Math 33B: Differential Equations

2022-23
Spring '23

Instructor: Chengxi Wang
Textbook: Polking et al. - Differential Equations (2^{nd} Edition - 2005)
Topics: Differential equations (1^{st} & 2^{nd} order), differential forms, Existence & Uniqueness Theorems, autonomous equations, systems of differential equations, higher-order linear systems

Table of Contents:

Review

4/3/23 Lecture 1 + 4/5/23 Lecture 2 + 4/7/23 Lecture 3 + 4/10/23 Lecture 4

Review: Partial Fractions

Given an expression $\frac{f(x)}{g(x)}$, we can alternatively express it as a series of fractions [assuming the degree of $f(x)$ is smaller than the degree of $g(x)$]:

Case 1: $g(x)$ can be expressed as a product of distinct linear factors $(x-\lambda)$ (i.e. $g(x) = (x-\lambda_1)\ldots(x-\lambda_n)$)

$$\rightarrow \frac{f(x)}{g(x)} = \frac{A}{x-\lambda_1} + \frac{B}{x-\lambda_2} + \ldots + \frac{N}{x-\lambda_n} \text{ [for some } A, B, \ldots, N \in \mathbb{R}]$$

↳ Can be alternatively expressed as $f(x) = A(x-\lambda_2)\ldots(x-\lambda_n) + B(x-\lambda_1)(x-\lambda_3)\ldots$

Case 2: $g(x)$ can be expressed as a product of distinct linear factors $(x-\lambda)^n$

$$\rightarrow \frac{f(x)}{g(x)} = \frac{A}{(x-\lambda_1)^1} + \frac{B}{(x-\lambda_1)^2} + \ldots + \frac{N}{(x-\lambda_1)^{n_1}} + \frac{M}{(x-\lambda_2)^1} + \ldots + \frac{Z}{(x-\lambda_n)^{n_n}} \text{ (e.g.)}$$

Case 3: $g(x)$ contains irreducible quadratic factors [polynomials of degree ≥ 2]

$$\rightarrow \frac{f(x)}{g(x)} = \frac{Ax+B}{h(x)} + \frac{C}{(x-\lambda)} \text{ (e.g.) [Repeated quadratics: same method as Case 2]}$$

Review: Linear Algebra

Intro to Differential Equations

4/10/23
Lecture 4
(cont.)

Def: Implicit Differential Equations (§3.1)

An implicit differential equation [of order r] is an equation that can be written in the form:

$$\boxed{F(t, y, y', y'', \dots, y^{(r)}) = 0}$$ [Implicit differential eq. of order r]

Definitions

- The order r of an equation is the order of the highest derivative $y^{(r)}$ in the equation
- A solution to an implicit differential equation of order r is a function $y: I \to \mathbb{R}$, differentiable at least r times, such that:

$$F(t, y(t), y'(t), \dots, y^{(r)}(t)) = 0 \ \forall\, t \in I$$

(*) A solution may also be referred to as an integral curve / soln. curve.

(*) I being expressed as an interval subset of $\mathbb{R}$

(*) Examples

1) A zeroth order equation is any equation containing no derivatives of y
   - Includes polynomials of y, e.g. $y^2 + y = 2$ [0^{th} order]
2) A first order equation is any equation $f(y,t) + g(y',t) + h(t) = 0$
   - (*) Logistic equation: $y' - y(b - cy) = 0 \to y(t) = C_1 \exp 2t + C_2 \exp t$

(*) $\exp f(t) = e^{f(t)}$

Normal Form (§3.2)

A differential equation of order r is said to be in normal form [or an explicit differential equation [of order r]] if it is in the form:

$$\boxed{y^{(r)} = F(t, y, y', y'', \dots, y^{(r-1)})}$$ [Normal form]

- A solution y is thus a function (differentiable at least r times) satisfying the equation.
- An implicit differential equation can be put in normal form if its highest derivative is expressible in terms of its lower derivatives

First-Order Differential Equations

4/12/23
Lecture 5

First-Order Differential Equations

Given an explicit first-order differential equation $y' = F(t, y)$, a solution is a differentiable function $y: I \to \mathbb{R}$ s.t. $y'(t) = F(t, y(t)) \; \forall \, t \in I$.

Definitions

- A general solution for an explicit first-order differential equation is the family of functions $y(t; C)$, dependent on a constant $C \in \mathbb{R}$, s.t.:

  1) $y(t; C_0)$ satisfies the equation for all valid parameters C_0

  2) Every solution is of the form $y(t; C_1)$ for some valid parameter C_1

→ • A particular solution is a function $y(t; C_0)$ for some fixed value C_0.

Initial Value Problems

Def: An initial value problem is a pair of two conditions any solution must satisfy:

1) A differential equation $y' = F(t, y)$

2) A specific point $y(t_0) = y_0$ [The initial condition]

(*) Direction Fields

A direction field is a plot where at each point (t_0, y_0), there is plotted a line segment with slope $y' = F(t_0, y_0)$.

Ex: $y' = y(3 - y) \longrightarrow$  (e.g.)

Def: 1st-Order Linear Differential Equation (§3.3)

A first-order linear differential equation is a differential equation that can be written in the form:

$$\boxed{y' + f(t)\,y = g(t)}$$ [First-Order Linear Differential Equation]

↳ (*) $f(t)$ and $g(t)$ are referred to as coefficient functions

Solutions of First-Order Linear Differential Equations

4/14/23
Lecture 6

Solving First-Order Linear Differential Equations

A first-order linear differential equation $y' + f(t) = g(t)$ can be solved as follows:

$$y' + f(t)y = g(t) \longrightarrow \text{Define } \underline{\text{integrating factor}}\ \mu(t) = \int f(t)\,dt$$

$$\rightarrow \mu(t)y' + \mu(t)f(t)y = \frac{d}{dt}\left(\mu(t)\,y\right)\ [= g(t)]$$

$$\boxed{\rightarrow y = \frac{1}{\mu(t)}\left(\int g(t)\,dt\right) = \frac{1}{\mu(t)}\int g(t)\,dt + \frac{C}{\mu(t)}}\ [\text{General solution}]$$

↳ Given an initial condition $y(t_0) = y_0$, we can find a <u>particular solution</u> $y(t; C_0)$.

(*) <u>Intervals of Existence</u>

Given a 1^{st}-order differential equation with coefficient functions $f: D \to \mathbb{R}$ and $g: E \to \mathbb{R}$ for some intervals D, E, and initial condition $y(t_0) = y_0$, the <u>interval of existence</u> of the particular solution satisfying the initial condition is the largest interval I such that:

1) $t_0 \in I$
2) $I \subseteq D$ and $I \subseteq E$ — (*) The interval of existence will always be a subset of the domains of the coefficient functions f and g.

(*) <u>Lemma</u>: Given a linear differential equation $y' + f(t) = g(t)$ satisfied by two differentiable functions $y_0, y_1 : I \to \mathbb{R}$ [possibly distinct]:

$\rightarrow$ 1) $\exists$ a constant C s.t. $y_0(t) = y_1(t) + \frac{C}{\mu(t)}\ \forall t \in I$

$\rightarrow$ 2) If $\exists\, t_0 \in I$ s.t. $y_0(t_0) = y_1(t_0)$, then $y_0 = y_1$

(*) <u>Observation</u>: If $\mu(t)$ is an integrating factor, any function $C\mu(t)$ $(C \neq 0)$ is also a valid integrating factor.

