Math 32B: Calculus of Several Variables (23W)

Instructor: Konstantinos Varvarezos
Textbook: Jon Rogawski - Multivariable Calculus (4th Edition) - 2019
Topics: Integration of multivariate functions, line integrals, surfaces & surface integrals, vector fields & associated operations, Green's/Stokes'/Divergence Theorems

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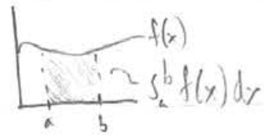
Double and Iterated Integrals

1/9/23
Lecture 1

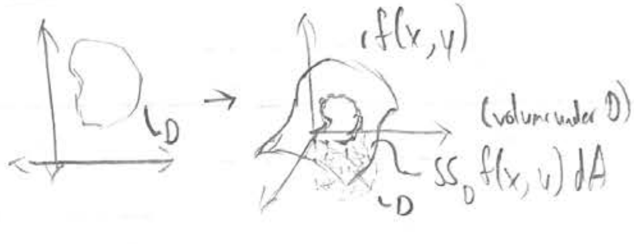
Double Integrals

$\int_a^b f(x)\,dx$ = area under f(x) on xy-plane from a to b



Double Integral

Given region D on the xy plane, the double integral of f(x,y) over D:
$\iint_D f(x,y)\,dA$ represents the volume under the surface f(x,y) in region D



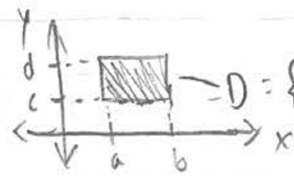
("Negative" volume (volume in regions where $f(x,y) < 0$) is counted as negative)

Iterated Integrals

An integrated integral is an expression of the following form:

$$\int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$$

is equivalent to a double integral over the region $\{(x,y) \mid a \le x \le b,\ c \le y \le d\}$
$= R = [a,b] \times [c,d]$ (notation for the rectangle)



$D = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

- Order of computation does not matter for iterated integrals

Iterated Integration for General Regions

1/11/23
Lecture 2

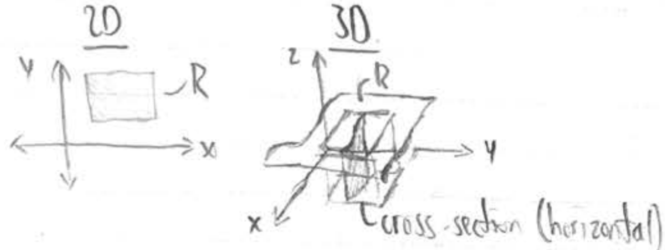
Fubini's Theorem (§16.2)

Given rectangular domain $R=[a,b]\times[c,d]$ and $z=f(x,y)$:

$$\iint_R f(x,y)\,dA = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

(*) $R=[a,b]\times[c,d]=\{(x,y)\mid a\le x\le b,\ c\le y\le d\}$

Reasoning: Each inner integral generates a cross-section of $f(x,y)$ at a given x-coordinate/y-coordinate; the integral of these cross-sections gives a volume.

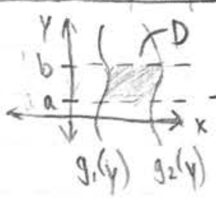


Iterated Integration for General Regions

We can use iterated integrals to calculate double integrals for two types of general region:

General Regions

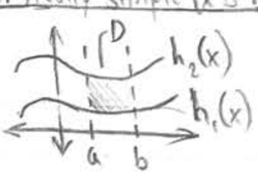
(1) Horizontally simple (y is bounded $\to a\le y\le b,\ g_1(y)\le x\le g_2(y)$)



$$\to \iint_D f(x,y)\,dA = \int_a^b\left(\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\right)dy$$

(x as a function of y)

(2) Vertically simple (x is bounded $\to a\le x\le b,\ h_1(x)\le y\le h_2(x)$)



$$\iint_D f(x,y)\,dA = \int_a^b\left(\int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\right)dx$$

(y as a function of x)

Notes on Double Integrals

1/13/23
Lecture 3

Properties of Double Integration

1) $\iint_D \lambda\, dA,\ \lambda \in \mathbb{R} = \lambda \cdot \text{area}(D)$

2) $\iint_D (f(x,y)+g(x,y))\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$

3) $\iint_D \lambda f(x,y)\, dA,\ \lambda \in \mathbb{R} = \lambda \iint_D f(x,y)\, dA$

4) Given $D = D_1 \cup D_2$, $D_1 \cap D_2 = \emptyset$ (D_1 and D_2 disjoint)*:

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

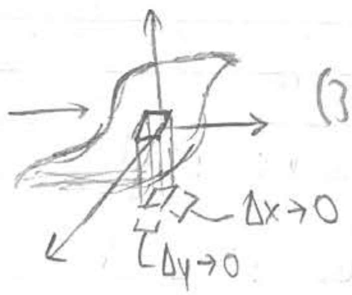
(*) Disjoint: non-overlapping

Notes

- 2D integrals can be seen as the limit of 2D Riemann Sums; similarly, in the context of 3D, 3D integrals (double integrals) can be seen as the limit of 3D Riemann Sums, i.e.:

$$\iint_R f(x,y)\, dA = \lim_{\Delta i,j \to 0} \sum_i \sum_j f(x_i, y_j)\, \Delta A_{ij}$$

(2D)



(3D)

Triple Integrals

1/18/23
Lecture 4

Triple Integrals (§16.3)
Given $f(x,y,z)$, and region W in 3D space ($\mathbb{R}^3$), we define the triple integral of $f(x,y,z)$ over W :=

