$$\begin{cases} -\frac{d}{dt}\left(\sin(t)\frac{dy}{dt}\right) + 2\sin(t)u(t) = 2\sin(2t) \\ u(0) = 1, \quad u(\pi) = -1 \end{cases}$$

$$\psi(t) = \frac{B-A}{b-a}(t-a) + A = \frac{-1-1}{\pi}(t-0) + 1 = \frac{-2t}{\pi} + 1$$

$p(t) = \sin(t)$; $r(t) = 2\sin(t)$; $f(t) = 2\sin(2t)$

(a) $A(\varphi_i, \varphi_i) = \int_0^{\pi} p(t)(\varphi_i'(t))^2 dt + \int_0^{\pi} r(t)(\varphi_i(t))^2 dt = a_{ii}$

$$= \int_{t_{i-1}}^{t_i} \sin(t)(\underbrace{\varphi_i'(t)}_{1/h})^2 dt + \int_{t_i}^{t_{i+1}} \sin(t)(\underbrace{\varphi_i'(t)}_{-1/h})^2 dt + 2\int_{t_{i-1}}^{t_i} \sin(t)(\varphi_i(t))^2 dt$$

$$+ 2\int_{t_i}^{t_{i+1}} \sin(t)(\varphi_i(t))^2 dt$$

$$\approx 1/h \sin(t_{i-1/2}) + 1/h \sin(t_{i+1/2}) + \frac{2h}{2}\left[\sin(t_{i-1})(\overbrace{\varphi_i(t_{i-1})}^{0})^2 + \sin(t_i)(\overbrace{\varphi_i(t_i)}^{1})^2\right]$$

$$+ \frac{2h}{2}\left[\sin(t_i)(\overbrace{\varphi_i(t_i)}^{1})^2 + \sin(t_{i+1})(\overbrace{\varphi_i(t_{i+1})}^{0})^2\right]$$

$$A(\varphi_i, \varphi_i) = 1/h \sin(t_{i-1/2}) + 1/h \sin(t_{i+1/2}) + 2h\sin(t_i) = a_{ii}$$

(b) $A(\varphi_{i-1}, \varphi_i) = A(\varphi_i, \varphi_{i-1}) = \int_0^{\pi} p(t)\varphi_{i-1}'(t)\varphi_i'(t) dt + \int_0^{\pi} r(t)\varphi_i(t)\varphi_{i-1}(t) dt$

$$= \int_{t_{i-1}}^{t_i} \sin(t)\varphi_{i-1}'(t)\varphi_i'(t) dt + \int_{t_{i-1}}^{t_i} 2\sin(t)\varphi_{i-1}(t)\varphi_i(t) dt$$

$$\approx \frac{-1}{h^2}\sin(t_{i-1/2}) \cdot h + \frac{2h}{2}\left[\sin(t)\overbrace{\varphi_{i-1}(t_{i-1})}^{=1}\overbrace{\varphi_i(t_{i-1})}^{=0} + \sin(t)\overbrace{\varphi_{i-1}(t_i)}^{=0}\overbrace{\varphi_i(t_i)}^{=1}\right]$$

$$A(\varphi_{i-1},\varphi_i) \approx \frac{-1}{h^2}\sin(t_{i-1/2})\cdot h + \frac{2h}{2}\Big[\sin(t_{i-1})\overbrace{\varphi_{i-1}(t_{i-1})}^{=1}\overbrace{\varphi_i(t_{i-1})}^{=0}\Big]$$

$$+ \frac{2h}{2}\Big[\sin(t_i)\overbrace{\varphi_{i-1}(t_i)}^{=0}\overbrace{\varphi_i(t_i)}^{=1}\Big]$$

$$A(\varphi_{i-1},\varphi_i) = \frac{-1}{h}\sin(t_{i-1/2}) = a_{i-1,i} = a_{i,i-1}$$

ⓒ $b_i = \langle f,\varphi_i\rangle - A(\varphi,\varphi_i)$

$$\langle f,\varphi_i\rangle = \int_0^{\pi} f(t)\varphi_i(t)\,dt = \int_{t_{i-1}}^{t_i} f(t)\varphi_i(t)\,dt + \int_{t_i}^{t_{i+1}} f(t)\varphi_i(t)\,dt$$

$$\approx \frac{2h}{2}\Big[\sin(2t_{i-1})\overbrace{\varphi_i(t_{i-1})}^{=0} + \sin(2t_i)\overbrace{\varphi_i(t_i)}^{=1}\Big] + \frac{2h}{2}\Big[\sin(2t_i)\overbrace{\varphi_i(t_i)}^{=1} + \sin(2t_{i+1})\overbrace{\varphi_i(t_{i+1})}^{=0}\Big]$$

