Error bound in case $\mu(A)=0$

(15) $$\|e^{(k)}(t)\| \leq \frac{1}{k!}\left(\left[\|B\|+b_0(t)\|F\|_0\right]\tilde{a}(t)\right)^k \underbrace{E(t)}_{=\max\limits_{\tau\in[0,t]}\|e^{(0)}(\tau)\|}$$

We derived the recursive inequality

(13) $$\|e^{(k+1)}(t)\| \leq \left[\|B\|+b_0(t)\|F\|_0\right]\int_0^t a(\tau)\exp\left(\mu(A)\int_\tau^t a(s)\,ds\right)\|e_\tau^{(k)}\|_0\,d\tau$$

We apply mathematical induction to prove (15). If we use $k=0$ and $\mu(A)=0$ in (13), we get

$$\|e^{(1)}(t)\| \leq \left[\|B\|+b_0(t)\|F\|_0\right]\int_0^t a(\tau)\underbrace{\|e_\tau^{(0)}\|_0}_{\leq E(t)}\,d\tau \leq$$

$$\leq \left[\|B\|+b_0(t)\|F\|_0\right]E(t)\underbrace{\int_0^t a(\tau)\,d\tau}_{=\tilde{a}(t)} = \left(\left[\|B\|+b_0(t)\|F\|_0\right]\tilde{a}(t)\right)^1\frac{1}{1!}E(t)$$

which shows (15) for $k=1$. We now assume (15) for a certain k and prove it for $k+1$. Then, from (13) with $\mu(A)=0$, we get

$$\|e^{(k+1)}(t)\| \leq \left[\|B\|+b_0(t)\|F\|_0\right]\int_0^t a(\tau)\,\|e_\tau^{(k)}\|_0\,d\tau.$$

Since $\frac{1}{k!}\left(\left[\|B\|+b_0(t)\|F\|_0\right]\tilde{a}(t)\right)^k E(t)$ is increasing as a function of t,

$$\|e_\tau^{(k)}\|_0 = \max_{\tau-\tau_0\leq s\leq\tau}\|e^{(k)}(s)\| \leq \max_{\tau-\tau_0\leq s\leq\tau}\frac{1}{k!}\left(\left[\|B\|+b_0(s)\|F\|_0\right]\tilde{a}(s)\right)^k E(s)$$

$\leq$

Therefore,

$$\|e^{(k+1)}(t)\| \leq \left[\|B\|+b_0(t)\|F\|_0\right]\int_0^t a(\tau)\frac{1}{k!}\left(\left[\|B\|+b_0(\tau)\|F\|_0\right]\tilde{a}(\tau)\right)^k E(\tau)\,d\tau$$

$$\leq \frac{\left[\|B\|+b_0(t)\|F\|_0\right]^{k+1}}{k!}E(t)\int_0^t a(\tau)\left(\tilde{a}(\tau)\right)^k d\tau$$

$$a(\tau) = \frac{d}{d\tau}\tilde{a}(\tau) = \frac{d}{d\tau}\int_0^\tau a(s)\,ds$$

$$\int_0^t a(\tau)\left(\tilde{a}(\tau)\right)^k d\tau = \int_0^t \left(\tilde{a}(\tau)\right)^k d\tilde{a}(\tau)$$

$$= \frac{[\|B\| + b_0(t)\|F\|_0]^{k+1}}{k!} E(t) \frac{1}{k+1} \underbrace{\left[\left(\tilde{a}(\tau)\right)^{k+1} \right]_{\tau=0}^{\tau=t}}_{\left(\tilde{a}(t)\right)^{k+1} - \left(\tilde{a}(0)\right)^{k+1}} = \frac{[\|B\| + b_0(t)\|F\|_0]^{k+1}}{(k+1)!} E(t) \left(\tilde{a}(t)\right)^{k+1}$$

$$\tilde{a}(0) = \int_0^0 a(s)\,ds = 0$$

which shows (15) with $k+1$. So, (15) is now proved by mathematical induction for all k.

Time integration methods of spectral semi-discrete systems

We use the grid-points $x_i = -L\cos(i\pi/N)$, $i = 0, 1, \cdots, N$, and the spectral differentiation matrix $D = [d_{ij}]_{i,j=0}^{N}$ to discretize the partial functional differential equation

$$\frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t) f(\underset{\substack{\downarrow \\ \text{function}}}{u_{(x,t)}}) + g(x,t)$$

in space and then we applied

Gauss-Seidel dynamic iterations

$$(8) \quad \frac{d v_i^{(k+1)}}{dt} = a(t) \sum_{j=1}^{i} d_{ij} v_j^{(k+1)}(t) + a(t) \sum_{j=i+1}^{N-1} d_{ij} v_j^{(k)}(t) + b(t) f((v_i^{(k)})_t) + g_i(t)$$

Jacobi dynamic iteration

$$(9) \quad \frac{d v_i^{(k+1)}}{dt} = a(t)\, d_{ii}\, v_i^{(k+1)}(t) + a(t) \sum_{\substack{j=1 \\ j \neq i}}^{N-1} d_{ij} v_j^{(k)}(t) + b(t) f((v_i^{(k)})_t) + g_i(t)$$

We can now apply time integration methods to integrate (8) and (9) in t. For example, if we apply the method

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) \longrightarrow y'(t) = f(t, y(t))$$

to (8), we get

$$v_{i,n+1}^{(k+1)} = v_{i,n}^{(k+1)} + h a(t_{n+1}) \sum_{j=1}^{i} d_{ij} v_{j,n+1}^{(k+1)} + h a(t_{n+1}) \sum_{j=i+1}^{N-1} d_{ij} v_{j,n+1}^{(k)} + h b(t_{n+1}) f\left(\left(v_i^{(k)}\right)_{t_{n+1}}\right) + h g_i(t_{n+1})$$

$$(1 - h a(t_{n+1}) d_{ii}) v_{i,n+1}^{(k+1)} = v_{i,n}^{(k+1)} + h a(t_{n+1}) \sum_{j=1}^{i-1} d_{ij} v_{j,n+1}^{(k+1)} + h a(t_{n+1}) \sum_{j=i+1}^{N-1} d_{ij} v_{j,n+1}^{(k)} + h b(t_{n+1}) f\left(\left(v_i^{(k)}\right)_{t_{n+1}}\right) + h g_i(t_{n+1})$$

<u>Recursive numerical scheme for the partial functional problem (1)</u>

$$v_{i,n+1}^{(k+1)} = (1 - h a(t_{n+1}) d_{ii})^{-1} \left[v_{i,n}^{(k+1)} + h a(t_{n+1}) \sum_{j=1}^{i-1} d_{ij} v_{j,n+1}^{(k+1)} + h a(t_{n+1}) \sum_{j=i+1}^{N-1} d_{ij} v_{j,n+1}^{(k)} + h b(t_{n+1}) f\left(\left(v_i^{(k)}\right)_{t_{n+1}}\right) + h g_i(t_{n+1}) \right]$$

$i = 1, 2, \dots, N-1$, $n = 0, 1, 2, 3, \dots, n_{max} - 1$, $k = 0, 1, 2, \dots, k_{max}$,

where $g_i(t) = g(x_i, t) + a(t) \left(d_{i0} \psi_-(t) + d_{iN} \psi_+(t) \right)$

boundary functions from the boundary conditions

$$v_{i,n}^{(k+1)} = \phi(x_i, t_n), \quad n = -n_0, -n_0 + 1, \dots, -2, -1, 0$$

from the inition condition

where $n_0 h = \tau_0$