

Question 1

$$a) \|A\|_1 = \|U\|_1 \|V\|_\infty$$

By definition, $\|A\|_1 = \max_j \left[\sum_i |a_{ij}| \right]$

Let $U = [u_1, u_2, \dots, u_m]$ and $V^T = [v_1, v_2, \dots, v_m]^T$

$$\|A\|_1 = \max_{1 \leq j \leq m} \left(\sum_{i=1}^m |u_i v_j| \right)$$

$$= \max_{1 \leq j \leq m} |v_j| \sum_{i=1}^m |u_i|$$

$$= \|V\|_\infty \|U\|_1$$

$$b) \|A\|_\infty = \|U\|_\infty \|V\|_1$$

By definition, $\|A\|_\infty = \max_{1 \leq i \leq m} \left[\sum_j |a_{ij}| \right]$

$$\|A\|_\infty = \max_{1 \leq i \leq m} \left[\sum_{j=1}^m |u_i v_j| \right]$$

$$= \max_{1 \leq i \leq m} |u_i| \sum_{j=1}^m |v_j|$$

$$= \|U\|_\infty \|V\|_1$$

$$c) \|A\|_F = \|U\|_2 \|V\|_2$$

By definition, $\|A\|_F = \left[\sum_{ij} |a_{ij}|^2 \right]^{1/2}$

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^m |u_i v_j|^2}$$

$$\leq \sqrt{\sum_{i=1}^m |u_i|^2} \times \sqrt{\sum_{j=1}^m |v_j|^2} \quad (\because \text{Cauchy-Schwarz Inequality})$$

$$= \|U\|_2 \|V\|_2$$

$$\|A\|_2 = \|U\|_2 \|V\|_2 \quad (4)$$

$$\|A\|_2 = \delta_{\max}(A) \leq \|A\|_F = \left[\sum_{i=1}^m \sum_{j=1}^m |a_{ij}|^2 \right]^{1/2}$$

where δ_{\max} is the maximum singular value of A .

$$\begin{aligned} \|A\|_2 &= \left[\sum_{i=1}^m \sum_{j=1}^n |u_{ij}v_{ji}| \right]^{1/2} \\ &\leq \left[\sum_{i=1}^m |u_{ii}|^2 \right]^{1/2} \cdot \left[\sum_{j=1}^n |v_{jj}|^2 \right]^{1/2} \quad (\because \text{Cauchy-Schwarz Inequality}) \\ &= \|U\|_2 \cdot \|V\|_2 \end{aligned}$$

Question 2

$$A = \begin{bmatrix} 4 & -2 \\ -8 & 19 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & -8 \\ -2 & 19 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -8 & 19 \end{bmatrix} = \begin{bmatrix} 260 & -180 \\ -180 & 365 \end{bmatrix}$$

Compute the eigen values and vectors of $A^T A$:

$$\det(A^T A - \lambda I) = 0$$

$$\det \left(\begin{bmatrix} 260 & -180 \\ -180 & 365 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

$$(260 - \lambda)(365 - \lambda) - (180)^2 = 0$$

$$\lambda^2 - 625\lambda + 62500 = 0 \quad \lambda_1 = 500, \lambda_2 = 125$$

Now we compute the eigen vectors of $A^T A$:

$$(A^T A - \lambda I) \vec{x} = \vec{0}$$

$$\lambda = 500$$

$$\begin{bmatrix} -240 & -180 \\ -180 & -135 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-240x_1 = 180x_2 \Rightarrow x_1 = -\frac{180}{240}x_2 = -\frac{3}{4}x_2$$

$$\vec{x} = \begin{bmatrix} -\frac{3}{4}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}$$

$$\text{If } x_2 = 1 ; \vec{x} = \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}$$

$$\text{Compute magnitude of } \vec{x} = \sqrt{(-\frac{3}{4})^2 + 1} = \frac{5}{4}$$

$$\text{Thus } \vec{v}_1 = \left(\frac{5}{4}\right)^{-1} \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} \\ \frac{1}{5} \end{bmatrix}$$

$$\lambda = 125$$

$$\begin{bmatrix} 135 & -180 \\ -180 & -230 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$135x_1 - 180x_2 = 0 \quad ; \quad x_2 = \frac{135x_1}{180} = \frac{3x_1}{4}$$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}; \text{ when } x_1 = 1 \quad \vec{x} = \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}$$

$$\text{magnitude of } \vec{x} = \sqrt{1 + (\frac{3}{4})^2} = \frac{5}{4}$$

$$\text{Thus } \vec{v}_2 = \left(\frac{5}{4}\right)^{-1} \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

