Finite element method : introduction

Definition $v : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if and only if

1° there exists v' almost everywhere in $[a,b]$

$$\downarrow$$

v' exists in $[a,b] \setminus A$, where A is of measure zero, that is,

$$\forall \varepsilon > 0 \ \exists \{(a_i, b_i)\}_{i=1}^{\infty} \quad A \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \quad \text{and} \quad \sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon$$

2° v' is integrable on $[a,b]$

3° $\int_a^t v'(s)\, ds = v(t) - v(a)$

Definition $H^1(a,b) = \{ v : [a,b] \to \mathbb{R} \mid v \text{ is absolutely continuous on } [a,b] \text{ and such that } v' \in L^2(a,b) \}$

Definition $L^2(a,b) = \{ v : [a,b] \to \mathbb{R} \mid \int_a^b |v(s)|^2 ds < \infty \}$

$\downarrow$ norm

$$\|v\|_2 = \left(\int_a^b |v(s)|^2 ds \right)^{1/2}$$

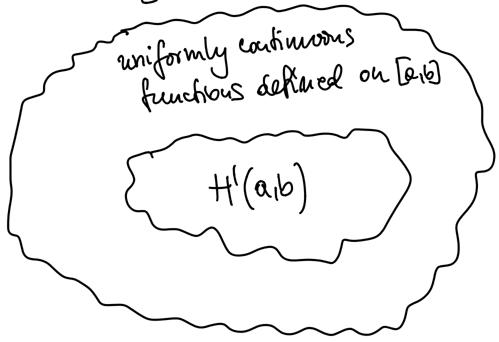
$H^1(a,b)$ is the Sobolev space of index 1.

Remark : $H^1(a,b) \subset \{ v : [a,b] \to \mathbb{R} \mid v \text{ is uniformly continuous} \}$

Reason :

$$|v(t) - v(s)| = \left| \int_t^s v'(\tau)\, d\tau \right| \leq$$

$$\leq |t-s|^{1/2} \cdot \underbrace{\left| \int_t^s |v'(\tau)|^2 d\tau \right|^{1/2}}_{\leq \|v'\|_2} \leq |t-s|^{1/2} \cdot \|v'\|_2$$



Let k be a positive integer

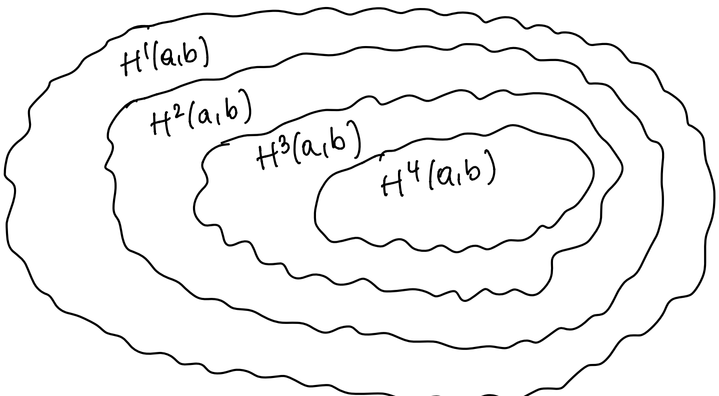
Definition $H^k(a,b) = \{ v \in H^{k-1}(a,b) \mid v, v', v'', \dots, v^{(k-1)}$ are absolutely continuous on $[a,b]$ and $v^{(k)} \in L^2(a,b) \}$

Sobolev space of index k

norm in $H^k(a,b)$: $\|v\|_{H^k(a,b)} = \left(\sum_{m=0}^{k} \|v^{(m)}\|_2^2 \right)^{1/2}$

Sobolev norm

convention $v^{(0)} = v$



For example, $H^2(a,b) = \{ v \in H^1(a,b) \mid v, v'$ are absolutely continuous on $[a,b]$ and $v'' \in L^2(a,b) \}$

So,

$H^k(a,b) = \{ v : [a,b] \to \mathbb{R} \mid v, v', v'', \dots, v^{(k-1)}$ are absolutely continuous on $[a,b]$ and $v', v'', \dots, v^{(k-1)}, v^{(k)} \in L^2(a,b) \}$

$$\|v\|_{H^k(a,b)} = \left(\sum_{m=0}^{k} \|v^{(m)}\|_2^2 \right)^{1/2} = \left(\sum_{m=0}^{k} \int_a^b |v^{(m)}(s)|^2 \, ds \right)^{1/2} =$$

$$= \left(\int_a^b |v(s)|^2 ds + \int_a^b |v'(s)|^2 ds + \dots + \int_a^b |v^{(k)}(s)|^2 ds \right)^{1/2}$$

Theorem : $C^k(a,b) \subset H^k[a,b]$