Computational math: Assignment 1

Thanks Ting Gao for her Latex file

1.1 Let B be a 4×4 matrix to which we apply the following operations:
1.double column 1,
2.halve row 3,
3.add row 3 to row 1,
4.interchange columns 1 and 4,
5.subtract row 2 from each of the other rows,
6.replace column 4 by column 3,
7.delete column 1.
(a) Write the result as a product of eight matrices.
(b) Write it again as a product ABC of three matrices.
Solution:
(a)

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

1.4 Let $f_1, \cdots, F_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \cdots, d_8$, there exists a set of coefficients $c_1, \cdots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \quad i = 1, \cdots, 8.$$

(a) Show by appealing to the theorems of this lecture that $d_1, \cdots, d_8$ determine $c_1, \cdots, c_8$ uniquely.
(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \cdots, d_8$ to coefficients $c_1, \cdots, c_8$. What's the i, j entry of A^{-1}?

Proof. (a)Since any vector in C^8 can be linearly expressed by the eight vectors

$$\begin{bmatrix} f_j(1) \\ f_j(2) \\ \vdots \\ f_j(8) \end{bmatrix}$$

where $(j = 1, \cdots, 8)$. Thus the range of matrix

$$\begin{bmatrix} f_1(1) & f_2(1) & \cdots & f_8(1) \\ f_1(2) & f_2(2) & \cdots & f_8(2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(8) & f_2(8) & \cdots & f_8(8) \end{bmatrix}$$

contains C^8. Since this matrix belongs to $C^{8\times 8}$, the range of this matrix is C^8. Thus, it has full rank. So it has an inverse, and therefore, any vector in C^8 can be expressed uniquely.

(b)

$$A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}, \begin{bmatrix} f_1(1) & f_2(1) & \cdots & f_8(1) \\ f_1(2) & f_2(2) & \cdots & f_8(2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(8) & f_2(8) & \cdots & f_8(8) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{bmatrix}$$

We can compute the linear operator A by

$$FA = FAI = F(AI) = I,$$

where I is the identity matrix.

Hence, the i, j entry of A^{-1} is $f_j(i)$. $\square$

2.3 Let $A \in C^{m\times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in C^m$ such that $Ax = \lambda x$ for some $\lambda \in C$, the corresponding eigenvalue.
(a) Prove that all eigenvalues of A are real.
(b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Proof. (a)Since A is hermitian, we have $A^* = A$. Hence for any eigenvalue λ of A and the corresponding eigenvector x , the following equation holds:

$$\overline{\lambda}x^*x = (\lambda x)^*x = (Ax)^*x = (A^*x)^*x = x^*Ax = x^*\lambda x = \lambda x^*x.$$

Since x^*x is a nonzero number, we obtain that $\overline{\lambda} = \lambda$, which implies λ is real.

(b) Let $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$, where $\lambda_1 \neq \lambda_2$. Then

$$x^*Ay = x^*\lambda_2 y = \lambda_2 x^*y.$$

On the other hand,

$$x^*Ay = x^*A^*y = (Ax)^*y = \overline{\lambda_1}x^*y = \lambda_1 x^*y.$$

As $\lambda_1 \neq \lambda_2$, we have $x^*y = 0$, which implies that x and y are orthogonal. $\square$

2.6 If u and v are $m-$vectors, the matrix $A = I + uv^*$ is known as a rank-one perturbation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and five an expression for α. For what u and v is A singular? If it is singular, what is $null(A)$?

Proof. If A is nonsingular, then $(I + uv^*)(I + \alpha uv^*) = I$. Hence,

$$\alpha uv^* + uv^* + uv^* \alpha uv^* = 0.$$

i.e.,

$$(\alpha + 1 + \alpha v^* u)uv^* = 0.$$

If $uv^* = 0$, $A = I$ is a trivial case. Hence, the number in the bracket should be zero. i.e.,

$$\alpha = -(1 + v^* u)^{-1} \quad (v^* u \neq -1).$$

If A is singular, then $Null(A) = span\{u\}$ and $u \neq 0$. Thus $A(ku) = 0$ for any $k \in \mathbb{C}$. It implies that $(k + k(v^* u))u = 0$. Therefore $v^* u = -1$.

The other method is used the results that $det(A) = 1 + v^* u$. So A is singular if and only if $det(A) = 0$. It shows that $v^* u = -1$.

To find the null(A), we first set $Ax = 0$ for some m-vector x. Then we have $(I + uv^*)x = 0$. i.e., $uv^* x = -x$. Since $v^* u = -1$, we can obtain $x = u$. Therefore, null(A)={the space spanned by u}. $\square$

3.2 Let $\| \cdot \|$ denote any norm on C^m and also the induced matrix norm on $C^{m \times m}$. Show that $\rho(A) \leq \| A \|$, where $\rho(A)$ is the spectral radius of A, i.e., the largest absolute value λ of an eigenvalue λ of A.

Proof. Let $\rho(A) = | \lambda |$, then there exists a corresponding eigenvector x such that $Ax = \lambda x$. Then,

$$| \lambda | \cdot \| x \| = \| \lambda x \| = \| Ax \| \leq \| A \| \cdot \| x \| .$$

Since $\| x \| \neq 0$, we have

$$\rho(A) = | \lambda | \leq \| A \| .$$

$\square$

3.3 Vector and matrix p-norms are related by various inequalities, often involving the dimensions m or n. For each of the following, verify the inequality

and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem x is an m-vector and A is an $m \times n$ matrix.
(a)$\| x \|_\infty \leq \| x \|_2$,
(b)$\| x \|_2 \leq \sqrt{m} \| x \|_\infty$,
(c)$\| A \|_\infty \leq \sqrt{n} \| A \|_2$,
(d)$\| A \|_2 \leq \sqrt{m} \| A \|_\infty$.

