Convergence of the numerical scheme

$$u_i^{n+1} = u_i^n + \frac{h}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \quad n = 0,1,2,\dots$$

Definition $\quad e_i^n = u_i^n - \underbrace{u(x_i, t_n)}_{\text{exact solution of the diffusion equation}}$

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

$$(9) \qquad \frac{u_i^{n+1} - u_i^n}{h} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

and

$$(10) \qquad \frac{u(x,t+h) - u(x,t)}{h} = \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{(\Delta x)^2} + \overset{\substack{\text{truncation error} \\ \uparrow}}{T(x,t)}$$

Substracting (10), where $x = x_i$ and $t = t_n$, from (9), we get

$$\frac{e_i^{n+1} - e_i^n}{h} = \frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{(\Delta x)^2} - T(x_i, t_n)$$

$$e_i^{n+1} = e_i^n + \frac{h}{(\Delta x)^2}\left(e_{i+1}^n - 2e_i^n + e_{i-1}^n\right) - hT(x_i, t_n)$$

$$= \underbrace{\left(1 - \frac{2h}{(\Delta x)^2}\right)}_{\substack{\geq \\ 0 \text{ if } \frac{h}{(\Delta x)^2} \leq \frac{1}{2}}} e_i^n + \frac{h}{(\Delta x)^2} e_{i+1}^n + \frac{h}{(\Delta x)^2} e_{i-1}^n - hT(x_i, t_n)$$

Notation:

$$E^n = \max\left\{ |e_i^n|,\ i = 0,1,2,\dots, i_{max} \right\}$$

$$|e_i^{n+1}| \leq \left(1 - \frac{2h}{(\Delta x)^2}\right)|e_i^n| + \frac{h}{(\Delta x)^2}|e_{i+1}^n| + \frac{h}{(\Delta x)^2}|e_{i-1}^n| + h|T(x_i, t_n)|$$

$$\leq \left(1 - \frac{2h}{(\Delta x)^2}\right)E^n + \frac{h}{(\Delta x)^2}E^n + \frac{h}{(\Delta x)^2}E^n + hT_{max}$$

$$= \underbrace{E^n + hT_{max}}_{\text{no } i}$$

$$T_{max} = \max_{(x,t) \in [0,1] \times [0,a]} |T(x,t)|$$

Taking maximum over i, we get

$$E^{n+1} \leq E^n + h\,T_{max}$$

Then,

$E^0 = 0$ (the initial error is zero because we use the initial function from the initial condition to start the numerical scheme)

$$E^1 \leq E^0 + hT_{max} = hT_{max}$$
$$E^2 \leq E^1 + hT_{max} \leq hT_{max} + hT_{max} = 2hT_{max}$$

....

$$E^n \leq nhT_{max}$$

because $E^{n+1} \leq E^n + hT_{max} \overset{\text{inductive assumption}}{\leq} nhT_{max} + hT_{max} = (n+1)hT_{max}$

So, by mathematical induction: $E^n \leq nhT_{max}$, for all $n = 0, 1, 2, \ldots$

$$0 \leq E^n \leq nhT_{max} = \underbrace{nh}_{nh \leq a} \underbrace{\left(\frac{h}{2} \max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq a}} \left| \frac{\partial^2 u}{\partial t^2}(x,t) \right| + \frac{(\Delta x)^2}{12} \max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq a}} \left| \frac{\partial^4 u}{\partial x^4}(x,t) \right| \right)}_{= T_{max}}$$

$$\leq \frac{h}{2} \cdot a \cdot \left(\max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq a}} \left| \frac{\partial^2 u}{\partial t^2}(x,t) \right| + \frac{1}{6 \underbrace{\frac{h}{(\Delta x)^2}}} \max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq a}} \left| \frac{\partial^4 u}{\partial x^4}(x,t) \right| \right)$$

if $\frac{h}{(\Delta x)^2} \leq \frac{1}{2}$ is constant, then

$$0 \leq E^n \leq \text{Constant} \cdot \frac{h}{2} \cdot a$$

and $\lim_{h \to 0} E^n = 0$