$\rightarrow := \iiint_W f(x,y,z)\,dV$

(*) V = volume

Fubini's Theorem for Triple Integrals
Given region $W_0 = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ p \le q\}$ for some constants/expressions a, b, c, d, p, q, we can write the expression:

$$\iiint_{W_0} f(x,y,z)\,dV = \int_a^b \left(\int_c^d \left(\int_p^q f(x,y,z)\,dz \right) dy \right) dx$$

(*) Order of integration can be changed, per Fubini's

Notes

- Region W is z-simple if: $\begin{cases} (x,y) \in D \text{ in } \mathbb{R}^2 \\ z_1(x,y) \le z \le z_2(x,y) \text{ for some } z_1(x,y), z_2(x,y) \end{cases}$
  - Given z-simple region W: $\iiint_W f(x,y,z)\,dV = \iint_D \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dA$
  - Similar corollaries exist for "x-simple" and "y-simple" regions
- Just like double integrals, triple integrals can be seen as the limit of [4D] Riemann Sums, i.e.: $\iiint_W f(x,y,z)\,dV = \lim_{i,j,k \to 0} \sum_i \sum_j \sum_k f(x,y,z)\,\Delta V_{i,j,k}$

Polar Integration

1/20/23
Lecture 5

Properties of Triple Integration

1) $\iiint_w \lambda \, dV, \lambda \in \mathbb{R} = \lambda V$ (Constants)

2) $\iiint_w \lambda f(x,y,z) \, dV = \lambda \iiint_w f(x,y,z) \, dV$ (Scalar Multiplication)

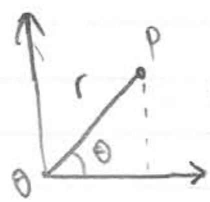
3) $\iiint_w (f(x,y,z) + g(x,y,z)) \, dV =$

$\to = \iiint_w f(x,y,z) \, dV + \iiint_w g(x,y,z) \, dV$ (Integral Sums)

4) Given region $w = w_1 \cup w_2$, $w_1 \cap w_2 = \emptyset$ (disjoint):

$\iiint_w f(x,y,z) \, dV = \iiint_{w_1} f(x,y,z) \, dV + \iiint_{w_2} f(x,y,z) \, dV$ (Disjoint Regions)

Polar Coordinates



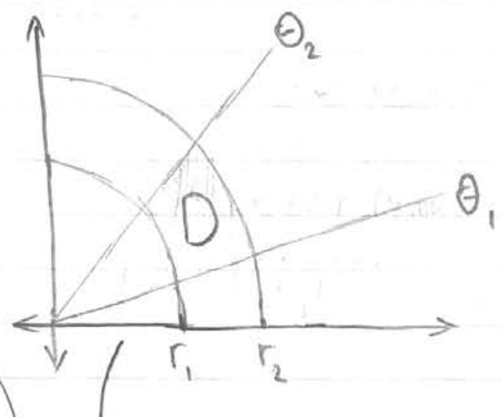
Polar coordinates: $\{(r, \theta) \mid r = |\overrightarrow{OP}|, \theta = \text{angle between x-axis and } \overrightarrow{OP}\}$

$\to x = r\cos\theta$, $y = r\sin\theta$ ($r = \sqrt{x^2+y^2}$, $\tan(\theta) = \frac{y}{x}$)

Polar Integration

Given region D in polar coordinates ($D =$
$\to = \{(r,\theta) \mid r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}$ for some expressions $r_1, r_2, \theta_1, \theta_2$, we write:

$$\iint_D f(x,y) \, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(x,y) \, r \, dr \, d\theta$$



Polar rectangle

$\left(D \begin{cases} \theta_1 \le \theta \le \theta_2 \mid \theta_1, \theta_2 \in \mathbb{R} \text{ (constant)} \\ r_1 \le r \le r_2 \mid r_1, r_2 \in \mathbb{R} \text{ (constant)} \end{cases} \right)$

(*) Polar integrals can also be thought of as the limits of Riemann Sums; however, the shape of the Riemann Sum slices differs in polar vs xy (vs), hence the inclusion of an addl. r in the expression for polar integration

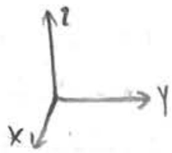
(*) Such a region D is "radially simple"

Cylindrical & Spherical Coordinates

1/23/23
Lecture 6

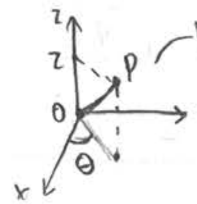
Types of 3D coordinate spaces:

1) Euclidean - x, y, z



2) Cylindrical - r, θ, z

$\rightarrow x = r\cos\theta$
$y = r\sin\theta$
$z = z$ (relative to x, y, z)



$|\overrightarrow{OP}| = r$

Given region $W: \begin{cases} \theta_1 \le \theta \le \theta_2 \\ r_1(\theta) \le r \le r_2(\theta) \\ z_1(r,\theta) \le z \le z_2(r,\theta) \end{cases}$ we can compute triple integrals in 3D space via cylindrical coordinates:

$$\rightarrow \iiint_W f(x,y,z)\,dV = \boxed{\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta}$$

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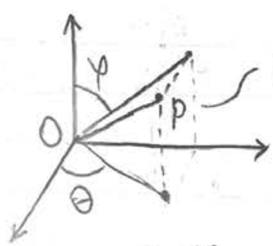
Lecture 7

3) Spherical - ρ, θ, φ

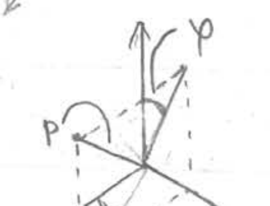
$\rightarrow x = \rho\sin(\varphi)\cos(\theta)$
$y = \rho\sin(\varphi)\sin(\theta)$
$z = \rho\cos(\varphi)$



$|\overrightarrow{OP}| = \rho$

Given region W:

