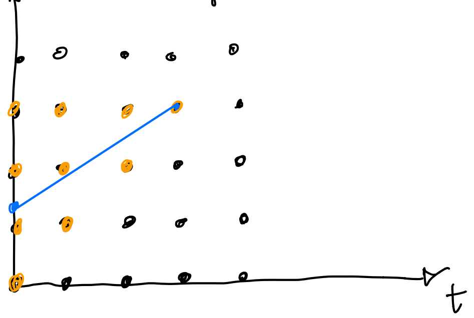
(1) $u_i^{n+1} = (1-\mu)\, u_i^n + \mu\, u_{i-1}^n$ $\rightsquigarrow$ $\frac{\partial u}{\partial t}(x,t) + a \frac{\partial u}{\partial x}(x,t) = 0$

$\mu = \frac{a h}{\Delta x}$

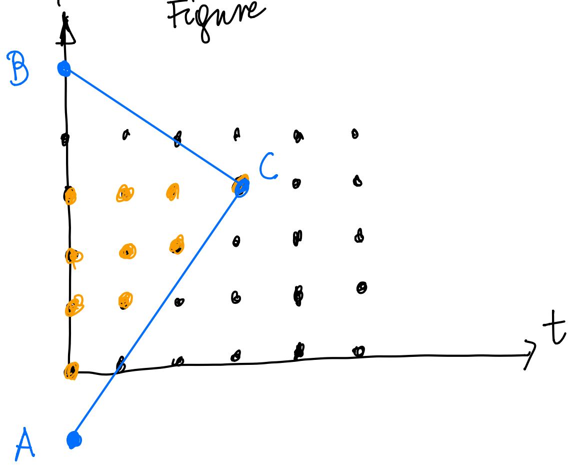
a: constant

Figure 1



Then, numerical scheme (1) is convergent to the exact solution $u(x,t)$

Figure



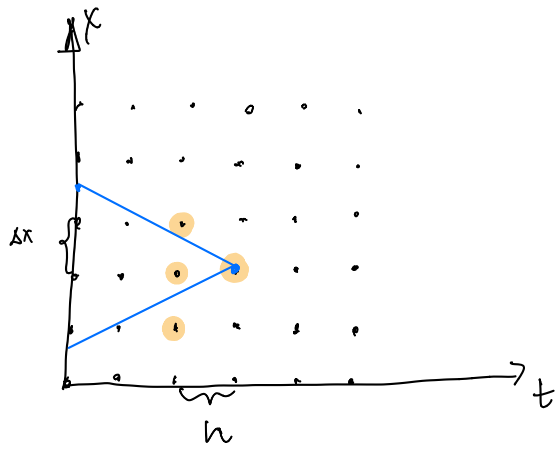
Then, numerical scheme (1) is <u>not</u> convergent to the exact solution $u(x,t)$

<u>Reason</u>: Suppose $h \to 0$ and $\Delta x \to 0$ is such a way that $\frac{h}{\Delta x}$ remains constant. Suppose initial data around A is change. Then, the exact solution at C is changed because it is constant along the characteristic AC. However, the numerical solution is computed based on the initial data that is <u>not</u> changed and therefore the numerical solution has no chances to converge to the exact solution at C even though $h \to 0$ and $\Delta x \to 0$.

<u>Conclusion</u>: Scheme (1) has no chances to converge if $a < 0$ (because in this cas the characteristics are like BC). If $a > 0$ then h and Δx have to satisfy the condition $\mu = \frac{ah}{\Delta x} \leq 1$ for (1) to converge (to eliminate characteristics like AC).

General constant a

$$\frac{\partial u}{\partial t}(x,t) + a\frac{\partial u}{\partial x}(x,t) = 0$$



Condition: $\frac{|a|h}{\Delta x} \leq 1$

$$|a| \leq \frac{\Delta x}{h}$$

is necessary, but not sufficient for stability

If $a>0$ then u_i^{n+1} should be computed from u_i^n and u_{i-1}^n

If $a<0$ then u_i^{n+1} should be computed from u_i^n and u_{i+1}^n

So,

$$\frac{u_i^{n+1} - u_i^n}{h} + a\,\frac{\overbrace{u_i^n - u_{i-1}^n}^{\text{backward difference operator}}}{\Delta x} = 0, \quad \text{if } a>0$$

$$\frac{u_i^{n+1} - u_i^n}{h} + a\,\underbrace{\frac{u_{i+1}^n - u_i^n}{\Delta x}}_{\text{forward difference operator}} = 0, \quad \text{if } a<0$$

$$(2)\quad \begin{cases} u_i^{n+1} = u_i^n - \frac{ah}{\Delta x}\left(u_i^n - u_{i-1}^n\right), & \text{if } a>0 \\ u_i^{n+1} = u_i^n - \frac{ah}{\Delta x}\left(u_{i+1}^n - u_i^n\right), & \text{if } a<0 \end{cases}$$

We use backward difference operator if $a>0$ and forward difference operator if $a<0$, therefore, (2) is called <u>upwind scheme</u>.