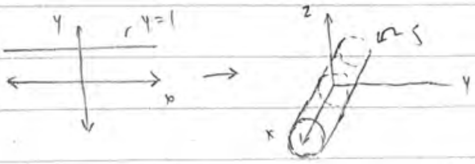
Def: Let $C \subseteq \mathbb{R}^2$ be a curve lying entirely above the x-axis $[y > 0 \ \forall (x,y) \in C]$; then the surface of revolution S is defined by:

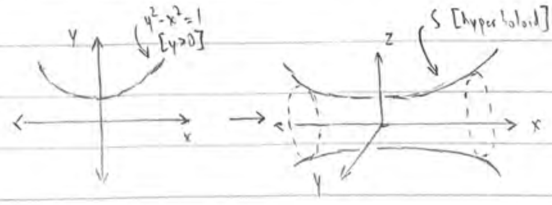
$$\boxed{S = \{(x,y,z) : (x, \sqrt{y^2+z^2}) \in C\}}$$

(*) Ex: $C = f^{-1}(c)$ for some $f: \mathbb{R}^2 \to \mathbb{R}$, regular val. $c \in \mathbb{R}$

→ $S = g^{-1}(c)$, $g(x,y,z) = f(x, \sqrt{y^2+z^2})$

- Obs: c is a regular value for g (i.e. c is smooth)
  [Proof via chain rule, using that $y > 0$]
- S corresponds to the revolution of C about the x-axis





(*) Revolving w/ pts. $y \leq 0$ results in a non-smooth $A \subseteq \mathbb{R}^3$

Tangent Spaces to Surfaces

10/30/24

Lecture 14 (cont.)

Tangent Spaces to Surfaces

Def: Let $S \subseteq \mathbb{R}^3$ be a smooth surface, and let $p \in S$. A vector (p,v) based at p is tangent to S if $\exists$ a smooth parametrized curve lying on S (i.e. $\gamma : (a,b) \to S$) such that for some $t_0 \in (a,b)$:

$$\text{(i) } \gamma(t_0) = p \quad \text{and} \quad \text{(ii) } \dot{\gamma}(t_0) = v$$

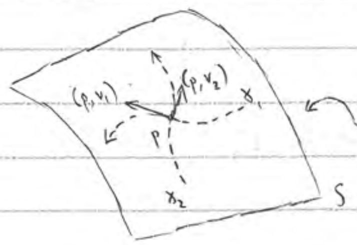
→ **Def:** The tangent space of S at p is the set $T_pS \subseteq \mathbb{R}^3_p$ defined by:

$$\boxed{T_pS = \{(p,v) : (p,v) \text{ is tangent to } S\}}$$

(*) (p,v) tangent to S

$\Updownarrow$

$p = \gamma(t_0)$, $v = \dot{\gamma}(t_0)$



tangent vectors to S at p (visualization)

Prop: For any surface $S \subseteq \mathbb{R}^3$ and $p \in S$, T_pS is a vector space with $\dim(T_pS) = 2$.

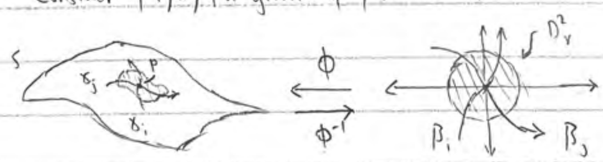
(*) **Proof:** We claim that $T_pS = \operatorname{span}\{(p, \dot{\gamma}(t_0)) \mid \gamma : (a,b) \to S,\ \gamma(t_0) = p\}$.

(i) $\subseteq$: Obvious

(ii) $\supseteq$: Let $(p,v) \in \operatorname{span}\{\ldots\}$ with $v = \lambda_1 \dot{\gamma}_1(t_1) + \ldots + \lambda_n \dot{\gamma}_n(t_n)$, where $\gamma_1, \ldots, \gamma_n$ are PCs $\gamma_i : (a_i, b_i) \to S$ satisfying $\gamma_i(t_i) = p$.

By definition of a surface, we know that $\exists\ r > 0$, $\Phi : D^2_r \to \mathbb{R}^3$ s.t. Φ parametrizes S near p and $\Phi(0,0) = p$.

$\longrightarrow$ Consider $\beta_1, \ldots, \beta_n$ given by $\beta_i = \Phi^{-1} \gamma_i$:



Looking at $\tilde{v} = \lambda_1 \dot{\beta}_1(t_1) + \ldots + \lambda_n \dot{\beta}_n(t_n)$, we can define a curve $\alpha : (a,b) \to D^2_r$ with $\alpha(t_0) = (0,0)$, $\dot{\alpha}(t_0) = \tilde{v}$

$\longrightarrow$ Then $\gamma := \Phi \circ \alpha$ is a curve satisfying $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$

$\longrightarrow$ $(p,v) = \lambda_1(p,v_1) + \ldots + \lambda_n(p,v_n) \in T_pS$.

[To see that $\dim(T_pS) = 2$, we observe that $\dim(D^2_r) = 2 \Rightarrow$ set of $\tilde{v}$'s spanned by 2 vectors.] ▨

Orientations for Surfaces

11/1/24

Lecture 15

(*) Recall (General chain rule): $\text{Jac}(f \circ g) = \text{Jac}(f)\,\text{Jac}(g)$

→ Notice: Let $S \subseteq \mathbb{R}^n$ a surface, $p \in S$, and $\phi : D_r^2 \to S$ a parametrization of S at p.

Let $\gamma : (a,b) \to D_r^2$ s.t. $\phi \circ \gamma$ is defined $(a,b) \to \mathbb{R}^n$; then:

(*) Notation: $(\phi \dot{\circ} \gamma)$

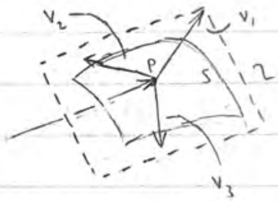
$$\frac{d}{dt}(\phi \circ \gamma)(t_0) = \text{Jac}(\phi)\,\dot{\gamma}(t_0)$$
$$= \text{Jac}(\phi)\left[c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}\right] \text{ for some } c_1, c_2 \in \mathbb{R}$$

⇒ T_pS is spanned by 2 vectors: (1) $\text{Jac}(\phi)\begin{pmatrix}1\\0\end{pmatrix} = \left(\frac{\partial \phi_1}{\partial x}(p) \; \frac{\partial \phi_2}{\partial x}(p) \; \dots \; \frac{\partial \phi_n}{\partial x}(p)\right)^T$

(2) $\text{Jac}(\phi)\begin{pmatrix}0\\1\end{pmatrix} = \left(\frac{\partial \phi_1}{\partial y}(p) \; \frac{\partial \phi_2}{\partial y}(p) \; \dots \; \frac{\partial \phi_n}{\partial y}(p)\right)^T$

→ Def: Let $S \subseteq \mathbb{R}^n$ a surface, and let $p \in S$.
Then the tangent plane of S at p is:

$$\boxed{P = \{p + v : (p,v) \in T_pS\}}$$



Prop. Let $S = g^{-1}(c)$ be a level set [$g : \mathbb{R}^3 \to \mathbb{R}$, c a regular value]. Then:

$$(p,v) \in T_pS \iff \nabla g(p) \perp v \longrightarrow T_pS = \{(p,v) : \nabla g(p) \cdot v = 0\}$$

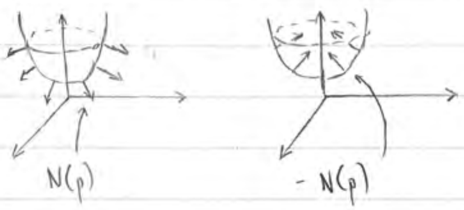
$\nabla g(p)$ a normal vector to tangent plane P

Orientation for Surfaces

Def: A vector (p,v) is orthogonal to a surface S if it is orthogonal to all vectors $(p,v_0) \in T_pS$ [i.e. $v \cdot v_0 = 0$].

