AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

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Question 1

(i) $\frac{dx}{dt} = |x|^{\frac{1}{2}}$
Case 1:$-x^{\frac{1}{2}}$
Case 2:$x^{\frac{1}{2}}$

$$\frac{dx}{dt} = -x^{\frac{1}{2}}$$

Applying seperation of variables techniques,

$$\frac{dx}{-x^{\frac{1}{2}}} = dt$$
$$\int -x^{\frac{1}{2}} dx = dt$$
$$\int -x^{\frac{1}{2}} dx = t + c$$
$$x = \sqrt{\frac{-t}{2} + c}$$

Taking similar steps for case 2, we shall obtain the second solution as; $x = \sqrt{\frac{t}{2} + c}$

(ii)

$$x\frac{dx}{dt} = t$$
$$xdx = tdt$$
$$\int xdx = \int tdt$$
$$\frac{x^2}{2} = \frac{t^2}{2} + c$$
$$x = \pm\sqrt{t^2 + c}$$

(2) (i) If a function $f(z)$ and its partial derivatives are continuous in domain of z, $i.e$ the function is defined, then the hypothesis of Picard-Lindelof's theorem are satisfied

$$\frac{dx}{dt} = |x|^{\frac{1}{2}}$$

Taking partial derivative of this fuction with respect to x yields;

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2\sqrt{x}}$$

The result is not continuous for $x = 0$, hence the hypothesis of Picard-Lindelof's theorem are not satisfied

(ii)

$$x\frac{dx}{dt} = t$$
$$x^{'} = \frac{t}{x}$$

The function is not continuous at $x = 0$, hence the hypothesis of the Picard-Lindelof theorem are not satisfied.

Question 2

$\frac{du}{dt} = Au \qquad u\left(0\right) = u_0$

(i)

$$A = \begin{pmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} -5-\lambda & -4 & 2 \\ -2 & -2-\lambda & 2 \\ 4 & 2 & 2-\lambda \end{pmatrix}$$

$$\left(-5-\lambda\right)\begin{bmatrix} -2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} + 4\begin{bmatrix} -2 & -4 \\ 4 & 2 \end{bmatrix} + 2\begin{bmatrix} -2 & -2-\lambda \\ 4 & 2 \end{bmatrix}$$

$$(\left(-5-\lambda\right)\left[\left(-2-\lambda\right)\left(-2-\lambda\right)-4\right] + 4[-2\left(2\right) - \left(4\right)\left(-4\right)] + 2[-2\left(2\right) - \left(4\right)\left(-2-\lambda\right)]$$

$$\left(-5-\lambda\right)\left(\lambda^2 - 8\right) + 4\left(-4+16\right) + 2\left(-4+8+4\lambda\right)$$
$$-\lambda^3 - 5\lambda^2 + 24\lambda = 0$$
$$\left(\lambda^2 + 5\lambda - 24\right) - \lambda = 0$$
$$\lambda_1 = 0 \quad \lambda_2 = -8 \quad \lambda_3 = 3$$

The eigen vectors attributed to $\lambda_1 = 0$ can be obtained as follows;

$$\begin{pmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-5x - 4y + 2z = 0 \tag{1}$$
$$-2x - 2y + 2z = 0 \tag{2}$$
$$4x + 2y + 2z = 0 \tag{3}$$

Resolving equations(1),(2)and (3) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_1 = 0 \qquad u_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

Similarly we can obtain the eigen vectors for the eigen value $\lambda_2 = 3$.

$$\begin{pmatrix} -5-(3) & -4 & 2 \\ -2 & -2-(3) & 2 \\ 4 & 2 & 2-(3) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -8 & -4 & 2 \\ -2 & -5 & 2 \\ 4 & -1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-8x - 4y + 2z = 0 \tag{4}$$
$$-2x - 5y + 2z = 0 \tag{5}$$
$$4x + 2y + -z = 0 \tag{6}$$

Resolving equations(4),(5)and (6) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_2 = 3 \qquad u_2 = \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix}$$

Similarly we can obtain the eigen vectors for the eigen value $\lambda_3 = -8$.

$$\begin{pmatrix} -5-(-8) & -4 & 2 \\ -2 & -2-(-8) & 2 \\ 4 & 2 & 2-(-8) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -4 & 2 \\ -2 & 6 & 2 \\ 4 & 2 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x - 4y + 2z = 0 \tag{7}$$
$$-2x + 6y + 2z = 0 \tag{8}$$
$$4x + 2y + 10z = 0 \tag{9}$$