$W \begin{cases} \theta_1 \le \theta \le \theta_2 \\ \varphi_1 \le \rho \le \varphi_2 \\ \rho_1(\theta,\varphi) \le \rho \le \rho_2(\theta,\varphi) \end{cases}$



We can compute triple integrals in 3D space via spherical coordinates with the formula:

$$\rightarrow \iiint_W f(x,y,z)\,dV = \boxed{\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_{\rho_1(\theta,\varphi)}^{\rho_2(\theta,\varphi)} f(\rho\sin(\varphi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), \rho\cos(\varphi))\,\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta}$$

Intro to Change of Variables

1/27/23
Lecture 8

Applications of Double & Triple Integrals (§16.5)

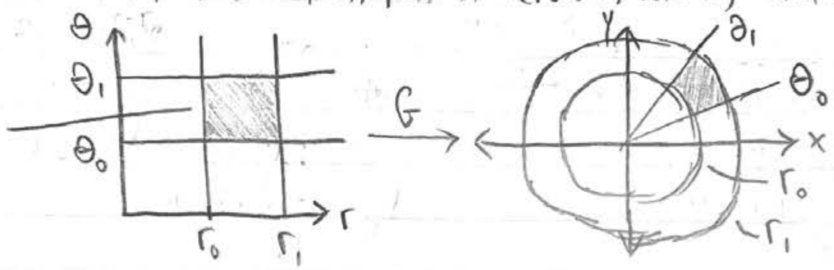
- Averages [over regions/volumes]: e.g. average of $f(x,y)$ over $D = \iint_D f(x,y)\,dA$
- Centroid of a region $D = (\bar{x}, \bar{y})$ = average of x-coords/y-coords $\in D \to \bar{x} = \frac{1}{\text{area}} \iint_D x\,dA$, $\bar{y} = \frac{1}{\text{area}} \iint_D y\,dA$
- Mass (from density): Given density function $p(x,y)$, total mass (over region D) $= \iint_D p(x,y)\,dA$
  - Center of mass: $(x_{cm}, y_{cm}) := x_{cm} = \frac{1}{\text{mass}} \iint_D x\,p(x,y)\,dA$, $y_{cm} = \frac{1}{\text{mass}} \iint_D y\,p(x,y)\,dA$
- Probability: Given probability density function $p(x,y)$ (i.e. $P(x=x_0, y=y_0) = p(x_0,y_0)$), region J s.t. $\iint_J p(x,y)\,dA = 1$,
  $\to P(a \le x \le b, c \le y \le d) = \int_a^b \int_c^d p(x,y)\,dy\,dx$ = probability that $a \le x \le b$, $c \le y \le d$ (for random x, y)

Change of Variables (§16.6)

Definition: A function $G: X \to Y$ is also called a map or a mapping (from X to Y)
- Given $x \in X$, $G(x)$ $(\in Y)$ is called the image of x
- The set of all images ($\{G(x) \mid x \in X\}$) is called the range or image of X, $G(X)$

(*) Notation: A function $G: \mathbb{R}^2 \to \mathbb{R}^2$ is written as $G(u,v) = (x(u,v), y(u,v))$
- Ex: $G(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$ (Polar $\to$ xy)

(Polar rectangle)



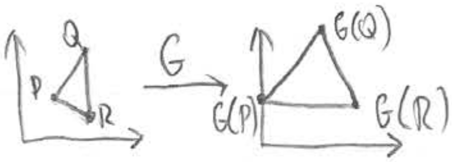
(*) Linear Maps

A map $G(u,v) \to (x(u,v), y(u,v))$ is called linear if $(x(u,v), y(u,v))$ takes the form $(Au+Bv, Cu+Dv)$ for some constants $A, B, C, D \in \mathbb{R}$

Properties of a Linear Map

1) $G(u_1+u_2, v_1+v_2) = G(u_1,v_1) + G(u_2,v_2)$
2) $G(cu, cv) = cG(u,v)$ $\forall$ constants c
3) A line between any two points $P = (u_1,v_1)$, $Q = (u_2,v_2)$ is mapped as a line between the points $G(P) = G(u_1,v_1)$ and $G(Q) = G(u_2,v_2)$

e.g.:



Change of Variables

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Lecture 9

The Jacobian

Given a map $G(u,v) \to (x(u,v), y(u,v))$, we define the Jacobian determinant (the Jacobian) of map G:

$$\mathrm{Jac}(G) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

(*) The Jacobian of a linear map $G(u,v) = (Au+Bv, Cu+Dv)$ is $AD-BC$ $\forall u,v$ (is constant)

Area and the Jacobian

Given a map G that maps a domain D to a codomain G(D), we can state that:

$$\mathrm{area}(G(D)) \approx |\mathrm{Jac}(G)(P)|\,\mathrm{area}(D)$$

(*) for some sample point $P \in D$

(i) The approximation grows more precise as $\mathrm{area}(D) \to 0$, i.e.:

$$\lim_{|D| \to 0} |\mathrm{Jac}(G)(P)|\,\mathrm{area}(D) = \mathrm{area}(G(D))$$

└ (size (diameter) of D)

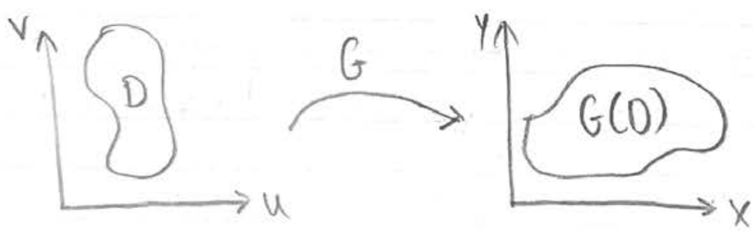
Change of Variables Formula (2D)

Given map $G(u,v) \to (x(u,v), y(u,v))$ that maps a given region D to G(D) one-to-one, we can say:

$$\iint_D f(u,v)\,dA = \iint_{G(D)} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Notes

(*) One-to-one: "$G(P) = G(Q)$" → "$P = Q$"
(*) Assumes $f(x,y)$ is continuous
(*) The act of integration sends the error of Jac(G) to zero (hence the use of "=")



Change of Variables (cont.)

