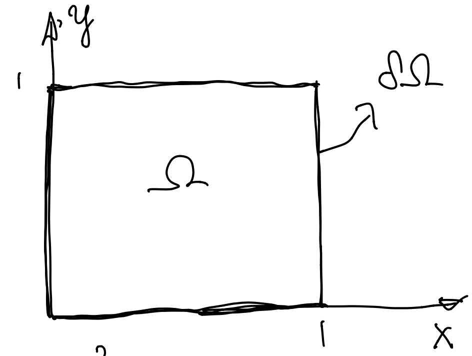
Elliptic equations

Boundary value problem:

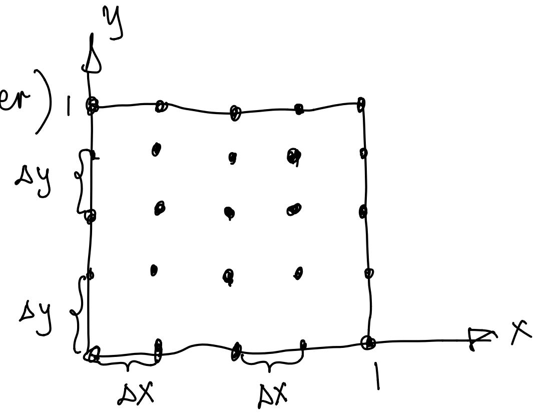
$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) + f(x,y) = 0, & (x,y) \in \Omega = (0,1)\times(0,1) \\ u(x,y) = 0, & (x,y) \in \underbrace{\delta\Omega}_{\text{boundary of } \Omega} \end{cases}$$



$$\delta\Omega = \{(x,y) \in \mathbb{R}^2 : x=0 \text{ or } x=1 \text{ or } y=0 \text{ or } y=1\}$$

Step-sizes: Δx in x-domain
Δy in y-domain

$\Delta x = \Delta y = \frac{1}{N}$, where $N \in \mathbb{N}$ (positive integer)



$x_i = i\Delta x$,
$y_j = j\Delta y$, $i,j = 0,1,2,\dots,N$

Approximations: $u_{ij} \approx \underbrace{u(x_i, y_j)}_{\substack{\text{exact solution} \\ \text{at grid-points}}}$

Example:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + \overbrace{f_{i,j}}^{=f(x_i,y_j)} = 0$$

(1) $$\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{(\Delta x)^2} + f_{i,j} = 0, \quad (x_i, y_j) \in \Omega \setminus \delta\Omega$$

interior points

Definition: operator

$$L u_{ij} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{(\Delta x)^2}$$

linear operator

Then, (1) is written in the form

$$(2) \qquad L u_{i,j} + f_{i,j} = 0$$

Truncation error

We substitute the exact solution $u(x_i, y_j)$ into equation (1) and get

$$(3) \quad \frac{1}{(\Delta x)^2}\Big(u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j)\Big) + f_{i,j} = \overbrace{T(x_i, y_j)}^{\text{truncation error}}$$

$$(4) \quad L u(x_i, y_j) + f_{i,j} = T(x_i, y_j)$$

Taylor expansion

From (3), we get

$$\frac{1}{(\Delta x)^2}\Big(u(x_i,y_j) + \frac{\Delta x}{1!}\frac{\partial u}{\partial x}(x_i,y_j) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}(x_i,y_j) + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}(x_i,y_j) + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}(x_i,y_j) + O((\Delta x)^5)$$

$$+ u(x_i,y_j) - \frac{\Delta x}{1!}\frac{\partial u}{\partial x}(x_i,y_j) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}(x_i,y_j) - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}(x_i,y_j) + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}(x_i,y_j) + O((\Delta x)^5)$$

$$+ u(x_i,y_j) + \frac{\Delta x}{1!}\frac{\partial u}{\partial y}(x_i,y_j) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial y^2}(x_i,y_j) + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial y^3}(x_i,y_j) + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial y^4}(x_i,y_j) + O((\Delta x)^5)$$

$$+ u(x_i,y_j) - \frac{\Delta x}{1!}\frac{\partial u}{\partial y}(x_i,y_j) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial y^2}(x_i,y_j) - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial y^3}(x_i,y_j) + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial y^4}(x_i,y_j) + O((\Delta x)^5)$$

$$- 4u(x_i,y_j)\Big) + f_{i,j}$$

$$= \frac{\partial^2 u}{\partial x^2}(x_i,y_j) + \frac{\partial^2 u}{\partial y^2}(x_i,y_j) + \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(x_i,y_j) + \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i,y_j) + O((\Delta x)^5) + f(x_i,y_j)$$

$$= \frac{(\Delta x)^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(x_i,y_j) + \frac{\partial^4 u}{\partial y^4}(x_i,y_j)\right) + O((\Delta x)^5)$$

$$= \frac{(\Delta x)^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(\xi,y_j) + \frac{\partial^4 u}{\partial y^4}(x_i,\eta)\right),$$ where ξ is between x_{i-1} and x_{i+1}, η is between y_{j-1} and y_{j+1}

$$= T(x_i, y_j)$$