→ Def: Let $S \subseteq \mathbb{R}^3$ a surface; then an orientation of S is a choice of unit normal vector field $N : S \to S \times \mathbb{R}^3$ s.t. $\forall\, p \in S$: $N(p) \perp S$, $\|N(p)\| = 1$.

(*) For $S = g^{-1}(c)$ level set in $\mathbb{R}^3$, $N(p) = (p, \nabla g(p)/\|\nabla g(p)\|)$ and $-N(p)$ are orientations of S.



The Gauss Map

11/1/24
Lecture 15
+ Lecture 16

Fact: If $S \subseteq \mathbb{R}^3$ is a connected surface, then S admits at most 2 orientations.

(*) Note: No guarantee that $\exists$ an orientation (unlike $\mathbb{R}^2$); a surface may have no orientations (ex: Möbius band)

Möbius band [surface w/ no orientation]

(*) Notice: If a surface S is non-orientable, then S cannot be expressed as a level set.

The Gauss Map

(*) Notation: Define $S^n := \{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_n^2 + x_{n+1}^2 = 1\}$ — can view as set of all unit vectors $v \in \mathbb{R}^{n+1}$

- In particular: S^1 = unit circle (centered at $(0,0)$) in $\mathbb{R}^2$, S^2 = unit sphere in $\mathbb{R}^3$

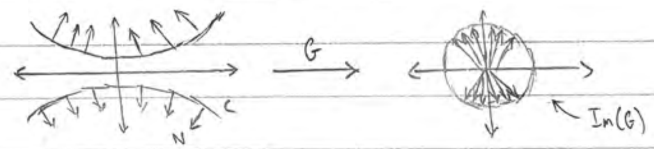
Def ($\mathbb{R}^2$): Given an oriented curve $C \subseteq \mathbb{R}^2$ with orientation N, the associated Gauss map is the function $G: C \to S^1$ satisfying:

$$\boxed{N(p) = (p, G(p)) \ \forall p \in C}$$

(*) Ex: $y^2 - x^2 = 1$

$N(p) = \left(p, (-2x, 2y)/\sqrt{4x^2+4y^2}\right)$

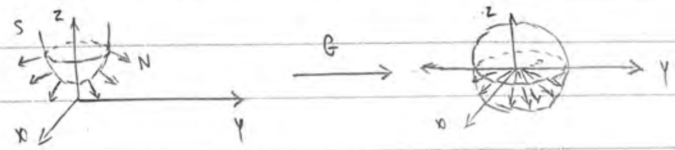
$\to G(p) = (-2x, 2y)/\sqrt{4x^2+4y^2}$



Def ($\mathbb{R}^3$): Given a surface $S \subseteq \mathbb{R}^3$ oriented by N, its Gauss map is given by $G: S \to S^2$ defined by:

$$\boxed{N(p) = (p, G(p)) \ \forall p \in C}$$

(*) Ex: $z = x^2 + y^2$



(*) Notice: A curve/surface oriented by $\nabla f / \|\nabla f\|$ will have Gauss map $G(p) = \nabla f(p) / \|\nabla f(p)\|$.

Spherical Image

11/4/28

Lecture 16 + Lecture 17 + Lecture 18

Def: The image of the Gauss map of a curve C/surface S is called the <u>spherical image</u> of C/S.

Thm: If a curve/surface $X = f^{-1}(c)$ is compact, then its Gauss map under orientation $N = \pm \nabla f / \|\nabla f\|$ is surjective.

(*) Proof: WLOG, assume X is connected.

Given $v \in S^1$ or $v \in S^2$, define $h(p) = p \cdot v$ $[\nabla h = v]$. By the EVT & Lagrange multipliers, know that $\exists\ p_1, p_2 \in X$ s.t. $h|_X$ (restriction of h to X) attains its minimum & maximum values at p_1 & p_2, and:

via Lagrange multipliers

$\nabla h(p_i) = \lambda_i \nabla f(p_i)$

$\Rightarrow \underline{\lambda_i = \pm 1/\|\nabla f(p_i)\|}$

Suppose, for contradiction, that $\left(\frac{\nabla f}{\|\nabla f\|}\right)(p_i) = -v$ at both p_1 & p_2; then $v \cdot \nabla f(p_1), v \cdot \nabla f(p_2) < 0$.

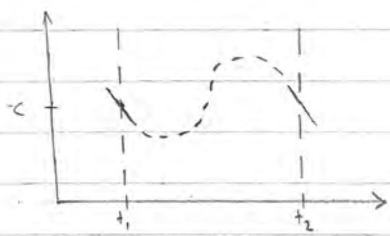
$\rightarrow$ Let $\gamma : (a,b) \to \mathbb{R}^n$ a curve s.t. $\gamma(t_1) = p_1$, $\gamma(t_2) = p_2$, $\dot\gamma(t_1) = \dot\gamma(t_2) = \underline{v}$, where $a < t_1 < t_2 < b$ are s.t. $\gamma(t) \notin X\ \forall\ t \in (t_1, t_2)$.

Looking at $f \circ \gamma$:

$\frac{d}{dt}(f \circ \gamma) = \nabla f(\gamma(t)) \dot\gamma(t)$ ...

$\rightarrow$ at $t = t_1$: $(f \dot\circ \gamma)(t_1) = \nabla f(p_1) \cdot v < 0$

$\rightarrow$ at $t = t_2$: $(f \dot\circ \gamma)(t_2) = \nabla f(p_2) \cdot v < 0$



By defn. of X, know that $f(\gamma(t_i)) = f(p_i) = c$ & $f(\gamma(t)) \neq c\ \forall\ t \in (c_1, c_2)$.

However, by IVT for $f \circ \gamma$, there must exist $t \in (t_1, t_2)$ s.t. $f(\gamma(t)) = c$. [contradiction] ∎

(*) <u>Heine-Borel</u>: A set $S \subseteq \mathbb{R}^n$ is <u>compact</u> <=> <u>S is closed and bounded</u>.

(*) <u>Note</u>: Any level set $S = f^{-1}(c)$ of a <u>smooth</u> function f, regular value c is closed.

$\rightarrow$ <u>Pf</u>: $\{c\}$ closed, f cts. $\Rightarrow$ inverse image under f of c $[f^{-1}(\{c\})]$ is closed.

Curves on Surfaces

11/8/24

Lecture 18

+ Lecture 19

Curves on Surfaces

We say a curve $\gamma : (a,b) \to \mathbb{R}^3$ is on a surface $S \subseteq \mathbb{R}^3$ if $\mathrm{Im}(\gamma) \subseteq S$

(*) Ex: $X = S^2$

$\to \gamma_1 = (\cos t, \sin t, 0)$, $\gamma_2 = (\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}})$ both lie on S

Rmks: (i) Let γ a curve on surface S; then $\dot{\gamma}(t) \in T_{\gamma(t)} S \ \forall t$

(ii) Let $S \subseteq \mathbb{R}^3$ surface, $P \subseteq \mathbb{R}^3$ plane w/ normal n. Usually, $S \cap P$ is a curve on S

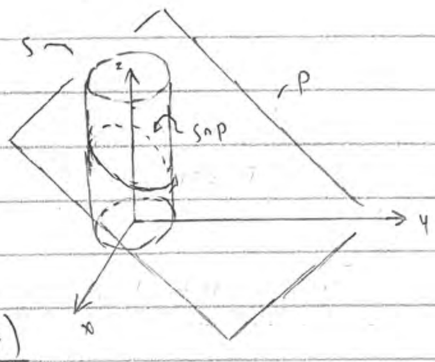
(*) Ex: $S = \{x^2+y^2=1\}$

P: plane through $(0,1,0)$

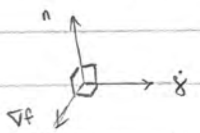
w/ normal $n = (0,1,1)$

$\to \underline{S \cap P = \left\{ \begin{matrix} x^2+y^2=1 \\ z = 1-y \end{matrix} \right\}}$

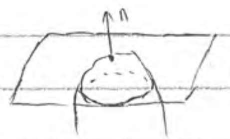
parametrization: $\underline{\gamma = (\cos t, \sin t, 1-\sin t)}$



Rmk: If $S = f^{-1}(c)$ level set and n is not parallel to ∇f on P, then $\dot{\gamma}(t)$ is orthogonal to S and to the normal vector n of P.