The vectors \vec{v}_1 and \vec{v}_2 will form the columns of matrix V .

$$V = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} ; \quad V^T = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$\text{The diagonal matrix } \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \text{ where } \sigma_1 = \sqrt{\lambda_1} \text{ and } \sigma_2 = \sqrt{\lambda_2}$$

$$\Sigma = \begin{bmatrix} 10\sqrt{5} & 0 \\ 0 & 5\sqrt{5} \end{bmatrix}$$

$$\text{since } A = U\Sigma V^T, \quad V^T V = I \text{ since } V \text{ and } V^T \text{ are orthogonal}$$

$$AV = U\Sigma V^T V, \quad V^T V = I$$

$$AV = U\Sigma$$

$$\text{Thus we have: } A\vec{v}_1 = \sigma_1 \vec{u}_1 \Rightarrow \vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1$$

$$A\vec{v}_2 = \sigma_2 \vec{u}_2 \Rightarrow \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = \frac{1}{10\sqrt{5}} \begin{bmatrix} 14 & -2 \\ -8 & 19 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \frac{1}{10\sqrt{5}} \begin{bmatrix} -10 \\ 20 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2 = \frac{1}{5\sqrt{5}} \begin{bmatrix} 14 & -2 \\ -8 & 19 \end{bmatrix} \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$= \frac{1}{5\sqrt{5}} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

\vec{u}_1 and \vec{u}_2 form the columns of the matrix U

$$U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

∴

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 10\sqrt{5} & 0 \\ 0 & 5\sqrt{5} \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

④

$$A^{-1} = \sqrt{\Sigma^{-1}} U^T$$

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{10\sqrt{5}} & 0 \\ 0 & \frac{1}{5\sqrt{5}} \end{bmatrix}, \quad U^T = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{5}} & 0 \\ 0 & \frac{1}{5\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{19}{250} & \frac{1}{125} \\ \frac{4}{125} & \frac{7}{125} \end{bmatrix}$$

(e)

$$\det(A - I\lambda) = 0$$

$$\det \begin{pmatrix} 14-\lambda & -2 \\ -8 & 19-\lambda \end{pmatrix} = 0$$

$$(14-\lambda)(19-\lambda) - 16 = 0$$

$$\lambda^2 - 33\lambda + 250 = 0$$

$$\lambda_1 = \frac{33 + \sqrt{89}}{2}, \quad \lambda_2 = \frac{33 - \sqrt{89}}{2}$$

(f)

$$\det(A) = \lambda_1 \lambda_2$$

$$= \frac{(33 + \sqrt{89})}{2} \times \frac{(33 - \sqrt{89})}{2}$$

$$= \frac{33^2 + 33\sqrt{89} - 33\sqrt{89} - 89}{4} = \frac{1000}{4} = 250$$

$$|\det(A)| = \sigma_1 \sigma_2$$

$$= 10\sqrt{5} \cdot 5\sqrt{5}$$

$$= 250$$

(g)

The area of an ellipse πab , where a and b are the lengths of the semi-major and minor axes. These lengths are the singular values.

$$\text{Area} = \pi \sigma_1 \sigma_2 = |\det(A)| \pi$$

$$= 250 \pi$$

Question 3

Let Q be orthogonal and lower triangular, such that it is expressed as:

$$Q = \begin{bmatrix} q_{11} & & \\ q_{21} & q_{22} & \\ \vdots & & \\ q_{n1} & \dots & \ddots & q_{nn} \end{bmatrix}$$

For simplicity, let Q be in $\mathbb{R}^{3 \times 3}$

$$Q = \begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

Let Q^T be unknown, such that

$$Q^T = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

To solve for the unknowns in Q^T , we solve the system $QQ^T = I$ using forward substitution.

