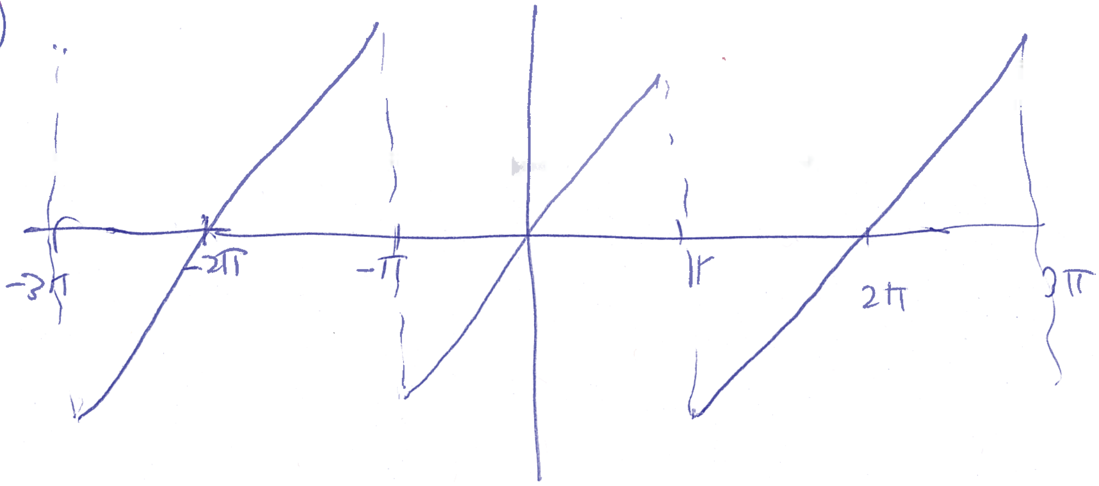
2

(a)



2

2

(b) $f(x)$ is odd so we can use $\frac{1}{2}$-range formula for Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\,dx .$$ 2

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\frac{\cos(nx)}{n}\,dx .$$

$$= \frac{-2\pi}{n\pi}(-1)^n + \frac{2}{\pi}\left[\cancel{\frac{\sin(nx)}{n^2}}\right]_0^{\pi}$$

$$= \frac{2}{n}(-1)^{n+1}$$ 2

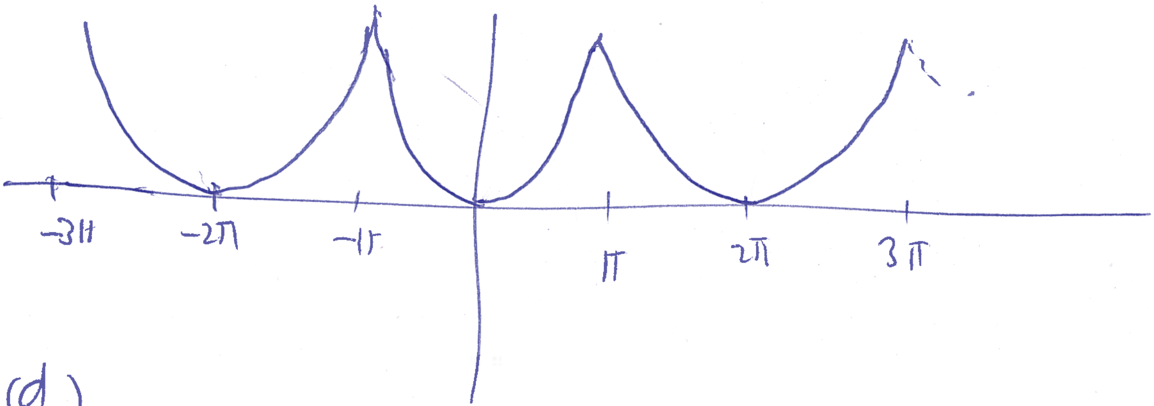
4

5

So $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$

(c) $g(x) = \frac{1}{2}x^2$ for $-\pi < x < \pi$. (1)
and is 2π-periodic.

So graph is



(2)

(d) $\boxed{3}$

We can always integrate the F.S. term by term.
So $g(x)$ has F.S.

$$g(x) = C + \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^{n+1}\frac{\cos(nx)}{n}.$$

$$= C + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos(nx)$$ (2)

$$C = \frac{1}{2}a_0 = \frac{1}{\pi}\int_0^{\pi} \frac{x^2}{2}\,dx = \frac{1}{\pi}\left[\frac{x^3}{6}\right] = \frac{\pi^2}{6}$$

So $g(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos(nx)$ (2)

$\boxed{4}$

(e) When $x=0$ $g(0)=0$ So.

$$0 = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n .$$

$$\Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

(f) $f(x)$ has a discontinuity at $x=(2k+1)\pi$, $k\in\mathbb{Z}$
so we cannot diff. the F.S. of
$f(x)$ to obtain that of $f'(x)$.

