Goal: construct Φ in such a way that the global error converges to zero as $h \to 0$

Definition of consistency

The method $y_{n+1} = y_n + h\,\Phi(t_n, y_n, h)$ is consistent with the differential equation $y'(t) = f(t, y(t)) \Longleftrightarrow$

$$\forall \varepsilon > 0\ \exists h_\varepsilon > 0\ \forall h \in (0, h_\varepsilon)\ \forall \left.\begin{matrix}(t_n, y(t_n)) \\ (t_{n+1}, y(t_{n+1}))\end{matrix}\right\} \in [a,b] \times [y_0 - C, y_0 + C]$$

$$\Big|\underbrace{\frac{y(t_{n+1}) - y(t_n)}{h} - \Phi(t_n, y(t_n), h)}_{T_n}\Big| < \varepsilon$$

Let $t_n = a + nh \xrightarrow[h \to 0]{n \to \infty} t \in [a,b]$. Then,

x_0

$$\lim_{h \to \infty} T_n = \lim_{n \to \infty}\left[\frac{y(t_{n+1}) - y(t_n)}{h} - \Phi(t_n, y(t_n), h)\right] = y'(t) - \Phi(t, y(t), 0)$$

continuous

$$\frac{y(t_n + h) - y(t_n)}{h}$$

So, if the method is consistent, then $y'(t) - \Phi(t, y(t), 0) = 0$

Since y is the exact solution, $y'(t) = f(t, y(t))$.

Therefore, if the method is consistent

$$\forall t, z \quad \Phi(t, z, 0) = f(t, z).$$

Assumption $\Phi(t, z, 0) = f(t, z)$