Question 1

a) $\|A\|_1 = \|u\|_1 \|v\|_\infty$

By definition, $\|A\|_1 = \max_j \left[\sum_i |a_{ij}|\right]$

Let $u = [u_1 \; u_2 \cdots u_m]$ and $v^T = [v_1 \; v_2 \cdots v_m]^T$

$$\|A\|_1 = \max_{1 \le j \le m} \left(\sum_{i=1}^{m} |u_i v_j|\right)$$

$$= \max_{1 \le j \le m} |v_j| \sum_{i=1}^{m} |u_i|$$

$$= \|v\|_\infty \|u\|_1$$

b) $\|A\|_\infty = \|u\|_\infty \|v\|_1$

By definition, $\|A\|_\infty = \max_{1 \le i \le m} \left[\sum_j^{m} |a_{ij}|\right]$

$$\|A\|_\infty = \max_{1 \le i \le m} \left[\sum_{j=1}^{m} |u_i v_j|\right]$$

$$= \max_{1 \le i \le m} |u_i| \sum_{j=1}^{m} |v_j|$$

$$= \|u\|_\infty \|v\|_1$$

c) $\|A\|_F = \|u\|_2 \|v\|_2$

By definition, $\|A\|_F = \left[\sum_{i,j}^{m} |a_{ij}|^2\right]^{1/2}$

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} |u_i v_j|^2}$$

$$\le \sqrt{\sum_{i=1}^{m} |u_i|^2} \times \sqrt{\sum_{j=1}^{m} |v_j|^2} \quad (\because \text{Cauchy-Schwarz Inequality})$$

$$= \|u\|_2 \|v\|_2$$

(d)

$$\|A\|_2 = \|u\|_2 \|V\|_2$$

$$\|A\|_2 = \sigma_{max}(A) \leq \|A\|_F = \left[\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^2\right]^{1/2}$$

where σ_{max} is the maximum singular value of A.

$$\|A\|_2 = \left[\sum_{i=1}^{m} \sum_{j=1}^{m} |u_i V_j|\right]^{1/2}$$

$$\leq \left[\sum_{i=1}^{m} |u_i|^2\right]^{1/2} \cdot \left[\sum_{j=1}^{m} |V_j|^2\right]^{1/2} \quad (\because \text{Cauchy-Schwarz Inequality})$$

$$= \|u\|_2 \cdot \|v\|_2$$

Question 2

$$A = \begin{bmatrix} 4 & -2 \\ -8 & 19 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 & -8 \\ -2 & 19 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -8 & 19 \end{bmatrix} = \begin{bmatrix} 260 & -180 \\ -180 & 365 \end{bmatrix}$$

Compute the eigen values and vectors of A^TA:

$$\det(A^TA - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 260 & -180 \\ -180 & 365 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$(260-\lambda)(365-\lambda) - (180)^2 = 0$$

$$\lambda^2 - 625\lambda + 62500 = 0 \quad \lambda_1 = 500, \ \lambda_2 = 125$$

Now we compute the eigen vectors of A^TA:

$$(A^TA - \lambda I)\vec{x} = \vec{0}$$

$\lambda = 500$

$$\begin{bmatrix} -240 & -180 \\ -180 & -135 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-240x_1 = 180x_2 \Rightarrow x_1 = -\frac{180}{240}x_2 = -\frac{3}{4}x_2$$

$$\vec{x} = \begin{bmatrix} -\frac{3}{4}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$$

If $x_2 = 1$; $\vec{x} = \begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$

Compute magnitude of $\vec{x} = \sqrt{(-3/4)^2 + 1} = 5/4$

Thus $\vec{V}_1 = (5/4)^{-1}\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$

$$= \frac{4}{5}\begin{bmatrix} -3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix}$$

$\lambda = 125$

$$\begin{bmatrix} 135 & -180 \\ -180 & -230 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$135x_1 - 180x_2 = 0$; $x_2 = \frac{135x_1}{180} = \frac{3x_1}{4}$

$\vec{x} = x_1 \begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$; when $x_1 = 1$ $\vec{x} = \begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$

magnitude of $\vec{x} = \sqrt{1 + (3/4)^2} = 5/4$

Thus $\vec{V}_2 = (5/4)^{-1} \begin{bmatrix} 1 \\ 3/4 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ 3/5 \end{bmatrix}$

The vectors $\vec{V}_1$ and $\vec{V}_2$ will form the columns of matrix V.