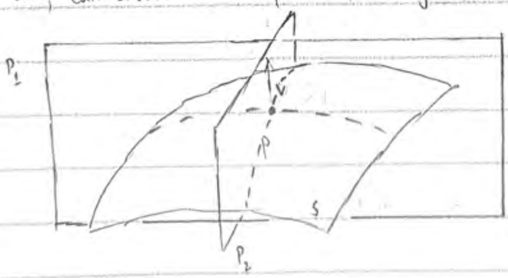
(*) $n \parallel P \to$ weird; ex:



[$S \cap P$ a single point]

Def: Let $S \subseteq \mathbb{R}^3$ a surface, $P \subseteq \mathbb{R}^3$ a plane, and let $p \in S \cap P$. We say that $\underline{P \text{ is orthogonal to } S \text{ at } p}$ if $\exists$ nonzero vector (p, v) orthogonal to S and tangent to P.

Rmk: If $S = f^{-1}(c)$, can choose $v = \nabla f(p)$ [(p,v) tangent to $P \iff p + v \in P \iff v \cdot n = 0$]



11/13/24
Lecture 19
(cont.)

Derivatives w.r.t. Vectors

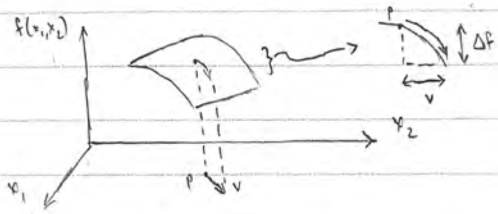
Derivatives w.r.t. Vectors

Def: Let $f:\mathbb{R}^n \to \mathbb{R}$ be a smooth function, and let $(p,v)\in\mathbb{R}^n_p$. Then the derivative of f with respect to (p,v) is given by:

$$\boxed{\nabla_{(p,v)} f = \frac{d}{dt}(f\circ\gamma)(t_0)}$$

where $\gamma:(a,b)\to\mathbb{R}^n$ is a parametrized curve satisfying $\gamma(t_0)=p$, $\dot\gamma(t_0)=v$ for some $t_0\in(a,b)$.

(*) Intuition: This corresponds to the idea of a "directional derivative", measuring the rate of change of f at p in the direction v.



(*) Alt. notation: $\nabla_{(p,v)}f = \nabla_v f = (f \mathbin{\dot\circ} \gamma)(t_0)$

Prop: $\boxed{\nabla_{(p,v)} f = \nabla f(p)\cdot v}$ (i.e. $\nabla_{(p,v)}f$ does not depend on choice of γ

(*) Proof: $\frac{d}{dt}(f\circ\gamma)(t_0) = \nabla f(\gamma(t_0))\cdot\dot\gamma(t_0) = \nabla f(p)\cdot v$

Corollary: $\nabla_{(p,v)}f$ is linear in v: $\nabla_{(p,av_1+bv_2)}f = a\nabla_{(p,v_1)}f + b\nabla_{(p,v_2)}f$

Can define derivatives of vector fields w.r.t. vectors similarly:

Def: Let $X: U\to U\times\mathbb{R}^n$ be a vector field on $U\subseteq\mathbb{R}^n$; then its derivative w.r.t. $(p,v)\in\mathbb{R}^n_p$ is given by (for γ P.C. w/ $\gamma(t_0)=p$, $\dot\gamma(t_0)=v$):

$$\boxed{\nabla_{(p,v)}X = (X \mathbin{\dot\circ} \gamma)(t_0)}$$

Rmk: In coordinates, $X(p) = (p,(x_1(p),\dots,x_n(p))) \to \nabla_{(p,v)}X = (p,(\nabla_{(p,v)}x_1,\dots,\nabla_{(p,v)}x_n))$
$= (p,(\nabla x_1(p)\cdot v,\dots,\nabla x_n(p)\cdot v))$

Prop: Let X, Y vector fields $U\to U\times\mathbb{R}^n$ & $f: U\to\mathbb{R}$ a function; then:

(i) $\nabla_{(p,v)}(X+Y) = \nabla_{(p,v)}X + \nabla_{(p,v)}Y$

(ii) $\nabla_{(p,v)}(fX) = \nabla_{(p,v)}f\cdot X(p) + f(p)\cdot\nabla_{(p,v)}X$

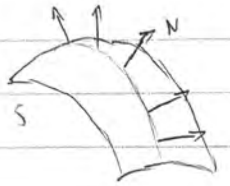
(iii) $\nabla_{(p,v)}(X\cdot Y) = \nabla_{(p,v)}X\cdot Y(p) + X(p)\cdot\nabla_{(p,v)}Y$

The Weingarten Map

11/15/24
Lecture 20

Previously, defined derivatives of vector fields w.r.t. directions $\left[\frac{d}{dt}(X\circ\gamma)(t_0)\text{, where } \gamma(t_0)=p \text{ \& } \dot\gamma(t_0)=v\right]$

→ Now: want to study rate of change ("turning") of the unit normal orientation N of a surface S



(*) Recall: For curves, had $\dot T = \kappa N$ and $\dot N = -\kappa T$ [Frenet, curvature]

→ Want to find "curvature for surfaces" by looking at $\dot N$

Lemma: Let $S \subseteq \mathbb{R}^3$ be an oriented surface with orientation N, and let $p \in S$, $(p,v) \in T_pS$.
Then $\nabla_{(p,v)} N \in T_pS$.

(*) Proof: $N\cdot N = 1 \xrightarrow{d/dt} 0 = \nabla_{(p,v)}(N\cdot N) = 2N(p)\cdot \nabla_{(p,v)} N$
$\Rightarrow N(p) \perp \nabla_{(p,v)} N \Rightarrow \nabla_{(p,v)} N$ is tangent to S at p. ∎

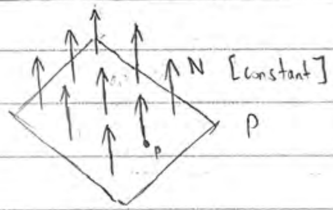
Def: Let $S\subseteq\mathbb{R}^3$ be a surface oriented by N, and let $p\in S$. Then the Weingarten map of S at p is the function $L_p : T_pS \to T_pS$ defined by:

$$\boxed{L_p(v) = -\nabla_{(p,v)} N} \quad \text{[Weingarten map]}$$

(*) Ex: Plane $P = \{ax+by+cz=d\}$ $[=f^{-1}(d),\ f=\text{LHS}]$

→ $N = \nabla f / \|\nabla f\| = (a,b,c)/\sqrt{a^2+b^2+c^2}$

→ for any $p\in S$, $v\in T_pS$: $L_p(v) = -\nabla_{(p,v)} N = \vec{0}$



Rmk: $\nabla_{(p,v)} N$ [for $(p,v)\in T_pS$] only depends on the value of N on S; in particular, if $N = \tilde N|_S$ (restriction to S) for some VF $\tilde N : \mathbb{R}^3 \to \mathbb{R}^3\times\mathbb{R}^3$, then $\nabla_{(p,v)} N = \nabla_{(p,v)}\tilde N$.

