Math 131AH: Analysis (Honors)

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Textbooks: W. Rudin - Principles of Mathematical Analysis

E. Copson - Metric Spaces

(*) K. Ross - Elementary Analysis

(*) J. Munkres - Topology

(*) Kaczor-Nowak - Problems in Mathematical Analysis I, II, & III

Topics: Real numbers, metric space topology, sequences & convergence, limits & continuity, sequences of functions, differentiation, Riemann integrals

Table of Contents

1) Review - 36

2) Real Numbers - 38

3) Metric Spaces - 43

i) Open & Closed Sets - 44

4) Sequences - 48

i) Convergent Sequences - 48

5) Metric Spaces (cont.) - 53

i) Completeness - 53

ii) Compactness - 54

iii) Connectedness - 59

6) Continuous Functions - 61

i) Limits - 62

ii) Uniform Continuity - 66

7) Sequences of Functions - 68

i) Uniform Convergence - 70

8) Differentiation - 73

i) Mean Value Theorem - 77

ii) Taylor's Theorem - 79

9) Riemann Integrals - 81

i) Cauchy Criterion for Integrability - 84

ii) Step Functions - 84

iii) Step Function Criterion for Integrability - 86

iv) Fundamental Theorem of Calculus - 90

(*) Change of Variables - 91

(*) Logarithm & Exponential Functions - 91

Review: Sets & Functions

9/29/23

Lecture 1

Analysis: a mathematical field developed after calculus, returning from calculus to more traditional mathematical roots

Review: Sets

Def: A set is, put simply, a collection of objects (called "elements").

Notable operations: Union ($A \cup B$), intersection ($A \cap B$), difference ($A - B$, $A \setminus B$), Cartesian product ($A \times B$)

Descriptions: complement (complement of $A = A^c = U - A$), disjoint, cardinality ($|A|$), subset ($X \subset Y \rightarrow$ "x in X implies x in Y", "all elements in X also in Y") [also $\subsetneq$]

(*) Note: cardinality = "# of elements" for finite sets; "complicated" for infinite sets

Review: Functions

Def: A function f from set A to B ($f: A \rightarrow B$) assigns elements $x \in A$ to elements $f(x) \in B$

(*) Note: The "f()" notation can be overloaded to refer to both sets ($f(A)$) vs elements ($f(a)$)

Notes: • A and B are called the domain and codomain, respectively; $f(A) = \{f(a) \in B \mid a \in A\}$ is called the range

• Given subset $A' \subset A$, we define the image of A' under f to be $f(A') = \{f(x) \mid x \in A'\} \subset B$

• Given subset $B' \subset B$, we define the preimage of B' to be $f^{-1}(B') = \{x \in A \mid f(x) \in B'\} \subset A$

Defs.: • A function $f: A \rightarrow B$ is called injective (one-to-one) if distinct elements of A are mapped to distinct elements of B ("$x, y \in A; x \neq y$" $\rightarrow$ "$f(x) \neq f(y)$"; $|f^{-1}(\{y\})| = 1 \ \forall y \in B$)

• A function $f: A \rightarrow B$ is called surjective (onto) if every element in B is mapped to by at least one element in A (range = codomain; $f^{-1}(\{y\}) \neq \emptyset \ \forall y \in B$)

• A function is called bijective (invertible) if it is both injective and surjective; implies $|A| = |B|$

Review: Equivalence Relations

10/2/23
Lecture 2

Common Sets

- Set of natural numbers $:= \mathbb{N} = \{1, 2, 3, \ldots\}$; whole numbers $:= \mathbb{W} = \{0, 1, 2, 3, \ldots\}$
- Set of integers $:= \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ [German: integer = "Zahlen"]

Review: Equivalence Relations

Def.: An equivalence relation is a relation $\sim$ that is reflexive ($p \sim p$), symmetric ($p \sim q$ → "$q \sim p$"), and transitive ($p \sim q, q \sim r \to p \sim r$).

If two objects p and q are related by an equivalence relation, we say $p = q$.

Def.: An equivalence class is any group of objects related to each other by an equivalence relation.

Constructing $\mathbb{Q}$

To construct a set of rational numbers $\mathbb{Q}$ from $\mathbb{Z}$, we first define the set A of all pairs (p, q) s.t. $p, q \in \mathbb{Z}$, $q \neq 0$.

↓

Next, we define the equivalence relation $\sim$, where $(p_1, q_1) \sim (p_2, q_2)$ if $p_1 q_2 = p_2 q_1$ $\left[\frac{p_1}{q_1} = \frac{p_2}{q_2}\right]$.
For an equivalence class (p, q), it is denoted $\frac{p}{q}$.

↓

We finally define a "rational number" to be any such equivalence class $\frac{p}{q}$, and the set of all rational numbers to be the set:

$$\boxed{\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}}$$

(*) Addition: $\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$; Multiplication: $\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2}$

(*) Absolute value ($\mathbb{Z}$): $|n| := \{n \text{ if } n \in \mathbb{N};\ 0 \text{ if } n = 0;\ -n \text{ if } -n \in \mathbb{N}\}$ $\left(\left|\frac{p}{q}\right| = \frac{|p|}{|q|}\right)$

Cauchy Sequences & Constructing $\mathbb{R}$

10/2/23
Lecture 2
(cont.)

Cauchy Sequences

Def.: Let $(a_n)_{n=0}^{+\infty}$ be a sequence of rational numbers s.t. $a_n \in \mathbb{Q}\ \forall n$. We call $(a_n)_{n=0}^{+\infty}$ a Cauchy sequence if, for every $r \in \mathbb{Q}_+$, $\exists N \in \mathbb{N}$ such that $|x_n - x_m| < r$ for all $n, m \geq N$ (i.e. $\lim_{n,m\to\infty} |a_n - a_m| = 0$).

Note: Cauchy sequences cannot in general be concluded to have a finite limit [converge]; however, for any space where not all Cauchy sequences converge, $\exists$ a higher space where all Cauchy sequences converge. (Such a space is called "complete"; this is often done in topology). (Ex: the Cauchy sequence tending towards π does not converge in $\mathbb{Q}$, but does in $\mathbb{R}$).

Constructing $\mathbb{R}$

Method 1 (Decimal expansion): Let $(a_n)_{n=0}^{+\infty} \subset \{0, 1, \ldots, 9\}$. Let $s_n = a_0 + \frac{a_1}{10} + \ldots + \frac{a_n}{10^n}$ (this is the decimal expansion $a_0.a_1a_2\ldots a_n$). Observe that for $n > m$, $s_n - s_m = \frac{a_{m+1}}{10^{m+1}} + \ldots + \frac{a_n}{10^n}$, and that $\frac{a_{m+1}}{10^{m+1}} + \ldots + \frac{a_n}{10^n} \leq \frac{9}{10^{m+1}} + \ldots + \frac{9}{10^n} = \frac{9}{10^{m+1}}\left(1 + \frac{1}{10} + \ldots + \frac{1}{10^{n-m-1}}\right)$.

Let $r = \frac{1}{10}$. Then $1 + \frac{1}{10} + \ldots + \frac{1}{10^{n-m-1}} = 1 + r + \ldots + r^{n-m-1} = \frac{1-r^{n-m}}{1-r} \leq \frac{1}{1-r}$ (if $0<r<1$) $= \frac{10}{9}$ ($r = \frac{1}{10}$).

$\to s_n - s_m = \frac{a_{m+1}}{10^{m+1}} + \ldots + \frac{a_n}{10^n} \leq \frac{9}{10^{m+1}}\left(1 + \frac{1}{10} + \ldots + \frac{1}{10^{n-m-1}}\right) = \frac{1}{10^m} \to \lim_{n,m\to\infty}(s_n - s_m) = 0$ [is a Cauchy sequence].

$\to$ Def.: We say that $(a_n)_{n=0}^{+\infty} \subset \{0, \ldots, 9\}$ and $(b_n)_{n=0}^{\infty} \subset \{0, \ldots, 9\}$ are equivalent if $\lim_{n\to\infty}\left|\left(a_0 + \frac{a_1}{10} + \ldots + \frac{a_n}{10^n}\right) - \left(b_0 + \frac{b_1}{10} + \ldots + \frac{b_n}{10^n}\right)\right| = 0$, and call the resulting equivalence classes "real numbers". (* Downside: this only defines the positive real numbers)

Method 2 (General): Let $(s_n)_{n=0}^{+\infty}$ and $(t_n)_{n=0}^{+\infty}$ be two Cauchy sequences. We say that $(s_n)_{n=0}^{+\infty}$ and $(t_n)_{n=0}^{+\infty}$ are equivalent if $\lim_{n\to\infty}|s_n - t_n| = 0$. We call these equivalence classes $(s_n)_{n=0}^{+\infty}$ "real numbers". (We have moved to a new, complete space: $\mathbb{R}$).

Operations in $\mathbb{R}$

10/4/23

Lecture 3

Cauchy Sequences & $\mathbb{R}$

Notation: If $a, b \in \mathbb{R}$ and $a-b \in \mathbb{R}_+$, we write $\underline{a>b}$. ("$a \geq b$" means "$a>b$ or $a=b$").

(*) Reminder: all Cauchy sequences (incl. all $x \in \mathbb{R}$) are bounded.

Operations: If $(s_n)_{n=0}^{+\infty}$ and $(t_n)_{n=0}^{+\infty}$ are Cauchy sequences in $\mathbb{Q}$, then $(s_n+t_n)_{n=0}^{+\infty}$ and $(s_nt_n)_{n=0}^{+\infty}$ are also Cauchy sequences in $\mathbb{Q}$. If $[(s_n)_{n=0}^{+\infty}]$ denotes the class of equivalence of (s_n), we define:

1) $[(s_n)_{n=0}^{+\infty}] + [(t_n)_{n=0}^{+\infty}] = [(s_n+t_n)_{n=0}^{+\infty}]$ (Addition in $\mathbb{R}$: $\underline{+}$)
2) $[(s_n)_{n=0}^{+\infty}] \cdot [(t_n)_{n=0}^{+\infty}] = [(s_nt_n)_{n=0}^{+\infty}]$ (Multiplication in $\mathbb{R}$: $\underline{\cdot}$)

Properties of $\mathbb{R}$

Let $a, b, c, d \in \mathbb{R}$:
(1) Commutativity of addition & multiplication
(2) Associativity of addition & multiplication
(3) Distributivity: $a(b+c) = ab + ac$
(4) $\exists$ distinct elements $0, 1$ s.t. $a+0=a$; $a \cdot 1 = a$
(5) $\exists$ an element $\in \mathbb{R}$ denoted $-a$ s.t. $a+(-a)=0$
(6) If $a \neq 0$, $\exists$ an element $\in \mathbb{R}$ denoted a^{-1} s.t. $a \cdot a^{-1} = 1$.

(*) We call any space satisfying (1)-(6) a field (so $(\mathbb{R}, +, \cdot)$ is a field).
(*) That $\mathbb{R}$ fulfills (1)-(6) can be concluded by its construction in $\mathbb{Q}$,
i.e.: (1)-(6) hold for $\mathbb{Z} \to$ hold for $\mathbb{Q} \to$ hold for $\mathbb{R}$

Upper & Lower Bounds

10/4/23
Lecture 3
(cont.)

(*) <u>Consequences of $\mathbb{R}$ as a Field</u>

(1) The element 1 is unique.
(2) a^{-1} is unique $\forall a \neq 0$ in $\mathbb{R}$.
(3) $a > b$, $b > c$ implies $a > c$.
(4) $a > b$, $c \geq d$ implies $a + c > b + d$
(5) $a > b > 0$, $c \geq d > 0 \rightarrow ac > bd$
(6) $a^2 \geq 0$ ($a^2 = 0$ iff $a = 0$)
(7) $\forall a, b \, \exists$ unique x s.t. $a + x = b$
(8) $a > 0$ implies $a^{-1} > 0$
(9) $a > b > 0$ implies $a^{-1} < b^{-1}$

<u>Def: Upper and Lower Bounds</u>

Let $S \subset \mathbb{R}$ be a non-empty set.

(i) We call $m \in \mathbb{R}$ an <u>upper bound</u> for S if $x \leq m \; \forall x \in S$.

(ii) We call $\bar{m}$ a <u>least upper bound</u> for S if: (1) $\bar{m}$ is an upper bound for S and
(2) $\bar{m} \leq m$ for any upper bound m for S.

(iii) We call $m \in \mathbb{R}$ a <u>lower bound</u> for S if $m \leq x \; \forall x \in S$.

(iv) We call $\bar{m}$ a <u>greatest lower bound</u> for S if: (1) $\bar{m}$ is a lower bound for S and
(2) $\bar{m} \geq m$ for any lower bound m for S.

<u>Notes on Upper & Lower Bounds</u>

- Any greatest lower bounds or least upper bounds, if they exist, are unique.
- Axiom: The existence of an upper bound in $\mathbb{R}$ implies the existence of a least upper bound (and likewise for lower bound - greatest lower bound).
  (see: Axiom of Choice & Zorn's lemma)

(*) <u>Characterization of Least Upper Bound</u>:

- Let $S \subset \mathbb{R}$ be a nonempty set which is bounded from above (i.e. has an upper bound). Then <u>$m = \text{l.u.b.}(S)$</u> iff: (1) m is an upper bound for S and
(2) <u>for every $\varepsilon > 0 \; \exists \, s_\varepsilon \in S$ s.t. $s_\varepsilon + \varepsilon > m$</u>.

- Notation: "least upper bound for S" = <u>$\text{l.u.b.}(S)$</u> = <u>$\sup(S)$</u> [supremum]
"greatest lower bound for S" = <u>$\text{g.l.b.}(S)$</u> = <u>$\inf(S)$</u> [infinum]

Notes: Epsilon of Room

10/5/23
Disc. 2

(*) "Epsilon of Room"

Prop: An inequality $X \leq Y$ can be proven by showing that $X < Y + \varepsilon$ for all $\varepsilon > 0$.

(*) Monotone Convergence Theorem

Reminder: $\lim_{n\to\infty} a_n = a$ if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that if $n > N_\varepsilon$, $|a_n - a| < \varepsilon$.

Lemma: If a sequence $(a_n)_{n=0}^{+\infty}$ is increasing and bounded above, then $\lim_{n\to\infty}(a_n) = \sup(a_n)$.

(*) Proof: Fix $\varepsilon > 0$. Then $\exists N$ such that $a_N > \sup(a_n) - \varepsilon$. Then $\sup(a_n) \geq \lim_{n\to\infty} a_n \geq a_n > \sup(a_n) - \varepsilon$
$\to \alpha + \varepsilon > \lim_{n\to\infty} a_n > \alpha - \varepsilon \to |\lim_n(a_n) - \sup(a_n)| < \varepsilon \to$ take ε to $0 \to \lim_{n\to\infty}(a_n) = \sup(a_n)$

(*) Supremum & Infimum

Lemma: Let $A \subset \mathbb{R}$ with $\inf(A) > 0$. Then $\sup(\frac{1}{A}) = \frac{1}{\inf(A)}$.

$\alpha = \inf(A)$

(*) Proof: Claim $a \geq \alpha \iff \frac{1}{a} \leq \frac{1}{\alpha}$. Fix $\varepsilon > 0$. Then $\exists a \in A$ w/ $\alpha + \varepsilon > a$. This implies $\frac{1}{\alpha+\varepsilon} < \frac{1}{a}$. Then $\sup(\frac{1}{A}) > \frac{1}{\alpha+\varepsilon} \to \lim_{\varepsilon\to 0}(\sup(\frac{1}{A}) > \frac{1}{\alpha+\varepsilon}) \to \sup(\frac{1}{A}) \geq \frac{1}{\alpha}$.

The Square Root

10/6/23
Lecture 4

(*) Axioms in $\mathbb{R}$

Reminder: Given any $x \in \mathbb{R}$, $\exists\ n \in \mathbb{N}$ s.t. $n \geq x$. Corollaries:

(i) Let $\varepsilon > 0$. Then $\exists\ n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

(ii) Let $x \in \mathbb{R}$. Then $\exists\ n \in \mathbb{Z}$ such that $n \leq x \leq n+1$.

(iii) Let $x \in \mathbb{R}$. Then for any $n \in \mathbb{Z}$ $\exists\ m \in \mathbb{Z}$ such that $\frac{n}{m} \leq x \leq \frac{n+1}{m}$.

Square Roots in $\mathbb{R}$

Def: If $\alpha \geq 0$, we call $x \in \mathbb{R}_+$ a square root of α if $x^2 = \alpha$.

(*) Properties of Square Roots in $\mathbb{R}$

- If $0 \leq x_1 < x_2$, $(x_1)^2 < (x_2)^2$. [If $\alpha = 0$, then $x = 0$.]
- Given $\alpha > 0$, $\exists$ a unique square root $x \in \mathbb{R}_+$ such that $x^2 = \alpha$.

(*) Proof: Existence of the Square Root

Claim: Let $S := \{x \in \mathbb{R}_+ : x^2 \leq \alpha\}$ for some $\alpha \geq 0$. Prove $\sup(S)$ exists & $\sup(S)^2 = \alpha$.

Proof: We claim that $m := \min(\alpha, 1) \in S$. (Proof: $m^2 = m \cdot m \leq m \cdot 1 \leq \alpha$).

Then the set S is non-empty.

We claim that S is bounded from above by $M = \max(\alpha, 1)$. Let $y > M$.

Then $y^2 = y \cdot y \geq y \cdot 1 = y \geq M \geq \alpha$; therefore $y \notin S$, and $S \subset (-\infty, M]$.

S is bounded from above; then $\sup(S)$ exists.

We know that $y_0^2 = \sup(S)^2 \leq \alpha$, and claim that $x_0^2 = \alpha$. To prove this, we prove $(x_0)^2 \nless \alpha$.

Proof: Assume $(x_0)^2 < \alpha$. Then $\alpha - (x_0)^2 > 0$. Let $\delta > 0$ be a number such that $(x_0 + \delta)^2 < \alpha$. Then $(x_0 + \delta) \in S$; this is a contradiction, therefore $(x_0)^2 \nless \alpha$.

