Finite element method — weak solutions

Definition $u \in H^1_E(a,b)$ such that

(6) $\forall v \in H^1_0(a,b) \quad A(u,v) = \langle f, v \rangle$

is called a weak solution of the boundary-value problem

$$(7) \quad \begin{cases} -\frac{d}{dt}\left[p(t)\frac{du}{dt}\right] + r(t)u(t) = f(t) \\ u(a) = A, \quad u(b) = B \end{cases}$$

and (6) is the weak formulation of problem (7).

Theorem on weak solutions

Suppose $u \in H^2(a,b) \cap H^1_E$ is a solution of problem (7). Then,

$$\forall v \in H^1_0(a,b) \quad A(u,v) = \langle f, v \rangle$$

(that is, u is a weak solution of (7)).

Proof. Let $u \in H^2(a,b) \cap H^1_E$ be a solution of (7). Then the differential equation

$$-\frac{d}{dt}\left[p(t)\frac{du}{dt}\right] + r(t)u(t) = f(t)$$

is satisfied almost everywhere in (a,b). Let $v \in H^1_0(a,b)$ be arbitrary. Then, we multiply both sides of the differential equation by $v(t)$ and integrate from a to b

$$\underbrace{-\int_a^b \frac{d}{dt}\left[p(t)\frac{du}{dt}\right] v(t)\,dt}_{\substack{= \left[p(t)\frac{du}{dt}v(t)\right]_{t=a}^{t=b} - \int_a^b p(t)\frac{du}{dt}\frac{dv}{dt}\,dt}} + \int_a^b r(t)u(t)v(t)\,dt = \int_a^b f(t)v(t)\,dt$$

$$= \left[p(t)\frac{du}{dt}v(t)\right]_{t=a}^{t=b} - \int_a^b p(t)\frac{du}{dt}\frac{dv}{dt}\,dt$$

integration by parts

$\left.\begin{array}{l} v(a)=0 \\ v(b)=0 \end{array}\right\}$ because $v \in H^1_0(a,b)$

So,

$$\underbrace{\int_a^b p(t)\,\frac{du}{dt}\,\frac{dv}{dt}\,dt + \int_a^b r(t)u(t)v(t)\,dt}_{=\mathcal{A}(u,v)} = \underbrace{\int_a^b f(t)v(t)\,dt}_{=\langle f, v\rangle}$$

Therefore,

$$\forall v \in H_0^1(a,b) \qquad \mathcal{A}(u,v) = \langle f, v\rangle,$$

which finishes the proof of the theorem.

<u>Remark</u> The opposite statement is not true, that is, it is not true that if u is a weak solution of problem (7) then it is a solution of (7).

<u>Conclusion</u>: by the theorem on equivalency of Rayleigh-Ritz and Galerkin methods and by the definition of weak solutions, we conclude that

$$u \in H_E^1(a,b) \text{ is a weak solution of (7)} \iff \mathcal{F}(u) = \min_{w \in H_E^1(a,b)} \mathcal{F}(w)$$