Objective: to derive an upper bound for e_n in terms of T_n

Theorem Suppose

1° $\Phi(\cdot,\cdot,\cdot)$ is continuous

2° Φ satisfies a Lipschitz condition with respect to the second argument on $[t_0, T] \times [y_0 - C, y_0 + C] \times [0, h_0]$

$$\forall t \in [t_0, T] \ \forall x, \tilde{x} \in [y_0 - C, y_0 + C] \ \forall h \in [0, h_0]$$

$$|\Phi(t, x, h) - \Phi(t, \tilde{x}, h)| \leq L|x - \tilde{x}|$$

3° the one-step method $y_{n+1} = y_n + h\Phi(t_n, y_n, h)$ produces approximations such that

$$\forall n \in \{1, 2, \dots, N\} \quad y_n \in [y_0 - C, y_0 + C]$$

Then, $\forall n \in \{1, 2, \dots, N\} \quad |e_n| \leq \frac{1}{L}\left(\exp(L(t_n - t_0)) - 1\right) \cdot \max_{0 \leq n \leq N-1} |T_n|$

Proof. From the definition of the truncation error, we get

$$y(t_{n+1}) = y(t_n) + h\Phi(t_n, y(t_n), h) + hT_n$$

From the formula of the method, we have

$$y_{n+1} = y_n + h\Phi(t_n, y_n, h).$$

Since $e_n = y_n - y(t_n)$, we get

$$e_{n+1} = e_n + h\left(\Phi(t_n, \underbrace{y(t_n)}_{\in [y_0 - C,\, y_0 + C]}, h) - \Phi(t_n, \overbrace{y_n}^{\in [y_0 - C,\, y_0 + C]}, h)\right) + hT_n$$

$$|e_{n+1}| \leq |e_n| + h\left|\Phi(t_n, y(t_n), h) - \Phi(t_n, y_n, h)\right| + h|T_n|$$

$$\underset{\text{Lipschitz condition}}{\leq} |e_n| + hL|e_n| + h|T_n| = (1 + hL)|e_n| + h|T_n|$$