Order of accuracy

Definition: Suppose $p \in \mathbb{N}$. The general one-step method $y_{n+1} = y_n + h\Phi(t_n, y_n, h)$, $n = 0,1,2,\dots,N-1$, where $y_0 = y(t_0)$, has order of accuracy $p \iff p$ is the largest positive integer such that for any f that defines a well-possed problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

there exists a constant K and an upper step-size h_0 such that

$\forall h \in (0, h_0)$ $\forall n \in \{0, 1, \dots, N-1\}$

$$|T_n| = \left| \frac{y(t_{n+1}) - y(t_n)}{h} - \Phi(t_n, y(t_n), h) \right| \leq K h^p$$

Trapezoidal rule : 2-nd order accuracy
implicit
one-step method

$$y'(t) = f(t, y(t)) \Rightarrow \int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$

$$(1) \quad \underbrace{y(t_{n+1})}_{\approx y_{n+1}} - \underbrace{y(t_n)}_{\approx y_n} = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt \approx \underbrace{(t_{n+1} - t_n)}_{=h} \frac{f(t_{n+1}, \overbrace{y(t_{n+1})}^{\approx y_{n+1}}) + f(t_n, \overbrace{y(t_n)}^{\approx y_n})}{2}$$

$$y_{n+1} = y_n + h \frac{f(t_{n+1}, y_{n+1}) + f(t_n, y_n)}{2}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_{n+1}, y_{n+1}) + f(t_n, y_n) \right) \rightarrow \text{trapezoidal rule}$$

Then, the truncation error is

$$(2) \quad T_n = \frac{y(t_{n+1}) - y(t_n)}{h} - \frac{1}{2}\left[f(t_{n+1}, y(t_{n+1})) + f(t_n, y(t_n)) \right]$$

For the trapezoidal rule applied to integration, we have

$$(3) \quad \left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)\left(f(a) + f(b)\right) \right| \leq \frac{(b-a)^3}{12} \max_{\xi \in [a,b]} |f''(\xi)| .$$

Then, from (1), (2), and (3), we get

$$|T_n| = \left| \frac{1}{h} \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt - \frac{1}{2}\left(f(t_{n+1}, y(t_{n+1})) + f(t_n, y(t_n)) \right) \right|$$

$$= \left| \frac{1}{h} \left(\int_{t_n}^{t_{n+1}} \underbrace{f(t, y(t))}_{y'(t)} dt - \frac{h}{2}\Big(f(t_{n+1}, y(t_{n+1})) + f(t_n, y(t_n)) \Big) \right) \right|$$

$$\leq \frac{1}{h} \cdot \frac{h^3}{12} \cdot \max_{\xi \in [t_n, t_{n+1}]} |y'''(\xi)| = \underbrace{\frac{h^2}{12} \max_{\xi \in [t_n, t_{n+1}]} |y'''(\xi)|}_{}$$

so the method is of order 2