Homework 4

Problem 6.3

Given $A \in C^{m\times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

If A^*A is nonsingular, its rank will be n and it has n nonzero eigenvalues. Then from Theorem 5.4, A has n nonzero singular values. So A has full rank.

Inversely if A has full rank, the number of nonzero singular values is n. Then A^*A also has n nonzero eigenvalues and does not have zero as eigenvalue. So A^*A is nonsingular.

Problem 6.4

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a) The orthogonal projector P onto range(A) is:

$$P = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} (\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} (\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

The image under P of the vector (1, 2, 3)* is:

$$y = Pv = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

b) The orthogonal projector P onto range(B) is:

$$P = B(B^*B)^{-1}B^* = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

The image under P of the vector (1, 2, 3)* is:

$$y = Pv = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Problem 6.5

$P \in C^{m \times m}$ is a nonzero projector.

- First: if $\|P\|_2 = 1$, P will be an orthogonal projector.

  Suppose $u \in Range(P), \quad v \in Null(P), \quad with \ \|v\|_2 = 1$. Make an orthogonal projection of u onto v, and then decompose u. We have: $u = r + v^*uv, \quad r = u - v^*uv$.

  So $\|u\|_2^2 = \|r\|_2^2 + \|v^*u\|_2^2$,

  $\Pr = P(u - v^*uv) = Pu = u$.

  So that $\dfrac{\|\Pr\|_2^2}{\|r\|_2^2} = \dfrac{\|u\|_2^2}{\|r\|_2^2} = \dfrac{\|r\|_2^2 + \|v^*u\|_2^2}{\|r\|_2^2} \le 1$, since $\|P\|_2 = 1$.

  Thus $\|v^*u\|_2^2 = 0$, this means that P is orthogonal.

- Then if P is an orthogonal projector, $\|P\|_2 = 1$.

  Suppose $u \in Range(P), \quad v \in Null(P), \quad w = u + v$. Then

$$\frac{\|Pw\|_2^2}{\|w\|_2^2} = \frac{\|P(u+v)\|_2^2}{\|u+v\|_2^2} = \frac{\|Pu\|_2^2}{\|u\|_2^2 + \|v\|_2^2 + u^* v + v^* u} = \frac{\|u\|_2^2}{\|u\|_2^2 + \|v\|_2^2} \le 1$$

On the other hand, $\|w\| = \|Pw\| \le \|P\| \|w\|$.

Thus we can get $\|P\|_2 = 1$.

Problem 7.1

a) Reduced QR factorization for A:

$$r_{11} = \|a_1\|_2 = \sqrt{2}, \qquad q_1 = \frac{a_1}{r_{11}} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix};$$

$$r_{12} = q_1^* a_2 = 0,$$

$$r_{22} = \|a_2 - r_{12} q_1\|_2 = 1, \qquad q_2 = \frac{a_2 - r_{12} q_1}{r_{22}} = a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Full QR factorization for A: (Assume a_3=[0 0 1]*)

$$r_{13} = q_1^* a_3 = \frac{\sqrt{2}}{2}, \qquad r_{23} = q_2^* a_3 = 0,$$

$$r_{33} = \|a_3 - r_{13} q_1 - r_{23} q_2\|_2 = \frac{\sqrt{2}}{2}, \qquad q_3 = \frac{a_3 - r_{13} q_1 - r_{23} q_2}{r_{33}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix};$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

b) Reduced QR factorization for B:

$$r_{11} = \|a_1\|_2 = \sqrt{2}, \qquad q_1 = \frac{a_1}{r_{11}} = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\0\\1\end{bmatrix};$$

$$r_{12} = q_1^{\,*} a_2 = \sqrt{2},$$

$$r_{22} = \|a_2 - r_{12}q_1\|_2 = \sqrt{3}, \qquad q_2 = \frac{a_2 - r_{12}q_1}{r_{22}} = \frac{\sqrt{3}}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix};$$

$$B = \begin{bmatrix}1 & 2\\0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ 0 & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3}\end{bmatrix}\begin{bmatrix}\sqrt{2} & \sqrt{2}\\ 0 & \sqrt{3}\end{bmatrix}.$$

Full QR factorization for A: (Assume b_3=[0 0 1]*)

$$r_{13} = q_1^{\,*} b_3 = \frac{\sqrt{2}}{2}, \qquad r_{23} = q_2^{\,*} b_3 = -\frac{\sqrt{3}}{3},$$

$$r_{33} = \|b_3 - r_{13}q_1 - r_{23}q_2\|_2 = \frac{\sqrt{6}}{6}, \qquad q_3 = \frac{b_3 - r_{13}q_1 - r_{23}q_2}{r_{33}} = \begin{bmatrix}-\frac{\sqrt{6}}{6}\\ \frac{\sqrt{6}}{3}\\ \frac{\sqrt{6}}{6}\end{bmatrix};$$

$$B = \begin{bmatrix}1 & 2\\0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3}\\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6}\end{bmatrix}\begin{bmatrix}\sqrt{2} & \sqrt{2}\\ 0 & \sqrt{3}\end{bmatrix}.$$

Problem 7.3

Give an algebraic proof of Hadamard s inequality: $|\det A| \le \prod_{j=1}^{m} \|a_j\|_2$.

Make a QR factorization of A. We get:

$$\det A = \det(QR) = \det Q \cdot \det R = \det R = \prod_{j=1}^{m} r_{jj} \ .$$

$$a_j = \sum_{i=1}^{j} r_{ij} q_j \ , \qquad \left\| a_j \right\|_2^2 = \left\| \sum_{i=1}^{j} r_{ij} q_j \right\|_2^2 = \sum_{i=1}^{j} \left\| r_{ij} \right\|_2^2 \geq \left\| r_{jj} \right\|_2^2 . \quad \textit{So}$$

$$\prod_{j=1}^{m} \left\| a_j \right\|_2^2 \geq \prod_{j=1}^{m} \left\| r_{jj} \right\|_2^2 .$$

Thus $\left| \det A \right| \leq \prod_{j=1}^{m} \left\| a_j \right\|_2$.

For m=3 case, the geometric interpretation of this result is: the volume of a parallelepiped is smaller or equal to that of a rectangular parallelepiped with the same side length. For m>3 case, it follows that: the volume of a super-parallelepiped is smaller or equal to that of a super-rectangular parallelepiped with the same side length.