Question 1

(a) STEP 1 : Upper triangular form

for $i = 1$ to $n-2$ do

$B_i = b_i / a_i$

$K_i = d_i / a_i$

$a_{i+1} = a_{i+1} - B_i C_i$

$c_{i+1} = c_{i+1} - B_i e_i$

$b_{i+1} = b_{i+1} - k_i c_i$

$a_{i+2} = a_{i+2} - k_i e_i$

$f_{i+1} = f_{i+1} - B_i f_i$

$f_{i+2} = f_{i+2} - k_i f_i$

end for

$$a_n = a_n - \frac{b_{n-1}}{a_{n-1}} C_{n-1}$$

$$f_n = f_n - \frac{b_{n-1}}{a_{n-1}} f_{n-1}$$

$x_n = f_n / a_n$

$x_{n-1} = (f_{n-1} - x_n C_{n-1}) / a_{n-1}$

STEP 2: Backward substitution.

for $i = n-2$ to 1 do

$$x_i = (f_i - x_{i+1} c_i - x_{i+2} e_i) / a_i$$

end for

(b) Computing the total FLOPS count

STEP 1:

$6(n-2) + 3$ ~~Additions~~ subtractions

$6(n-2) + 3$ Multiplications

$2(n-2) + 4$ Divisions

Total for step 1 $= 14(n-2) + 10 = 14n - 18$

STEP 2:

$2(n-2)$ subtractions

$2(n-2)$ Multiplications

$(n-2)$ Divisions

Total for step 2 $= 5(n-2) = 5n - 10$

Therefore, total number of FLOPS $= 14n - 18 + 5n - 10$

$$= 19n - 28$$

$$\in O(n)$$

Question 3

(a) $Ay = c$ have a fast algorithm. We want to obtain a fast algorithm for solving $(A - uv^T)x = b$.

$Bx = b$

where $B = (A - uv^T)$

$$x = B^{-1}b$$

$$x = (A^{-1} + \frac{1}{\alpha} A^{-1}uv^T A^{-1})b$$

$$x = A^{-1}b + \frac{1}{\alpha} A^{-1}\cancel{b}uv^T A^{-1}b \quad — (1)$$

Since we can solve $A^{-1}b$ using our fast algorithm, let $A^{-1}b = P_1$. Then (1) becomes

$$x = P_1 + \frac{1}{\alpha} A^{-1}uv^T P_1$$

Similarly, we can solve $A^{-1}u$ with our fast algorithm

$$x = P_1 + \frac{1}{\alpha} P_2 v^T P_1 \quad — (2)$$

$$\alpha = 1 - v^T \underbrace{A^{-1}u}_{= P_2} = 1 - v^T P_2 \quad — (3)$$

From (3), since P is a vector, the scalar product $v^T p \in O(n)$ and the result from evaluating (3) is a scalar

From (2)

$$x = P_1 + \frac{1}{\alpha} P_2 \underbrace{v^T P_1}_{= \text{scalar product } (\in O(n))} \quad — (4)$$

(3b) $(A - uv^T)x = b$

where $u \in \mathbb{R}^{n \times 1}$ and $v \in \mathbb{R}^{1 \times n}$.

Therefore, $uv^T = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \cdots & u_nv_n \end{bmatrix}$

Comparing this matrix with what is given in the question, we can deduce that

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad v^T = [2, 2, 2, \cdots, 2]$$

Question 4

(b) $Ax = b$

$A = D - L - U \quad \Rightarrow \quad U = D - L - A$

$(D-L)x^{k+1} = Ux^k + b$

$x^{k+1} = \underbrace{(D-L)^{-1}U}_{= R_G} x^k + (D-L)^{-1}b$

$R_G = (D-L)^{-1}U = (D-L)^{-1}(D-L-A)$

$= (D-L)^{-1}(D-L) - (D-L)^{-1}A$

$= I - (D-L)^{-1}A$

4a

$$A = D - L - U$$

$$Ax = b$$

$$Dx^{i+1} = (L+U)x^{i} + b$$

$$x^{i+1} = D^{-1}(L+U)x^{i} + D^{-1}b$$

Let $R_{g} = D^{-1}(L+U)$

where D is of the form
$$\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & 0 & \ddots & \\ & & & a_{nn} \end{bmatrix}$$

L is of the form
$$\begin{bmatrix} 0 & & & & 0 \\ a_{21} & 0 & & & \\ \vdots & \ddots & 0 & & \\ \vdots & & \ddots & \ddots & \\ a_{n1} & \cdots & \cdots & a_{n,n-1} & 0 \end{bmatrix}$$

and
$$U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \ddots & \vdots \\ & & 0 \ddots & \vdots \\ 0 & & \ddots & a_{n-1,n} \\ & & & 0 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/a_{11} & & & 0 \\ & 1/a_{22} & & \\ & 0 & \ddots & \\ & & & 1/a_{nn} \end{bmatrix}$$

$$R_J = D^{-1}(L+U)$$

$$R_J = \begin{bmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & \\ & & \ddots & \\ & & & 1/a_{nn} \end{bmatrix} \left(\begin{bmatrix} 0 & & & O \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn-1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \ddots & \vdots \\ O & & \ddots & a_{n-1n} \\ & & & 0 \end{bmatrix} \right)$$

$$R_J = \begin{bmatrix} 0 & a_{12}/a_{11} & \frac{a_{13}}{a_{11}} & \cdots & a_{1n}/a_{11} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & & a_{2n}/a_{22} \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \frac{a_{n-1n}}{a_{n-1n-1}} \\ \frac{a_{n1}}{a_{nn}} & \cdots & & \frac{a_{nn-1}}{a_{nn}} & 0 \end{bmatrix}$$

We know that the Jacobi method requires the matrix to be diagonally dominant, i.e the magnitude of the diagonal element in a row be greater than or equal to the sum of the magnitudes of all other non-diagonal elements in that row for each row of the matrix. Also, we note that each row of R_J is scaled or divided by the diagonal element. So $\frac{a_{ij}}{a_{ii}} < 1$; $i \neq j$; $i,j = 1, 2, \ldots, n$

Therefore, $\|R_J\|_\infty < 1$