Differential Forms

4/17/23
Lecture 7

Implicit Equations (§3.4)

Def: An <u>implicit equation</u> is an equation that can be written in the form: $\boxed{F(t,y)=0}$

Def: Given a function $F(t,y)$ and a constant $C\in\mathbb{R}$, we call an equation a <u>level set</u> of F if it is on the form:

$$\boxed{F(t,y)=C} \text{ [Level set of F]}$$

(*) Also considered an implicit equation

Ex: $t^2+y^2=r^2$ [circle of radius r]

Explicit variants have discontinuities; this does not

Differential Forms

Motivation: How we can take the derivative of an implicit equation?

Def: Differential Forms

A <u>differential form</u> is an equation of the form:

$$\boxed{P(t,y)\,dt+Q(t,y)\,dy} \text{ [Differential form]},$$

where dy, dt are placeholders associated with y and t, called <u>differentials</u>

Differentials of Equations

Given a function $F(t,y)$ the <u>differential</u> of F is:

$$\boxed{dF=\frac{\partial F}{\partial t}(t,y)\,dt+\frac{\partial F}{\partial y}(t,y)\,dy} \text{ [Differential of } F(t,y)\text{]}$$

Can be used to solve for dy/dt, e.g.

Separable Equations

4/19/23
Lecture 8

Obtaining Differential Forms (§3.5)

Given an explicit 1^{st}-order linear differential equation $y'\ [=\frac{dy}{dt}] = f(t,y)$, it can be rewritten as a differential form equation as follows:

$$\frac{dy}{dt} = f(t,y) \xrightarrow{\times dt} dy = f(t,y)\,dt$$

$$\xrightarrow{-f(t,y)dt} -f(t,y)\,dt + dy = 0$$

(*) Meant to make a more readily integrable equation

$$\xrightarrow{\times\mu(t,y)} -f(t,y)\mu(t,y)\,dt + \mu(t,y)\,dy = 0$$ [for some integrating factor μ]

Separable Differential Equations

Def.: A separable differential equation is an equation of the form:

$$\boxed{\frac{dy}{dt} = f(t)g(y)}$$ [Separable differential equation]

↳ Can be rearranged into $-f(t)\,dt + \frac{dy}{g(y)} = 0\ [= dF]$

$$\longrightarrow F(t,y) = \int(-f(t))\,dt + \int\left(\frac{1}{g(y)}\right)dy + C$$

Solutions of Separable Equations

Given a separated equation $P(t)\,dt + Q(y)\,dy = 0$:

1) Define function $F(t,y) := \int P(t)\,dt + \int Q(y)\,dy$

$\rightarrow$ s.t. $dF = P(t)\,dt + Q(y)\,dy\ [=0]$

2) $dF = 0 \longrightarrow \boxed{F(t,y) = C}$ [General solution], for arbitrary $C \in \mathbb{R}$

↳ Alt.: $\int P(t)\,dt + \int Q(t)\,dt = C$

↳ Can find values for C to make a particular solution

Exact Differential Equations

4/21/23
Lecture 9

Exact Differential Equations

Def: An <u>exact differential equation</u> is an equation of the more general form: $\boxed{y' = f(t,y)}$
and can be solved by rearranging the terms and multiplying by an
<u>integrating factor</u> $\mu(t)$ into the form:

$$-f(t,y)\mu(t,y)\,dt + \mu(t,y)\,dy = \boxed{P(t,y)\,dt + Q(t,y)\,dy = 0}$$ [Differential form]

Potential Functions

To find the integrating factor $\mu(t,y)$, we first find a <u>potential function</u>.
Def: A potential function is a function $F(t,y)$ such that:

$$\boxed{\partial F = P(t,y)\,dt + Q(t,y)\,dy} \rightarrow \begin{cases} \frac{\partial F}{\partial t} = P(t,y) \\ \frac{\partial F}{\partial y} = Q(t,y) \end{cases}$$ [Potential function]

Def: Given two-variable functions $P, Q: D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$, the differential form
$P\,dt + Q\,dy$ is considered <u>exact</u> if $\exists$ a continuously differentiable function $F: D \to \mathbb{R}$ s.t.
$\partial F = P\,dt + Q\,dy$. [$\exists$ a potential function]

↳ Condition: We assume P and Q are continuously differentiable.
⟶ F must have continuous second-order partial derivatives.

$$\longrightarrow \frac{\partial F}{\partial y \partial t} = \frac{\partial F}{\partial t \partial y} \text{ [per Clairaut's]} \longrightarrow \boxed{\frac{\partial Q}{\partial t} = \frac{\partial P}{\partial y}}$$

⟶ <u>Closedness</u>

Given $P, Q: D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$, we say the differential form $P\,dt + Q\,dy$
is <u>closed</u> if:

$$\boxed{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial t} = 0}$$ [Closedness]

THM
Let $P, Q: I \times J \to \mathbb{R}$ be continuously differentiable functions across some intervals
I, J [s.t. $I \times J$ is a rectangle.] Then:

$\boxed{\text{"}P\,dt + Q\,dy \text{ is closed."} \leftrightarrow \text{"}P\,dt + Q\,dy \text{ is exact."}}$

Existence & Uniqueness Theorems

4/24/23
Lecture 10

The Integrating Factor

The role of the integrating factor is to make non-exact equations, exact.

Def: Given continuous functions $P, Q: D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^2$), we say that a function $\mu: D \to \mathbb{R}$ is an integrating factor for the differential form equation $P(t,y)dt + Q(t,y)dy = 0$ if: 1) $\mu(t,y) \neq 0$ for every $(t,y) \in D$; and
2) $\mu(t,y)P(t,y)dt + \mu(t,y)Q(t,y)dy$ is exact.

$\longrightarrow$ 2) satisfied iff: $\boxed{\frac{\partial}{\partial t}(\mu(t,y)P(t,y)) = \frac{\partial}{\partial y}(\mu(t,y)Q(t,y)) = 0}$ [for I.F. μ]

Existence Theorem (§1.6)

Let $f: I \times J \to \mathbb{R}$ be a continuous two-variable function defined on a rectangle $I \times J$ for intervals I, J. Then, given any point $(t_0, y_0) \in I \times J$, the initial value problem $\{y' = f(t,y);\ y(t_0) = y_0\}$ has a solution defined on some interval $I' \subseteq I$ [$t_0 \in I'$].
The solution is defined at least until the solution curve $t \mapsto f(t, y(t))$ leaves the rectangle $I \times J$.

Uniqueness Theorem

Let $f: I \times J \to \mathbb{R}$ be a continuous two-variable function defined on a rectangle $I \times J$ for intervals I, J. Furthermore, suppose $\frac{\partial f}{\partial y}$ exists and is continuous on all of $I \times J$. Let $(t_0, y_0) \in I \times J$, and suppose we have two solutions $y(t), \tilde{y}(t)$ to the same IVP:

1) $y'(t) = f(t, y(t))$ and $\tilde{y}'(t) = (t, \tilde{y}(t))$ for every t, and
2) $y(t_0) = y_0$ and $\tilde{y}(t_0) = y_0$.

