Question 1

1a

From the definition of the truncation error

$$T_n = \frac{y(x_n+h) - y(x_n)}{h} - \Phi(x_n, y(x_n), h) \quad -① $$

Where the continuous function $\Phi(x_n, y(x_n), h) = \frac{1}{2}(k_1 + k_2)$

From Taylor Series expansion:

$$y(x_n+h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + O(h^4)$$

① can thus be re-written as:

$$T_n = \frac{1}{h}\left[y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) - y(x_n) + O(h^4)\right] - \Phi(x_n, y(x_n), h)$$

Here

$$y'(x_n) = f(x_n, y(x_n)) = f$$

$$y''(x_n) = \frac{\partial f}{\partial x}(x_n, y(x_n)) + \frac{\partial f}{\partial y}(x_n, y(x_n)) \cdot \overbrace{y'(x_n)}^{f(x_n, y(x_n))}$$

$$= f_x + f_y y' = f_x + f_y \cdot f$$

$$y'''(x_n) = \frac{\partial^2 f}{\partial x^2}(x_n, y(x_n)) + \frac{\partial^2 f}{\partial y \partial x}(x_n, y(x_n))y'(x_n) + \left[\frac{\partial^2 f}{\partial y \partial x}(x_n, y(x_n)) + \frac{\partial^2 f}{\partial y^2}(x_n, y(x_n))\right]y'$$

$$+ \frac{\partial f}{\partial y}(x_n, y(x_n))y''(x_n)$$

$$= f_{xx} + f_{xy}y' + f_{xy}y' + f_{yy}[y']^2 + f_y y''$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y[f_x + f_y \cdot f]$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + [f_y]^2 \cdot f$$

1b

From 1

$$\Phi(x_n, y(x_n), h) = \tfrac{1}{2}(k_1 + k_2)$$
$$= \tfrac{1}{2} f(x_n, y_n) + \tfrac{1}{2} f(x_n + h, y_n + h f(x_n, y_n))$$
$$= \tfrac{1}{2} f(x_n, y_n) + \tfrac{1}{2}\Big[f(x_n, y_n) + \frac{h}{1!}\frac{\partial f}{\partial x}(x_n, y_n) + \frac{h}{1!} f(x_n, y_n)\frac{\partial f}{\partial y}(x_n, y_n)$$
$$+ \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}(x_n, y_n) + \frac{2h^2}{2!} f(x_n, y_n)\frac{\partial^2 f}{\partial x \partial y}(x_n, y_n) + \frac{(h f(x_n, y_n))^2}{2!}\frac{\partial^2 f}{\partial y^2}(x_n, y_n)$$
$$+ O(h^3)\Big]$$
$$= \frac{1}{2} f + \frac{1}{2} f\left[f + h f_x + h f f_y + \frac{h^2}{2} f_{xx} + h^2 f f_{xy} + \frac{h^2}{2} f^2 f_{yy}\right] + O(h^3)$$

Substituting this result into 1

$$T_n = f + \frac{1}{2} h [f_x + f f_y] + \frac{1}{6} h^2 [f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_x f_y + f_y^2 f]$$
$$- \frac{1}{2} f - \frac{1}{2}[f + h f_x + h f f_y + h^2/2 f_{xx} + h^2 f f_{xy} + h^2/2 f^2 f_{yy}] + O(h^3)$$

Simplifying the terms:

$f - 1/2 f - 1/2 f = 0$

$1/2 h f_x - 1/2 h f_x = 0$

$1/2 h f f_y - 1/2 h f f_y = 0$

$\frac{1}{6} h^2 f_{xx} - \frac{h^2}{4} f_{xx} = \frac{-1}{12} h^2 f_{xx}$

$\frac{h^2 f f_{xy}}{3} - \frac{h^2 f f_{xy}}{2} = \frac{-h^2 f f_{xy}}{6}$

$\frac{h^2}{6} f^2 f_{yy} - \frac{h^2}{4} f^2 f_{yy} = \frac{-1}{12} h^2 f^2 f_{yy}$

Thus;

$$T_n = \frac{h^2}{6} f_x f_y + \frac{h^2}{6}[f_y]^2 f - \frac{h^2}{12} f_{xx} - \frac{h^2}{6} f f_{xy} - \frac{h^2}{12} f^2 f_{yy}$$
$$= \frac{h^2}{6}\left[f_x f_y + [f_y]^2 f - 1/2 (f_{xx} + 2 f f_{xy} + f^2 f_{yy})\right]$$

