# Math 61: Introduction to Discrete Structures (23W)

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Textbook: Richard Johnsonbaugh - Discrete Mathematics ($8^{th}$ Edition - 2017)
Topics: Sets, proofs, functions, relations & equivalence relations, counting (incl. permutations, combinations, Binomial Theorem), recurrence relations, graphs (incl. paths & cycles, isomorphisms), trees (incl. binary trees), number theory*

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# Intro to Sets

1/9/23
Lecture 1

## Sets (§1.1)

A set is a fundamental mathematical concept, denoting a collection of objects (known as elements/members).

A set is denoted by the use of curly braces ({}), e.g. {1,2,3}

## Properties of Sets

(*) The order of elements in a set does not matter.
(*) A set is defined uniquely by the elements that it contains.
(*) Anything can be an element of a set, even other sets.
(*) Duplicate elements in a set are ignored.

A set can be described by:
- Listing all of its elements (e.g. {1, 2, 3})
- Listing conditions for membership in the set, e.g. $\{x \mid x \text{ is an integer}\}$ is a set consisting of all items fulfilling the conditions (being an integer)

## Set Notation

$|X|$: cardinality of set X (number of distinct elements in [finite set] X)
$\alpha \in X$: $\alpha$ is an element of (belongs to) X; $\alpha \notin X$ is the converse
$X \subseteq Y$: X is a subset of Y (every element in X is also in Y)
(*) $X \subset Y$: X is a proper subset of Y ($X \subseteq Y$, but $X \neq Y$)
(*) $P(X)$: $P(X)$ (power set of X) is the set containing all subsets of X
$X = Y$: X and Y have the same elements; identical to ($X \subseteq Y$ && $Y \subseteq X$)

Alt: "$\subset$" = "subset"
"$\subsetneq$" = "proper subset"

Common Sets: $\emptyset = \{\}$ = empty set (set with no elements) → subset of all sets
$\mathbb{Z} = \{x \mid x \text{ is an integer}\}$ (set of all integers)
$\mathbb{Q} = \{x \mid x \text{ is a rational number}\}$ (set of all rational numbers) $= \{\frac{a}{b} \mid a, b \in \mathbb{Z}\}$
$\mathbb{R} = \{x \mid x \text{ is a real number}\}$ (set of all real numbers)

# Set Theory

1/10/23
Disc. 1

A set represents a collection of things (of any form)

The natural numbers (1, 2, 3, ...) can be redefined as
the cardinality of various sets, e.g.: "2" := |{thing 1, thing 2}|
"is defined as"

Addition can also be redefined as the cardinality of the set obtained from the union of non-overlapping sets, e.g. "1+2" := |{thing 1} ∪ {thing 2, thing 3}|
= |{thing 1, thing 2, thing 3}| = "3"

Multiplication can be defined via associations through
the Cartesian product of sets ($X \times Y$), e.g.: given $X = \{a, b\}$, $Y = \{c, d\}$,
$X \times Y = \{(a,c), (a,d), (b,c), (b,d)\}$

Integer multiplication: Given $m = |A|$ and $n = |B|$, $m \cdot n = |A \times B|$

Subtraction can be defined as the inverse of addition (and division, as inv. of multiplication)

Negative numbers can be defined with subtraction

# Set Operations & Intro to Proofs

1/11/23
Lecture 2

## Set Operations

Union $(X \cup Y)$: $\{a \mid a \in X \text{ or } a \in Y\}$
Intersection $(X \cap Y)$: $\{a \mid a \in X \text{ and } a \in Y\}$
 * If $X \cap Y = \emptyset$, X and Y are disjoint (no overlap)
Difference $(X - Y)$: $\{a \mid a \in X \text{ and } a \notin Y\}$

* "U" denotes the universal set
 • Exact contents depend on context (e.g. if X = all evens, U = all integers)
* $U - X$ = complement of $X = \overline{X}$
 └ Also written $U \setminus X$

## Notation

• Given collection of sets $\{A_i\}$ ($\{A_i\}_{i=1}^{n}$), $\bigcup_i A_i$ denotes the union of all sets $A_i$
 - $\bigcup_i A_i = \bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for some } i\} = (A_1 \cup A_2 \cup \ldots \cup A_n)$
 - Similarly, $\bigcap_i A_i$ = intersection of all sets $A_i$
• Given set X and $S = \{A_i \mid A_i \subseteq X\}$, S is a partition of X if $\bigcup_i A_i = X$, $A_i \cap A_j = \emptyset\ \forall i, j$
• Cartesian product of sets X and Y $= X \times Y = \{(a, b) \mid a \in X, b \in Y\}$
 - e.g. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$
 └ produces ordered pairs

## Proofs (§2)

A proof is an argument establishing the truth of a mathematical statement.

A mathematical statement is usually composed of a hypothesis (P) leading to a conclusion (Q).

• A proof is usually ended with the symbols "$\square$" and "Q.E.D."

* Can also be written as $P \to Q$

## Types of Proof (Proving $P \to Q$) - §2.2, 2.4

1. Direct proof: Show directly that "P is true" implies "Q is true"
2. Proof by contradiction: To show $P \to Q$, assume P is true and Q is false, then prove that these assumptions lead to a contradiction (impossible statement)

# Methods of Proof

1/13/23
Lecture 3

## Types of Proof (cont.)

3) Proof by contrapositive: to show $P \to Q$, show the equivalent statement $!Q \to !P$
   e.g. to show "$x$ = irrational" ($P$) → "$\sqrt{x}$ is irrational" ($Q$):
   $\sqrt{x}$ = rational ($!Q$) → $\sqrt{x} = \frac{a}{b}$, $a, b \in \mathbb{Z} \to x = \frac{a^2}{b^2} \to x$ = rational ($!P$)

4) Proof by cases: to show $P \to Q$, break $P$ down into smaller cases $P_1 \dots P_n$ and show $P_i \to Q \; \forall i$
   e.g. to show $n(n+1)$ is even for all $n \in \mathbb{Z}$:
   case 1: $n$ is even $\to n(n+1) = (2k)(2k+1)$, $k \in \mathbb{Z} = 2(2k^2+k)$ = even
   case 2: $n$ is odd $\to n(n+1) = (2k-1)(2k)$, $k \in \mathbb{Z} = 2(2k^2-k)$ = even

5) Proving "if and only if" ("iff") statements: $P$ iff $Q = \{P \to Q, Q \to P\}$ = {only if, if}
   e.g. to show $n$ = even ($P$) iff $n^2$ = even ($Q$), $n \in \mathbb{Z}$:
   $P \to Q$: $n = 2k \to n^2 = 4k^2 = 2(2k^2)$
   $Q \to P$: $n = 2k+1$ ($!P$) $\to n^2 = 4k^2+4k+1$ ($!Q$), therefore $Q \to P$