$$\langle f,\varphi_i\rangle = 2h\sin(2t_i)$$

$$A(\varphi,\varphi_i) = \underbrace{\int_0^{\pi}\sin(t)\varphi'(t)\varphi_i'(t)\,dt}_{*} + \underbrace{\int_0^{\pi}2\sin(t)\varphi(t)\varphi_i(t)\,dt}_{**} \quad -①$$

from *

$$\int_0^{\pi} \sin(t)\varphi'(t)\varphi_i'(t)\,dt = \int_{t_{i-1}}^{t_i} \sin(t)\left(-2/\pi\right)\cdot 1/h\,dt + \int_{t_i}^{t_i} \sin(t)\left(-2/\pi\right)\left(-1/h\right)dt$$

$$\approx \frac{h}{2}\cdot\frac{1}{h}\left(\frac{-2}{\pi}\right)\left[\sin(t_i)+\sin(t_{i-1})\right] + \frac{h}{2}\left(\frac{-1}{h}\right)\left(-2/\pi\right)\left[\sin(t_{i+1})+\sin(t_i)\right]$$

$$= \frac{1}{\pi}\left[\sin(t_{i+1})-\sin(t_{i-1})\right]$$

from **

$$\int_0^{\pi} 2\sin(t)\varphi(t)\varphi_i(t)\,dt = \int_{t_{i-1}}^{t_i} 2\sin(t)\varphi(t)\varphi_i(t)\,dt + \int_{t_i}^{t_{i+1}} 2\sin(t)\varphi(t)\varphi_i(t)\,dt$$

$$\approx \frac{h}{2}\Big[2\sin(t_i)\varphi_i \underbrace{\varphi_i(t_i)}_{=1} + 2\sin(t_{i+1})\varphi_{i+1}\underbrace{\varphi_i(t_{i+1})}_{=0}\Big] + \frac{h}{2}\Big[2\sin(t_i)\varphi_i\underbrace{(\varphi_i(t_i))}_{=1}$$
$$+ 2\sin(t_{i-1})\varphi_{i-1}\underbrace{\varphi_i(t_{i-1})}_{=0}\Big]$$

$$\int_0^{\pi} 2\sin(t)\varphi(t)\varphi_i(t)\,dt = 2h\sin(t_i)\varphi_i$$

$$A(\varphi,\varphi_i) = \frac{1}{\pi}\left[\sin(t_{i+1})-\sin(t_{i-1})\right] + 2h\sin(t_i)\varphi_i$$

Substituting * and ** into 4

$$b_i = \langle f,\varphi_i\rangle - A(\varphi,\varphi_i)$$

$$= \cancel{\frac{1}{\pi}\left(\sin(t_{i+1})-\sin(t_{i-1})\right)}$$

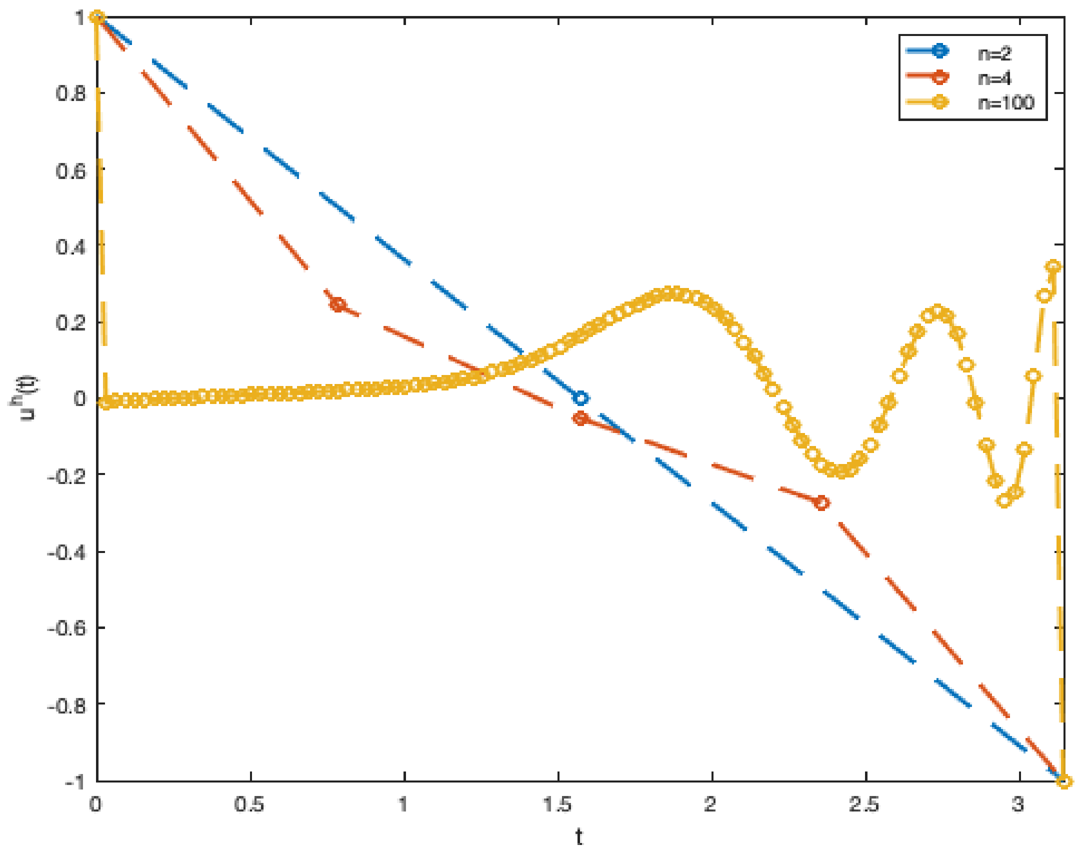
$$b_i = 2h\sin(2t_i) + \frac{1}{\pi}\left[\sin(t_{i-1})-\sin(t_{i+1})\right] - 2h\sin(t_i)\varphi_i$$

