Zero-stability

Definition: Let $\{y_0, y_1, y_2, \dots, y_{K-1}\}$ and $\{\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{K-1}\}$ be any different sets of starting values for the general linear K-step method:

$$(1) \qquad \sum_{j=0}^{K} \alpha_j z_{n+j} = h \sum_{j=0}^{K} \beta_j f(t_{n+j}, z_{n+j})$$

and y_n, $\tilde{y}_n$ be computed from (1) using the following two formulas:

$$\sum_{j=0}^{K} \alpha_j y_{n+j} = h \sum_{j=0}^{K} \beta_j f(t_{n+j}, y_{n+j}) \leftarrow y_n$$

$$\sum_{j=0}^{K} \alpha_j \tilde{y}_{n+j} = h \sum_{j=0}^{K} \beta_j f(t_{n+j}, \tilde{y}_{n+j}) \leftarrow \tilde{y}_n$$

for $n = 0, 1, 2, 3, \dots$

Then, method (1) is zero-stable $\Leftrightarrow$ $\exists K > 0$ $\forall t_n \in [a,b]$ $\forall h > 0$

$$|y_n - \tilde{y}_n| \leq K \cdot \max_{0 \leq i \leq K-1} |y_i - \tilde{y}_i|$$

(that is, small perturbations at initial values stay small during the process of computations).

Notation

$$\rho(x) = \sum_{j=0}^{K} \alpha_j x^j \qquad \text{first characteristic polynomial}$$

$$\sigma(x) = \sum_{j=0}^{K} \beta_j x^j \qquad \text{second characteristic polynomial}$$

Theorem on successive approximations

Suppose 1° $K \in \mathbb{N}$, $K \geq 1$, $l \in \mathbb{N}$, $1 \leq l \leq K$

2° x_r, where $1 \leq r \leq l$, are distinct roots of the first characteristic polynomial (that is, $\rho(x_r) = 0$), where

$$\rho(x) = \alpha_K x^K + \alpha_{K-1} x^{K-1} + \dots + \alpha_1 x + \alpha_0, \quad \alpha_K \neq 0, \ \alpha_0 \neq 0$$

3° $m_r \geq 1$ is the multiplicity of the r-th root x_r and

$m_1 + m_2 + \ldots + m_\ell = k$

4° $\{y_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ is such that, for all $n = 0, 1, 2, \ldots$

(2) $\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \ldots + \alpha_1 y_{n+1} + \alpha_0 y_n = 0$.

Then,

$$y_n = \sum_{r=1}^{\ell} p_r(n)\, x_r^n, \quad \text{for } n = 0, 1, 2, \ldots$$

where $p_r(n)$ are polynomials of degree $m_r - 1$, respectively, for all $1 \leq r \leq \ell$ (if x_r is a simple root with multiplicity 1, then $p_r(n) = p_r$ is a constant).