7) Proof by induction: To prove $P(n)$ is true for all $n \geq n_0$, prove that $P(n_0)$ is true and that if $P(n)$ is true for some $n$, then $P(n+1)$ is also true
   e.g. Given $P(n)$ = "sum of the first $n$ positive odd integers is equal to $n^2$":
   Base case ($n=1$): $1 = 1^2$ (true)
   Inductive step ($P(n) \to P(n+1)$): Assume $P(n)$ ($\sum_{j=1}^{n} 2j-1 = n^2$).
   case $n+1 \to \sum_{j=1}^{n+1} 2j-1 = n^2 + 2n + 1 = (n+1)^2$ ($P(n)$ implies $P(n+1)$)
   e.g. To show $4^n - 1 = 3m$, $m \in \mathbb{Z} \; \forall n \in \mathbb{Z}^+$: ($4^n-1$ is a multiple of 3 for all $n \in \mathbb{Z}^+$)
   Base case ($P(0)$, $n=0$): $4^0 - 1 = 3 = 3(1)$ = multiple of three
   Inductive step: Assume $4^n - 1 = 3m$ for some $n, m$
   $\to 4^{n+1} - 1 = 4(3m+1) - 1 = 12m + 3 = 3(4m+1)$ = multiple of 3

(*) Induction can also be proven for $n \leq n_0$ by proving the $n_0$ and $(n-1)$ cases instead of $n_0$ and $(n+1)$

# Functions

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Lecture 4

| Functions (§3.1) | Notation |
|---|---|
| A function from set $X$ to set $Y$ ($f: X \to Y$) is an assignment of an element $y \in Y$ to an element $x \in X$ (i.e. $f(x) = y$). | $f: X$ (domain [of f]) $\to Y$ (codomain [of f]) Range [of f] - set of all possible values of $f(x)$, i.e. $\{y \in Y \mid y = f(x) \text{ for some } x \in X\}$ |

(*) Given function $f: X \to Y$, $f$ represents a subset of Cartesian product $X \times Y$
- i.e. $f = \{(x, f(x)) \mid x \in X, f(x) \in Y\} = \{(x,y) \mid x \in X, y \in Y, y = f(x)\} \subseteq \{(x,y) \mid x \in X, y \in Y\} = X \times Y$

## Definitions

- $f$ is injective/one-to-one if $(f(x_1) = f(x_2)) \to (x_1 = x_2)$
- $f$ is surjective/onto if $\forall\, y \in Y\ \exists\, x \in X$ s.t. $f(x) = y$
  - Equivalent statement: "codomain = range"
- $f$ is bijective if $f$ is both injective and surjective
  - If $f$ is bijective, $\exists\, f^{-1}: Y \to X$ s.t. given $y = f(x)$, $f^{-1}(y) = x\ \forall\, x, y$
  - If $f$ is bijective, it can be concluded that $|X| = |Y|$
- Given function $f: X \to Y$ and function $g: Y \to Z$, we define the composition of functions $f$ and $g$ $g \circ f: X \to Z$, such that $f(x) = y$, $g(y) = z$ implies $g(f(x)) = g \circ f(x) = z$

## Common Functions

Floor: for $x \in \mathbb{R}$, floor of $x = \lfloor x \rfloor$ = largest integer $n \in \mathbb{Z}$ s.t. $n \leq x$
Ceiling: for $x \in \mathbb{R}$, ceiling of $x = \lceil x \rceil$ = smallest integer $n \in \mathbb{Z}$ s.t. $n \geq x$

# Sequences & Strings

1/20/23
Lecture 5

## Sequences (§3.2)

A <u>sequence</u> is a function with domain $I \subseteq \mathbb{Z}$, such that sequence $s: I \to X$ for some $X$.

Given sequence $s$, the value $s(n)$, $n \in I$ can also be written $s_n$, where $n$ represents an <u>index</u> of the sequence.

(*) $I$ is typically either $\in \mathbb{Z}/>0$ or $\in \mathbb{Z}/\geq 0$

(*) Def: A <u>finite sequence</u> refers to a sequence $s: I \to Y$, where $I$ is a finite set ($|I| \neq \infty$)

(*) Def: A <u>closed formula</u> for a sequence $s$ is an expression of $s_n$ as a function of $n$.

- A <u>subsequence</u> of a sequence $s$ is a sequence formed by deleting terms of $s$
  - subseq.$(s): I_2 \to X$, $I_2 \subset I$
  - A subsequence can be described via <u>indices of $s$</u>: subseq$(s) = \{s_{n_k}\}_{k \geq 1}$, where the term $s_{n_k}$ represents the $k^{th}$ index (of $s$) included in the subsequence ($s_{n_k} \in I \ \forall k$)

## Operations

Given terms $\{s_i\}_{i=n}^{m} = \{s_n, \dots, s_m\}$, we define:

1) Addition (Summation): $\sum_{i=n}^{m} s_i = s_n + s_{n+1} + \dots + s_m$

2) Multiplication: $\prod_{i=n}^{m} s_i = s_n \cdot s_{n+1} \cdot \dots \cdot s_m$

## Definitions

- Increasing sequence: $\underbrace{s_{n+1} \geq s_n}$ $\forall n \in I$
  - Strictly increasing: $s_{n+1} > s_n$
- Decreasing sequence: $\underbrace{s_{n+1} \leq s_n}$ $\forall n \in I$
  - Strictly decreasing: $s_{n+1} < s_n$

## Strings

A <u>string</u> is any finite sequence of characters (of any type); a <u>substring</u> is a subsequence of a string, composed of only consecutive elements of the string.

## Definitions

(*) All characters $\in X \to$ string is over $X$

(*) $|\alpha|$ = <u>length</u> of $\alpha$

(*) If $|\alpha| = 0$, then $\alpha$ is <u>null</u>

(*) $\alpha + \beta$ = <u>concatenation</u> $\alpha\beta$ (of $\alpha$ and $\beta$)

# Relations

1/23/23
Lecture 6

**Relations (§3.3,4,5)**

Given sets $X$ and $Y$, we define a relation $R$ from $X$ to $Y$ as a subset of the Cartesian product $X \times Y$ ($R \subseteq X \times Y$).

If $(x \in X, y \in Y) \in R$, we write $x$ is related to $y$ ($xRy$, $x \sim y$).

Examples

1) $X = \mathbb{R}$, $R \subseteq (X \times X)$, $a \sim b \to a = b$
$\to R = \{(a,b) \mid a, b \in X,\ a = b\}$

2) $X = \mathbb{R}$, $R \subseteq (X \times X)$, $a \sim b \to a < b$
$\to R = \{(a,b) \mid a, b \in X,\ a < b\}$

(*) The expression defining a relation can take virtually any form

Definitions

Given set $X$, relation $R$ on $X$ ($R \subseteq X \times X$):

1) $R$ is reflexive if $x \sim x$ $\forall x \in X$
2) $R$ is symmetric if $a \sim b \to b \sim a$ $\forall a, b \in X$
3) $R$ is transitive if $a \sim b,\ b \sim c \to a \sim c$ $\forall a, b \in X$

**Equivalence Relations**

An equivalence relation on $X$ is any relation $R$ [on set $X$] that is reflexive, symmetric, and transitive

(*) Has similar properties to the operator "="