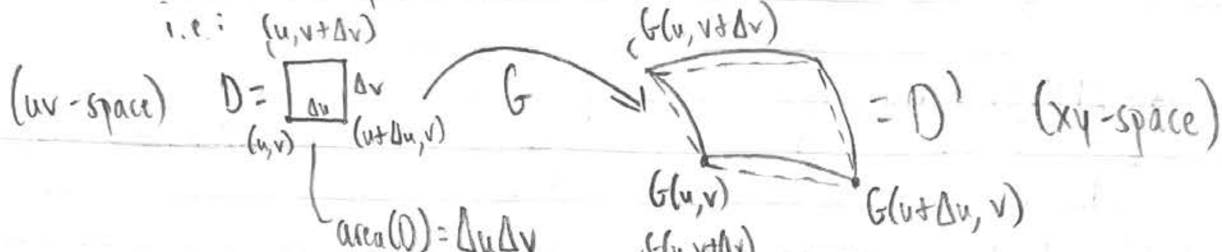
The Jacobian

2/1/23
Lecture 10

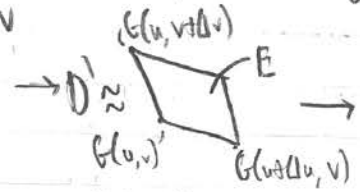
Justification for the inclusion of the term $|\mathrm{Jac}(G)|$ when changing variables:

The Jacobian accounts for the change of the area of regions (i.e. of dA) when mapped from one coordinate space to another.

i.e.:



(*) Changing variables allows us to map difficult-to-integrate regions into more integration-friendly ones (eg. an ellipse → a unit circle)



$$\rightarrow \text{area}(E) = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix} \Delta u + \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix} \Delta v = \mathrm{Jac}(G)\,\Delta u \Delta v$$

(*) Given $G: D' \to D$, $G^{-1}: D \to D'$:

$\mathrm{Jac}(G) = (\mathrm{Jac}(G^{-1}))^{-1}$ (assuming there exists a function G^{-1})

Change of Variables for Triple Integrals:

2/3/23
Lecture 11

Change of variables in 3D is virtually the same as for 2D, i.e.:

Change of Variables Formula (3D)	The Jacobian (3D)
Given map $G(u,v,w) \to (x(u,v,w), y(u,v,w), z(u,v,w))$ that maps a 3D region D to $G(D)$ one-to-one, we can say: $\iiint_{G(D)} f(x,y,z)\,dV = \iiint_D f(G(u,v,w))\,\lvert\mathrm{Jac}(G)\rvert\,dV$	$\mathrm{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ (3D)

(*) The Jacobian for maps to polar, cylindrical, and spherical coordinates are $\mathrm{Jac}(G) = r$, $\mathrm{Jac}(G) = r$, and $\mathrm{Jac}(G) = \rho^2 \sin\theta$, respectively (this can be proven via the determinant definition)

Intro to Vector Fields

2/3/23
Lecture 11

Vector Fields

Vector Fields

A vector field is a type of function that assigns to each point $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ a vector output

$$\vec{F}(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \longrightarrow$$

$$\longrightarrow = \langle F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n) \rangle$$

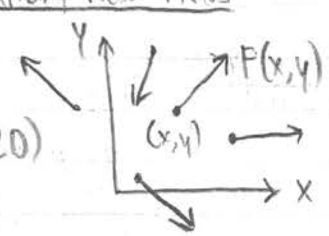
Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$

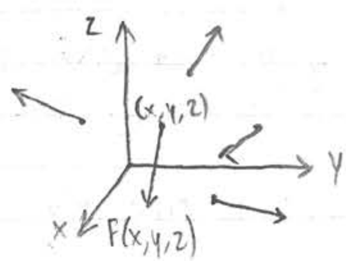
$\mathbb{R}^2$: $\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle$
$= F_1\hat{i} + F_2\hat{j}$

$\mathbb{R}^3$: $\vec{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$
$= F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Visualizing Vector Fields

(2D)





(3D)

(*) Vector fields akin to "vectors living in a plane"

Types of Vector Fields

- Constant vector field: $\vec{F}(x, y\, [, z])$ constant for all (x,y,z)
- Unit vector field: $\vec{F}(P)$ such that $\|F(P)\| = 1$ for all points P
- Radial vector field: $\vec{F}(P)$ such that $F(P)$ is pointing away from the origin ($\overrightarrow{OP} \parallel F(P)$) and $\|F(P)\|$ depends only on the distance from the origin for all points P
  - Unit radial vector field ($\mathbb{R}^2$): $\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$

Operations on Vector Fields

∇ (del operator) $= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ ("Quasi-vector of partial derivatives" (used for notation))

- ∇f (scalar function f) $= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle =$ gradient of f
- Divergence of vector field F: $\mathrm{div}(F) = \nabla \cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
- Curl of vector field $F = \mathrm{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$

Vector Fields (cont.)

