Convergence theorem Suppose

1° $f: [a,b]\times[y_0-C, y_0+C] \to \mathbb{R}$ satisfies the assumptions of the theorem that guarantees that the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(\underbrace{a}_{t_0}) = y_0 \end{cases}$$ is well-possed

2° $\forall h \in (0, h_0)$ $\forall n = 0, 1, 2, \dots, N-1$ $y_{n+1} = y_n + h\Phi(t_n, y_n, h) \in [y_0 - C, y_0 + C]$

$t_n = a + nh$

3° $\Phi: [a,b]\times[y_0-C, y_0+C]\times[0,h_0] \to \mathbb{R}$ is continuous

4° $\forall t \in [a,b]$ $\forall x, \tilde{x} \in [y_0-C, y_0+C]$ $\forall h \in [0, h_0]$

$$|\Phi(t, x, h) - \Phi(t, \tilde{x}, h)| \leq L|x - \tilde{x}|$$

5° $\forall t \in [a,b]$ $\forall x \in [y_0-C, y_0+C]$ $\Phi(t, x, 0) \equiv f(t, x)$

Then,
$$\lim_{\substack{h\to 0\\ n\to\infty}} y_n = y(t), \quad \text{where } t = \lim_{\substack{h\to 0\\ n\to\infty}} t_n \in [a,b]$$

$$t_n = a + nh, \quad h = \frac{b-a}{\underbrace{N}_{\text{number of subintervals in } [a,b]}}$$

$N \in \mathbb{N}$ is such that $h \leq h_0$

Proof. By Theorem 1,

$$|e_n| \leq \frac{1}{L}\left(\exp(L(t_n - a)) - 1\right) \cdot \max_{0\leq i\leq N-1} |T_i|$$

and, since $t_n \leq b$ for $n = 1, 2, 3, \dots, N$, we get

(1) $$|e_n| \leq \frac{1}{L}\left(\exp(L(b-a)) - 1\right) \cdot \max_{0\leq i\leq N-1} |T_i| .$$

Also, $|e_0| = 0$. Here,

$$T_i = \frac{y(t_{i+1}) - y(t_i)}{h} - \Phi(t_i, y(t_i), h) + \overbrace{\Phi(t_i, y(t_i), 0)}^{=f(t_i, y(t_i))} - \Phi(t_i, y(t_i), 0)$$

$$= \underbrace{\frac{y(t_{i+1}) - y(t_i)}{h}}_{= \frac{y'(\xi_i)(t_{i+1} - t_i)}{h}} - \underbrace{f(t_i, y(t_i))}_{= y'(t_i)} + \Phi(t_i, y(t_i), 0) - \Phi(t_i, y(t_i), h)$$

$$T_i = y'(\xi_i) - y'(t_i) + \Phi(t_i, y(t_i), 0) - \Phi(t_i, y(t_i), h)$$

Since $t_{i+1} - t_i = h$

here $\xi_i \in [t_i, t_{i+1}]$