Then, for every t s.t. $(t, y(t))$ and $(t, \tilde{y}(t))$ remain in $I \times J$, we have: $\boxed{y(t) = \tilde{y}(t)}$

$\hookrightarrow$ $\boxed{\text{Consequence: Different solution curves cannot cross.}}$

Autonomous Equations

4/26/23
Lecture 11

Autonomous Equations (§3.7)

Def: A 1^{st}-order differential equation is called <u>autonomous</u> if it can be written in the form: $\boxed{y' = f(y)}$
(Alt: Equation does not depend on t)

↳ Can be used to study the <u>qualitative properties</u> of solutions, e.g. via <u>direction fields</u>.

Properties of Autonomous Equations

1) Since autonomous equations do not depend on t, the direction field of an autonomous equation will not change left-to-right, only up/down.
2) Let $y_0(t)$ be a particular solution and $C \in \mathbb{R}$ be a constant. Then $y_0(t+C)$ is also a solution; in fact:

$$(y_0(t+C))' = y_0'(t+C) = f(y_0(t+C))$$

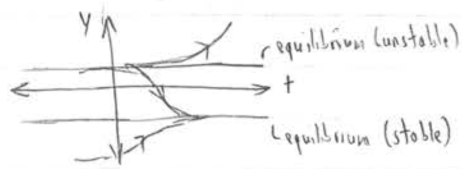
3) Suppose $y_0 \in \mathbb{R}$ is such that $f(y_0) = 0$. Then the constant function $y(t) := y_0$ for all t is a solution to $y' = f(y)$. Such a number y_0 is called an <u>equilibrium point</u> and the constant function $y(t) := y_0$ is called an <u>equilibrium solution</u>.

Non-Equilibrium Solutions

Any non-equilibrium solution will be asymptotic to one of the equilibrium solutions. Furthermore, it will be either <u>strictly increasing</u> or <u>strictly decreasing</u>.

Def: Given an autonomous equation with initial condition (t_0, y_0), it is called <u>strictly increasing</u> if $y' = f(y_0) > 0$. Conversely, it is called <u>strictly decreasing</u> if $y' = f(y_0) < 0$.

($f(y_0) = 0 \rightarrow$ equilibrium solution) Ex:



The qualitative behavior of solutions to an autonomous equation can notably be visualized using a <u>phase line</u>.

Autonomous Equations (cont.)

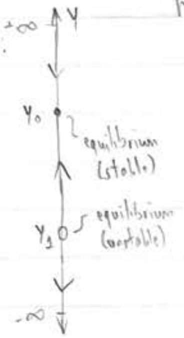
4/26/23
Lecture 11
(cont.)

Autonomous Equations (cont.)

Phase Lines

Def: A phase line for an equation $y' = f(y)$ is a plot of the y-axis (displayed horizontally) with the features:

Ex:



1) At every equilibrium point y_0 (where $f(y_0) = 0$), there is a dot.
2) In each region between two equilibrium points, or between an equilibrium point and $\pm\infty$:
   i) If $f(y) > 0$ in that region, there is an arrow to the right [y strictly increasing].
   ii) Conversely, if $f(y) < 0$, there is an arrow to the left [y is strictly decreasing].
3) At each equilibrium point y_0, if the arrows to its side are pointing toward y_0, then the dot at y_0 is filled in; otherwise, the dot is not filled in.

Equilibrium Stability

Def: Let $y_0 \in \mathbb{R}$ be an equilibrium point for an autonomous equation $y' = f(y)$. Then, we say that y_0 is:

1) asymptotically stable if, given a solution passing through $(t_0, y_0 + \varepsilon)$ for some very small ε ($|\varepsilon| < 1$, e.g.), that solution will asymptotically approach the solution $y(t) = y_0$.
2) unstable otherwise, i.e. if y_0 is not asymptotically stable.

Analogy: Asymptotically stable equilibrium points "attract" solutions; unstable points repel them.

First-Derivative Test for Stability

Let y_0 be an equilibrium point for autonomous equation $y' = f(y)$, and suppose the function f is differentiable. Then:

1) if $f'(y_0) < 0$, then f is strictly decreasing at y_0 and y_0 is asymptotically stable.
2) if $f'(y_0) > 0$, then f is strictly increasing at y_0 and y_0 is unstable.
3) if $f'(y_0) = 0$, the test is inconclusive.

(*) Stability Application: "Given a slight measurement error, will this still be the equilibrium?

Second-Order Linear Equations

4/28/23
Lecture 12

Second-Order Linear Equations (§4.1)

Def: A <u>second-order linear differential equation</u> is a differential equation which can be expressed in the form:

$$\boxed{y''(t) + p(t)y' + q(t)y = g(t)} \quad [2^{nd}\text{-order linear diff. equation}]$$

<u>Notes</u>:

- p, q, g referred to as <u>coefficient functions</u>; $g(t)$ referred to as <u>the forcing term</u>
- A 2^{nd}-order linear eq. is said to be <u>homogenous</u> if it can be expressed in the form: $y''(t) + p(t)y' + q(t)y = 0$ [if $g(t) = 0$]

(*) <u>Existence and Uniqueness Theorems</u>

THM: Suppose $p, q, g : I \to \mathbb{R}$ are continuous functions with domain interval $I \subseteq \mathbb{R}$. Then, given $t_0 \in I$ and any two real numbers $y_0, y_1 \in \mathbb{R}$ there is a unique function $y : I \to \mathbb{R}$ satisfying the IVP: (i) $y'' + p(t)y' + q(t)y = g(t)$; (ii) $y(t_0) = y_0$; $y'(t_0) = y_1$

<u>Linear Combinations</u>

Given functions $y_1, y_2, \ldots, y_n : I \to \mathbb{R}$, a <u>linear combination</u> of $y_1, y_2, \ldots, y_n$ is any function of the form:

$$\boxed{C_1 y_1 + C_2 y_2 + \ldots + C_n y_n : I \to \mathbb{R}} \quad \text{for constants } C_1, C_2, \ldots, C_n \in \mathbb{R}$$

<u>Proposition: Linear Combinations</u>

Prop.: Suppose $y_1(t)$, $y_2(t)$ are solutions to the homogenous second-order differential equation $y'' + p(t)y' + q(t)y = 0$. Then, for any $C_1, C_2 \in \mathbb{R}$, the function <u>$C_1y_1 + C_2y_2$ is also a solution.</u>

4/28/23
Lecture 12
(cont.)

Second-Order Linear Equations (cont.)

Linear Independence

Def: Suppose $y_1, y_2 : I \to \mathbb{R}$ are functions defined on an interval $I \subseteq \mathbb{R}$. We say that y_1, y_2 are <u>linearly independent</u> if: for every $C_1, C_2 \in \mathbb{R}$, if

$$\boxed{C_1 y_1(t) + C_2 y_2(t) = 0 \text{ for all } t \in I \longrightarrow C_1, C_2 = 0}, \quad \text{[Linear Independence]}$$

i.e. if the <u>only</u> linear combination satisfying $C_1 y_1 + C_2 y_2 = 0$ for all t in the domain is the trivial linear combination $0y_1 + 0y_2 = 0$.

