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Question 1

Given

$$D_2u(\bar{x}) = \frac{\alpha u(\bar{x}) + \beta u(\bar{x}-h) + \gamma u(\bar{x}-2h)}{h}$$

We are interested in finding the values of the constants; α, β, γ.

$$\begin{aligned}
\beta u(\bar{x}-h) &= \beta \left[u(\bar{x}) - hu'(\bar{x}) + \frac{h^2}{2}u''(\bar{x}) - \frac{h^3}{6}u'''(\bar{x}) + \frac{h^4}{24}u''''(\bar{x}) - \frac{h^5}{120}u'''''(\bar{x}) + 0(h^6)\right] \\
\gamma u(\bar{x}-2h) &= \gamma \left[u(\bar{x}) - 2hu'(\bar{x}) + 2h^2u''(\bar{x}) - \frac{4h^3}{3}u'''(\bar{x}) + \frac{2h^4}{3}u''''(\bar{x}) - \frac{4h^5}{15}u'''''(\bar{x}) + 0(h^6)\right] \\
\alpha u(\bar{x}) &= \alpha u(\bar{x})
\end{aligned}$$

Thus we shall have;

$$\begin{aligned}
u(\bar{x}-h) + u(\bar{x}-2h) + \alpha u(\bar{x}) = (\alpha+\beta+\gamma)u(\bar{x}) + h(-\beta-2\gamma)u'(\bar{x}) + h^3(-\frac{1}{6}\beta - \frac{4}{3}\gamma)u'''(\bar{x}) \\
D_2u = (\bar{x})\frac{1}{h}\left((\alpha+\beta+\gamma)u(\bar{x}) + h(-\beta-2\gamma)u'(\bar{x}) + h^2\left(\frac{\beta+4\gamma}{2}\right)\right)
\end{aligned}$$

Since we want to approximate the first derivative, we can obtain a system of equations given by

$$\begin{cases}
\alpha+\beta+\gamma &= 0 \\
-\beta-2\gamma &= 1 \\
\beta+4\gamma &= 0
\end{cases}$$

By solving the system of equations simultaneously, we obtain

$$\alpha = \frac{3}{2}, \quad \beta = -2, \quad \gamma = \frac{1}{2}$$

Question 2

(1) We want to show that

$$Du(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{h}$$

We know that the central scheme is obtained by averaging the forward and the backward schemes, that is;

$$Du(x_{i+\frac{1}{2}}) = \frac{1}{2}\left(D_{-}u(x_{i+\frac{1}{2}}) + D_{+}u(x_{i+\frac{1}{2}})\right)$$

But

$$D_{-}u(x_{i+\frac{1}{2}}) = \frac{u(x_{i+\frac{h}{2}}) - u(x_i)}{\frac{h}{2}} \approx u'(x_{i+\frac{1}{2}})$$
$$D_{+}u(x_{i+\frac{1}{2}}) = \frac{u(x_{i+\frac{h}{2}+\frac{h}{2}}) - u(x_{i+\frac{h}{2}})}{\frac{h}{2}} \approx u'(x_{i+\frac{1}{2}})$$

Since $x_{i+\frac{1}{2}} = x_{i+\frac{h}{2}}$, we shall then have;

$$Du(x_{i+\frac{1}{2}}) = \frac{1}{2}\left(\frac{u(x_{i+\frac{1}{2}}) - u(x_i)}{\frac{h}{2}} + \frac{u(x_{i+1}) - u(x_{i+\frac{1}{2}})}{\frac{h}{2}}\right)$$
$$Du(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{h} \approx u'(x_{i+\frac{1}{2}})$$

Hence shown!

(2) We want to find the approximate of $v(x_{i+\frac{1}{2}})$. Given

$$v(x) = K(x)u'(x)$$

Therefore;

$$v\left(x_{i+\frac{1}{2}}\right) = K\left(x_{i+\frac{1}{2}}\right)u'\left(x_{i+\frac{1}{2}}\right) \tag{1}$$

But

$$u'\left(x_{i+\frac{1}{2}}\right) = \frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{h}$$

Thus equation1, becomes;

$$v\left(x_{i+\frac{1}{2}}\right) = K\left(x_{i+\frac{1}{2}}\right)\frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{h} \tag{2}$$

(3) We want to show $Dv(x_i)$ is a central approximation of $v'(x_i)$, where;

$$Dv(x_i) = \frac{v\left(x_{i+\frac{1}{2}}\right) - v\left(x_{i-\frac{1}{2}}\right)}{h}$$

