

# Proof of Green's theorem Math 131 Multivariate Calculus

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September 25, 2020

## 1 Introduction

We'll show why Green's theorem is true for elementary regions  $D$ . These regions can be patched together to give more general regions. First, suppose that  $D$  is a region of that is, it can be described by inequalities  $a \leq x \leq b$  and  $\gamma(x) \leq y \leq \delta(x)$  where  $\gamma$  and  $\delta$  are  $C^1$  where functions. First, we'll show that

$$\iint_D -\frac{\partial M}{\partial y} dA = \oint_{\partial D} M(x, y) dx$$

We can directly integrate the left integral as a double integral

$$\begin{aligned} \iint_D -\frac{\partial M}{\partial y} dA &= \int_a^b \int_{\gamma(x)}^{\delta(x)} -\frac{\partial M}{\partial y} dy dx \\ &= \int_a^b -M(x, y) \Big|_{y=\gamma(x)}^{\delta(x)} dx \\ &= \int_a^b (M(x, \delta(x)) - M(x, \gamma(x))) dx \\ &= \int_a^b M(t, \delta(t)) dt - \int_a^b M(t, \gamma(t)) dt \end{aligned}$$

We're almost done. The first integral

$$\int_a^b M(t, \gamma(t)) dt$$

is the path integral along  $y = \gamma(x)$  from the left to right, that is, it is  $\int_{\gamma} M(x, y(t))$  likewise the second integral  $\int_a^b M(t, \delta(t)) dt$  is the parameterization along the curve

$y = \delta(x)$  from left to right, but that

portion of the boundary  $\partial D$  should go with from right to left, and the minus sign reverses the

orientation. The two vertical sides  $x = a$  and  $x = b$  of  $D$  form the other two parts of  $\partial D$ . since  $\frac{dx}{dt}$  on those vertical paths, therefore

$$\int M(x, y) dx = \int M(x, y) \frac{dx}{dt} dt = \int 0 dt = 0$$

along them. therefore,

$$\int_a^b M(t, \gamma(t)) dt - \int_a^b M(t, \partial(t)) dt = \oint_{\partial D} M(x, y) dx$$

Likewise, if  $D$  is a region of “type 2,” that is, one bounded between horizontal lines, then

$$\iint_D \frac{\partial D}{\partial N} dA = \oint_{\partial D} N(x, y) dy$$

where the minus sign is dropped because the symmetry exchanging  $y$  for  $x$  reverses orientation. Adding these two equations gives Green’s theorem for  $D$

$$\iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_{\partial D} M(x, y) dx + N(x, y) dy$$

That takes care of the case when the region is of both type 1 and type 2. But regions that can be decomposed into a finite number of these can be patched together to take care of the general case. q.e.d. There are other proofs that are more inclusive to show that some regions that are a union of an infinite number of these regions also satisfy Green’s theorem.