$$\begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

first column of the identity matrix

$\xrightarrow{\hspace{1cm}}$ first column of Q^T

$$x_{11}q_{11} = 1$$

$$x_{11} = 1/q_{11}$$

$$x_{11}q_{21} + q_{22}x_{21} = 0$$

$$x_{21} = -\frac{q_{21}}{q_{11}q_{22}} = 0 \quad (\text{Suppose } Q^T \text{ is upper triangular because } Q \text{ is lower triangular})$$

$$\text{Similarly, } x_{31} = 0$$

$$\begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$q_{11}x_{21} = 0$$

$$x_{12} = 0$$

$$x_{12}q_{21} + q_{22}x_{22} = 1 \Rightarrow x_{22} = 1/q_{22}$$

$$x_{12}q_{31} + x_{22}q_{32} + q_{33}x_{32} = 0 \Rightarrow x_{32} = -\frac{q_{31}}{q_{33}q_{22}} = 0 \quad \begin{array}{l} \text{Suppose } Q^T \text{ is} \\ \text{upper triangular} \end{array}$$

Finally, for the last column of Q^T , we have;

$$\begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_{13}q_{11} = 0 \Rightarrow x_{13} = 0$$

$$q_{21}x_{13} + q_{22}x_{23} = 0 \Rightarrow x_{23} = 0$$

$$q_{31}x_{13} + q_{32}x_{23} + q_{33}x_{33} = 1 \Rightarrow x_{33} = 1/q_{33}$$

In general, we observe that all the elements of Q^T are zeros except at the diagonal. Thus we can conclude that for any arbitrary $Q \in \mathbb{R}^{n \times n}$ and Q^T of unknowns (i.e. $x_{11}, x_{12}, \dots, x_{nn}$)

$$x_{ij} = \frac{1}{q_{ij}}, \text{ if } i=j$$

$$x_{ij} = 0 \quad \text{if } i > j$$

$$x_{ij} = 0 \quad \text{if } j > i$$

Therefore $Q^T = \begin{bmatrix} \frac{1}{\sqrt{q_{11}}} & \frac{1}{\sqrt{q_{22}}} & 0 \\ 0 & \ddots & \vdots \\ & & \frac{1}{\sqrt{q_{nn}}} \end{bmatrix}$

(3b)

The diagonal entries of Q are ± 1 . The column vectors of Q form an orthonormal set i.e. each column vector has length one and is orthogonal to all the other column vectors.

Question 4

Let F be the m -by- m matrix that flips $x = [x_1, x_2, \dots, x_n]^T$ to $x = [x_n, x_{n-1}, \dots, x_1]$

so,

$$F \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix}$$

It follows that F is a permutation matrix of the form

$$F = \begin{bmatrix} 0 & & & 1 \\ & \ddots & 1 & \\ & & \ddots & 0 \\ 1 & & & \end{bmatrix}$$

F is unary and symmetric, so $F = F^{-1}$ and $FF^{-1} = I$

Question 4

$$a) E = \frac{1}{2}(I+F)$$

If E is a projector, $E = E^2$

$$E^2 = \frac{1}{4}(I^2 + 2IF + F^2) = \frac{1}{4}(I + 2F + F^2)$$

$$E^2 = \frac{1}{4}(I + 2F + FF^{-1}) \quad (\because \text{since } F \text{ is unitary})$$

$$= \frac{1}{4}(2I + 2F) = \frac{1}{2}(I + F)$$

Since $E = E^2$, E is a projector

For an orthogonal projector, $F = F^*$

$$F^* = \frac{1}{2}(I+F)^*$$

$$= \frac{1}{2}(I^*+F^*)$$

$$F^* = \frac{1}{2}(I+F)$$

Thus F is orthogonal.

(b)

$$T = \frac{1}{2}(I-F)$$

$$= \frac{1}{4}(I^2 - 2IF + F^2)$$

$$= \frac{1}{4}(I - 2F + F(F^{-1})) = \frac{1}{4}(2I - 2F)$$

$$T = \frac{1}{2}(I-F)$$

Therefore T is a projector

$$T^* = \frac{1}{2}(I^* - F^*) \quad (\because F = F^*)$$

$$= \frac{1}{2}(I - F)$$

Thus T is orthogonal.