If y_1, y_2 are not linearly independent [i.e. $\exists$ a pair of constants C_1, C_2 not both 0 s.t. $C_1 y_1 + C_2 y_2 = 0 \ \forall t \in I$], then they are called <u>linearly dependent</u>; meaning that one of y_1, y_2 is a constant multiple of the other.

Wronksian

Def: Suppose $u, v : I \to \mathbb{R}$ are two differentiable functions defined on an interval $I \subseteq \mathbb{R}$. Then we define the <u>Wronksian</u> of u and v to be the function $W : I \to \mathbb{R}$ defined by:

[Wronksian]
$$\boxed{W(t) := \det \begin{bmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{bmatrix} := u(t)v'(t) - v(t)u'(t)}$$

Proposition: Wronksian

Prop.: Suppose $p, q, u, v : I \to \mathbb{R}$ are functions defined on an interval $I \subseteq \mathbb{R}$, such that u and v are solutions to: $y'' + p(t)y' + q(t)y = 0$.

$\to$ Let W be the Wronksian $W(t)$ of u and v. Then exactly one of the following is true:

i) (Case 1) $W(t) = 0$ for all $t \in I$, or

ii) (Case 2) $W(t) \neq 0$ for all $t \in I$

Second-Order Linear Equations (cont.)

5/1/23
Lecture 13 Second-Order Linear Equations (cont.)

(*) Proposition: Wronksian

Prop: Suppose $p, q, u, v: I \to \mathbb{R}$ are functions defined on interval $I \subseteq \mathbb{R}$ s.t. u, v are solns. to the equation $y'' + p(t)y' + q(t)y = 0$. Let $W(t)$ be the Wronksian of u and v; then: i) (Case 1) if $\exists\, t_0 \in I$ s.t. $W(t_0) = 0$, u and v are linearly dependent;
ii) (Case 2) if $\exists\, t_0 \in I$ s.t. $W(t_0) \neq 0$, u and v are linearly independent

Theorem: Solutions

THM: Suppose y_1, y_2 are the solutions to the homogenous equation $y'' + p(t)y' + q(t)y = 0$. Then the general solution is:

$$\boxed{y(t; C_1, C_2) = C_1 y_1(t) + C_2 y_2(t)}, \; C_1, C_2 \in \mathbb{R} \text{ [General soln.]}$$

Fundamental Solutions

Def: A fundamental set of solutions to a homogenous second-order linear differential equation $y'' + p'(t)y + q(t)y = 0$ is a pair y_1, y_2 of linearly independent solutions to the equation.

→ (*) Any other solutions will be variations of the general formula including y_1 and y_2, i.e, will be of the form $C_1y_1 + C_2y_2$ [$= y(t; C_1, C_2)$].
(*) y_1, y_2 form a basis of the solution space [Linear algebra]

Homogenous 2^{nd}-Order Linear Equations w/ Constant Coefficients

5/1/23
Lecture 13
(cont.)

Homogenous 2^{nd}-Order Linear Equations w/ Constant Coefficients (§4.2)

Def: A homogenous 2^{nd}-order linear differential equation is a differential equation that can be expressed in the form $y'' + py' + qy = 0$; $p, q \in \mathbb{R}$ [constants].

Characteristic Polynomials

Def: Given a homogenous 2^{nd}-order linear equation $y'' + py' + qy = 0$ ($p, q \in \mathbb{R}$), the associated characteristic polynomial is the quadratic polynomial

$$\boxed{f(\lambda) = \lambda^2 + p\lambda + q} \quad \text{[Characteristic polynomial]};$$

a root of the characteristic polynomial is called a characteristic root.

Characteristic Roots

The roots of a characteristic polynomial can be used to find a general solution, depending on the nature of the roots: λ_1, λ_2:

Case 1: Distinct real roots ($\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$):

$$\longrightarrow \boxed{\text{General solution: } y(t; C_1, C_2) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}} \quad (\lambda_1, \lambda_2 \in \mathbb{R}; \lambda_1 \neq \lambda_2)$$

Case 2: Repeated real roots ($\lambda_1 = \lambda_2$; $\lambda_1, \lambda_2 \in \mathbb{R}$):

$$\longrightarrow \boxed{\text{General solution: } y(t; C_1, C_2) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t}} \quad (\lambda_1, \lambda_2 \in \mathbb{R}; \lambda_1 = \lambda_2)$$

Case 4: Distinct complex roots ($\lambda_1 = a + bi$, $\lambda_2 = a - bi$; $\lambda_1, \lambda_2 \in \mathbb{C}$):

$$\longrightarrow \boxed{\text{General solution: } y(t; C_1, C_2) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}} \quad (\lambda_1 = \overline{\lambda_2}; \lambda_1, \lambda_2 \in \mathbb{C})$$

↳ (*) Given complex soln. $z(t)$, its conjugate $\overline{z(t)}$ is also a solution.

Complex Numbers Review

5/3/23
Lecture 14

Complex Numbers Review

Def: A complex number is a number of the form $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{(-1)}$ is the imaginary unit. We denote the set of all complex numbers by $\mathbb{C}$.

Given a complex number $z = a + bi$, we define its real part to be $\operatorname{Re}(z) := a$ and its imaginary part to be $\operatorname{Im}(z) := b$.

Given a complex number $z = a + bi$, we define its complex conjugate $\bar{z} = a - bi$.

Complex functions (i.e. functions $z: D \to \mathbb{C}$) can also be decomposed into real and imaginary parts, i.e. $z(t) = x(t) + iy(t)$.

i) $\overline{z(t)} = x(t) - iy(t)$ [Complex conjugate]

ii) $\frac{d}{dt} z(t) = \frac{d}{dt} x(t) + i \frac{d}{dt} y(t)$ [Derivative]

$$\boxed{\text{Euler's formula: } e^{a+bi} = e^{a}(\cos b + i \sin b)}$$

(*) Can be used to split solns. $C_1 e^{(a_1+\lambda i)t} + C_2 e^{(a_2-\lambda i)t}$ into real solns. $C_1^* \cos(\lambda t) + C_2^* \sin(\lambda t)$, e.g.

(*) Given a polynomial $f(\lambda) = \lambda^2 + p\lambda + q$ with real coefficients p, q and complex root $\lambda_1 = a + bi$, $\lambda_2 = \bar{\lambda}_1 = a - bi$ is also a complex root (complex roots of real polynomials occur in complex conjugate pairs):

(*) Complex Characteristic Roots

Let $z(t)$ be a complex-valued function which is a solution to $y'' + py' + qy = 0$ $(p, q \in \mathbb{R})$.
Then: $\overline{z(t)}$ is also a solution.

[$z(t) = x(t) + iy(t)$ is soln. $\to$ $\overline{z(t)} = x(t) - iy(t)$ is also a solution]

And since the set of all solutions is closed under linear combinations:

i) $\operatorname{Re}(z(t)) = x(t)$ is a real-valued solution; and
ii) $\operatorname{Im}(z(t)) = y(t)$ is a real-valued solution

Inhomogenous Equations

5/8/23
Lecture 15

Inhomogenous Equations (§4.3)

Recall: An inhomogenous 2^{nd}-order linear differential equation is an equation of the form:
$y''+p(t)y'+q(t)y=g(t)$, where $g(t)$ is not 0 for all t.