**Equivalence Classes**

Given equivalence relation $R$ on $X$, we can define equivalence class $[a]$ of $X$ ($a \in X$):

$$[a] = \{b \mid b \in X,\ aRb\}$$

(*) All elements in $X$ are elements of exactly 1 equivalence class of $X$

# Functions Review

1/24/23
Disc 3

## Definitions

Surjective - every element in the codomain is mapped to by at least one element in the domain (codomain = range)

Implies: $|\text{codomain}| \le |\text{domain}|$

Injective - every element in the codomain is mapped to by at most one element in the domain

Implies: $|\text{domain}| \le |\text{codomain}|$

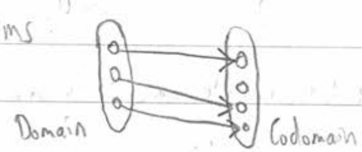
Bijective - every element in the codomain is mapped to exactly one element in the domain

Implies: $|\text{domain}| = |\text{codomain}|$

- If a function $f: A \to B$ is bijective, there must also exist a function $f^{-1}: B \to A$ s.t. $f^{-1}(f(a)) = a$, $f(f^{-1}(b)) = b$
  [f is invertible]

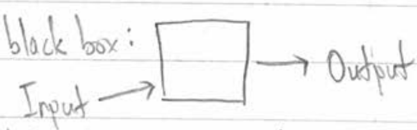
## Describing Functions

1) Arrow diagrams




(Suitable for small sets)

2) Functions as a black box:




3) Functions as labels: Each element in the domain maps to exactly one element of the codomain; that element in the codomain can thus be thought of as a label for the element in the domain
e.g. f("apple") = "red", f("banana") = "yellow"

4) Bijective functions as translation: f("Uno") = "One", $f^{-1}$("One") = "Uno"

# Properties of Equivalence Relations

1/25/23
Lecture 7

## Equivalence Classes

Given set $X$, equivalence relation $R$ on $X$, the set of equivalence classes $\{[a] \mid a \in X\}$ form a partition of $X$ (i.e. every element $b \in X$ is contained in exactly 1 equivalence class)

- Conversely, a partition of $X$ can be used to form a unique equivalence relation, i.e. given set $X$, $A_1 \dots A_n$ (partition of $X$), $R_{\{A_1, \dots, A_n\}} = a \sim b$ iff $\exists i$ s.t. $a, b \in A_i$

→ Proof: $a \sim a \; \forall a \in X \rightarrow a \in [a]$ (each element belongs to at least 1 equivalence class)

$a \sim b \rightarrow a, b \in [a] \rightarrow a \sim b, b \sim c \; (b \in [c]) \rightarrow a \sim c \rightarrow c \in [a] \rightarrow [a] = [c]$

(each element belongs to at most 1 equivalence class)

## Modular Arithmetic

Notation: Given $a, b, c \in \mathbb{Z}$, if $a = b \cdot c \rightarrow$ "b divides a", "$b \mid a$", "a divisible by b"

→ Given $m \in \mathbb{Z} > 0$, $a, b \in \mathbb{Z}$, $m \mid a - b \rightarrow$ "m is congruent to b modulo m" (m divides $a - b$)

→ "$a \equiv b \pmod{m}$" represents an equivalence relation with $m-1$ equivalence classes

## Matrices of Relations

Given set $X = \{x_1, \dots, x_n\}$ ($|X| = n$), relation $R$ on $X$, we can depict $R$ as an $n \times n$ matrix,

i.e.: $M_R = \begin{array}{c|c} & x_1 \; \cdots \; x_n \\ \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} & \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array} = \{M_{ij}\} \begin{cases} M_{ij} = 1 \text{ if } x_i \sim x_j \\ M_{ij} = 0 \text{ if } x_i \not\sim x_j \end{cases}$

### Properties

1) $R$ is reflexive →: matrix diagonal will be all 1s ($M_{ii} = 1 \; \forall i$)
2) $R$ is symmetric →: $M$ will be symmetric over its diagonal ($A_{ij} = A_{ji} \; \forall i, j$, $M = M^T$)
3) $R$ is transitive →: "$(A^2)_{ij} \neq 0 \rightarrow A_{ij} \neq 0$" ($\leftrightarrow$ transitive)

(*) Properties only hold if row order [of elements] = column order

# Intro to Counting

1/27/23
Lecture 8

## Counting (§6)

**Counting**

Counting is the process of determining the number of elements contained within a set (i.e. the size of the set), such that a bijection can be established between a set of size n and the set of positive integers $\{1, 2, \ldots, n\}$

Ex: Given set $Y$ ($|Y| = n$), a bijection $f: X \to Y$ can be made between $Y$ and the set $X \subseteq \mathbb{R}$ ($|X| = n$), such that every $i \in X$ is mapped to an element $x_i \in Y$ (i.e. $f(i) = x_i$)

### Principles of Counting

- Multiplication Principle - If an event/activity can be broken into $t \in \mathbb{R}$ independent steps, with $n_i$ ways to do a given step $i$, the total number of different possible events/activities is $n_1 \cdot n_2 \cdot \ldots \cdot n_t$

  Ex: Creating a sequence of 4 English letters = 4 steps of picking 1 character each
  $\to$ total # of possible strings = 26 (step 1) · 26 (step 2) · 26 (step 3) · 26 (step 4)
  $\to$ total # (without repetition) = 26 (step 1) · 25 (step 2) · 24 (step 3) · 23 (step 4)

- Addition Principle - Given sets $X_1, \ldots, X_t$ ($|X_i| = n_i$) if $X_i \cap X_j = \emptyset \; \forall i \neq j$ ($X_i$ and $X_j$ pairwise disjoint), the number of possible elements that can be chosen from either $X_1$ or $X_2$ or ... or $X_t = n_1 + n_2 + \ldots + n_t$ (Alt: The union $X_1 \cup \ldots \cup X_t$ contains $\sum_{j=1}^{t} n_j$ elements)

- Inclusion/Exclusion Principle (for $t=2$) - Given finite sets $X$ and $Y$, $|X \cup Y| = |X| + |Y| - |X \cap Y|$
  (*) Used in place of the Addition Principle for non-disjoint sets
  (*) $t=3$: Given sets $X, Y, Z$, $|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|$

# Permutations and Combinations

1/30/23
Lecture 9

## Permutations

Definition: A permutation of $n$ distinct elements $x_1, \ldots, x_n$ is an ordering of the elements $x_1, \ldots, x_n$.

- Given $n$ elements, there are $n!$ possible permutations of those $n$ elements.

Ex: $\underbrace{\{A, B, C\}}_{\text{starting elements}} \rightarrow \underbrace{ABC, ACB, BAC, BCA, CAB, CBA}_{\text{diff. permutations of the 3 elements}}$ ($6$ total $= 3!$)

Definition: An $r$-permutation of $n$ elements $x_1, \ldots, x_n$ is an ordering of an $r$-element subset of the set of $n$ elements $X = \{x_1, \ldots, x_n\}$ (i.e. an $r$-permutation is an ordering of any $r$ elements selected from the original $n$ elements).

