**AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES**

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# Question 1

(a) For $y_1(x)$ and $y_2(x)$ to be independent, their wronskian given by; $W[y_1(x), y_2(x)](x) \neq 0$ for some values of $x$.

Given;

$$\tilde{y_1}(x) = ay_1 + ay_2 \tag{1}$$
$$\tilde{y_2}(x) = cy_2 + dy_2 \tag{2}$$
$$\tag{3}$$

$W[\tilde{y_1}(x), \tilde{y_2}(x)]$=$\tilde{y_1}(x)\tilde{y_2}'(x)$ -$\tilde{y_2}(x)\tilde{y_1}'(x)$,

Where:

$$\tilde{y_1}'(x) = ay_1' + by_2' \tag{4}$$
$$\tilde{y_2}'(x) = cy_2' + dy_2' \tag{5}$$

Therefore:

$$W[y_1\tilde{(}x), \tilde{y_2}(x)](x) = (ay_1 + ay_2)(cy_2' + dy_2') - (cy_2 + dy_2)(ay_1' + by_2') \tag{6}$$
$$= (ad - bc)(y_1y_2' - y_2y_1') \tag{7}$$
$$= (ad - bc)W[y_1(x), y_2(x)] \tag{8}$$

From our Previous statement, we have established that for $y_1$ and $y_2$ to be independent, $W[y_1(x), y_2(x)](x) \neq 0$. From equation(8) we can deduce that;

$W[y_1\tilde{(}x), \tilde{y_2}(x)]$, $\therefore$ (ad-bc) $\neq 0$

(b) The Green's function $G(x, s)$ is given by;

$$G(x, s) = \frac{\tilde{y_1}(s)\tilde{y_2}(x) - \tilde{y_2}(s)\tilde{y_1}(x)}{W[\tilde{y_1}, \tilde{y_2}](s)},$$

Substituting the values of $\tilde{y_1}$ and $\tilde{y_2}$ given in equations(2,3) and the expression for the Wronskian given in equation(8). We shall obtain;

$$
\begin{aligned}
G(x,s) &= \frac{(ad-bc)(y_1(S)y_2(x) - y_2(S)y_1(x))}{(ad-bc)W[y_1,y_2](s)} \\
&= \frac{y_1(S)y_2(x) - y_2(S)y_1(x)}{W[y_1,y_2](s)} \\
&= G(x,s)
\end{aligned}
$$

In conclusion, the Green function obtained from $\tilde{y_2}(x)$ and $\tilde{y_1}(x)$, is the same as that of $y_1$ and $y_2$.

(c)

$$
y'' - 4y' + 4y = r(x) \qquad\qquad y(0) = 0, \quad y' = 0
$$

To solve the equation above, we first consider the homogeneous case given below;

$$
y'' - 4y' + 4y = 0
$$

By assuming that $y = e^{mx}$ is a solution, we obtain the following characteristics equation.

$$
m^2 - 4m + 4 = 0 \tag{9}
$$

Solving the quadratic equation given in equation(9) yields $m_1 = 2$ and $m_2 = 2$, the solution of the homogeneous part is given by;

$$
y_1 = e^{2x} \qquad y_2 = xe^{2x}
$$

The Wronskian is given by $W[y_1(x), y_2(x)] = y_2'y_1 - y_1'y_1$
Where;

$$
\begin{aligned}
y_1' &= 2e^{2x} \\
y_2' &= e^{2x} + 2xe^{2x}
\end{aligned}
$$

$\therefore W[y_1(x), y_2(x)] = (e^{2x} + 2xe^{2x})e^{2x} - 2e^{2x}(xe^{2x}) = e^{4x}$.

The Green function $G(x,s)$ is given by;

$$
\begin{aligned}
G(x,s) &= \frac{y_1(s)y_2(x) - y_2(s)y_1(x)}{W[y_1,y_2](s)} \\
&= \frac{e^{2s}xe^{2x} - se^{2s}e^{2x}}{e^{4s}} \\
&= xe^{2x}e^{-2s} - se^{-2s}e^{2x} \\
&= (x-s)e^{2x-2s}
\end{aligned}
$$

The particular solution $y_p$ of the differential equation can be obtained by using the following approach.

$$\begin{aligned}
y_p &= \int\limits_{x=0}^{s} r(s)G(x,s)ds \\
&= \int\limits_{x=0}^{s} se^{2s}(x-s)e^{2x-2s}ds \\
&= xe^{2x}\int\limits_{x=0}^{s} sds - e^{2x}\int\limits_{x=0}^{s} s^2ds \\
&= xe^{2x}(\frac{x^2}{2}) - e^{2x}(\frac{x^3}{3}) \\
&= \frac{x^3e^{2x}}{6}
\end{aligned}$$