$V = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & 3/5 \end{bmatrix}$; $V^T = \begin{bmatrix} \frac{-3}{5} & 4/5 \\ \frac{4}{5} & 3/5 \end{bmatrix}$

The diagonal matrix $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ where $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$

$$\Sigma = \begin{bmatrix} 10\sqrt{5} & 0 \\ 0 & 5\sqrt{5} \end{bmatrix}$$

Since $A = U\Sigma V^T$

$AV = U\Sigma V^T V$, $V^T V = I$ Since V and V^T are orthogonal

$AV = U\Sigma$

Thus we have :

$A\vec{V}_1 = \sigma_1 \vec{u}_1 \Rightarrow \vec{u}_1 = \frac{1}{\sigma_1} A\vec{V}_1$

$A\vec{V}_2 = \sigma_2 \vec{u}_2 \Rightarrow \vec{u}_2 = \frac{1}{\sigma_2} A\vec{V}_2$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{10\sqrt{5}} \begin{bmatrix} 14 & -2 \\ -8 & 19 \end{bmatrix} \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \frac{1}{10\sqrt{5}} \begin{bmatrix} -10 \\ 20 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{5\sqrt{5}} \begin{bmatrix} 14 & -2 \\ -8 & 19 \end{bmatrix} \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$= \frac{1}{5\sqrt{5}} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$\vec{u}_1$ and $\vec{u}_2$ form the columns of the matrix $\mathcal{U}$

$$\mathcal{U} = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$\therefore$

$$A = \mathcal{U} \Sigma V^T$$

$$A = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 10\sqrt{5} & 0 \\ 0 & 5\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{-3}{5} & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

(d)

$$A^{-1} = V\Sigma^{-1}U^T$$

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{10\sqrt{5}} & 0 \\ 0 & \frac{1}{5\sqrt{5}} \end{bmatrix}, \quad U^T = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{5}} & 0 \\ 0 & \frac{1}{5\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{19}{250} & \frac{1}{125} \\ \frac{4}{125} & \frac{7}{125} \end{bmatrix}$$

(e)

$$\det(A - I\lambda) = 0$$

$$\det\left(\begin{bmatrix} 14-\lambda & -2 \\ -8 & 19-\lambda \end{bmatrix}\right) = 0$$

$$(14-\lambda)(19-\lambda) - 16 = 0$$

$$\lambda^2 - 33\lambda + 250 = 0$$

$$\lambda_1 = \frac{33+\sqrt{89}}{2}, \quad \lambda_2 = \frac{33-\sqrt{89}}{2}$$

(f)

$$\det(A) = \lambda_1 \lambda_2$$

$$= \frac{(33+\sqrt{89})}{2} \times \frac{(33-\sqrt{89})}{2}$$

$$= \frac{33^2 + 33\sqrt{89} - 33\sqrt{89} - 89}{4} = \frac{1000}{4} = 250$$

$$|\det(A)| = \sigma_1 \sigma_2$$

$$= 10\sqrt{5} \cdot 5\sqrt{5}$$

$$= 250$$

(g)

The area of an ellipse πab, where a and b are the lengths of the semi major and minor axes. These lengths are the singular values.

$$\text{Area} = \pi \sigma_1 \sigma_2 = |\det(A)|\,\pi$$

$$= 250 \times \pi$$

Question 3

Let Q be orthogonal and lower triangular, such that it is expressed as:

$$Q = \begin{bmatrix} q_{11} & & & \\ q_{21} & q_{22} & & 0 \\ \vdots & & \ddots & \\ q_{n1} & \cdots & \cdots & q_{nn} \end{bmatrix}$$

For simplicity, let Q be in $\mathbb{R}^{3\times 3}$

$$Q = \begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & \cdot \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

Let Q^T be unknown, such that

$$Q^T = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

To solve for the unknowns in Q^T, we solve the system $QQ^T = I$ using forward substitution.

$$\begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

first column of the identity matrix

first column of Q^T

$x_{11} q_{11} = 1$

$x_{11} = 1/q_{11}$

$x_{11} q_{21} + q_{22} x_{21} = 0$

$x_{21} = \dfrac{-q_{21}}{q_{11} q_{22}} = 0$ (Suppose Q^T is upper triangular because Q is lower triangular)

Similarly, $x_{31} = 0$

$$\begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$q_{11} x_{21} = 0$

$x_{12} = 0$

$x_{12} q_{21} + q_{22} x_{22} = 1, \Rightarrow x_{22} = 1/q_{22}$

$x_{12} q_{31} + x_{22} q_{32} + q_{33} x_{32} = 0 \Rightarrow x_{32} = \dfrac{-q_{32}}{q_{33} q_{22}} = 0$ Suppose Q^T is ? (upper triangular)

Finally, for the last column of Q^T, we have;

$$\begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_{13} q_{11} = 0 \Rightarrow x_{13} = 0$

$q_{21} x_{13} + q_{22} x_{23} = 0 \Rightarrow x_{23} = 0$

$q_{31} x_{13} + q_{32} x_{23} + q_{33} x_{33} = 1 \Rightarrow x_{33} = 1/q_{33}$