Resolving equations(4),(5)and (6) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_3 = -8 \qquad u_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

From the given equation;

$$\begin{aligned} \frac{du}{dt} &= Au \\ \frac{du}{u} &= Adt \\ \int \frac{du}{u} &= \int Adt \\ \ln |u| &= At + c \\ u &= ce^{tA} \end{aligned}$$

Applying the initial value condition $u\,(0) = u_0$ we shall obtain;

$$\begin{aligned} u_0 &= ce^0 \\ c &= u_0 \end{aligned}$$

$$u_1(t) = e^{\lambda_1 t} u_1 = e^0 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$
$$u_2(t) = e^{\lambda_2 t} u_1 = e^{3t} \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix}$$
$$u_3(t) = e^{\lambda_3 t} u_1 = e^{-8t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$
$$u\,(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}$$

$u\left(t\right)=c_1\begin{pmatrix}-2\\3\\1\end{pmatrix}+c_2e^{3t}\begin{pmatrix}1\\6\\16\end{pmatrix}+c_3e^{-8t}\begin{pmatrix}-2\\-1\\1\end{pmatrix}$

At $t=0\quad u\left(0\right)=u_0$

$$\begin{pmatrix}2\\1\\2\end{pmatrix}=\begin{pmatrix}-2&1&-2\\3&6&-1\\1&16&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\tag{10}$$

$$-2c_1+-c_2+-2c_3=2\tag{11}$$
$$3c_1+6c_2+-c_3=1\tag{12}$$
$$c_1+16c_2+c_3=2\tag{13}$$

Resolving equations (11),(12)and (13) simultaneously, we shall obtain the value of the constants as;

$$c_1=\frac{-1}{4}\quad c_2=\frac{2}{11}\quad c_3=\frac{-29}{44}$$

The general solution is given as;

$u\left(t\right)=\frac{-1}{4}\begin{pmatrix}-2\\3\\1\end{pmatrix}+\frac{2}{11}e^{3t}\begin{pmatrix}1\\6\\16\end{pmatrix}+\frac{-29}{44}e^{-8t}\begin{pmatrix}-2\\-1\\1\end{pmatrix}$

(ii)

$$A=\begin{pmatrix}4&-3&0\\3&4&0\\5&10&10\end{pmatrix}$$

We want to determine the eigen values of the matrix A.

$$\begin{pmatrix}4-\lambda&-3&0\\3&4-\lambda&0\\5&10&10-\lambda\end{pmatrix}$$

We want to obtain the obtain the determinant the of the matrix A.

$$\left(4-\lambda\right)\left(\left(4-\lambda\right)\left(10-\lambda\right)\right)+3\left(3\left(10-\lambda\right)\right)=0$$
$$-\lambda^3+18\lambda^2-105\lambda+250=0$$
$$\left(\lambda_1-10\right)\left(\lambda_2-4+3i\right)\left(\lambda_3-4-3i\right)$$
$$\lambda_1=10\quad\lambda_2=4-3i\quad\lambda_3=4+3i$$

Now we want to determine the eigen vectors associated with the eigen values;
$\lambda_1=10$

$$\begin{pmatrix}4-(10)&-3&0\\3&4-(10)&0\\5&10&10-(10)\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix} -6 & -3 & 0 \\ 3 & -6 & 0 \\ 5 & 10 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-6x - 3y + 0 = 0 \tag{14}$$
$$3x + 6y + 0 = 0 \tag{15}$$
$$5x + 10y + 0 = 0 \tag{16}$$

Resolving equations(14),(15)and (16) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_1 = 10 \qquad u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Similarly for $\lambda_2 = 4 + 3i$

$$\begin{pmatrix} 4-(4+3i) & -3 & 0 \\ 3 & 4-(4+3i) & 0 \\ 5 & 10 & 10-(4+3i) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By resolving the above matrix, we shall obtain;

$$\lambda_2 = 4 + 3i \qquad u_1 = \begin{pmatrix} 12 - 9i \\ -9 + 12i \\ 25 \end{pmatrix}$$