1C

The method $y_{n+1} = y_n + \frac{1}{2}h(K_1 + K_2)$; $K_1 = f(x_n, y_n)$, $K_2 = f(x_n + h, y_n + hK_1)$ is consistent with the differential equation $y'(x) = f(x, y(x))$ if and only ~~if~~ if

$\forall \varepsilon > 0 \ \exists h_\varepsilon > 0 \ \forall h \in (0, h_\varepsilon) \ \forall \left. \begin{matrix}(x_n, y(x_n)) \\ (x_{n+1}, y(x_{n+1}))\end{matrix} \right\} \in [a,b] \times [y_0 - c, y_0 + c]$

$$\underbrace{\left| \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n), h) \right|}_{T_n} < \varepsilon \ ; \ \Phi(x_n, y(x_n), h) = \frac{1}{2}(K_1 + K_2)$$

$$\frac{1}{6}h^2 \left| f_x f_y + f(f_y)^2 - \frac{1}{2}(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + O(h^3) \right| < \varepsilon$$

Let $x_n = a + nh \xrightarrow[h \to 0]{n \to \infty} x \in [a,b]$. Then,

$$\lim_{\substack{n \to \infty \\ h \to 0}} |T_n| = \lim_{\substack{n \to \infty \\ h \to 0}} \frac{h^2}{6} \left| f_x f_y + f(f_y)^2 - \frac{1}{2}(f_{xx} + 2ff_{xy} + f^2 f_{yy}) \right| < \varepsilon$$

$0 < \varepsilon$.

Question 2

(2a)

By applying repeated integration by part to $\int_{x_n}^{x_{n+1}} (x-x_{n+1})(x-x_n)y'''(x)\,dx$, show that

$$y(x_{n+1})-y(x_n)=\frac{h}{2}\left[f(x_n,y(x_n))+f(x_{n+1},y(x_{n+1}))\right]-\frac{1}{2}\int_{x_n}^{x_{n+1}}(x_{n+1}-x)(x-x_n)\,y'''(x)\,dx$$

Let $u=(x-x_{n+1})(x-x_n)=x^2-xx_n-xx_{n+1}+x_nx_{n+1}$

$du=(2x-x_n-x_{n+1})dx$

$dv=y'''(x)\,dx\ ;\ v=y''(x)$

$$\int u\,dv=uv-\int v\,du$$

$$=(x-x_{n+1})(x-x_n)y''(x)\Big|_{x_n}^{x_{n+1}}-\int_{x_n}^{x_{n+1}}y''(x)\cdot(2x-x_n-x_{n+1})\,dx \quad — *$$

Again, we integrate the second term of * by parts

$\int_{x_n}^{x_{n+1}} y''(x)(2x-x_n-x_{n+1})\,dx$

Let : $u=2x-x_n-x_{n+1}$

$du=2$

$dv=y''(x)\,dx\ ;\ v=y'(x)$

$$\int_{x_n}^{x_{n+1}}(2x-x_n-x_{n+1})y''(x)\,dx=(2x-x_n-x_{n+1})y'(x)\Big|_{x_n}^{x_{n+1}}-2\int_{x_n}^{x_{n+1}}y'(x)$$

$$=(2x-x_n-x_{n+1})y'(x)\Big|_{x_n}^{x_{n+1}}-2y(x_{n+1})+2y(x_n)$$

$$=(x_{n+1}-x_n)y'(x_{n+1})-(x_n-x_{n+1})y'(x_n)-2y(x_{n+1})+2y(x_n)$$

$$=(x_{n+1}-x_n)\left[y'(x_{n+1})+y'(x_n)\right]-2y(x_{n+1})+2y(x_n) \quad — *^2$$

Substituting $*^2$ in *

$$\int_{x_n}^{x_{n+1}}(x-x_{n+1})(x-x_n)y'''(x)\,dx=\overbrace{(x-x_{n+1})(x-x_n)y''(x)}\Big|_{x_n}^{x_{n+1}}-\Big[(x_{n+1}-x_n)\big(y(x_{n+1})+y(x_n)\big)$$
$$-2y(x_{n+1})+2y(x_n)\Big]$$

