Question 1

Let B be a 4X4 matrix to which we can carry out the following operations.

1. Double Column 1.

$$B * \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Halve row 3.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * B$$

3. Add row 3 to row 1

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * B$$

4. Interchange columns 1 and 4

$$B * \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

5. Subtract row 2 from each of the other rows

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} * B$$

6) Replace column 4 by column 3

$$B * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7) Delete column 1.

$$B * \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a)

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 2

Consider A to be an $m \times n$ matrix of rank 1. This implies that A has only one linearly independent column or row vector.

Let $A = [a_1 \; a_2 \; a_3 \; \cdots \; a_n]$, with $a_1, a_2, \ldots, a_n$ being column vectors in $\mathbb{R}^m$. Thus, we can express $a_1, a_2, \ldots, a_n$ as a linear combination of some vectors $u_1, u_2, \ldots, u_m$ and some scalars $c_1, c_2, \ldots c_m, d_1, d_2, \ldots, d_m, \ldots, g_1, g_2, \ldots, g_m$.

$$
\begin{aligned}
a_1 &= c_1 u_1 + c_2 u_2 + \cdots + c_m u_m \\
a_2 &= d_1 u_1 + d_2 u_2 + \cdots + d_m u_m \\
&\vdots \\
a_n &= g_1 u_1 + g_2 u_2 + \cdots + g_m u_m
\end{aligned}
$$

Thus, A can be expressed as:

$$A = [c_1 u_1 + c_2 u_2 + \cdots + c_m u_m \quad d_1 u_1 + d_2 u_2 + \cdots + d_m u_m \quad \cdots \quad g_1 u_1 + g_2 u_2 + \cdots + g_m u_m]$$

Let $V = \begin{vmatrix} c_1 & d_1 & \cdots & g_1 \\ c_2 & d_2 & \cdots & g_2 \\ \vdots & \vdots & & \vdots \\ c_m & d_m & \cdots & g_m \end{vmatrix}$

We can then rewrite A as:

$$A = \underbrace{[u_1 \; u_2 \; \cdots \; u_m]}_{u}[c_1 \; c_2 \cdots c_m \quad d_1 \; d_2 \cdots d_m \quad \cdots \quad g_1 \; g_2 \cdots g_m]$$

$$A = uV^T$$

Thus, $A = uV^T$ for some vectors $u, V \in \mathbb{R}^m$

Question 2b

$A = uv^T$ is a matrix of m rows and colums.
The rank of a matrix is the number of linearly independent rows or columns. Since u and v are scalar multiples of each other, the columns of A are linearly dependent. Thus A has rank one.

Question 3

(a–b)

Suppose $u, v \neq 0$ and A is non singular, then $AA^{-1} = I$ (identity)

$$(I + uv^T)(I + \beta uv^T) = I$$

$$I + \beta uv^T + uv^T + \beta uv^T uv^T = I$$

$$I + uv^T(1 + \beta + \beta v^T u) = I$$

$$uv^T(1 + \beta + \beta v^T u) = 0 \quad \cdots\cdots \quad (*)$$

$(*)$ holds true if: $1 + \beta + \beta v^T u = 0$

$$\Rightarrow \beta(1 + v^T u) = -1$$

$$\beta = -1(1 + v^T u)^{-1} \qquad \text{for } v^T u \neq -1$$

(c)

$\det(A) = 1 + v^T u$

However, A is singular ~~iff~~ if and only if $\det(A) = 0$, implying that $v^T u = -1$.

(d) Suppose A is singular, there exists a non-zero vector $x \in \mathbb{C}^m$ such that:

$$Ax = 0$$

$$(I + uv^T)x = 0 \Rightarrow uv^T x = -x \qquad (**)$$

Let α be a non zero scalar such that $\alpha \in \mathbb{C}$ and $x = \alpha u$

Substituting $(x = \alpha u)$ in $(**)$

$$uv^T(\alpha u) = -\alpha u$$

$$\alpha(v^T u)u = -\alpha u \Rightarrow v^T u = -1$$

Therefore, $\text{null}(A) = \text{span}(u)$.

Question 4 (a)

In order to compute Ax, the following steps are taken;

$$A = I + uv^T \quad ; \text{ where } u, v \in \mathbb{R}^m$$

$$Ax = Ix + u(v^T x)$$

$$Ax = x + u(v^T x) \quad - \quad \circledast$$

We evaluate how many floating point operations are required to compute the right hand side of $\circledast$

$u(v^T x)$: First, the inner product $v^T x$ requires $2m-1$ FLOPS, then the scalar resulting from $v^T x$ when multiplied by the vector u has m FLOPS. In total, we get $2m-1+m = 3m-1$ FLOPS.

$x + u(v^T x)$: The sum operation between the two vectors has m FLOPS.

In general, we have : $(m+3m-1)$ FLOPS $= 4m-1 = O(m)$.

