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Question 1

From Bayes theorem;

(a)

$$f_{X|Y} = \frac{f_{Y|X}(Y|X)f_X(X)}{f_Y(Y)} \tag{1}$$

$$f_{X|Y} \propto f_{Y|X}(Y|X)f_X(X) \tag{2}$$

But;

$$f_{Y|X} \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{y_i - x}{\tau}\right)^2\right]$$

$$f_{Y|X} \propto \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{N}\frac{y_i^2 - 2y_i x}{\tau^2} + \frac{Nx^2}{\tau^2}\right)\right]$$

Similarly,

$$f_X(X) = \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

Applying equation2, we shall have;

$$f_{X|Y} \propto \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{N}\frac{y_i^2 - 2y_i x}{\tau^2} + \frac{Nx^2}{\tau^2}\right)\right] \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$f_{X|Y} \propto \left[-\frac{1}{2}\left(\sum_{i=1}^{N}\frac{y_i^2}{\tau^2} + \frac{\mu^2}{\sigma^2} - \frac{2y_i x}{\tau^2} + \frac{2\mu x}{\sigma^2} + \frac{x^2}{\sigma^2} + \frac{Nx^2}{\tau^2}\right)\right]$$

$$f_{X|Y} \propto \left[-\frac{1}{2}\left(\left(\sum_{i=1}^{N}\frac{y_i^2}{\tau^2} + \frac{\mu^2}{\sigma^2}\right) - 2x\left(\sum_{i=1}^{N}\frac{y_i}{\tau^2} + \frac{\mu}{\sigma^2}\right) + x^2\left(\frac{N}{\tau^2} + \frac{1}{\sigma^2}\right)\right)\right]$$

$$f_{X|Y} \propto \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{N}\frac{y_i^2}{\tau^2}+\frac{\mu^2}{\sigma^2}\right)\right]\exp\left[-\frac{1}{2}\left(-2x\left(\sum_{i=1}^{N}\frac{y_i}{\tau^2}+\frac{\mu}{\sigma^2}\right)+x^2\left(\frac{N}{\tau^2}+\frac{1}{\sigma^2}\right)\right)\right] \tag{3}$$

We shall ignore the first term of our exponent in equation3 because it is independent of x. Therefore;

$$f_{X|Y} \propto \left[-\frac{1}{2}\left(-2x\left(\sum_{i=1}^{N}\frac{y_i}{\tau^2}+\frac{\mu}{\sigma^2}\right)+x^2\left(\frac{N}{\tau^2}+\frac{1}{\sigma^2}\right)\right)\right] \tag{4}$$

(b) In order to use the completing the squares method, we make some simplification to equation4. So let;

$$Q = \left(\frac{N}{\tau^2}+\frac{1}{\sigma^2}\right)$$
$$R = \left(\sum_{i=1}^{N}\frac{y_i}{\tau^2}+\frac{\mu}{\sigma^2}\right)$$

Equation4 now becomes;

$$f_{X|Y} \propto \exp\left[-\frac{1}{2}\left(Qx^2 - 2Rx\right)\right]$$
$$f_{X|Y} \propto \exp\left[-\frac{Q}{2}\left(x^2 - \frac{2Rx}{Q}\right)\right]$$

Now by applying completing the squares to the simplified distribution, we shall have;

$$f_{X|Y} \propto \exp\left[-\frac{Q}{2}\left(\left(x-\frac{R}{Q}\right)^2 - \left(\frac{R}{Q}\right)^2\right)\right]$$
$$f_{X|Y} \propto \exp\left[-\frac{Q}{2}\left(x-\frac{R}{Q}\right)^2\right]\exp\left[\frac{Q}{2}\left(\frac{R}{Q}\right)^2\right]$$

Since the second exponential is independent of x, it is assumed to be a constant in this situation, Hence;

$$f_{X|Y} \propto \exp\left[-\frac{Q}{2}\left(x-\frac{R}{Q}\right)^2\right]$$
$$\propto \exp\left[\frac{-\frac{1}{2}\left(x-\frac{R}{Q}\right)^2}{\frac{1}{Q}}\right]$$

$f_{X|Y}$, is a gaussian distribution with mean $\frac{R}{Q}$, therefore;

$$\mu^* = \frac{R}{Q} = \frac{\left(\sum_{i=1}^{N}\frac{y_i}{\tau^2}+\frac{\mu}{\sigma^2}\right)}{\left(\frac{N}{\tau^2}+\frac{1}{\sigma^2}\right)}$$
$$= \left(\frac{N}{\tau^2}+\frac{1}{\sigma^2}\right)^{-1}\left(\sum_{i=1}^{N}\frac{y_i}{\tau^2}+\frac{\mu}{\sigma^2}\right)$$

The variance is given by;

$$
\begin{aligned}
Var(X|Y) &= \frac{1}{Q} \\
&= \left(\frac{N}{\tau^2} + \frac{1}{\sigma^2}\right)^{-1}
\end{aligned}
$$

(c) We have;

