Introduction to numerical methods for differential equations

initial value problem
$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \le t \le T, \\ y(t_0) = y_0, \end{cases}$$

where $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The exact solution $y: [t_0, T] \to \mathbb{R}$.

Goal: to compute approximations to $y(t)$, for $t_0 \le t \le T$.

One step methods: $y_{n+1} \to$ computed by using $y_n \approx y(t_n)$

K-step methods $K \geq 2$: $y_{n+1} \to$ computed by using $y_n, y_{n-1}, \dots, y_{n-k+1}$

Grid-points: $t_n = t_0 + nh$, $h = \frac{T - t_0}{N}$, where N is the number of subintervals in $[t_0, T]$.

$$y(\underbrace{t_n + h}_{= t_{n+1} = t_n + h}) = y(t_n) + h f(t_n, y(t_n)) + \mathcal{O}(h^2)$$

$$y(t_n + h) = y(t_n) + \underbrace{\frac{y'(t_n)}{1!}}_{= f(t_n, y(t_n))} h + \underbrace{\frac{y''(t_n)}{2!} h^2 + \frac{y'''(t_n)}{3!} h^3 + \dots}_{= \mathcal{O}(h^2)}$$

$= \mathcal{O}(h^2) \to$ can be omitted

$h \to 0$

Euler's method:

$y_{n+1} = y_n + h f(t_n, y_n)$, because $y(t_n) \approx y_n$

y_0 — given at t_0 from the initial condition

Generalization

$y_{n+1} = y_n + h \Phi(t_n, y_n, h)$

$\Phi \to$ continuous

For Euler's method $\Phi(t_n, y_n, h) = f(t_n, y_n)$

Definition of global error: $e_n = y(t_n) - y_n$

Definition of truncation error

$$T_n = \frac{y(t_{n+1}) - y(t_n)}{h} - \Phi(t_n, y(t_n), h)$$

$T_n = \mathcal{O}(h^2)$ for Euler's method

Goal: to get an error bound for e_n in terms of T_n