Metric Spaces

10/9/23
Lecture 5

Def: Metric Spaces (Chapter 3)

Let E be a non-empty set and let $d: E \times E \to [0, +\infty)$ be a function. We call d a metric (or a distance function) on E, and (E, d) a metric space, if the following hold $\forall\, p, q, r \in E$:

(i) $d(p,q) = 0$ iff $p = q$

(ii) $d(p,q) = d(q,p)$

(iii) $d(p,r) \le d(p,q) + d(q,r)$ [Triangle inequality]

(*) Example Metric Spaces

- Let $E = \mathbb{R}^n$ and $d(p,q) = |p-q| \to (E,d)$ [euclidean space $\mathbb{R}^n$] is a metric space.
- For any arbitrary set E, we can find metric space (E,d) with metric $d(p,q) = \{0 \text{ if } p = q;\ 1 \text{ if } p \ne q\}$.
- Let $n \in \mathbb{N}$, $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}$; $x, y \in \mathbb{R}^n$. Define metric spaces $(\mathbb{R}^n, d)$, where d is:
  - (i) $d_1(x,y) = |x_1 - y_1| + \dots + |x_n - y_n| = \|x-y\|_{\ell_1}$ [norm of $x-y$]
  - (ii) $d_\infty(x,y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\} = \|x-y\|_{\ell_\infty}$ [infinity norm]
  - (iii) $d_p(x,y) = (|x_1-y_1|^p + \dots + |x_n - y_n|^p)^{1/p} = \|x-y\|_{\ell_p}$ $(1 \le p < +\infty)$ [
    - (*) $\lim_{p\to\infty} d_p(x,y) = d_\infty(x,y)$

(*) **Proof: $d_2(x,y)$ is metric**

Reminder: Given $x, y \in \mathbb{R}^n$, let the set inner product $\langle x, y\rangle = x_1 y_1 + \dots + x_n y_n$. Then $\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{\langle x, x\rangle}$.

$\boxed{\to \text{Cauchy-Schwarz inequality: } |\langle x, y\rangle| \le \|x\|_2 \cdot \|y\|_2}$

(*) Proof: Let $\alpha, \beta \in \mathbb{R}$. Have $0 \le \|\alpha x + \beta y\|_2^2 = \alpha^2 x^2 + \beta^2 y^2 + 2\alpha\beta\langle x, y\rangle$.

Set $\alpha = \pm\|y\|_2$, $\beta = \pm\|x\|_2 \to 0 \le 2\|x\|_2^2\|y\|_2^2 \pm 2\|x\|_2\|y\|_2\langle x, y\rangle$

$\to \pm\langle x, y\rangle \le \|x\|_2 \cdot \|y\|_2$. □

$\boxed{(\ast)\ \text{Corollary: } \|x+y\|_2 \le \|x\|_2 + \|y\|_2}$

Then to prove d_2 is metric, we need only use the corollary to prove the triangle inequality.

Open & Closed Sets

10/9/23

Lecture 5 (cont.)

Notes on Metric Spaces

- Notation: We call the elements of a metric space (E,d) <u>points</u>, and $d(p,q)$ the <u>distance</u> from p to q.
- Def: Given metric space (E,d), then for any $S\subset E$ we call (S,d) a <u>[metric] subspace</u> of (E,d).
- Reminder: Given $x,y\in\mathbb{R}^n$, define <u>inner product</u> $\langle x,y\rangle = x_1y_1+\ldots+x_ny_n$
  - (*) <u>Cauchy-Schwarz inequality</u>: $|\langle x,y\rangle| \le \|x^2\|\cdot\|y^2\|$
  - (*) <u>Corollary</u>: $\|x+y\|_2 \le \|x\|_2+\|y\|_2$
- (*) To prove $d_p(x,y)$ is metric, need Young's inequality: $a,b\in\mathbb{R}\to ab\le \frac{a^p}{p}\cdot\frac{b^{\frac{p}{p-1}}}{\frac{p}{p-1}}$ [$\frac{p}{p-1}$: dual conjugate of p]
- (*) Def: Let (E,d) be a metric space; $K\in\mathbb{R}$. We call a function $f:E\to\mathbb{R}$ <u>K-Lipschitz</u> or <u>Lipschitz continuous</u> if $|f(p)-f(q)|\le K\cdot d(p,q)\ \forall p,q\in E$
  - (*) Ex: The function $f(p)=d(p,r)$ [$p,r\in E$] is <u>1</u>-Lipschitz

Open & Closed Sets (§2)

Def: Let $p_0\in E$ and $r\ge 0$. Then:

(i) The <u>open ball</u> in E of center p_0 and radius r is the set <u>$D(p_0,r)=D_r(p_0)=\{p\in E; d(p,p_0)<r\}$</u>.

(ii) The <u>closed ball</u> in E of center p_0 and radius r is the set <u>$B(p_0,r)=B_r(p_0)=\{p\in E; d(p,p_0)\le r\}$</u>.

Ex: $E=\mathbb{R}^2$; $p_0=(0,0)$ $r=1$ $\rightarrow$  $[D(p_0,1)]$

metric d_1 metric d_2 metric d_∞

> **Def: <u>Open & Closed Sets</u>**
>
> Call subset $S\subset E$ an <u>open set</u> if either $S=\emptyset$ or for every $p_0\in S$ $\exists$ $r>0$ such that $D(p_0,r)\subset S$. Then we call the complement $E\setminus S$ a <u>closed set</u>.
>
> (*) <u>Corollary</u>: A subset $S\subset E$ is <u>closed</u> if its complement $E\setminus S$ is open.

Open & Closed Sets (cont.)

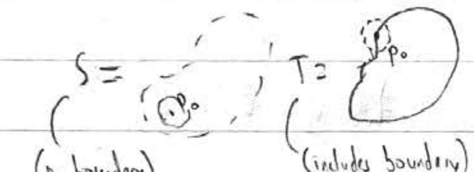
10/11/23
Lecture 6

Notes on Open/Closed Sets

- The definition of open/closed sets would be unchanged if written to specify $B(p_0, r)$ instead of $D(p_0, r)$
- For any arbitrary set E, E is open under the metric $d(p,q) = \{0 \text{ if } p=q;\ 1 \text{ if } p \neq q\}$

Openness/Closedness Examples

- Let $E = \mathbb{R}$, $d(p,q)$ → any interval (a,b) in $\mathbb{R}$ is equivalent to open ball $D\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$ $[b > a]$
- Let S, T be two sets:



→ Then for any $p_0 \in S$, p_0 not on boundary(S) → can draw a ball; S is open

↳ A $p_0 \in T$ may be on boundary(T) → a ball @ $p_0 \not\subset T$; T is closed

Remarks on Openness & Closedness

(i) Let (E, d) be a metric space. Then E is open.

(ii) Any union of open sets is open.

(*) Proof: Let $O = \bigcup_{i \in I} O_i$. Then for any $p_0 \in O$, $D(p_0, r) \subset O_i \subset O$ (for some $i \in I$).

(iii) The intersection of open sets need not be open.

(*) Proof: Let $O_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$. Then $\bigcap_{i \in I} O_i = \{0\}$ is not open.

(iv) The empty set $\emptyset$ is closed in any space (E, d).

(*) Proof: $(\emptyset)^c = E$ is open per (i); therefore, $\emptyset$ must be closed.

(v) The intersection of closed sets is closed.

(*) Proof: Let $S = \bigcap_{i \in I} S_i$ (S_i closed). Then $S^c = \left(\bigcap_{i \in I} S_i\right)^c = \bigcup_{i \in I} (S_i^c)$ [DeMorgan's Law], and the union of open sets is open per (ii).

(vi) The finite union of closed sets is closed.

Open & Closed Sets (cont.)

10/11/23
Lecture 6 (cont.)

(*) <u>Exercises on Openness & Closedness</u>

(i) Exercise: Show that if $p_0 \in E$ and $r > 0$, then $D(p_0, r)$ is an open subset of E.

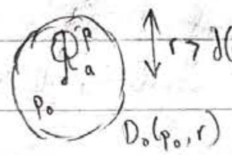
Proof: We want to show that for any $a \in D(p_0, r)$, $\exists$ some $\varepsilon > 0$ such that $\underline{D(a,\varepsilon) \subset D(p_0, r)}$.

To do this, we want to prove that for any $p \in D(a, \varepsilon)$, $\underline{d(p_0, a) + d(a, p) < r}$.

$\rightarrow$ Take any $a \in D(p_0, r)$. Set $\varepsilon = r - d(p_0, a)$. Then for any $p \in D(a, \varepsilon)$,

$d(p, a) < \varepsilon = r - d(p_0, a) \Rightarrow \underline{d(p, p_0) \le d(p, a) + d(p_0, a) < r}$.

Then $p \in D(p_0, r)$, therefore $D(a, \varepsilon) \subset D(p_0, r)$.



$\updownarrow r > d(p, p_0) \updownarrow \Rightarrow p \in D(p_0, r)$ [Drawing: $E = \mathbb{R}^2$]

(ii) Exercise: Show that if $p_0 \in E$ and $r > 0$, then $B(p_0, r)$ is a closed subset of E.

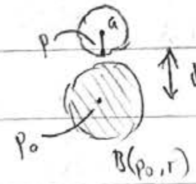
Proof: To prove that $B(p_0, r)$ is closed, we prove that the complement $(B(p_0,r))^c$ is open.

Let $a \in (B(p_0, r))^c = E \setminus B(p_0, r)$, such that $d(p_0, a) > r$. Set $\varepsilon = d(p_0, a) - r$.

If $p \in D(a, \varepsilon)$ then $d(p, a) < \varepsilon = d(p_0, a) - r \Rightarrow r < d(p_0, a) - d(p, a) \le d(p_0, p)$.

Then $p \notin B(p_0, r)$; therefore, for any $a \in (B(p_0, r))^c$, $\exists\, \varepsilon > 0$ such that

$D(a, \varepsilon) \subset (B(p_0, r))^c$, therefore $(B(p_0, r))^c$ is open.



$\updownarrow d(p, p_0) > r \updownarrow \Rightarrow p \notin B(p_0, r)$ [Drawing: $E = \mathbb{R}^2$]

<u>Def: Bounded Sets</u>

Let (E, d) be a metric space. Then we call $S \subset E$ <u>bounded</u> if $\exists\, r > 0$ and $p_0 \in E$ such that $S \subset D(p_0, r)$, i.e. $d(s, p_0) < r \;\forall s \in S$.

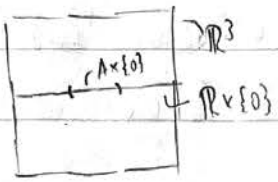
(*) Openness

10/12/23
Discussion 3
+ Disc 6

Openness

"A set is both open and closed" - openness is relative to the space being considered.

Ex: $A = (-1, 1)$ is open relative to $\mathbb{R}$, but $A \times \{0\}$ is closed in $\mathbb{R}^2$



Let (E, d) a metric space; $A \subset E$. Then an open set O of A is of the form of an open set O' in E, intersection A [i.e. $O = O' \cap A$, where $O' \cap A$ open in E].

(*) A subset $S \subset E$ is open iff it can be expressed/covered by a union of open balls $D(p \in S, r_p)$.

(*) For any metric space (E, d), the sets E and $\emptyset$ are considered both closed and open in E.

Ex: $(-1, 1]$ is open relative to its own topology, but is closed in $\mathbb{R}$.

(*) Arithmetic-Geometric Inequality

Arithmetic-Geometric Inequality: $A_n \geq G_n \geq H_n$ [AM-GM]

Arithmetic mean: $A_n = \frac{a_1 + \dots + a_n}{n}$

Geometric mean: $G_n = (a_1 \cdot \dots \cdot a_n)^{1/n}$ (may be used for means across orders of magnitude)

Harmonic mean: $H_n = \frac{1}{1/a_1 + \dots + 1/a_n}$

(*) Metric Space Topology

(*) A set A is open if the set of interior points of A is equal to A [$\text{interior}(A) = A$]

(*) Interior point: p is an interior point if $\exists\ \varepsilon > 0$ s.t. $D(p, \varepsilon) \subset S$

(*) A set B is closed if B contains all of its limit points; alt: closure of B: $\text{cl} B = B$

(*) Closure of B: $B \cup \{\text{limit points of } B\} = \text{cl} B$

Convergent Sequences

10/17/23
Lecture 7

(*) Exercises: Openness and Closedness

Lemma: Let $S \subset \mathbb{R}$ be a nonempty closed set. Then if S is bounded from above, S has a maximum element.

(*) Proof: We know $\sup(S)$ exists, and claim $\sup(S) \in S$. Assume $\sup(S) = M \notin S$; then $M \in S^c$.
Since S^c open, $\exists\, M-\varepsilon \in S^c$; then $M-\varepsilon$ is an upper bound of S, $M-\varepsilon < M$. [Contradiction]

Lemma: Let $S \subset E$ be a nonempty set. Then the following are equivalent: (i) S is open and (ii) S is a union of open balls.

(*) Proof: (ii)$\to$(i) is trivial; for (i)$\to$(ii), "S is open" implies for every $p_0 \in S$, $\exists\, r>0$ s.t. $D(p_0, r) \subset S$. ∎

Def: Convergent Sequences (§3)

Given a function $\phi: \mathbb{N} \to E$ or $\phi: \mathbb{N} \cup \{0\} \to E$ [(E,d) a metric space], we call ϕ a sequence in E and we write $(\phi(n))_{n=1}^{+\infty} \subset E$. It is customary to set $x_n = \phi(n)$ and use notation $(x_n)_{n=1}^{+\infty}$ without showing ϕ.

(i) If the set $\{\phi: 1, 2, \ldots\}$ is bounded, then we say $(\phi(n))_{n=1}^{\infty}$ is bounded.

(ii) Given $x \in E$, we say that $(x_n)_{n=1}^{+\infty}$ converges to x and write $\lim_{n\to\infty}(x_n) = x$ if, for every $\varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon \;\forall\, n \geq N_\varepsilon$.

Exercises on Convergence

Let $(x_n) \subset E$ be a sequence.

(i) Prop: (x_n) cannot have 2 distinct limits.

Proof: Assume (x_n) converges to x and y, $x \neq y$. Set $\varepsilon = \frac{1}{2} d(x,y)$. Then $\exists\, N_\varepsilon^1$ such that $d(x_n, x) < \varepsilon \;\forall\, n \geq N_\varepsilon^1$; and $\exists\, N_\varepsilon^2$ such that $d(x_n, y) < \varepsilon \;\forall\, n \geq N_\varepsilon^2$. Set $N_\varepsilon = \max(N_\varepsilon^1, N_\varepsilon^2)$.
Then, for every $n \geq N_\varepsilon$, $d(x,y) \leq d(x_n, x) + d(x_n, y) < 2\varepsilon = d(x,y)$. [Contradiction] ∎

(*) Sequence vs set of a sequence:

- Sequence $\phi(n)$: $(\phi(n))_{n=1}^{+\infty} = (0, 1, 0, 1, \ldots)$ [ex]
- Set of $\phi(n)$: $\{\phi: 1, 2, \ldots\} = \{0, 1\}$ [ex]

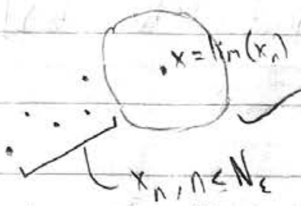
Subsequences

10/13/23
Lecture 7
(cont.)

Exercises on Convergence (cont.)

(ii) **Prop**: If (x_n) is convergent, then (x_n) is bounded.

Proof: Let $x = \lim_{n\to\infty}(x_n)$. Then:



(Visualization in $\mathbb{R}^2$)

Set $\varepsilon = 1$; then $\exists\, N \in \mathbb{N}$ such that $d(x_n, x) < 1 \ \forall\, n \geq N$.

Set $M = (A = d(x, x_1) + \dots + d(x, x_{N-1})) + 1$.

Then $d(x_n, x) = \{ M < 1 \text{ if } n \geq N;\ m < A \text{ if } n \leq N \}$

Then $\{x_n : n \in \mathbb{N}\} \subset B(x, M)$. ∎

(*) Note: Boundedness does not imply convergence (ex: $(1, -1, 1, -1, \dots)$ bounded, but not convergent)

> **Def: Subsequences**
>
> Let $\phi : \mathbb{N} \to E$ be a sequence in E, and let $\varphi : \mathbb{N} \to \mathbb{N}$ be strictly monotone [i.e. a function where "$x < y$" implies $\varphi(x) < \varphi(y)$]. We call $\phi \circ \varphi$ a subsequence of ϕ.

Exercise: Convergent Subsequences

Prop: Suppose (x_n) converges to x. Then any subsequence $(x_{n_k})_{n_k=1}^{\infty}$ of (x_n) converges to x.

Proof: Given $\varepsilon > 0$, $\exists\, N_\varepsilon$ such that $d(x, x_n) < \varepsilon \ \forall\, n \geq N_\varepsilon$. We claim that $n_k \geq k \ \forall\, k$.

Proof: $\varphi(1) \geq 1$. Then it must be that $\varphi(2) > \varphi(1) \geq 1$; then $\varphi(2) \geq 2$. This can be extended to all k.

$n_k \geq k \ \forall\, k$. If $k \geq N_\varepsilon$, then $n_k \geq N_\varepsilon$, therefore $d(x_n, n) < \varepsilon \ \forall\, k \geq N_\varepsilon$. ∎

Limits of Sequences

10/18/23
Lecture 8

Characterization of Closed Sets
Let $S \subset E$ be a nonempty set. Then the following are equivalent:
(i) S is a closed set
(ii) For any sequence $(x_n) \subset S$ converging to $x \in E$, $x \in S$ [i.e. any limit point of S, belongs to S]

(*) Exercises w/ Limits

Prop: Suppose $E = \mathbb{R}$, $d(p,q) = |q-p|$. Suppose $(a_n)_n, (b_n)_n \subset \mathbb{R}$ converge to $a, b \in \mathbb{R}$. Then:
(i) $\lim_{n\to\infty}(a_n + b_n) = a+b$; $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$
(ii) If $b \neq 0$, then for large enough n, $b_n \neq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Proof (ii):
$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - a b_n}{b_n b}\right| = \frac{|(a_n - a)b + a(b - b_n)|}{|b_n b|} \leq |a_n - a|\frac{1}{|b|} + \left|\frac{a}{b}\right|\frac{1}{|b_n|}|b_n - b|$$
Let $\varepsilon = \frac{1}{3}|b|$; then $\exists N_\varepsilon$ s.t. $|b_n - b| < \varepsilon \ \forall n \geq N_\varepsilon$
$\rightarrow |a_n - a|\frac{1}{|b|} + \left|\frac{a}{b}\right|\frac{1}{|b_n|}|b_n - b| \leq |a_n - a|\frac{3}{2}\frac{1}{|b|} + \left|\frac{a}{|b|}\right|\frac{3}{2}\frac{1}{|b|}|b_n - b| \leq A\left(|a_n - a| + |b_n - b|\right)$,
where $A = \frac{3}{2}\frac{1}{|b|}\left(1 + \left|\frac{a}{b}\right|\right)$.
$\rightarrow$ Given $\varepsilon > 0$, then $\frac{\varepsilon}{2A} > 0 \rightarrow \exists N_1$ s.t. $|a_n - a| < \varepsilon/2A \ \forall n \geq N_1$;
$\exists N_2$ s.t. $|b_n - b| < \varepsilon/2A \ \forall n \geq N_2$.
Set $N = \max(N_1, N_2) \rightarrow |a_n - a|, |b_n - b| < \varepsilon/2A \ \forall n \geq N \rightarrow \left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \varepsilon \ \forall n \geq N \rightarrow$ limit $= \frac{a}{b}$. ∎

Claim: Let $(a_n)_n, (b_n)_n \subset \mathbb{R}$ be sequences converging to $a, b \in \mathbb{R}$. If $a_n \leq b_n \ \forall n$, then $a \leq b$.
Proof: $a - b = \lim_{n\to\infty}(a_n - b_n)$. We know $(a_n - b_n)_n \subset (-\infty, 0]$. Let $S = (-\infty, 0]$.
S is closed, therefore: since $(a_n - b_n)_n \subset S$, then $\lim_{n\to\infty}(a_n - b_n) \in S$
$\Rightarrow (a-b) \in S \Rightarrow a \leq b$. ∎

Monotonic Sequences

10/16/23

Lecture 8

(cont.)

