Midterm Exam Last name: ______________________

MATH 567 First name: ______________________

Show your work for each problem

1. (*10 points*) Consider the numerical method

$$\begin{cases} y_{n+1} = y_n + \frac{1}{2}h(k_1 + k_2), \\ k_1 = f(x_n, y_n), \quad k_2 = f(x_n + h, y_n + hk_1) \end{cases} \tag{1}$$

applied to the initial value problem

$$\begin{cases} y'(x) &= f(x, y(x)), \\ y(x_0) &= y_0, \end{cases} \tag{2}$$

where $f = f(x, y)$ is a smooth function of two variables.

(a) Show that

$$y'' = f_x + f_y y' = f_x + f_y f \tag{3}$$

and

$$\begin{aligned} y''' &= f_{xx} + f_{xy}y' + f_y y'' + (f_{xy} + f_{yy}y')y' \\ &= f_{xx} + f_{xy}f + f_x f_y + f(f_y)^2 + f_{yy}f^2, \end{aligned} \tag{4}$$

where y is the solution of (1) and

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad \text{etc...}$$

(b) Apply (3) and (4) to show that the truncation error of the numerical method (1) is defined by

$$T_n = \frac{1}{6}h^2\Big(f_x f_y + f(f_y)^2 - \frac{1}{2}(f_{xx} + 2ff_{xy} + f^2 f_{yy})\Big) + \mathcal{O}(h^3), \tag{5}$$

where all functions are evaluated at $(x_n, y(x_n))$.

(c) Apply (5) to show that for any $\varepsilon > 0$ there exists $h(\varepsilon) > 0$ such that the truncation error of method (1) satisfies the inequality $|T_n| < \varepsilon$ for any step-size $0 < h < h(\varepsilon)$ and any points $(x_n, y(x_n))$, $(x_{n+1}, y(x_{n+1}))$.

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MATH 567 **First name:** ______________________

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2. (*10 points*) Let T_n be the truncation error and $e_n = y(x_n) - y_n$ be the global error of the implicit method
$$y_{n+1} = y_n + \frac{1}{2}h\Big(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\Big)$$
(where $h = x_{n+1} - x_n$) for the numerical solution of
$$\begin{cases} y'(x) &= f(x, y(x)), \\ y(x_0) &= y_0. \end{cases}$$

   (a) By applying repeated integration by parts to $\displaystyle\int_{x_n}^{x_{n+1}} (x - x_{n+1})(x - x_n)y'''(x)dx$, show that
$$y(x_{n+1}) - y(x_n) = \frac{h}{2}\Big(f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\Big) - \frac{1}{2}\int_{x_n}^{x_{n+1}} (x_{n+1} - x)(x - x_n)y'''(x)dx$$

   (b) Apply the property saying that there exists $\xi_n \in (x_n, x_{n+1})$ such that
$$\int_{x_n}^{x_{n+1}} (x_{n+1} - x)(x - x_n)y'''(x)dx = y'''(\xi_n)\int_{x_n}^{x_{n+1}} (x_{n+1} - x)(x - x_n)dx$$
and use the substitution $x = x_n + sh$ to the right-hand side of the above equation to show that
$$T_n = -\frac{1}{12}h^2 y'''(\xi_n).$$

   (c) Use parts (a) and (b) to derive the formula
$$e_{n+1} = e_n + \frac{h}{2}\Big(f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y_{n+1})\Big) + \frac{h}{2}\Big(f(x_n, y(x_n)) - f(x_n, y_n)\Big) + hT_n.$$

   (d) Let $L > 0$ be a Lipschitz constant for the function f with respect to y, that is,
$$|f(x, y) - f(x, \tilde{y})| \le L|y - \tilde{y}|,$$
for all $x, y, \tilde{y} \in \mathbb{R}$, and let $M > 0$ be a constant such that $|y'''(x)| \le M$, for all $x \in \mathbb{R}$. Use part (c) to show that
$$|e_{n+1}| \le |e_n| + \frac{Lh}{2}\Big(|e_{n+1}| + |e_n|\Big) + \frac{M}{12}h^3.$$

   (e) Let $y_0 = y(x_0)$ and $h > 0$ be such that $hL < 2$. Use part (d) to derive the formula
$$|e_n| \le \frac{h^2 M}{12L}\left[\left(\frac{2 + hL}{2 - hL}\right)^n - 1\right].$$

3. (*10 points*) Apply the 4th order Runge-Kutta method

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\ k_3 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2) \\ k_4 &= f(x_n + h, y_n + hk_3) \end{aligned}$$

to the differential equation $y'(x) = \lambda y(x)$, where λ is a constant, and show the following properties.

(a) Show that

$$y_{n+1} = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{3!}h^3\lambda^3 + \frac{1}{4!}h^4\lambda^4\right)y_n.$$

(b) Show that, for the exact solution $y(x)$,

$$y(x_{n+1}) = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{3!}h^3\lambda^3 + \frac{1}{4!}h^4\lambda^4 + \dots\right)y(x_n).$$

(c) Comment on the accuracy of the numerical approximations.