Hyperbolic equations

advection equation: $\frac{\partial u}{\partial t}(x,t) + a(x,t)\frac{\partial u}{\partial x}(x,t) = 0$

Method of characteristics

characteristic $x = x(t)$ are such that $\frac{dx}{dt} = a(x(t),t)$

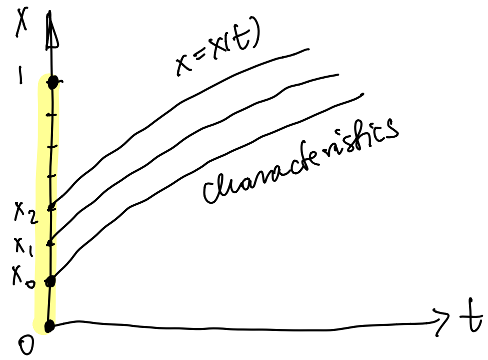
$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial x}(x(t),t)\cdot \underbrace{\frac{dx}{dt}}_{=a(x(t),t)} + \frac{\partial u}{\partial t}(x(t),t)$$

$$= \frac{\partial u}{\partial t}(x(t),t) + a(x(t),t)\cdot\frac{\partial u}{\partial x}(x(t),t) = 0$$

So, $\frac{d}{dt}u(x(t),t) = 0$ and $u(x(t),t)$ is a constant function of t.

from the initial condition

$u^0(x)$



Each characteristic satisfies the initial-value problem

$$(1)\quad \begin{cases} \frac{dx}{dt} = a(x(t),t) \\ x(0) = u^0(x_i), \quad i = 0,1,2,\dots,i_{max} \end{cases}$$

grid point

If we find the solution $x(t)$ of (1), then $u(x(t),t) = u^0(x_i)$.

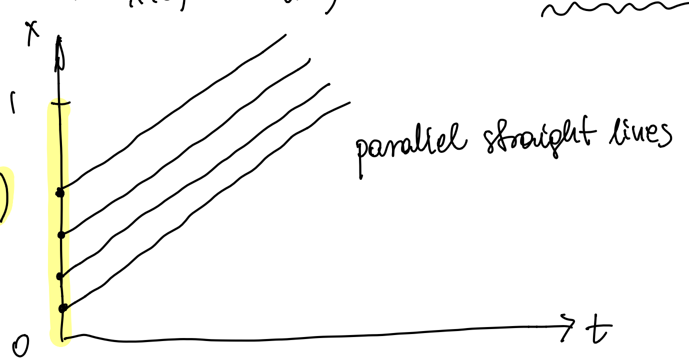
Note: if $a(x,t)$ is continuous and Lipschitz continuous with respect to x, then the characteristics $x(t)$ don't cross each other.

Example $a(x,t) \equiv a$ (constant). Then, $\frac{dx}{dt} = a \Rightarrow dx = a\,dt$

$\Rightarrow x = x(t) = at + c$, where c is a constant of integration.

$\Rightarrow x(0) = u^0(x_i) = c \Rightarrow \underbrace{x(t) = at + u^0(x_i)}$

$u^0(x_i)$



Then, $u(x(t),t) = u(at + u^0(x_i), t) = u^0(x_i) = x(t) - at$

So, $u(x(t),t) = x(t) - at$.

Finite difference scheme for $\frac{\partial u}{\partial t}(x,t) + a\frac{\partial u}{\partial x}(x,t) = 0$

a → constant

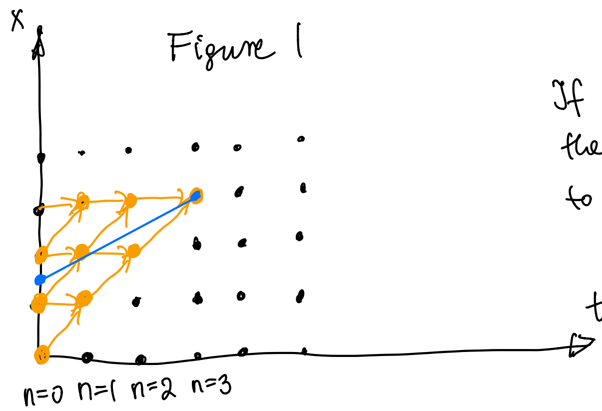
$$\frac{u_i^{n+1} - u_i^n}{h} + a\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Then,

$$u_i^{n+1} = u_i^n - \underbrace{\frac{ah}{\Delta x}}_{=\mu}\left(u_i^n - u_{i-1}^n\right) = u_i^n - \mu u_i^n + \mu u_{i-1}^n$$

(1) $$u_i^{n+1} = (1-\mu)\,u_i^n + \mu\, u_{i-1}^n$$

Figure 1



If the characteristic is within the grid-points, then the numerical scheme (1) is convergent to the exact solution $u(x,t)$.

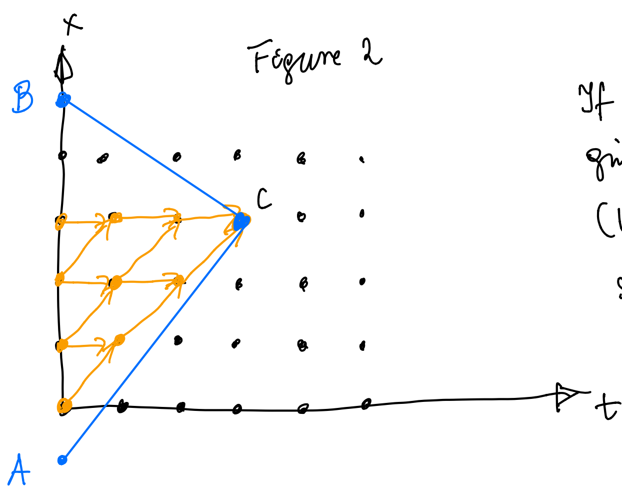
u_i^3 needs $u_i^2,\ u_{i-1}^2$

$u_i^2 \to u_i^1,\ u_{i-1}^1$; $u_{i-1}^2 \to u_{i-1}^1,\ u_{i-2}^1$

$u_i^1 \to u_i^0,\ u_{i-1}^0$; $u_{i-1}^1 \to u_{i-1}^0,\ u_{i-2}^0$; $u_{i-2}^1 \to u_{i-2}^0,\ u_{i-3}^0$

u_i^n needs: $u_i^0, u_{i-1}^0, u_{i-2}^0, \dots, u_{i-n}^0$

Figure 2



If the characteristic is outside of the grid-points needed for u_i^n, then (1) is not convergent to the exact solution $u(x,t)$.