Def: Monotonic Sequences

We call a sequence $(a_n)_{n=1}^{+\infty} \subseteq \mathbb{R}$ monotonic (alt: "monotone") if one of the following holds for all $n \in \mathbb{N}$:

(i) $a_n \leq a_{n+1} \Rightarrow (a_n)$ is monotone non-decreasing

(ii) $a_n \geq a_{n+1} \Rightarrow (a_n)$ is monotone non-increasing

(*) Monotone Convergence Theorem

Theorem: Let $(a_n)_{n=1}^{+\infty} \subset \mathbb{R}$ be monotonic and bounded. Then (a_n) converges to either $\sup(S)$ or $\inf(S)$, where $S := \{a_n : n \in \mathbb{N}\}$.

Proof: (Replace a_n with $-a_n$ as needed). Assume (a_n) is monotone non-decreasing. Set $a = \sup(S)$. Set $\varepsilon > 0$; then $\exists N$ such that $|a_n - a| < \varepsilon \ \forall n \geq N$. Then $a < a_n + \varepsilon \ \forall n \geq N$. Then $|a - a_n| < \varepsilon \Rightarrow \lim_{n\to\infty}(a_n) = a$.

(*) Exercise: Let $a \in (-1, 1)$. Show that $\lim_{n\to\infty}(a^n) = 0$.

Proof: (i) $a \in [0,1)$. Set $x_n = a^n$; we know x_n is monotone non-decreasing, and therefore converges to some $x \in [0,1]$. We have $x = \lim_{n\to\infty}(x_{n+1}) = \lim_{n\to\infty}(a x_n) = x^2$. Then $x = 0$ or $x = 1$; therefore $x = 0$.

(ii) $a \in (-1, 0)$. Then we can divide the sequence (x_n) $[x_n = a^n]$ into two subsequences $y_n = a^{2n}$, $z_n = a^{2n+1}$. Then both $(y_n)_n$, $(-z_n)_n$ are sequences as described by (i); therefore $\lim_{n\to\infty}(y_n) = \lim_{n\to\infty}(z_n) = 0 \Rightarrow \lim_{n\to\infty}(x_n) = 0$.

Notes on Cauchy Sequences

10/18/23

Lecture 9

Notes on Cauchy Sequences

Reminder: A sequence $(x_n)_{n=1}^{+\infty}$ is called <u>Cauchy</u> if, for every $\varepsilon>0$, $\exists N\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ $\forall n,m\geq N$.

(*) Alt: $\lim_{(n,m)\to(\infty,\infty)} d(x_n,x_m)=0$

Lemma: Every convergent sequence $\subset E$ is Cauchy, but not every Cauchy sequence is necessarily convergent.

(*) Ex: The sequence $(x_n)_n=\frac{1}{n}$ is Cauchy, but not convergent, on $E=(0,\infty)$ $[\lim(x_n)=0\notin E]$.

Prop: Let $(x_n)_{n=1}^{+\infty}\subset E$ be a Cauchy sequence. Then:

(i) (x_n) is bounded

(ii) Every subsequence of (x_n) is Cauchy

(*) Proof (i): Given $\varepsilon=1$, choose $N\in\mathbb{N}$ such that $d(x_n,x_n)<\varepsilon$ $\forall n\geq N$.
Set $M=d(x_1,x_N)+\dots+d(x_{N-1},x_N)+1$; then $(x_n)_n\subset B(x_N,M)\Rightarrow(x_n)$ bounded. ∎

(*) Proof (ii): Let $\phi:\mathbb{N}\to\mathbb{N}$ be strictly increasing, and let $y_n=x_{\phi(n)}$ such that (y_n) is a subsequence of (x_n). Given $\varepsilon>0$, $\exists N\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ $\forall x_n,x_m\geq N$.
Fix ε. Then $d(y_n,y_m)<\varepsilon$ $\forall y_n,y_m\geq N$, therefore (y_n) is Cauchy. ∎

(*) Notes on Sequences

(*) Notation (Subsequences): We can define a subsequence as follows: let $\phi:\mathbb{N}\to\mathbb{N}$ be strictly monotone increasing. Let $(x_n)_{n=1}^{+\infty}$ be a sequence; then the sequence $(y_n)_{n=1}^{+\infty}$, where $y_n=x_{\phi(n)}$, is a subsequence of $(x_n)_{n=1}^{+\infty}$. Any subsequence of (x_n) can be expressed in this form.

(*) A sequence can be Cauchy but not convergent if the space occupied is too small, s.t. $\lim_{n\to\infty}(x_n)\notin E$.
See the example above: $\frac{1}{n}\in(0,+\infty)$ $\forall n>0$, but the space is not big enough to contain $\lim_{n\to\infty}(x_n)=0$.

Completeness

10/18/23 Lecture 9 (cont.)

Def. Completeness (§5)

Def. Given a metric space (E, d), we say that (E, d) is complete if every Cauchy sequence $[\subset E]$ converges $[$in $E]$.

Notes on Completeness

Remark: If (E,d) is complete and $S \subset E$ is closed, then S is complete.

(*) Proof: Let $(x_n)_{n=1}^{+\infty}$ be a Cauchy sequence in S. Since E is complete, then (x_n) converges to some $x \in E$. Since $(x_n) \subset S$, and S is closed, then $x \in S$ for any $(x_n) \subset S$.

Theorem: If $E = \mathbb{R}$, $d(p,q) = |p-q|$ $\forall p, q \in \mathbb{R}$, then $\mathbb{R}$ is complete.

(*) Proof: We know that (i) $\mathbb{R}$ is an ordered set and (ii) if $S \subset \mathbb{R}$ is bounded above, then $\sup(S) \in \mathbb{R}$ [for any such $S \subset \mathbb{R}$]. We want to prove that this implies $\mathbb{R}$ is complete.

Let $(a_n)_{n=1}^{+\infty} \subset \mathbb{R}$ be Cauchy. We want to show that (a_n) converges in $\mathbb{R}$.

Set $S := \{x \in \mathbb{R} : x \leq a_n$ for an infinite number of $n\}$. Since (a_n) is Cauchy, then it (a_n) is bounded and has a lower bound m & an upper bound n. Then $m \in S$ and n is an upper bound for S. Therefore $\alpha = \sup(S)$ exists in $\mathbb{R}$.

Let $\varepsilon > 0$. Since $\alpha + \varepsilon/2 \notin S$, $\exists N_1 \in \mathbb{N}$ such that $a_n < \alpha + \varepsilon/2$ $\forall n \geq N_1$.

Since (a_n) is Cauchy, $\exists N_2$ such that $d(a_n, a_m) < \varepsilon/2$ $\forall n, m \geq N_2$. Then, since $\alpha - \varepsilon/2 \in S$, $\alpha - \varepsilon/2 < a_n$ for infinitely many n: then for at least one $n_0 \geq N_1$ & $n_0 \geq N_2$, we have $\alpha - \varepsilon/2 \leq a_{n_0}$.

$\rightarrow$ For any $n \geq \max(N_1, N_2)$, $|a_n - a| \leq |a_n - \alpha| + |a_{n_0} - \alpha| \Rightarrow |a_n - \alpha| \leq \varepsilon$, therefore (a_n) converges to α (and we know $\alpha \in \mathbb{R}$). ∎

(*) Corollary: For $n \geq 1$, $\mathbb{R}^n$ is complete under the metric $d_{e_p}(x,y)$.

Reasoning: $|x_i - y_i| \leq d_{e_p}(x,y)$ under this metric

Compactness

10/20/23

Lecture 10

Def: Compactness

Let $S \subset E$. Then:

(i) A collection $(O_i)_{i \in I}$ of subsets of E is called a cover of E if $E = \bigcup_{i \in I} O_i$. The cover of E is open if each O_i is open; it is finite if $|\{O_i : i \in I\}|$ is finite.

(ii) We say that S is compact if, for every open cover $(O_i)_{i \in I}$ of S, there exists a finite $J \subseteq I$ such that $S = \bigcup_{i \in J} O_i$ ($\exists$ finite subcover $(O_i)_{i \in J}$ of $(O_i)_{i \in I}$).

Compactness

Prop: (i) If $S \subset E$ is closed and E is compact, then S is compact.

(ii) If $S \subset E$ is compact, then S is bounded.

(*) Proof (i): Suppose E compact, S closed and $(O_i)_{i \in I}$ is an open cover of S. We want to show that we can find a finite subcover of $(O_i)_{i \in I}$ [and thus S is compact].

We know $S = \bigcup_{i \in I} (O_i)$. We note that $F := \{S^c\} \cup \{O_i\}$ is an open cover for E.

Since E is compact, then we can find a finite subcover $O_1', \ldots, O_k', S^c$ of F, where $\{O_1', \ldots, O_k'\} \subset \{O_i : i \in I\}$. Then $E = (\bigcup_{i=1}^{k} O_i') \cup \{S^c\}$; since $S = E \setminus S^c$, then $S = \bigcup_{i=1}^{k} O_i'$. Then $\{O_i' : 1 \le i \le k\}$ is a finite subcover of $(O_i)_{i \in I}$. $\blacksquare$

(ii): Assume S is compact. We know S can be expressed as the union of open balls $S \subset \bigcup_{p \in E} D(p, 1)$.

Since S is compact, we can find a finite subcover $\rightarrow$ finite union of bounded sets is bounded.

Def: Cluster Points

Let $S \subset E$. Then:

We call $p \in E$ a cluster point [limit point] of S if, for every $\varepsilon > 0$, $(D(p,\varepsilon) \cap S) \setminus \{p\} \neq \emptyset$.

Equivalently: $D(p, \varepsilon)$ contains infinitely many points of S.

10/23/23
Lecture 11

Compactness (cont.)

Prop. (Nested Sets Property): Assume (E,d) is compact and that $(S_n)_{n=1}^{+\infty}$ is a sequence of closed subsets such that $S_{n+1} \subset S_n \ \forall n \in \mathbb{N}$. If $S_n \neq \emptyset \ \forall n$, then $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$. [A]

(*) Equivalently: For any sequence $(F_n)_{n=1}^{\infty}$ of closed subsets of E such that $\bigcap_{n \in I} F_n \neq \emptyset$ for any finite $I \subset \mathbb{N}$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. [A']

(*) Proof:

$A \to A'$: Let $S_n = \bigcap_{j=1}^{n} F_j$. Then if $S_n \neq \emptyset \ \forall n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

$A' \to A$: Assume, for the sake of contradiction, that $\bigcap_{n=1}^{\infty} S_n = \emptyset$. Set $O_n = S_n^c$. Then O_n is open, therefore $\bigcup_{n=1}^{\infty} O_n = \bigcup_{n=1}^{\infty} (S_n^c) = \left(\bigcap_{n=1}^{\infty} S_n\right)^c = \emptyset^c = E$. Since E is compact, then $\exists$ finite subcover $O_1, \dots, O_{n_1}$ such that $O_1 \cup \dots \cup O_{n_1} = E$. Then $S_{n_1}^c = (S_1 \cap \dots \cap S_{n_1})^c = \bigcup_{n=1}^{n_1} (S_n^c) = E$. Then $S_{n_1} = \emptyset$ [contradiction].

Nested sets: $S_3 \subset S_2 \subset S_1$

Prop. E is compact $\iff$ every sequence in E has a convergent subsequence.

Thm. If E is compact and $S \subset E$ has infinitely many elements, then S has a cluster point.

(*) Proof: Assume, for the sake of contradiction, that S has no cluster point. Then for any $p \in S$, p is not a cluster point, i.e. $\exists r_p > 0$ such that $D(p, r_p) \cap S$ has finitely many elements; call the number of elements n_p. Since E is compact, $E = \bigcup_{p \in E} D(p, r_p)$. Then $\exists$ $p_1, \dots, p_k$ such that $E = D(p_1, r_{p_1}) \cup \dots \cup D(p_k, r_{p_k})$. Since $S = S \cap E$, then $S = S \cap (D(p_1, r_{p_1}) \cup \dots \cup D(p_k, r_{p_k}))$. Then $|S| \leq n_{p_1} + \dots + n_{p_k}$; since $n_{p_1}, \dots, n_{p_k}$ are finite, therefore $|S|$ is finite. [contradiction].

Compactness (cont.)

10/23/23
Lecture 11
(cont.)

Corollary (from previous)

Suppose E is compact. Then the following are true:

(i) Every sequence in E has a convergent subsequence. [E is sequentially compact]

(ii) E is complete.

(iii) If $S \subset E$ is compact, then S is closed.

(*) Proof

(i) Suppose $(x_n)_n \subset E$. Set $S := \{x_n : n \in \mathbb{N}\}$.

(a) Case 1: $|S|$ is finite. Then $\exists\, p \in S$, $n_1 < n_2 < \dots$ such that $p = x_{n_i}\ \forall\, i \in \mathbb{N}$.
Then $(x_{n_i}) \subset (x_n)$ is a subsequence converging to p.

(b) Case 2: $|S| = +\infty$. Then we know S has a cluster point; then we can find a convergent subsequence by simply shrinking a ball around it.

(ii) Assume $(x_n)_n \subset E$ is Cauchy. By (i), we know $(x_n)_n$ has a convergent subsequence; then $(x_n)_n$ also converges in E.

(iii) Let $(x_n)_n \subset S$ be a sequence converging in E. We want to show that $\lim_{n \to \infty} (x_n) \in S$.
Since S is compact, then S is sequentially compact; then $\exists$ a subsequence $(y_n)_n \subset S$ of $(x_n)_n$ converging in S. Since $(y_n)_n$ is a subsequence of $(x_n)_n$, then $(x_n)_n$ must converge to the same limit; then $(x_n)_n$ converges to limit in S.

Heine - Borel Theorem

10/25/23
Lecture 12

Theorem (Heine-Borel)
A set $K \subset \mathbb{R}$ is compact if and only if K is bounded and closed.

($\Leftarrow$) Proof

We know that every compact set is closed and bounded, then it remains to show the converse. Since we know K is bounded, then $\exists\, a, b \in \mathbb{R}$ such that $a < b$ and $K \subset [a,b]$. We know that if $[a,b]$ is compact, then K being a closed subset of $[a,b]$ would imply K is compact. We claim $[a,b]$ is compact.

Proof: Let $(O_i)_{i\in I}$ be an open cover of $[a,b]$; we want to show $\exists$ a finite subcover. We know $\{a\}$ is compact; then $[a,a]$ is compact.

Let $F := \{x \in [a,b] : \exists\, I(x) \subset I,\ I(x) \text{ finite, such that } [a,x] \in \bigcup_{i\in I(x)} O_i\}$. We know $a \in F$, thus $F \neq \emptyset$; since $F \subset [a,b]$, then F is bounded [from above]. Then $M = \sup(F)$ exists and $M \leq b$; we want to show $M = b$.

(i) Claim: $M \in F$. Since $[a,b] \subset \bigcup_{i\in I} O_i$ and $M \in [a,b]$, $\exists\, \bar{i} \in I$ such that $M \in O_{\bar{i}}$.
Since $O_{\bar{i}}$ is open, we can find $\varepsilon > 0$ such that $(M-\varepsilon, M+\varepsilon) \in O_{\bar{i}}$. Since $M = \sup(F)$, $\exists\, x \in F \cap (M-\varepsilon, M+\varepsilon)$. We have $[a,M] = [a,x] \cup [x,M]$.
$[a,x] \cup [x,M] = \left(\bigcup_{i\in I(x)} O_i\right) \cup (M-\varepsilon, M+\varepsilon) = \left(\bigcup_{i\in \bar{I}(x)} O_i\right)$, where $\bar{I} = I(x) \cup \{\bar{i}\}$.
We have a finite subcover for $[a,M]$ from $(O_i)_{i\in I}$; then $M \in F$.

(ii) Claim: $M = b$. Assume, for the sake of contradiction, that $M < b$.
- Write $[a,M] = \bigcup_{i\in I(M)} O_i$, where $I(M) \subset I$ and $I(M)$ finite. Choose $\bar{i} \in I(M)$ such that $M \in O_{\bar{i}}$ and choose $\varepsilon > 0$ such that $(M-\varepsilon, M+\varepsilon) \subset O_{\bar{i}}$. Then $[a, \min\{b, M+\varepsilon\}] = \bigcup_{i\in I(M)} O_i$. But $M < \min\{b, M+\varepsilon\} \Rightarrow$ contradiction.

Heine-Borel Theorem (cont.)

10/25/23
Lecture 12
(cont.)

Sequential Compactness

Def: A set $S \subset E$ is sequentially compact if every sequence in S has a convergent subsequence with limit in S.

Prop: If $S \subset E$ is compact, then S is sequentially compact.

(*) The converse is true for metric spaces.

Corollary: Every bounded, closed subset of $\mathbb{R}^n$ is compact.

Heine-Borel (cont.)

Corollary: Every bounded, closed subset of $\mathbb{R}^n$ is compact.