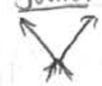
2/6/23
Lecture 12

Properties of Divergence

Given vector field $\vec{F}$ at a point P:

1) If $\text{div}(\vec{F}) > 0$ at P, we say P is a source (vectors out > vectors in)
2) If $\text{div}(\vec{F}) < 0$ at P, we say P is a sink (vectors out < vectors in)
3) If $\text{div}(\vec{F}) = 0$ at P, P is neither a source nor a sink
   (*) If $\text{div}(\vec{F}) = 0$ for all points P, we say $\vec{F}$ is incompressible

Source ($\text{div}(\vec{F}) > 0$)



Sink ($\text{div}(\vec{F}) < 0$)

Properties of Curl

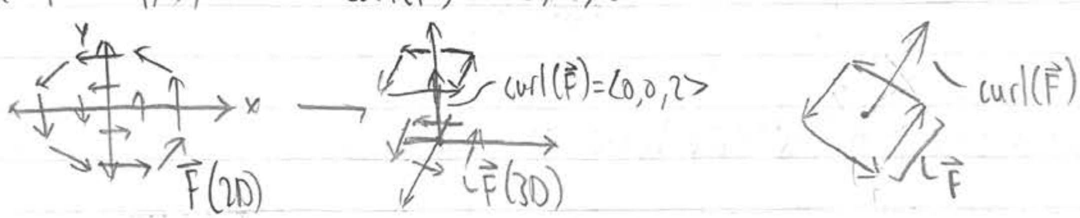
$$\boxed{\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

(*) Cross product (and thus curl) is only defined for three dimensions
(*) Curl gives a vector [field] as output

Theorem: The direction of $\text{curl}(\vec{F})$ is the direction of the axis of rotation of $\vec{F}$;*
the magnitude of $\text{curl}(\vec{F})$ is proportional to the rate of rotation of $\vec{F}$.

(*) as given by the right-hand rule

Ex: $\vec{F} = \langle -y, x, 0 \rangle \rightarrow \text{curl}(\vec{F}) = \langle 0, 0, 2 \rangle$:



Conservative Vector Fields

(Math 32A: Given scalar function $f(x,y,z)$, $\text{grad}(f) = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$ [vector field])

Conservative Vector Fields

A vector field $\vec{F}$ is conservative [a conservative vector field] if $\vec{F} = \nabla f$ for some scalar function f (called the "potential of $\vec{F}$" / "potential function for $\vec{F}$").

$\longrightarrow$ i.e. $\vec{F} = \nabla f = \langle F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z} \rangle$

A vector field $\vec{F}$ can only be conservative if it does not curl (i.e. $\text{curl}(\vec{F}) = 0$ $\forall$ points $P \in \mathbb{R}^3$)

$\rightarrow$ 2D: $\text{curl}\langle F_1, F_2, 0 \rangle = 0$

$\rightarrow$ 3D: $\text{curl}\langle F_1, F_2, F_3 \rangle = 0$

(*) 0 curl does not necessarily imply a vector field is conservative

Scalar Line Integrals

2/7/23
Lecture 13

Conservative Vector Fields

The potential function f of a vector field F can be found through integration, i.e.:

$F_1 = \frac{\partial f}{\partial x} \to \int F_1\,dx = h(x,y) + g(y) +$ ($g(y)$ = some function not dependent on x)

$\hookrightarrow F_2 = \frac{\partial f}{\partial y} = h'(x,y) + g'(y) \to \int F_2\,dy = f(x,y) + C$ (e.g.)

Line Integrals (§17.2)

A scalar line integral represents the integral of a scalar function $f: \mathbb{R}^n \to \mathbb{R}$ over a curve $C \in \mathbb{R}^n$ (i.e. area of the sheet between C and the surface of f).

Line Integral of f(x,y,z) over C

$\to = \int_C f(x,y,z)\,ds$

(*) "ds" represents arc length,

$ds \to \lim_{\Delta s \to 0} \sum_{i=1}^{N} f(P)\Delta s$

Curves for line integrals can most easily be expressed via parametrization, i.e. $C = \vec{r}(t)$ [function of time]

Computing Scalar Line Integrals

Given a curve C with parametrization $\vec{r}(t)$, $a \le t \le b$, and continuous functions $f(x,y,z)$ and $r'(t)$, the line integral of C over f is:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,\|r'(t)\|\,dt$$

(*) $ds \to \|r'(t)\|\,dt$

Vector Line Integrals

2/10/20
Lecture 14

Properties of Scalar Line Integrals

1) $\int_C 1\, ds$ = arc length of C
2) Given $C = C_1 + C_2$ ($C_1 \cup C_2 = \emptyset$) $\rightarrow \int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds$
3) The value of $\int_C f\, ds$ is independent of the choice of parametrization $\vec{r}(t)$ of C

Vector Line Integrals

Definition: Given vector field $\vec{F}$ and oriented curve (i.e. curve w/ associated direction) C, the line integral of $\vec{F}$ along C is:

$$\longrightarrow \boxed{\int_C \vec{F}\, dr = \int_C (\vec{F} \cdot \vec{T})\, ds}$$

(*) $\vec{T}$ = unit vector tangent to C, at point P
$\rightarrow$ (*) $\vec{F} \cdot \vec{T}$ = tangential component of F (at P)

- Vector line integrals, unlike scalar line integrals, depend on the direction along the curve
  - (*) Intuition: Travelling in a direction opposing the wind is more work ($\rightarrow$ larger integral) than travelling in the same direction as the wind
  - (*) The "positive" direction is defined as the direction [orientation] along C; negative = direction against

An oriented curve can be parametrized (expressed in the form $C = \vec{r}(t)$) for easier computation.
(!) The parametrization $\vec{r}(t)$ must travel in the same direction as the original curve.

