Question 1

a)

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \quad -① $$

We apply Gaussian elimination on ① and obtain

$$\begin{bmatrix} I & A \\ 0 & A^T A \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ A^T b \end{bmatrix} \quad -② $$

from (2) we obtain the following results;

$$r + Ax = b$$

$$A^T A x = A^T b$$

$$x = (A^T A)^{-1} A^T b$$

Thus x minimizes $\|Ax - b\|_2$.

$r = b - Ax$, which is the residual.

(b)

$$A = \begin{bmatrix} I & U\Sigma V^T \\ V\Sigma^T U^T & 0 \end{bmatrix}$$

Next, let $P = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$

and $P^T A P = \underbrace{\begin{bmatrix} I & \Sigma \\ \Sigma^T & 0 \end{bmatrix}}_{\tilde{A}}$; $\Sigma^T = \Sigma$

Expressing $\tilde{A}$ as blocks of 2×2 matrices have;

$\tilde{A}_1 = \begin{bmatrix} 1 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}$, $\tilde{A}_2 = \begin{bmatrix} 1 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}$, ..., $\tilde{A}_n = \begin{bmatrix} 1 & \sigma_n \\ \sigma_n & 0 \end{bmatrix}$

If $A \in \mathbb{R}^{m\times n}$ and $m \geq n$, we shall have some extra form expressed as

$\tilde{A}_{m-n} = I_{m-n}$.

In general, we can express $\tilde{A}_i$, $i = 1, 2, \cdots, n$ in a compact form as:

$$\tilde{A}_i = \begin{bmatrix} 1 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}$$

we now compute the determinant of $\tilde{A}_i$

$$\begin{bmatrix} 1-\lambda_i & \sigma_i \\ \sigma_i & -\lambda_i \end{bmatrix} = 0$$

$$\lambda_i^2 - \lambda_i - \sigma_i^2 = 0 \Rightarrow \lambda_i = \frac{1 \pm \sqrt{1+4\sigma_i^2}}{2}, \; i = 1, 2, \cdots, n$$

$$\tilde{A}_{m-n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

determinant of $\tilde{A}_{m-n}$:

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0 \Rightarrow \lambda = 1$$

The 2-norm condition number of A is given by

$$K(A) = \frac{|\text{eigenvalue max }(A)|}{|\text{eigenvalue min }(A)|}$$

maximum eigenvalue of $A = \lambda_1 = \dfrac{1+\sqrt{1+4\sigma_1^2}}{2}$

minimum eigen value of $A = \lambda_n = \dfrac{1-\sqrt{1+4\sigma_n^2}}{2}$

$$K(A) = \frac{|1+\sqrt{1+4\sigma_1^2}|}{|1-\sqrt{1+4\sigma_n^2}|}$$

Since $1-\sqrt{1+4\sigma_n^2} < 1$, we can re-arrange terms to incorporate the absolute sign.

$$K(A) = \frac{1+\sqrt{1+4\sigma_1^2}}{\sqrt{1+4\sigma_n^2}-1}$$

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$$K(A) = \frac{1+\sqrt{1+4\sigma_1^2}}{\sqrt{1+4\sigma_n^2}-1} \quad \text{— (1)}$$

$$K(A^TA) = \frac{\sigma_1^2}{\sigma_n^2} \quad \text{— (2)}$$

By comparing (1) and (2), we can observe that (2) will always be bigger for a given σ_1 and σ_n.

Question 3

We want to show mathematically that;

$$\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ \vdots & 0 & \ddots & & & \\ \vdots & \vdots & 1 & & & \\ \vdots & \vdots & l_{i+1,i} & 1 & & \\ & & l_{i+2,i} & 0 & \ddots & \\ & & \vdots & \vdots & & \\ 0 & 0 & l_{n,i} & 0 & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ \vdots & 0 & \ddots & & & \\ \vdots & \vdots & & 1 & & \\ \vdots & \vdots & & -l_{i+1,i} & 1 & \\ 0 & 0 & & -l_{i+2,i} & 0 & \ddots & 1 \\ & & & \vdots & \vdots & & \\ & & & -l_{n,i} & 0 & & \end{bmatrix} \quad — ①$$

Let $L_K = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{K+1,K} & 1 & & \\ & & \vdots & & \ddots & \\ & & -l_{n,K} & & & 1 \end{bmatrix}$, $K = 1, 2, \dots, n$

The multipliers in L_K can be expressed as: $l_{jK} = \dfrac{a_{jK}}{a_{KK}}$, $j = K+1, \dots, n$

Let $l_K = \begin{bmatrix} 0 \\ \vdots \\ l_{K+1,K} \\ \vdots \\ l_{n,K} \end{bmatrix}$

We can then express L_K as: — ②

$L_K = I - l_K e_K$

where $e_K = \begin{bmatrix} 0 & \cdots & 0 & & & \\ \vdots & & 1 & \ddots & & \\ \vdots & & \vdots & & \ddots & \\ 0 & & 1 & & & 0 \end{bmatrix}$

from (2), It follows that $L_k^{-1} = (I + l_k e_k)$.

For (1) to hold, then $L_k L_k^{-1} = I$.

$$L_k L_k^{-1} = (I - l_k e_k)(I + l_k e_k)$$
$$= I + l_k e_k - l_k e_k - l_k \underbrace{e_k l_k}_{=0} e_k$$
$$= I - l_k e_k l_k e_k$$

But the inner product $e_k l_k = 0$ because the non-zero term in e_k get multiplied with a zero term in l_k. The non-zero terms of l_k begins from index $k+1$. Hence $e_k l_k = 0$.

Therefore:

$$L_k L_k^{-1} = I - 0 = I$$

Question 4

We apply induction to show that the growth factor of A is exactly : $g_n(A) = 2^{n-1}$ i.e for $A \in \mathbb{R}^{n \times n}$. $g_n(A) = \frac{\max_{i,j}|u_{ij}|}{\max_{i,j}|a_{ij}|}$.

Let us consider a few base cases that are easy to visualize

When $n = 2$:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow[\text{Decomposition}]{LU} U = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

$$g_2(A) = \frac{2^1}{1} = 2.$$

When $n = 3$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Decomposition}]{LU} U = \begin{bmatrix} 1 & 0 & 2^0 \\ -1 & 1 & 2^1 \\ -1 & -1 & 2^2 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$g_3(A) = \frac{2^2}{1} = 2^2$$

Let us assume that the growth factor holds for $A \in \mathbb{R}^{n \times n}$ where $n = K$

That is, $g_K(A) = 2^{K-1}$

Now we shall show that it is true for $n = K+1$.

Let $A \in \mathbb{R}^{n \times n}$ for $n = K+1$.

$$A = \begin{bmatrix} 1 & & O & 1 \\ -1 & 1 & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \\ -1 & \cdot & \cdot & \cdot\ -1\ 1 \end{bmatrix} \xrightarrow[\text{Decomposition}]{LU} U = \begin{bmatrix} 1 & O & 2^0 \\ -1 & & 2^1 \\ \cdot & \cdot & 2^2 \\ \cdot & & \cdot \\ \cdot & \cdot & \cdot \\ -1 \cdots -1 & & 2^{(K+1)-1} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & \cdots & -1 & 1 \end{bmatrix}$$

$$g_{k+1}(A) = \frac{2^{(k+1)-1}}{1} = 2^{(k-1)+1} = g_k(A) \cdot 2 = 2^k.$$

Therefore $g_n(A) = 2^{n-1}$.