Question 2 (a)

$P(z) = z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$.

$$A = \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix} \quad \text{for } p(z) = z^2 + c_1 z + c_0 \quad -①$$

We need to show that $p(z) = \det(zI - A)$ for general co-efficients c_1 and c_0.

$$\det(zI - A) = \det\left[z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix}\right]$$

$$= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} z & -1 \\ c_0 & z+c_1 \end{bmatrix} \right)$$

$$= z(z+c_1) + c_0$$

$$\det(zI - A) = z^2 + zc_1 + c_0$$

$$\therefore \quad P(z) = \det(zI - A)$$

Extra Credit.

In order to compute the exact solution of $Ax=b$ using GMRES we would need the minimum number of iteration to be the dimension of the Krylov subspace formed by $\{b, Ab, A^2b, \ldots, A^{k-1}b\}$

Question 5 (a)

Given $AX - XB = C$ — (i)

Where $A = S_A^{-1} \Lambda_A S_A$ and $B = S_B^{-1} \Lambda_B S_B$. — (ii)

Substituting (ii) into (i)

$$S_A^{-1} \Lambda_A S_A X - X S_B^{-1} \Lambda_B S_B = C$$

Multiply S_A from the left side

$$\underbrace{S_A S_A^{-1}}_{=I} \Lambda_A S_A X - S_A X S_B^{-1} \Lambda_B S_B = S_A C$$

$$\Lambda_A S_A X - S_A X S_B^{-1} \Lambda_B S_B = S_A C \quad \text{— (iii)}$$

Multiply (iii) by S_B^{-1} from the right

$$\Lambda_A S_A X S_B^{-1} - S_A X S_B^{-1} \Lambda_B \underbrace{S_B S_B^{-1}}_{=I} = S_A C S_B^{-1}$$

$$\Lambda_A S_A X S_B^{-1} - S_A X S_B^{-1} \Lambda_B = S_A C S_B^{-1} \quad \text{— (iv)}$$

Let:

$$\hat{C} = S_A C S_B^{-1}, \quad \hat{X} = S_A X S_B^{-1}.$$

(iv) now becomes;

$$\Lambda_A \hat{X} - \hat{X} \Lambda_B = \hat{C} \quad \text{— (v)}$$

, as required.

To solve (v), we obtain the ith column entries of each component of the equation. Thus, we can then express (v) as:

$$(\Lambda_A - \Lambda_B(i,i) I)\hat{X}_i = S_A C S_{B_i}^{-1} \quad \text{— (vi)}$$

where I is the identity matrix.

so long the left hand side i.e $(\Lambda_A - \Lambda_B(i,i))$ is not zero, then the solution of x_i will exist. Therefore $x_i = S_A^{-1} \hat{x} S_{Bi}$