We know that a central approximation is defined by;

$$Du(x_i) = \frac{1}{2}\left(D_-u(x_i) + D_+u(x_i)\right)$$

Thus

$$\begin{aligned}
D_+v(x_i) &= \frac{v\left(x_{i+\frac{h}{2}}\right) - v\left(x_i\right)}{\frac{h}{2}} \approx v'(x_i) \\
D_-v(x_i) &= \frac{v\left(x_i\right) - v\left(x_{i-\frac{h}{2}}\right)}{\frac{h}{2}} \approx v'(x_i) \\
Dv(x_i) &= \frac{1}{2}\left(\frac{v\left(x_{i+\frac{h}{2}}\right) - v\left(x_i\right)}{\frac{h}{2}} + \frac{v\left(x_i\right) - v\left(x_{i-\frac{h}{2}}\right)}{\frac{h}{2}}\right) \\
&= \frac{1}{2}\left(\frac{v\left(x_{i+\frac{1}{2}}\right) - v\left(x_i\right)}{\frac{h}{2}} + \frac{v\left(x_i\right) - v\left(x_{i-\frac{1}{2}}\right)}{\frac{h}{2}}\right) \\
Dv(x_i) &= \frac{v\left(x_{i+\frac{1}{2}}\right) - v\left(x_{i-\frac{1}{2}}\right)}{h} \approx v'(x_i)
\end{aligned}$$

Hence shown!

(4) We want to show that

$$D^2u(x_i) = \frac{1}{h^2}\left(\left(K\left(x_{i-\frac{1}{2}}\right)u\left(x_i - 1\right) - K\left(x_{i-\frac{1}{2}}\right) + K\left(x_{i+\frac{1}{2}}\right)\right)u(x_i) + K\left(x_{i+\frac{1}{2}}\right)u(x_{i+1})\right)$$

is the approximate of $(kv')'\left(x_i\right)$

we know that;

$$\begin{aligned}
v(x_i) &= k(x_i)u'(x_i) \\
v(x_i)' &= (k(x)u')'(x_i)
\end{aligned}$$

But

$$v(x_i)' \approx Dv(x_i) = \frac{v\left(x_{i+\frac{1}{2}}\right) - v\left(x_{i-\frac{1}{2}}\right)}{h}$$

From equation2, we already obtained $v\left(x_{i+\frac{1}{2}}\right)$, from which we can also infer for $v\left(x_{i-\frac{1}{2}}\right)$

$$\begin{aligned}
v(x_i)' &= \frac{1}{h}\left(K\left(x_{i+\frac{1}{2}}\right)\frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{h} - K\left(x_{i-\frac{1}{2}}\right)\frac{u\left(x_i\right) - u\left(x_{i-1}\right)}{h}\right) \\
&\frac{1}{h^2}\left[K\left(x_{i+\frac{1}{2}}\right)u\left(x_{i+1}\right) - \left(K\left(x_{i+\frac{1}{2}}\right) + K\left(x_{i-\frac{1}{2}}\right)\right)u\left(x_i\right) + K\left(x_{i-\frac{1}{2}}\right)u\left(x_{i-1}\right)\right]
\end{aligned}$$

Hence shown!

(5)

$$PH\begin{cases}\frac{1}{h^2}\left(-K_{i-\frac{1}{2}}U_{i-1}+\left(K_{i-\frac{1}{2}}+K_{i+\frac{1}{2}}\right)U_i-K_{i+\frac{1}{2}}U_{i+1}\right) & =f(x_i) \qquad \forall 1\leq i\leq N-1\\ U_0=\alpha; \quad U_N=\beta\end{cases}$$

We want to express PH in the matrix form $AU=F$, where A, U and F have to be determined.

$$-K_{i-\frac{1}{2}}U_{i-1}+\left(K_{i-\frac{1}{2}}+K_{i+\frac{1}{2}}\right)U_i-K_{i+\frac{1}{2}}U_{i+1}=h^2f(x_i)$$
$$for \quad i=1: -K_{\frac{1}{2}}U_0+\left(K_{\frac{1}{2}}+K_{\frac{3}{2}}\right)U_1-K_{\frac{3}{2}}U_2=h^2f(x_1)$$
$$for \quad i=2: -K_{\frac{3}{2}}U_1+\left(K_{\frac{3}{2}}+K_{\frac{5}{2}}\right)U_2-K_{\frac{5}{2}}U_3=h^2f(x_2)$$
$$for \quad i=3: -K_{\frac{5}{2}}U_2+\left(K_{\frac{5}{2}}+K_{\frac{7}{2}}\right)U_3-K_{\frac{7}{2}}U_4=h^2f(x_3)$$
$$\vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$for \quad i=N-1: -K_{N-\frac{3}{2}}U_{N-2}+\left(K_{N-\frac{3}{2}}+K_{N-\frac{1}{2}}\right)U_{N-1}-K_{N-\frac{1}{2}}U_N=f(x_{N-1})$$