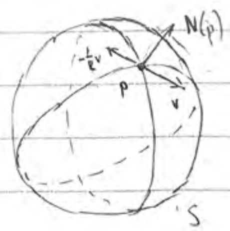
[by defn, $\nabla_{(p,v)} X$ defined w.r.t. a curve γ on S]

(*) Ex: Sphere radius R, $S = \{x^2+y^2+z^2=R^2\}$, outward $N = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$

→ for $p\in S$, $v\in T_pS$: $L_p(v) = \left(p, \left(-\nabla\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right]\cdot v, \dots\right)\right) \underset{\text{by Rmk}}{=} \left(p, \left(-\nabla \frac{x}{R}\cdot v, \dots\right)\right)$

→ $\boxed{L_p(v) = \left(p, -\frac{1}{R} v\right)}$

[Rec.: $\kappa(\text{circle}) = 1/R$]



The Weingarten Map (cont.)

11/18/24
Lecture 21

Fact: The Weingarten map is linear: $L_p(au+v) = aL_p(u) + L_p(v)$.

Thm: Let $S \subseteq \mathbb{R}^3$ oriented by N, and let $\gamma : (a,b) \to S$ be a parametrized curve on S with $\gamma(t_0) = p \in S$, $\dot{\gamma}(t_0) = v \in \mathbb{R}^3$ for any $t_0 \in (a,b)$. Then:

$$\boxed{L_p(v) \cdot v = \ddot{\gamma}(t_0) \cdot N(p)}$$

(*) **Proof:** Know that: (i) $N(\gamma(t)) \perp S$, and (ii) $\dot{\gamma}(t)$ tangent to S [$\forall\, t \in (a,b)$]

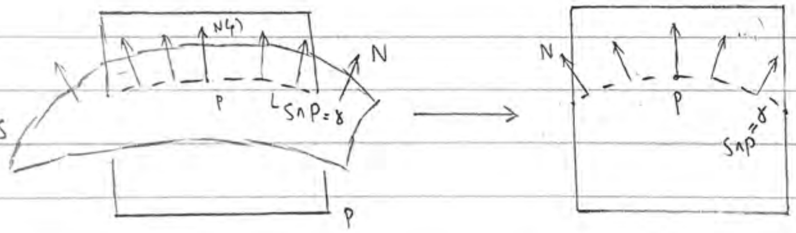
$\rightarrow \forall\, t \in (a,b)$: $0 = \dot{\gamma}(t) \cdot N(\gamma(t))$

$\left(\frac{d}{dt}\big|_{t=t_0}\right) \longrightarrow 0 = \ddot{\gamma}(t_0) \cdot N(\gamma(t_0)) + \dot{\gamma}(t_0) \cdot \frac{d}{dt}(N \circ \gamma)(t_0)$

$\longrightarrow \ddot{\gamma}(t_0) \cdot N(p) = -\dot{\gamma}(t_0) \cdot \frac{d}{dt}(N \circ \gamma)(t_0) = L_p(v) \cdot v$.

Observation: The Weingarten map is related to curvature - let P a plane orthogonal to S at p (S oriented by N), γ a unit-speed parametrization of $S \cap P$; then:

$$\underline{\kappa(p) = \ddot{\gamma}(t_0) \cdot N(p)} \Rightarrow \underline{\kappa(p) = L_p(v) \cdot v} \quad \text{← } \kappa = \text{curvature of } \gamma$$



$[\kappa(p) = L_p(v) \cdot v]$

Prop.: Let $S = f^{-1}(c)$ be oriented by $N = \nabla f / \|\nabla f\|$, and let $p \in S$. Then for any $v, w \in T_pS$:

$$\boxed{L_p(v) \cdot w = L_p(w) \cdot v} \quad [L_p \text{ is } \underline{\text{self-adjoint}}]$$

(*) Proof on next page

<u>(*) Recall (Linear Algebra)</u>

1. A linear operator T is self-adjoint $\Rightarrow$ for any orthonormal basis β, $[T]_\beta$ is symmetric
2. <u>Thm (Spectral Thm)</u>: Let $T: V \to V$ linear self-adjoint; then $\exists$ an orthonormal basis β for V consisting of eigenvectors of $T \longrightarrow [T]_\beta$ is a diagonal matrix.

Curvatures of Surfaces

11/18/24
Lecture 21
+ Lecture 22

(*) Proof $[L_p(v)\cdot w = L_p(w)\cdot v]$

Via computation: $L_p(v)\cdot w = -\nabla_v\left(\nabla f/\|\nabla f\|\right)\cdot w$

$$= -\nabla_v\left(\frac{1}{\|\nabla f\|}\cdot \nabla f\right)\cdot w$$

$$= -\left(\nabla_v\left(\frac{1}{\|\nabla f\|}\right)\cdot\nabla f + \frac{1}{\|\nabla f\|}\cdot\nabla_v[\nabla f]\right)\cdot w$$

Notice: (i) w tangent to S
(ii) ∇f orthogonal to S
$\rightarrow w \perp \nabla f$

$$= -\frac{1}{\|\nabla f\|}\nabla_v(\nabla f)\cdot w$$

$$= \frac{-1}{\|\nabla f\|}\left(\nabla\frac{\partial f}{\partial x_1}\cdot v, \ldots, \nabla\frac{\partial f}{\partial x_n}\cdot v\right)\cdot w$$

$$= \frac{-1}{\|\nabla f\|}\sum_{i,j=1}^{n} v_i \frac{\partial^2 f}{\partial x_i \partial x_j} w_j = \frac{-1}{\|\nabla f\|}\nabla_w(\nabla f)\cdot v = L_p(w)\cdot v.$$ ▨

Curvatures of Surfaces

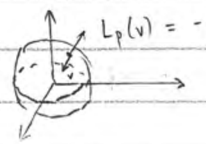
Def: Let $S \subseteq \mathbb{R}^3$ be a surface oriented by N. Let $p \in S$, $(p,v) \in T_pS$ unit vector $[\|v\| = 1]$; then the normal curvature of S at p in direction v is given by:

$$\boxed{k(p,v) = L_p(v)\cdot v}$$

(i) Ex: $S = \{x^2+y^2+z^2 = R^2\}$ [sphere radius R]

$N = \nabla f/\|\nabla f\| = \frac{1}{R}(x,y,z)$

$\rightarrow L_p(v) = -\frac{1}{R}v \rightarrow k(p,v) = -\frac{1}{R}\underbrace{\|v\|^2}_{=1} = -1/R$



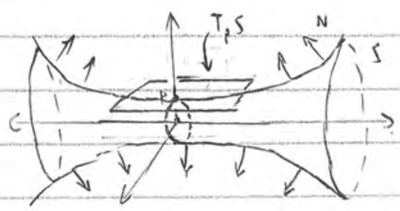
(ii) Ex: $S = \{-x^2+y^2+z^2 = 1\}$ [hyperboloid]

- $N = \nabla f/\|\nabla f\| = (-x,y,z)/\sqrt{1+2x^2}$
- $p = (0,0,1) \rightarrow N(p) = (0,0,1)$

$\downarrow$

$L_p(v) = -((-1,0,0)\cdot v, (0,1,0)\cdot v, (0,0,1)\cdot v)$
$= (v_1, -v_2, 0)$ $[v_3 = 0]$

$\rightarrow k(p,v) = v_1^2 - v_2^2$



Curvatures of Surfaces (cont.)