$\downarrow$

Computing Vector Line Integrals

$$\int_C \vec{F}\, dr = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt^{*}$$

$$\left(C = \vec{r}(t) \;\; (t_0 \le t \le t_1) \right)$$

(*) Alt. Notation: $= \int_{t_0}^{t_1} (F_1\, dx + F_2\, dy + F_3\, dz)\, dt$

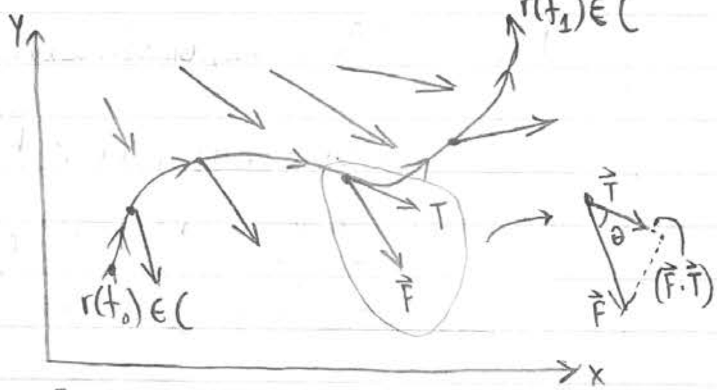


Figure: Visualization of an Oriented Curve $C = \vec{r}(t)$ in a Vector Field $\vec{F}$ (2D)

$\hookrightarrow$ Proof: $C = \vec{r}(t) \rightarrow \vec{T} = \frac{r'(t)}{\|r'(t)\|}$ $\quad ds = \|r'(t)\|\, dt$

$$\rightarrow \int_C (\vec{F} \cdot \vec{T})\, ds = \int_{t_0}^{t_1} \left(\vec{F} \cdot \frac{r'(t)}{\|r'(t)\|}\right) \|r'(t)\|\, dt = \int_{t_0}^{t_1} F \cdot r'(t)\, dt$$

Conservative Vector Fields

2/13/23
Lecture 15

Vector Line Integrals

Applications: A vector line integral can be used to calculate the work done by a particle moving along a curve C in vector field $\vec{F}$, i.e.: $\underline{W = \int_C \vec{F} \cdot d\vec{r}}$

Notations: Given an oriented curve C, we write $-C$ to denote the curve C in the opposite direction

(*) $\boxed{\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}}$

Conservative Vector Fields (§17.3)

Definition: A vector field is path-independent if $\int_C \vec{F} \cdot d\vec{r}$ is constant for all curves C with the same start and endpoints, i.e. $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ (C_1 curve $P \to Q$, C_2 curve $P \to Q$)

Fundamental Theorem of Line Integrals

Given conservative vector field $\vec{F} = \nabla f$ for some scalar function f, any curve C [path r] from point P to point Q in the domain of $\vec{F}$, we can say:

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P)}$$

(Conservative vector fields are path-independent)

(*) Conversely: On an open, connected domain, all path-independent vector fields are conservative

Definition: A curve C is closed if its start and end points are the same (i.e. $P = Q$)

Notation (Line Integrals for Closed Curves): $\int_C \vec{F} \cdot d\vec{r}$ (C closed) $= \underline{\oint_C \vec{F} \cdot d\vec{r}}$

Integrals of Closed Curves

Given a conservative vector field $\vec{F}$, the following expression is true for all closed curves C:

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = 0}$$

Conservative Vector Fields (cont.)

2/15/23
Lecture 11

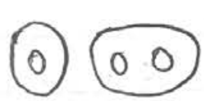
Conservative Vector Fields

(!) Note: ($\text{curl}(\vec{F}) = \vec{0}$) does not necessarily imply conservativity in all cases

(*) $\vec{F}$ cons. $\to$ $\text{curl}(\vec{F}) = \vec{0}$ $\forall \vec{F}$, but $\text{curl}(\vec{F}) = 0 \to \vec{F}$ cons. only on simply connected domains
(*) $\oint_C \vec{F} \cdot d\vec{r} = 0$ also does not imply conservativity, unless proven for all closed curves
(*) Finding a potential function works regardless of domain

Domains

Simply Connected Regions
A region D is considered simply connected if it is connected (i.e. one single region, not multiple parts) and if every loop in D [any loop] can be contracted continuously within D to a single point (i.e. D has no holes).

Ex:  $\mathbb{R}^2$ ¦  $(\mathbb{R}^2 - (0,0))$

✓ ✓ ✓ ¦ X X X

- $\text{curl}(\vec{F}) = 0$ does imply $\vec{F}$ is conservative if the domain of $\vec{F}$ is simply-connected

(*) $\text{curl}(\vec{F}) = 0$ is true of all conservative vector fields, even in non-simply-connected domains; it is only true of only conservative v.f.s on simply-connected domains

Ex: $\text{curl}(\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle) = 0$, but it is not conservative

Intro to Surfaces

2/17/23
Lecture 17

Domains

- The domain $(\mathbb{R}^2 - (0,0))$ is not simply connected $[\mathbb{R}^2 -$ a 1D figure$]$
- The domain $(\mathbb{R}^3 - (0,0,0))[\mathbb{R}^3 -$ a 1D figure$]$ is simply connected (any loop around $(0,0,0)$ can just be pulled "up" the z-axis and then contracted, e.g.)
- The domain $(\mathbb{R}^3 - \{(x,y,z) \mid x,y=0\})$ $[\mathbb{R}^3 -$ a line$]$ is not simply connected
- Generally, any domain $\mathbb{R}^n$ would require an $(n-2)$D removal to stop being simply connected
- A disconnected domain of simply connected domains can be treated like a piecewise domain
  - Ex: $D = D_1 \cup D_2 \to \vec{F} = \nabla f,\ f = \begin{cases} f_1 & (P \in D_1) \\ f_2 & (P \in D_2) \end{cases}$
- A vector field can be conservative on some parts of its domain, but not others
  - Ex: If f $(\vec{F} = \nabla f)$ is not defined on parts of the domain of F

Summary

$\vec{F}$ conservative $\leftrightarrow \vec{F} = \nabla f$

$\vec{F}$ conservative $\longleftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0\ \forall C$