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$$\int_{x_n}^{x_{n+1}}(x-x_{n+1})(x-x_n)y'''(x)\,dx=(x_n-x_{n+1})\big(y'(x_{n+1})+y'(x_n)\big)+2y(x_{n+1})-2y(x_n)$$

multiply through by -1

$$\int_{x_n}^{x_{n+1}}(x_{n+1}-x)(x-x_n)y'''(x)\,dx=(x_{n+1}-x_n)\big(y'(x_{n+1})+y'(x_n)\big)-2y(x_{n+1})+2y(x_n)$$

$$2y(x_{n+1}) - 2y(x_n) = (x_{n+1}-x_n)\left(y'(x_{n+1}) + y'(x)\right) - \int_{x_n}^{x_{n+1}} (x_{n+1}-x_n)(x-x_n)\, y'''(x)\, dx$$

But $y(x_{n+1}) = f(x_{n+1}, y(x_{n+1}))$; $y(x_n) = f(x_n, y(x_n))$ and $x_{n+1} - x_n = h$

$$y(x_{n+1}) - y(x_n) = \frac{h}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right] - \frac{1}{2}\int_{x_n}^{x_{n+1}} (x_{n+1}-x)(x-x_n)\, y'''(x)\, dx$$

2b

$$y(x_{n+1}) - y(x_n) = \frac{h}{2}\left(f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right) - \frac{1}{2}\int_{x_n}^{x_{n+1}} (x_{n+1}-x)(x-x_n)\, y'''(x)\, dx$$

$$\underbrace{\frac{y(x_{n+1}) - y(x_n)}{h} - \frac{1}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right]}_{T_n} = \frac{-1}{2h}\int_{x_n}^{x_{n+1}} (x_{n+1}-x)(x-x_n)\, y'''(x)\, dx$$

$$T_n = \frac{-1}{2h} y'''(\xi_n) \int_{x_n}^{x_{n+1}} (x_{n+1}-x)(x-x_n)\, dx$$

Since: $\int_{x_n}^{x_{n+1}} (x_{n+1}-x)(x-x_n)\, y'''(x)\, dx = y'''(\xi_n) \int_{x_n}^{x_{n+1}} (x_{n+1}-x)(x-x_n)\, dx$

$x = x_n + sh$; $x_n = x - sh$; $x_{n+1} = x - sh + h$; $dx = h\,ds$

$$T_n = -\frac{1}{2h} h^3 y'''(\xi_n) \int_{x_n}^{x_{n+1}} (1-s)(s)\, ds$$

Now we perform a change of variable for the limits. of the integral:

$x_n = x - sh$; $s = \frac{x - x_n}{h}$; if $x = x_n \Rightarrow s = \frac{x_n - x_n}{h} = 0$

$x_{n+1} = x - sh + h$; $s = \frac{x - x_{n+1} + h}{h}$, if $x = x_{n+1} \Rightarrow s = \frac{x_{n+1} - x_{n+1} + h}{h} = 1$

$$T_n = \frac{-h^2}{2} y'''(\xi_n) \int_0^1 (s - s^2)\, ds = \frac{-h^2}{2} y'''(\xi_n) \left[\frac{s^2}{2} - \frac{s^3}{3}\right]_0^1$$

$$T_n = \frac{-h^2}{12} y'''(\xi_n)$$

2c

By definition of the truncation error;

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \tfrac{1}{2}\left[f(x_{n+1}, y(x_{n+1})) + f(x_n, y(x_n))\right]$$

$$y(x_{n+1}) - y(x_n) = hT_n + \frac{1}{2}\left[f(x_{n+1}, y(x_{n+1})) + f(x_n, y(x_n))\right] \quad — (1)$$

and the Implicit method is expressed as

$$y_{n+1} = y_n + \tfrac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] \quad — (2)$$

$$e_{n+1} = y(x_{n+1}) - y_{n+1} \;;\; e_n = y(x_n) - y_n$$

Subtract (2) from (1)

$$y(x_{n+1}) - y_{n+1} = y(x_n) - y_n + \frac{h}{2}\left[f(x_n, y(x_n)) - f(x_n, y_n)\right] + \frac{h}{2}\left[f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y_{n+1})\right] + hT_n$$

$$e_{n+1} = e_n + \frac{h}{2}\left[f(x_n, y(x_n)) - f(x_n, y_n)\right] + \frac{h}{2}\left[f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y_{n+1})\right] + hT_n$$