(b)

What are the diagonal entries of A?

$$A = I + uv^T$$

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$, $v^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}$$

$$A = \begin{bmatrix} 1+u_1v_1 & u_1v_2 & \cdots & u_1v_m \\ u_2v_1 & 1+u_2v_2 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \cdots & 1+u_mv_m \end{bmatrix}$$

Therefore, the diagonal entries of A are $1+u_1v_1, 1+u_2v_2, \dots, 1+u_mv_m$.

More generally, $\operatorname{diag}(A) = 1+u_iv_i \, , \; i = 1, 2, \dots, m$

Question 5

(a)

```
format long

A=[4,-1,2;1,2,3;-1,7,-5];

norm_1=norm(A,1); % 1- norm of A
norm_2=norm(A,2);  % 2- norm of A
norm_inf=norm(A,"inf"); % inf- norm of A
norm_frob=norm(A,"fro"); % Frobenius norm of A


spectral_radius=max(abs(eig(A)));
```

(b)

```
n=size(A,1);

if(1/sqrt(n)*norm_2 <=norm_1 && norm_1<=sqrt(n)*norm_2)
    disp("True")
end
```

```
True
```

```

if(1/sqrt(n)*norm_2 <=norm_inf && norm_inf<=sqrt(n)*norm_2)
    disp("True")
end
```

```
True
```

```

if(1/n*norm_inf <=norm_1&& norm_1<=n*norm_inf)
    disp("True")
end
```

```
True
```

```

if(norm_1 <=norm_frob&& norm_frob<=sqrt(n)*norm_2)
    disp("True")
end
```

```
True
```

```

A_norm=[norm_1,norm_2,norm_inf];
```

```
for i=1:length(A_norm)
    if(spectral_radius<=A_norm(i))
        fprintf("True \n")
    end
end
```

```
True
True
True
```

Question 6

(b)

```
lis=[2^4,2^5,2^6,2^7];
table(lis',Implementation(lis), 'VariableNames', {'matrix size','Norm of Difference'})
```

ans = 4×2 table

	matrix size	Norm of Difference
1	16	0
2	32	7.904787935331115e-14
3	64	4.298783551348606e-13
4	128	2.703615109567181e-12

```
function Diff=Implementation(lis)
n=length(lis);
Diff=zeros(n,1);
for i=1:n
    M=rand(lis(i),lis(i));
    N=rand(lis(i),lis(i));
    strass_n=strass(M,N);
    reg=M*N;
    norm_diff=norm(strass_n-reg,"inf");
    Diff(i)=norm_diff;
end

end



function c = strass(a,b)
 nmin = 16;
 [~,n] = size(a);
 if n <= nmin
```

```
    c = a*b;
else
     m = n/2; u = 1:m; v = m+1:n;
     p1 = strass(a(u,u)+a(v,v),b(u,u)+b(v,v));
     p2 = strass(a(v,u)+a(v,v),b(u,u));
     p3 = strass(a(u,u),b(u,v)-b(v,v));
     p4 = strass(a(v,v),b(v,u)-b(u,u));
     p5 = strass(a(u,u)+a(u,v),b(v,v));
     p6 = strass(a(v,u)-a(u,u),b(u,u)+b(u,v));
     p7 = strass(a(u,v)-a(v,v),b(v,u)+b(v,v));
     c = [p1+p4-p5+p7,p3+p5; p2+p4, p1-p2+p3+p6];
 end
end
```

Question 6a

$P_1 = (A_{11} + A_{22})(B_{11} + B_{22})$

$P_2 = (A_{21} + A_{22})B_{11}$

$P_3 = A_{11}(B_{12} - B_{22})$

$P_4 = A_{22}(B_{21} - B_{11})$

$P_5 = (A_{11} + A_{12})B_{22}$

$P_6 = (A_{21} - A_{11})(B_{11} + B_{12})$

$P_7 = (A_{12} - A_{22})(B_{21} + B_{22})$

$C_{11} = P_1 + P_4 - P_5 + P_7$

$C_{12} = P_3 + P_5$

$C_{21} = P_2 + P_4$

$C_{22} = P_1 - P_2 + P_3 + P_6$

$$C_{11} = (A_{11} + A_{22})(B_{11} + B_{22}) + A_{22}(B_{21} - B_{11}) - (A_{11} + A_{12})B_{22} + (A_{12} - A_{22})(B_{21} + B_{22})$$

$$= A_{11}B_{11} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{11}} - \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}}$$

$$+ A_{12}B_{21} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{22}}$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$

$$C_{12} = A_{11}(B_{12} - B_{22}) + (A_{11} + A_{12})B_{22}$$

$$= A_{11}B_{12} - A_{11}B_{22} + A_{11}B_{22} + A_{12}B_{22}$$

$$= A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = (A_{21} + A_{22})B_{11} + A_{22}(B_{21} - B_{11})$$