- Notation: The number of $r$-permutations of a set of $n$ distinct elements is denoted $P(n, r)$ ($P(n,r) = \frac{n!}{(n-r)!}$)

Ex: $P(3,2)$ for set $\{A, B, C\} = |\{AB, BA, AC, CA, BC, CB\}|$ ($=6$)

## Combinations

Definition: An $r$-combination ([r-]arrangement) of a set $X$ containing $n$ distinct elements is an unordered selection of $r$ elements of $X$ (i.e. is an $r$-element subset of $X$ (i.e. = set $Y$, $Y \subseteq X$, $|Y| = r$)).

- Notation: The number of $r$-combinations of a set of $n$ elements is denoted $C(n,r)$ or $\binom{n}{r}$ ("$n$ choose $r$") ($C(n,r) = \frac{n!}{(n-r)! \cdot r!}$)

Ex: $C(n,r) = \binom{n}{r}$ for set $\{A, B, C\} = |\{\{A, B\}, \{A, C\}, \{B, C\}\}|$ ($=3$)

# Equivalence Relations Review

1/31/23
Disc 4

## Equivalence Relations

- For every set X, there exists a bijection between the set of equivalence relations on X and the set of partitions of X (i.e. each equivalence relation is associated with a distinct partition, and vice versa)
  - The number of equivalence relations on X (in terms of pairings of elements of X) is thus equal to the number of possible partitions of X

- An equivalence relation encodes some quality of "sameness" (equality), shared between all elements of an equivalence class
  - Relations (more generally) represent forms of "relationships" between elements

## Proving Statements

- In mathematics, a statement (e.g. "relation R on X is transitive") is held as true unless there exists a counterexample
  - A statement that is held to be true solely due to a lack of any examples to either prove or disprove it (e.g. "$x \sim y$ if $x = y+1$ is reflexive on the empty set" [because there are no elements in $\emptyset$ to disprove this]) are described as being "vacuously true"

- When proving transitivity for a relation (i.e. "$a \sim b$, $b \sim c \rightarrow a \sim c$"), the elements a, b, c do not need to be distinct

# Counting (cont.)

2/7/23
Lecture 10

## Generalized Permutations & Combinations (§6.3)

**THM** *Counting*: Given a sequence $S$ of $n$ items containing $n_1$ identical objects of type 1, $n_2$ of type 2, ..., $n_t$ of type $t$ → the number of orderings of $S$ is:

$$\#\text{ of orderings of } S = \frac{n!}{n_1!\,n_2!\ldots n_t!}$$

(*) "Stars and Bars"

Problem: How many solutions are there for the equation $x+y+z=9$? ($x,y,z \in \mathbb{Z}^+$)

Solution: We think of a solution to $x+y+z=9$ as a string of nine 1s and 2 slashes (where each slash demarcates a variable)

i.e.: $\underbrace{1\,1\ldots 1}_{x}/\underbrace{1\,1\ldots 1}_{y}/\underbrace{1\,1\ldots 1}_{z}$ ↳ the values of $x, y, z$ being determined by the placement of the slashes

→ The number of solutions = the number of distinct possible strings = $\binom{11}{2}$

(*) Case: A restriction (e.g. $x \geq 2$) can be factored into the equation, i.e.: $(x-2)+y+z=7$

↓

**THM** *Counting*: Given set $X$ ($|X| = t$), the number of possible unordered selections of $k$ elements from $X$ (allowing repetition) is:

$$\#\text{ of possible } k\text{-element selections}: \binom{k+t-1}{t-1} = C(k+t-1, t-1)$$

↳ (*) This can be seen as an extension of "stars and bars", where ~~each~~ the value of the variables $x, y, z, \ldots$ represent the indices of the selection, i.e. $1\,1/1/11 \rightarrow \{x_3, x_4\}$

# Binomial Coefficients

2/6/23
Lecture 11

## Binomial Coefficients (§6.7)

**The Binomial Theorem**

Given two numbers $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, then:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$ where the numbers $\binom{n}{k}$ are referred to as <u>binomial coefficients</u>.

↳ Justification: The term $a^k b^{n-k}$ is obtained by picking $k$ $a$'s out of $n$ choices $(a+b)$, hence the coefficient $\binom{n}{k}$.

(*) Note: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ (can be proven)

**Pascal's Triangle**

$$\begin{array}{lccccccccc}
n=0 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & ]-(a+b) \\
n=2 & & & 1 & & 2 & & 1 & & ]-(a+b)^2 \\
n=3 & & 1 & & 3 & & 3 & & 1 & ]-(a+b)^3 \\
n=4 & 1 & & 4 & & 6 & & 4 & 1 & ]-(a+b)^4
\end{array}$$

Pascal's Triangle is a triangle of numbers, where the $n^{th}$ row contains the coefficients for the expression $(a+b)^n$.

Each number in the triangle is a sum of the two numbers directly above it.

↳ $\boxed{\text{Theorem: } \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k+1}}$

## Polynomial Coefficients

**The Trinomial Theorem**

Given $a, b, c \in \mathbb{R}$, $n \in \mathbb{Z}^+$:

$$(a+b+c)^n = \sum_{\substack{i,j,k \\ (i+j+k=0)}}^{n} \frac{n!}{i!\,j!\,k!} a^i b^j c^k$$

(*) Can be generalized for higher-dimensional polynomials

# The Pigeonhole Principle + Intro to Recurrence Relations

2/8/23
Lecture 12

## The Pigeonhole Principle (§6.8)

If $\geq n+1$ pigeons fly into $n$ pigeonholes, at least one pigeonhole contains at least two pigeons.

↓

## The Pigeonhole Principle (V2)

Given a function $f: X \to Y$, if $|X| = n$ and $|Y| = m$, there are at least $\lceil \frac{n}{m} \rceil$ elements of $X$ that map to the same element of $Y$.

i.e. $a_1, a_2, \ldots, a_k \ [k = \lceil \frac{n}{m} \rceil] \to f(a_1) = f(a_2) = \ldots = f(a_k)$

## Recurrence Relations (§7.1, 7.2)

Definition: A recurrence relation for sequence $\{a_k\}_{k=0}$ is a relation that defines the value of each element $a_i$ based on the values of the preceding elements.

i.e. $a_i = f(a_1, a_2, \ldots, a_{i-1})$

Notation: Typically, a certain number of elements at the beginning of the sequence are given explicit values in order to "start up" the sequence. Those elements are referred to as initial conditions.

