Runge-Kutta methods

(1) $\begin{cases} y_{n+1} = y_n + h(ak_1 + bk_2), & \text{where } k_1 = f(t_n, y_n) \\ & k_2 = f(t_n + \alpha h, y_n + \beta h k_1) \end{cases}$

$a, b, \alpha, \beta \in \mathbb{R}$ parameters

Here,

(2) $\Phi(t_n, y_n, h) = a f(t_n, y_n) + b f(t_n + \alpha h, y_n + \beta h k_1)$

consistency condition

$\forall t, x \quad \Phi(t, x, 0) \equiv f(t, x)$

$$a f(t,x) + b f(t + \alpha \cdot 0, x + \beta \cdot 0) = (a+b) f(t,x) \equiv f(t,x)$$

$$\Updownarrow$$

$$a + b = 1$$

order of accuracy

from the definition of the truncation error

$$T_n = \frac{y(t_n+h) - y(t_n)}{h} - \Phi(t_n, y(t_n), h)$$

$$= \frac{1}{h}\left[y(t_n) + \frac{1}{1!} y'(t_n) h + \frac{1}{2!} y''(t_n) h^2 + \frac{1}{3!} y'''(t_n) h^3 + \mathcal{O}(h^4) - y(t_n) \right]$$

$$- \Phi(t_n, y(t_n), h) =$$

(3) $= y'(t_n) + \frac{h}{2!} y''(t_n) + \frac{h^2}{3!} y'''(t_n) + \mathcal{O}(h^3) - \Phi(t_n, y(t_n), h)$

Here,

$$y'(t_n) = f(t_n, y(t_n)) = f$$

$$y''(t_n) = \frac{\partial f}{\partial t}(t_n, y(t_n)) + \frac{\partial f}{\partial y}(t_n, y(t_n)) \cdot \overbrace{y'(t_n)}^{= f(t_n, y(t_n))} = f_t + f_y \cdot f$$

$$y'''(t_n) = \frac{\partial^2 f}{\partial t^2}(t_n, y(t_n)) + \frac{\partial^2 f}{\partial y \partial t}(t_n, y(t_n)) \cdot y'(t_n) + \left(\frac{\partial^2 f}{\partial y \partial t}(t_n, y(t_n)) + \right.$$

$$\left. + \frac{\partial^2 f}{\partial y^2}(t_n, y(t_n)) \cdot y'(t_n) \right) \cdot y'(t_n) + \frac{\partial f}{\partial y}(t_n, y(t_n)) \cdot y''(t_n) =$$

$$= f_{tt} + f_{ty} \cdot f + f_{ty} \cdot f + f_{yy} [f]^2 + f_y [f_t + f_y f]$$

$$= f_{tt} + 2 f f_{ty} + f^2 f_{yy} + f_t \cdot f_y + [f_y]^2 f$$

From (2)

$$\Phi(t_n, y_n, h) = af(t_n, y_n) + bf(t_n+\alpha h, y_n+\beta h f(t_n, y_n)) =$$

$$= af(t_n, y_n) + b\Big[f(t_n, y_n) + \frac{\alpha h}{1!}\frac{\partial f}{\partial t}(t_n, y_n) + \frac{\beta h f(t_n, y_n)}{1!}\frac{\partial f}{\partial y}(t_n, y_n)$$

$$+ \frac{(\alpha h)^2}{2!}\frac{\partial^2 f}{\partial t^2}(t_n, y_n) + 2\frac{(\alpha h)\,\beta h\, f(t_n, y_n)}{2!}\frac{\partial^2 f}{\partial t \partial y}(t_n, y_n) +$$

$$+ \frac{[\beta h f(t_n, y_n)]^2}{2!}\cdot\frac{\partial^2 f}{\partial y^2}(t_n, y_n) + \mathcal{O}(h^3)\Big] =$$

$$= af + b\left[f + \alpha h f_t + \beta h f f_y + \tfrac{1}{2}\alpha^2 h^2 f_{tt} + \alpha\beta h^2 f f_{ty} + \tfrac{1}{2}\beta^2 h^2 f^2 f_{yy}\right]$$

$$+ \mathcal{O}(h^3)$$

Therefore, from (3)

$$T_n = f + \tfrac{1}{2}h\left[f_t + f f_y\right] + \tfrac{1}{6}h^2\left[f_{tt} + 2 f f_{ty} + f^2 f_{yy} + f_t f_y + f_y^2 f\right]$$

$$- af - b\left[f + \alpha h f_t + \beta h f f_y + \tfrac{1}{2}\alpha^2 h^2 f_{tt} + \alpha\beta h^2 f f_{ty} + \tfrac{1}{2}\beta^2 h^2 f^2 f_{yy}\right]$$

$$+ \mathcal{O}(h^3)$$

$= 0$ because $a+b=1$

Conclusion

$$\tfrac{1}{2}h f_t + \tfrac{1}{2}h f f_y - b\alpha h f_t - b\beta h f f_y = 0$$

$$\Updownarrow$$

$$\left(\tfrac{1}{2} - b\alpha\right) f_t + \left(\tfrac{1}{2} - b\beta\right) f f_y = 0$$

$$\Updownarrow$$

$b\alpha = 1/2$ and $b\beta = 1/2$

So, $b\alpha = b\beta$ and $\alpha = \beta \neq 0$