Question 2 (a)

$P(z) = z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$.

$$A = \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix} \quad \text{for } p(z) = z^2 + c_1 z + c_0 \quad -(1)$$

We need to show that $p(z) = \det(zI - A)$ for general co-efficients c_1 and c_0.

$$\det(zI - A) = \det\left[z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix}\right]$$

$$= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} z & -1 \\ c_0 & z + c_1 \end{bmatrix} \right)$$

$$= z(z + c_1) + c_0$$

$$\det(zI - A) = z^2 + z c_1 + c_0$$

$\therefore \quad P(z) = \det(zI - A)$

Extra Credit.

In order to compute the exact solution of $Ax=b$ using GMRES we would need the minimum number of iteration to be the dimension of the Krylov subspace formed by $\{b, Ab, A^2b, \dots, A^{k-1}b\}$

Question 5 ⓐ

Given $AX - XB = C$ — ①

Where $A = S_A^{-1}\Lambda_A S_A$ and $B = S_B^{-1}\Lambda_B S_B$. — ⑪

Substituting ⑪ into ①

$$S_A^{-1}\Lambda_A S_A X - X S_B^{-1}\Lambda_B S_B = C$$

Multiply S_A from the left side

$$\underbrace{S_A S_A^{-1}}_{=I}\Lambda_A S_A X - S_A X S_B^{-1}\Lambda_B S_B = S_A C$$

$$\Lambda_A S_A X - S_A X S_B^{-1}\Lambda_B S_B = S_A C \quad \text{— (iii)}$$

Multiply (iii) by S_B^{-1} from the right

$$\Lambda_A S_A X S_B^{-1} - S_A X S_B^{-1}\Lambda_B \underbrace{S_B S_B^{-1}}_{=I} = S_A C S_B^{-1}$$

$$\Lambda_A S_A X S_B^{-1} - S_A X S_B^{-1}\Lambda_B = S_A C S_B^{-1} \quad \text{— (iv)}$$

Let:

$$\hat{C} = S_A C S_B^{-1}, \quad \hat{X} = S_A X S_B^{-1}.$$

(iv) Now becomes;

$$\Lambda_A\hat{X} - \hat{X}\Lambda_B = \hat{C} \quad \text{— (v)}$$

, as required.

To solve (v), we obtain the ith column entries of each component of the equation. Thus, we can then express ⓥ as:

$$(\Lambda_A - \Lambda_B(i,i)I)\hat{X}_i = S_A C S_{B_i}^{-1} \quad \text{— (vi)}$$

where I is the identity matrix.

so long the left hand side i.e $(\Lambda_A - \Lambda_B(i,i))$ is not zero; then the solution of x_i will exist. Therefore $x_i = S_A^{-1} \hat{X} S_{B i}$

```
clc;
clear all;
format long;
```

QUESTION 1

```
tol = 1e-10;
N = 200;
A = 2*eye(N) + 0.5*randn(N)/sqrt(N);
b=ones(N,1);
```

```
[x,count,res] = gmres(A,b,tol);
```

```
Number_of_iterations = count
```

```
Number_of_iterations =
    17
```

```
Relative_residual = res
```

```
Relative_residual =
     4.001613056528999e-11
```

```
x
```

```
x = 200×1
   0.384555256382748
   0.625841897275675
   0.180404306372180
   0.373943620432330
   0.517029276633928
   0.376626553293669
   0.478962197072671
   0.596480920115364
   0.548677881907637
   0.368488253780582
       ⋮
```

QUESTION 2

```
%%% 2b %%%%%%
```

```
q = @(z) (1/16)*(231*z.^6 - 315*z.^4 + 105*z.^2 -5);
```

```
p = @(z)(16/231)*q(z)
```

```
p = function_handle with value:
    @(z)(16/231)*q(z)
```

```

```

```
syms z
X = companion_matrix(p(z))
```

```
X = 6×6
                   0   1.000000000000000                   0                   0···
                   0                   0   1.000000000000000                   0
                   0                   0                   0   1.000000000000000
                   0                   0                   0                   0
                   0                   0                   0                   0
   0.021645021645022                   0  -0.454545454545455                   0
```

```
eiggs = eig(X)
```

```
eiggs = 6×1
  -0.932469514203151
  -0.661209386466264
   0.932469514203153
   0.661209386466265
  -0.238619186083197
   0.238619186083197
```

```
%% sorted roots
sort(eiggs)
```

```
ans = 6×1
  -0.932469514203151
  -0.661209386466264
  -0.238619186083197
   0.238619186083197
   0.661209386466265
   0.932469514203153
```

```
q(eiggs) % Verify that they are actual roots
```

ans = 6×1

10^{-13} ×

```
  -0.106581410364015
   0.008881784197001
   0.044408920985006
  -0.004440892098501
  -0.006106226635438
  -0.003885780586188
```

```