# Solving Recurrence Relations

2/10/23
Lecture 13

## Solving Recurrence Relations (§6.2)

1) Iteration - continuously expressing a term $a_n$ in terms of its predecessors, until it can eventually be expressed in terms of the initial condition (e.g. $a_i$)

Ex: $a_n = 2a_{n-1} = 2(2a_{n-2}) = 2^3 a_{n-3} = \dots$

$\longrightarrow = 2^{n-1} a_1$ (for initial condition $a_1$)

2) Substitution - substituting a certain sequence for another may make a given expression easier to compute

Ex: $q_n = 2q_{n-1} + 1 \rightarrow$ We define $a_n = q_n + 1$

$= 2q_{n-1} + 1 + 1 = 2q_{n-1} + 2$

$= 2(q_{n-1} + 1) = 2a_{n-1} \rightarrow a_n = 2a_{n-1}$

$\longrightarrow q_n = a_n - 1 = \boxed{2^{n-1}(a_1) - 1}$

(*) 3) Linear homogenous recurrence relations with constant coefficients, of order $k$ (i.e. sequences $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$, $c_k \neq 0$, $c_1, \dots, c_k \in \mathbb{R}$) can be solved algebraically, e.g.:

Ex: $R: a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $n \geq 2$, w/ initial conditions $a_0, a_1$ (order = 2)

$\longrightarrow$ Assume that for the equation $x^2 - c_1 x + c_2 = 0$, $\exists$ two distinct solns. $r_1 \neq r_2$

$\longrightarrow \exists$ two constants $\alpha, \beta$ such that $\boxed{a_n = \alpha r_1^n + \beta r_2^n \ \forall n \geq 0}$

($\alpha$ and $\beta$ being computable from the initial conditions)

# Solving Recurrence Relations (cont.)

2/13/23
Lecture 14 + Lecture 15
2/15/23

**General Solns. for Linear Homogeneous Recurrence Relations (w/ Constant Coefficients):**

Given $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

→ Characteristic polynomial $= x^k - x^{k-1}c_1 - x^{k-2}c_2 - \ldots - xc_{k-1} - c_k$

→ Use roots to compute soln.

Ex: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$ (order 3) $\rightarrow x^3 - c_1 x^2 - c_2 x - c_3$ [$=0$] → 3 roots $r_1, r_2, r_3$

→ 3 cases: 1) Distinct roots $r_1, r_2, r_3 \rightarrow a_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$

2) 2 distinct roots $r_1, r_2 = r_3 \rightarrow a_n = \alpha r_1^n + \beta r_2^n + \gamma n r_3^n$ [one repeated root]

3) 1 distinct root $r_1 = r_2 = r_3$ [$=r$] $\rightarrow a_n = \alpha r^n + \beta n r^n + \gamma n^2 r^n$ [two repeated roots]

(*) Inclusion of addl. terms ($n, n^2, \ldots$) expands space of solutions [satisfying the relation]

**(*) Notes on Solutions**

- Solutions can take many forms (e.g. periodic)

- Complex roots may appear during the polynomial-solving process - are solved just like real roots
  - (-) $i$ can be "eliminated" by being reexpressed (e.g. define $\delta = i\alpha + i\beta$, then find a real # value for $\delta \rightarrow i$ disappears)

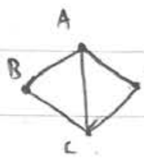
$$\boxed{\begin{array}{c}\underline{\text{Euler's Formula}} \\ e^{ix} = \cos(x) + i\sin(x)\end{array}}$$

# Intro to Graphs

2/17/23
Lecture 16

## Graphs

A graph $G(V,E)$ is defined as a set of vertices $V(G)$ and associated edges between vertices $E(G)$ ($E(G) \subseteq \{(a,b) \mid a,b \in V(G)\}$).

Ex: 

$= G \begin{cases} V(G) = \{A, B, C, D\} & \text{(points [vertices])} \\ E(G) = \{(A,B), (A,C), (B,C), (C,D), (D,A)\} & \text{(lines [edges])} \end{cases}$

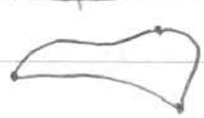
## Definitions

- Edge $e = (u,v) \rightarrow e$ is incident to vertices $u$ and $v$
- $u, v \in V(G)$, $\exists e = (u,v) \in E(G)$ [$u$ and $v$ connected by an edge] $\rightarrow$ $u$ and $v$ are adjacent
  - $\rightarrow$ a vertex $v_0$ not adjacent to any other vertices is called isolated

## Types of Graphs

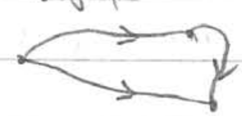
- Graph (undirected graph): graph where edge $(a,b) = (b,a)$ [edges unordered], vs:
- Digraph (directed graph): graph where edge $(a,b) \neq (b,a)$ [pairs of vertices ordered]

Graph



vs.

Digraph



- Simple graph: graph with no loops (edges $(a,a) \in E(G)$) and no parallel edges (i.e. any two vertices connected by at most one edge)

Loop

Parallel edges

- Complete graph with N vertices ($K_n$): A graph with $n$ vertices ($|V(K_n)| = n$), such that all points are adjacent to all other points ($(a,b) \in E(K_n)\ \forall a,b \in V(K_n)$)

$K_1$: •  $K_2$:   $K_3$:   $K_4$:  etc...

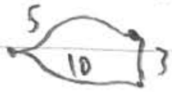
$\boxed{(* \ |E(K_n)| = \binom{n}{2})}$

# Intro to Graphs (cont.)

2/17/23
Lecture 16 (cont.)

## Types of Graphs (cont.)

- **Weighted graph**: a graph where each edge has an associated number (weight/label)
  - An unweighted graph assigns weight 1 to every edge by default

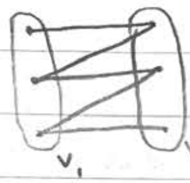


(Weighted graph)

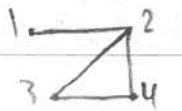
- **Bipartite graph**: a graph where $\exists$ a partition of $V$, $V(G) = V_1 \cup V_2$ such that every edge in $E(G)$ is incident to one vertex in $V_1$ and one vertex in $V_2$

($*$ $(a,b) \in E(G) \rightarrow a \in V_1,\ b \in V_2$)

Ex:




($*$) The graph shown below is not bipartite.




Proof: Assume the graph is bipartite $\rightarrow V(G) = V_1 \cup V_2$.
$\rightarrow$ each vertex 2, 3, 4 belongs to either $V_1$ or $V_2$
$\rightarrow$ at least two of $\{2,3,4\}$ will be in the same set (per Pigeonhole)
$\rightarrow \exists$ an edge between two points in the same set $\rightarrow$ $\boxed{\text{not bipartite}}$

- **Complete bipartite graph on m & n vertices ($K_{m,n}$)**: a bipartite graph, such that:
  - ($*$) $V(K_{m,n}) = V_1 \cup V_2$, $|V_1| = m$, $|V_2| = n$
  - ($*$) $E(K_{m,n}) = \{(a,b) \mid a \in V_1, b \in V_2\}$ (all points in $V_1$ connected to all points in $V_2$)

Ex:


($K_{2,2}$)


($K_{3,2}$)


($K_{3,3}$)

# Paths and Cycles

2/22/23
Lecture 17

## Paths

A path from vertex v to vertex w on graph G is defined as a sequence of vertices $(v_0, v_1, \dots, v_n)$ [length $= n$], where $v_0 = v$, $v_n = w$, and $\forall$ pairs of vertices $v_i$ and $v_{i+1} \rightarrow (v_i\ v_{i+1}) \in E(G)$ ($v_i, v_{i+1}$ adjacent)

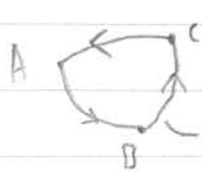
(*) On a weighted graph, the length of a path is defined as the sum of the weights of the edges traversed

