Partial Differential Equations

Problem sheet 1

Answers

1. The Green's function for the IVP

$$y'' + p(x)y' + q(x)y = r(x), \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0 \tag{1}$$

is given by

$$G(x,s) = \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{W[y_1,y_2](s)} \tag{2}$$

where y_1 and y_2 are independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

(a) (2 marks) The functions

$$\tilde{y}_1(x) = ay_1(x) + by_2(x)$$
$$\tilde{y}_2(x) = cy_1(x) + dy_2(x)$$

are also solutions of (3). Give a condition on a, b, c and d which makes $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ independent solutions.

Solution to 1(a): Differentiating the equations for $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ one obtains

$$\tilde{y}_1'(x) = ay_1'(x) + by_2'(x)$$
$$\tilde{y}_2'(x) = cy_1'(x) + dy_2'(x)$$

Substituting this into $W[\tilde{y}_1, \tilde{y}_2]$ gives after a small amount of algebra

$$W[\tilde{y}_1, \tilde{y}_2] = (ad - bc)(y_1y_2' - y_2y_1') = (ad - bc)W[y_1, y_2]$$

Since y_1 and y_2 are independent solutions we know that $W[y_1, y_2](x) \neq 0$. Hence

$$W[\tilde{y}_1, \tilde{y}_2] \neq 0 \Leftrightarrow ad - bc \neq 0$$

Note that this is simply the condition that the linear transformation relating $\tilde{y}_1$, $\tilde{y}_2$ to y_1, y_2 has non-zero determinant and is therefore invertible.

(b) (2 marks) Calculate the Green's function using independent functions $\tilde{y}_1(x)$ and $\tilde{y}_2(x)$ as given by the above equation and show that it is identical to the Green's function given by equation (2).

Solution 1(b) Using the above equations we calculate that

$$\tilde{y}_1(s)\tilde{y}_2(x) - \tilde{y}_1(x)\tilde{y}_2(s) = (ad - bc)\{y_1(s)y_2(x) - y_1(x)y_2(s)\}$$

We already know that

$$W[\tilde{y}_1, \tilde{y}_2](s) = (ad - bc)W[y_1, y_2](s)$$

So that using formula (2) but with $\tilde{y}_1$ and $\tilde{y}_2$ gives:

$$\begin{aligned}
\frac{\tilde{y}_1(s)\tilde{y}_2(x)-\tilde{y}_1(x)\tilde{y}_2(s)}{W[\tilde{y}_1,\tilde{y}_2](s)} &= \frac{(ad-bc)\{y_1(s)y_2(x)-y_1(x)y_2(s)\}}{(ad-bc)W[y_1,y_2](s)} \\
&= \frac{y_1(s)y_2(x)-y_1(x)y_2(s)}{W[y_1,y_2](s)} \qquad \text{Since } (ad-bc)\neq 0 \\
&= G(x,s)
\end{aligned}$$

So that the Green's function constructed from $\tilde{y}_1$ and $\tilde{y}_2$ is **identical** to that constructed from y_1 and y_2.

(c) (6 marks) Calculate the Green's function for the IVP

$$y''-4y'+4y=r(x) \quad \text{with } y(0)=0 \text{ and } y'(0)=0 \tag{4}$$

and use this to obtain the solution to the above problem when $r(x)=xe^{2x}$.

Solution 1(c)

The homogeneous equation is

$$y''-4y'+4y=0$$

with *characteristic equation*

$$m^2-4m+4=0$$

This is just $(m-2)^2=0$ so that the solution is $m=2$ (twice). We may therefore take $y_1(x)=e^{2x}$ and $y_2(x)=xe^{2x}$ as independent solutions of the homogeneous equation. The Wronskian is given by

$$W[y_1,y_2](x)=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x}\end{vmatrix}=e^{4x}$$

Hence substituting into equation (2) gives:

$$G(x,s)=\frac{e^{2s}xe^{2x}-e^{2x}se^{2s}}{e^{4s}}=xe^{2x}e^{-2s}-e^{2x}se^{-2s}$$

Note (although we will not use this fact below) that we can write the above as $G(x,s)=(x-s)e^{2(x-s)}$ so that $G(x,s)$ can be written as a function of $(x-s)$.

To solve the equation $y''-4y+4=xe^{2x}$ with $y(0)=0$ and $y'(0)=0$ we use the Green's function calculated above and $r(x)=xe^{2x}$

$$\begin{aligned}
y(x) &= \int_{s=0}^{x} G(x,s)r(s)ds \\
&= \int_{s=0}^{x} (xe^{2x}e^{-2s}-e^{2x}se^{-2s})se^{2s}ds \\
&= xe^{2x}\int_{s=0}^{x} sds - e^{2x}\int_{s=0}^{x} s^2ds \\
&= xe^{2x}(x^2/2)-e^{2x}(x^3/3) \\
&= (x^3/6)e^{2x}
\end{aligned}$$

So that the required solution is $y(x)=(x^3/6)e^{2x}$.

