Chapitre 1

Finite difference Quotient

1.1 Introduction

1.1.1 What is a PDE ?

A partial differential equation (PDE) is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function.
Examples : Transport equation $u_t + cu_x = 0$, heat equation $u_t - k\Delta u = 0$, wave equation $u_{tt} - c^2\Delta u = 0$, N-S equations, $\cdots$

The first thing when we are dealing with PDEs in general is the so-called "Well-posedness" : that is :

- Solutions must exist for all right-hand expressions (in equations and conditions) ;
- Solution must be unique ;
- Solution must depend on this right-hand expressions continuously.

1.1.2 Taylor series expansion

The solution of PDEs by means of FD is based on approximating derivatives of continuous functions. A FDM proceeds by replacing the derivatives in the differential equations with finite difference approximations. This gives a large but finite algebraic system of equations to be solved in place of differential equation, something that can be done on a computer.
Let $u(x)$ represents a function of one variable which assume to be smooth, meaning that we can differentiate it several times and each derivative is well-defined.
The Taylor-series expansion for $u(x_0+h)$, away from the point x_0 by a small amount h (sometimes here also denoted by Δx), gives

$$u(x_0 + h) = u(x_0) + \sum_{k=1}^{n} u^{(k)}(x_0)\frac{h^k}{k!} + O(h^{n+1}).$$

Here, $O(h^{n+1})$ indicates that the full solution would require additional terms of order h^{n+1}, h^{n+2}, and so on. O is called the truncation error.
Suppose we want to approximate $u'(x)$
• **Forward approximation** :

$$D_+u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x})}{h}$$

for some small value of h.
• **Backward approximation** :

$$D_-u(\bar{x}) = \frac{u(\bar{x}) - u(\bar{x}-h)}{h}.$$

• **Centered approximation** :

$$D_0u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h} = \frac{1}{2}(D_+u(\bar{x}) + D_-u(\bar{x})).$$

This is a slope of the line interpolating u at $\bar{x}-h$ and $\bar{x}+h$.
We set the operator error by $E_h(\bar{x}) = Du(\bar{x}) - u'(\bar{x})$.

Remark 1.1.1 *If the error $E(h)$ behaves like $E(h) \approx Ch^p$, then $log|E(h)| \leq log|C| + plogh$. So on a log-log scale, the error behaves linearly with the slope that is equal to p, the order of accuracy.*

1.1.3 Truncation errors

The standard approach to analyzing the error in FDA is to expand each of the function values of u in a Taylor series about the point $\bar{x}$.

$$u(\bar{x}+h) = u(\bar{x}) + hu'(\bar{x}) + \frac{h^2u''(\bar{x})}{2} + \frac{h^3u^3(\bar{x})}{3!} + \frac{h^4u^4(\bar{x})}{4!} + \cdots$$
$$u(\bar{x}-h) = u(\bar{x}) - hu'(\bar{x}) + \frac{h^2u''(\bar{x})}{2} - \frac{h^3u^3(\bar{x})}{3!} + \frac{h^4u^4(\bar{x})}{4!} + \cdots$$

$$\begin{aligned} D_+u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x})}{h} &= u'(\bar{x}) + h/2u''(\bar{x}) + 1/6h^2u^3(\bar{x}) + \cdots \\ &= u'(\bar{x}) + \varepsilon. \end{aligned}$$

ε is called the **truncation error**.

Definition 1.1.1 *1) The truncation error is the error made by approximating the derivative operator by the finite difference approximation.*
2) The lowest power of h that appears in the truncation error is called the **order of accuracy** *of the corresponding difference approximation.*

Exercise 1.1.1 *Given* $D_3u(\bar{x}) = \frac{1}{6h}[2u(\bar{x}+h) + 3u(\bar{x}) - 6u(\bar{x}-h) + u(\bar{x}-2h)]$*. Show that* $D_3u(\bar{x})$ *is an approximation of* $u'(\bar{x})$*. What is the order of accuracy of* D_3 *?*

Our goal in what follows is to develop systematic way to derive such formulas and to analyse their accuracy.

Example 1 *find the approximation of* $u'(\bar{x})$ *based on* $u(\bar{x}+h)$, $u(\bar{x})$, $u(\bar{x}-h)$ *and* $u(\bar{x}-2h)$*.*

1.1.4 Second order derivatives

The standard second order centered approximation is given by

$$\begin{aligned} D_0^2u(\bar{x}) &= \frac{1}{h^2}\left(u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)\right) \\ &= u''(\bar{x}) + \frac{h^2}{12}u^{(4)}(\bar{x}) + \cdots . \end{aligned}$$

Note that $D_0^2u(\bar{x}) = D_+(D_-u(\bar{x})) = D_-(D_+u(\bar{x}))$.

