Theorem on necessary and sufficient conditions for zero-stability

Suppose the function f is such that the initial-value problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

is well-possed. Then, a linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f(t_{n+j}, y_{n+j})$$

is zero stable $\iff$ all roots z_r of the first characteristic polynomial

$$\rho(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \ldots + \alpha_1 z + \alpha_0$$

are such that $|z_r| \leq 1$ and if $|z_r| = 1$ then z_r is a simple root.

Proof $\boxed{\Rightarrow}$ We suppose that the method is zero-stable and we apply it to $y' = 0$ (here, $f \equiv 0$). Then,

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \ldots + \alpha_1 y_{n+1} + \alpha_0 y_n = 0$$

By the theorem on successive approximations, we conclude that

$$(*) \quad y_n = \sum_{r=1}^{l} p_r(n) z_r^n,$$

where $p_r(n)$ is a polynomial of degree $m_r - 1$, for all $1 \leq r \leq l \leq k$, and m_r is the multiplicity of the root z_r. Suppose, by contradiction, that $|z_r| > 1$, for a certain r. Then,

$$z_r^n = \left[d_r(\cos\varphi_r + i\sin\varphi_r)\right]^n = d_r^n(\cos\varphi_r + i\sin\varphi_r)^n =$$

$$= d_r^n\left(\cos(n\varphi_r) + i\sin(n\varphi_r)\right),$$

where $d_r = |z_r| = \sqrt{(Re(z_r))^2 + (Im(z_r))^2} > 1$,

$$|z_r^n| = d_r^n\sqrt{\cos^2(n\varphi_r) + \sin^2 n\varphi_r} = d_r^n \xrightarrow[n\to\infty]{} \infty.$$

Therefore, from $(*)$, $y_n \sim p_r(n) z_r^n$ as $n \to \infty$, where r is such that $|z_r| > 1$ is the largest and the initial values $y_0, y_1, y_2, \ldots, y_{k-1}$ are such that $p_r(n)$ is non-zero.

Then, $\lim_{n\to\infty} y_n = \infty$, where nh is fixed and $h \to 0^+$.

On the other hand, by taking $\tilde{y}_0 = \tilde{y}_1 = \tilde{y}_2 = \ldots = \tilde{y}_{k-1} = 0$, we get $\tilde{y}_n = 0$, for all $n \in \mathbb{N}$. So, we get

$$\lim_{\substack{h\to 0^+\\ n\to\infty}} |y_n - \tilde{y}_n| = \infty,$$

that contradicts the assumption that the method is zero-stable.

In case $|z_r| = 1$ and $m_r > 1$, from (*), we conclude that y_n includes a component with the factor $n^{m_r - 1} \xrightarrow[n\to\infty]{} \infty$ and $\lim_{\substack{n\to\infty\\ h\to 0}} y_n = \infty$. So, the method is not zero-stable.

Therefore, $\forall r$ $|z_r| \leq 1$ and if $|z_r| = 1$ then z_r is a simple root.