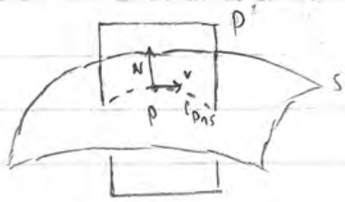
11/20/24
Lecture 22

Recall: Let γ a parametrized curve on S, $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v \rightarrow \ddot{\gamma}(t_0) \cdot N(p) = L_p(v) \cdot v = \overset{\text{if } \|v\| = 1}{k(p,v)}$

$\rightarrow$ **Corollary**: Let $S \subseteq \mathbb{R}^3$ be a surface oriented by N. Let $p \in S$ and P a plane through p orthogonal to S at p $\rightarrow$ then: $\underline{P \cap S \text{ is (near } p) \text{ a smooth curve}}$, and:

curvature of $P \cap S \longrightarrow$ $\boxed{\kappa_{P\cap S}(p) = k(p,v)}$ $[v \in P,\ v \in T_pS,\ \|v\| = 1]$

(*) Proof: To show that $P \cap S$ is locally a curve, use that:



(i) S a surface $\rightarrow$ S locally a level set $f^{-1}(c)$ around p

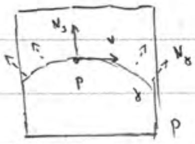
(ii) P a plane $\rightarrow$ $P = \mathrm{Im}(g)$ for some $g : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$\downarrow$

$\underline{P \cap S = g((f \circ g)^{-1}(c))}$ ← parametrization of $P \cap S$

$$\nabla(f \circ g) = \nabla f \cdot \underbrace{\mathrm{Jac}(g)}_{[\,v\ \ N(p)\,]} = (\nabla f \cdot v,\ \underbrace{\nabla f \cdot N(p)}_{\neq 0}) \quad [v,\ N(p) \text{ span } P]$$

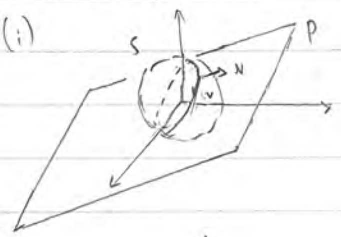
$P \cap S$ a smooth curve $\rightarrow$ can parametrize $P \cap S$ by unit-speed γ [w/ $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$]



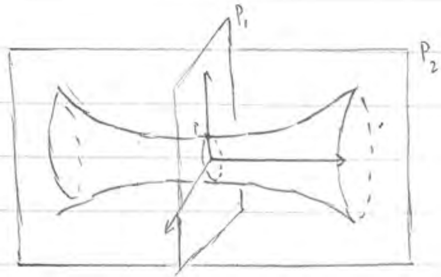
$$\rightarrow \kappa(p) = \ddot{\gamma} \cdot N_\gamma(p) \Big/ \underbrace{\|\dot{\gamma}(t_0)\|^2}_{=1} = \ddot{\gamma}(t_0) \cdot N_S(p) = k(p,v).$$ ☐

(*) Ex (Revisited)

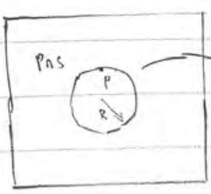
(i)



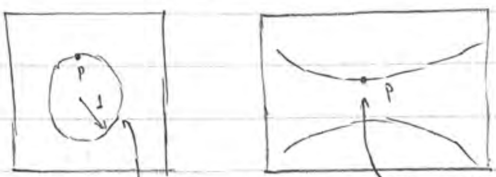
(ii)



$\downarrow$ $\downarrow$



circle radius R
$\Downarrow$
$\underline{\kappa(p) = -1/R}$



P_1 $\kappa = -1/R = -1$ $\quad$ P_2 κ(hyperbola) $= 1$

Principal Curvatures

11/25/24

Lecture 23

<u>Recall</u>: L_p is self-adjoint & linear $\Rightarrow$ by the Spectral Theorem, $\exists$ an orthonormal basis $\{v_1, v_2\}$ for T_pS consisting of eigenvectors of L_p [w/ eigenvalues λ_1, λ_2]

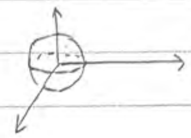
<u>Def</u>: Let $S \subseteq \mathbb{R}^3$ be a surface oriented by N, and let $p \in S$. Let $\{v_1, v_2\}$ be an ON basis (for T_pS) of eigenvectors of L_p w/ eigenvalues λ_1, λ_2; then the <u>principal curvatures of S at p</u> are defined by:

$$\boxed{k_1(p) = k(p, v_1) = \lambda_1} \qquad \boxed{k_2(p) = k(p, v_2) = \lambda_2}$$

(*) <u>Ex</u>:

(i) Sphere radius R. $k(p,v) = -\frac{1}{R}$

$$\rightarrow L_p(v) = \begin{pmatrix} -1/R & 0 \\ 0 & -1/R \end{pmatrix} v \rightarrow k_1(p), k_2(p) = -1/R$$

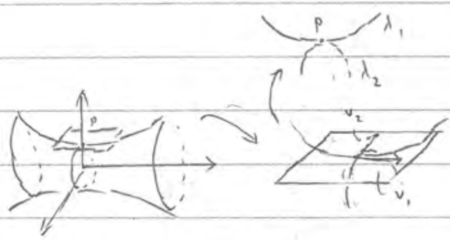


(ii) $S = \{-x^2+y^2+z^2 = 1\}$, $p = (0,0,1)$

$\rightarrow L_p(a,b,c) = (a, -b, 0)$

- $v_1 = (1,0,0) \rightarrow k_1(p) = 1$
- $v_2 = (0,1,0) \rightarrow k_2(p) = -1$

$; [L_p]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



<u>Prop</u>: Let $V \subseteq \mathbb{R}^n$ a 2D vector subspace of $\mathbb{R}^n$, and let $T: V \to V$ a self-adjoint linear operator with eigenvalues $\underline{\lambda_1 \le \lambda_2}$. Then for any $v \in V$ with $\|v\| = 1$:

$$\underline{\lambda_1 \le T(v) \cdot v \le \lambda_2}$$

(*) <u>Proof</u>: $\underset{\|v_1\|=\|v_2\|=1}{v_1, v_2}$ eigenvectors of T corresponding to λ_1, λ_2 $\rightarrow$ can write v (uniquely) as $\underline{v = av_1 + bv_2}$, where $\underline{\|av_1\|^2 + \|bv_2\|^2 = a^2+b^2 = 1}$ [$= \|v\|^2$].

Then:

$$\begin{aligned} T(v) \cdot v &= \lambda_1 \|av_1\|^2 + \lambda_2 \|bv_2\|^2 \\ &= \lambda_1 a^2 + \lambda_2 b^2 \\ &= \lambda_1 a^2 + \lambda_2 (1-a^2) \quad [0 \le a^2 \le 1] \Rightarrow T(v)\cdot v \in [\lambda_1, \lambda_2]. \end{aligned}$$

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Lecture 23
+ Lecture 24

Gauss Curvature

<u>Corollary</u>: Let $S \subseteq \mathbb{R}^3$ oriented surface, $p \in S$; then the principal curvatures $k_1(p), k_2(p)$ are the min & max of the normal curvatures $k(p,v)$ at p.

$$k_1(p) = \min_v k(p,v) \quad ; \quad k_2(p) = \max_v k(p,v)$$

<u>Def</u>: Let $S \subseteq \mathbb{R}^3$ an oriented surface, and let $p \in S$. Then the <u>Gauss curvature</u> of S at p is defined by:

$$\boxed{K(p) = k_1(p) \cdot k_2(p)}$$

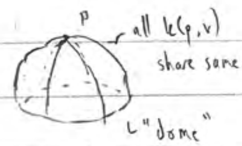
(*) <u>Ex</u>: (i) Sphere radius R: $k_1(p) = k_2(p) = -1/R$
$\rightarrow K(p) = 1/R^2 \ \forall p$

(ii) $S = \{-x^2+y^2+z^2 = 1\}$, $p = (0,0,1)$: $k_1(p) = 1$, $k_2(p) = -1$
$\rightarrow K(0,0,1) = -1$

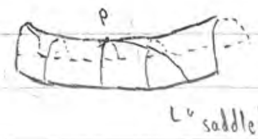
<u>Remarks</u>: (i) The Gauss curvature does not depend on choice of orientation (i.e. N vs. $-N$)
(ii) Notice: $K(p) = \det(L_p)$

Geometrically: [locally]

$K(p) > 0$

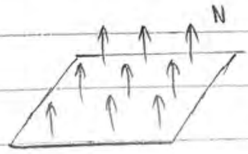


$K(p) < 0$

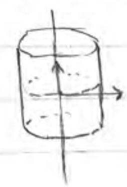


$\rightarrow K = 0$?