2d

$$e_{n+1} = e_n + \frac{h}{2}\left[f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y_{n+1})\right] + \frac{h}{2}\left[f(x_n, y(x_n)) - f(x_n, y_n)\right] + hT_n$$

$$|e_{n+1}| = |e_n| + \frac{h}{2}\left|f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y_{n+1})\right| + \frac{h}{2}\left|f(x_n, y(x_n)) - f(x_n, y_n)\right| + h|T_n|$$

$$|e_{n+1}| \le |e_n| + \frac{Lh}{2}|e_{n+1}| + \frac{Lh}{2}|e_n| + h\left|-\frac{h^2}{12}y'''(\xi_n)\right|$$

$$|e_{n+1}| \le |e_n| + \frac{Lh}{2}\left(|e_{n+1}| + |e_n|\right) + \frac{Mh^3}{12}$$

2e

$$|e_{n+1}| \le |e_n| + \frac{Lh}{2}\left(|e_{n+1}| + |e_n|\right) + \frac{Mh^3}{12}$$

for $n = 0$

$$|e_1| \le |e_0| + \frac{Lh}{2}\left(|e_1| + |e_0|\right) + \frac{Mh^3}{12}$$

Since $y(x_0) = y_0$, $e_0 = 0$

$$|e_1| \le \left(\frac{2}{2 - hL}\right)\frac{Mh^3}{12}$$

for $n = 1$

$$|e_2| \le |e_1| + \frac{hL}{2}|e_1| + \frac{hL}{2}|e_2| + \frac{Mh^3}{12}$$

$$|e_2| \le \left(\frac{2+hL}{2-hL}\right)|e_1| + \left(\frac{2}{2-hL}\right)\frac{Mh^3}{12}$$

$$|e_2| \le \frac{Mh^3}{12}\left[\frac{2+hL}{2-hL} + 1\right]\left(\frac{2}{2-hL}\right)$$

for $n = 2$

$$|e_3| \le |e_2| + \frac{hL}{2}|e_2| + \frac{hL}{2}|e_3| + \frac{Mh^3}{12}$$

$$|e_3| = \left(\frac{2+hL}{2-hL}\right)|e_2| + \left(\frac{2}{2-hL}\right)\frac{Mh^3}{12}$$

$$= \left(\frac{2+hL}{2-hL}\right)\left[\frac{Mh^3}{12}\left(\frac{2+hL}{2-hL} + 1\right)\left(\frac{2}{2-hL}\right)\right] + \frac{2}{2-hL}\cdot\frac{Mh^3}{12}$$

$$= \frac{Mh^3}{12}\left[\left(\frac{2+hL}{2-hL}\right)^2 + \left(\frac{2+hL}{2-hL}\right) + 1\right]\left(\frac{2}{2-hL}\right)$$

Let $x = \frac{2+hL}{2-hL}$

$$|e_3| = \frac{Mh^3}{12}\left[x^2 + x + 1\right]\left(\frac{2}{2-hL}\right)$$

we re-write x^2+x+1 as $\frac{1-x^3}{1-x}$

Proof: $\frac{1-x^3}{1-x} = \frac{-(x-1)(x^2+x+1)}{1-x} = x^2+x+1$ ◻

$$|e_3| = \frac{Mh^3}{12}\left[\frac{1 - \left(\frac{2+hL}{2-hL}\right)^3}{1 - \left(\frac{2+hL}{2-hL}\right)}\right]\cdot\left(\frac{2}{2-hL}\right)$$

$$= \frac{Mh^3}{12}\left[\frac{1 - \left(\frac{2+hL}{2-hL}\right)}{\frac{-2hL}{2-hL}}\right]\cdot\left(\frac{2}{2-hL}\right)$$

$$|e_3| \leq \frac{Mh^3}{12}\left[\left(\frac{2+hL}{2-hL}\right)^3 - 1\right]\frac{2-hL}{2hL}\cdot\frac{2}{2-hL}$$

$$|e_3| \leq \frac{Mh^2}{12L}\left[\left(\frac{2+hL}{2-hL}\right)^3 - 1\right]$$

Let us suppose that

$$|e_n| \leq \frac{h^2M}{12L}\left[\left(\frac{2+hL}{2-hL}\right)^n - 1\right]; \text{ then}$$

$$|e_{n+1}| \leq |e_n| + \frac{hL}{2}\left(|e_{n+1}| + |e_n|\right) + \frac{Mh^3}{12}$$

$$|e_{n+1}| \leq \left(\frac{2+hL}{2-hL}\right)|e_n| + \frac{2}{2-hL}\cdot\frac{Mh^3}{12}$$