# Question 2

$$\frac{d^2y}{dx^2} = \lambda y \tag{10}$$

$$y'(1) = 0 \tag{11}$$

$$y(0) = 0 \tag{12}$$

(a)

$$\int\limits_0^1 (y_m(x)y''_n - y_n(x)y''_m dx = 0 \tag{13}$$

By applying integration by parts, $\int uv' = uv - \int vdu$.
Where;
$u_1 = y_m(x)$, $v_1' = y''_n$, $v_1 = y'_n$ and $du_1 = y'_m dx$
$u_2 = y_n(x)$ and $v_2' = y''_m$, $v_1 = y'_m$ and $du_1 = y'_n dx$

$$\begin{aligned}
\int\limits_0^1 (y_m(x))y''_n(x) - (y_n(x))y''_m(x) &= u_1v_1 - \int v_1du_1 - (u_2v_2 - \int v_2du_2) \\
&= y_m(x)y'_n(x)|_0^1 - \int\limits_0^1 y'_n(x)y'_m dx - (y_n(x)y'_m(x)|_0^1 - \int\limits_0^1 y'_m(x)y'_n(x)dx \\
&= |y_m(x)y'_n(x) - y_n(x)y'_m(x)|_0^1 - \int\limits_0^1 y'_n(x)y'_m(x)dx + \int\limits_0^1 y'_n(x)y'_m(x)d \\
&= |y_m(x)y'_n(x) - y_n(x)y'_m(x)|_0^1
\end{aligned}$$

Applying the boundary conditions; $y'(1) = 0$ and $y(0) = 0$

$$\int_0^1 (y_m(x))y''_n - (y_n(x))y''_m = |y_m(x)y'_n(x) - y_n(x)y'_m(x)|_0^1$$
$$= 0$$

From equation(10), $y''_m = \lambda_n y_m$ and $y''_n = \lambda_m y_n$. Now inserting these values into equation(13), we shall obtain;

$$\int_0^1 (y_m(x)y''_n - y_n(x)y''_m dx = 0 \tag{14}$$

$$0 = \int_0^1 (\lambda_n y_m y_n - \lambda_m y_n(x)y_m(x))dx \tag{15}$$

$$= (\lambda_n - \lambda_m)\int_0^1 y_m(x)y_n(x)dx \tag{16}$$

Since $(\lambda_n - \lambda_m) \neq 0$ for $n \neq m$ then we can conclude that $\int_0^1 y_m(x)y_n(x)dx$ must be zero.

$$\int_0^1 y_m(x)y_n(x)dx = 0 \qquad\qquad m \neq n \tag{17}$$

(b) Considering the ordinary differential equation given in equation(10) and its associated boundary conditions. We shall investigate the solutions to the ODE depending on the values of $\lambda$. It happens that $\lambda$ can either be negative *i.e* $\lambda < 0$, positive *i.e* $\lambda > 0$ or zero *i.e* $\lambda = 0$

- CASE 1 $\lambda = 0$
  The ODE now becomes; $\frac{d^2y}{dx^2} = 0$, and the solution is given by;
  $y_1 = A \quad y_2 = Bx$
  The general solution is given by; $y(x) = A + Bx$. Now applying the boundary conditions $y'(1) = 0, \quad y(0) = 0$, we shall obtain;
  $y(x) = A + Bx$, $y(0) = A + B(0) = 0 \therefore A = 0$
  $y'(x) = B$, $y'(0) = B = 0$. Since $A$ and $B$ are both zero, the only solution is to the equation is the trivial one $y(x) = A + Bx$.
- CASE 2 $\lambda > 0$
  The ODE now becomes; $\frac{d^2y}{dx^2} - \lambda y = 0$. In order to be sure that $\lambda > 0$, we make a substitution by letting $\lambda = p^2$ where $p \neq 0$. By making this substituiton in the ODE, we now obtain; $\frac{d^2y}{dx^2} - p^2y = 0$. The general solution of this ODE is given by $y(x) = Ae^{px} + Be^{-px}$. Now applying the boundary conditions $y'(1) = 0, \quad y(0) = 0$, we shall obtain;
  $y(x) = Ae^{px} + Be^{-px}$, $y(0) = A + B = 0 \therefore B = -A$. Now we can rewrite the general

solution to the ODE as $y(x) = A(e^{px} + e^{-px})$.
$y'(x) = Ap(e^{px} + e^{-px}) = 2Ap(cosh(px)) =$, $y'(0) = 2Ap(cosh(px)) = 0$. We have initially established the fact that $p \neq 0$, hence $A$ must be zero. Now since $A = -B$, this means that $B$ is also zero, thus leaving us with only the trivial solution.

- CASE 2 $\lambda < 0$
  The ODE now becomes; $\frac{d^2y}{dx^2} + \lambda y = 0$. In order to be sure that $\lambda < 0$, we make a substitution by letting $\lambda = -p^2$ where $p \neq 0$. Making this substituiton in the ODE, we now obtain; $\frac{d^2y}{dx^2} + p^2 y = 0$. The general solution of this ODE is given by $y(x) = A\cos(px) + B\sin(px)$. Now applying the boundary conditions $y'(1) = 0, \quad y(0) = 0$, we shall obtain;
  $y(x) = A\cos(px) + B\sin(px)$, $y(0) = A\cos(0) + B\sin(0) = 0$, $\therefore A = 0$. Now we can rewrite the general solution to the ODE as $y(x) = B\sin(px)$.
  $y'(x) = Bp\cos(px)$, $y'(0) = Bp\cos(px) = 0$. Since $p \neq 0$, it implies that $B$ is zero, which then yields the trivial solution. However we are not interested in the trivial solution. To maneuver through the solution, we set $p\cos(px) = 0$.