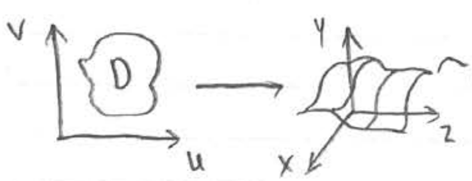
$\vec{F}$ conservative $\to \operatorname{curl}(\vec{F}) = 0$

$\operatorname{curl}(\vec{F}) = 0 \to \vec{F}$ conservative (for simply connected domains)

Surfaces (§17.4)

Parametrized Surfaces

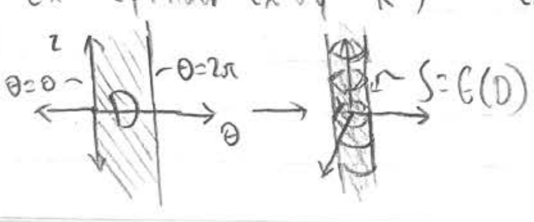
A parametrized surface is a set of points $S \subseteq \mathbb{R}^3$, which can be expressed in the form $G(u,v) = (x(u,v), y(u,v), z(u,v))$ for some region D in the uv-plane.



$\sim G(D) = S$ (Parametrization of S by G in parameter domain D)

Ex: Cylinder $(x^2+y^2=R^2) \to (x,y,z) = G(\theta, z)$

$= (R\cos\theta, R\sin\theta, z)$, $D = \begin{cases} 0 \le \theta \le 2\pi \\ -\infty \le z \le \infty \end{cases}$



Surface Integrals

2/22/23
Lecture 18

Example Surfaces

- Sphere $(x^2+y^2+z^2=R^2) \longrightarrow G(\theta, \phi) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$ $(0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi)$
- Graph of $f(x,y) \longrightarrow G(x,y) = (x, y, f(x,y))$ $(-\infty \le x, y \le \infty)$

Tangent & Normal Vectors

Given surface S parametrized by $G(u,v) = (x(u,v), y(u,v), z(u,v))$, we define the vectors tangent to S at a point $P = (u_0, v_0)$:

$$\vec{T}_u(P) = \frac{\partial G}{\partial u}(u_0, v_0) = \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right\rangle$$

$$\vec{T}_v(P) = \frac{\partial G}{\partial v}(u_0, v_0) = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right\rangle$$

(!) Significance: $\vec{T}_u$ and $\vec{T}_v$ span the tangent plane of S at P; from then, we can define:

Normal Vector of a Surface

Given tangent vectors $\vec{T}_u(P)$ and $\vec{T}_v(P)$ of a surface S at a point P, we can find the normal vector to the tangent plane of S at P, normal to the surface S:

$$\boxed{\vec{N} = \vec{N}(P) = \vec{N}(u_0, v_0) = \vec{T}_u(P) \times \vec{T}_v(P)} \quad \text{(Normal to surface } S\text{)}$$

(*) T_u, T_v, and N do not necessarily need to be unit vectors (and T_u and T_v need not be orthogonal)

Surface Integrals

A parametrization G is regular at a point P if $\vec{N}(P) \neq \vec{0}$.

* $\iint_S 1\, dS$ = surface area (S)

Surface Integrals

Given a surface S parametrized by $G(u,v)$ over some parameter domain, where G is continuously differentiable, 1:1, and regular. Then we can express the surface integral of a scalar function $f(x,y,z)$ over S as follows:

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(G(u,v)) \|\vec{N}\|\, dA} \quad \text{(Integral of } f \text{ over } S\text{)}$$

Vector Surface Integrals

2/24/23
Lecture 19

Surface Integrals for Vector Fields (§17.5)

Given a surface S, an orientation on S is a continuous choice of [direction of] unit normal vector $\hat{n}(P)$ for all points $P \in S$

(*) $\exists$ two normal directions at all points

(*) Given an orientation $\hat{n}(P)$, the opposite orientation is denoted $-\hat{n}(P)$

$\hat{n}(P)$

Vector Surface Integrals

Given an orientable and connected surface S with orientation $\hat{n}(P)$ and parametrization $G(u,v)$ on domain D + a vector field $\vec{F}$, the vector surface integral of F over S is:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \hat{n})\, dS = \iint_D (\vec{F}(G(u,v)) \cdot \vec{N}(u,v))\, dA} \quad [\vec{N} = \vec{T}_u \times \vec{T}_v]$$

Vector Surface Integrals

- The vector surface integral $\iint_S \vec{F} \cdot d\vec{S}$ represents the flux of $\vec{F}$ through S, or the integral of the normal components of $\vec{F}$ on S
- The vector surface integral is only defined for surfaces that are orientable, i.e. on which a continuous orientation $\hat{n}(P)$ can be defined
  - (*) The Möbius strip is an example of an unorientable surface

(*) Notes

- $\hat{n}(P) = \frac{\vec{N}}{\|\vec{N}\|}$ [normal vectors]
- $\iint_S (\vec{F} \cdot (-\hat{n}))\, dS = -\iint_S (\vec{F} \cdot \hat{n})\, dS$ [opposite orientation]

Green's Theorem

2/27/23
Lecture 20

Green's Theorem (§18.1)

Definition: A curve C is *simple* if it has no self-intersections.

Notation: Given a region D in the xy-plane, ∂D denotes the *boundary* of D.

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Lecture 21

Green's Theorem (2D)

Given a region D with a simple closed curve boundary ∂D (oriented counterclockwise) and vector field $\vec{F}$ with continuously differentiable components [on D]:

then: $$\boxed{\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_{\partial D} \vec{F} \cdot d\vec{r}}$$ $[2D: \vec{F} = \langle F_1, F_2 \rangle]$

↳ Can be used to convert hard-to-compute double integrals into line integrals (or vice versa)

Ex: D = unit circle $\longrightarrow \partial D: \vec{r}(t) = \langle \cos(t), \sin(t) \rangle,\ 0 \le t \le 2\pi$

$$\longrightarrow \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_0^{2\pi} (\vec{F} \cdot \vec{r}\,'(t))\, dt$$

(*) Computing Area

$$\text{Area}(D) = \iint_D 1\, dA \longrightarrow [\vec{F} = \langle 0, x \rangle] = \int_{\partial D} x\, dy \quad (\text{e.g.})$$

Domains w/ Multiple Boundaries

Given a domain D with boundary ∂D consisting of multiple simple closed curves $C_1, \dots, C_n$ ($\partial D = C_1 + \dots + C_n$), the boundary orientation on each curve is such that the region D is always on the *left* when following the orientation direction.

