Question 1

Let B be a 4X4 matrix to which we can carryout the following operations.

1. Double column 1.

$$B * \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Halve row 3.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * B$$

3. Add row 3 to row 1

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * B$$

4. Interchange columns 1 and 4

$$B * \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

5. Subtract row 2 from each of the other rows

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} * B$$

6) Replace column 4 by column 3

$$B * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7) Delete column 1.

$$B * \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a)

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 2

Consider A to be an $m \times n$ matrix of rank 1. This implies that A has only one linearly independent column or row vector.

Let $A = [a_1 \ a_2 \ a_3 \ \cdots \ a_n]$, with $a_1, a_2, \ldots, a_n$ being column vectors in $\mathbb{R}^m$. Thus, we can express $a_1, a_2, \ldots, a_n$ as a linear combination of some vectors $u_1, u_2, \ldots, u_m$ and some scalars $c_1, c_2, \ldots c_m, d_1, d_2, \ldots, d_m, \ldots, g_1, g_2, \ldots, g_m$.

$$\begin{aligned} a_1 &= c_1 u_1 + c_2 u_2 + \cdots + c_m u_m \\ a_2 &= d_1 u_1 + d_2 u_2 + \cdots + d_m u_m \\ \vdots \ & \quad \ \vdots \qquad \ \vdots \quad \cdots \quad \vdots \\ a_n &= g_1 u_1 + g_2 u_2 + \cdots + g_m u_m \end{aligned}$$

Thus, A can be expressed as:

$$A = [c_1 u_1 + c_2 u_2 + \cdots + c_m u_m \quad d_1 u_1 + d_2 u_2 + \cdots + d_m u_m \quad \cdots \quad g_1 u_1 + g_2 u_2 + \cdots + g_m u_m]$$

Let $V = \begin{vmatrix} c_1 & d_1 & \cdots & g_1 \\ c_2 & d_2 & \cdots & g_2 \\ \vdots & \vdots & & \vdots \\ c_m & d_m & \cdots & g_m \end{vmatrix}$

We can then rewrite A as:

$$A = \underbrace{[u_1 \ u_2 \ \cdots \ u_m]}_{u}[c_1 \ c_2 \cdots c_m \quad d_1 \ d_2 \cdots d_m \ \cdots \ g_1 \ g_2 \cdots g_m]$$

$$A = uV^T$$

Thus, $A = uV^T$ for some vectors $u, V \in \mathbb{R}^m$

Question 2b

$A = uv^T$ is a matrix of m rows and colums.
The rank of a matrix is the number of linearly independent rows or columns. Since u and v are scalar multiples of each other, the columns of A are linearly dependent. Thus A has rank one.

<u>Question 3</u>

(a-b)

Suppose $u, v \neq 0$ and A is non singular, then $AA^{-1} = I$ (identity)

$$(I + uv^T)(I + \beta uv^T) = I$$

$$I + \beta uv^T + uv^T + \beta uv^T uv^T = I$$

$$I + uv^T(1 + \beta + \beta v^T u) = I$$

$$uv^T(1 + \beta + \beta v^T u) = 0 \quad \ldots \quad (*)$$

$(*)$ holds true if: $1 + \beta + \beta v^T u = 0$

$$\Rightarrow \beta(1 + v^T u) = -1$$

$$\beta = -1(1 + v^T u)^{-1} \quad \text{for } v^T u \neq -1$$

(c)

$\det(A) = 1 + v^T u$

However, A is singular ~~iff~~ if and only if $\det(A) = 0$, implying that $v^T u = -1$.

(d) Suppose A is singular, there exists a non-zero vector $x \in \mathbb{C}^m$ such that:

$$Ax = 0$$

$$(I + uv^T)x = 0 \Rightarrow uv^T x = -x \quad \text{--- } (**)$$

Let α be a non zero scalar such that $\alpha \in \mathbb{C}$ and $x = \alpha u$

Substituting $(x = \alpha u)$ in $(**)$

$$uv^T(\alpha u) = -\alpha u$$

$$\alpha(v^T u)u = -\alpha u \quad \Rightarrow v^T u = -1$$

Therefore, null(A) = span(u).

Question 4 (a)

In order to compute Ax, the following steps are taken;

$A = I + uv^T$; where $u, v \in \mathbb{R}^m$

$Ax = Ix + u(v^Tx)$

$Ax = x + u(v^Tx)$ - ⊛

We evaluate how many floating point operations are required to compute the right hand side of ⊛

$u(v^Tx)$: First, the inner product v^Tx requires $2m-1$ FLOPS, then the scalar resulting from v^Tx when multiplied by the vector u has m FLOPS. In total, we get

$2m-1+m = 3m-1$ FLOPS.

$x+u(v^Tx)$: The sum operation between the two vectors has m FLOPS.

In general, we have : $(m+3m-1)$ FLOPS $= 4m-1 = O(m)$.

(b)

What are the diagonal entries of A?

$A = I + uv^T$

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$, $v^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}$$

$$A = \begin{bmatrix} 1+u_1v_1 & u_1v_2 & \cdots & u_1v_m \\ u_2v_1 & 1+u_2v_2 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \cdots & 1+u_mv_m \end{bmatrix}$$

Therefore, the diagonal entries of A are $1+u_1v_1, 1+u_2v_2, \ldots, 1+u_mv_m$.

More generally, $\operatorname{diag}(A) = 1+u_iv_i$, $i = 1, 2, \ldots, m$