Initial-boundary value problems for parabolic partial differential equations

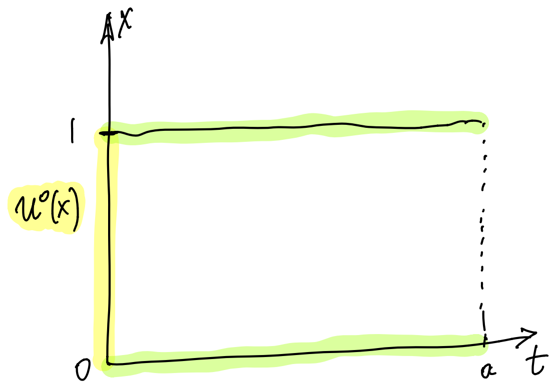
(1) $\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial u}{\partial x}(x,t)\right) + g(x,t)\, u(x,t) + f(x,t)$,

where D, g, f are given functions and $D(x,t) > 0$ for all x and t.

Initial condition

(2) $u(x,0) = u^0(x)$,

where $u^0(x)$ is a given function



Boundary conditions

(3) $u(0,t) = b_0(t)$, $u(1,t) = b_1(t)$,

where b_0, b_1 are given functions.

Example:

(4) $$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), & x \in (0,1),\ t > 0, \\ u(x,0) = u^0(x), & x \in (0,1), \\ u(0,t) = 0,\ u(1,t) = 0, & t > 0, \end{cases}$$

h: step-size in time

Δx: step-size in space

grid-points: $t_n = nh$, $n = 0,1,2,\ldots$

$x_i = i\Delta x$, $i = 0,1,2,\ldots, i_{max}$

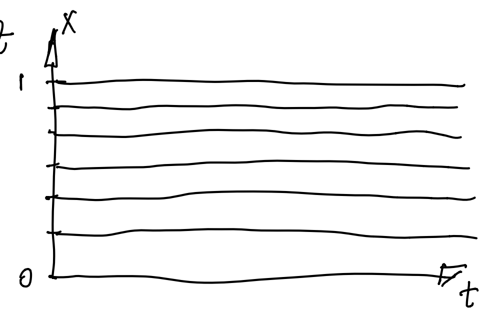
$\downarrow$

$h\, i_{max} = 1$

$i_{max} \in \mathbb{N}$

Semi-discretization process

We discretize in x keeping continuous t



$$\underbrace{u(x_i,t)}_{\text{exact solution}} \approx \underbrace{u_i(t)}_{\text{approximation}}$$

Finite difference operator

$$\frac{\partial^2 u}{\partial x^2}(x_i,t) \approx \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{(\Delta x)^2}$$

Semi-discrete scheme

$$\frac{\partial u}{\partial t}(x_i,t) \approx \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{(\Delta x)^2}$$

$$(5) \begin{cases} \dfrac{d u_i}{dt} = \dfrac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{(\Delta x)^2}, & i = 1, 2, \dots, imax-1 \\ u_0(t) = 0, \quad u_{imax}(t) = 0 & \text{(from the boundary conditions)} \\ u_i(0) = u^0(x_i) \end{cases}$$

If the initial condition (2) is consistent with the boundary conditions (3), then $u^0(0) = u^0(1) = 0$ (so that the exact solution doesn't have discontinuities at $(0,0)$ and $(1,0)$.

We can apply time integration methods to (5). For example, if we apply $y_{n+1} = y_n + h f(t_n, y_n)$, where $f = (f_1, f_2, \dots, f_{imax-1})$
$f_i(t,y) = f_i(t, y_1, y_2, \dots, y_{imax-1}) = \frac{1}{(\Delta x)^2}(y_{i+1} - 2y_i + y_{i-1})$

then we get

$$(6) \begin{cases} u_i^{n+1} = u_i^n + \dfrac{h}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), & n = 0, 1, 2, \dots \\ u_i^0 = u^0(x_i), & i = 1, 2, \dots, imax-1, \end{cases}$$

where $u_0^n = 0$, $u_{imax}^n = 0$, and

$$\underbrace{u_i^n}_{\substack{\text{approximations}\\ \text{by the time}\\ \text{integration}\\ \text{methods}}} \approx \underbrace{u_i(t_n)}_{\substack{\text{approximation}\\ \text{by the semi-discrete}\\ \text{systems}}} \approx \underbrace{u(x_i, t_n)}_{\text{exact solution}}$$