(*) Proof

It suffices to study the case of the cube $K = [-R, R]^n \subset \mathbb{R}^n$. We want to show that K is sequentially compact. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence $(x_k)_k \subset K$, and write each term $x_k = (x_k^{(1)}, \dots, x_k^{(n)})$. We want to find $x \in K$ and subsequence $(x_{k_i})_{i=1}^{+\infty}$ of $(x_k)_k$ such that $0 = \lim_{i\to\infty} d^2(x_{k_i}, x) = \lim_{i\to\infty}\left(|x_{k_i}^{(1)} - x^{(1)}|^2 + \dots + |x_{k_i}^{(n)} - x^{(n)}|^2\right)$.

Observe that $|x_k^{(1)}|, \dots, |x_k^{(n)}| \leq R$ by definition. Since $[-R, R]$ is compact, then we can find a sequence $(k_i)_{i=1}^{+\infty}$ such that $(x_{k_i}^{(1)})_{i=1}^{+\infty}$ converges to some $x^{(1)} \in [-R, R]$. Then we can find a subsequence of $(k_i)_i$ such that $(x_{k_i}^{(2)})_{i=1}^{+\infty}$ converges to some $x^{(2)} \in [-R, R]$. We can repeat this process for all n; then we will obtain a convergent subsequence $(x_{k_i})_{i=1}^{+\infty}$ with limit $(x^{(1)}, x^{(2)}, \dots, x^{(n)})$.

(*) This proof can be extended to any $E^1 \times \dots \times E^n$ [$E^1, \dots, E^n$ compact], not just $\mathbb{R}^n$; however, the argument fails in infinite dimensions (though the results still hold).

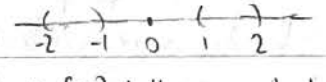
Connectedness

10/27/21

Lecture 13

Reminder: A set $O \subset S \subset E$ is open (relative to S) if and only if $\exists\ V \subset E$ such that V is open (relative to E) and $O = S \cap V$. Conversely, $F \subset S$ is closed in S if $\exists\ K \subset E$ such that K is closed (relative to E) and $F = S \cap K$ [(*) Proof: $S \setminus F = S \cap V$].

(*) Openness: "O open" $\iff O = \bigcup_{i \in I} B_S(p_i, r_i)$ $[B_S(p_i, r_i) = B_E(p_i, r_i) \cap S]$

Remark: Let $S = (-2,-1) \cup \{0\} \cup (1,2)$, i.e.:  $[S \subset \mathbb{R}]$

Observe: $D_S(0, \tfrac{1}{2}) = O = B_S(0, \tfrac{1}{2}) \Rightarrow \{0\}$ both open and closed in S.

Connectedness (§6)

Def: We call E connected if any subset of E which is both open and closed, is either E or $\emptyset$.

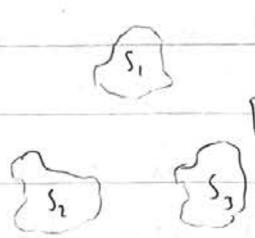
Thm: E is not connected [disconnected] if and only if $\exists$ open sets $A, B \subset E$ such that $E = A \cup B$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$.

(*) Proof

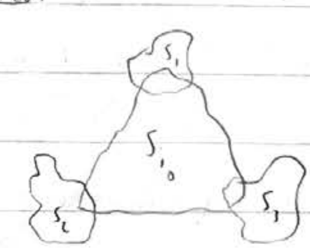
($\Leftarrow$) Suppose two such sets A and B exist. Then $A^c = B$, $B^c = A \Rightarrow A, B$ are both open and closed in E. Then E is not connected.

($\Rightarrow$) Suppose E is not connected. Let $A \neq \emptyset, E$ be a set that is both open and closed in E. Let $B = A^c$. Then $A \cup B = E$, $A \cap B = \emptyset$, and $B \neq \emptyset$.

Prop: Suppose $(S_i)_{i \in I}$ is a collection of connected subsets of E and $\exists\ i_0 \in I$ such that $S_i \cap S_{i_0} \neq \emptyset\ \forall i \in I$. Then $C = \bigcup_{i \in I} S_i$ is connected.



[Union disconnected] $\Rightarrow$



[Union connected]

Connectedness (cont.)

10/27/23 Lecture 13 (cont.)

(*) Proof of Prop

Assume C disconnected. Then $\exists\, A, B \subset C$ open such that $A \cup B = C$, $A \cap B = \emptyset$, and A, B nonempty. We have that $S_{i_0} \cap A$, $S_{i_0} \cap B$ are open subsets of S_{i_0}. Then $(S_{i_0} \cap A) \cup (S_{i_0} \cap B) = S_{i_0}$ and $(S_{i_0} \cap A) \cap (S_{i_0} \cap B) = \emptyset$. Since S_{i_0} is connected, therefore either $S_{i_0} \cap A = \emptyset$ or $S_{i_0} \cap B = \emptyset$. Assume $S_{i_0} \cap A \neq \emptyset \Rightarrow S_{i_0} \cap A = S_{i_0}$. Using similar argument for all S_i: either $S_i \cap A = S_i$ or $S_i \cap B = S_i$. But $S_i \cap B \neq S_i$, since $S_{i_0} \cap B = \emptyset$. Then $S_i \cap A = S_i\ \forall S_i$. Then $C = \bigcup_{i \in I} S_i = \bigcup_{i \in I} (S_i \cap A) = (\bigcup_{i \in I} S_i) \cap A = C \cap A = A$. But then $A^c = B$ must be $\emptyset$. ∎

Remark: $S \subset \mathbb{R}$ is an interval [convex set] iff $\forall\, a, b \in S$ where $a < b$, $[a, b] \subset S$.

Prop: If $S \subset \mathbb{R}$ and S is not an interval (i.e. $\exists\, a, b \in S$ [$a < b$], $c \in \mathbb{R}$ such that $a < c < b$ but $c \notin S$), then S is not connected.

(*) Proof: Set $A = (-\infty, c) \cap S$, $B = (c, +\infty) \cap S$. A, B are open in $\mathbb{R}$; therefore they are open in S. Then $S = A \cup B$, $A \cap B = \emptyset$, and A, B nonempty. ∎

Thm: If $S \subset \mathbb{R}$ is an interval, then S is connected.

(*) Proof

Assume on the contrary, that S is not connected. Then $\exists\, A, B$ open in S such that $S = A \cup B$, $A \cap B = \emptyset$, and A, B nonempty. Let $a \in A$, $b \in B$; assume WLOG that $a < b$. Then $[a, b] \subset S$. Set $A_1 = [a, b] \cap A$, $B_1 = [a, b] \cap B$. Then $a \in A_1$, $b \in B_1$, $A_1 \cup B_1 = [a, b]$, $A_1 \cap B_1 = \emptyset$. Since B_1 is open in $[a, b]$, then $B_1^c = A_1$ is closed. A_1 is closed, nonempty, and bounded; therefore $c = \sup(A_1)$ exists and $c \in A_1$. Since $b \in B_1$, $c < b$. Since we know A_1 is an open subset of $\mathbb{R}$, then $\exists$ open subset $O \subset \mathbb{R}$ such that $c \in A_1 = [a, b] \cap O$. Choose $\varepsilon > 0$ such that $c + \varepsilon < b$ and $[a, b] \cap (c - \varepsilon, c + \varepsilon) \subset A_1$. Then $[c, c + \varepsilon) \subset A_1$; but we said c was the l.u.b. ∎

Continuous Functions

10/30/23

Lecture 14

Continuous Functions (§1)

Def. Let $f: E \to E'$ be a function and $x_0 \in E$. We say f is continuous at x_0 if, for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that $d(f(x), f(x_0)) < \varepsilon$ $\forall\, x \in E$ where $d(x, x_0) < \delta$.

Alternatively: for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that $D(x_0, \delta) \subset f^{-1}(D(f(x_0), \varepsilon))$.

Continuous Functions

Def: We say a function f is continuous if f is continuous at every $x_0 \in E$.

Theorem: Let $f: E \to E'$. Then the following are equivalent:

(i) f is continuous.

(ii) If $V \subset E'$ is open, then $f^{-1}(V)$ is open.

(*) Proof

(i)→(ii): Let $V \subset E'$ be open and let $x_0 \in f^{-1}(V)$. Since $f(x_0) \in V$ and V is open, $\exists\, \varepsilon > 0$ such that $D_{E'}(f(x_0), \varepsilon) \subset V$. Then per (i), we can find $\delta > 0$ such that $D(x_0, \delta) \subset f^{-1}(D_{E'}(f(x_0), \varepsilon)) \subset f^{-1}(V)$. Since this is true for any arbitrary $x_0 \in f^{-1}(V)$, therefore $f^{-1}(V)$ is open. □

(ii)→(i): Let $x_0 \in E$; fix $\varepsilon > 0$. Since $D_{E'}(f(x_0), \varepsilon) \subset E'$ is open, then $f^{-1}(D(f(x_0), \varepsilon))$ is an open subset of E containing x_0. Then $\exists\, \delta > 0$ such that $D(x_0, \delta) \subset f^{-1}(D(f(x_0), \varepsilon))$. □

Example: Let $E = E' = \mathbb{R}$, $f(x) = x^2$. We want to show f is continuous at x_0, i.e. $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $|f(x_0) - f(x)| < \varepsilon$ $\forall\, x \in B(x_0, \delta)$. Fix $\varepsilon > 0$. We know $|f(x_0) - f(x)| = |x_0 + x||x_0 - x|$. It suffices to find $\delta \in (0,1)$ to prove continuity; then $|x_0 + x| < 1 + 2|x_0| \Rightarrow |x_0 + x||x_0 - x| \le \delta(2|x_0| + 1)$. Then we need only find a δ such that $\delta \le \frac{\varepsilon}{2|x_0|+1} \Rightarrow |x - x_0| \le \frac{\varepsilon}{2|x_0|+1}$. $\delta = \min\{1, \frac{\varepsilon}{2|x_0|+1}\}$ suffices. □

Limits

10/30/23 Lecture 14 (cont.)

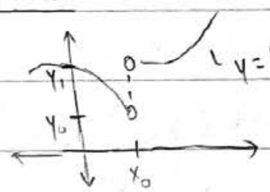
(*) Composition of Continuous Functions

Theorem: If $f: E \to E'$ is continuous at $x_0 \in E$ and $g: E' \to E''$ is continuous at $f(x_0) \in E'$, then their composition $g \circ f: E \to E''$ is continuous at x_0.

(*) Proof: We know $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$, i.e. $x \in (g \circ f)^{-1}(B) \iff x \in f^{-1}(g^{-1}(B))$.
Fix $\varepsilon > 0$. Since g is continuous at $f(x_0)$, $\exists\, a > 0$ such that $D_{E'}(f(x_0), a) \subset g^{-1}(D_{E''}(g \circ f(x_0), \varepsilon))$.
Since f is continuous at x_0, $\exists\, \delta > 0$ such that $D_E(x_0, \delta) \subset f^{-1}(D_{E'}(f(x_0), a))$. Then
$D_E(x_0, \delta) \subset f^{-1}(D_{E'}(f(x_0), a)) \subset g^{-1}(f^{-1}(D_{E''}(g \circ f(x_0), \varepsilon))) = (g \circ f)^{-1}(D(g \circ f(x_0), \varepsilon))$. $\square$

(*) Corollary: If $f: E \to E'$ and $g: E' \to E''$ are continuous, then $g \circ f: E \to E''$ is continuous.

Limits



$\Rightarrow$ We claim the function f does not need to be defined at x_0 to find its limit at x_0 (if one exists), and that if f has a limit at x_0, then that limit is either y_0 or y_1.

Def: Limit (§2)

Let x_0 be a cluster point of E. Then we say $f: (E \setminus \{x_0\}) \to E'$ has $p_0 \in E'$ as its limit when x tends to x_0 ($\lim_{x \to x_0} f(x) = p_0$) if the function $g: E \to E'$ defined by:

$$g(x) = \begin{cases} f(x) & x \neq x_0 \\ p_0 & x = x_0 \end{cases}$$

is continuous at x_0.

Equivalently: (1) For every $\varepsilon > 0$, $\exists\, \delta > 0$ such that $(D_E(x_0, \delta) \setminus \{x_0\}) \subset f^{-1}(D_{E'}(p_0, \varepsilon))$.
(2) For every $\varepsilon > 0$, $\exists\, \delta > 0$ such that $0 < d(x, x_0) < \delta \Rightarrow d'(f(x), p_0) < \varepsilon$.

Continuous Functions (cont.)

11/1/23

Lecture 1

Limits (cont.)

<u>Prop</u> Let $f: E \to E'$, $a \in E$. Then the following are equivalent:

(i) f is continuous at a; and

(ii) for every sequence $(x_n)_n$ converging to a, $f(a) = \lim_{n\to\infty} f(x_n)$.

(*) <u>Proof</u>: (i) $\Rightarrow$ (ii): Assume f is continuous at a. Let $(x_n)_n \subset E$ be a sequence converging to a. Since f is continuous at a, for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that $D(a,\delta) \subset f^{-1}(D(f(a),\varepsilon))$. Since $(x_n)_n$ converges to a, $\exists\ N \in \mathbb{N}$ such that $x_n \in D(a,\delta) \subset f^{-1}(D(f(a),\varepsilon))\ \forall\ n \geq N$.

(ii) $\Rightarrow$ (i): Assume (ii). Assume, for the sake of contradiction, that $\exists\ \varepsilon > 0$ such that for any $\delta > 0$, $D_E(a,\delta) \not\subset f^{-1}(D_{E'}(f(a),\varepsilon))$. In particular, if $\delta = 1$, then $\exists\ x_1 \in D(a,1)$ such that $x_1 \notin f^{-1}(D(f(a),\varepsilon))$. Letting $\delta = \min\{\frac{1}{2}, d(x,a)\}$, we can find x_2 closer to $f(a)$ fulfilling the condition. Then eventually, we obtain $x_n \in D(a, r_n)$, $x_n \notin f^{-1}(D(a,\varepsilon))$, $r_n = \min\{\frac{1}{n}, d(x_{n-1}, a)\}$. Then $d(x_n, a) < r_n < \frac{1}{n}$, but $d(f(x_n), f(a)) \geq \varepsilon$. Then $\lim_{n\to\infty}(x_n) = a$, but $f(x_n)$ does not converge to $f(a)$ [contradiction].

Operations on sets of continuous functions (§7)

<u>Prop</u>. Let $f, g: E \to \mathbb{R}$ be continuous at $a \in E$. Then:

(i) $f+g$, fg are continuous at a

(ii) if $g(a) \neq 0$, then f/g is well-defined near a and continuous at a.

(*) <u>Proof</u>: (ii) If $g(a) \neq 0$, then $\varepsilon = |g(a)| > 0$. Since g is continous at a, $\exists\ \delta > 0$ such that $|g(x) - g(a)| < \varepsilon\ \forall\ x \in D(a,\delta)$. Hence, f/g is well-defined in $D(a,\delta)$.

To prove f/g is continuous, consider some $(x_n)_n \subset D(a,\delta)$ converging to a. Then $\lim_{n\to\infty} \frac{f}{g}(x_n) = \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = f(a)/g(a) = \frac{f}{g}(a)$. Hence f/g is continuous at a.

Continuous Functions (cont.)

11/1/23

Lecture 15 (cont.)

Operations on sets of continuous functions (cont.)

<u>Corollary</u>: Let $a \in E$ be a cluster point of E, and let $f, g : E \setminus \{a\} \to \mathbb{R}$. Then:

(i) $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$; $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} g(x) \neq 0 \Rightarrow \lim_{x \to a} f/g\,(x) = \lim_{x \to a} f(x) / g(x)$.

(*) <u>Proof</u>: Set $\bar{f}(x) = \{f(x) \text{ if } x \neq a;\ \lim_{x \to a} f(x) \text{ if } x = a\}$. Set $\bar{g}(x)$ similarly. Then $\bar{f}(x)$ and $\bar{g}(x)$ are continuous at a, therefore the properties hold for $\bar{f}(x), \bar{g}(x)$. □

(*) Ex: Let $d \in \mathbb{N}$, define $\pi^i : \mathbb{R}^d \to \mathbb{R}$ by $\underline{\pi^i(x_1, \dots, x_d) = x_i}$. Then π^i is continuous; indeed, if $(x_n)_n \subset \mathbb{R}^d$ converges to a, then $|\pi^i(x_n) - \pi^i(a)| \leq \|x_n - a\|$.

$\Rightarrow$ <u>Corollary</u>: A function $f : E \to \mathbb{R}^d$ is continuous at $x \in E$ if and only if <u>$\pi^i \circ f$ is continuous at x</u> for each $i = 1, \dots, d$.

(*) Proof: ($\Rightarrow$) If f is continuous at x, since π^i is continuous at $f(x)$, then we conclude that $\pi^i \circ f$ is continuous at x.

($\Leftarrow$) Let $(x_n)_n \subset E$ be a sequence converging to x. Then we have $\|f(x_n) - f(x)\|^2 = \sum_{i=1}^{d} (\pi^i \circ f(x_n) - \pi^i \circ f(x))^2$. Since $\|f(x_n) - f(x)\|$ goes to 0, therefore $\sum_{i=1}^{d} (\pi^i \circ f(x_n) - \pi^i \circ f(x))^2$ goes to $0 \Rightarrow (\pi^i \circ f(x_n) - \pi^i \circ f(x))^2$ goes to 0 for all $i = 1, \dots, d$. □

Continuous Functions (cont.)

11/6/23

Lecture 16

Continuous Functions on Compact Sets (§5)

Thm: If E is compact and $f: E \to E'$ is continuous, then $f(E) = \{f(p) : p \in E\}$ is compact.

(*) Proof: Let $(V_i')_{i \in I}$ be an open cover for $f(E)$, where each $V_i' = V_i \cap f(E)$ for some open set V_i in E'. We want to show $\exists$ a finite subcover of $(V_i')_{i \in I}$ for $f(E)$.
We know $E \subset f^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} (f^{-1}(V_i))$. Since V_i' are open in E' and $f: E \to E'$ is continuous, then $f^{-1}(V_i')$ is open. Since E is compact and $(f^{-1}(V_i'))_{i \in I}$ is an open cover for E, then $\exists$ finite $J \subset I$ such that $E = \bigcup_{i \in J} f^{-1}(V_i')$. Then $f(E) = \bigcup_{i \in J} V_i' \Rightarrow (V_i')_{i \in J}$ is a finite subcover for $f(E)$. $\square$

Remark: Let $E = (0,1)$, $E' = \mathbb{R}$, $f(x) = x \Rightarrow f: E \to E'$ is continuous.
Remark: Let $E = (0,1)$, $E' = \mathbb{R}$, $f(x) = \frac{1}{x} \Rightarrow f(E) = (0, +\infty)$ is not bounded.