Ex:

(!) Green's Theorem still holds for such domains (can be expressed by decomposing/splitting D, e.g.)

Stokes' Theorem

3/3/23
Lecture 22

Green's Theorem (cont.)

Given the integral of vector field $\vec{F}$ over closed curved boundary $\partial D = \partial D_1 + \partial D_2 + \dots + \partial D_n$ for non-overlapping regions $D_1, \dots, D_n$ intersecting only on their boundaries [or not intersecting]:

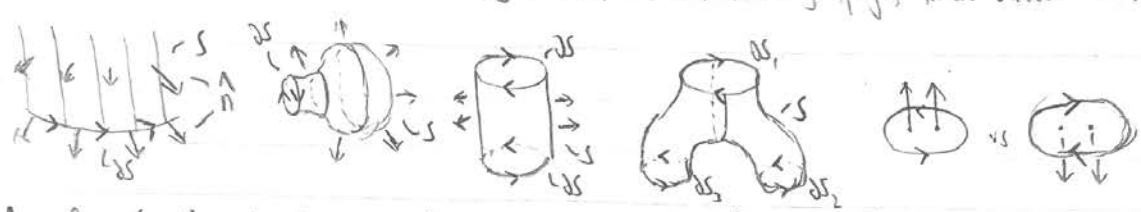
$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D_1} \vec{F} \cdot d\vec{r} + \dots + \oint_{\partial D_2} \vec{F} \cdot d\vec{r} \quad \text{[Additivity of Closed Curves]}$$

(*) ∂D should composed of only simple closed curves (no subtracting single points, e.g.)

3/6/23
Lecture 23

Stokes' Theorem (§18.2)

Definition: Given a surface S oriented by $\hat{n}$ with boundary ∂S, we define a boundary orientation of ∂S such that the surface S is on the left when following the orientation and "standing upright" in the direction of $\hat{n}$.

Ex: 

(*) A surface S with no boundary is called a closed surface $\to \partial S = \emptyset$. Ex: $\partial(\ominus) = \emptyset$
(finite)

$\downarrow$

Stokes' Theorem

Given a surface S with oriented boundary ∂S, and vector field $\vec{F}$ with continuously differentiable components $(\vec{F}(x,y,z))$ on S:

$$\boxed{\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}}$$

(*) Notes on Stokes' Theorem

- Green's Theorem represents a 2D instance of Stokes' Theorem
- If S is closed, then $\iint_S \operatorname{curl}(F) \cdot d\vec{S} = 0$
- Given two surfaces S_1, S_2 s.t. $\partial S_1 = \partial S_2$
  $\to \iint_{S_1} \operatorname{curl}(F) \cdot dS = \iint_{S_2} \operatorname{curl}(F) \cdot dS$ [Only depends on the boundary]
- Can be used to convert hard surface integrals to line integrals (or vice versa)
  (*) Can also "chain" Stokes' Theorem (e.g. hard surface $\to$ line $\to$ easier surface)

Divergence Theorem

3/8/23
Lecture 24

(*) Notes on Stokes' Theorem (cont.)

- A boundary is defined (loosely) as the edge of a surface
- Alternate formulation of Stokes' Theorem: If a vector field $\vec{F}$ can be expressed as the curl of another vector field G (i.e. F = curl(G)), then:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{G}\cdot d\vec{r}} \; [F = \text{curl}(G)] \qquad \text{(Stokes' Theorem)}$$

- Reminder: given an oriented curve C, all surfaces S with $\partial S = C$ will share the same integral of curl under Stokes' Theorem

* In the case that $\text{curl}(\vec{F}) = 0$ [e.g. for conservative vector fields], $\oint_C \vec{F}\cdot d\vec{r} = 0$ for all curves $C = \partial S$ for some surface S

* "$C = \partial S$ for some surface S" only holds true for all curves C on simply-connected regions

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Lecture 25

Divergence Theorem (§18.3)

Definition: Given a 3D region W, the boundary ∂W of W is the surface that encloses W

Ex: W =  $\to \partial W =$  $W: x^2+y^2+z^2 \le 4 \;\downarrow\; \partial W: x^2+y^2+z^2 = 4$

W:  $\to \partial W$: 

Divergence Theorem

Given 3D region W in $\mathbb{R}^3$ enclosed by surface ∂W, where ∂W is piecewise smooth and oriented with normal vectors pointing outside W, then for any vector field F(x,y,z) with partial derivatives continuous in W:

$$\boxed{\iint_{\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \text{div}(\vec{F})\, dV} \qquad [\text{Divergence Theorem}]$$

(*) The symbol $\oiint$ can be used to denote integration over a closed surface, i.e. $\iint_S \vec{F}\cdot d\vec{S} \to \oiint_S \vec{F}\cdot d\vec{S}$

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Lecture 26

(*) <u>3D Regions w/ Multiple Boundaries</u>

Given a 3D region W, if ∂W is composed of multiple surfaces (i.e. $\partial W = S_1 + S_2 + \ldots + S_n$), then we orient each surface S_i such that the normal vector $\hat{n}$ points away from W.

Ex: $W\{1 \leq x^2+y^2+z^2 \leq 2\} \rightarrow W =$ 

[Orientation]