Last time:

$$(1) \quad \frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t) f\left(u_{(x,t)}\right) + g(x,t)$$

$$(4) \quad \frac{d u_i}{dt}(t) = \frac{a(t)}{(\Delta x)^2}\left[u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\right] + b(t) f\left((u_i)_t\right) + g(i\Delta x, t)$$

Gauss-Seidel dynamic iterations

$$(5) \quad \frac{d}{dt} u_i^{(k+1)}(t) = \frac{a(t)}{(\Delta x)^2}\left[u_{i+1}^{(k)} - 2u_i^{(k+1)} + u_{i-1}^{(k+1)}\right] + b(t) f\left(\left(u_i^{(k)}\right)_t\right) + g(i\Delta x, t)$$

Jacobi dynamic iterations

$$(6) \quad \frac{d}{dt} u_i^{(k+1)}(t) = \frac{a(t)}{(\Delta x)^2}\left[u_{i+1}^{(k)} - 2u_i^{(k+1)} + u_{i-1}^{(k)}\right] + b(t) f\left(\left(u_i^{(k)}\right)_t\right) + g(i\Delta x, t)$$

Numerical schemes (5) and (6) are supplemented by the initial and boundary conditions:

$$u_i^{(k+1)}(t) = \phi(i\Delta x, t), \quad t \in [-r_0, 0], \qquad u_{\pm M}^{(k+1)}(t) = \psi_{\pm M}(t)$$
$$t \in [0, T].$$

Spectral methods

The idea is to approximate the exact solution u of problem (1) by a truncated expansion

$$v(x,t) = \sum_{i=0}^{N}{}' \alpha_i(t)\varphi_i(x), \quad t \in [0,T], \; x \in [-L, L],$$

where $\varphi_i(x)$ are some trial functions. For example, $\varphi_i(x)$ can be Chebyshev polynomials, that is,

$$\varphi_i(x) = T_i\left(\frac{x}{L}\right) = \cos(i \arccos x/L), \quad i = 0, 1, 2, \ldots, N.$$

To get the approximations $v(x,t)$ to $u(x,t)$, we apply the grid-points $x_i = -L\cos(i\pi/N)$, $i = 0, 1, 2, \ldots, N$ and a differentiation matrix $D = [d_{ij}]_{i,j=0}^{N}$ such that

$$\begin{bmatrix} \frac{\partial^2 v}{\partial x^2}(x_0,t) \\ \frac{\partial^2 v}{\partial x^2}(x_1,t) \\ \cdots \\ \frac{\partial^2 v}{\partial x^2}(x_N,t) \end{bmatrix} = \begin{bmatrix} & & \\ & D & \\ & & \end{bmatrix} \begin{bmatrix} v(x_0,t) \\ v(x_1,t) \\ \cdots \\ v(x_N,t) \end{bmatrix}$$

If $v_i(t) = v(x_i,t)$, then

$$\frac{\partial^2 v}{\partial x^2}(x_i,t) = \sum_{j=0}^{N}{}' d_{ij}\, v_j(t)$$

and we get the following scheme (1)

$$\text{(7)} \quad \frac{dv_i}{dt}(t) = a(t) \sum_{j=1}^{N-1}{}' d_{ij}\, v_j(t) + b(t) f\big((v_i)_t\big) + g_i(t),$$

where $g_i(t) = g(x_i,t) + a(t)\big(d_{i0}\Psi_-(t) + d_{iN}\Psi_+(t)\big)$, $i = 1,2,\dots,N-1$. System (7) is supplemented by the initial condition

$$v_i(t) = \Phi(x_i,t), \quad t \in [-\tau_0, 0], \quad i = 1,2,\dots,N-1.$$

Gauss-Seidel dynamic iterations

$$\text{(8)} \quad \frac{dv_i^{(k+1)}}{dt}(t) = a(t) \sum_{j=1}^{i}{}' d_{ij}\, v_j^{(k+1)}(t) + a(t) \sum_{j=i+1}^{N-1}{}' d_{ij}\, v_j^{(k)}(t) + b(t) f\big((v_i^{(k)})_t\big) + g_i(t)$$

Jacobi dinamic iterations

$$\text{(9)} \quad \frac{dv_i^{(k+1)}}{dt}(t) = a(t)\, d_{ii}\, v_i^{(k+1)}(t) + a(t) \sum_{\substack{j=1 \\ j\neq i}}^{N-1}{}' d_{ij}\, v_j^{(k)}(t) + b(t) f\big((v_i^{(k)})_t\big) + g_i(t)$$

Schemes (8) and (9) are supplemented by the initial condition

$$v_i^{(k+1)}(t) = \Phi(x_i,t), \quad t \in [-\tau_0, 0], \quad i = 1,2,\dots,N-1.$$