Corollary: Suppose E is compact and $f: E \to E'$ is continuous. Then:

(i) f is bounded

(ii) if $E' = \mathbb{R}$, then f achieves its maximum, minimum values on E

(*) Proof

(i) E compact, f continuous $\Rightarrow f(E)$ is compact. Then $f(E)$ is closed and bounded.

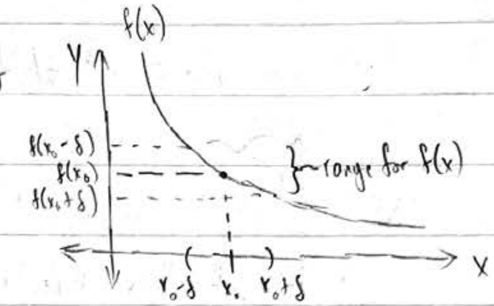
(ii) Since $f(E) \subset \mathbb{R}$ and $f(E)$ is compact, then $f(E)$ contains its upper and lower bounds. Then $\exists\, x_0, x_1 \in E$ such that $\sup(f(E)) = f(x_1)$ and $\inf(f(E)) = f(x_0)$. Then $f(x_0)$ is a minimum value of f and $f(x_1)$ is a maximum value of f. $\square$

Uniform Continuity

11/6/23
Lecture 16
(cont.)

Uniform Continuity

Example: Let $f(x) = \frac{1}{x}$, $x \in (0,1)$. Let's say that we want to find $f(x_0)$ for some x_0, but make an error in the x direction of size δ (i.e find a value in $(x_0-\delta, x_0+\delta)$). Then what is the impact on the error on $f(x)$?



In this case: $\frac{1}{x_0+\delta} < f(x) < \frac{1}{x_0-\delta}$. Then $\frac{1}{x_0} - \frac{1}{x_0+\delta} < |f(x)-f(x_0)| < \frac{1}{x_0-\delta} - \frac{1}{x}$. We can simplify this to $|f(x)-f(x_0)| < \frac{\delta}{x_0(x_0-\delta)}$. Particularly, $\sup_{x_0\in(0,1)}\left(\sup_{x\in(x_0-\delta,x_0+\delta)} f(x)\right) = +\infty$. Since the error $|f(x)-f(x_0)|$ is dependent on x_0 (increases as $x_0 \to 0$), we say f is not uniformly continuous.

> **Def: Uniform Continuity**
>
> We call $f: E \to E'$ uniformly continuous if one of the following equivalent conditions holds:
>
> (i) for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ whenever $d(x,x_0) < \delta$; $x, x_0 \in E$.
>
> (ii) $D(\delta) := \sup_{x,x_0\in E}\left(\{d'(f(x), f(x_0)) : d(x,x_0) < \delta\}\right) \to 0$ as $\delta \to 0$.

(*) Continuity vs. Uniform Continuity:

Continuity: for every $\varepsilon > 0$ and $x_0 \in E$, $\exists\, \delta > 0$ s.t. $d(x,x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon$ $[\delta(\varepsilon, x_0)]$

Unif. Cont.: for every $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. for every $x_0, x \in E$, $d(x,x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon$ $[\delta(\varepsilon)]$

Remark: Every uniformly continuous function is continuous.

(*) Ex: $f(x) = \frac{1}{x}$ is continuous on $(0,1)$, but not uniformly continuous.

(*) Preview (Mean value theorem): A function $f: E \to E'$ is uniformly continuous if its derivative is bounded.

11/16/23
Lecture 16
+ Lecture 17

Continuity (cont.)

Uniform Continuity (cont.)

Theorem: If E is compact and $f: E \to E'$ is continuous, then f is uniformly continuous.

(*) Proof: Let $\varepsilon > 0$; then we want to find $\delta > 0$. If $x_0 \in E$, then since f is continuous, $\exists$ $\delta(x_0, \varepsilon) > 0$ such that $D_E(x_0, \delta) \subset f^{-1}(D_{E'}(f(x_0), \frac{\varepsilon}{2}))$. Since $E = \bigcup_{x_0 \in E} D_E(x_0, \frac{\delta(x_0)}{2})$ and E is compact, then $\exists$ $x_1, \dots, x_k \in E$ (k finite) such that $E = \bigcup_{i=1}^{k} D_E(x_i, \frac{\delta(x_i)}{2})$. Set $\delta = \frac{1}{2}\min\{\delta(x_1), \dots, \delta(x_k)\}$. If $x, y \in E$, $\exists$ $i \in \{1, \dots, k\}$ such that $x \in D_E(x_i, \frac{\delta(x_i)}{2})$. If $d(x,y) < \delta$, then $d(x,y) \leq d(x_i, x) + d(x_i, y) < \delta(x_i)$. Then $d'(f(x_i), f(y)) < \frac{\varepsilon}{2}$ and $d(f(x_i), f(x)) < \frac{\varepsilon}{2} \Rightarrow d(f(x), f(y)) \leq d(f(x), f(x_i)) + d(f(x_i), f(y)) < \varepsilon$. $\square$

Continuous Functions on Connected Sets (§6)

Thm: If E is connected and $f: E \to E'$ is continuous, then $f(E)$ is connected.

(*) Proof: Assume WLOG that $E' = f(E)$. Assume, for the sake of contradiction, that $f(E)$ is disconnected. Then $\exists$ $A, B \subset f(E)$ open such that $A \cup B = f(E)$, $A \cap B = \phi$, and A, B are nonempty. Since A and B are open, then $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty open sets in E, such that $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(f(E)) = E$. We know $f^{-1}(A) \cap f^{-1}(B)$, since A, B share no elements. Then $f^{-1}(A), f^{-1}(B) \subset E$ are nonempty open sets such that $f^{-1}(A) \cup f^{-1}(B) = E$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Then E is disconnected. $\square$

Corollary: Let E be connected, E' disconnected. Let $A, B \subset E'$ be the nonempty open sets such that $A \cup B = E$, $A \cap B = \phi$. If a function $f: E \to E'$ is continuous, then $f(E) \subset A$ or $f(E) \subset B$.

Sequences of Functions

11/8/23
Lecture 17
(cont.)

Continuity (cont.)

Theorem (Intermediate value theorem): If $a, b \in \mathbb{R}$, $a < b$, and $f : [a,b] \to \mathbb{R}$ is continuous, then for every γ between $f(a)$ and $f(b)$, $\exists\ c \in (a,b)$ such that $\gamma = f(c)$.

(*) "Between $f(a)$ and $f(b)$": $f(a) < \gamma < f(b)$ if $f(a) < f(b)$, $f(b) < \gamma < f(a)$ if $f(a) > f(b)$

(*) Proof: Assume $f(a) > \gamma > f(b)$. Since $f([a,b])$ is a connected subset of $\mathbb{R}$, it is an interval containing both $f(a)$ and $f(b)$. Thus it contains $[f(a), f(b)]$, i.e. $[f(a), f(b)] \subset f([a,b])$. Since $\gamma \in [f(a), f(b)]$, then $\gamma \in f([a,b])$. Then $\exists\ c \in [a,b]$ such that $\gamma = f(c)$. Hence, $c \in (a,b)$. $\square$

(*) Note: If a set $O \subset E$ is connected by arc (i.e. $\forall\ x, y \in O$ $\exists$ a path [arc] $x \to y$ in O), then O is connected. (The converse is true only if O is open).

(*) Remark: If $a, b \in \mathbb{R}$ and $f : [a,b] \to E'$ is continuous, then $R(f) = \{f(t) : t \in [a,b]\}$ is connected.

Sequences of Functions (§6)

Def: Let $f_n : E \to E'$ $(n \in \mathbb{N})$ and $f : E \to E'$ be functions. We say that $(f_n)_n$ converges to f [pointwise/simply] if, for every $x \in E$, $(f_n(x))_n \subset E'$ converges to $f(x) \in E'$, i.e. $\lim_{n\to\infty} f_n(x) = f(x)\ \forall\ x \in E$.

Def: Uniform Convergence

We say that $(f_n)_n$ coverges uniformly to f if either of the following equivalent conditions hold:

(i) $\lim_{n\to\infty} a_n = 0$, where $a_n = \sup_{x \in E} d(f_n(x), f(x))$

(ii) for every $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon\ \forall\ n \geq N, x \in E$.

Sequences of Functions (cont.)

11/8/23
Lecture 17
+Disc 7

(*) Pointwise & Uniform Convergence

Set $\bar{a}_n(x) = d'(f_n(x), f(x))$; $a_n = \sup_{x \in E} \bar{a}_n(x)$.

$\Rightarrow$ $\bar{a}_n(x) \to 0$ implies pointwise convergence; $a_n \to 0$ implies uniform convergence.

Remark: Every uniformly convergent sequence is pointwise convergent.

Theorem: Uniform Convergence & Continuity

If a sequence $(f_n : E \to E')_n$ of continuous functions converges uniformly to $f : E \to E'$, then f is continuous.

(*) Proof: Set $a_n = \sup_{x \in E} d'(f_n(x), f(x))$. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $a_n \leq \varepsilon/3$ $\forall$ $n \geq N$. Let $x_0 \in E$; we want to show f is continuous at E. We have $d'(f(x), f(x_0)) \leq$
$\leq d'(f(x), f_n(x)) + d'(f_n(x), f_n(x_0)) + d'(f_n(x_0), f(x)) = 2\varepsilon/3 + d'(f_n(x), f_n(x_0))$.
Since f_n is continuous at x_0, $\exists$ $\delta > 0$ such that $D(x_0, \delta) \subset f_n^{-1}(D(f_n(x_0), \varepsilon/3))$. Then $d(x, x_0) < \delta \Rightarrow d(f_n(x), f_n(x_0)) < \varepsilon/3$. Then $d'(f(x), f(x_0)) < \varepsilon$
$\Rightarrow D_E(x_0, \delta) \subset f^{-1}(D_{E'}(f(x_0), \varepsilon))$. $\square$

(*) Cauchy Criterion for Uniform Convergence

Let $E \in \mathbb{R}$ (not necessarily complete), and let $(f_n : E \to \mathbb{R})_n$ be a sequence. Then $(f_n)_n$ is uniformly convergent $\Leftrightarrow$ for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\sup_{x \in E} |f_n(x) - f_m(x)| \leq \varepsilon$ $\forall$ $n, m \geq N$.

(*) Proof: ($\Rightarrow$) Know: for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon$ $\forall$ $n \geq N$. Fix N for $\varepsilon/2$.
Then $\forall$ $n, m \geq N$: $d(f_n(x) - f_m(x)) \leq d(f_n(x), f(x)) + d(f(x), f_m(x)) < \varepsilon$.

($\Leftarrow$) Fix $x \in E$; then f is pointwise convergent at x. Fix $\varepsilon > 0$; then $\exists$ $N \in \mathbb{N}$ such that if $n, m \geq N$, $|f_n(x) - f_m(x)| \leq \varepsilon$. Taking the limit: $\varepsilon \geq \lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|$.

Uniform Convergence

11/13/23

Lecture 18 Given $f, g: E \to E'$: define $\underline{D(f,g) = \sup_{x \in E} d'(f(x), g(x))}$.

(*) $D(f,g) \in [0, \infty]$; since it may achieve ∞, then it is not a metric in the general case for the set $\underline{F(E, E')}$ of functions $f: E \to E'$.

(*) Remark: Let $(f_n)_n \subset F(E,E')$, $f(x) \in F(E,E')$. If $(f_n)_n$ converges uniformly to f, then $\lim_{n\to\infty} D(f_n, f) = 0$.

> <u>Cauchy Criterion for Uniform Convergence</u>
> Let $(f_n)_n$ be a sequence in $F(E, E')$. We say that $(f_n)_n$ is <u>Cauchy for uniform convergence</u> if, for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $D(f_n, f_m) < \varepsilon \ \forall n, m \geq N$.

<u>Observation</u>: Let $f, g: E \to E'$ be continuous functions. Then $\underline{F: x \mapsto d(f(x), g(x))}$ is continuous.

<u>Proposition</u>: Let E' be a complete metric space, $(f_n)_n \subset F(E, E')$. Then TFAE:

(i) $(f_n)_n$ converges uniformly to some $f \in F(E, E')$.

(ii) $(f_n)_n$ is Cauchy for uniform convergence.

(*) <u>Proof</u>: $(\Rightarrow)$ Assume (i). Then, for each $x \in E$: $d'(f_n(x), f_m(x)) \leq d'(f_m(x), f(x)) + d'(f(x), f_n(x)) \leq D(f_m, f) + D(f, f_n)$. Hence $D(f_n, f_m) \leq D(f_m, f) + D(f, f_n)$. [Triangle inequality for D]. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be a number such that $D(f_n, f) \leq \frac{\varepsilon}{2} \ \forall n \geq N$. Then for every $n, m \geq N$: $D(f_n, f_m) < \varepsilon$.

$(\Leftarrow)$ Assume (ii). Let $x \in E$. Since $d'(f_n(x), f_m(x)) \leq D(f_n, f_m)$ and $(f_n)_n$ is Cauchy, then $(f_n(x))_n$ converges in E' to some $f(x) \in E'$, i.e. $\lim_{n\to\infty} d'(f(x), f_n(x)) = 0$. Since $(f_n)_n$ is Cauchy, then for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $D(f_n, f_m) < \varepsilon \ \forall n, m \geq N$. Then $d'(f_n(x), f_m(x)) < \varepsilon \ \forall n, m \geq N, x \in E \Rightarrow d'(f(x), f_n(x)) < \varepsilon \ \forall m \geq N, x \in E$. Then $D(f, f_n) < \varepsilon$ $\forall n \geq N$. $\square$

Uniform Convergence (cont.)

11/13/23
Lecture 18
(cont.)

<u>Alt. proof ($(f_n)_n$ Cauchy $\Rightarrow$ $(f_n)_n$ converges uniformly to f)</u>: Let $x \in E$. Then we know for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $d'(f_m(x), f_n(x)) < \varepsilon \; \forall \; n, m \geq \varepsilon$. Additionally, we know $(f_n(x))_n$ converges to some $f(x) \in E'$. We know, also, that $\lim_{n\to\infty} d'(f_n(x), f_m(x))$ exists. Then we can take its limit: $\lim_{n\to\infty} d'(f_n(x), f_m(x)) =$
$= d'(f(x), f_m(x)) < \varepsilon$. Since this is true for any $x \in E \Rightarrow D(f, f_n) < \varepsilon \; \forall \; n \geq N$. $\square$

<u>Lemma</u>: If $f, g : E \to E'$ are continuous functions, then $F : x \mapsto d'(f(x), g(x))$ is a continuous function $E \to \mathbb{R}$.

(*) <u>Proof</u>: Let $x_0 \in E$. We want to show F is continuous at x_0. We have that for $x \in E$,

$$|F(x) - F(x_0)| = |d'(f(x) - g(x))| - |d'(f(x_0), g(x_0))|$$
$$= |d'(f(x), g(x))| - d'(f(x), g(x_0)) + d'(f(x), g(x_0)) - |d'(f(x_0), g(x_0))|$$
$$\leq |d'(g(x_0), g(x)) + d'(f(x), f(x_0))|.$$

Given $\varepsilon > 0$, $\exists \, \delta_1, \delta_2 > 0$ such that if $d(x_0, x) < \delta_1, \delta_2$, then $d'(f(x), f(x_0)) < \varepsilon/2$ and $d'(g(x), g(x_0)) < \varepsilon/2$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$, and $|F(x) - F(x_0)| < \varepsilon$ whenever $d(x_0, x) < \delta$. $\square$

(*) In cases where a limit for an expression is not known to exist, we can generally take $\overline{\lim}$ [limsup] instead. Limsup always exists; and if limsup $\to 0$, then the limit exists.

(*) Misc. Notes

MT Review

Misc. Notes

Sequences

Prop. Any sequence in $\mathbb{R}$ has a monotonic subsequence.

(*) Proof: Given a sequence, keep taking either next-least or next-greatest terms

↳ one or the other must always exist

→ Theorem (Bolzano-Weierstrass). Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Prop. Given a sequence $(x_n)_n \subset \mathbb{R}$ w/ $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} \frac{\sum_{i=0}^{n} x_i}{n} = x$.

Metric Spaces

Def. A set is called totally bounded if, for any $\varepsilon > 0$, the set can be covered by a finite number of balls of radius ε.

Prop. A set is totally bounded iff every sequence in the set has a Cauchy subsequence.

→ Prop. Let E be a metric space; then TFAE: i) E is compact

ii) E is sequentially compact

iii) E is totally bounded and complete

Continuity

one-to-one

Prop. For any function $f: E \to E'$, $V \subset E$, $f(V^c) = (f(V))^c$.

→ Prop. For any function $f: E \to E'$ (E, E' metric spaces) s.t. E compact, f continuous & bijective, $f^{-1}: E' \to E$ is continuous & bijective.

$\{f: E \to \mathbb{R} : f \text{ cts.}\}$

Prop. For any compact metric space E, $C(E)$ is complete & normed under the supremum metric.

Differentiability

11/15/23

Lecture 19

Recall: $f, g \in F(E, E') \Rightarrow$ define $D(f,g) = \sup_{x \in E} d'(f(x), g(x))$

Define $F_B(E, E') = \{f: E \to E' : f \text{ bounded}\}$.

Theorem: D is a metric on $F_b(E, E')$. In particular, if E is compact, D is a metric on $C(E, E')$, set of continuous functions $f: E \to E'$.

Theorem: If E is compact and E' is complete, then $C(E, E')$ is a complete space under the metric D.

(*) Proof: Let $(f_n)_n \subset C(E, E')$ be a Cauchy sequence. Since E' is complete and $(f_n)_n$ Cauchy, therefore $(f_n)_n$ converges uniformly to some $f \in F(E, E')$. Since $(f_n)_n$ is a sequence of continuous functions, then f is continuous $\Rightarrow f \in C(E, E')$. □

Differentiation of Real-Valued Functions (Ch. 5)

Derivatives defined (generally) for functions: $f: \mathcal{O} \subset H \to \mathbb{R}$ [$\mathcal{O}$ open; H a Hilbert space].