Theorem: Solutions to Inhomogenous Equations

THM: Suppose $y_p(t)$ is a particular solution to the inhomogenous equation

$$\text{(A)} \quad y''+p(t)y'+q(t)y=g(t),$$

and y_1, y_2 form a fundamental set of solutions to the corresponding homogenous equation

$$\text{(B)} \quad y''+p(t)y'+q(t)y=0,$$

then the general solution to the inhomogenous equation (A) is of the form:

$$\boxed{y(t)=y(t;C_1,C_2)=C_1y_1(t)+C_2y_2(t)+y_p(t)}$$

Method of Undetermined Coefficients

Given an inhomogenous equation $y''+py'+qy=g(t)$, where:

i) p and q are constant functions ($p,q\in\mathbb{R}$), and

ii) $g(t)$ is a "nice enough" function, and not a solution to the homogenous equation $y''+py'+qy=0$, then:

$\rightarrow$ We can find a particular solution to the inhomogenous equation $y''+py'+qy=g(t)$ via guessing a trial solution, based on the nature of $g(t)$.

(*) Ex: $y''+4y=\cos(3t) \rightarrow g(t)=\cos(3t)$, trial soln: $y_p(t)=a\cos(3t)+b\sin(3t)$

$$\rightarrow y_p''+4y_p=-5a\cos(3t)-5b\sin(3t)$$
$$=g(t)=\cos(3t)$$
$$\rightarrow a=-\tfrac{1}{5};\ b=0$$

$\rightarrow$ Particular solution: $y_p(t)=-\frac{1}{5}\cos(3t)+0\sin(3t)$

Solutions to Inhomogenous Equations

5/8/23
Lecture 15
(cont.)

Inhomogenous Equations (cont.)

(*) Trial Solutions

Define constants $A, B, a, b, r, w \in \mathbb{R}$; polynomial $P(t)$, where $p_0(t)$, $p_1(t)$ are polynomials of the same degree as P. Then, based on the form of the forcing function $g(t)$, the trial solutions are as follows:

$g(t)$	Trial Solution $y_p(t)$
i) $= e^{rt}$	$= ae^{rt}$
ii) $= A\cos(wt) + B\sin(wt)$	$= a\cos(wt) + b\sin(wt)$
iii) $= P(t)$	$= p_0(t)$
iv) $= P(t)\cos(wt)$ or $P(t)\sin(wt)$	$= p_0(t)\cos(wt) + p_1(t)\sin(wt)$
v) $= e^{rt}\cos(wt)$ or $e^{rt}\sin(wt)$	$= e^{rt}(a\cos(wt) + b\sin(wt))$
vi) $= e^{rt}P(t)\cos(wt)$ or $e^{rt}P(t)\sin(wt)$	$= e^{rt}(p_0(t)\cos(wt) + p_1(t)\sin(wt))$

→ (*) If $g(t)$ is a solution to $y'' + py' + qy$, then use the trial solution $y_p(t) = tg(t)$; if that fails, try $y_p(t) = t^2 g(t)$.

Superposition Principle

Suppose $y_f(t)$ is a particular solution to the equation $y'' + p(t)y' + q(t)y = f(t)$; and $y_g(t)$ is a particular solution to the equation $y'' + p(t)y' + q(t)y = g(t)$.

→ Then, for constants $a, b \in \mathbb{R}$, the combined function $y(t) := ay_f(t) + by_g(t)$ is a solution to the equation

$$\boxed{y'' + p(t)y' + q(t)y = af(t) + bg(t) \quad [\text{Soln}: y(t) = ay_f(t) + by_g(t)]}$$

(*) Ex: $g(t) = 2t + \cos(2t)$ $[P(t) + A\cos(wt)]$

$$\rightarrow y_p(t) = \underbrace{(a_0 t + b_0)}_{p_0(t)} + (a_1\cos(2t) + b_1\sin(2t))$$

Variation of Parameters

5/12/23
Lecture 16

Variation of Parameters (§4.4)

(*) Systems of Equations

Suppose $a, b, c, d, e, f \in \mathbb{R}$ are numbers such that:

$$W := \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0.$$

→ Then the system $\{ax+by=e;\ cx+dy=f\}$ has unique solution:

$$\boxed{x = \frac{de-bf}{W}, \quad y = \frac{-ce+af}{W}}$$ [Unique solution]

Variation of Parameters

Suppose $y_1(t), y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$; and define Wronksian $W(t) := y_1 y_2' - y_1' y_2 \neq 0$ for all t. Then the inhomogenous equation $y'' + p(t)y' + q(t)y = g(t)$ has, as a particular solution, the following:

$$\boxed{y_p(t) = y_1 \int \frac{-y_2(t)g(t)}{W(t)}\,dt + y_2 \int \frac{y_1(t)g(t)}{W(t)}\,dt}$$ [Particular solution]

(*) Ex: $y'' + \frac{1}{t}y' - \frac{1}{t^2}y = \frac{\ln(t)}{t}$; $y_1(t) = t$, $y_2(t) = \frac{1}{t}$

$\longrightarrow W(t) = \frac{-2}{t}$

$$\longrightarrow v_1(t) = \int \frac{-y_2 g(t)\,dt}{W(t)} = \int \frac{-(1/t)(\ln t)}{-(2/t)}\,dt = \frac{(\ln(t))^2}{4}$$

$$v_2(t) = \int \frac{y_1(t)g(t)\,dt}{W(t)} = \int \frac{t(\ln t)}{-(2/t)}\,dt = \frac{t^2(2\ln(t)-1)}{8}$$

$$y_p(t) = y_1 v_1 + y_2 v_2 = \frac{t(\ln t)^2}{4} - \frac{t(2\ln(t)-1)}{8}$$ [Particular solution]

Review: Linear Algebra

5/15/22
Lecture 17

Linear Algebra Review (§5.1)

Def.: A matrix of size $m \times n$ ($m, n \in \mathbb{N}_{>0}$) is a rectangular array of real numbers with m rows and n columns, composed of entries/components a_{ij} ($i \in \{1, \dots, m\}$; $j \in \{1, \dots, n\}$).

(*) An entry a_{ij} can also be referred to as the (i,j)-entry of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \quad [m \times n \text{ matrix } A]$$

$$A = (a_{ij})_{1 \le i \le m,\, 1 \le j \le n} \; ; \; A = (a_{ij}) \quad [\text{Alternate notation}]$$

(*) $\mathrm{Mat}_{m \times n}(\mathbb{R})$ denotes the set of all real-number-entried $m \times n$ matrices (alt: $\mathbb{R}^{m \times n}$)

A matrix with only one column (matrix $\in \mathrm{Mat}_{m \times 1}(\mathbb{R})$) is referred to as a column vector. ($\vec{v} \in \mathbb{R}^m$)

For each $m, n \ge 1$, we define zero matrix $O_{m \times n}$ [$O_{m \times n} \in \mathrm{Mat}_{m \times n}(\mathbb{R})$].

