PARTIAL DIFFERENTIAL EQUATIONS

Practice Quiz 3a, Time 2 hours

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1. This question carries [20 MARKS] in total.

Let $f(x)$ be given by

$$f(x) = x, \qquad \text{for } 0 \leqslant x \leqslant \pi.$$

The function is then defined on the interval $-\pi \leqslant x \leqslant \pi$ by extending it as an **even** function. It is then defined on the whole real line by extending it as an 2π-periodic function.

(a) [3 MARKS] Sketch the graph of $f(x)$ between -3π and 3π.

(b) [4 MARKS] Calculate the Fourier series of $f(x)$.

(c) [3 MARKS] Let $g(x) = f'(x)$. Sketch the graph of $g(x)$ between -3π and 3π.

(d) [4 MARKS] Differentiate the Fourier series of $f(x)$ term by term to obtain the Fourier series of $g(x)$.Briefly explain why this is possible.

(e) [4 MARKS] By evaluating the Fourier series of $g(x)$ at $x = 0$ show that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

(f) [2 MARKS] Is it possible to differentiate the Fourier series of $g(x)$ term by term to obtain the Fourier series of $g'(x)$? Give brief reasons for your answer.

See next page for question 2

2. This question carries [25 MARKS] in total.

The displacement $y(x,t)$ of an elastic string on the interval $[0,\pi]$ vibrating with damping is governed by a modified wave equation given by

$$\frac{\partial^2 y}{\partial t^2} + 4\frac{\partial y}{\partial t} + 4y = \frac{\partial^2 y}{\partial x^2}, \quad 0 \leqslant x \leqslant \pi,$$

together with the boundary conditions

$$\frac{\partial y}{\partial x}(0,t) = 0, \quad \frac{\partial y}{\partial x}(\pi,t) = 0\,.$$

(a) [15 MARKS] Use the method of separation of variables to show that the general solution is given by

$$y(x,t) \;=\; (A_0+B_0 t)e^{-2t}+\sum_{n=1}^{\infty}\cos(nx)e^{-2t}\left[A_n\cos(nt)+B_n\sin(nt)\right]\,.$$

where A_n and B_n are constants.

(b) [10 MARKS] Find the solution which also satisfies the initial conditions

$$y(x,0) \;=\; 0\,, \qquad \text{and} \qquad \frac{\partial y}{\partial t}(x,0) = \cos(3x)\,.$$

See next page for question 3

3. This question carries [25 MARKS] in total.

The temparature $u(r,t)$ inside a sphere of radius one is given by a spherically symmetric solution of the diffusion equation and therefore satisfies the equation,

$$\frac{1}{\kappa^2}\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \quad 0 \le r \le 1 \tag{1}$$

where κ is a constant.

The temperature is *finite* everwhere inside the sphere and also satisfies the boundary condition:

$$u(1,t) = 0 \quad \text{for all } t > 0, \tag{2}$$

and the initial condition

$$u(r,0) = 1 - r \quad \text{for all } 0 \le r \le 1. \tag{3}$$

(a) [3 MARKS] Look for a separated solution to the equation of the form $u(r,t) = R(r)T(t)$ and obtain the differential equations satisfied by $R(r)$ and $T(t)$.

(b) [3 MARKS] Show that by making a change of variable to $S(r) = rR(r)$ the spatial equation simplifies to

$$S''(r) - \lambda S(r) = 0$$

where λ is the separation constant from part (a).
What are the boundary conditions satisfied by $S(r)$?

(c) [6 MARKS] Solve the resulting eigenvalue and eigenfunction equation for λ and $S(r)$ and hence obtain the corresponding expression for $R(r)$.

(d) [5 MARKS] Solve the T equation using the values of λ found in part (c), and hence obtain a formula for the separated solution of the partial differential equation. Hence show that the general solution of (1) satsfying the boundary condtions is given by

$$u(r,t) = \sum_{n=1}^{\infty} \frac{A_n}{r}\sin(n\pi r)e^{-n^2\pi^2\kappa^2 t}$$

(e) [8 MARKS] Use the initial condition to obtain the values of A_n and hence obtain the required solution to equations (1), (2) and (3).

See next page for Formulae for Fourier Series

Formulae for Fourier Series

A function $f(x)$ of period $2L$ has Fourier series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

where the Fourier coeficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{2}$$

If the function $f(x)$ is **even** then it has a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{3}$$

where the Fourier coeficients are given by the *half-range* formula

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{4}$$

If the function $f(x)$ is **odd** then it has a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{5}$$

where the Fourier coeficients are given by the *half-range* formula

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{6}$$