Representing the above in matrix form, we obtain;

$$AU=F$$

$$\begin{bmatrix} K_{\frac{1}{2}}+K_{\frac{3}{2}} & -K_{\frac{3}{2}} & 0 & 0 & \cdots & 0 \\ -K_{\frac{3}{2}} & K_{\frac{3}{2}}+K_{\frac{5}{2}} & -K_{\frac{5}{2}} & 0 & \cdots & 0 \\ 0 & -K_{\frac{5}{2}} & K_{\frac{5}{2}}+K_{\frac{7}{2}} & -K_{\frac{7}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & \cdots & 0 & \cdots & -K_{N-\frac{3}{2}} & K_{N-\frac{3}{2}}+K_{N-\frac{1}{2}} \end{bmatrix}\begin{bmatrix}U_1\\U_2\\U_3\\\vdots\\U_{N-1}\end{bmatrix}$$
$$=\begin{bmatrix}\alpha K_{\frac{1}{2}}+h^2f(x_1)\\h^2f(x_2)\\h^2f(x_3)\\\vdots\\\beta K_{N-\frac{1}{2}}+h^2f(x_{N-1})\end{bmatrix}$$

(2) For a matrix to be symmetric,then it must be equal to its transpose. The obtained matrix A is equal to its transpose.

$$A=A^T$$

$$A^T=\begin{bmatrix} K_{\frac{1}{2}}+K_{\frac{3}{2}} & -K_{\frac{3}{2}} & 0 & 0 & \cdots & 0 \\ -K_{\frac{3}{2}} & K_{\frac{3}{2}}+K_{\frac{5}{2}} & -K_{\frac{5}{2}} & 0 & \cdots & 0 \\ 0 & -K_{\frac{5}{2}} & K_{\frac{5}{2}}+K_{\frac{7}{2}} & -K_{\frac{7}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & \cdots & 0 & \cdots & -K_{N-\frac{3}{2}} & K_{N-\frac{3}{2}}+K_{N-\frac{1}{2}} \end{bmatrix}$$

Since $A=A^T$, thus matrix A is symmetric.

To prove that matrix A is positive define, we perform the operation (Av, v) and check if it is greater than zero for all $v \in \mathbb{R}^{N-1}$ and $(Av, v) = 0$ if and only if $v = 0$

$$Av = \begin{bmatrix} K_{\frac{1}{2}} + K_{\frac{3}{2}} & -K_{\frac{3}{2}} & 0 & 0 & \cdots & 0 \\ -K_{\frac{3}{2}} & K_{\frac{3}{2}} + K_{\frac{5}{2}} & -K_{\frac{5}{2}} & 0 & \cdots & 0 \\ 0 & -K_{\frac{5}{2}} & K_{\frac{5}{2}} + K_{\frac{7}{2}} & -K_{\frac{7}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & \cdots & 0 & \cdots & -K_{N-\frac{3}{2}} & K_{N-\frac{3}{2}} + K_{N-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

$$Av = \begin{bmatrix} \left(K_{\frac{1}{2}} + K_{\frac{3}{2}}\right) v_1 - K_{\frac{3}{2}} v_2 \\ -K_{\frac{3}{2}} v_1 + \left(K_{\frac{3}{2}} + K_{\frac{5}{2}}\right) v_2 - K_{\frac{5}{2}} v_3 \\ -K_{\frac{5}{2}} v_2 + \left(K_{\frac{5}{2}} + K_{\frac{7}{2}}\right) v_3 - K_{\frac{7}{2}} v_4 \\ \vdots \\ -K_{N-\frac{3}{2}} v_{N-2} + \left(K_{N-\frac{3}{2}} + K_{N-\frac{1}{2}}\right) v_{N-1} \end{bmatrix}$$