2. Let λ_n, $n = 1, 2, 3, \ldots$ be the eigenvalues, and $y_n(x)$ be the corresponding eigenfunctions, for the boundary value problem

$$\frac{d^2y}{dx^2} = \lambda y, \qquad y(0) = 0, \quad \frac{dy}{dx}(1) = 0, \quad x \in [0, 1] \tag{5}$$

(a) (4 marks) **Without calculating** the $y_n(x)$ use integration by parts to show that

$$\int_0^1 \left(y_m(x)y_n''(x) - y_n(x)y_m''(x)\right) dx = 0 \tag{6}$$

Hence show (**again without calculating the** $y_n(x)$) that

$$\int_0^1 y_m(x)y_n(x)dx = 0, \qquad \text{for } m \neq n \tag{7}$$

Solution 2(a)

$$\begin{aligned}
\int_0^1 \left(y_m(x)y_n''(x) - y_n(x)y_m''(x)\right) dx &= \left[y_m(x)y_n'(x) - y_n(x)y_m'(x)\right]_0^1 \\
&\quad + \int_0^1 \left(-y_m'(x)y_n'(x) + y_n'(x)y_m'(x)\right) dx \\
&= \left[y_m(x)y_n'(x) - y_n(x)y_m'(x)\right]_0^1 \\
&= 0 \qquad \text{Since } y'(1) = 0 \text{ and } y(0) = 0
\end{aligned}$$

Which gives equation (6). We now use the fact that $y_m''(x) = \lambda_m y_m(x)$ and $y_n''(x) = \lambda_n y_n(x)$. Substituting into (6) we get

$$\begin{aligned}
0 &= \int_0^1 \left(y_m(x)y_n''(x) - y_n(x)y_m''(x)\right) dx \\
&= \int_0^1 \left(\lambda_n y_m(x)y_n(x) - \lambda_m y_n(x)y_m(x)\right) dx \\
&= (\lambda_n - \lambda_m) \int_0^1 y_m(x)y_n(x)dx
\end{aligned}$$

Now for $m \neq n$ we have $\lambda_n - \lambda_m \neq 0$ hence we must have

$$\int_0^1 y_m(x)y_n(x)dx = 0, \qquad \text{for } m \neq n$$

which is equation (7).

(b) (6 marks) Calculate the eigenvalues and eigenfunctions for the BVP given by equation (5). Substitute in the values you have obtained for $y_n(x)$ and $y_m(x)$ into (7) to obtain an "orthogonality condition" for certain trigonometric functions.

Solution 2(b) To calculate the eigenvalues and eigenfunctions we need to solve the BVP

$$y'' - \lambda y = 0, \qquad y(0) = 0, \;\; y'(1) = 0$$

For this ODE there are three cases: $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

Case (i) $\lambda > 0$ For this case we write $\lambda = \mu^2$ (where $\mu \neq 0$). Then the general solution is $y(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary condition $y(0) = 0$ gives $0 = A + B$, so that $B = -A$. Hence we may write $y(x) = A(e^{\mu x} - e^{-\mu x})$. To apply the boundary condition $y'(1) = 0$ we must differentiate $y(x)$ to obtain $y'(x) = A\mu(e^{\mu x} + e^{-\mu x}) = 2A\mu\cosh(\mu x)$. Hence $y'(1) = 0$ gives $0 = 2A\mu\cosh(\mu)$. Since $\mu \neq 0$ this is only possible if $A = 0$. Hence the boundary conditions give $y(x) = 0$ for all x. So case (i) only gives the *trivial solution*.

Case (ii) $\lambda = 0$. In this case the DE becomes $y'' = 0$ with general solution $y(x) = A + Bx$. The boundary condition $y(0) = 0$ gives $0 = A$ so that $y(x) = Bx$. Differentiating gives $y'(x) = B$ so the boundary condition $y'(0) = 0$ gives $B = 0$ and again we get the *trivial solution*.