The eigen vectors associated with the eigen value$\lambda_3 = 4 - 3i$ is the conjugate of the eigen vectors of $\lambda_2 = 4 + 3i$

$$\lambda_3 = 4 - 3i \qquad u_1 = \begin{pmatrix} 12 + 9i \\ -9 - 12i \\ 25 \end{pmatrix}$$

$$u_1(t) = e^{\lambda_1 t} u_1 = e^{10t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$u_2(t) = e^{\lambda_2 t} u_1 = e^{4t} \left(\cos(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} - \sin(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} \right)$$
$$u_3(t) = e^{\lambda_2 t} u_1 = e^{4t} \left(\cos(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} + \sin(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} \right)$$

$$u(t) = c_1 e^{\lambda_1 t} u_1 = e^{10t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{4t} \left(\cos(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} - \sin(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} \right)$$
$$+ c_3 e^{4t} \left(\cos(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} + \sin(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} \right)$$
At $t = 0 \quad u(0) = u_0$

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -12 & -9 \\ 0 & -9 & 12 \\ 1 & 25 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{17}$$

$$-12c_2 + -9c_3 = 2 \tag{18}$$
$$-9c_2 + 12c_3 = 1 \tag{19}$$
$$c_1 + 25c_2 = 2 \tag{20}$$

Resolving equations (18),(19)and (20) simultaneously, we shall obtain the value of the constants as;

$$c_1 = \frac{17}{3} \quad c_2 = \frac{-11}{75} \quad c_3 = \frac{-2}{75}$$

Our general solution now becomes;

$$u(t) = \tfrac{17}{3} e^{\lambda_1 t} u_1 = e^{10t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \tfrac{-11}{75} e^{4t} \left(\cos(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} - \sin(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} \right)$$
$$+ \tfrac{-2}{75} e^{4t} \left(\cos(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} + \sin(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} \right)$$

Question 3

(i)

$$\begin{aligned}
\frac{du}{dt} &= Au + F\left(u,t\right) \qquad on \quad [a,b] \\
e^{-At}\frac{du}{dt} &= e^{-At}\left(Au + F\left(u,t\right)\right) \\
\frac{du}{dt}\left(ue^{At}\right) &= e^{-At}F\left(u,t\right) \\
ue^{-At}|_{tn}^{t_{n+1}} &= \int\limits_{t_n}^{tn+1} e^{-At}F\left(u,t\right) \\
u(t_{n+1})e^{At_{n+1}} - u(t_n)e^{At_n} &= \int\limits_{t_n}^{tn+1} e^{-At}F\left(u,t\right) \\
u(t_{n+1})e^{At_{n+1}} &= u(t_n)e^{At_n} + \int\limits_{t_n}^{tn+1} e^{-At}F\left(u,t\right)
\end{aligned}$$

Dividing through by $e^{A(t_{n+1})}$

$$\begin{aligned}
u(t_{n+1}) &= u(t_n)e^{At_n}e^{-At_{n+1}} + \int\limits_{t_n}^{tn+1} e^{At_{n+1}}e^{-At}F\left(u,t\right) \\
u(t_{n+1}) &= u(t_n)e^{A(-t_n+t_{n+1})} + \int\limits_{t_n}^{tn+1} e^{A(t_{n+1}-t_n)}F\left(u,t\right)
\end{aligned}$$

Setting $h = t_{n+1} - t_n$, $t = t_n + \tau$ and $\tau = t - t_n$ we shall obtain the desired solution.

$$u(t_{n+1}) = u(t_n)e^{A(-t_n+t_{n+1})} + \int\limits_0^h e^{-(\tau-h)A}F(u(t_n+\tau), t_n+\tau)d\tau$$

(ii) .

$F\left(u_n, t_n\right) = F_n$

$$
\begin{aligned}
u(t_{n+1}) &= u(t_n)e^{A(-t_n+t_{n+1})} + \int\limits_0^h e^{-(\tau-h)A} F\left(u\left(tn+\tau\right), t_n+\tau\right) d\tau \\
&= e^{hA}u_n + \int\limits_0^h e^{-(\tau-h)A} F(u(t_n+\tau), t_n+\tau) d\tau \\
&= u(t_n)e^{hA} + \int\limits_0^h e^{-(\tau-h)A} F_n d\tau \\
&= u(t_n)e^{hA} - A^{-1}F_n e^{-(\tau-h)}|_0^h \\
&= u(t_n)e^{hA} - A^{-1}F_n\left(I - e^{hA}\right) \\
&= e^{hA}u(t_n) + A^{-1}\left(e^{hA} - I\right)F_n
\end{aligned}
$$