PARTIAL DIFFERENTIAL EQUATIONS

Practice Quiz 3b, Time 2 hours

Instructor: James Vickers

Date: Friday 22nd November

---

1. This question carries [20 MARKS] in total.

The function $f(x)$ is defined on the interval $0 \leq x \leq \pi$ as

$$f(x) = \pi - x\,, \quad 0 \leq x \leq \pi\,.$$

(a) [2 MARKS] Sketch the graph of $f(x)$ between 0 and π.

(b) [10 MARKS] Let $g(x)$ be the even 2π -periodic extension of $f(x)$. Sketch the graph of $g(x)$ between -3π and 3π . Calculate the Fourier series of g(x).

(c) [3 MARKS] By evaluating the Fourier series of g(x) at x = 0 show that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

(d) [3 MARKS] Explain why it is possible to differentiate the Fourier series of $g(x)$ term by term to obtain the Fourier series of $g'(x)$. Use this result to obtain the Fourier series of $g'(x)$.

(e) [2 MARKS] Sketch the graph of $g(x)$ between -3π and 3π. Is it possible to differentiate the Fourier series of $g'(x)$ term by term to give the Fourier series of $g''(x)$? Give brief reasons for your answer.

See next page for question 2

2. This question carries [25 MARKS] in total.

(a) [8 MARKS] By considering all possible real values of λ, show that the non-trivial solutions of the eigenvalue problem

$$X''(x) - \lambda X(x) = 0, \quad 0 \le x \le 1,$$

$$X'(0) = 0\,, \qquad X(1) = 0$$

are

$$X_n(x) = A_n \cos\left[\left(n + \tfrac{1}{2}\right)\pi x\right], \quad n = 0, 1, 2, 3, \ldots$$

where A_n is an arbitrary constant.

(b) [11 MARKS] Using the results of part (a) or otherwise, use separation of variables to find the general solution $u(x,t)$ of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad 0 \le x \le 1, \tag{1}$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 = u(1,t). \tag{2}$$

(c) [6 MARKS] Determine the solution to equation (1) subject to the boundary conditions (2) and the initial condition

$$u(x,0) = \cos\left(\frac{\pi x}{2}\right).$$

See next page for question 3

3. This question carries [25 MARKS] in total.

In plane polar coordinates Laplace's equation takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{3}$$

We want to solve the problem in the region *outside* the circle $r = 2$. The solution required is finite in the region $r \geq 2$, and be periodic in θ with period 2π. We also want to find a solution that is an odd function of θ i.e.

$$u(r, -\theta) = -u(r, \theta),$$

(a) [17 MARKS] Use separation of variables to show that the general solution may be written in the form

$$u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta).$$

(b) [8 MARKS] Find the solution of (3) which also satisfies the boundary condition on $r = 2$, that

$$u(2, \theta) = \frac{1}{2}\sin^{(} 3\theta) - \frac{1}{4}\sin^{(} 5\theta)\,.$$

See next page for Formulae for Fourier Series

Formulae for Fourier Series

A function $f(x)$ of period $2L$ has Fourier series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{4}$$

where the Fourier coeficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{5}$$

If the function $f(x)$ is **even** then it has a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{6}$$

where the Fourier coeficients are given by the *half-range* formula

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{7}$$

If the function $f(x)$ is **odd** then it has a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{8}$$

where the Fourier coeficients are given by the *half-range* formula

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{9}$$