PARTIAL DIFFERENTIAL EQUATIONS

Quiz 3, Time 2 hours

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Date: Friday 22nd November

Bonus Question

The forced heat equation

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = F(x) \tag{1}$$

is defined on the domain $x \in [-\pi, \pi]$ and for $t > 0$ and is subject to the Neumann boundary conditions

$$\left.\frac{\partial y}{\partial x}\right|_{x=-\pi} = 0, \qquad \text{and} \qquad \left.\frac{\partial y}{\partial x}\right|_{x=\pi} = 0. \tag{2}$$

4(a) [15 MARKS] Assuming that the forcing term $F(x)$ is given by

$$F(x) = \begin{cases} \pi + x & x < 0, \\ \pi - x & x > 0 \end{cases}$$

show that the general solution of the heat equation (1) is

$$y(x,t) = \frac{\pi t}{2} + C_0 + \sum_{n=1}^{\infty} \left[\frac{2}{\pi n^4} \left(1 - (-1)^n\right) + C_n e^{-n^2 t} \right] \cos(nx). \tag{3}$$

4(b) [10 MARKS] If in addition the initial conditions are

$$y(x,0) = 0,$$

show that the solution can be written as

$$y(x,t) = \frac{\pi t}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^4} \left(1 - e^{-(2k-1)^2 t}\right) \cos\left((2k-1)x\right).$$

Solution:
We immediately note that the problem is reflection symmetric about $x = 0$ and the boundary conditions are such that, when expressed in the standard Fourier Series like form, we expect a cosine series. Therefore we make the ansatz that

$$y(x,t) = \frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(nx).$$

In addition we make the ansatz

$$F(x) = \frac{1}{2}F_0 + \sum_{n=1}^{\infty} F_n \cos(nx).$$

We then use the Euler formulas, noting that F is even and so using the half range series, to compute

$$F_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,\mathrm{d}x = \pi$$

and (more lengthily, but straightforwardly)

$$F_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi n^2}\left(1 - (-1)^n\right).$$

Substituting our ansatzes in to the heat equation gives

$$\frac{1}{2}\dot{T}_0 + \sum \dot{T}_n \cos(nx) = \sum -T_n n^2 \cos(nx) + \frac{\pi}{2} + \sum F_n \cos(nx).$$

Noting that this must hold for all values of t we note that this is a Fourier series at any fixed value of t, which allows us to use orthogonality to get

$$\begin{aligned} \dot{T}_0 &= \pi, \\ \dot{T}_n + n^2 T_n &= F_n \end{aligned}$$

which immediately implies (from the linear equation for T_n) that

$$\begin{aligned} T_0 &= \pi t + C_0, \\ T_n &= \frac{F_n}{n^2} + C_n e^{-n^2 t} \end{aligned}$$

where the C_n are integration constants.

Plugging these results back in gives the claimed general solution. We set $t = 0$ to give

$$0 = 0 + \sum \left[\frac{2}{\pi n^4} \left(1 - (-1)^n\right) + C_n \right] \cos(nx).$$

We can immediately use orthogonality or the uniqueness of Fourier series to note that the term in square brackets must vanish. This implies that C_n is given by

$$C_n = -\frac{2}{\pi n^4} \left(1 - (-1)^n\right)$$

from which it follows that

$$y(t, x) = \frac{\pi t}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^4} \left(1 - (-1)^n\right) \left(1 - e^{-n^2 t}\right) \cos(nx).$$

We note that this vanishes for even n; setting $n = 2k - 1$ gives the stated result.