(*) Hilbert space: a vector space w/ associated inner product $\langle x, y \rangle \in H$ & norm $\|x\| = \sqrt{\langle x, x \rangle}$

(*) Stipulate $\mathcal{O}$ open because derivative only defined for interior points

Def: Differentiability (General)

Let $x_0 \in \mathcal{O}$. Define:

$$r(x) = \frac{|f(x) - f(x_0) + \langle y_0, x - x_0 \rangle|}{\|x - x_0\|}$$

Then f is differentiable at x_0 if, for some y_0, $\lim_{x \to x_0} r(x) = 0$.

Differentiability (cont.)

11/15/23
Lecture 19
(cont.)

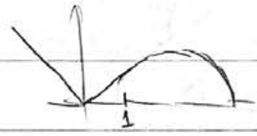
Def: Differentiability ($\mathbb{R}$)

Let $I \subset \mathbb{R}$ open, $x_0 \in I$. We say that $f: I \to \mathbb{R}$ is <u>differentiable at x_0</u> if

$$\boxed{f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} \quad \left[f'(x_0) = \frac{df}{dx}(x_0) \right]$$

exists. Then we call $f'(x_0)$ the <u>derivative of f at x_0</u>.

(*) Ex: Define $f(x) = \begin{cases} |x| & -1 < x < 1 \\ -x^2 + 3x - 1 & 1 \le x \le 3 \end{cases}$



$\Rightarrow$ differentiable?

(ex) Case 1: $x_0 = 0$. Want to find an <u>affine function</u> [tangent line] $\underline{A(x) = y_0(x - 0) + f(0)}$ such that $\underline{\lim_{x \to x_0} \frac{f(x) - A(x)}{x - x_0} = 0}$. But $A(x) = y_0 x \Rightarrow \frac{f(x) - A(x)}{x - x_0} = \frac{|x| - y_0 x}{x}$. There is no value of y_0 such that $\lim_{x \to x_0^+} \frac{|x| - y_0 x}{x} = \lim_{x \to x_0^-} \frac{|x| - y_0 x}{x} = 0 \to$ <u>not differentiable at x_0</u>.

(ex) Case 2: $x_0 \in (1, 3)$. Set $y_0 = -2x_0 + 3$. Define affine function $A(x) = f(x_0) + y_0(x - x_0)$.
Then $\frac{f(x) - A(x)}{x - x_0} = \frac{-x^2 + 3x - 1 - (-x_0^2 + 3x_0 - 1) - (-2x_0 + 3)(x - x_0)}{x - x_0} = -x_0 - x + 3 + 2x_0 - 3 = x_0 - x$. Then $\lim_{x \to x_0} \frac{f(x) - A(x)}{x - x_0} = 0$.

<u>Operations on Sets of Differentiable Functions</u>

<u>Def</u>: We say a function $f: I \to \mathbb{R}$ is <u>differentiable</u> if <u>$f'(x_0)$ exists $\forall x_0 \in I$</u>.

<u>Lemma</u>: Let $f, g: I \to \mathbb{R}$ be differentiable at x_0. Then:

(i) $f+g$, fg are differentiable at x_0:

(a) $\underline{(f+g)'(x_0) = f'(x_0) + g'(x_0)}$

(b) $\underline{(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)}$ [Product Rule]

(ii) <u>f, g are continuous at x_0</u>

(iii) if $g(x_0) \ne 0$, then f/g is well defined near x_0 and $\underline{(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}}$.

(*) Proof: (ii) Since f is differentiable at x_0, $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$ exists. Then
$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0) \right) = f'(x_0) \cdot 0 + f(x_0) = f(x_0)$.

Differentiability (cont.)

11/15/23 Lecture 19 + Lecture 20

(*) Proof (Quotient Rule): If $g(x_0) \neq 0$, since g is continuous at x_0, therefore $g(x) \neq 0$ for x close to x_0. Then:

$$\frac{f}{g}(x) - \frac{f}{g}(x_0) = \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}$$

$$= \frac{1}{g(x)g(x_0)}\left(\left(f(x_0) + \frac{f(x)-f(x_0)}{x-x_0}(x-x_0)\right)g(x_0) - f(x_0)\left(g(x_0) + \frac{g(x)-g(x_0)}{x-x_0}(x-x_0)\right)\right)$$

$$\rightarrow \frac{f/g(x) - f/g(x_0)}{x-x_0} = \frac{1}{g(x)g(x_0)}\left(\frac{f(x)-f(x_0)}{x-x_0}g(x_0) - f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right)$$

$$\xrightarrow{\text{goes to}} \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2} \text{ as } x \rightarrow x_0. \quad \square$$

(*) Fact: $(x^n)' = nx^{n-1}$ $\forall n \in \mathbb{N}$ [proof via induction]

Theorem: Chain Rule

Thm: Let $J \subset \mathbb{R}$ be an open set, $f: I \rightarrow J$ differentiable at $x_0 \in I$, $g: J \rightarrow \mathbb{R}$ differentiable at $f(x_0) \in J$. Then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.

(*) Proof: Want to show: $$\frac{g\circ f(x) - g \circ f(x_0)}{x-x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x)-f(x_0)} \cdot \frac{f(x)-f(x_0)}{x-x_0} \text{ as } x \rightarrow x_0$$

Set: $$A(x) = \begin{cases} \frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)} & x \neq x_0 \\ g'(f(x_0)) & x = x_0 \end{cases} \Longrightarrow \lim_{x \rightarrow x_0} A(x) = A(x_0) = g'(f(x_0))$$

Then:

$$\lim_{x\rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x)-f(x_0)} = \lim_{x \rightarrow x_0} A(x)\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\rightarrow x_0} \frac{g(f(x))-g(f(x_0))}{x-x_0} \cdot \frac{f(x)-f(x_0)}{x-x_0}. \quad \square$$

(*) Dini's Theorem

11/16/23
Disc 8

Theorem (Dini's Theorem)

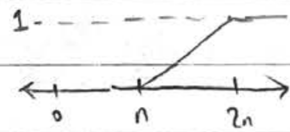
Let (a) K be a compact set, (b) f a continuous function $K \to \mathbb{R}$, $(f_n : K \to \mathbb{R})_n$ sequence of continuous functions converging pointwise to f, and (c) $f_n(x) \geq f_{n+1}(x)$.

$\to$ Then $(f_n)_n$ converges uniformly to f on K.

Note: All conditions [(a), (b), (c)] are required. Otherwise:

(*) Counterexamples

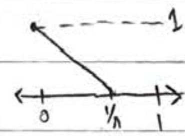
(i) Assume !(a). Let $K = \mathbb{R}$ [not compact]. Set $f_n =$



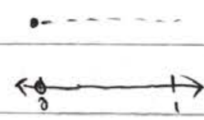
$\Rightarrow \lim_{n\to\infty} f_n = 0$ [$f = 0$ is continuous] and $f_n(x) \geq f_{n+1}(x)$,

but $\sup_{x\in K} |f_n(x) - f(x)| = 1$ always [not unif. convergent].

(ii) Assume !(b). Let $K = [0,1]$. Set $f_n =$

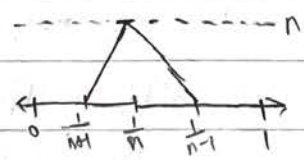


$\Rightarrow \lim_{n\to\infty} f_n =$



$\to$ f not continuous, but $f_n \geq f_{n+1}$. $\forall n \in \mathbb{N}: \sup_{x\in K} |f_n(x) - f(x)| = 1$ [not unif. convergent].

(iii) Assume !(c). Let $f_n =$



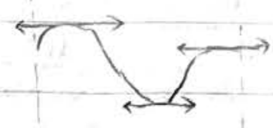
converges to $f = 0$ non-uniformly.

Mean Value Theorem

11/17/23

Lecture 20

<u>Thm.</u> Suppose $f: I \to \mathbb{R}$ is differentiable at x_0 and achieves its <u>maximum/minimum at x_0</u>. Then <u>$f'(x_0) = 0$</u>.



(*) <u>Proof</u>: Assume, e.g., f achieves its maximum at x_0. Then: $\frac{f(x_0+h)-f(x_0)}{h} \leq 0$ if $h > 0$, ≥ 0 if $h < 0$. Thus $f'(x_0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(x_0+h)-f(x_0)}{h} \leq 0 \leq \lim_{\substack{h\to 0\\ h<0}} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$. □

<u>Rolle's Theorem</u>

<u>Thm.</u> Let $a, b \in \mathbb{R}$ s.t. $a < b$, and suppose that $f: [a,b] \to \mathbb{R}$ is continuous and $f: (a,b) \to \mathbb{R}$ is differentiable. If $f(a) = f(b)$, then $\exists$ <u>$c \in (a,b)$ such that $f'(c) = 0$</u>.

(*) <u>Proof</u>: Since $[a,b]$ is compact and $f: [a,b] \to \mathbb{R}$ is continuous, therefore $f([a,b])$ is compact. Then f attains its minimum, maximum on $\mathbb{R}$, i.e. $\exists\ x_0, x_1 \in [a,b]$ such that $f(x_0) \leq f(x) \leq f(x_1)\ \forall\ x \in [a,b]$.

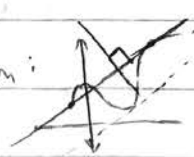
Case 1: $f(x_0) = f(x_1)$. Then $f(x) = f(x_0)\ \forall\ x \in (a,b) \Rightarrow f'(x) = 0\ \forall\ x \in (a,b)$.

Case 2: $f(x_0) \neq f(x_1)$. Then it cannot be that both $f(x_0) = f(a)$ and $f(x_1) = f(a)$. Assume, e.g., that $f(x_0) = f(a)$. Then $x_1 \in (a,b) \Rightarrow f'(x_1) = 0$. □

<u>Mean Value Theorem (§3)</u>

<u>Thm.</u> Let $a, b \in \mathbb{R}$ s.t. $a < b$. Suppose $f: [a,b] \to \mathbb{R}$ is continuous and that $f: (a,b) \to \mathbb{R}$ is differentiable. Then $\exists$ <u>$c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$</u>.

Proof Intuition: Use Rolle's Theorem:



"Shift axes" such that $f(a) = f(b)$

$\to$ follows from Rolle's Theorem

Mean Value Theorem (cont.)

11/17/23

Lecture 20 (cont.)

(*) Proof (Mean Value Theorem)

Set $F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then we know that $F: [a,b] \to \mathbb{R}$ is continuous, $F: (a,b) \to \mathbb{R}$ is differentiable, $F(a) = f(a)$, and $F(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$. Then per Rolle's Theorem, $\exists\, c \in (a,b)$ such that $F'(c) = 0$, where $F'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$. $\square$

Mean Value Theorem

Corollary: Let $f: [a,b] \to \mathbb{R}$ be a continuous function such that $f'(x) = 0\ \forall\, x \in (a,b)$. Then $f(x) = f(a)\ \forall\, x \in (a,b)$.

(*) Proof: Let $x \in (a,b)$. Then $f: [a,x] \to \mathbb{R}$ is continuous and $f: (a,b) \to \mathbb{R}$ is differentiable; then per the MVT, $\exists\, c \in (a,x)$ s.t. $f'(c) = \frac{f(x)-f(a)}{x-a}$. But $f'(c) = 0 \Rightarrow f(x) - f(a) = 0$. $\square$

Corollary: Let $f: [a,b] \to \mathbb{R}$ be continuous and $f: (a,b) \to \mathbb{R}$ be differentiable. Then:

(i) If $f' > 0$ on (a,b), then f is (strictly) increasing.

(ii) $f' \geq 0$ on (a,b) iff f is monotone non-decreasing.

(*) Proof: Let $x_1, x_2 \in [a,b]$ s.t. $a \leq x_1 < x_2 \leq b$. Then $\exists\, c \in (x_1, x_2)$ s.t. $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

But $f'(c) > 0 \,/\, f'(c) \geq 0 \Rightarrow f(x_2) - f(x_1) > 0 \,/\, f(x_2) - f(x_1) \geq 0$. $\square$

(ii) ($\Leftarrow$) Assume f is monotone non-decreasing. Then for any $\varepsilon > 0$, $\frac{f(x+\varepsilon) - f(x)}{(x+\varepsilon) - x} \geq 0$.

(*) Note: The ($\Leftarrow$) relation fails for (i) [strictly increasing $\not\Rightarrow f' > 0$].

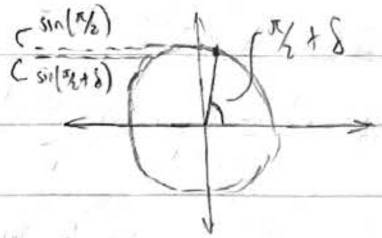
Ex: Let $f(x) = x^3$. Then f is strictly increasing on $\mathbb{R}$, but $\exists\, x \in \mathbb{R}$ s.t. $f'(x) = 0$.

Taylor's Theorem

11/20/23

Lecture 21

Motivation: Taylor's Theorem (§4)



Consider: $f(x) = \sin(x)$. We know $f(x)$ @ $x = 0, \frac{\pi}{2}, \ldots$

Want to find $f(x)$ at some x near $\pi/2$ $(x = \pi/2 + \delta)$

$\rightarrow$ Take $f(x) = f(\frac{\pi}{2} + \delta) = f(\frac{\pi}{2}) + \delta f'(\pi/2) + \frac{\delta^2}{2} f''(\frac{\pi}{2}) \ldots$

Def: Higher-Order Derivatives

Let $f: I \to \mathbb{R}$ be differentiable. If f' is differentiable, we say that f is twice differentiable, and we write $f'' = (f')' = f^{(2)}$. We call $f^{(2)}$ the second-order derivative of f. By induction, the n^{th}-order derivative of f is the derivative of the $(n-1)^{th}$-order derivative, and is denoted $f^{(n)}$.

Notation: We write $f \in C^n(I)$ to indicate that $f^{(n)}$ exists and is continuous. If the n^{th}-order derivative of f exists $\forall n \in \mathbb{N}$, then we write $f \in C^\infty(I)$.

(*) Note: $f \in C^n(I) \Rightarrow$ all derivatives $f, f^{(1)}, \ldots, f^{(n-1)}, f^{(n)}$ exist.

Observation: Let $P: I \to \mathbb{R}$ be a polynomial of degree n, and let $a \in I$. Then $P(x) = \sum_{i=0}^{n} \frac{P^{(i)}(a)}{i!}(x-a)^i$.

Lemma. Suppose I is an interval and $f: I \to \mathbb{R}$ is $(n+1)$-times differentiable. For $a, b \in I$, we define $R_n(b,a) = f(b) - (f(a) + \frac{b-a}{1!} f^{(1)}(a) + \ldots + \frac{(b-a)^n}{n!} f^{(n)}(a))$.

Then $\frac{d}{dx} R_n(b, x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n$.

Goal: Want to show that for any function $f: I \to \mathbb{R}$:

$$\left| f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \right| \leq \frac{M_{n+1}}{(n+1)!}|x-a|^{n+1}, \quad \text{where } M_{n+1} = \sup_{x \in I} \left| f^{(n+1)}(x) \right|,$$

i.e. that the error is bounded above by some term going to 0 as $n \to \infty$.

Taylor's Theorem (cont.)

11/20/23

Lecture 21

(cont.)

(*) Proof: $\frac{d}{dx} R_n(b,x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n$

$$\frac{d}{dx} R_n(b,x) = \frac{d}{dx}\left(f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!}(b-x)^i\right)$$

$$= f'(x) + \sum_{i=0}^{n}\left(\frac{f^{(i)}(x)}{i!}(b-x)^i + \frac{f^{(i)}(x)}{i!}(i)(-1)(b-x)^{i-1}\right)$$

$$= f'(x) + \sum_{i=0}^{n}\left(\frac{f^{(i+1)}(x)}{i!}(b-x)^i\right) - \sum_{i=1}^{n}\left(\frac{f^{(i)}(x)}{(i-1)!}(b-x)^{i-1}\right) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n. \square$$

Theorem (Taylor's Theorem)

Let I be an open interval, and let $f: I \to \mathbb{R}$ be $(n+1)$-times differentiable. Then for any $a, b \in I$ such that $a < b$, $\exists\, c \in (a,b)$ such that:

$$\boxed{f(b) = \left(f(a) + \frac{f'(a)}{1!}(b-a) + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n\right) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}.$$

→ Consequence: We can approximate $f(x)$ with a polynomial of degree n.

(*) Proof

We want to show $\exists\, c \in (a,b)$ such that $R_n(b,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$. Define $K = \frac{R_n(b,a)}{(b-a)^{n+1}}(n+1)!$; then we want to show $\exists\, c \in (a,b)$ such that $f^{(n+1)}(c) = K$.

Set $\varphi(x) = R_n(b,x) - K\frac{(b-x)^{n+1}}{(n+1)!}$. We observe that φ is differentiable on I, and that $\varphi(a), \varphi(b) = 0$.

Then per Rolle's Theorem, $\exists\, c \in (a,b)$ such that $0 = \varphi'(c) = \frac{d}{dx}R_n(b,x)\Big|_{x=c} - K(n+1)(-1)\frac{(b-x)^n}{(n+1)!}\Big|_{x=c}$.

Per our previous lemma: $\frac{d}{dx}R_n(b,x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n \Rightarrow -\frac{f^{(n+1)}(c)}{n!}(b-c)^n + K\frac{(b-c)^n}{n!} = 0$. Since $b \neq c$, then (dividing by $\frac{(b-c)^n}{n!}$) $\Rightarrow K - f^{(n+1)}(c) = 0 \Rightarrow f^{(n+1)}(c) = K$. $\square$

(*) **Def: Analytic Functions**

Let $f \in C^\infty(I)$. Then we say f is real analytic if, for any $a \in I$ near x ($|x-a| < r$, $r > 0$):

$$\boxed{f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i}$$

Riemann Integrals

11/22/23

Lecture 22

Riemann Sums (§6)

Def: Let $a, b \in \mathbb{R}$ such that $a < b$. Then we define:

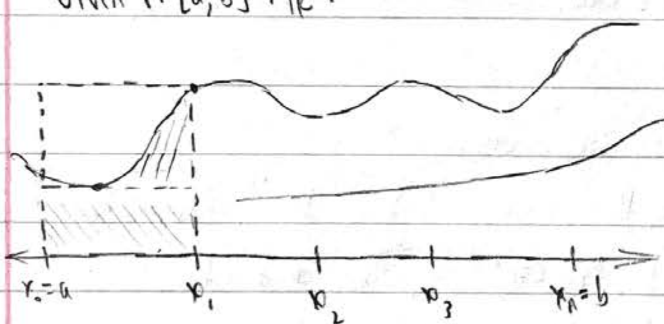
(i) A partition of $[a,b]$ is a sequence $x_0 = a < x_1 < \ldots < x_{n-1} < x_n = b$.