(C)

Since F flips a matrix, it is a permutation matrix of the form:

$$F = \begin{bmatrix} 0 & 0 & \dots & \dots & 1 \\ 0 & \dots & \dots & 1 & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix}$$

$$E = \frac{1}{2} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \ddots \\ 0 & \vdots \\ \vdots & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & \ddots & 0 \\ \vdots & \ddots & 0 \end{bmatrix} \right\}$$

$$E = \begin{bmatrix} \frac{1}{2} & \dots & 0 & \dots & \frac{1}{2} \\ 0 & \dots & \frac{1}{2} & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \frac{1}{2} & \dots & 0 & \dots & \frac{1}{2} \end{bmatrix}$$

$$T = \frac{1}{2}(I - F)$$

$$= \frac{1}{2} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \ddots \\ 0 & \vdots \\ \vdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & \ddots & 0 \\ \vdots & \ddots & 0 \end{bmatrix} \right\} = \begin{bmatrix} \frac{1}{2} & \dots & & & \frac{1}{2} \\ 0 & \dots & \frac{1}{2} & 0 & \frac{1}{2} \\ \vdots & \ddots & 0 & \ddots & \vdots \\ -\frac{1}{2} & \dots & 0 & \ddots & -\frac{1}{2} \end{bmatrix}$$

Question 5 @

Show that $\|P\|_2 \geq 1$ for any non-zero projector $P \in \mathbb{C}^{m \times m}$.
 For P to be a projector, $P^2 = P$. Since P is non-zero, then
 $\|P\|_2 \neq 0$.

$$\begin{aligned}\|P\|_2 &= \|P^2\|_2 \quad (\because P^2 = P) \\ &\Rightarrow \|P\|_2^2 \geq \|P\|_2 \\ &\Rightarrow \|P\|_2 \geq 1\end{aligned}$$

Let P be an orthogonal projector such that $P = P^*$
 The SVD of $P = U\Sigma V^*$

$$\text{Since } P = P^2$$

$$\begin{aligned}P &= P^*P = (U\Sigma V^*)^*(U\Sigma V^*) \\ &= V\Sigma^* U^* U\Sigma V^* \\ P &= V\Sigma^* \Sigma V^* = V\Sigma^2 V^*\end{aligned}$$

From theorem 5.3,

$$\|P\|_2 = \|\Sigma^2\|_2 = \max\{\sigma_j^2\} = \sigma_1^2, \quad j=1, 2, \dots, r. \quad -①$$

where r is rank(P) of P .

However,

$$\begin{aligned}\|P\|_2 &= \|U\Sigma V^*\|_2 \\ &= \|\Sigma\|_2 \\ &= \max\{\sigma_j\} = \sigma_1, \quad j=1, 2, \dots, r \quad -②\end{aligned}$$

where σ_1 is the biggest singular value of Σ

Equating (2) and (1)

$$\sigma_1^2 = \sigma_1$$

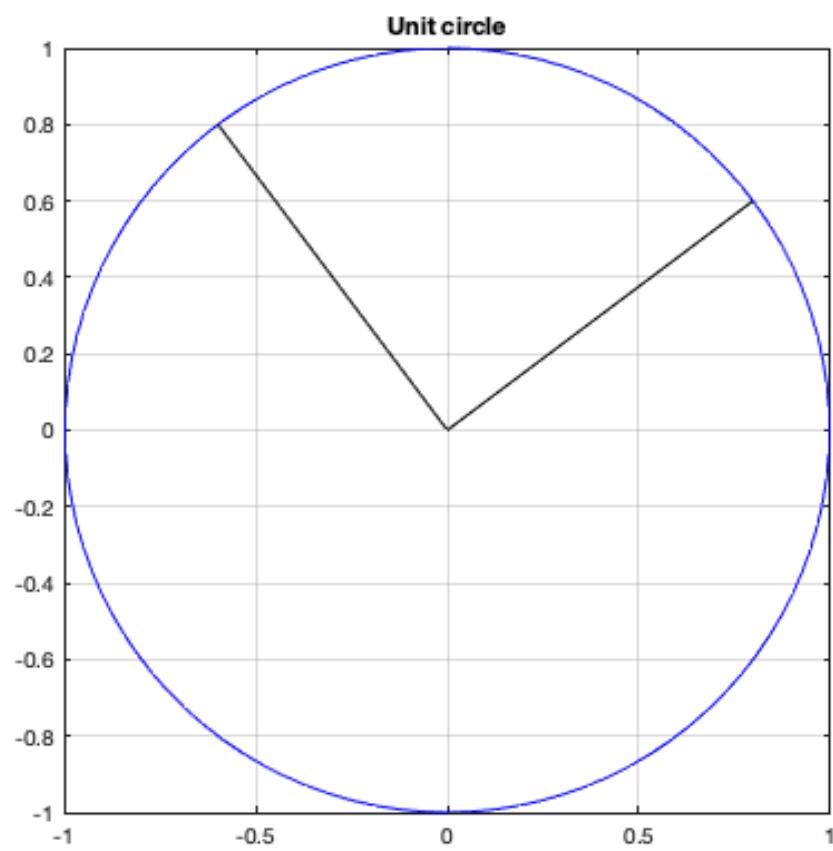
$$\sigma_1 = 1, \text{ thus } \|P\|_2 = 1$$