(*) Alt. Notation: · Paths as a sequence of edges $((v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n))$

· Paths as a sequence of vertices + edges $(v_0, e_1, v_1, \dots, e_n, v_n)$


## Definitions

- A connected graph is a graph where $\exists$ a path between any two vertices $v, w \in V(G)$
- Given a graph $G(V, E)$, we define a subgraph $G'(V', E')$ of G to be any graph such that 1. $V' \subseteq V$, 2. $E' \subseteq E$, 3. $\forall (v, w) \in E' \rightarrow v, w \in V'$ [no dangling edges]
  - Given a vertex $v \in V(G)$, the component of G containing V is the subgraph G' of G containing all vertices & edges of G expressible as an element of at least 1 path beginning at v
- A simple path is a path that contains no repeated vertices
- A cycle is a path [of nonzero length] from v to v, with no repeated edges
  - A simple cycle is a cycle containing no repeated vertices (except for the beginning/end)
  - An Eulerian cycle (Euler cycle/circuit/walk) is a cycle that visits every edge and vertex
    - A graph G contains an Eulerian cycle iff G is connected and contains only vertices of even degree
- The degree $\delta(v)$ of a vertex v is the number of edges incident to v
  - (*) For digraphs, we distinguish between the in-degree and out-degree




Cycle (A, B, C, A)

# Paths and Cycles (cont.)

2/24/23
Lecture 18

## Paths and Cycles (cont.)

- Given a graph $G(V,E)$ with vertices $V(G)=\{v_1,\dots,v_n\}$, the sum of the degrees of $V(G)$ is even, i.e. $\sum_{i=1}^{n} \delta(v_i) = 2|E(G)|$
  - (*) Corollary: the number of vertices with odd weights is even
- An Eulerian path in graph G is defined as a path from v to w ($v \neq w$), containing all edges and vertices of G
  - An Eulerian path exists if G is connected and v to w are the only vertices $\in V(G)$ of odd degree
- Reminder: A simple cycle is a cycle $v \to v$ with no repeated vertices, except for v itself
  - If there exists a cycle in a graph G, there also exists a simple cycle in G

## Hamiltonian Cycles (§8.3)

- A Hamiltonian cycle in a graph G is a cycle that contains the initial vertex exactly twice, and all other vertices exactly one time
- Travelling Salesman Problem - finding the shortest Hamiltonian cycle in a graph (usually weighted)
  - (*) Length in terms of the sum of weights of the edges traversed
  - (*) If any given vertex can be proven to be visited more than once, a Hamiltonian cycle does not exist
  - (*) If $\delta(v_i)+\delta(v_j) \geq |V(G)|$ for any non-adjacent vertices $v_i$, $v_j$ in $V(G)$, then graph G contains a Hamiltonian cycle (if, not iff)

# Dijkstra's & Intro to Isomorphisms

2/27/23
Lecture 19

## Dijkstra's Shortest-Path Algorithm (§18.4)

Given a weighted graph $G$, Dijkstra's algorithm returns the shortest path from $a \in V(G) \to z \in V(G)$.

Notation: Given edge $(i,j) \in E(G)$, $w(i,j)$ denotes the weight of $(i,j)$

Mechanism: Algorithm gives each vertex $i \in V$ a label $L(i)$, representing length of shortest path $a \to i$.

Algorithm: 1. $L(x) = \infty \ \forall x \in V, x \neq a$ ; $L(a) = 0$

2. Define $T$ = set of all vertices [w/ undetermined $L$]

while($z \in T$): a. choose $v \in T$ w/ smallest $L(v)$

b. $T = T - v$

c. for each $x$ adjacent to $v$, $L(x) = \min(L(x), L(v) + w(v,x))$

Logic: If a path $(v_1, v_2, \ldots, v_n)$ is the shortest path $v_1 \to v_n$, $(v_1, v_2, \ldots, v_{n-1})$ must have been the shortest path $v_1 \to v_{n-1}$.

Variations: A* (Dijkstra's with an added heuristic for guiding queries toward destination)

Negative-weight Dijkstra's (Dijkstra's only works for non-negative weights by default)

(*) Shortest-path algorithms are not restricted to abstract graphs (e.g. can be used for speedrunning)

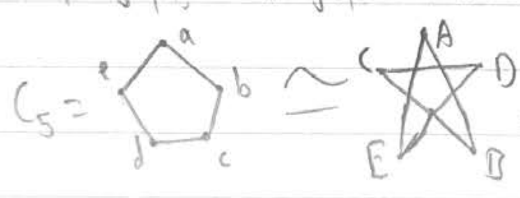
## Isomorphisms of Graphs (§8.6)

Definition: Two graphs $G_1, G_2$ are isomorphic if every vertex/edge $\in G_1$ can be mapped to a corresponding vertex/edge in $G_2$, i.e. $\exists$ bijective functions $f: V(G_1) \to V(G_2)$ and $g: E(G_1) \to E(G_2)$, such that $g((v,w)) = (f(v), f(w))$.

Notation: $G_1$ and $G_2$ isomorphic $\to G_1 \simeq G_2$

$\to$ The pair of functions $(f,g)$ is an isomorphism of $G_1$ onto $G_2$.

(*) Definition: Cyclic graph $C_n$ is the graph of $n$ vertices, where $E(C_n) = \{(v_i, v_{i+1}) \mid 0 \le i \le n-1\} + (v_n, v_0)$




$$\begin{cases} f(a) = A,\ f(b) = B, \ldots \\ g((a,b)) = (A,B),\ g((b,c)) = (D,C), \ldots \end{cases}$$

# Planar Graphs

3/1/23
Lecture 20

## Invariants

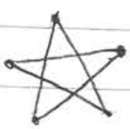
Definition: An invariant is any property that is preserved under isomorphisms (i.e. $G_1 \simeq G_2 \rightarrow G_1, G_2$ share invariants).

(*) Contrapositive: If two graphs $G_1, G_2$ do not share invariants, $G_1 \not\simeq G_2$

Example Invariants: 1) # of vertices/edges in a graph
2) Connectedness
3) Degree sequence (* same deg seq. does not imply isomorphism)
4) Having simple cycles of length k

## Planar Graphs (§8.7)

Definition: A graph G is considered planar if it can be drawn on a plane without any of its edges intersecting.

Ex:  (not planar),  (planar - $C_5$). (*) $K_{3,3}$, $K_5$ non-planar across isomorphisms

When drawing connected planar graphs in the plane, the edges of the graph divide the plane into regions, known as faces. Each face is defined by the cycle of edges forming its boundary.