Thus the sccalar product (Av, v) is given by;

$$\begin{bmatrix} (K_{\frac{1}{2}} + K_{\frac{3}{2}}) v_1 - K_{\frac{3}{2}} v_2 \\ -K_{\frac{3}{2}} v_1 + (K_{\frac{3}{2}} + K_{\frac{5}{2}}) v_2 - K_{\frac{5}{2}} v_3 \\ -K_{\frac{5}{2}} v_2 + (K_{\frac{5}{2}} + K_{\frac{7}{2}}) v_3 - K_{\frac{7}{2}} v_4 \\ \vdots \\ -K_{N-\frac{3}{2}} v_{N-2} + (K_{N-\frac{3}{2}} + K_{N-\frac{1}{2}}) v_{N-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

Which then yields;

$$\begin{aligned} (Av, v) &= (K_{\frac{1}{2}} + K_{\frac{3}{2}}) v_1^2 - K_{\frac{3}{2}} v_1 v_2 - K_{\frac{3}{2}} v_1 v_2 + (K_{\frac{3}{2}} + K_{\frac{5}{2}}) v_2^2 - K_{\frac{5}{2}} v_3 v_2 - K_{\frac{5}{2}} v_2 v_3 + (K_{\frac{5}{2}} + K_{\frac{7}{2}}) v_3^2 \\ &\quad - K_{\frac{7}{2}} v_4 v_3 + \cdots + -K_{N-\frac{3}{2}} v_{N-2} v_{N-1} + (K_{N-\frac{3}{2}} + K_{N-\frac{1}{2}}) v_{N-1}^2 \\ &= K_{\frac{1}{2}} v_1^2 + K_{\frac{3}{2}} (v_1 - v_2)^2 + K_{\frac{5}{2}} (v_1 - v_2)^2 + K_{\frac{7}{2}} (v_3 - v_2)^2 + \cdots + K_{N-\frac{3}{2}} (v_{N-2} - v_{N-1})^2 \end{aligned}$$

For $i, j \in \mathbb{R}$, Since $(v_i - v_j)^2$ and v_i^2 or v_j^2 is always greater than or equal to zero for $i \neq j$, and given the fact that $K = x^2 \geq 0$, we can thus conclude that $(Av, v) \geq 0$ and so A is positive definite.

APPLICATION

(1) Given $a = 0, \quad b = 1,\ K(x) = x^2$ and $u(x) = x(1+x)$

$$\begin{aligned} u(x) &= x(1+x) \qquad x \in (0,1) \\ u(0) &= u_0 = 0 \\ u(1) &= u_1 = 2 \end{aligned}$$

Therefore $\alpha = 0, \quad \beta = 2.$

$$\begin{aligned} u' &= 1 + 2x \\ -(ku)'(x) &= f(x) \\ \implies f(x) &= -(x^2(1+2x))' \\ &= -2x - 6x^2 \end{aligned}$$

(3) The figure below shows the plot of the exact and approximate solution of (PH)

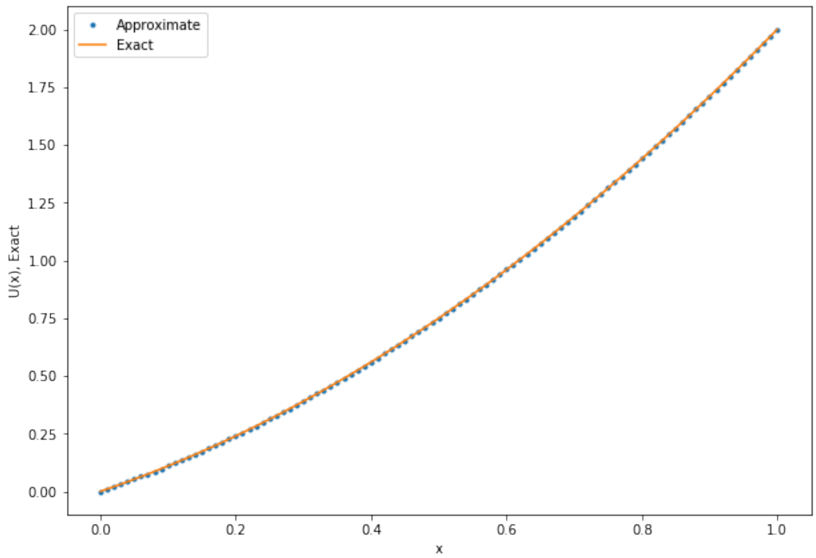

Figure 1: Approximate Vs Exact Solution