```

QUESTION 3

```
%%% 3a %%%%%%%
```

```
function [v,lam,count] = rqi(A,x0,ep)
N =size(A,2);
v = x0;
I = eye(N);
lam = v'*A*v;
count = 0;
while true
    p = A - lam*I;
    w = p\v;
    v_new = w/norm(w,2);
    lam_new = v_new'*A*v_new;

    nmw = norm(A*v_new - lam_new*v_new);

    count = count + 1;
    if nmw < ep
        break
    end

    v = v_new;
    lam = lam_new;

end
end
```

```
%%% 3b %%%%%%%
```

```
%rqi(A,x0,ep)
N = 6;
a1 = -2*ones(N,1);
a2 = ones(N-1,1);
A = diag(a2,-1)+diag(a1)+diag(a2,1);
x0 =  (1/sqrt(6)).*ones(N,1);
```

```
ep = 1e-10;
[v,lam,iteration_count] = rqi(A,x0,ep)
```

```
v = 6×1
   0.231939176669697
   0.417913061018633
   0.521107370659655
   0.521107370659655
   0.417913061018633
   0.231939176669697
lam =
```

```
  -0.198062265752671
iteration_count =
     3
```

```
% verify results
```

```
[vn,d] = eig(A)
```

```
vn = 6×6
   0.231920613924330  -0.417906505941275  -0.521120889169602   0.521120889169603 ···
  -0.417906505941275   0.521120889169603   0.231920613924330   0.231920613924330
   0.521120889169603  -0.231920613924330   0.417906505941275  -0.417906505941275
  -0.521120889169603  -0.231920613924330  -0.417906505941275  -0.417906505941275
   0.417906505941275   0.521120889169602  -0.231920613924330   0.231920613924330
  -0.231920613924330  -0.417906505941275   0.521120889169602   0.521120889169602
d = 6×6
  -3.801937735804839                   0                   0                   0 ···
                   0  -3.246979603717467                   0                   0
                   0                   0  -2.445041867912629                   0
                   0                   0                   0  -1.554958132087371
                   0                   0                   0                   0
                   0                   0                   0                   0
```

```
lam_err = abs(d(end,end) - lam)
```

```
lam_err =
     1.557509121674627e-09
```

```
v_error = abs(vn(:,end)-(-v))
```

v_error = 6×1

10^{-4} ×

```
   0.185627453672121
   0.065550773586942
   0.135185099473523
   0.135185099472412
   0.065550773586942
   0.185627453672399
```

```

```

QUESTION 4

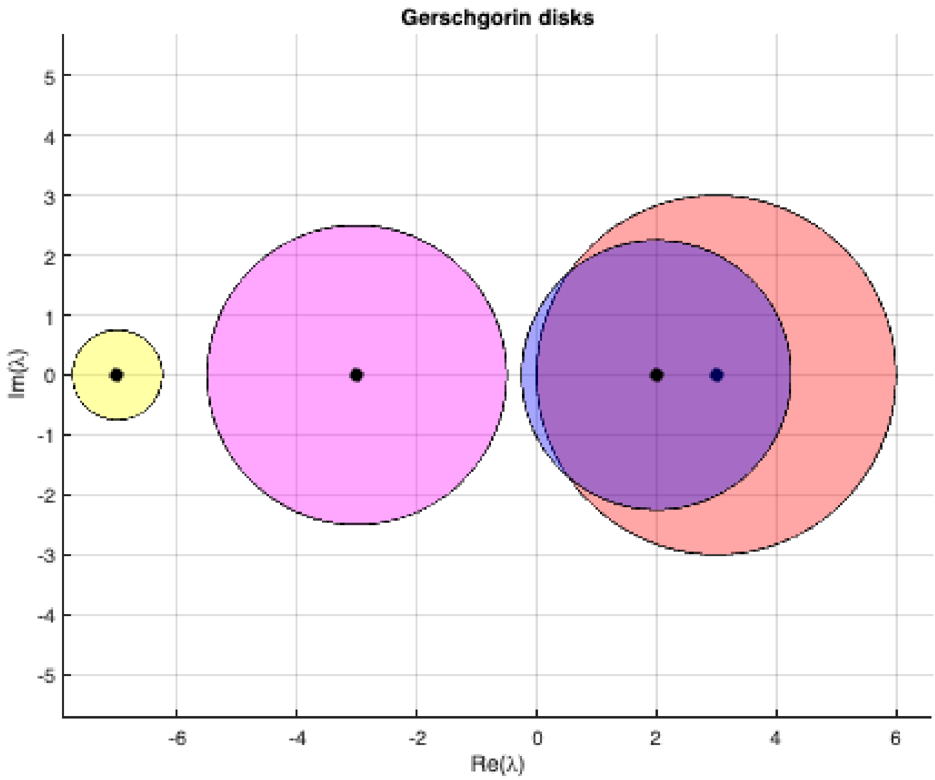
```
% set up matrix

A = [3 -1 -1 1;
    -1 2 -1/4 1 ;
    -1 -1 -3 1/2;
    -1/2 -1/4 0 -7];
```

```
%centre
c1 = A(1,1);
c2 = A(2,2);
c3 = A(3,3);
c4 = A(4,4);

%radius
r1 = 1 + 1+ 1; r2 = 1 + 1/4 +1;
r3 = 1 +1 + 1/2; r4 = 1/2 +1/4;

true_eigen_values = eig(A)
```

```
true_eigen_values = 4×1
   3.645698826149831
   1.533195532698929
  -3.279302662649521
  -6.899591696199242
```

```
close all;


figure(1)
p = nsidedpoly(1000, 'Center', [c1 0], 'Radius', r1);
plot(p, 'FaceColor', 'r')
axis equal
grid on
hold on
plot(c1,0,'k.','MarkerSize',20)
hold on
p2 = nsidedpoly(1000, 'Center', [c2 0], 'Radius', r2);
plot(p2, 'FaceColor', 'b')
axis equal
hold on
plot(c2,0,'k.','MarkerSize',20)
hold on
p3 = nsidedpoly(1000, 'Center', [c3 0], 'Radius', r3);
plot(p3, 'FaceColor', 'm')
axis equal
hold on
plot(c3,0,'k.','MarkerSize',20)
hold on
p4 = nsidedpoly(1000, 'Center', [c4 0], 'Radius', r4);
plot(p4, 'FaceColor', 'y')
axis equal
hold on
plot(c4,0,'k.','MarkerSize',20)
xlabel("Re(\lambda)"); ylabel("Im(\lambda)");
```

```
title("Gerschgorin disks")
```