From our expression of the posterior mean;

$$\mu^* = \frac{\left(\sum_{i=1}^{N} \frac{y_i}{\tau^2} + \frac{\mu}{\sigma^2}\right)}{\left(\frac{N}{\tau^2} + \frac{1}{\sigma^2}\right)}$$

We shall simplify expressions in μ^* by making the following computation;

$$\sum_{i=1}^{N} \frac{y_i}{\tau^2} = \sum_{i=1}^{N} \frac{N}{N}\frac{y_i}{\tau^2} = \frac{N}{\tau^2}\sum_{i=1}^{N}\frac{y_i}{N} = \frac{N}{\tau^2}\bar{y}$$

Therefore;

$$
\begin{aligned}
\mu^* &= \frac{\frac{N}{\tau^2}\bar{y} + \frac{\mu}{\sigma^2}}{\frac{N}{\tau^2} + \frac{1}{\sigma^2}} \\
&= \frac{N\bar{y}\sigma^2 + \mu\tau^2}{N\sigma^2 + \tau^2} \\
&= \frac{N\bar{y}\sigma^2 + \mu\tau^2 + \mu(N\sigma^2 + \tau^2) - \mu(N\sigma^2 + \tau^2)}{N\sigma^2 + \tau^2} \\
&= \mu + \frac{N\bar{y}\sigma^2 + \mu\tau^2 - \mu(N\sigma^2 + \tau^2)}{N\sigma^2 + \tau^2} \\
&= \mu + \frac{N\sigma^2(\bar{y} - \mu)}{N\sigma^2 + \tau^2} \\
\mu^* &= \mu + k(\bar{y} - \mu)
\end{aligned}
$$

(cii) Given;

$$\mu^* = \mu + k(\bar{y} - \mu)$$

By expansion, we obtain;

$$
\begin{aligned}
\mu^* &= \mu + \frac{N\sigma^2}{N\sigma^2 + \tau^2} \cdot \bar{y} - \frac{N\sigma^2}{N\sigma^2 + \tau^2} \cdot \mu \\
&= \left(1 - \frac{N\sigma^2}{N\sigma^2 + \tau^2}\right)\mu + \frac{N\sigma^2}{N\sigma^2 + \tau^2} \cdot \bar{y} \\
&= \frac{\tau^2}{N\sigma^2 + \tau^2} \cdot \mu + \frac{N\sigma^2}{N\sigma^2 + \tau^2} \cdot \bar{y}
\end{aligned}
$$

(d) The posterior mean is given by;

$$\mu^* = \mu + \frac{N\sigma^2(\bar{y} - \mu)}{N\sigma^2 + \tau^2} \tag{5}$$

d(i) As $\tau^2 \to \infty$, then the equation of the posterior mean μ^* becomes;

$$\mu^* \approx \mu$$

That is, the posterior mean becomes equivalent to the prior mean.

d(ii) As $N \to \infty$, and the uncertainty of τ is fixed, then the equation of the posterior mean μ^* becomes;

$$\mu^* \approx \bar{y}$$

That is, the posterior mean becomes equivalent to the mean of all the observations.

(e) We have seen that the Var$(X|y)$ is given by

$$\begin{aligned} Var(X|y) &= \left(\frac{N}{\tau^2} + \frac{1}{\sigma^2}\right)^{-1} \\ &= \frac{\tau^2\sigma^2}{N\sigma^2 + \tau^2} \\ &= \frac{\tau^2}{N\sigma^2 + \tau^2}\sigma^2 \end{aligned}$$

We have been given the expression that $k = \frac{N\sigma^2}{N\sigma^2+\tau^2}$, thus;

$$(1-k)\sigma^2 = \left(1 - \frac{N\sigma^2}{N\sigma^2 + \tau^2}\right)\sigma^2 \tag{6}$$

$$= \left(\frac{\tau^2}{N\sigma^2 + \tau^2}\right)\sigma^2 \tag{7}$$

Hence, we have shown that Var$(X|Y) = (1-k)\sigma^2$
From equation7, as $\tau^2 \to \infty$, then $Var(X|Y) = \sigma^2$. However, as $N \to \infty$, $Var(X|Y) = 0$

Question 2

(a) The parametres in the kalman filter formular are;

$$\begin{aligned} M_t &= 0.7\mathbf{I}_1 \\ H_t &= 1 \\ \Sigma_t &= 1 \\ R_t &= 0.1 \\ Q_t &= 0.5 \end{aligned}$$

(b) The dimensions are

$$\begin{aligned} \mathbf{K}_t &\in \mathbb{R} \\ \boldsymbol{\mu}_{t|t-1} &\in \mathbb{R} \\ \boldsymbol{\Sigma}_{t|t-1} &\in \mathbb{R} \\ \boldsymbol{\mu}_{t|t} &\in \mathbb{R} \\ \boldsymbol{\Sigma}_{t|t} &\in \mathbb{R} \end{aligned}$$

