Fourier analysis

the idea is to use the substitutions

$$u_j^n = \lambda^n e^{ik(j\Delta x)} \quad i\text{: the imaginary unit}$$

$$u_j^{n+1} = \lambda^{n+1} e^{ik(j\Delta x)}$$

λ, k - parameters

for the numerical scheme

$$u_j^{n+1} = u_j^n + \frac{h}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$\lambda^{n+1} e^{ik(j\Delta x)} = \lambda^n e^{ik(j\Delta x)} + \frac{h}{(\Delta x)^2}\left(\lambda^n e^{ik(j+1)\Delta x} - 2\lambda^n e^{ikj\Delta x} + \lambda^n e^{ik(j-1)\Delta x}\right)$$

We now divide both sides of the above equation by λ^n and $e^{ik(j\Delta x)}$ and get

$$\lambda = \lambda(k) = 1 + \frac{h}{(\Delta x)^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) =$$

$$= 1 + \frac{h}{(\Delta x)^2}\left(\cos(k\Delta x) + \underset{\sim\sim}{i\sin(k\Delta x)} - 2 + \cos(k\Delta x) - \underset{\sim\sim}{i\sin(k\Delta x)}\right)$$

$$= 1 - 2\frac{h}{(\Delta x)^2}\left(1 - \cos(k\Delta x)\right)$$

$$= 1 - 4\frac{h}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right)$$

$\cos 2\alpha = 1 - 2\sin^2\alpha$
$2\sin^2\alpha = 1 - \cos 2\alpha$
$4\sin^2\alpha = 2(1 - \cos 2\alpha)$

Then, for $k = m\pi$, $\lambda(m\pi) = 1 - 4\frac{h}{(\Delta x)^2}\sin^2\left(\frac{m\pi\Delta x}{2}\right)$

$$(11) \qquad u_j^n = \sum_{m=-\infty}^{\infty} b_m \underbrace{e^{im\pi(j\Delta x)}}_{\text{bounded}} \left[\lambda(m\pi)\right]^n$$

to compare with the exact solution of the diffusion equation

Using the method of separation of variables, we get the exact solution

$$(12) \qquad u(x,t) = \sum_{m=1}^{\infty} a_m e^{-(m\pi)^2 t} \underbrace{\sin(m\pi x)}_{\text{bounded}}, \qquad a_m = 2\int_0^1 u^0(x)\sin(m\pi x)\,dx$$

$\downarrow$ for comparison with (11)

To compare (11) and (12), we use the series expansions for

$$e^{-k^2h} = \underset{\sim\sim}{1 - k^2h} + \underline{\underline{\tfrac{1}{2}k^4h^2}} - \tfrac{1}{3!}k^6h^3 + \dots$$

$$\lambda(k) = 1 - 4\frac{h}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right) =$$

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \dots$$

$$= 1 - 4\frac{h}{(\Delta x)^2}\left[\frac{k\Delta x}{2} - \frac{1}{3!}\left(\frac{k\Delta x}{2}\right)^3 + \frac{1}{5!}\left(\frac{k\Delta x}{2}\right)^5 - \frac{1}{7!}\left(\frac{k\Delta x}{2}\right)^7 + \dots\right]^2$$

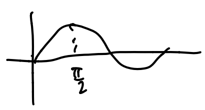
$$= 1 - 4\frac{h}{(\Delta x)^2}\left[\frac{(k\Delta x)^2}{4} - \frac{2(k\Delta x)^4}{2\cdot 3!\cdot 2^3} + \dots\right]$$

$$= 1 - k^2 h + \underline{\underline{\frac{k^4 h(\Delta x)^2}{12}}} + \dots$$

Conclusion : we get at least first order accuracy and if $\frac{1}{2}k^4 h^2 = \frac{1}{12}k^4 h(\Delta x)^2$, that is, $6h = (\Delta x)^2$, then we get second order accuracy

We concluded before that if $\frac{h}{(\Delta x)^2} \leq \frac{1}{2}$ then the numerical scheme is convergent. Now, if we take $\frac{h}{(\Delta x)^2} > \frac{1}{2}$ and $k\Delta x = \pi$, then

$$\lambda(k) = 1 - 4\frac{h}{(\Delta x)^2}\underbrace{\sin^2\left(\frac{k\Delta x}{2}\right)}_{=\sin\frac{\pi}{2}=1} = 1 - 4\frac{h}{(\Delta x)^2} < 1 - 4\cdot\frac{1}{2} = 1 - 2 = -1$$



$$-\frac{h}{(\Delta x)^2} < -\frac{1}{2}$$
$$-\frac{4h}{(\Delta x)^2} < -\frac{1}{2}\cdot 4 = -2$$
$$1 - \frac{4h}{(\Delta x)^2} < 1 - 2 = -1$$

So, the modes $[\lambda(k)]^n$ grow unboundedly as n increases leading to instabilities.

Therefore, the method is stable for $\frac{h}{(\Delta x)^2} \leq \frac{1}{2}$ and unstable for $\frac{h}{(\Delta x)^2} > \frac{1}{2}$.

For stability, we require that there exists a constant M such that

$$\forall k \ \forall n \quad nh \leq a \Rightarrow |[\lambda(k)]|^n \leq M$$