2

$\boxed{2}$

(3) $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

Let $u(x,t) = X(x)\,T(t)$.

Then. $X''T = \frac{1}{c^2} X \ddot{T}$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = \lambda = \text{const}.$$

fct of x ↘ ↙ fct of t.

const.

(2)

$$\Rightarrow \quad X'' - \lambda X = 0 \quad -①$$
$$\ddot{T} - \lambda c^2 T = 0 \quad -②$$

(1)

B.C. for u give.

$0 = \frac{\partial u}{\partial x}(0,t) = X'(0)\,T(t) \Rightarrow X'(0) = 0$ for non-trivial solution.

$0 = \frac{\partial u}{\partial x}(L,t) = X'(L)\,T(t) \Rightarrow X'(L) = 0$ "

(1)

So we need to solve the eigenvalue problem.

$$X'' - \lambda X = 0, \quad X'(0) = 0, \quad X'(L) = 0.$$

3 cases:

(1) $\lambda > 0$. Let $\lambda = \mu^2$ $(\mu \neq 0)$ then $X'' - \mu^2 X = 0$

$$\Rightarrow X(x) = Ae^{\mu x} + Be^{-\mu x}.$$
$$X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}.$$
$$X'(0) = 0 \Rightarrow A\mu - B\mu = 0 \Rightarrow A = B.$$

So $X'(x) = A\mu(e^{\mu x} - e^{-\mu x}) = 2A\mu\sinh(\mu x)$

$X'(L) = 0 \Rightarrow 0 = 2A\mu\sinh(\mu L) \Rightarrow A = 0.$

(1)

So only the trivial solution for $\lambda > 0$.
(alternatively quote appropriate Sturm-Liouville theorem).

(ii) $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = A + Bx$.

$X'(x) = B$.

$X'(0) = 0 \Rightarrow B = 0$.

$X'(L) = 0 \Rightarrow B = 0$. (1)

So when $\lambda = 0$ $X_0(x) = A_0 = \text{const}$. is non-trivial soln.

(iii) $\lambda < 0$. Let $\lambda = -\mu^2$ ($\mu \neq 0$).

Then $X'' + \mu^2 X = 0 \Rightarrow X(x) = A\cos(\mu x) + \cancel{B\sin(\mu x)}$. (2)

$X'(x) = -A\mu\sin(\mu x) + \cancel{B\mu\cos(\mu x)}$.

$X'(0) = 0 \Rightarrow B = 0$.

$X'(L) = 0 \Rightarrow 0 = -A\mu\sin(\mu L)$.

$\Rightarrow \mu L = k\pi, \quad k \in \mathbb{Z}$

$\Rightarrow \mu = \frac{k\pi}{L}, \quad k \in \mathbb{Z}\setminus\{0\}$

So eigenvalues are $\lambda_k = -\left(\frac{k\pi}{L}\right)^2$

eigenfunctions are. $X_k(x) = A_k\cos\left(\frac{k\pi x}{L}\right)$ $\quad k \in \mathbb{Z}\setminus\{0\}$.

The negative integers give nothing new so we can take $k \in \mathbb{N}$. (2)

For the T eqn. when $\lambda = 0$ we have.

$\ddot{T} = 0 \Rightarrow T_0 = C_0 + D_0 t$.

From the initial condition $u(x,0) = 0$ we want $T(0) = 0 \Rightarrow A_0 = 0$ so $T_0(t) = D_0 t$. (1)

When $\lambda = -\left(\frac{n\pi}{L}\right)^2$ we get.

$$\ddot{T} + \left(\frac{n\pi c}{L}\right)^2 T = 0$$

$$\Rightarrow T_n(t) = C_n \cos\left(\frac{n\pi c t}{L}\right) + D_n \sin\left(\frac{n\pi c t}{L}\right)$$

$T_n(0) = 0 \Rightarrow C_n = 0$.

So $T_n(t) = D_n \sin\left(\frac{n\pi c t}{L}\right)$. (2)

So $u_0(x,t) = B_0 t$

$$u_n(x,t) = B_n \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right)$$

Hence by the principle of superposition.

$$u(x,t) = B_0 t + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right) \quad (2)$$

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(b) $$\frac{\partial u}{\partial t} = D_0 + \sum_{n=1}^{\infty} \frac{n\pi c D_n}{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right) \quad (2)$$

When $t=0$ we get.

$$\frac{\partial u}{\partial t}(x,0) = L - x = D_0 + \sum_{n=1}^{\infty} \frac{n\pi c D_n}{L} \cos\left(\frac{n\pi x}{L}\right)$$

So $D_0 = \frac{a_0}{2}$ & $\frac{n\pi c}{L} D_n = a_n$ where (1)

$a_0, a_1, \ldots$ are the Fourier Coeffs. of the Cosine series for $L-x$.