(ii) Given a partition $P = (x_0, \ldots, x_n)$ of $[a,b]$, let $C = (x_1', \ldots, x_n')$ be a set of middle points of P, s.t. $x_i' \in [x_{i-1}, x_i] \ \forall \ 1 \le i \le n$. Let $f : [a,b] \to \mathbb{R}$; then we define the Riemann sum of f over $[a,b]$ with respect to partition P, set of middle points C to be:

$$\boxed{S(f, P, C) = \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1})}$$

Intuition: Integrability

Given $f : [a,b] \to \mathbb{R}$:



Picking largest, smallest values for f on $[x_0, x_1]$, we observe: setting $A_{min} = f(x_{min})(x_1 - x_0)$, $A_{max} = f(x_{max})(x_1 - x_0)$, then for the area A under f on $[x_0, x_1]$:

$$A_{min} \le A \le A_{max}$$

integral exists $\Leftarrow$ want: A_{min}, A_{max} converge as $x_1 - x_0 \to 0$

Def: Riemann Integrals

Let $f : [a,b] \to \mathbb{R}$. We say that the Riemann integral of f over $[a,b]$ exists and is $I \in \mathbb{R}$ if, for every $\varepsilon > 0$, $\exists \ \delta > 0$ s.t. $|S(f, P, C) - I| \le \varepsilon \ \forall \ P$ partition of $[a,b]$ and C set of middle points of P s.t. $|x_i - x_{i-1}| < \delta \ \forall \ x_i \in P = \{x_0, \ldots, x_n\}$.

Riemann Integrals (cont.)

11/22/23
Lecture 22
(cont.)

Integrability

Notation: Define $\|P\| = \max_{i \in \{1,\dots,n\}} |x_i - x_{i-1}|$

Def: Define $s(\delta) := \sup_{P,C} \{ |S(f,P,C) - I| : P$ partition of $[a,b]$, $\|P\| \leq \delta$, C set of middle points$\}$.

Then $f:[a,b] \to \mathbb{R}$ is Riemann-integrable if $\lim_{\delta \to 0} s(\delta) = 0$; then $I = \int_a^b f(x)\,dx = \int_a^b f$.

(*) Ex: Non-Integrable

Consider $f:[0,1] \to \mathbb{R}$ defined by: $f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$

Let $P = (x_0, \dots, x_n)$.

Let $C' = \{x_1', \dots, x_n'\}$, where $x_i' \in [x_{i-1}, x_i] \cap \mathbb{Q}$; let $C'' = \{x_1'', \dots, x_n''\}$, where $x_i'' \in [x_{i-1}, x_i] \setminus \mathbb{Q}$.

Then $S(f,P,C') = 1$, but $S(f,P,C'') = 0$.

If $I = \int_0^1 f(x)\,dx$ existed, would have $I \leq |S(f,P,C') - S(f,P,C'')| \leq |S(f,P,C') - I| + |I - S(f,P,C'')|$. But take $\varepsilon = 1/3$; then $1 \leq |S(f,P,C') - I| + |I - S(f,P,C'')| \leq 2/3$. $\square$

Claim. The Riemann integral of f over $[a,b]$, if one exists, is unique.

(*) Proof: Let $I, I' \in \mathbb{R}$ be two values for the Riemann integral of f over $[a,b]$. Assume, for the sake of contradiction, that $I \neq I'$. Fix $\varepsilon = |I - I'|/3$. Then $\exists\, \delta$ s.t. $|S(f,P,C) - I|, |S(f,P,C) - I'| < \varepsilon$ $\forall P$ satisfying $\|P\| < \delta$. Fix δ. Then $|I - I'| \leq |I - S(f,P,C)| + |S(f,P,C) - I'| < 2\varepsilon = \frac{2|I-I'|}{3}$. [contra]. $\square$

(*) Alternative proof: Take as middle points the sets $C' = \{x_1', \dots, x_n'\}$, $C'' = \{x_1'', \dots, x_n''\}$, where $x_i' = \inf[x_{i-1}, x_i]$ and $x_i'' = \sup[x_{i-1}, x_i]$. [Defined relative to some $\varepsilon > 0$, $\delta(\varepsilon) > 0$, partition P ($\|P\| < \delta$)]. Then $S(f,P,C') \leq S(f,P,C) \leq S(f,P,C'')$ for any other set of middle points. (?)

Properties of Riemann Integrals

11/27/23

Lecture 23

Properties of the Integral (§2)

Recalling the definition of the Riemann sum: we see clearly that $S(f,P,C)+S(g,P,C)=S(f+g,P,C)$.

→ **Thm.** Suppose $f,g:[a,b]\to\mathbb{R}$ are Riemann-integrable and $c\in\mathbb{R}$. Then $f+g$, cf are Riemann-integrable and:

(i) $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

(ii) If $f\le g$, then $\int_a^b f \le \int_a^b g$. In particular: if $f\ge 0$, then $\int_a^b f \ge 0$.

(*) Proof: (i) Let $I=\int_a^b f+\int_a^b g$. Fix ε. Since f,g are Riemann-integrable, $\exists\, \delta_1,\delta_2>0$ s.t. $|S(f,P,C)-\int_a^b f|<\varepsilon/2$, $|S(g,P,C)-\int_a^b g|<\varepsilon/2$ $\forall$ partitions P of $[a,b]$ s.t. $\|P\|<\delta_1,\delta_2$ and sets of middle points C of P. Fix $\delta=\min(\delta_1,\delta_2)$, P, C; then $|S(f+g,P,C)-\int_a^b f-\int_a^b g|$ $=|S(f,P,C)-\int_a^b f+S(g,P,C)-\int_a^b f|<|S(f,P,C)-\int_a^b f|+|S(g,P,C)-\int_a^b g|<\varepsilon$. □

(ii) Assume $f\le g$. Set $x_i^n=a+\frac{b-a}{n}i$, $i=0,\dots,n$. Set $P^{(n)}=(x_0^n,\dots,x_i^n)$, $C^{(n)}=(x_1^n,\dots,x_n^n)$. Since f,g are Riemann-integrable, then $\lim_{n\to\infty} S(f,P^{(n)},C^{(n)})$, $\lim_{n\to\infty} S(g,P^{(n)},C^{(n)})$ exist. Then $\int_a^b f=\lim_{n\to\infty} S(f,P^{(n)},C^{(n)})\le \lim_{n\to\infty} S(g,P^{(n)},C^{(n)})=\int_a^b g$. □

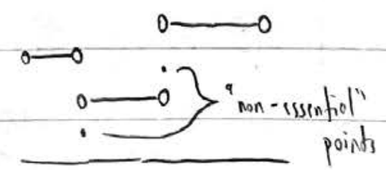
Corollary: Suppose $f:[a,b]\to[m,n]$ is Riemann-integrable. Then $m(b-a)\le \int_a^b f\le n(b-a)$.

(*) Weaknesses of the Riemann Integral [131C]

Key Weakness: Riemann sums allow for taking any point as a middle point

Consequences: (i) If f_n is a sequence converging to f, it is not always true that $\int_a^b f=\lim_{n\to\infty}\int_a^b f_n$.

(ii)



Solution: Lebesgue integrals restrict middle points under consideration to "essential" points [Lebesgue measure > 0]

Integrability

11/27/23

Lecture 23 (cont.)

Existence of the Riemann Integral (§3)

Recall (Limits): $(x_n)_n \subset \mathbb{R}$ converges iff $(x_n)_n$ is Cauchy. [Can define Cauchy criterion for integrals]

Cauchy Criterion for Integrability

Thm. Let $f : [a,b] \to \mathbb{R}$. Then TFAE:

(i) f is Riemann-integrable over $[a,b]$.

(ii) for every $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. for any P_1, P_2 partitions of $[a,b]$ such that $\|P_1\|, \|P_2\| < \delta$ with middle points C_1, C_2: $|S(f, P_1, C_1) - S(f, P_2, C_2)| < \varepsilon$.

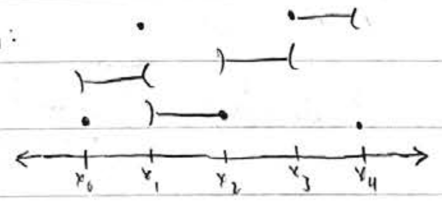
(*) Ex (Non-integrable: $f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$ $\rightsquigarrow$ can always find P_1, P_2 s.t. $S(f, P_1, C_1) - S(f, P_2, C_2) = 1$ $\to$ not integrable

(*) Proof: ($\Leftarrow$) Assume (ii). Set $x_i^{(n)} = a + \frac{b-a}{n} i$ $(0 \le i \le n)$, $P^{(n)} = (x_0^n, \dots, x_n^n)$, $C^{(n)} = (x_1^n, \dots, x_n^n)$. Set $I^{(n)} = S(f, P^{(n)}, C^{(n)})$. Then $(I^{(n)})_n \subset \mathbb{R}$ is Cauchy, per (ii) $\Rightarrow (I^{(n)})_n$ converges to $I \in \mathbb{R}$. Fix $\varepsilon > 0$; set $\delta > 0$ to be s.t. $|S(f, P_1, C_1) - S(f, P_2, C_2)| < \varepsilon$ whenever $\|P_1\|, \|P_2\| < \delta$. Let P be a partition of $[a,b]$ w/ middle points C s.t. $\|P\| < \delta$; then $\exists$ s.t. $\frac{b-a}{n} < \delta$ and $|I^{(n)} - I| < \varepsilon \Rightarrow |S(f, P, C) - I^{(n)}| < \varepsilon$. Hence $|S(f, P, C) - I| \le |S(f, P, C) - I^{(n)}| + |I^{(n)} - I| < 2\varepsilon$. $\square$

Def: Step Functions

We call $f : [a,b] \to \mathbb{R}$ a step function if $\exists$ partition $P = (x_0, \dots, x_n)$ of $[a,b]$ such that $f|_{(x_{i-1}, x_i)}$ is constant $\forall\, 1 \le i \le n$.

(*) Ex:



$\Rightarrow$ step function

Integrals of Step Functions

11/29/23

Lecture 24

Def: Characteristic Function

Let $a, b \in \mathbb{R}$ s.t. $a < b$, and let $A \subset [a,b]$. Then we define the characteristic function of A to be $\chi_A : [a,b] \to \mathbb{R}$ defined by $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$.

χ = "Chi"

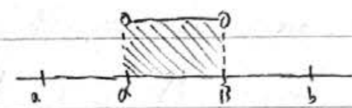
Ex (Riemann-Integrable):

ξ = "Xi"

(i) Let $\xi \in [a,b]$ and set $f = \chi_{\{\xi\}}$.
→ Then $\int_a^b f = 0$.

$\int_a^b f = 0$

(ii) Let $\alpha, \beta \in [a,b]$ s.t. $\alpha < \beta$, and set $f = \chi_{(\alpha,\beta)}$. Then $\int_a^b f = \beta - \alpha$.

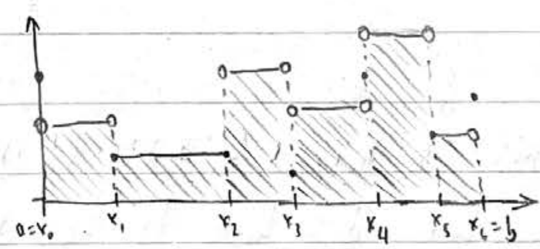


$\int_a^b f = \beta - \alpha$

Prop. Let f be a step function on $[a,b]$; then $\exists$ partition $P = (x_0, \ldots, x_n)$ of $[a,b]$ and $c_1, \ldots, c_n \in \mathbb{R}$ s.t. $f(x) = c_i \ \forall x \in (x_{i-1}, x_i)$. Then:

(i) f is Riemann-integrable

(ii) $\int_a^b f = \sum_{i=1}^n c_i (x_i - x_{i-1})$.



(*) Proof: Set $g = \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)}$, $h = \sum_{i=0}^n f(x_i) \chi_{\{x_i\}}$. Observe: $f = g + h$.

Then f is Riemann-integrable and
$$\int_a^b f = \int_a^b g + \int_a^b h$$
$$= \sum_{i=1}^n c_i \int_a^b \chi_{(x_{i-1}, x_i)} + \sum_{i=0}^n \int_a^b f(x_i) \chi_{\{x_i\}}$$
$$= \sum_{i=1}^n c_i (x_i - x_{i-1}). \quad \square$$

11/29/23
Lecture 24
+ Lecture 25

Theorem: Integrability

Thm. Let $f:[a,b]\to\mathbb{R}$. Then TFAE:

(i) f is Riemann-integrable

(ii) for every $\varepsilon>0$, $\exists$ step functions $f_1, f_2:[a,b]\to\mathbb{R}$ s.t. $f_1\le f\le f_2$ and $\int_a^b (f_2-f_1)<\varepsilon$.

(*) Proof: (ii) $\to$ (i).

Per (ii), $\exists$ step functions $f_1, f_2:[a,b]\to\mathbb{R}$ s.t. $f_1\le f\le f_2$ and $\int_a^b(f_2-f_1)<\varepsilon/6$. Since f_1, f_2 are Riemann-integrable, $\exists\ \delta$ [δ_1, δ_2 min] such that for any partitions P, Q where $\|P\|, \|Q\|<\delta$, $|S(f_1,P,C)-S(f_1,Q,D)|<\varepsilon/6$ (& likewise for f_2) and $|\int_a^b(f_2-f_1)-S(f_2-f_1,P,C)|<\varepsilon/6$.

Then $|S(f,P,C)-S(f,Q,D)| = |[S(f,P,C)-S(f_1,P,C)]+[S(f_1,P,C)-S(f_1,Q,D)]+$
$+[S(f_1,Q,D)-S(f_2,Q,D)]+[S(f_2,Q,D)-S(f,Q,D)]|$
$= |S(f-f_1,P,C)+S(f_1-f_2,Q,D)-S(f-f_2,Q,D)+(S(f_1,P,C)-S(f_1,Q,D))|$
$\le |S(f-f_1,P,C)+S(f_1-f_2,Q,D)-S(f_1-f_2,Q,D)+(S(f_1,P,C)-S(f_1,Q,D))|\le\varepsilon/2$.

(*) Proof: (i) $\to$ (ii)

Per (i), f is Riemann-integrable; then for every $\varepsilon>0$, $\exists\ \delta>0$ s.t. $|S(f,P,C)-S(f,P',C')|<\varepsilon$ whenever $\|P\|, \|P'\|<\delta$. In particular: if $P=P'$, then $|S(f,P,C)-S(f,P,C')|<\varepsilon$. We can use this definition to construct step functions f_1, f_2.

Claim 1: f is bounded

Proof: Fix $\varepsilon>0$. Set $\delta>0$ s.t. for any $P=(x_0,\dots,x_n)$ partition s.t. $\|P\|<\delta$, we have $|\sum_{i=1}^n (f(x_i')-f(x_i''))(x_i-x_{i-1})|<\varepsilon/2$ for $x_i', x_i''\in[x_{i-1},x_i]$. Fix $i_0\in\{1,\dots,n\}$ and set $x_i''=x_i'$ when $i\ne i_0$. Then $|(f(x_{i_0}')-f(x_{i_0}''))(x_{i_0}-x_{i_0-1})|<\varepsilon/2 \Rightarrow$
$\to |(f(x_{i_0}')-f(x_{i_0}''))|<\frac{\varepsilon}{2(x_{i_0}-x_{i_0-1})}$. Set $x_{i_0}''=x_{i_0}$; then $|f(x_{i_0}')|\le|f(x_{i_0})|+\frac{\varepsilon}{2(x_{i_0}-x_{i_0-1})}$.

Integrability (cont.)

12/1/23

Lecture 25

(cont.)

(*) Proof (cont.)

Proof: f is bounded (cont.)

$|f(x_{i_0}')| \le |f(x_{i_0})| + \frac{\varepsilon}{2(x_{i_0}-x_{i_0-1})}$ $\forall\ x_{i_0}' \in [x_{i_0-1}, x_{i_0}]$. Set $N = \max_{i \in \{1,\dots,n\}} \left(\frac{\varepsilon}{2(x_i - x_{i-1})} + |f(x_i)|\right)$;

then $|f(x)| \le N$ $\forall\ x \in [a,b]$. $\square$

Define $m_i = \inf_{[x_{i-1}, x_i]} f$, $M_i = \sup_{[x_{i-1}, x_i]} f$.

Claim 2: m_i, M_i exist.

Proof: Write $m_i = \lim_{k \to +\infty} f(x^{(i,k)})$ for sequence $(x^{(i,k)})_k \subset [x_{i-1}, x_i]$. Write $M_i = \lim_{k \to +\infty} f(y^{(i,k)})$ for $(y^{(i,k)})_k \subset [x_{i-1}, x_i]$. Then $\left|\sum_{i=1}^{n} (m_i - M_i)(x_{i-1} - x_i)\right| \le \varepsilon/2$. $\square$

Then we have $m_i \le f(x) \le M_i$ $\forall\ x \in [x_{i-1}, x_i]$. Set $f_1 = \sum_{i=1}^{n} m_i \chi_{(x_{i-1}, x_i)} + \sum_{i=0}^{n} f(x_i) \chi_{\{x_i\}}$, $f_2 = \sum_{i=1}^{n} M_i \chi_{(x_{i-1}, x_i)} + \sum_{i=0}^{n} f(x_i) \chi_{\{x_i\}}$. Then f_1, f_2 are step functions $f_1 \le f \le f_2$.

Claim: $\int_a^b (f_2 - f_1) < \varepsilon$.

$$\int_a^b (f_2 - f_1) = \int_a^b (M_i - m_i)\chi_{(x_{i-1}, x_i)} = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \le \varepsilon/2 < \varepsilon. \ \square$$

Remark: A function must be bounded to be Riemann-integrable. [flaw]

(*) Ex: $f(x) = \begin{cases} 1/\sqrt{x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$ is Lebesgue-integrable [Leb: $\int_0^1 f(x) = \frac{1}{2}$], but not Riemann-integrable.

Corollary: If $f : [a,b] \to \mathbb{R}$ is continuous, then f is Riemann-integrable.

