Math 566 **Homework 2** **Due Feb. 3, 2022**
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- Problems marked NLA (Numerical Linear Algebra) are from the book.

1. (**Norms of rank one matrices**, 12 pts) Suppose $A = \mathbf{u}\mathbf{v}^T$ for some vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$. Show the following identities for the norms of A
   (a) $\|A\|_1 = \|\mathbf{u}\|_1 \|\mathbf{v}\|_\infty$
   (b) $\|A\|_\infty = \|\mathbf{u}\|_\infty \|\mathbf{v}\|_1$
   (c) $\|A\|_F = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$
   (d) $\|A\|_2 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$

2. (**Getting familiar with the SVD**, 21 pts) Consider the matrix

$$A = \begin{bmatrix} 14 & -2 \\ -8 & 19 \end{bmatrix}$$

   (a) Determine, using pen and paper (and symbolic algebra systems where needed), a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of negative signs in U and V. Your final answer should be exact and not involve any decimal values.
   (b) The MATLAB code below plots the unit ball (circle) in $\mathbb{R}^2$ and it image under multiplication by a real 2-by-2 matrix A. Use this code to plot a labeled picture of the unit circle, and its image under the matrix A given above. Include in your plot of the unit circle the two right singular vectors (columns of V) and in your plot of the image under A plot the left singular vectors (columns of U) scaled by the singular values σ_1 and σ_2, respectively.

```
% Matrix
A = [4 1; 1 3];
% Set-up the vectors for the unit circle
n = 100; z = exp(1i*(0:n)*2*pi/n);
x = [real(z);imag(z)];
% Image of the unit circle under A
y = A*x;

% Plot of the unit circle
figure(1);
plot(x(1,:),x(2,:),'b-'), axis square, title('Unit circle')

% Plot of the image
figure(2)
plot(y(1,:),y(2,:),'r-'), axis square, title('Image under $A$ circle');
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   (c) What are the two- and Frobenius-norms of A? (Use the singular values)?
   (d) Compute A^{-1} using the SVD (do not find it directly).
   (e) Find the eigenvalues λ_1, and λ_2 of A
   (f) Verify that $\det(A) = \lambda_1\lambda_2$ and $|\det(A)| = \sigma_1\sigma_2$.

(g) What is the area of the ellipsoid created by the image of the unit circle under the A?

3. (**A property of orthogonal matrices**, 8 pts) Two special types of matrices that we discussed in class and that we will use in several places in the course are orthogonal and unitary matrices:

   - A real square matrix Q is orthogonal if $Q^{-1} = Q^T$.
   - A complex square matrix is unitary if $Q^{-1} = Q^*$.

   Orthogonal and unitary matrices have the property that all of their eigenvalues, λ_j, satisfy $|\lambda_j| = 1$. This problem deals with orthogonal matrices.

   (a) Show that if a matrix Q is orthogonal and lower triangular, then it is diagonal.

   Hint: One way to proceed is to note that $QQ^T = I$, where I is the identity matrix. Perform forward substitution on this system, treating Q^T as an unknown, and show you can't get an upper triangular matrix.

   (b) What are the entries on the diagonal of Q in the case that it is diagonal?

4. (**Are they projectors?**, 15 pts) Every vector $\mathbf{x} \in \mathbb{R}^m$ can be decomposed into an even and odd part defined as follows:

$$\mathbf{x} = \underbrace{\frac{1}{2}(I+F)\mathbf{x}}_{\text{even}} + \underbrace{\frac{1}{2}(I-F)\mathbf{x}}_{\text{odd}},$$

   where I is the m-by-m identity matrix and F is the m-by-m matrix that flips $\mathbf{x} = [x_1 \quad \cdots \quad x_n]^T$ to $\mathbf{x} = [x_n \quad \cdots \quad x_1]^T$.

   (a) Let $E = \frac{1}{2}(I+F)$ be the matrix that "extracts" the even part of $\mathbf{x}$. Is E a projector? If it is, determine whether it is an orthogonal or oblique projector.

   (b) Repeat part (a), but for the matrix $T = \frac{1}{2}(I-F)$ that extracts the odd part.

   (c) What are the entries of E and T?

5. (**Norms of projectors**, 9 pts) Show that $\|P\|_2 \geq 1$ for any non-zero projector $P \in \mathbb{C}^{m\times m}$. Furthermore, show that $\|P\|_2 = 1$ only if P is an orthogonal projector.