In general, we observe that all the elements of Q^T are zeros except at the diagonal. Thus we can conclude that for Q^T of unknowns (ie $x_{11}, x_{12}, \cdots, x_{nn}$) and any arbitrary $Q \in \mathbb{R}^{n \times n}$

$x_{ij} = \dfrac{1}{q_{ij}}$, if $i = j$

$x_{ij} = 0$ if $i > j$

$x_{ij} = 0$ if $j > i$

Therefore $Q^T = \begin{bmatrix} 1/q_{11} & & & 0 \\ & 1/q_{22} & & \\ & & \ddots & \\ 0 & & & 1/q_{nn} \end{bmatrix}$

(3b)

The diagonal entries of Q are ± 1. The columns vectors of Q form an orthonormal set i.e each column vector has length one and is orthogonal to all the other column vectors.

<u>Question 4</u>

Let F be the n-by-n matrix that flips $x = [x_1, x_2, \dots, x_n]^T$ to $x = [x_n, x_{n-1}, \dots x_1]^T$

So,

$$F \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix}$$

It follows that F is a permutation matrix of the form

$$F = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{bmatrix}$$

F is unitary and symmetric, so $F = F^{-1}$ and $FF^{-1} = I$

Question 4

a) $E = \frac{1}{2}(I+F)$

If E is a projector, $E = E^2$

$E^2 = \frac{1}{4}(I^2 + 2IF + F^2) = \frac{1}{4}(I + 2F + F^2)$

$E^2 = \frac{1}{4}(I + 2F + FF^{-1})$ ($\because$ Since F is symmetric and unitary)

$= \frac{1}{4}(2I + 2F) = \frac{1}{2}(I+F)$

Since $E = E^2$, E is a projector

For an orthogonal projector, $E = E^*$

$E^* = \frac{1}{2}(I+F)^*$

$= \frac{1}{2}(I^* + F^*)$

$E^* = \frac{1}{2}(I+F)$

Thus E is orthogonal.

(b)

$T = \frac{1}{2}(I-F)$

$= \frac{1}{4}(I^2 - 2IF + F^2)$

$= \frac{1}{4}(I - 2F + F(F^{-1})) = \frac{1}{4}(2I - 2F)$

$T = \frac{1}{2}(I-F)$

Therefore T is a projector

$$T^* = \frac{1}{2}(I^* - F^*) \qquad (\because F = F^*)$$
$$= \frac{1}{2}(I - F)$$

Thus T is orthogonal.

(c)

Since F flips a matrix, it is a permutation matrix of the form:

$$F = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$E = \frac{1}{2}\left\{ \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \right\}$$

$$E = \begin{bmatrix} 1/2 & & 0 & & 1/2 \\ & \ddots & & \iddots & \\ 0 & 1/2 & 1 & 1/2 & 0 \\ & \iddots & & \ddots & \\ 1/2 & & 0 & & 1/2 \end{bmatrix}$$

$$T = \frac{1}{2}(I - F)$$
$$= \frac{1}{2}\left\{ \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \right\} = \begin{bmatrix} 1/2 & & & & -1/2 \\ & \ddots & & \iddots & \\ & -1/2 & 0 & 1/2 & \\ & \iddots & & \ddots & \\ -1/2 & & & & 1/2 \end{bmatrix}$$

Question 5 (a)

Show that $\|P\|_2 \geq 1$ for any non-zero projector $P \in \mathbb{C}^{m \times m}$.

For P to be a projector, $P^2 = P$. Since P is non zero, then $\|P\|_2 \neq 0$.

$$\|P\|_2 = \|P^2\|_2 \quad (\because P^2 = P)$$

$$\Rightarrow \|P\|_2^2 \geq \|P\|_2$$

$$\Rightarrow \|P\|_2 \geq 1$$

(b)

Let P be an orthogonal projector such that $P = P^*$

The SVD of $P = U\Sigma V^*$

Since $P = P^2$

$$P = P^* P = (U\Sigma V^*)^*(U\Sigma V^*)$$

$$= V\Sigma^* U^* U \Sigma V^*$$

$$P = V\Sigma^*\Sigma V^* = V\Sigma^2 V^*$$

From theorem 5.3,

$$\|P\|_2 = \|\Sigma^2\|_2 = \max\{\sigma_j^2\} = \sigma_1^2, \quad j = 1, 2, \ldots, r. \quad -(1)$$

where r is rank(P) of P.

However,

$$\|P\|_2 = \|U\Sigma V^*\|_2$$

$$= \|\Sigma\|_2$$

$$= \max\{|\sigma_j|\} = \sigma_1, \quad j = 1, 2, \ldots, r \quad -(2)$$

where σ_1 is the biggest singular value of Σ

Equating (2) and (1)

$$\sigma_1^2 = \sigma_1$$

$$\sigma_1 = 1, \quad \text{thus } \|P\|_2 = 1$$