(i) Plane $ax+by+cz = d$
$\rightarrow L_p(v) = 0 \rightarrow k_1, k_2 = 0$
$\downarrow$
$K = 0$



(ii) Cylinder radius R: $x^2+y^2 = R^2$ — $N = \nabla f/\|\nabla f\| = \frac{1}{R}(x,y,0)$
$\rightarrow v_1 = (0,0,1)$ $[\lambda_1 = 0]$,
$+ \ v_2 = (-y,x,0)$ $[\lambda_2 = -\frac{1}{R}]$



$\hookrightarrow K(p) = 0 \cdot -1/R = 0$

(*) In general: flat surface/slice of flat cylinder (locally) $\rightarrow K(p) = 0$

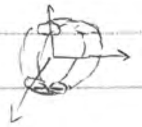
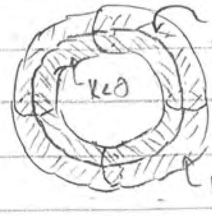
Gauss Curvature (cont.)

11/27/24

Lecture 24

+Lecture 25

(*) Ex: $S = \{x^2 + (\sqrt{y^2+z^2} - 2)^2 = 1\}$

 → [torus] $\xrightarrow{\text{side view}}$ 

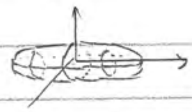
(*) Fact: The Gauss curvature $K(p)$ varies smoothly with p.

(*) Lemma: Let $S \subseteq \mathbb{R}^3$ a surface oriented by N & X a vector field s.t. $\underline{X/\|X\| = N}$. Let $p \in S$ & $\{v_1, v_2\}$ any basis for T_pS; then:

$$\underline{K(p) = \det\begin{pmatrix} \nabla_{(p,v_1)}X \\ \nabla_{(p,v_2)}X \\ X(p) \end{pmatrix} \cdot \frac{1}{\|X(p)\|^2 \det\begin{pmatrix} v_1 \\ v_2 \\ X(p) \end{pmatrix}}}$$ [alternative formula]

↳ $N = \nabla f / \|\nabla f\|$ → can use $X = \nabla f$ instead

(*) Ex $S = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ [ellipsoid]



→ $\nabla f = (2x/a^2, 2y/b^2, 2z/c^2\}$ → choose $X = (x/a^2, y/b^2, z/c^2)$ [$= \frac{1}{2}\nabla f$]

Let $p \in S$, $p = (x \neq 0, y, z)$ → take $\underline{v_1 = (-y/b^2, x/a^2, 0)}$ and $\underline{v_2 = (-z/c^2, 0, x/a^2)}$

$$\to K(p) = \begin{vmatrix} -y/a^2b^2 & x/a^2b^2 & 0 \\ -z/a^2c^2 & 0 & x/a^2c^2 \\ x/a^2 & y/b^2 & z/c^2 \end{vmatrix} \Bigg/ \left[\|X\|^2 \cdot \begin{vmatrix} -y/b^2 & x/a^2 & 0 \\ -z/c^2 & 0 & x/a^2 \\ x/a^2 & y/b^2 & z/c^2 \end{vmatrix} \right] = \underline{\frac{x}{a^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)}$$

↳ take $a = b = c = 1$ → obtain sphere formula

Curvatures of Compact Surfaces

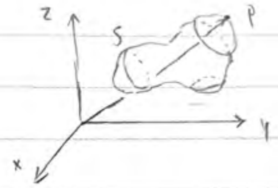
12/2/24
Lecture 26

Thm: Let $S \subseteq \mathbb{R}^3$ be a compact oriented surface; then $\exists\ p \in S$ such that either (i) $k(p,v) > 0$ $\forall\ v$ $[\|v\| = 1]$, or (ii) $k(p,v) < 0$ $\forall\ v$ $[\|v\|=1]$.

(*) Proof:

For simplicity, assume $S = f^{-1}(c)$ level set w/ $N = \nabla f / \|\nabla f\|$.

Define $g : \mathbb{R}^3 \to \mathbb{R}$ by: $g(x,y,z) = x^2 + y^2 + z^2$ [L2 dist.]



$\to$ by compactness of S + EVT: $\exists\ p \in S$ where g attains its max on S

By Lagrange multipliers, $\nabla f(p) \parallel \nabla g(p) \Rightarrow \nabla f(p) = \lambda \nabla g(p)$ for some $\lambda \in \mathbb{R}$, $\nabla g \perp S$ at p.

We know g:

$$\to \nabla g(p) = (2x, 2y, 2z) = 2p \Rightarrow p = \tfrac{\lambda}{2} \nabla f(p) \Rightarrow N(p) = \pm \frac{p}{\|p\|} \quad [\text{WLOG, assume } +]$$

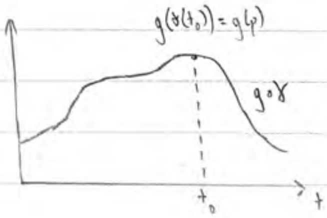
$\uparrow$ since $N \parallel \nabla f \parallel p$ & $\|N\| = 1$ unit VF

Let $(p,v) \in T_pS$ w/ $\|v\| = 1$, and let $\gamma : (a,b) \to S$ s.t. $\gamma(t_0) = p$, $\dot\gamma(t_0) = v$.

Know: g attains its max on S at p

$$\Rightarrow g(\gamma(t)) \le g(\gamma(t_0)) \quad \forall\ t \in (a,b)$$

$\Rightarrow g \circ \gamma$ is maximized at t_0



$$\begin{aligned}
\Rightarrow 0 &\ge \frac{d^2}{dt^2}(g \circ \gamma)(t_0) \\
&= \frac{d}{dt}\Big|_{t_0} \left(\nabla g(\gamma(t)) \cdot \dot\gamma(t_0)\right) \\
&= \frac{d}{dt}\Big|_{t_0} \left(2\gamma(t) \cdot \dot\gamma(t_0)\right) \\
&= 2\left(\dot\gamma(t_0) \cdot \dot\gamma(t_0) + \gamma(t_0) \cdot \ddot\gamma(t_0)\right) \\
&= 2\left(v \cdot v + p \cdot \ddot\gamma(t_0)\right) \\
&= 2\underbrace{\|v\|^2}_{1} + \|p\| \underbrace{N(p) \cdot \ddot\gamma(t_0)}_{L_p(v) \cdot v = k(p,v)}
\end{aligned}$$

$$\to 0 \ge 1 + \|p\| k(p,v) \Rightarrow k(p,v) \le -\frac{1}{\|p\|} < 0$$

$\blacksquare$

Corollary: Let $S \subseteq \mathbb{R}^3$ be a compact oriented surface; then $\exists\ p \in S$ with Gauss curvature $K(p) > 0$.