Proof. (a)$\| x \|_\infty = \max\limits_{1 \leq i \leq m} | x_i | = \sqrt{(\max\limits_{1 \leq i \leq m} | x_i |)^2} \leq \sqrt{\sum_{i=1}^{m} | x_i |^2} = \| x \|_2$.
For example, let

$$x_{m \times 1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T,$$

Then,$\| x \|_\infty = 1$ and $\| x \|_2 = 1$.

(b)$\| x \|_2 = \sqrt{\sum_{i=1}^{m} | x_i |^2} \leq \sqrt{m \cdot (\max\limits_{1 \leq i \leq m} | x_i |)^2} = \sqrt{m} \| x \|_\infty$.
For example, let

$$x_{m \times 1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T,$$

Then,$\| x \|_\infty = 1$ and $\| x \|_2 = \sqrt{m}$.

(c)

$$\sqrt{n} \| A \|_2 = \sqrt{n} \sup_{\|x\|_2 \leq 1} \| Ax \|_2 = \sqrt{n} \sup_{\|x\|_2 \leq 1} \sqrt{\sum_{i=1}^{m} | \sum_{j=1}^{n} a_{ij} x_j |^2}$$

$$\geq \sqrt{n} \sup_{\|x\|_2 \leq 1} \sqrt{\max_{1 \leq i \leq m} | \sum_{j=1}^{n} a_{ij} x_j |^2}$$

Let

$$x = \begin{cases} \dfrac{| a_{ij} |}{\sqrt{n} a_{ij}}, & a_{ij} \neq 0, \\ 0, & a_{ij} = 0. \end{cases}$$

Hence,

$$\sqrt{n} \| A \|_2 \geq \sqrt{n} \cdot \sqrt{\max_{1 \leq i \leq m} (\sum_{j=1}^{n} | a_{ij} | \cdot \frac{1}{\sqrt{n}})^2} = \sqrt{n} \cdot \sqrt{\max_{1 \leq i \leq m} (\sum_{j=1}^{n} | a_{ij} |)^2 \cdot \frac{1}{n}} = \| A \|_\infty .$$

For example,

$$A_{m\times n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then, $\| A \|_\infty = n$ and $\| A \|_2 = \sqrt{\rho(A^*A)} = \sqrt{n}$.

(d)

$$\| A \|_2 = \sup_{\|x\|_2=1} \| Ax \|_2 = \sup_{\|x\|_2=1} \sqrt{\sum_{i=1}^m \left| \sum_{j=1}^n a_{ij}x_j \right|^2}$$

$$\le \sup_{\|x\|_2=1} \sqrt{\sum_{i=1}^m (\sum_{j=1}^n | a_{ij} |^2)(\sum_{j=1}^n | x_j |^2)} = \sqrt{\sum_{i=1}^m \sum_{j=1}^n | a_{ij} |^2}$$

$$\le \sqrt{m \cdot \max_{1\le i\le m} (\sum_{j=1}^n | a_{ij} |)^2} \le \sqrt{m} \max_{1\le i\le m} \sum_{j=1}^n | a_{ij} | = \sqrt{m} \| A \|_\infty .$$

For example,

$$A_{m\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

then, $\| A \|_2 = \sqrt{\rho(A^*A)} = \sqrt{m}$ and $\| A \|_\infty = 1$. $\square$

4.1 Determine the SVDs of the following matrices.

(b) Solution: Now we want to apply the geometric interpretation to solve this SVD problem. The image of the unit disk under the transformation of A is the ellipse with semi-axes $[0,1]^T$ and $[1,0]$ with the lengths 3 and 2 respectively. Thus we have the U and Σ as follows

$$U = [u_1\, u_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

To find V, we solve the two linear equations $Av_1 = 3u_1, Av_2 = 2u_2$. The solution is $V = [v_1\, v_2] = U$. Thus the SVD is

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = U\Sigma V^*$$

(d) Solution:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

First, find the image of a unit disk after applying the transformation A: it is a degenerate ellipse, a line segment between the points $(-\sqrt{2}, 0)$ and $(0, \sqrt{2})$. Thus, we get $u_1 = [1\ 0]^T$ and $\sigma_1 = \sqrt{2}$. Since u_2 must be perpendicular to u_1, we choose $u_2 = [0\ 1]^T$. Then we solve the linear systems $Av_1 = \sqrt{2}u_1, Av_2 = 0$, we have the orthonormal basis of $\mathbb{R}^2$

$$v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T,\ v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T.$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now we can write that

$$A = U\Sigma V^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

4.2 Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper. Do A and B have the same singular values?

Proof. Suppose

$$B_{n\times m} = [A_{m\times n}]^T \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{m\times m} \triangleq A^T C$$

and

$$A = U\Sigma V^*$$

where U and V are both unitary. Then we have

$$B = (U\Sigma V^*)^T C = \overline{V}\Sigma^T U^T C.$$

Moreover,

$$\overline{V}^*\overline{V} = V^T\overline{V} = (V^*V)^T = I^T = I$$

and

$$(U^TC)^*(U^TC) = C^*\overline{U}U^TC = C^*(UU^*)^TC = C^*C = I.$$

This implies that both $\overline{V}$ and U^TC are unitary. Moreover, since Σ and Σ^T have the same nonnegative eigenvalues, B and A have the same singular values.

The other method is to show that σ is the singular value of A if and only if σ is the singular value of B. First, we suppose that σ is the singular value of A, i.e., $Av = \sigma u$. Since $CB^T = A$, we have $Av = CB^Tv = \sigma u$. It implies that $v^TB = \sigma u^TC$ because of $C^TC = I$. Therefore σ is the singular value of B. In the other way, we can also verify the results in the same manner.

$\square$