Exercise 1.1.2 *1) Compute* $D_0D_+D_-u(\bar{x})$ *and check the order of accuracy. 2) Do the same with* $D_+D_0^2u(\bar{x})$*.*

1.1.5 Some examples

1. Central second derivative, fourth order :

$$\frac{\partial^2 u}{\partial x^2}(x_i) = \frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12h^2} + O(h^4)$$

2. Central third derivative, second order :

$$\frac{\partial^3 u}{\partial x^3}(x_i) = \frac{-u_{i+2} + 2u_{i+1} - 2u_{i-1} + u_{i-2}}{2h^3} + O(h^2)$$

3. Central fourth derivative, second order :

$$\frac{\partial^4 u}{\partial x^4}(x_i) = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{h^4} + O(h^2)$$

1.1.6 Derivatives with variable coefficients

Let us consider the following form

$$\frac{\partial}{\partial x}(K(x)\frac{\partial u}{\partial x}) \tag{1.1.1}$$

where $K(x)$ is the function of the space. The form (1.1.1) can be approximating in many ways.

1. The following, common approximations are inadequate to maintain second order accuracy :

$$\frac{\partial}{\partial x}(K(x)\frac{\partial u}{\partial x})(x_i) = \frac{K_{i+1}\frac{u_{i+1}-u_i}{h} - K_i\frac{u_i-u_{i-1}}{h}}{h}; \tag{1.1.2}$$

$$\frac{\partial}{\partial x}(K(x)\frac{\partial u}{\partial x})(x_i) = K_i\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} \tag{1.1.3}$$

2. To maintain the second order accuracy of the central difference approach for second derivatives is used

$$\frac{\partial}{\partial x}(K(x)\frac{\partial u}{\partial x})(x_i) = \frac{K_{i+1/2}\frac{u_{i+1}-u_i}{h} - K_{i-1/2}\frac{u_i-u_{i-1}}{h}}{h} + O(h^2)$$

   where $K_{i+1/2}$ are evaluated between x_i and x_{i+1}.

Note : If K has strong jumps from one grid point to another that are not aligned with the grid-nodes, most second-order methods will show first order accuracy at best.

1.2 Extension to the 2-dimension

The approach presented above can be generalized to the multi-dimensional problems. To illustrate this, we will use the 2-dimensional Laplacian operator. Such operator appears for example in the diffusion equation with constant diffusion coefficient :

$$\frac{\partial u}{\partial t} = K\Delta u.$$

The standard second order five-point stencils for the centered difference on the uniform grid. Note $\Delta u(x_i, y_j) = u_{xx}(x_i, y_j) + u_{yy}(x_i, y_j)$ and if we fix y_j $u_{xx}(x_i, y_j)$ is seen like $u''(x_i)$ in one dimension so using the 3-point stencils in the direction of x-axis with uniform grid with size d_x, we have

$$D^2_{xx}u(x_i, y_j) = \frac{1}{d_x^2}(u_{i-1j} - 2u_{ij} + u_{i+1j})$$

Applying the same on the y-direction,

$$D^2_{yy}(x_i, y_j) = \frac{1}{d_y^2}(u_{ij-1} - 2u_{ij} + u_{ij+1})$$

The skewed stencil with $d_x = d_y = d$

$$\Delta_h u(x_i, y_j) = \frac{1}{(\sqrt{2}d)^2}(u_{i-1j-1} - 2u_{ij} + u_{i+1j+1}) + \frac{1}{(\sqrt{2}d)^2}(u_{i+1j-1} - 2u_{ij} + u_{i-1j+1})$$

Chapitre 2

Finite Difference schemes

Introduction

A finite difference schemes can be defined as a finite difference equation that approximates term-by-term a differential equation. We can find approximation to each term of differential equation and we have already seen that the error of such approximation can be made as small as desired.
The method of deriving finite difference schemes is very simple. However the analysis of finite difference schemes to determine if they are useful approximations to the differential equation requires some powerful mathematical tools.
Attention You might think that if we have a finite difference equation F, that is constructed by writing down a good approximation to each term of differential equation D, then the solution of F will be a useful approximation of the solution of D. Unfortunately, it is not always the case. The approximation F needs to satisfy certain conditions : Consistency, stability and convergence.

The first step in the finite differences method is to construct a grid with points on which we are interested in solving the equation (this is called discretization). The next step is to replace the continuous derivatives with their finite difference approximations.

2.1 1D-Advection equation

Advection is the process by which the wind moves things around. It is also sometimes call transport or atmospheric dispersion. Advection equations are also solved in atmospheric models to predict the transport of pollutants, water vapour and volcanic ash. Non-linear advection is the process by which the wind advects itself (the advection term in the momentum equation (1)). In this section, we will focus on numerical methods for solving the linear advection equation in one spatial dimension, x, without sources or