(*) Theorem (Euler)

Given a connected planar graph G, with v vertices, e edges, and f faces:

# vertices - # edges + # faces = $\boxed{v - e + f = 2}$

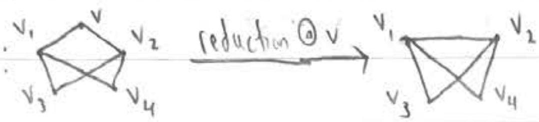
(*) Proving non-planarity: Assume planarity, then prove-by-contradiction w/ the above theorem

# Planar Graphs (cont.)

3/6/23
Lecture 21

## Planar Graphs

Definition: Given a vertex $v$ with $\delta(v) = 2$ incident to edges $(v_1, v)$ and $(v, v_2)$, we say edges $(v_1, v)$ and $(v, v_2)$ are in <u>series</u>. A <u>series reduction</u> deletes $v$ and replaces edges $(v_1, v), (v, v_2)$ with a single edge $(v_1, v_2)$.

Ex: 


Definition: Graphs $G_1, G_2$ are <u>homeomorphic</u> if $G_1, G_2$ can be reduced (by series reductions) to isomorphic graphs.

Ex:  is homeomorphic to 

↓

| | |
|---|---|
| Kuratowski's Theorem | A graph $G$ is planar <u>iff</u> $G$ does not contain a subgraph homeomorphic to $K_{3,3}$ or $K_5$. |

(*) <u>Applications of Planar Graphs (Platonic Solids)</u>

Definition: A <u>polyhedron</u> is a solid where all faces are polygons, and all edges contained in exactly two faces.

→ The formula "$v-e+f=2$" holds for polyhedrons. (Test: "project" into 2D, e.g.  → )

Definition: A <u>regular polyhedron</u> is a polyhedron where all faces are regular polygons with $n$ sides and all vertices are of the same degree $(r)$.

$\rightarrow rV = 2E = nF,\ V+F-E=2 \rightarrow \frac{2E}{R} - E + \frac{2E}{n} = 2,\ E, n, r \in \mathbb{Z}^+$ (A Diophantine eq.)

$\rightarrow \frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \rightarrow$ both $r, n$ cannot be $\geq 4 \rightarrow \min(r,n) \leq 3$, but $\begin{cases} r \geq 3 & \text{(no "2-gon")} \\ n \geq 3 & \text{(solid)} \end{cases}$

$\rightarrow$ either $r$ or $n=3$, $\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \rightarrow$ The only possible combos $(r,n)$ are for a tetrahedron, icosahedron, cube, octahedron, dodecahedron (5 <u>Platonic solids</u>)

# Intro to Trees

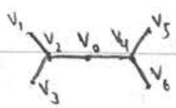
3/8/23
Lecture 22

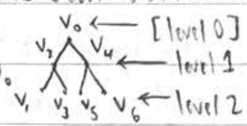
## Trees (§9.1)

A **tree** $T$ is a [simple] graph where, for any vertices $v, w \in V(T)$, there exists exactly one unique simple path in $T$ from $v$ to $w$.

### Definitions

- A rooted tree is a tree where one vertex has been defined as the "root" of the tree

Ex: [Tree] $T$: 

→ [Rooted tree] $T_{v_0}$ 

(*) Any node in a tree can be the root

- The level of a vertex $w$ in a rooted tree is defined as the length of the unique path from the root of the tree to $w$
  - The height of the tree is the largest level number in the tree

## Properties of Trees (§9.2)

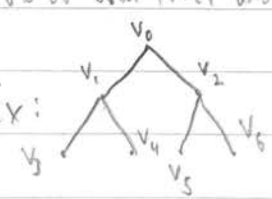
Definition: A graph $G$ is acyclic if it does not contain any cycles.

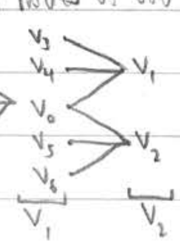
(*) Given a graph $G$ with $n$ vertices, the following statements are equivalent:

i. $G$ is a tree
ii. $G$ is a connected, acyclic graph
iii. $G$ is a connected graph with $(n-1)$ edges
iv. $G$ is an acyclic graph with $(n-1)$ edges

All trees are connected, acyclic graphs with $(n-1)$ edges

(*) All [rooted] trees are valid bipartite graphs $V(T) = V_1 \cup V_2$, where $V_1$ contains all nodes of even level and $V_2$ all nodes of odd level

Ex: 

→ 


(*) All trees are also planar

# Intro to Trees (cont.)

3/8/23
Lecture 22
(cont.)

## Tree Terminology

Given a rooted tree $T$ rooted at $v_0$, let $x, y, z \in V(T)$ [be vertices of $T$] and let $(v_0, v_1, \ldots, v_k)$ be a simple path $v_0$ to $v_k$ in $T$.

i. $v_{n-1}$ is the parent of $v_n$; conversely, $v_n$ is the child of $v_{n-1}$

(*) A vertex may also be described as having children in the plural case.

ii. $v_0, v_1, \ldots, v_{k-1}$ are the ancestors of $v_k$.

→ iii. If a vertex $x$ is an ancestor of $y$, it can also be said that $y$ is a descendant of $x$.

iv. If two vertices $x$ and $y$ are both children of a third vertex $z$, $x$ and $y$ are called siblings.

## Definitions (cont.)

- A vertex with no children is called a leaf or terminal vertex.
  - (*) All leaves are of degree 1; the only vertices in a tree of degree 1 are the leaves, and potentially the root [which is not called a leaf].
- Any vertex that is not a leaf is called a branch or internal vertex.
- The subtree of $T$ rooted at $x$ (for some tree $T$ with vertex $x$) is the graph

$$T_x(V', E') : V' = \{x\} \cup \{\text{descendants of } x\},$$
$$E' = \{\text{all edges } e \text{ contained in a path from } x \text{ to some vertex } v \in V'\}$$

# (*) Properties of Trees (Proof)

3/10/23

Lecture 23

Reminder: Given graph $G$ with $n$ vertices, TFAE: (1) $G$ is a tree

(2) $G$ is a connected, acyclic graph

(3) $G$ is a connected graph with $(n-1)$ edges

(4) $G$ is an acyclic graph with $(n-1)$ edges

Proof: (1) → (2): By definition of a tree (Endpoints of longest path in $G$)

(2) → (3): (*) Lemma: Graph $G$ w/ $|V(G)| \geq 2$ contains at least two vertices of degree 1

→ Proof by Induction: $T$ $(|V(T)| = 1) \rightarrow |E(T)| = 0$

$(n) \rightarrow (n+1)$: $T$ has $(n+1)$ vertices → define subgraph $T'$ (removing one leaf)

$\rightarrow |V(T')| = n$, $|E(T')| = n-1$ (per assumption) $= |E(T')| - 1 \rightarrow |E(T)| = n$ . □

(3) → (4): Proof by Contradiction: Assume $G$ is cyclic → removing one edge of the cycle still gives a connected graph w/ $(n-2)$ edges, $n$ vertices → impossible (contradiction). □

(4) → (1): (a) $T$ connected: Assume $T$ consists of $k$ connected components $T_1, \ldots, T_k$,

where $v_i = |V(T_i)|$, $n = v_1 + v_2 + \ldots + v_k$

All $T_i$ connected, acyclic (per assumption) $\rightarrow |E(T_i)| = n_i - 1$ . □

$\rightarrow |E(T)| = (n_1 - 1) + \ldots + (n_k - 1) = n - k = n - 1$ (per (4)) $\rightarrow k = 1$ [$T$ connected]

(b) ∃ a unique path between any two vertices:

i. $T$ is connected → # paths $(v \rightarrow w) \geq 1$

ii. $T$ acyclic → # paths $(v \rightarrow w) < 2$

→ # paths = 1 . □

↓

(1) → (2) → (3) → (4) → (1)

↓

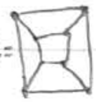
(1) ⇔ (2) ⇔ (3) ⇔ (4) [are equivalent]

# Spanning Trees

3/10/23
Lecture 23
(cont.)