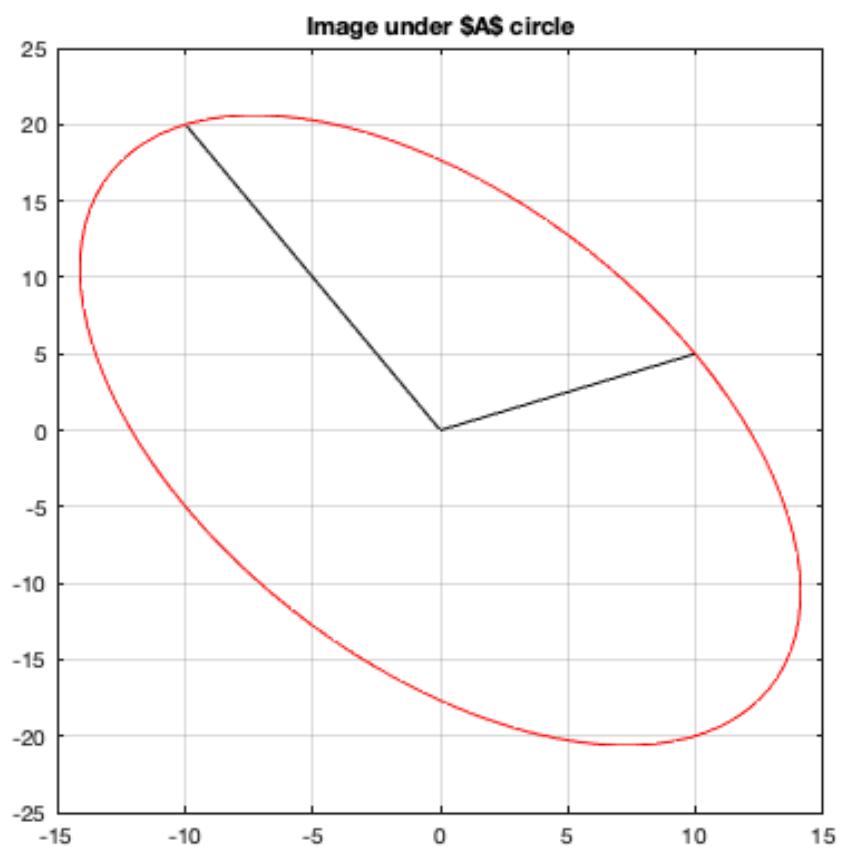
```
clear all;close all;clc

A = [14 -2; -8 19];
[U,S,V]=svd(A);

S=diag(S);
n = 100; z = exp(1i*(0:n)*2*pi/n);
x = [real(z);imag(z)];
% Image of the unit circle under A
y = A*x;
% Plot of the unit circle
figure(1);
plot(x(1,:),x(2,:),'b-'), axis square, title('Unit circle'), hold on
plot([0,V(1,1)],[0,V(2,1)],'k')
plot([0,V(1,2)],[0,V(2,2)],'k')
grid on

figure(2)
plot(y(1,:),y(2,:),'r-'), axis square, title('Image under $A$ circle'),hold on
plot([0,S(1)*U(1,1)],[0,S(1)*U(2,1)],'k')
plot([0,S(2)*U(1,2)],[0,S(2)*U(2,2)],'k')
grid on
```





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