Question 2

a) $x, y \in \mathbb{R}^m$

$x^Tx = y^Ty$, $v = \frac{y-x}{\|y-x\|}$, $H = I - 2vv^T$

Show that $Hx = (I - 2vv^T)x = y$

soln

$$Hx = (I - 2vv^T)x$$

$$= x - 2vv^Tx = x - \frac{2(y-x)(y-x)^Tx}{\|y-x\|^2}$$

$$= x - \frac{2(y-x)(y-x)^T}{\|y-x\|^2} + (y-y)$$

$$= \frac{-(y-x)}{\|y-x\|^2}\left[\|y-x\|^2 + 2(y-x)^Tx\right] + y$$

$$= \frac{-(y-x)}{\|y-x\|^2}\left[(y-x)^T(y-x) + 2(y-x)^Tx\right] + y$$

$$= \frac{-(y-x)}{\|y-x\|^2}\left[y^Ty - y^Tx - x^Ty + x^Tx + 2y^Tx - 2x^Tx\right] + y$$

\# Since $x^Tx = y^Ty$, then $y^Tx = x^Ty$

$$Hx = \frac{-(y-x)}{\|y-x\|^2}\left[2x^Tx - 2x^Tx + 2y^Tx - 2y^Tx\right] + y$$

$$= \frac{-(y-x)}{\|y-x\|^2}.[0] + y$$

$$Hx = y$$

2bi

$H = I - 2vv^T$

$Tr(H) = Tr(I) - 2Tr(vv^T)$

for an identity matrix in $\mathbb{R}^{m \times m}$, $Tr(I) = m$

$$vv^T = \begin{bmatrix} v_1^2 & v_1v_2 & \cdots & v_1v_m \\ v_1v_2 & v_2^2 & & \\ \vdots & & \ddots & v_m^2 \\ v_1v_m & \cdots & & \end{bmatrix}$$

$Tr(vv^T)$ is simply the inner product v^Tv. Since v is a unit vector,

$Tr(vv^T) = 1$

Therefore

$Tr(H) = m - 2 \times 1 = m - 2$ — ①

We know that H is both symmetric and orthogonal so the ~~magnitude~~ absolute value of its eigenvalues is 1 (i.e $\|\lambda\| = 1$)

Also, $Tr(H) = \sum_{i=1}^{m} \lambda_i$, $i = 1, 2, \cdots, m$ — ②

where λ_i are the eigenvalues of H

Equating ① and ②

$\lambda_1 + \lambda_2 + \cdots + \lambda_m = m - 2$

Let $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_{m-1} = \lambda$

$(m-1)\lambda + \lambda_m = m - 2$

$m\lambda - \lambda + \lambda_m = m - 2$

Equating co-efficients

$m\lambda = m \Rightarrow \lambda = 1$

$\lambda_m - \lambda = -2$

but $\lambda = 1$

$\lambda_m = -2 + 1 = -1$

Thus $\lambda = 1$ with multiplicity of $m - 1$ and $\lambda = -1$ with multiplicity of 1.

2b

(ii)

$$H = I - 2vv^T$$

$$Hv = (I - 2vv^T)v$$

$$= v - 2v(v^Tv)$$

But $v^Tv = 1$

$$Hv = v - 2v$$

$$= -v$$

$$H = I - 2vv^T$$

$$Hu = u - 2vv^Tu$$

Since u is orthogonal to v (i.e $v^Tu = 0$) — $*$

$$Hu = u$$

From $*$, $v \perp u$. Since the dim span$(v) = 1$, dim span$(v)^{\perp} = m-1$

Thus, $\lambda = 1$ is an eigen value with multiplicity of $m-1$.

Additionally, $Hv = -v$ shows that the remaining eigenvalue is -1.

2biii

The determinant of a matrix is the same as the product of its eigenvalues, that is;

$$\det(H) = \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_m$$

where $\lambda_i, i = 1, 2, \ldots, m$ are the eigenvalues of H

Therefore, $\det(H) = -1(1)^{m-1} = -1$