Error bounds for elliptic equations

Last time, we derived the truncation error

$$T(x_i, y_j) = \frac{(\Delta x)^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(\xi, y_j) + \frac{\partial^4 u}{\partial y^4}(x_i, \eta)\right), \quad \text{where}$$

$(\xi, y_j), (x_i, \eta) \in \Omega$

Conclusion :

$$|T(x_i, y_j)| \leq \frac{(\Delta x)^2}{12}\left(\max_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1}}\left|\frac{\partial^4 u}{\partial x^4}(x,y)\right| + \max_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1}}\left|\frac{\partial^4 u}{\partial y^4}(x,y)\right|\right) = T_{max}$$

We now subtract (4) from (2) and

$$L u_{i,j} + f_{i,j} - \left(L u(x_i, y_j) + f_{i,j}\right) = -T(x_i, y_j).$$

Since L is linear,

$$L\left[u_{i,j} - u(x_i, y_j)\right] = -T(x_i, y_j)$$

Definition

$$e_{i,j} = \underset{\substack{\uparrow \\ \text{approximate} \\ \text{solution}}}{u_{i,j}} - \underset{\substack{\uparrow \\ \text{exact solution} \\ \text{at the grid-points}}}{u(x_i, y_j)}, \qquad i,j = 0, 1, 2, \ldots, N$$

Then,

$$L[e_{i,j}] = -T(x_i, y_j)$$

Note : since $u_{i,j} = u(x_i, y_j)$ for i, j such that $(x_i, y_j) \in \delta\Omega$, (that is, $i=0$ or $i=1$ or $j=0$ or $j=1$)

$e_{i,j} = 0$ on the boundary of Ω

Goal: to have an error bound for $e_{i,j}$

We apply the maximum principle to the discrete function

$$\varphi_{i,j} = e_{i,j} + \frac{1}{4} T_{max} \left[\left(x_i - \frac{1}{2}\right)^2 + \left(y_j - \frac{1}{2}\right)^2 \right].$$

Then, we get

$$L\varphi_{i,j} = \underbrace{L e_{i,j}}_{=-T(x_i,y_j)} + \frac{1}{4} T_{max} \underbrace{L\left[\left(x_i - \frac{1}{2}\right)^2 + \left(y_j - \frac{1}{2}\right)^2 \right]}_{\text{Taylor expansion}}$$

By Taylor expansion

$$L\left[\left(x_i - \frac{1}{2}\right)^2 + \left(y_j - \frac{1}{2}\right)^2 \right] = \underbrace{\frac{\partial^2}{\partial x^2}\left(x-\frac{1}{2}\right)^2 \Big|_{x=x_i}}_{=2} + \underbrace{\frac{\partial^2}{\partial y^2}\left(y-\frac{1}{2}\right)^2 \Big|_{y=y_j}}_{=2} +$$

$$+ \frac{(\Delta x)^2}{12} \left(\underbrace{\frac{\partial^4}{\partial x^4}\left(x-\frac{1}{2}\right)^2 \Big|_{x=x_i}}_{=0} + \underbrace{\frac{\partial^4}{\partial y^4}\left(y-\frac{1}{2}\right)^2 \Big|_{y=y_j}}_{=0} \right) = 4.$$

So,

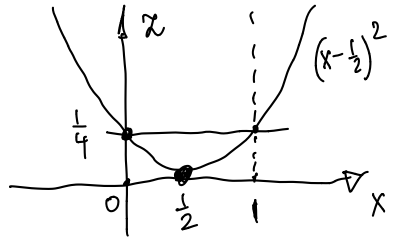
$$L\varphi_{i,j} = -T(x_i,y_j) + \frac{1}{4} T_{max} \cdot 4 = -T(x_i,y_j) + T_{max} \geq 0,$$

for all $\underbrace{(x_i,y_j) \in \Omega \setminus \delta\Omega}_{\text{interior points}}$.

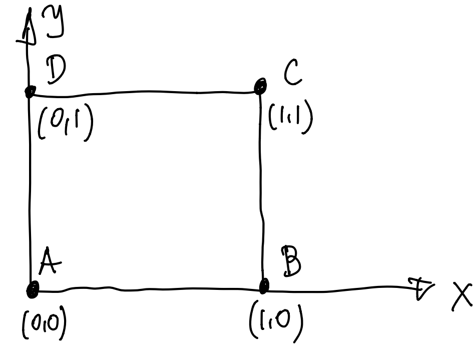
Since $L\varphi_{i,j} = \frac{1}{(\Delta x)^2}\left[\varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1} - 4\varphi_{i,j} \right] \geq 0$, by the maximum principle, $\varphi_{i,j}$ cannot be greater than the values of all its neighbours. So, the discrete function φ attains its positive maximum value at a point on the boundary $\delta\Omega$. Since

$$\varphi_{i,j} = e_{i,j} + \frac{1}{4} T_{max} \left[\left(x_i - \frac{1}{2}\right)^2 + \left(y_j - \frac{1}{2}\right)^2 \right] \text{ and } \underbrace{e_{i,j} = 0}_{\text{on } \delta\Omega},$$

$$\varphi_{i,j} = \frac{1}{4}T_{max}\left[\left(x_i-\frac{1}{2}\right)^2+\left(y_j-\frac{1}{2}\right)^2\right], \quad \text{for } (x_i,y_j)\in\delta\Omega.$$



The function $z(x,y)=(x-\frac{1}{2})^2+(y-\frac{1}{2})^2$ has maximum value $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ at the points A, B, C, D



Conclusion The maximum value of φ on $\delta\Omega$ is $\frac{1}{4}T_{max}\cdot\frac{1}{2}$
$= \frac{1}{8}T_{max}$

So, by the maximum principle,

$$\forall (x_i,y_j)\in\Omega \qquad \varphi_{i,j}\leq \frac{1}{8}T_{max}.$$

Therefore, from the definition of $\varphi_{i,j}$, for all $(x_i,y_j)\in\Omega$,

$$(*) \qquad e_{i,j}\leq\varphi_{i,j}=e_{i,j}+\frac{1}{4}T_{max}\left[\left(x_i-\frac{1}{2}\right)^2+\left(y_j-\frac{1}{2}\right)^2\right]\leq\frac{1}{8}T_{max}$$

We now want to get an upper bound for $-e_{ij}$ so that we get an upper bound for $|e_{i,j}|$. In a similar way, we can demonstrate that

$$(**) \qquad -e_{ij}\leq\frac{1}{8}T_{max}$$

Combining $(*)$ and $(**)$, we get

$$|e_{i,j}|\leq\frac{1}{8}T_{max}$$

and from the definition of T_{max},

$$|e_{i,j}|\leq\frac{1}{8}\cdot\frac{(\Delta x)^2}{12}\left(\max_{\substack{0\leq x\leq 1\\0\leq y\leq 1}}\left|\frac{\partial^4 u}{\partial x^4}(x,y)\right|+\max_{\substack{0\leq x\leq 1\\0\leq y\leq 1}}\left|\frac{\partial^4 u}{\partial y^4}(x,y)\right|\right).$$