Matrix Arithmetic

Sum: Given two matrices $A, B \in \mathrm{Mat}_{m \times n}(\mathbb{R})$, we define the matrix sum $A + B \in \mathrm{Mat}_{m \times n}(\mathbb{R})$ as the $m \times n$ matrix $(a_{ij} + b_{ij})_{1 \le i \le m,\, 1 \le j \le n}$.

Scalar Multiplication: Given a matrix $A \in \mathrm{Mat}_{m \times n}(\mathbb{R})$ and a scalar $\lambda \in \mathbb{R}$, we define the scalar multiple of A by λ ($\lambda A \in \mathrm{Mat}_{m \times n}(\mathbb{R})$) as the $m \times n$ matrix $(\lambda a_{ij})_{1 \le i \le m,\, 1 \le j \le n}$.

(*) Rules

i) $(A+B)+C = A+(B+C)$
ii) $O_{m \times n} + A = A$
iii) $(-1)A + A = O_{m \times n}$
iv) $A + B = B + A$
v) $\lambda(A+B) = \lambda A + \lambda B$
vi) $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$
vii) $(\lambda_1 \lambda_2)A = \lambda_1(\lambda_2 A)$
viii) $1 \cdot A = A$; $0 \cdot A = O_{m \times n}$

Linear Combinations

Def: A linear combination of column vectors $v_1, \dots, v_m \in \mathbb{R}^n$ ($n \ge 1$) is an expression of the form:

$$\boxed{\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m} \quad [\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}] \quad [\text{Linear combination}]$$

Review: Matrix Equations

5/17/23
Lecture 19

Matrix Equations (§5.2)

Def: Given a matrix $A \in \mathbb{R}^{m\times n}$ and vector $\vec{x} \in \mathbb{R}^n$, we define the <u>product</u> of A and $\vec{x}$ to be the column vector $A\vec{x} \in \mathbb{R}^n$, where $(A\vec{x})_{i,1} = \sum_{k=1}^{n} A_{i,k} x_k$.

(Alt: $A = [\vec{a}_1\ \vec{a}_2 \dots \vec{a}_n] \rightarrow A\vec{x} =$ linear combination: $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$)

(*) The product $A\vec{x}$ only defined if # of columns in A = # of rows in $\vec{x}$

(*) Properties of Matrix Multiplication

Given $A \in \mathbb{R}^{m\times n}$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$:

(1) $A(\alpha\vec{x}) = \alpha A\vec{x}$ (Scalar multiplication)

(2) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ (Commutativity)

Systems of Equations

Systems of equations can be alternatively expressed as matrix equations, i.e.:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

(System of equations)

$$\downarrow$$

Coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(*) Can be solved via augmented matrix, e.g.

$$\downarrow$$

$$Ax = b \quad \text{(Matrix equation)}$$

(*) Alt. notation:

$$A = [\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n]$$

$$\rightarrow x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = b$$

Review: Spans, Spaces, and Dependence

5/17/23
Lecture 19
(cont.)

Definitions (§5.3)

Def: Given a matrix equation $Ax=b$ (§5.2), we say the equation is homogeneous if $b=\vec{0}$; otherwise, if $b \neq \vec{0}$, then the equation is inhomogeneous.

Definition: Null Space

Def.: Let $A \in \mathbb{R}^{m \times n}$. We define the null space of A to be the set of all solutions $\vec{x}$ to the homogeneous equation $A\vec{x}=\vec{0}$ [the set of all vectors $\vec{x}$ s.t. $A\vec{x}=\vec{0}$]. More formally:

$$\boxed{N(A) = \text{null}(A) := \{x \in \mathbb{R}^n ; A\vec{x}=0\} \subseteq \mathbb{R}^n}$$ [Null space of A]

(*) null(A) is a subspace of $\mathbb{R}^n$ [i.e null(A) is closed under addition, scalar multi.]

Definition: Span

Def.: Given vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{R}^n$, the span of $\vec{x}_1, \dots, \vec{x}_k$ is the set of all linear combinations of $\vec{x}_1, \dots, \vec{x}_k$.

$$\boxed{\text{span}(\vec{x}_1, \dots, \vec{x}_k) = \{\alpha_1 x_1 + \dots + \alpha_k \vec{x}_k ; \alpha_1, \dots, \alpha_k \in \mathbb{R}\}}$$ [Span]

Definition: Linear Independence

Def.: Given vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{R}^n$, we say $\vec{x}_1, \dots, \vec{x}_k$ are linearly independent if, for constants $c_1, \dots, c_k \in \mathbb{R}^n$, $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_k\vec{x}_k = 0$ only when $c_1 = c_2 = \dots = c_k = 0$. In other words, if given $A = [\vec{x}_1\ \vec{x}_2 \dots \vec{x}_k]$, the homogenous equation $A\vec{c}=0$ has exactly one solution $\vec{c}=\vec{0}$.

Def.: Otherwise, we say that $\bar{x}_1, \dots, \bar{x}_k$ are linearly dependent, i.e. when $\exists$ $c_1, c_2, \dots, c_k \in \mathbb{R}$ ($c_1, \dots, c_k$ not all zero) s.t. $c_1 x_1 + \dots + c_k x_k = 0$. In this case, such a linear combination $c_1 x_1 + \dots + c_k x_k = 0$ is called a nontrivial dependence relation.

(*) As opposed to the trivial relation $c_1 = \dots = c_k = 0$.

Review: Bases and Dimension

5/17/23
Lecture 19
(cont.)

Definitions (§5.3, cont.)

Definition: Basis

Def.: Given a subspace V of $\mathbb{R}^n$, a basis of V is a collection of linearly independent basis vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ that span V (i.e. $V = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$). Additionally, we define the dimension of V to be the number of vectors in a basis of V.

(*) Rank-Nullity Theorem

THM: Given a matrix A with n columns: $\boxed{\text{rank}(A) + \dim(\text{null}(A)) = n}$

Proposition: Solutions

Prop.: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ [$b \neq \vec{0}$]. Given the inhomogeneous equation $A\vec{x} = b$, with a particular solution $x_p \in \mathbb{R}^n$, the set of all solutions to $A\vec{x} = b$ is the set:

$$\boxed{\{x_p + x_h ; x_h \in \text{null}(A)\}}, \text{ [set of all solutions]}$$

consisting of the particular solution $\vec{x}_p$ and a homogeneous solution $\vec{x}_h \in \text{null}(A)$.

Definition: Consistency

Def.: Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$; we say the matrix equation $Ax = b$ is inconsistent if $\nexists$ any solutions $\vec{x}$ to the equation $Ax = b$. Otherwise, we say the equation is consistent.

(*) Notes on Consistency

Given a consistent matrix equation $Ax = b$, there is a unique solution to $Ax = b$ iff $\text{null}(A) = \{\vec{0}\}$. Otherwise, if $\text{null}(A) \neq \{\vec{0}\}$, there are infinitely many solutions to the matrix equation $Ax = b$.

(*) Note on Bases

Given a subspace V of $\mathbb{R}^n$, V may have infinitely many bases; however, all bases of V will be of the same size (i.e. $\dim(V)$ does not depend on choice of basis).

Review: Determinants

5/19/23
Lecture 20 Square Matrices and Determinants (§5.4)
Def.: A matrix A is called square if $A \in \mathbb{R}^{n \times n}$ for some $n \geq 1$ (i.e. # rows = # columns).