(c) Figure1 shows the simulation of our process model for $t = 1, ..., 100$

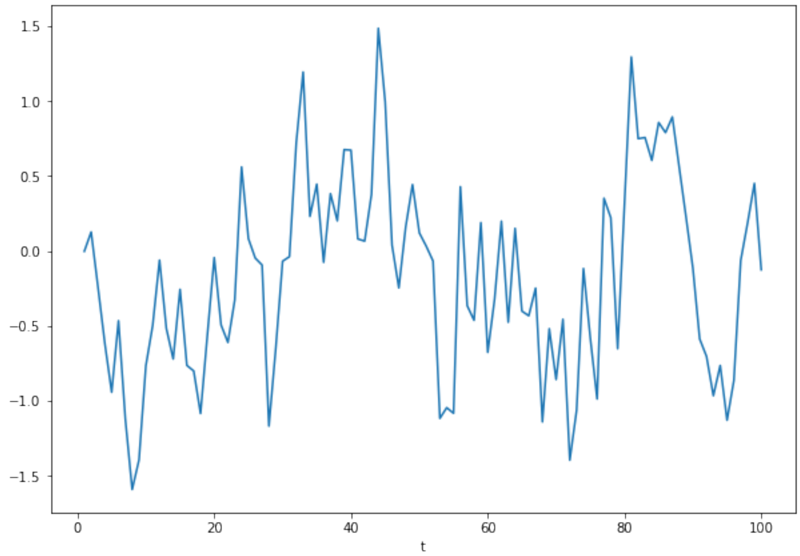

Figure 1: Process Model Simulation

(d) .
Figure2, shows a plot of the truth x_t and data y_t for $t = 1, ..., 100$, excluding points $40, .., 43$ and $80, ..., 83$

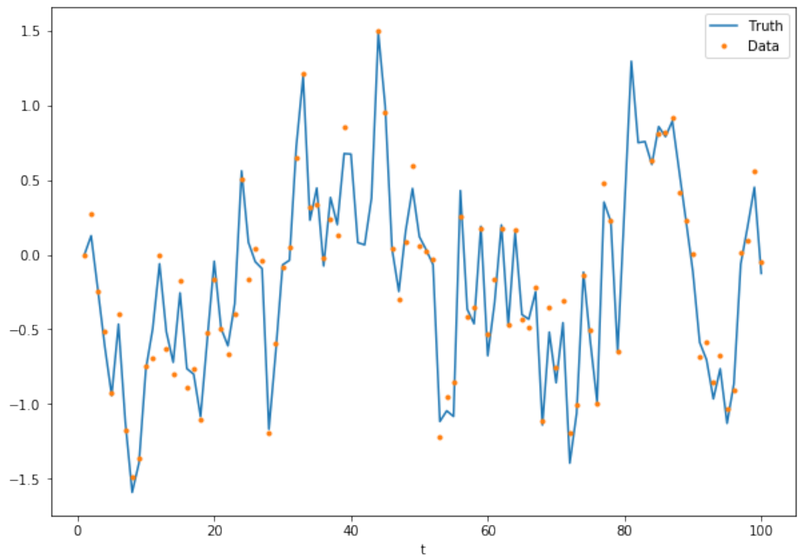

Figure 2: Simulation of the Process Model and Data

(e) Figure3 below shows the plot of the kalman estimate, process model, and the data. Carefully inspecting the plot, one can observe that the kalman estimates are often lying between the noise observations and the truth states. However from $t = 40, .., 43$ and $t = 80, .., 83$, we observe that the kalman estimates are far away from the truth states, this is because data is not available within those points

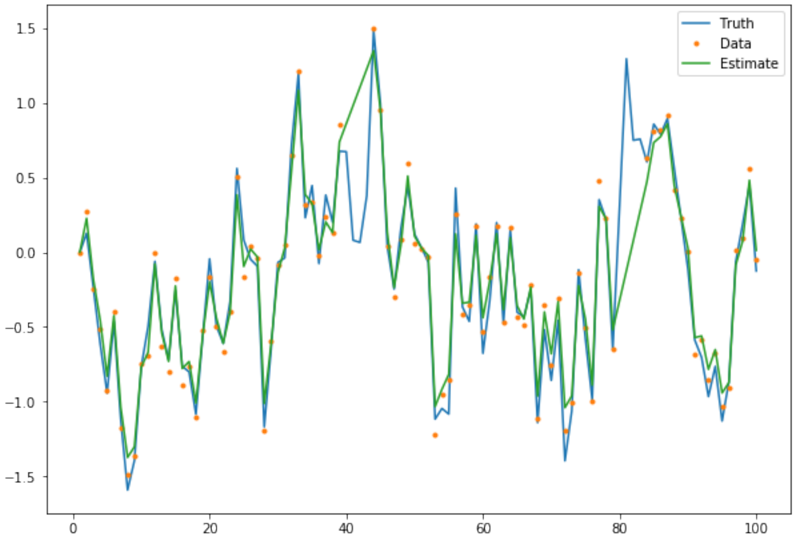

Figure 3: Simulation of the Process Model, Data and Kalman Estimate