## Spanning Trees (§9.3)

Definition: A subgraph T of a graph G is called a spanning tree of G if T is a tree visiting all vertices of G (i.e. $V(T) = V(G)$)

Ex: G=  $\longrightarrow$ T=  , e.g.

(*) A graph G contains a spanning tree iff G is connected

## (*) Methods for Finding Spanning Trees

### 1) Breadth-First Search (BFS)

Given a connected graph G with vertices ordered $v_1, \ldots, v_n$:

→ Starting from $v_1$, add all children of $v_1$ to a queue (sorted by order) & add all edges to the children, to E' [add children to V']

→ while ($V' \neq V(G)$):

for each vertex in the queue, add all of its children to V' iff they have not already been, and add all edges to added children to E'

once the queue has been cleared, add all newly-added children to the queue

### 2) Depth-First Search (DFS)

Given a connected graph G with vertices ordered $v_1, \ldots, v_n$:

→ (Starting at $v_1$): while there is at least one edge incident to the current vertex that would not create a cycle if added, add the edge to the vertex of least ordered value (first in ordering) and move to that edge

→ If there are no such edges, backtrack until there are; if the backtracking returns to $v_1$, and no such edges are present at $v_1$, terminate

# Binary Trees

3/15/26

Lecture 24

## Binary Trees (§9.5)

Definition: A binary tree is a rooted tree where each vertex has 0, 1, or 2 children. For internal vertices, we distinguish between a left and right child.

Ex: 


(*) Def: A full binary tree is a binary tree where each vertex has 0 or 2 children.

↳ 

**THM** Let $T$ be a full binary tree with $i$ internal vertices. $T$ has $i+1$ terminal vertices.

↳ (*) Corollary: $|V(T)| = 2i + 1$

↳ (*) Proof: $V(T) = \{root\} \cup \{v \in V \mid v \text{ is a child of another vertex}\}$

$\rightarrow |V(T)| = 1 + 2i$

**THM** Let $T$ be a binary tree of height $h$. The number of terminal vertices $t \leq 2^h$.

↳ (*) Can be proven by induction

(*) Applications of Binary Trees

- Binary search trees are binary tree-based structures that can be traversed and modified easily [in a computer science context].
- Binary trees can be used to model/describe the process of deciphering Morse code.

# (*) Intro to Number Theory

3/15/23
Lecture 24
(cont.)

## Number Theory (§5)

Def: The study of numbers (integers/whole numbers; elements $\in \mathbb{Z}$).

Branches:

1) Analytic Number Theory - real/complex analysis (the study of real/complex numbers, functions, series, etc.)
2) Elementary Number Theory - study of elementary numbers
3) Algebraic Number Theory - study of algebra & algebraic structures

(*) Example Theorems

- Goldbach conjecture - every even integer $\geq 4$ can be expressed as the sum of two primes
  (* Verified up to $10^{14}$)
- Fermat's Last Theorem - $\nexists$ integers $x, y, z \neq 0$ s.t. $x^n + y^n = z^n$ (for any $n \geq 3$)
  (* Proven by Andrew Wiles in 1994; first stated in 1637)
- Helfgott's Theorem - every odd integer $\geq 7$ can be expressed as the sum of 3 primes

## Intro to Number Theory (§5.1)

Def: An integer ($\geq 2$) is said to be prime iff its only [integer] divisors are 1 and itself; otherwise, it is said to be composite.

| Fundamental Theorem of Arithmetic ☆ |
|---|
| Every integer $\geq 2$ can be expressed as a product of prime numbers. |
| ⟶ The prime factorization of an integer is unique. |

Def: Given two integers a and b, the greatest common denominator (gcd) of a and b is the largest integer n ($\leq a, b$) s.t. $n \mid a$ and $n \mid b$.

# (*) Intro to Number Theory (cont.)

3/17/23

Lecture 25 Euclid's Algorithm

To find $\gcd(a,b)$ $[a,b \in \mathbb{Z}^+;\ a<b]$: $\rightarrow$ find $r$ s.t. $b = qa + r$, $0 \leq r \leq a$ $[q \in \mathbb{Z}^+]$

$\longrightarrow$ case $r = 0$: $\rightarrow \gcd(a,b) = a$

case $r \neq 0$: $\rightarrow \gcd(a,b) = \gcd(r,a)$ [repeat]

Lemma: Given $a, b \in \mathbb{Z}$ ($a$ and $b$ not both $0$), $\exists\, u, v \in \mathbb{Z}$ s.t. $\gcd(a,b) = ua + vb$

Lemma: Given prime $p \in \mathbb{Z}$ s.t. $p | ab$ ($a, b \in \mathbb{Z}$), $p$ divides at least one of $a, b$

$\rightarrow$ if $a$ and $b$ are coprime, $\gcd(a,b) = 1 = ua + vb = ua \pmod b$

Fermat's Little Theorem: $a \in \mathbb{Z}_{\geq 1} \longrightarrow a^p \equiv a \pmod p$ $[*$ ($a, p$ coprime) $\rightarrow a^{p-1} \equiv 1 \pmod p]$

Proof: Define equiv. classes of ($x \equiv y \pmod p$): $S = \{[0], [1], \ldots, [p-1]\}$

$\rightarrow$ all elements of $a \cdot S$ are distinct $\rightarrow a \cdot S = S$

$\longrightarrow \prod_{[i] \in S} [i] = \prod_{i \in aS} [i] \longrightarrow (p-1)! = a^{p-1}(p-1)! \longrightarrow 1 = a^{p-1}$

(*) Corollary: Given $N = pq$ ($p \neq q$; $p, q$ prime) and $x \in \mathbb{Z}$ ($x, N$ coprime):

$\longrightarrow x^{(p-1)(q-1)} \equiv 1 \pmod N$

## (*) RSA Public Key Cryptosystem (Rivest-Shamir-Adleman)

To send a message $m$ such that anyone can see it, but only the intended recipient can decipher it:

1) Recipient puts out a public key $(N, e)$ and stores a private key $d$
2) Sender puts out $c = m^e \pmod N$ $[m < N]$, recipient receives $c$
3) $N = pq$ [$p, q$ prime], $e \cdot d = 1 \bmod ((p-1)(q-1))$ [$p, q$ large]

$\rightarrow c^d = m^{ed} \equiv m^{k \cdot (p-1)(q-1)+1} \rightarrow$ [eliminate extra terms] $\rightarrow m$

↳ Others cannot see the message because they do not know $d$, cannot decrypt $c$

(*) RSA outdated due to cost of storing large primes ($\sim 2^{4000}$ bits/key)