(*) The Identity Matrix
Def.: An identity matrix is a square matrix $I \in \mathbb{R}^{n \times n}$, where: $(I)_{i,j} \begin{cases} = 1 \text{ if } i = j \\ = 0 \text{ if } i \neq j \end{cases}$ ($\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, e.g.)
(*) Note: $I\vec{v} = \vec{v} \ \forall \vec{v} \in \mathbb{R}^n$

Definition: Determinant
Def.: Every square matrix $A \in \mathbb{R}^{n \times n}$ is associated with a number $\det(A) \in \mathbb{R}$, known as the determinant of A (i.e. there exists a function $\det: \mathbb{R}^{n \times n} \to \mathbb{R}$). Notably, the determinant can can be used to determine if $null(A) = \{\vec{0}\}$ (i.e if $\exists$ a solution $\vec{v} \neq \vec{0}$ to the equation $A\vec{v} = 0$).

(*) Computing the Determinant
Def.: Given square matrix $A \in \mathbb{R}^{n \times n}$, $i, j \in \{1, \dots, n\}$, we define the ij-cofactor matrix of A to be the matrix $\tilde{A}_{ij} \in \mathbb{R}^{(n-1) \times (n-1)}$, obtained from removing the i^{th} row and j^{th} column of A.

→ We can use so-called cofactor expansion to compute the determinant of A as follows:

$$\boxed{\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}),}$$ for any $1 \leq j \leq n$ (*: can also expand by columns, instead)

→ the determinant of a 1×1 matrix is just the contained value.

(*) Notes on Determinants:
Given a matrix $A \in \mathbb{R}^{n \times n}$, $null(A) = \{\vec{0}\}$ iff $\det(A) \neq 0$.

Also: (1) $\det(I_{n \times n}) = 1$
(2) $\det(\alpha A)$ $[\alpha \in \mathbb{R}] = \alpha^n A$

Review: Eigenvalues and Eigenvectors

5/19/23
Lecture 20
(cont.)

Eigenvalues and Eigenvectors (§5.5)

Def.: Given square matrix $A \in \mathbb{R}^{n \times n}$ and constant $\lambda \in \mathbb{R}$, we say that λ is an eigenvalue for A if $\exists$ nonzero vector $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$. Such a vector $\vec{v}$ is called an eigenvector of A associated with λ.

(*) Eigenvalue Theorem
Given $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$, the following are equivalent: $\boxed{\lambda \text{ is an eigenvalue of } A \Leftrightarrow \det(A - \lambda I) = 0}$

The Characteristic Polynomial

Def.: Given square matrix $A \in \mathbb{R}^{n \times n}$, the characteristic polynomial of A is the polynomial $p(\lambda) := \det(A - \lambda I)$ [alt: $(\lambda I - A)$], and the equation $p(\lambda) = 0$ is the characteristic equation [of A].

$\rightarrow$ The eigenvalues of A are the zeroes of its characteristic polynomial: $\boxed{p(\lambda) = \det(A - \lambda I) = 0}$

Definition: Eigenspaces and Eigenbases

Def.: Given $A \in \mathbb{R}^{n \times n}$ with eigenvalue λ, the eigenspace of λ is the set of all eigenvectors associated with λ [+ $\vec{0}$], i.e.:

$$\boxed{E_\lambda := \text{null}(A - \lambda I)} \quad [\text{Eigenspace of } \lambda]$$

Def.: Given $A \in \mathbb{R}^{n \times n}$, an eigenbasis of A is a basis of $\mathbb{R}^n$ composed of eigenvectors of A, i.e. a set of vectors $\vec{v}_1, \dots, \vec{v}_n$ s.t.:

(1) $A\vec{v}_i = \lambda\vec{v}_i$ for some λ for each $i = 1, \dots, n$;
(2) $\mathbb{R}^n = \text{span}(v_1, \dots, v_n)$, or: $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent

(*) Forming Eigenbases
Given $A \in \mathbb{R}^{n \times n}$ with distinct eigenvalues $\lambda_1, \dots, \lambda_k$, for some $k \leq n$. If, given bases $B_1, \dots, B_k$ of the eigenspaces $E_{\lambda_1}, \dots, E_{\lambda_k}$, $|B_1| + |B_2| + \dots + |B_k| = n$ [# of vectors in bases adds up to n], then $B := B_1 \cup B_2 \cup \dots \cup B_k$ is an eigenbasis of A.

(*) If $k = n$, $B_1 \cup \dots \cup B_k$ is always an eigenbasis.

Systems of Differential Equations

5/22/23
Lecture 21

Systems of Differential Equations (§6.1)

Def.: A homogeneous linear system of differential equations (with constant coefficients) is a set of differential equations of the form:

$$\left.\begin{aligned} x_1'(t) &= a_{11}x_1(t)+a_{12}x_2(t)+\dots+a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t)+a_{22}x_2(t)+\dots+a_{2n}x_n(t) \\ &\vdots \\ x_n'(t) &= a_{n1}x_1(t)+a_{n2}x_2(t)+\dots+a_{nn}x_n(t) \end{aligned}\right] \rightarrow \boxed{\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \boxed{\rightarrow x' = Ax}}$$

→ A homogeneous linear system can thus be rewritten in the form of a matrix equation, with solutions in the form of vector-valued functions $x: \mathbb{R} \to \mathbb{R}^n$.

Proposition: Solutions

Prop.: Suppose $A \in \mathbb{R}^{n\times n}$ and $\vec{x} = \{x_1(t), x_2(t), \dots, x_k(t)\}$ are solutions to the system $x' = Ax$. Then, for every $c_1, \dots, c_k \in \mathbb{R}$:

$$c_1x_1(t) + c_2x_2(t) + \dots + c_kx_k(t) \text{ is also a solution.}$$

Definition: Linear Independence

Def.: Let $x_1(t), x_2(t), \dots, x_k(t): \mathbb{R} \to \mathbb{R}^n$ be vector-valued functions. We say that $x_1(t), x_2(t), \dots, x_k(t)$ are linearly independent if the linear combination $c_1x_1(t) + \dots + c_kx_k(t)$ equals zero ($=\vec{0}$) only when $c_1 = c_2 = \dots = c_k = 0$. Otherwise, we say that $x_1(t), x_2(t), \dots, x_k(t)$ are linearly dependent.

(*) Fact: Solutions

Suppose $x_1(t), x_2(t), \dots, x_k(t)$ are solutions to $x' = Ax$. If there is some fixed t_0 where $x_1(t_0), x_2(t_0), \dots, x_k(t_0)$ are linearly dependent, then the functions $x_1(t), x_2(t), \dots, x_k(t)$ are linearly dependent; conversely, if there is some fixed t_0 where $x_1(t_0), x_2(t_0), \dots, x_k(t_0)$ are linearly independent, then the functions $x_1(t), x_2(t), \dots, x_k(t)$ are linearly independent.

Homogeneous Linear Systems w/ Constant Coefficients

5/22/23
Lecture 21
(cont.)

Homogeneous Linear Systems with Constant Coefficients (§6.1, cont.)