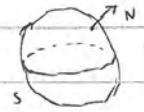
Mean Curvature + Fundamental Forms

12/2/24
Lecture 26
+ Lecture 27

Def: Let $S \subseteq \mathbb{R}^3$ an oriented surface, and let $p \in S$. Then the mean curvature of S at p is:

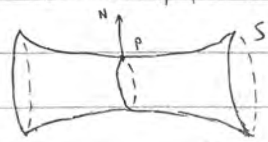
$$\boxed{H(p) = \frac{1}{2}\left(k_1(p) + k_2(p)\right)}$$

(*) Ex: (i) Sphere radius R, outward N



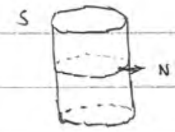
$k_1 = k_2 = -1/R \Rightarrow H(p) = -1/R \;\; \forall p \in S$

(ii) $S = \{-x^2 + y^2 + z^2 = 1\}$, $p = (0,0,1)$, $N = \nabla f / \|\nabla f\|$



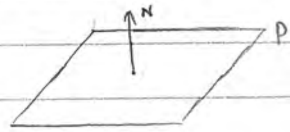
$k_1(p) = -1$, $k_2(p) = 1 \rightarrow H(p) = 0$

(iii) $S = \{x^2 + y^2 = R^2\}$, $N = \nabla f / \|\nabla f\|$



$k_1 = 0$, $k_2 = -1/R \rightarrow H(p) = -1/2R$ $(\forall p \in S)$

(iv) Plane $\{ax + by + cz = d\}$



$k_1 = 0$, $k_2 = 0 \rightarrow H(p) = 0 \;\; \forall p \in S$

(*) "Fun fact": Minimal surfaces (i.e. surfaces trying to minimize surface area under certain boundary constraints) are surfaces with $H(p) = 0$ everywhere (& vice versa).

Fundamental Forms of a Surface

Def: Let $V \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$, and let $T: V \to V$ be a self-adjoint linear map. Then the quadratic form associated with T is the map $Q: V \to \mathbb{R}$ given by:

$$\boxed{Q(v) = T(v) \cdot v}$$

Def: Let $S \subseteq \mathbb{R}^3$ an oriented surface, and let $p \in S$. Then the 2nd fundamental form of S at p is the function $\mathcal{S}_p: T_pS \to \mathbb{R}$ defined by:

$$\boxed{\mathcal{S}_p(v) = L_p(v) \cdot v}$$

(*) This is the quadratic form associated with the Weingarten map L_p

Fundamental Forms of Surfaces

12/4/24

Lecture 27 (cont.)

Fundamental Forms of a Surface (cont.)

(*) <u>Def</u>: Given an oriented surface $S \subseteq \mathbb{R}^3$ and $p \in S$, the <u>1^{st} fundamental form of S at p</u> is the function $\ell_p : T_pS \to \mathbb{R}$ given by:

$$\boxed{\ell_p(v) = v \cdot v = \|v\|^2}$$

<u>Notice</u>: The 1^{st} and 2^{nd} fundamental forms of S at p are both quadratic forms:

- 1^{st}: $\ell_p(v) = v \cdot v \longrightarrow$ quadratic form associated with Id identity on T_pS
- 2^{nd}: $\rho_p(v) = L_p(v) \cdot v \longrightarrow$ quadratic form associated with L_p Weingarten map of S

<u>Def</u>: A quadratic form $Q: V \to \mathbb{R}$ is called:

(i) <u>Positive-definite</u> [P.D.] if $Q(v) > 0 \ \forall \ v \neq 0$

(ii) <u>Negative-definite</u> [N.D.] if $Q(v) < 0 \ \forall \ v \neq 0$

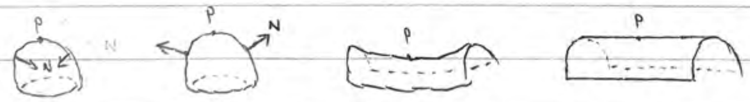
(iii) <u>Indefinite</u> if it is neither positive- nor negative-definite

Fundamental Forms and Curvature

<u>Observe</u>: Let $(p,v) \in T_pS$, $v \neq 0 \Rightarrow$ then: $\rho_p(v) = L_p(v) \cdot v = \|v\|^2 \left(L_p\big(\overbrace{v/\|v\|}^{\text{unit vector}}\big) \cdot v/\|v\| \right)$

$$\Rightarrow \boxed{\rho_p(v) = \|v\|^2 k(p,v)}$$

In particular:



ρ_p	P.D.	N.D.	Indefinite	Indef. [PSD/NSD]
$K(p)$	>0	>0	<0	$=0$

<u>Rec</u>: $S \subseteq \mathbb{R}^3$ compact oriented surface $\Rightarrow \exists \ p \in S$ with $k(p,v) > 0$ or $k(p,v) < 0 \ \forall \ v \in T_pS$ [$\|v\| = 1$]

$\rightarrow$ <u>Corollary</u>: Let $S \subseteq \mathbb{R}^3$ a <u>compact oriented surface</u>; then <u>$\exists \ p \in S$ s.t. $K(p) > 0$</u> [$\Leftrightarrow \rho_p$ definite]

Isometries

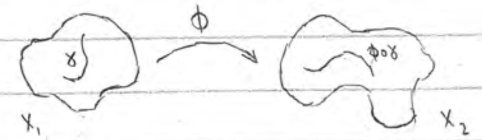
12/4/24
Lecture 27 (cont.)

Isometries

<u>Def</u>: Let $X_1, X_2 \subseteq \mathbb{R}^n$, and let $\phi: X_1 \to X_2$ be a diffeomorphism (smooth bijection with smooth inverse) from X_1 to X_2. We call ϕ an <u>isometry</u> if $\forall\, \gamma: (a,b) \to X_1$, the arc length of γ is equivalent to the arc length of $\phi \circ \gamma$, i.e.:

$$\boxed{\int_a^b \|\dot{\gamma}(t)\|\,dt = \int_a^b \|\dot{\phi \circ \gamma}(t)\|\,dt}$$

(*) <u>Note</u>: There is no guarantee that a diffeomorphism $\phi: X_1 \to X_2$ even exists! (e.g. if X_1, X_2 have different cardinalities, no such ϕ exists)



(*) <u>Ex</u>: $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ given by $\underline{\phi(x,y) = \phi(-y,x)}$ is an isometry

corresponds to a rotation by 90° CCW about origin

Can check:

$$\int_a^b \|\dot{\phi \circ \gamma}\|\,dt = \int_a^b \|(-\dot{\gamma}_2(t), \dot{\gamma}_1(t))\|\,dt = \int_a^b \|\dot{\gamma}(t)\|\,dt \quad \checkmark$$

(*) <u>Note</u>: This ϕ is also an isometry between any 2 subsets $X, \phi(X) \subseteq \mathbb{R}$

[In general, for 2D: only isometries are rotations / translations / reflections] ← isometries analogous / almost equiv. to idea of a "rigid motion"

Isometries and Curvature for Curves

<u>Q</u>: Is curvature "<u>intrinsic</u>" to curves? (i.e. <u>preserved under isometries</u>)

<u>A</u>: In general, <u>no</u>!

<u>Prop</u>: Let $C \subseteq \mathbb{R}^n$ be a smooth curve; then C is locally isometric to an interval $(a,b) \subseteq \mathbb{R}$

(*) <u>Proof</u> Let $p \in C$; then we can parametrize C near p by a PC $\beta: (a,b) \to \mathbb{R}^n$ (in particular, by β unit-speed).