$$\varphi_i = -\frac{2}{\pi}t_i + 1$$

$$b_i = 2h\sin(2t_i) + \frac{1}{\pi}\left(\sin(t_{i-1})-\sin(t_{i+1})\right) + \left(\frac{4ht_i}{\pi}-2h\right)\sin(t_i)$$

```
clc;clear all;close all


n=[2,4,100]; % n values being considered

for i=1:length(n)
    [U,t]=Nscheme(n(i));
    plot(t,U,'--o',LineWidth=2),hold on

end
legend('n=2','n=4','n=100')
xlabel('t')
ylabel('u^h(t)')
axis tight



function [U,t]=Nscheme(n)

h=pi/n; % step size


% Entries of matrix A
for i=2:n
    ti=(i-1)*h; % i=1,2,3,...n-1

    % diagonal entries
    A(i-1,i-1)=(1/h)*(sin(ti-h/2)+sin(ti+h/2))+2*h*sin(ti);

    % upper diagonal entries
    A(i-1,i)=-1/h*(sin(ti-h/2));

    % lower diagonal entries
    A(i,i-1)=A(i-1,i);
end

%Reshaping matrix A to by of size n-1 by n-1
A=A(1:n-1,1:n-1);

% creating the time vector
for i=1:n+1
    t(i)=(i-1)*h; % i = 0,1,2,....n
end

% Initializing the b vector

b=zeros(1,size(A,1))';
for i=2:n
    b(i-1)=2*h*sin(2*t(i))+(1/pi)*(sin(t(i-1))-sin(t(i+1)))+(4*h*t(i)/
pi-2*h)*sin(t(i));
end
```

```matlab
c=A\b; % coefficients of the Galerkin approximation


% initializing the the numerical approximation storage space
U=zeros(1,n+1);

% Boundary Values
U(1)=1;U(end)=-1;


% Implementation of equation (2)
for k=2:length(t)-1

    for i=2:length(t)-1
        U(k)=U(k)+hat(i,t,t(k),h)*c(i-1);
    end

    U(k)=U(k)+(hat0(k,t,h)-hatn(k,t,h));

end

end


%%%% The hat functions %%%%%
function phi=hat(i,t,t_i,h)
    phi=0;
    % instances where phi is non-zero
    if (t(i-1)<=t_i) && (t_i<=t(i))
        phi=(t_i-t(i-1))/h;
    elseif (t(i)<=t_i) && (t_i<=t(i+1))
        phi=(t(i+1)-t_i)/h;
    end
end

function phin=hatn(i,t,h)
    phin=0;
     % instance where phin is non-zero
    if (t(end-1)<=t(i)) && (t(i)<=t(end))
        phin=(t(i)-t(end-1))/h;
    end
end


 function phi0=hat0(i,t,h)
    phi0=0;
    % instance where phi0 is non-zero
     if (t(1)<=t(i)) && (t(i)<=t(2))
         phi0=(t(2)-t(i))/h;
      end
 end
```



Published with MATLAB® R2022a