(*) Proof: Fix $\varepsilon > 0$. f cts., $[a,b]$ compact $\Rightarrow$ f uniformly continuous; then $\exists\ \delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$ $\forall\ x, y \in [a,b]$ s.t. $|x-y| < \delta$. Let $P = (x_0, \dots, x_n)$ partition of $[a,b]$ w/ $\|P\| < \delta$; $\forall\ 1 \le i \le n$, set $x_i', x_i'' \in [x_{i-1}, x_i]$ s.t. $f(x_i') = \min_{[x_{i-1}, x_i]} f$, $x_i'' = \max_{[x_{i-1}, x_i]} f$. Set $f_1 = \sum_{i=1}^{n} f(x_i') \chi_{(x_{i-1}, x_i)} + \sum_{i=0}^{n} f(x_i) \chi_{\{x_i\}}$, $f_2 = \sum_{i=1}^{n} f(x_i'') \chi_{(x_{i-1}, x_i)} + \sum_{i=0}^{n} f(x_i) \chi_{\{x_i\}}$. Then $f_1 \le f \le f_2$ step fns., and $\int_a^b (f_2 - f_1) = \sum_{i=1}^{n} \left((f(x_i'') - f(x_i'))(x_i - x_{i-1}) \right) \le \sum_{i=1}^{n} \frac{\varepsilon}{2(b-a)} (x_i - x_{i-1}) = \varepsilon/2$. $\square$

(*) Dini's & Taylor's Theorems

11/30/23

Disc 9

Dini's Theorem

Recall (Dini's Thm): If (a) K compact, (b) $f: K \to \mathbb{R}$ is continuous, $(f_n: K \to \mathbb{R})_n$ sequence of continuous fns. converging pointwise to f, and (c) $f_n(x) \geq f_{n+1}(x) \Rightarrow f_n$ converges unif. to f on K.

(*) Proof: Set $g_n = f_n - f$; then g_n cts, goes to 0 pointwise, and $g_n \geq g_{n+1}$. WTS: $g_n \xrightarrow{\text{unif}} 0$. Fix $\varepsilon > 0$, and define $O_n = \{x : g_n(x) < \varepsilon\}$. We see that $O_n \subset O_{n+1}$, $\bigcup_n^\infty O_n = K$. Take $x \in K$; then $\exists\, n$ s.t. $g_n(x) < \varepsilon \Rightarrow x \in O_n$. Observe: $O_n = g_n^{-1}((-\infty, \varepsilon))$ is pre-image of an open set, therefore O_n is open. O_n open, $\bigcup_n^\infty O_n = K$, K compact $\Rightarrow$ can find finite subcover $O_1 \cup \dots \cup O_N$ of K. Since $O_n \subset O_{n+1}$, therefore $O_N = K$. Then $0 = g_n(x) < \varepsilon\ \forall\, x \in K,\ n \geq N$; then $\sup_{x\in K} |g_n| = \sup_{x \in K} |f_n - f| < \varepsilon\ \forall\, n \geq N$. □

Taylor's Theorem

Lagrange form: Let $f: I \to \mathbb{R}$ be $(n+1)$-times differentiable. Take $a, b \in I$; then $\exists\ c \in (a,b)$ w/ $f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \dots + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$. [Used for proofs].

Peano form: Let $f: I \to \mathbb{R}$ be (n)-times differentiable. Let $P_a^{(x)} = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$. Let $a, x \in I$. Then $f(x) = P_a(x) + h_n(x) \cdot (x-a)^n$, where $\lim_{x \to a} h_n(x) = 0$. [Used for applications]

Problem: Let $f: I \to \mathbb{R}$; suppose f'' exists. Show $\lim_{h \to 0} (f(x+h) - 2f(x) + f(x-h))/h^2 = f''(x)$.

Integrability (cont.)

12/4/23

Lecture 26

<u>Remark</u>: Let $f, g : [a,b] \to \mathbb{R}$ be step functions. Then:

(i) For any $c \in [a,b]$, $\int_a^b f = \int_a^c f + \int_c^b f$.

(ii) $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

<u>Theorem</u>. Let $f : [a,b] \to \mathbb{R}$, $c \in [a,b]$. Then TFAE:

(i) f is Riemann-integrable

(ii) $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Riemann-integrable

(*) <u>Proof</u>: (ii) → (i). Fix $\varepsilon > 0$. We want to find step functions f_1, f_2 s.t. $f_1 \le f \le f_2$ and $\int_a^b f_2 - f_1 < \varepsilon$. Since $f|_{[a,c]}$ is Riemann-integrable, $\exists$ step functions $g_1, g_2 : [a,c] \to \mathbb{R}$ s.t. $g_1 \le f|_{[a,c]} \le g_2$ and $\int_a^c (g_2 - g_1) < \varepsilon/2$. We can find similar functions $h_1, h_2 : [c,b] \to \mathbb{R}$ s.t. $h_1 \le f|_{[c,b]} \le h_2$ and $\int_c^b (h_2 - h_1) < \varepsilon/2$. Set:

$$f_1 = \begin{cases} g_1(x) & x \in [a,c] \\ h_1(x) & x \in (c,b] \end{cases} \quad \text{and} \quad f_2 = \begin{cases} g_2(x) & x \in [a,c] \\ h_2(x) & x \in (c,b] \end{cases}$$

Then f_1, f_2 are step functions $[a,b] \to \mathbb{R}$ s.t. $f_1 \le f \le f_2$, $\int_a^b (f_2 - f_1) = \int_a^c (g_2 - g_1) + \int_c^b (h_2 - h_1) < \varepsilon$.

(*) <u>Claim</u>: $\int_a^b f = \int_a^c f + \int_c^b f$.

<u>Proof</u>: Let f, f_1, f_2 as before. We know that $\int_a^b f \le \int_a^b f_2 = \int_a^b (f_2 - f_1 + f_1) < \varepsilon + \int_a^b f_1$ $= \varepsilon + \int_a^c f_1 + \int_c^b f_1 \le \varepsilon + \int_a^c f + \int_c^b f$. Similarly, $\int_a^b f \ge \int_a^b f_1 = \int_a^b (f_1 - f_2 + f_2) > -\varepsilon + \int_a^b f_2$ $= -\varepsilon + \int_a^c f_2 + \int_c^b f_2 \ge -\varepsilon + \int_a^c f + \int_c^b f$. Then $\left| \int_a^b f - \left(\int_a^c f + \int_c^b f \right) \right| < \varepsilon$. □

(i) → (ii). Let $\varepsilon > 0$. Then $\exists$ step functions $f_1, f_2 : [a,b] \to \mathbb{R}$ s.t. $f_1 \le f \le f_2$ and $\int_a^b (f_2 - f_1) < \varepsilon$. We observe that $f_1|_{[a,c]}, f_2|_{[a,c]}$ and $f_1|_{[c,b]}, f_2|_{[c,b]}$ are functions satisfying the conditions for integrability on $[a,c]$ and $[c,b]$, respectively. □

Fundamental Theorem of Calculus

12/4/23

Lecture 26 + Lecture 27

<u>Def.</u> Let $f : [a,b] \to \mathbb{R}$ be Riemann-integrable, and let $c \in [a,b]$. Then $\underline{\int_c^c f = 0}$ and $\underline{\int_b^a f = -\int_a^b f}$.

<u>Corollary.</u> If $f : [a,b] \to \mathbb{R}$ is Riemann-integrable and $\alpha, \beta, \gamma \in [a,b]$, then $\underline{\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f}$.

<u>Fundamental Theorem of Calculus (§4)</u>

<u>Thm.</u> Let $a, b \in \mathbb{R}$ such that $a < b$, and let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then $\underline{F : x \mapsto \int_a^x f \text{ is differentiable on } [a,b]}$ with $\underline{F' = f}$.

(*) <u>Proof:</u> Let $x_0 \in (a,b)$. Then we want to compute $F'(x_0)$, if it exists.

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\int_a^x f - \int_a^{x_0} f}{x - x_0} = \lim_{x \to x_0} \frac{\int_{x_0}^x f}{x - x_0} = \lim_{x \to x_0} \frac{\int_{x_0}^x (f - f(x_0)) + (x - x_0) f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\int_{x_0}^x f - f(x_0)}{x - x_0} + f(x_0).$$

Fix $\varepsilon > 0$. Since f is continuous and $[a,b]$ compact, then f is uniformly continuous; then $\exists\, \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon \;\forall\, x, y \in [a,b]$ s.t. $|x - y| < \delta$. Then:

$$\lim_{x \to x_0} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \lim_{x \to x_0} \left| \frac{\int_{x_0}^x f - f(x_0)}{x - x_0} \right| \leq \lim_{x \to x_0} \frac{\int_{\min\{x,x_0\}}^{\max\{x,x_0\}} |f - f(x_0)|}{|x - x_0|}.$$

$\rightarrow$ For $|x - x_0| < \delta$: $\displaystyle\lim_{x \to x_0} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq \lim_{x \to x_0} \frac{\int_{\min\{x,x_0\}}^{\max\{x,x_0\}} |f - f(x_0)|}{|x - x_0|} < \frac{\delta \varepsilon}{\delta} = \varepsilon.$ $\square$

<u>Corollary:</u> Let U be an open interval in $\mathbb{R}$, and let $F : U \to \mathbb{R}$ be a continuous function with continuous derivative. Let $a, b \in U$; then $\underline{F(b) - F(a) = \int_a^b F'}$.

(*) <u>Proof:</u> By the Fundamental Theorem of Calculus, $\frac{d}{dx}\left(F(x) - \int_a^x F'\right) = F' - F' = 0$, so $x \mapsto F(x) - \int_a^x F'$ is constant on every interval in U containing a, b. In particular, $F(b) - \int_a^b F' = F(a) - \int_a^a F' = F(a)$. $\square$

Further Topics

12/6/23

Lecture 27

(cont.)

(*) Change of Variables (Integration)

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, "f\circ\varphi"'] & V \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$\nabla\varphi = \left(\frac{\partial \varphi_i}{\partial x_j}\right)_{i,j}$

$\sim$ Integral of $f\circ\varphi$: $\int_U f\circ\varphi\,|\det(\nabla\varphi)| = \int_{\varphi(U)} f(y)N(y)\,dy$,

where $N(y) = |\{x\in U : \varphi(x) = y\}|$.

If φ is 1:1, $N(y)=1$ [or 0] $\to \int_U f\circ\varphi \det(\nabla\varphi) = \int_{\varphi(U)} f$

When $U, V \subset \mathbb{R}$, 1:1 subtlety disappears

$\to$ Theorem (Change of Variables). Let U, V be 2 nonempty open intervals of finite length and let $\varphi: U\to V$ be continuous with differentiable derivative. For every $a,b\in U$ s.t. $a<b$, $f: V\to\mathbb{R}$ continuous,
$\int_a^b f\circ\varphi\,\varphi' = \int_{\varphi(a)}^{\varphi(b)} f$.

Logarithm & Exponential Functions (§5)

Def. For $x>0$, define $\log(x) = \int_1^x \frac{dt}{t}$ for $x>1$, $\log(x) = \int_x^1 \frac{dt}{t}$ for $x<1$.

Thm. The logarithm is differentiable on $(0,+\infty)$ with derivative $\frac{d}{dx}(\log x) = 1/x$. Furthermore, if $x,y>0$: (i) $\log(xy) = \log(x)+\log(y)$

(ii) $\log(x/y) = \log(x) - \log(y)$

(*) Proof: By Fundamental Theorem of Calculus: $\frac{d}{dx}(\log x) = \frac{d}{dx}\left(\int_1^x \frac{dt}{t}\right) = \frac{1}{x}$.

(i). Fix $a>0$. Then $\frac{d}{dt}\log(at) = a\cdot\frac{1}{at} = \frac{1}{t} \to \log(ax) - \log(a) = \int_1^x \frac{dt}{t} = \log(x)$.

Setting $y=a$, $\log(xy) - \log(y) = \log(x) \Rightarrow \log(xy) = \log(x)+\log(y)$.

(ii). It suffices to show that $\log(1/y) = -\log(y)$: $0 = \log(1) = \log(\frac{y}{y}) = \log(y) + \log(\frac{1}{y})$. $\square$

Logarithm & Exponential (cont.)

12/8/23

Lecture 28 Logarithm & Exponential Functions (cont.)

Want to show: logarithm is continuous & range(log) = $\mathbb{R}$ → define $\log^{-1}: \mathbb{R} \to (0, +\infty)$. [exponential]

↳ Continuity derived from differentiability

Lemma: range(log) = $\mathbb{R}$.

(*) Proof: Since log is continuous and increasing on $(0, +\infty)$, it suffices to show that $\lim_{x\to 0} \log(x) = -\infty$ and $\lim_{x\to+\infty} \log(x) = +\infty$. We can look at $\lim_{n\to\infty} \log\left(\frac{1}{2^n}\right)$, $\lim_{n\to\infty} \log(2^n)$.
$\lim_{n\to\infty} \log\left(\frac{1}{2^n}\right) = \lim_{n\to\infty} n\log\left(\frac{1}{2}\right)$; $\lim_{n\to\infty} \log(2^n) = \lim_{n\to\infty} n\log(2)$. We can show $\log(2) > 0$ directly; then
$\lim_{x\to 0} \log(x) = \lim_{n\to\infty} \log\left(\frac{1}{2^n}\right) = \lim_{n\to\infty} -n\log(2) = -\infty$, $\lim_{x\to+\infty} \log(x) = \lim_{n\to\infty} \log(2^n) = \infty$.

Def. Define the exponential function $\exp: \mathbb{R} \to (0, +\infty)$ as $\exp = (\log)^{-1}$, i.e. $y = \exp(x)$ if $x = \log(y)$.

Lemma. The exponential function $x \mapsto \exp(x)$ is continuous.

(*) Proof: Let $x_0 \in \mathbb{R}$, $\varepsilon > 0$. We want to find $\delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |\exp(x) - \exp(x_0)| < \varepsilon$,
i.e. $\exp(x_0) - \varepsilon < \exp(x) < \exp(x_0) + \varepsilon$. Applying log: $\log(-\varepsilon + \exp(x_0)) < x < \log(\varepsilon + \exp(x_0))$.
$\to \log(-\varepsilon + \exp(x_0)) - x_0 < x - x_0 < \log(\varepsilon + \exp(x_0)) - x_0$

Observe: for $\varepsilon < \exp(x_0)$, $\log(-\varepsilon + \exp(x_0)) < \log(\exp(x_0)) = x_0 < \log(\varepsilon + \exp(x_0))$
$\to$ set $\delta = \min\{\log(\varepsilon + \exp(x_0)) - x_0,\ x_0 - \log(-\varepsilon + \exp(x_0))\}$; $\delta > 0$.

For x s.t. $|x - x_0| < \delta$: $-\delta + x_0 < x < \delta + x_0$. Know: $\delta + x_0 \le \log(\varepsilon + \exp(x_0))$,
$-\delta + x_0 \ge \log(-\varepsilon + \exp(x_0)) \to -\varepsilon + \exp(x_0) < \exp(x) < \varepsilon + \exp(x_0)$. □

Logarithm & Exponential (cont.)

12/8/23
Lecture 28
(cont.)
FINAL

Logarithm & Exponential Functions (cont.)

Remark. $(\log)^{-1} = \exp$ & $\log, \exp$ continuous $\Rightarrow \lim_{n\to\infty} \exp(x_n) = \exp(x)$ iff $\lim_{n\to\infty} x_n = x$.

Theorem. The exponential function is differentiable on $\mathbb{R}$ with $\frac{d}{dx}\exp(x) = \exp(x)$.

For $x, y \in \mathbb{R}$: (i) $\exp(x+y) = \exp(x)\exp(y)$

(ii) $\exp(x-y) = \exp(x)/\exp(y)$.

(*) **Proof:** Let $x_0 \in \mathbb{R}$, and set $y_0 = \exp(x_0)$. For any $x \in \mathbb{R}$, set $y = \exp(x)$.

$$\frac{d}{dx}\exp(x)\Big|_{x=x_0} = \lim_{x\to x_0} \frac{\exp(x)-\exp(x_0)}{x-x_0} = \lim_{y\to y_0} \frac{y-y_0}{\log y - \log y_0} = \lim_{y\to y_0} \frac{1}{\frac{\log y - \log y_0}{y-y_0}} = \frac{1}{(\log y)'|_{y=y_0}}$$

$$\rightarrow = \frac{1}{\frac{1}{y_0}} = y_0 = \exp(x_0).$$

For (i): $\log(\exp(x+y)) = x+y = \log(\exp(x)) + \log(\exp(y)) = \log(\exp(x)\exp(y))$. Same arg. for (ii).

Corollary: Let $n \in \mathbb{Z}$, $x \in \mathbb{R}$. Then $\exp(nx) = (\exp(x))^n$.

(*) **Proof:** $\exp(nx) = \exp((n-1)x)\exp(x) = \dots = (\exp(x))^n$ by induction, for $n \in \mathbb{N}$.

For $n \notin \mathbb{N}$: $(-n) \in \mathbb{N} \rightarrow \exp(nx) = \exp(-n(-x)) = (\exp(-x))^{-n} = (\exp(x))^n$. $\square$

Def (Exponent). For $a > 0$, $x \in \mathbb{R}$, define $a^x = \exp(x \log(a))$.

Def (Euler's Number). Define $e = \exp(1)$.

Remark. (i) For $a > 0$ and $x, y \in \mathbb{R}$, $a^{x+y} = \exp((x+y)\log a) = a^x a^y$.

(ii) For $x \in \mathbb{R}$, $e^x = \exp(x \log(e)) = \exp(x)$.

(*) Misc. Notes

Final Review

Misc. Notes

Differentiation

Prop. For any function f differentiable on $[a,b]$, for any $\gamma \in \mathbb{R}$ s.t. either $f(a) < \gamma < f(b)$ or $f(b) < \gamma < f(a)$, $\exists\ c \in (a,b)$ s.t. $f'(c) = \gamma$.

└ proof: look at $g(x) = f(x) - \gamma x$

Integration

Prop. For any function f integrable on $[a,b]$, $|f|$ is integrable on $[a,b]$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Prop. For any functions f, g integrable on $[a,b]$, fg is integrable on $[a,b]$.

Corollary: f^n is integrable on $[a,b]$ $\forall\ n \in \mathbb{N}$.

└ Can use corollary to prove Prop. using Cauchy Criterion

Prop. For any function f continuous on $[a,b]$, $\exists\ c \in [a,b]$ s.t. $\int_a^b f = f(c) \cdot (b-a)$