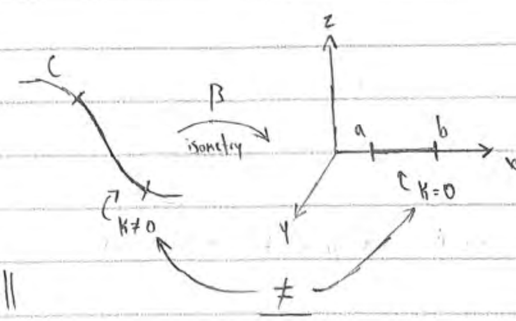
Then $\beta: (a,b) \to \mathrm{Im}(\beta)$ is an isometry:

- Take $\gamma: (c,d) \to (a,b)$; then:

$$\ell(\gamma) = \int_c^d |\dot{\gamma}|\,dt$$

$$\ell(\beta \circ \gamma) = \int_c^d \|\dot{\beta \circ \gamma}\| = \int_c^d \|\dot{\gamma} \cdot \dot{\beta}(\gamma(t))\|$$

$$= \int_c^d |\dot{\gamma}| \cdot \underbrace{\|\dot{\beta}(\gamma(t))\|}_{=1} = \ell(\gamma)$$



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Intrinsic Properties of Surfaces

12/06/24
Lecture 28

<u>Recall</u>: A function $\phi: X_1 \to X_2$ is an <u>isometry</u> if: (1) <u>ϕ is a diffeomorphism</u>, and
(2) <u>ϕ preserves arc length</u>

(*) Intuitively: ϕ isometry $\iff$ X_1, $\phi(X_1) = X_2$ identical to an ant walking along the surface
- Intrinsic properties (i.e. preserved by isometry): measurable by an ant on the surface (e.g. arc length)
- Extrinsic properties (not preserved by isometry): require "zooming out" to measure (eg. curvature of a curve)

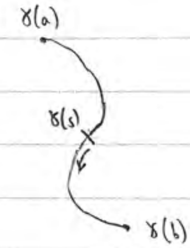
<u>Arc Length & the 1st Fundamental Form</u>

<u>Notice</u>: Let $S \subseteq \mathbb{R}^3$ a surface, $\gamma: (a,b) \to S$ curve on S

$$\Rightarrow \ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\,dt$$
$$= \int_a^b \sqrt{\dot{\gamma}(t)\cdot\dot{\gamma}(t)}\,dt = \int_a^b \sqrt{\mathbb{I}_{\gamma(t)}(\dot{\gamma}(t))}\,dt$$

arc length is "determined" by the 1st fundamental form

Can show the converse (i.e. the 1st fundamental form is determined by arc length):



(*) <u>Proof</u>: Let $(p,v) \in T_pS$, and let $\gamma: (a,b) \to S$ s.t. $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$.
Define $L: (a,b) \to \mathbb{R}$ by:

$$L(s) = \int_a^s \|\dot{\gamma}(t)\|\,dt \quad [\text{Arc length of } \gamma,\ a \to s]$$

$\to$ By the Fundamental Theorem of Calculus:

$$\left.\frac{dL}{ds}\right|_{s=t_0} = \|\dot{\gamma}(t_0)\| = \|v\| = \sqrt{\mathbb{I}_p(v)} \Rightarrow \mathbb{I}_p(v) = \left[\left.\frac{dL}{ds}\right|_{s=t_0}\right]^2$$

can compute $\mathbb{I}_p$ from arc length

$\blacksquare$

$\to$ Intuitively: isometries should preserve the 1st fundamental form

<u>Def</u>: Let $S_1, S_2 \subseteq \mathbb{R}^3$, $\phi: S_1 \to S_2$ smooth. Given $(p,v) \in T_pS$, the <u>push-forward</u> of (p,v) is:

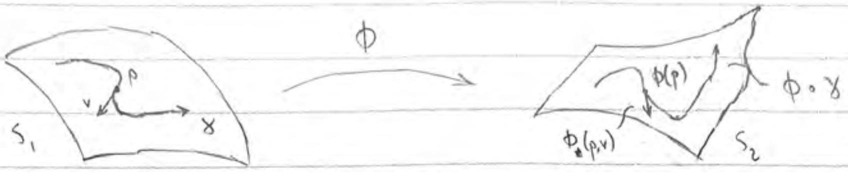
$$\Big(\phi(p),\ \phi_*(p,v)\Big), \text{ where } \phi_*(p,v) = \left.\frac{d}{dt}(\phi\circ\gamma)\right|_{t=t_0}$$

for γ smooth curve s.t. $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$.

$\to$ We say that ϕ preserves the 1st Fundamental Form if <u>$\mathbb{I}_p(v) = \mathbb{I}_{\phi(p)}(\phi_*(p,v))$</u>.

Intrinsic Properties of Surfaces (cont.)

12/06/24
Lecture 28
(cont.)



Thm: Let $S_1, S_2 \subseteq \mathbb{R}^3$ surfaces, and let $\Phi : S_1 \to S_2$ a diffeomorphism. Then:

$$\Phi \text{ is an isometry} \iff \Phi \text{ preserves the } 1^{st} \text{ fundamental form}$$

Angles & the 1st Fundamental Form

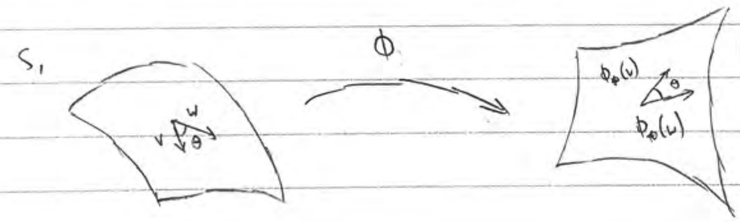
Notice: Let $v, w \in \mathbb{R}^n \to (v+w)\cdot(v+w) = v\cdot v + 2v\cdot w + w\cdot w$.

In particular:

$$[\|v\|\cdot\|w\|\cos\theta =]\ v\cdot w = \tfrac{1}{2}\left[\|v+w\|^2 - \|v\|^2 - \|w\|^2\right]$$

$$\to \theta = \cos^{-1}\left(\frac{v\cdot w}{\|v\|\cdot\|w\|}\right) = \cos^{-1}\left(\frac{\frac{1}{2}[I_p(v+w) - I_p(v) - I_p(w)]}{\sqrt{I_p(v)\, I_p(w)}}\right)$$ determined by the 1st fundamental form

Consequence: Isometries preserve dot product, angles



Idea: S_1, S_2 (in some sense) "geometrically equivalent" under Φ

Theorem (Theorema Egregium, Gauss)

Let $S_1, S_2 \subseteq \mathbb{R}^3$ surfaces, and let $\Phi : S_1 \to S_2$ an isometry:

$$\to \forall p \in S_1 : \boxed{K(p) = K(\Phi(p))}$$

(*) Proof idea: Can express K solely via dot products $\Rightarrow$ by above: K preserved by isometry

12/06/24

Lecture 28 (cont.)

FINAL

Intrinsic Properties of Surfaces (cont.)

Theorema Egregium - Applications

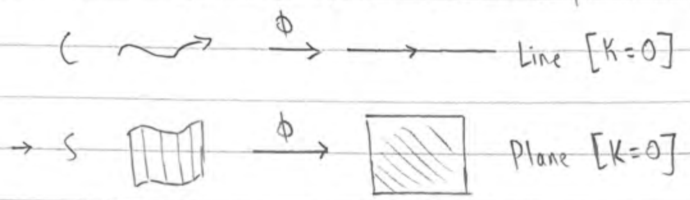
Gauss curvature is an intrinsic property of surfaces (Theorema Egregium)

→ Applications:

(1) Map projections: All maps of Earth are false/distorted

- Earth has $K \neq 0$, but every 2D map [plane] has $K = 0$ everywhere
  - → any f'n ϕ: Earth $\mapsto$ map is not an isometry (distorts arc length)

(2) Surfaces: All (generalized) cylinders have $K = 0$ everywhere (alt. proof)



(*) Intuitively: isometries akin to bending/folding shapes (e.g. a piece of paper) w/o deformation