Finite element method – the minimum of the functional $\mathcal{F}$ over a subset $H_E^1(a,b)$

$u \in H_E^1(a,b)$ is a weak solution of the boundary-value problem

$$(7)\quad \begin{cases} -\frac{d}{dt}\left[p(t)\frac{du}{dt}\right] + r(t)u(t) = f(t) \\ u(a) = A, \qquad u(b) = B \end{cases}$$

if and only if

$$(8)\qquad \mathcal{F}(u) = \min_{w \in H_E^1(a,b)} \mathcal{F}(w)$$

The idea of the finite element method: find u such that

$$\mathcal{F}(u) = \min_{w \in S_E^h} \mathcal{F}(w),$$

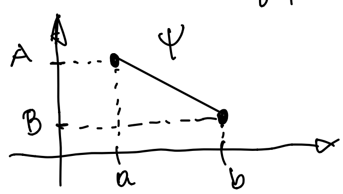
where $S_E^h \subset H_E^1(a,b)$.

For example, $n \geq 2$

$$S_E^h = \left\{ v^h \in H_E^1(a,b) : \forall t \in [a,b]\ v^h(t) = \psi(t) + \sum_{i=1}^{n-1} c_i \varphi_i(t),\ (c_1, c_2, \dots, c_{n-1}) \in \mathbb{R}^{n-1} \right\}$$

where

$$\psi(t) = \frac{B-A}{b-a}(t-a) + A$$



$\psi(a) = A$
$\psi(b) = B$

$\psi \in H_E^1(a,b)$

and

$\varphi_1, \varphi_2, \dots, \varphi_{n-1} \in H_0^1(a,b)$ are linearly independent.

Then, $S_E^h \subset H_E^1(a,b)$.

Therefore, for (8), we want to find $u^h \in S_E^h$ such that

$$(9)\qquad \mathcal{F}(u^h) = \min_{w^h \in S_E^h} \mathcal{F}(w^h)$$

Theorem on equivalence of Ritz and Galerkin methods

Ritz method $\left\{ \begin{array}{l} u^h \in S_E^h \text{ is such that} \\ \mathcal{J}(u^h) = \min\limits_{w^h \in S_E^h} \mathcal{J}(w^h) \end{array} \right\} \iff \underbrace{\forall v^h \in S_0^h \ \mathcal{A}(u^h, v^h) = \langle f, v^h \rangle}_{\text{Galerkin method}},$

where $S_0^h = \{ v^h \in H_0^1(a,b) : \forall t \in [a,b] \ \ v^h(t) = \sum_{i=1}^{n-1} c_i \varphi_i(t), \ (c_1, c_2, \dots, c_{n-1}) \in \mathbb{R}^{n-1} \}$

Proof: similar to the proof of the previous theorem

Since $u^h \in S_E^h$, $\quad u^h = \Psi(t) + \sum_{i=1}^{n-1} c_i \varphi_i(t)$

constants that have to be determined

Therefore,

$$\langle f, \varphi_i \rangle = \mathcal{A}(u^h, \varphi_i) = \mathcal{A}\Big(\Psi(t) + \sum_{j=1}^{n-1} c_j \varphi_j(t), \varphi_i(t)\Big) =$$

$$= \mathcal{A}(\Psi, \varphi_i) + \sum_{j=1}^{n-1} c_j \underbrace{\mathcal{A}(\varphi_j, \varphi_i)}$$

$$= a_{ji} = a_{ij} = \mathcal{A}(\varphi_i, \varphi_j) \rightarrow \text{because } \mathcal{A} \text{ is symmetric}$$

definition

$$\sum_{j=1}^{n-1} c_j \underbrace{\mathcal{A}(\varphi_j, \varphi_i)}_{= a_{ij}} = \underbrace{\langle f, \varphi_i \rangle - \mathcal{A}(\Psi, \varphi_i)}_{b_i}, \quad i = 1, 2, \dots, n-1$$

$$c_1 a_{i1} + c_2 a_{i2} + \dots + c_{n-1} a_{in-1} = b_i$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n-1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-11} & a_{n-12} & a_{n-13} & \cdots & a_{n-1n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}}_{b} \iff Ac = b$$

symmetrix metrix