$$\leq \frac{2+hL}{2-hL}\left[\frac{h^2M}{12L}\left(\left(\frac{2+hL}{2-hL}\right)^n - 1\right)\right] + \frac{2}{2-hL}\cdot\frac{Mh^3}{12}$$

$$= \frac{Mh^2}{12L}\left[\left(\frac{2+hL}{2-hL}\right)^{n+1} - \left(\frac{2+hL}{2-hL}\right) + \frac{2Lh}{2-hL}\right]$$

$$= \frac{Mh^2}{12L}\left[\left(\frac{2+hL}{2-hL}\right)^{n+1} + \frac{2Lh-2-Lh}{2-hL}\right]$$

$$= \frac{Mh^2}{12L}\left[\left(\frac{2+hL}{2-hL}\right)^{n+1} \cancel{+} - \left(\frac{2-Lh}{2-Lh}\right)\right]$$

$$|e_{n+1}| \leq \frac{Mh^2}{12L}\left[\left(\frac{2+hL}{2-hL}\right)^{n+1} - 1\right]$$

which show (2e) for all n, and $|e_n| \leq \frac{h^2M}{12L}\left[\left(\frac{2+hL}{2-hL}\right)^n - 1\right]$

which completes the ~~proof~~ derivation.

Question 3

3a

Show that

$$y_{n+1} = (1 + h\lambda + 1/2\, h^2\lambda^2 + 1/3!\, h^3\lambda^3 + 1/4!\, h^4\lambda^4)\, y_n$$

given $y'(x) = \lambda y(x)$, $y'(x) = f(x, y(x))$

$$k_1 = f(x_n, y_n) = \lambda y_n$$

$$k_2 = f(x_n + 1/2 h, y_n + 1/2 h(\lambda y_n)) = \lambda[y_n + \tfrac{h}{2}\lambda y_n] = \lambda y_n + \lambda^2 \tfrac{h}{2} y_n$$

$$k_3 = f(x_n + 1/2 h, y_n + 1/2 h[\lambda y_n + h/2\, \lambda^2 y_n]) = \lambda y_n + \tfrac{h}{2}\lambda^2 y_n + \tfrac{h}{4}\lambda^3 y_n$$

$$k_4 = f(x_n + h, y_n + h[\lambda y_n + h/2\, \lambda^2 y_n + h/4\, \lambda^3 y_n]) = \lambda y_n + h\lambda^2 y_n + \tfrac{h^2}{2}\lambda^3 y_n + \tfrac{h^3}{4}\lambda^4 y_n$$

$$y_{n+1} = y_n + \frac{h}{6}\Big[\lambda y_n + 2[\lambda y_n + \lambda^2 \tfrac{h}{2} y_n] + 2[\lambda y_n + \tfrac{h}{2}\lambda^2 y_n + h/4\, \lambda^3 y_n] + \lambda y_n + h\lambda^2 y_n + \tfrac{h^2}{2}\lambda^3 y_n + \tfrac{h^3}{4}\lambda^4 y_n\Big]$$

$$y_{n+1} = \Big[1 + h\lambda + \frac{h^2}{2}\lambda^2 + \frac{1}{3!}h^3\lambda^3 + \frac{h^4\lambda^4}{4!}\Big] y_n$$

3b

From Taylor series expansion

$$y(x_{n+1}) = y(x_n) + \frac{h}{1!}y'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + \frac{h^4}{4!}y^{iv}(x_n) + \cdots$$

$$y'(x_n) = \lambda y(x_n)$$

$$y''(x_n) = \lambda^2 y(x_n)$$

$$y'''(x_n) = \lambda^3 y(x_n)$$

$$y^{iv}(x_n) = \lambda^4 y(x_n)$$

Thus;

$$y_h(x_{n+1}) = y(x_n) + h\lambda y(x_n) + \frac{h^2}{2}\lambda^2 y(x_n) + \frac{h^3}{3!}\lambda^3 y(x_n) + \frac{h^4}{4!}\lambda^4 y(x_n) + \cdots\cdots$$

$$= \left(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{3!} + \frac{1}{4!}h^4\lambda^4 + \cdots\cdots\right) y(x_n)$$

3c

From the exact solution, we observe that successive terms from the fifth ~~power~~ derivative of the taylor expansion can be omitted. Clearly, we see that the approximate solution is very close to the exact as those omitted terms are too small to make any significant difference.