According to Gerschgorin's theorem, the matrix A has eigenvalues in the union of the Gerschgorin disks $|\lambda - 3| < 3$, $|\lambda - 2| < 2.25$, $|\lambda + 3| < 2.5$ and $|\lambda + 7| < 0.75$. Since $|\lambda - 3| < 3$, $|\lambda - 2| < 2.25$ are disjoint from the other two, the Gerschgorin's theorem tells us that two of the eigenvalues must be in the two disk with the remaining two in the union of the other two disks.

QUESTION 5

```
%%% a
```

```
%%% b
```

```
SA = [4 7 -6 10 9;
    4 -6 4 9 5;
    -2 4 6 10 3;
    -4 6 3 -3 7;
    -1 8 0 6 2];

UA = zeros(5,5); UA(1,1) = 4; UA(2,2) = 1; UA(3,3) = 3; UA(4,4) = 9; UA(5,5) = 10;

SB = [6 6 -1 5;
```

```
    8 7 -6 -6;
    -3 3 -5 10;
    -6 -6 -9 -7];

UB = zeros(4,4); UB(1,1) = -7; UB(2,2) = -4; UB(3,3) = -3; UB(4,4) = -5;

C = [-9 10 6 -7;
    -8 -2 -5 3;
    -7 0 -6 5;
    -8 9 0 8;
    -4 -9 -4 5];
```

```
x= silvester(SA,SB,UA,UB,C) % solution
```

```
x = 5×4
   7.164257638512401  10.901274644160024  -0.358273182187601   2.112026215055829
   2.277357282833669   2.038219188954689   0.346773493076260  -0.755989882624110
   3.164640265518682   5.588511648918816  -1.286321203219333   2.534960382610655
  -2.473528353258586  -0.290393814006969  -1.064736389326259   2.264702503984480
  -1.340482562043512  -0.388265945321812  -0.949257360228529   1.772782710005899
```

```
function [x,count,R_norm] = gmres(A,b,tol)
m = size(A,1);
q = zeros(m,m);
h = zeros(m,m);
norm_b = norm(b);
q(:,1) = b/norm_b;

for n = 1:m
    v = A*q(:,n);
    for j = 1:n
        h(j,n) = q(:,j)'*v;
        v = v - h(j,n)*q(:,j);
    end
    h(n+1,n) = norm(v);
    q(:,n+1) = v/h(n+1,n);
    H = h(1:n+1,1:n);

    bb = norm_b*speye(n+1,1);
    y = H\bb;

    xn = q(:,1:n)*y;

    r = A*xn - b;

    R_norm = norm(r)/norm_b;

    count = n;

    if (norm(r)< norm_b*tol)
```

```
        break
    end
end

x = xn;
end


function X = companion_matrix(p)
coeff = coeffs(p,'all');
c = fliplr(coeff);

N =  polynomialDegree(p);
It = eye(N-1,N-1);

X = [zeros(N-1,1),It];
X = [X; -double(c(1:N))];
end


function [v,lam,count] = rqi(A,x0,ep)
N =size(A,2);
v = x0;
I = eye(N);
lam = v'*A*v;
count = 0;
while true
    p = A - lam*I;
    w = p\v;
    v_new = w/norm(w,2);
    lam_new = v_new'*A*v_new;

    nmw = norm(A*v_new - lam_new*v_new);

    count = count + 1;
    if nmw < ep
        break
    end

    v = v_new;
    lam = lam_new;

end
end

function x= silvester(SA,SB,UA,UB,C)

N = size(C,2);
n = size(C,1);
```

```
I = diag(diag(ones(n)));

invers_SA = inv(SA);
invers_SB = inv(SB);

xht = zeros(n,N);
for j = 1:N
    A = UA - UB(j,j)*I;
    cht = SA*C*invers_SB(:,j);
    xht(:,j) = A\cht;
end
x = zeros(n,N);
for i = 1:N
    x(:,i) = invers_SA*xht*SB(:,i);
end
end
```