$$= A_{21}B_{11} + A_{22}B_{11} + A_{22}B_{21} - A_{22}B_{11}$$

$$= A_{21}B_{11} + A_{22}B_{21}$$

$$
\begin{aligned}
C_{22} &= (A_{11}+A_{22})(B_{11}+B_{22}) - (A_{21}+A_{22})B_{11} + A_{11}(B_{12}-B_{22}) + (A_{21}-A_{11})(B_{11}+B_{12}) \\
&= \cancel{A_{11}B_{11}} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{11}} + A_{22}B_{22} - \cancel{A_{21}B_{11}} - \cancel{A_{22}B_{11}} + \cancel{A_{11}B_{12}} - \cancel{A_{11}B_{22}} + \cancel{A_{21}B_{11}} \\
&\quad + A_{21}B_{12} - \cancel{A_{11}B_{11}} - \cancel{A_{11}B_{12}} \\
&= A_{21}B_{12} + A_{22}B_{22}
\end{aligned}
$$

6c

$7 \cdot 7^m - 6 \cdot 4^m < 2n^3 - n^2$, where $n = 2^m$

$7 \cdot 7^m - 3 \times 2^{m+1} < 2(2^{3m}) - 2^{2m}$

$7 \cdot 7^m - 3 \times 2^{2m+1} < 2^{3m+1} - 2^{2m}$ — (*)

Solving (*) with wolfram Alpha, we obtain $m \geq 10$. Thus, $n = 2^{10} = 1024$.

Strassen's algorithm is practical for large n. i.e $n \geq 1024$.

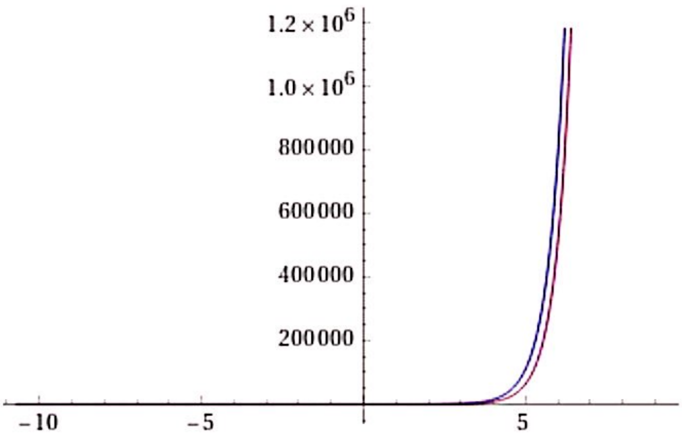
Input

$7 \times 7^m - 6 \times 4^m < 2\left(2^m\right)^3 - \left(2^m\right)^2$

Result

$7^{m+1} - 3 \times 2^{2m+1} < 2^{3m+1} - 2^{2m}$

Inequality plot



Enlarge | Data | Customize

Alternate forms

$7^{m+1} < 4^m\left(2^{m+1} + 5\right)$

$7^{m+1} < 5 \times 2^{2m} + 2^{3m+1}$

$7^{m+1} - 3 \times 2^{2m+1} < 2^{2m}\left(2^{m+1} - 1\right)$

Number line

Real solutions

Exact forms | More digits

$m < 0$

$m > 9.35321$

6d

Base case $m = 1, n = 2^1$

$7 \cdot 7^m - 6 \cdot 4^m$

$7 \cdot 7^1 - 6 \cdot 4^1 = 25$

By Analysing the Strassen's algorithm for the case of a 2x2 matrix, we indeed observe that the FLOPs count is 25. Hence we conclude that the base case is true.

Inductive Hypothesis $(m = K)$ Assume the statement is true for $m = K$

$$7 \cdot 7^K - 6 \times 4^K = A \quad - \circledast$$

Inductive Step $(m = K+1)$

$7 \cdot 7^{K+1} - 6 \cdot 4^{K+1}$

$= 7(7 \cdot 7^K) - 6 \cdot 4^{K+1}$

from $*$ $7 \cdot 7^K = A + 6 \cdot 4^K$

$= 7(A + 6 \cdot 4^K) - 4 \times 6 \cdot 4^K$

$= 7A + 6 \cdot 4^K(7-4)$

$= 7A + 3 \cdot 6 \cdot 4^K$

$= 7(7 \cdot 7^K - 6 \cdot 4^K) + 3 \cdot 6 \cdot 4^K$

$= 7^{K+1} + (3-7)\, 6 \cdot 4^K$

$= 7^{K+1} - 4 \cdot 6 \cdot 4^K = 7^{K+1} - 6 \cdot 4^{K+1}$

Therefore, the total arithmetic operation count for Strassen's algorithm is $7 \cdot 7^m - 6 \cdot 4^m$ for $\forall\, m \geq 1$