Last time:

$$D_+\|e^{(k+1)}(t)\| \leq \overbrace{\mu(a(t)A)}^{=a(t)\mu(A)}\|e^{(k+1)}(t)\| + a(t)\|B\|\,\|e^{(k)}(t)\| + |b(t)|\,\|F\|_0\cdot\|e^{(k)}_t\|_0,$$

where D_+ is the right-hand side derivative

By the comparison theorem, we conclude that

$$(13)\quad \left\{\begin{aligned}
\|e^{(k+1)}(t)\| &\leq \int_0^t \exp\Big(\mu(A)\int_\tau^t a(s)\,ds\Big)\, a(\tau)\|B\|\cdot\|e^{(k)}(\tau)\| + |b(\tau)|\,\|F\|_0\|e^{(k)}_\tau\|_0\,d\tau\\
&\leq \int_0^t \exp\Big(\mu(A)\int_\tau^t a(s)\,ds\Big)\Big[a(\tau)\|B\| + |b(\tau)|\,\|F\|_0\Big]\|e^{(k)}_\tau\|_0\,d\tau\\
&\leq \int_0^t a(\tau)\exp\Big(\mu(A)\int_\tau^t a(s)\,ds\Big)\Big[\|B\| + \underbrace{\frac{|b(\tau)|}{a(\tau)}}_{\leq \max\left\{\frac{|b(\tau)|}{a(\tau)}\,:\,\tau\in[0,t]\right\} \overset{\text{definition}}{=} b_0(t)}\|F\|_0\Big]\|e^{(k)}_\tau\|_0\,d\tau\\
&\leq \Big[\|B\| + b_0(t)\|F\|_0\Big]\int_0^t a(\tau)\exp\Big(\mu(A)\int_\tau^t a(s)\,ds\Big)\|e^{(k)}_\tau\|_0\,d\tau
\end{aligned}\right.$$

We want to use (13) to prove the error bounds:

$$(14)\quad \|e^{(k)}(t)\| \leq \left(\frac{\|B\|+b_0(t)\|F\|_0}{-\mu(A)}\right)^k \alpha_k\big(-\mu(A)\tilde{a}(t)\big)\max_{\tau\in[0,t]}\|e^{(0)}(\tau)\|,$$

for $\mu(A)\neq 0$ and

$$(15)\quad \|e^{(k)}(t)\| \leq \frac{1}{k!}\Big(\big[\|B\|+b_0(t)\|F\|_0\big]\tilde{a}(t)\Big)^k\max_{\tau\in[0,t]}\|e^{(0)}(\tau)\|,$$

for $\mu(A)=0$,

where $\tilde{a}(t)=\int_0^t a(s)\,ds$, $\quad \alpha_k(\tau)=1-e^{-\tau}\sum_{j=0}^{k-1}\frac{\tau^j}{j!}$

Let $E(t)=\max_{\tau\in[0,t]}\|e^{(0)}(\tau)\|$, $\mu(A)\neq 0$. Then, (13) implies that (14) is satisfied for $k=0$. To apply mathematical induction, we assume (14) for a certain k and prove it for $k+1$.

From (13), we get

$$\|e^{(k+1)}(t)\| \leq \big[\|B\|+b_0(t)\|F\|_0\big]\int_0^t a(\tau)\exp\Big(\mu(A)\underbrace{\int_\tau^t a(s)\,ds}_{\substack{=\int_\tau^0 a(s)\,ds+\int_0^t a(s)\,ds\\ =-\tilde{a}(\tau)+\tilde{a}(t)}}\Big)\|e^{(k)}_\tau\|_0\,d\tau$$

$$e^{\mu(A)\tilde{a}(t)}\cdot e^{-\mu(A)\tilde{a}(\tau)}$$

$$= \left[\|B\| + b_0(t)\|F\|_0\right] e^{\mu(A)\tilde{a}(t)} \int_0^t a(\tau)\, e^{-\mu(A)\tilde{a}(\tau)} \underbrace{\|e_\tau^{(k)}\|_0}\, d\tau$$

$$\|e_\tau^{(k)}\|_0 \le \left(\frac{\|B\| + b_0(\tau)\|F\|_0}{-\mu(A)}\right)^k \alpha_k\left(-\mu(A)\tilde{a}(\tau)\right) E(\tau)$$

because the right-hand side function is increasing on τ
and we use the inductive assumption

$$\le \left(\frac{\|B\| + b_0(t)\|F\|_0}{-\mu(A)}\right)^{k+1} e^{\mu(A)\tilde{a}(t)} E(t) \int_0^t -\mu(A)\, a(\tau)\, e^{-\mu(A)\tilde{a}(\tau)} \underbrace{\alpha_k\left(-\mu(A)\tilde{a}(\tau)\right)}_{= 1 - e^{\mu(A)\tilde{a}(\tau)} \sum_{j=0}^{k-1} \frac{(-\mu(A)\tilde{a}(\tau))^j}{j!}} d\tau$$

$$= \left(\frac{\|B\| + b_0(t)\|F\|_0}{-\mu(A)}\right)^{k+1} e^{\mu(A)\tilde{a}(t)} E(t) \int_0^t -\mu(A)\, a(\tau) \left[e^{-\mu(A)\tilde{a}(\tau)} - \sum_{j=0}^{k-1} \frac{(-\mu(A)\tilde{a}(\tau))^j}{j!} \right] d\tau$$

$$= \left(\frac{\|B\| + b_0(t)\|F\|_0}{-\mu(A)}\right)^{k+1} e^{\mu(A)\tilde{a}(t)} E(t) \left[e^{-\mu(A)\tilde{a}(\tau)} - \sum_{j=0}^{k-1} \frac{(-\mu(A)\tilde{a}(\tau))^{j+1}}{(j+1)!} \right]_{\tau=0}^{\tau=t}$$

$$= \left(\frac{\|B\| + b_0(t)\|F\|_0}{-\mu(A)}\right)^{k+1} e^{\mu(A)\tilde{a}(t)} E(t) \left[e^{-\mu(A)\tilde{a}(t)} \underbrace{- 1 - \underbrace{\sum_{j=0}^{k-1} \frac{(-\mu(A)\tilde{a}(t))^{j+1}}{(j+1)!}}_{= \sum_{j=1}^{k} \frac{(-\mu(A)\tilde{a}(t))^j}{j!}}}_{= - \sum_{j=0}^{k} \frac{(-\mu(A)\tilde{a}(t))^j}{j!}} \right]$$

$$= \left(\frac{\|B\| + b_0(t)\|F\|_0}{-\mu(A)}\right)^{k+1} E(t) \left[\underbrace{1 - e^{\mu(A)\tilde{a}(t)} \sum_{j=0}^{k} \frac{(-\mu(A)\tilde{a}(t))^j}{j!}}_{= \alpha_{k+1}\left(-\mu(A)\tilde{a}(t)\right)} \right]$$