Theorem: Solutions

THM.: Suppose $A \in \mathbb{R}^{n\times n}$ and $x_1(t), x_2(t), \dots, x_n(t)$ are n linearly independent solutions to the equation $x' = Ax$. Then $x_1(t), \dots, x_n(t)$ form a fundamental set of solutions, i.e. any arbitrary solution $x_0(t)$ can be expressed uniquely in the form:

$$\boxed{x_0(t) = c_1 x_1(t) + c_2 x_2(t) + \dots c_n x_n(t)} \text{ for every } t \ (c_1, \dots, c_n \in \mathbb{R}).$$

Proposition: Finding Solutions

Prop.: Suppose $A \in \mathbb{R}^{n\times n}$, λ is an eigenvalue of A, and $\vec{v}$ is an eigenvector associated with λ. Then:

$$\boxed{x(t) := e^{\lambda t}\vec{v}} \quad [\text{Solution to } x' = Ax]$$

is a solution to the system $x' = Ax$ satisfying initial condition $x(0) = \vec{v}$.

→ We can construct a fundamental set of solutions for a system $x' = Ax$ from the vectors of an eigenbasis $\mathcal{B}$ of A, i.e.

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, where each vector $\vec{v}_i$ is associated with eigenvalue λ_i.

→ Any solution $x_0(t)$ can be expressed in the form:

$$x_0(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \dots + c_n e^{\lambda_n t}\vec{v}_n.$$

Planar Systems

5/24/23
Lecture 22

Planar Systems (§6.2)

In the 2×2 system case, there are three possible classes of solutions for the characteristic polynomial:

Case 1: Distinct real roots

THM.: Suppose $A \in \mathbb{R}^{2\times 2}$ has two distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ ($\lambda_1 \neq \lambda_2$). Additionally, suppose v_1 is an eigenvector associated with λ_1, and v_2 an eigenvector associated with λ_2. Then the general solution to $x' = Ax$ is:

$$\boxed{x(t; C_1, C_2) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2}$$ [General solution: distinct real roots]

Case 2: Complex conjugate roots

THM.: Suppose $A \in \mathbb{R}^{2\times 2}$ has complex conjugate eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}$, and w is an eigenvector associated with λ. Then $\bar{w}$ is an eigenvector associated with $\bar{\lambda}$; thus, the general solution has the forms

(i) Complex: $$\boxed{x(t; C_1, C_2) = C_1 e^{\lambda t} w + C_2 e^{\bar{\lambda} t}\bar{w}}$$ [Complex-valued solution]

(ii) Real: (per Euler's) $$\boxed{\begin{aligned} x(t; C_1, C_2) = {} & C_1 e^{at}(\cos(bt)v_1 - \sin(bt)v_2) + \\ & C_2 e^{at}(\sin(bt)v_1 + \cos(bt)v_2), \end{aligned}}$$ [Real-valued solution]

where $\lambda = a + bi$ and $\vec{w} = \vec{v}_1 + i\vec{v}_2$

Case 3(i): Double real roots (easy)

THM.: Suppose $A \in \mathbb{R}^{2\times 2}$ has one eigenvalue λ (of multiplicity 2). Additionally, suppose we can find two linearly independent eigenvectors of A associated with λ. Then the general solution is $x' = Ax$ is:

$$\boxed{x(t; C_1, C_2) = C_1 e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix} + C_2 e^{\lambda t}\begin{bmatrix}0\\1\end{bmatrix}}$$ [General solution: double real roots (easy case)]

Higher-Order Linear Equations

5/26/23
Lecture 23

Planar Systems (§6.2, cont.)

Case 3(ii): Double real roots (hard)

THM.: Suppose $A \in \mathbb{R}^{2\times 2}$ has one eigenvalue λ (of multiplicity 2). Additionally, suppose we can find only one linearly independent eigenvector v_1 of A associated with λ. Then the general solution to $x' = Ax$ is:

$$\boxed{x(t; C_1, C_2) = C_1 e^{\lambda t}\vec{v}_1 + C_2 e^{\lambda t}(\vec{v}_2 + t\vec{v}_1),}$$ [General solution: double real roots (hard)]

where v_2 is any particular solution to the equation $(A - \lambda I)\vec{v}_2 = \vec{v}_1$.

Higher-Order Linear Equations (§6.3)

Given a homogeneous n^{th}-order linear equation with constant coefficients:

$$\boxed{y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = 0,}$$ with $a_1, \ldots, a_n \in \mathbb{R}$,

we can reexpress it as an $n\times n$ linear system $\underline{x' = Ax}$:

$$\boxed{\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_{(n-1)}'(t) \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}}$$ $\left[\begin{array}{l} \text{where } x_i(t) := y^{(i-1)}(t), \\ x_{(i+1)}(t) = x_i'(t) \end{array}\right]$

(*) The matrix A, in this case, is called the <u>companion matrix</u>

Theorem: Solutions

THM: Suppose $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues λ_i, with v_i being an eigenvector of A associated with λ_i. Then the general solution of the equation $\underline{x' = Ax}$ is:

$$\boxed{x(t; C_1, \ldots, C_n) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2 + \ldots + C_n e^{\lambda_n t}\vec{v}_n,}$$

where the <u>topmost row $y(t; C_1, \ldots, C_n) = x_1(t)$</u> of x is the solution to the original equation.

Higher-Order Linear Equations (cont.)

5/31/23
Lecture 24

Higher-Order Linear Equations (§6.3, cont.)

Theorem: Solutions

THM.: Consider the n^{th}-order homogeneous linear differential equation w/ constant coefficients:

$$\boxed{y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0 \quad (a_1, \dots, a_n \in \mathbb{R}),}$$

with associated linear system $x' = Ax$.

(1) The following are equivalent:

(a) $y(t)$ is a solution to the original equation

(b) the vector-valued function

$$x(t) = \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix},$$

is a solution to $x' = Ax$.

(2) Suppose $y_1(t), \dots, y_n(t)$ are solutions to the original equation. The following are equivalent:

(a) $y_1(t), \dots, y_n(t)$ are linearly independent

(b) For some t_0:

$$\det \begin{bmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

(c) For every t:

$$\det \begin{bmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Higher-Order Linear Equations (cont.)

5/31/23
Lecture 24
(cont.)

Higher-Order Linear Equations (§6.3, cont.)

Definition: Wronksian

Def.: Let $y_1, \dots, y_n : I \to \mathbb{R}$ be real-valued functions on some interval $I \subseteq \mathbb{R}$. We define the Wronksian of $y_1, \dots, y_n$ to be the function:

$$\boxed{W(t) := \det \begin{bmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{bmatrix}}$$

[Wronksian of $y_1, \dots, y_n$]

Proposition: Solutions to Higher-Order Linear Equations

Prop.: Suppose $y_1(t), \dots, y_n(t)$ are solutions to:

$$\boxed{y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_{n-1} y'(t) + a_n y(t) = 0 \quad (a_1, \dots, a_n \in \mathbb{R})}$$

Then $y_1(t), \dots, y_n(t)$ are linearly independent iff $W(t) \neq 0$ iff $W(t_0) \neq 0$ for some fixed t_0. If so, then the general solution is:

$$\boxed{y(t) = C_1 y_1(t) + C_2 y_2(t) + \dots + C_n y_n(t) \quad (C_1, \dots, C_n \in \mathbb{R})}$$