Case (iii) $\lambda < 0$. In this case we take $\lambda = -\mu^2$ (with $\mu \neq 0$). Then the DE becomes $y'' + \mu^2 y = 0$ with general solution $y(x) = A\cos(\mu x) + B\sin(\mu x)$. The boundary condition $y(0) = 0$ gives $0 = A$ and therefore $y(x) = B\sin(\mu x)$. Differentiating gives $y'(x) = B\mu\cos(\mu x)$. We now apply the boundary condition $y'(1) = 0$. This gives $0 = B\mu\cos(\mu)$. For non-trivial solutions this is only possible if

$$\cos(\mu) = 0 \quad \Rightarrow \quad \mu = \frac{\pi}{2} + k\pi, \;\; k \in \mathbb{Z}$$

Hence the eigenvalues are given by

$$\lambda_k = -\left(\frac{(2k+1)\pi}{2}\right)^2 = -\frac{(2k+1)^2\pi^2}{4}, \quad k \in \mathbb{Z}$$

The corresponding eigenfunctions are given by

$$y_k(x) = B\sin\left(\frac{(2k+1)\pi x}{2}\right), \quad k \in \mathbb{Z}$$

Note however that $k = 0$ and $k = -1$ give the same eigenvalues and eigenfunctions (at the expense of changing the sign of B). Similarly $k = 1$ and $k = -2$ give the same eigenvalues and eigenfunctions, and in general $n = n$ and $k = -(n + 1)$ give the same eigenvalues and eigenfunctions for $n \in \mathbb{N}$. Remember that Sturm-Liouville theory tells us that there is a minimum vale of (with my convention) $-\lambda$. So that the **distinct** set of eigenvalues and eigenfunctions are given by

$$\lambda_n = -\frac{(2n+1)^2\pi^2}{4} \quad n \in \mathbb{N}$$
$$y_n(x) = B\sin\left(\frac{(2n+1)\pi x}{2}\right), \quad n \in \mathbb{N}$$

Substituting the expressions for $y_n(x)$ into equation (7) and ignoring the arbitrary constant B we get:

$$\int_0^1 \sin\left(\frac{(2m+1)\pi x}{2}\right)\sin\left(\frac{(2n+1)\pi x}{2}\right)dx = 0 \quad \text{for } m, n \in \mathbb{N} \text{ and } m \neq n$$

This is the required "orthogonality condition".

Note without restricting m, n to be in $\mathbb{N}$ rather than $\mathbb{Z}$ the above result would not be true. For example the above would not be true if we chose $m = 1$ and $n = -2$ since it would be the integral of $\sin^2(3\pi x/2)$ which integrates to $1/2$, rather than 0.

3. The function $f(x)$ is 2π-periodic and is defined for $-\pi \leqslant x \leqslant \pi$ by

$$f(x) = \begin{cases} -1 & -\pi \leqslant x < 0 \\ 1 & 0 \leqslant x < \pi \end{cases} \tag{8}$$

(a) (2 marks) Plot the graph of the function $f(x)$ for $-3\pi \leqslant x \leqslant 3\pi$ using Python.

(b) (2 marks) Explain why $f(x)$ has a Fourier Sine series (i.e. the cosine terms in the Fourier series all vanish).

Solution 3(b) The function is 2π-periodic so according to the theory of Fourier series one can write it as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} \sin(nx)$$

where

$$a_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n \in \mathbb{N}$$
$$b_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad n \in \mathbb{N}^*$$

To calculate a_n we note that in our case we have $f(-x) = -f(x)$ and $\cos(-nx) = \cos(nx)$ so that the integrand $f(x)\cos(nx)$ is an **odd** function. Hence the integral on $-\pi \leqslant x \leqslant 0$ exactly cancels the integral on $0 \leqslant x \leqslant \pi$ and thus $a_n = 0$. Alternatively just calculate the integral to show $a_n = 0$ for $n \in \mathbb{N}$.

(c) (6 marks) Calculate the Fourier series for $f(x)$.

Soltion 2(c) We are left with the calculation of the b_n. Since $f(x)$ has a different formula for $x < 0$ and $x > 0$ we need to split the integral for b_n into two parts as shown below:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)dx \\
&= \frac{1}{\pi}\left[\frac{1}{n}\cos(nx)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_{0}^{\pi} \\
&= \frac{1}{n\pi}(\cos(0) - \cos(-n\pi)) - \frac{1}{n\pi}(\cos(n\pi) - \cos(0))
\end{aligned}$$

Using the fact that $\cos(0) = 1$, $\cos(-n\pi) = \cos(n\pi) = (-1)^n$ we get

$$b_n = \frac{2}{n\pi}(1 - (-1)^n)$$

So that the Fourier series is given by

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n}\sin(nx)$$

Noting that $(1 - (-1)^n)$ is zero for n even and is equal to 2 for n odd. We may make sure that n is odd by writing $n = 2m - 1$ for $m \in \mathbb{N}^*$ which enables us to write the above in a slightly neater way as

$$f(x) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin((2m-1)x).$$