10

$$a_0 = \frac{2}{L}\int_0^L (L-x)\,dx = \frac{2}{L}\left[Lx - \tfrac{1}{2}x^2\right]_0^L = L$$

$$\text{So } D_0 = \frac{L}{2}$$

(2)

$$a_n = \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\frac{L a_n}{2} = \cancel{\left[(L-x)\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L} + \int_0^L \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \left[\frac{-L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{L^2}{n^2\pi^2}\left(1-(-1)^n\right)$$

(3)

$$\text{So } a_n = \frac{2L}{n^2\pi^2}\left(1-(-1)^n\right) \Rightarrow D_n = \frac{L a_n}{n\pi c} = \frac{2L^2}{(n\pi)^3 c}\left(1-(-1)^n\right)$$

$$\text{So } u(x,t) = \frac{Lt}{2} + \sum_{n=1}^{\infty} \frac{2L^2}{(n\pi)^3 c}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

(2)

10

$$= \frac{Lt}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4L^2}{(n\pi)^3 c}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

(4) $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Let $u(r,\theta) = R(r)T(\theta)$

$$R''T + \frac{1}{r}R'T = -\frac{1}{r^2}RT''$$

$$\Rightarrow r^2R''T + rR'T = -RT''$$

$$\Rightarrow \underbrace{\frac{r^2R'' + rR'}{R}}_{\text{fct of } r} = \underbrace{-\frac{T''}{T}}_{\text{fct of } t} = \lambda = \text{const.} \quad (2)$$

$$\Rightarrow r^2R'' + rR' - \lambda R = 0 \quad -(1)$$

$$T'' + \lambda T = 0 \quad -(2) \quad (1)$$

We need to solve

$T'' + \lambda T = 0$ with 2π-periodic B.C's. (1)

3 cases $\lambda < 0$ $T'' - \mu^2 T = 0$ where $\lambda = -\mu^2$.

No periodic solutions. (1)

$\lambda = 0$ $T'' = 0 \Rightarrow T = A + Bt$

So $T_0(\theta) = A_0$ is a (constant) periodic soln (1)

$\lambda > 0$ $T'' + \mu^2 T = 0$ when $\lambda = \mu^2$ $\mu \neq 0$.

$T(t) = A\cos(\mu\theta) + B\sin(\mu\theta)$

For 2π-periodic solutions we need $\mu \in \mathbb{Z} \setminus \{0\}$.

But negative integers give nothing new so take

$\lambda_n = +n^2$, $T_n(t) = A_n\cos(n\theta) + B_n\sin(n\theta)$ (2)

Putting $\lambda_n = +n^2$ in ① gives.

$$r^2R'' + rR' - n^2R = 0$$

This is an Euler eqn. Put $R = r^k$.

$$(k^2 - n^2)r^k = 0$$

So $k = \pm n$. and solutions are

$$R_n(r) = C_n r^n + D_n r^{-n}$$

For solutions <u>finite</u> at $r=0$ we must have $D_n = 0$

so $R_n(r) = C_n r^n, \quad n = 1, 2, \ldots$ ③

For $n=0$ we have $r^2R'' + rR' = 0$

$$\Rightarrow (rR')' = 0$$

$$\Rightarrow rR' = D_0$$

$$\Rightarrow R' = \frac{D_0}{r}$$

$$\Rightarrow R = D_0 \ln r + C_0$$

For solution <u>finite</u> at $r=0$ we must have $D_0 = 0$.

So $u_0(r,\theta) = A_0$ ②

$$u_n(r,\theta) = r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

So the G.S. is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right].$$

②

$\boxed{15}$

(b) Putting $r=1$ in $u(r,\theta)$ we get.

$$|\theta| = u(1,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

Now $|\theta|$ is even for $-\pi < \theta < \pi$.

So B_n vanish and A_n are given by the ½-range formula.

(2)

$$A_0 = \frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} \theta \, d\theta = \frac{1}{\pi}\left[\frac{\theta^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

(2)

$$A_n = a_n = \frac{2}{\pi}\int_0^{\pi} \theta \cos(n\theta)\, d\theta.$$

$$= \frac{2}{\pi}\left[\cancel{\theta \frac{\sin(n\theta)}{n}}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \frac{\sin(n\theta)}{n}\, d\theta.$$

$$= \frac{2}{\pi}\left[\frac{\cos(n\theta)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

(4)

$$\text{So } u(r,\theta) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^2} r^n \cos(n\theta)$$

(2)

$\boxed{10}$