$$p\cos(px) = 0 \tag{18}$$

The values of $p$ for which equation(18) is zero is given by;

$$p = ((\frac{\pi}{2}) + k\pi) \qquad k \in \mathbb{Z}$$

From our initial substititution, $\lambda = -p^2$. Where $\lambda$ is the eigen value.

$$\begin{aligned}\lambda_n &= -p^2 \\ &= -((\frac{\pi}{2}) + k\pi)^2\end{aligned}$$

Thus, the corresponding eigen funtion is given as;

$$\begin{aligned}y(x) &= B\sin(px) && (19)\\ &= B\sin((\frac{\pi}{2} + k\pi)x) && (20)\\ &= B\sin(\frac{(2k+1)\pi x}{2}) && (21)\end{aligned}$$

From equation(21), we can obtain $y_n(x) = B\sin(\frac{(2n+1)\pi x}{2})$. Similarly,$y_m(x) = B\sin(\frac{(2m+1)\pi x}{2})$.
Now substitutting $y_m(x)$ and $y_m(x)$ into equation(17), we shall obtain;

$$\begin{aligned}0 &= \int_0^1 y_m(x)y_n(x)dx \\ &= B^2 \int_0^1 \sin(\frac{(2n+1)\pi x}{2})\sin(\frac{(2m+1)\pi x}{2}dx \qquad m \neq n \in \mathbb{N} \\ &= B^2 \cdot 0 \\ &= 0\end{aligned}$$

This is the required orthogonality condition.

# Question 3

(a) .

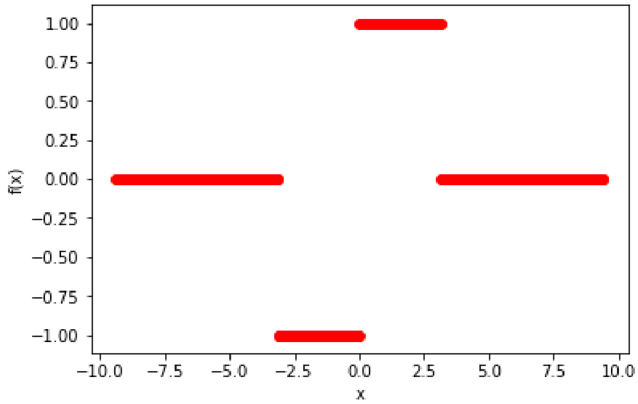


Figure 1: A plot of $f(x)$ against $x$

(b) Since the funtion is periodic, we can express it in terms of fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) \tag{22}$$

Where:

$$a_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$b_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

We want to show that $a_n(x) = 0$ and $a_0 = 0$, so that the function f(x) is expressed only in sines.

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}[\int_{-\pi}^{0} -\cos(nx)dx + \int_{0}^{\pi}\cos(nx)dx] \\
&= \frac{1}{\pi}[|\frac{-\sin(nx)}{n}|_{-\pi}^{0} + |\frac{\sin(nx)}{n}|_{0}^{\pi}] \\
&= 0
\end{aligned}$$

In a similar way, we compute $a_0$;

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}[\int\limits_{-\pi}^{0} -1dx + \int\limits_{0}^{\pi} 1dx] \\
&= \frac{1}{\pi}[|-x|_{-\pi}^{0} + |x|_{0}^{\pi}] \\
&= 0
\end{aligned}$$

Since $a_0$ and $a_n$ have been proven to be zero, equation(22) now becomes;

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

(c) The fourier series for f(x) is given by;

$$\begin{aligned}
b_n(x) &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}[\int\limits_{-\pi}^{0} f(x)\sin(nx)dx + \int\limits_{0}^{\pi} \sin(nx)dx] \\
&= \frac{1}{\pi}[\int\limits_{-\pi}^{0} -\sin(nx)dx + \int\limits_{0}^{\pi} \sin(nx)dx] \\
&= \frac{1}{\pi}[|\frac{\cos(nx)}{n}|_{-\pi}^{0} + |\frac{-\cos(nx)}{n}|_{0}^{\pi}] \\
&= \frac{1}{n\pi}(\cos(0) - \cos(-n\pi)) - \frac{1}{n\pi}(\cos(-n\pi) - \cos(0))
\end{aligned}$$

$\cos(n\pi) = (-1)^n$, $\cos(-n\pi) = (-1)^n$, and $\cos(0) = 1$.

$$\begin{aligned}
b_n &= \frac{1}{n\pi}(1 - (-1)^n - (-1)^n + 1) \\
&= \frac{1}{n\pi}(2 - 2(-1)^n) \\
&= \frac{2}{n\pi}(1 - (-1)^n).
\end{aligned}$$

The fourier series of f(x) can be expressed as;

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin(nx) \\
&= \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - (-1)^n)\sin(nx)
\end{aligned}$$