Last time:

$$y_{n+1} = y_n + h(ak_1 + bk_2), \quad \text{where} \quad \begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + \alpha h, y_n + \beta h k_1) \end{aligned}$$

$a, b, \alpha, \beta \in \mathbb{R}$ parameters

Example

$$\begin{cases} y_{n+1} = y_n + \frac{1}{6} h (k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 = f(t_n, y_n) \\ k_2 = f(t_n + h/2, y_n + \frac{h}{2} \cdot k_1) \\ k_3 = f(t_n + h/2, y_n + \frac{h}{2} \cdot k_2) \\ k_4 = f(t_n + h, y_n + h k_3) \end{cases}$$

method of order 4

this method needs 4 function evaluations per each step

General linear multistep methods

Example $\quad y'(s) = f(s, y(s)), \quad y(t_0) = y_0$

after integrating both sides of the differential equation over $[t_{n-1}, t_{n+1}]$

$$\int_{t_{n-1}}^{t_{n+1}} y'(s)\,ds = \int_{t_{n-1}}^{t_{n+1}} f(s, y(s))\,ds \underset{\text{Simpson's rule}}{\approx} \frac{t_{n+1} - t_{n-1}}{6}\Big[f(t_{n-1}, y(t_{n-1})) + 4 f\big(\underbrace{\tfrac{t_{n+1}+t_{n-1}}{2}}_{=t_n}, y(\underbrace{\tfrac{t_{n+1}+t_{n-1}}{2}}_{=t_n})\big) + f(t_{n+1}, y(t_{n+1}))\Big]$$

$$y(t_{n+1}) - y(t_{n-1}) \approx \frac{2h}{6}\Big[f(t_{n-1}, y(t_{n-1})) + 4 f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\Big]$$

$$y(t_{n+1}) \approx y(t_{n-1}) + \frac{h}{3} f(t_{n-1}, y(t_{n-1})) + \frac{4h}{3} f(t_n, y(t_n)) + \frac{h}{3} f(t_{n+1}, y(t_{n+1}))$$

Two-step method;

$$y_{n+1} = y_{n-1} + \underbrace{\frac{h}{3} f(t_{n-1}, y_{n-1}) + \frac{4h}{3} f(t_n, y_n) + \frac{h}{3} f(t_{n+1}, y_{n+1})}_{\text{3 function evaluations per step}}$$

General linear k-step methods

(1) $$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j \underbrace{f(t_{n+j}, y_{n+j})}_{=f_{n+j}}, \quad \text{where } \left.\begin{array}{l}\alpha_0, \dots, \alpha_k \in \mathbb{R} \\ \beta_0, \dots, \beta_k \in \mathbb{R}\end{array}\right\} \text{parameters}$$

$n = 0, 1, 2, \dots$

$\alpha_k \neq 0$ and either $\alpha_0 \neq 0$ or $\beta_0 \neq 0$

$\alpha_0^2 + \beta_0^2 \neq 0$

If $\beta_k = 0$ then method (1) is explicit.

If $\beta_k \neq 0$ then method (1) is implicit.

(1) needs k starting values $y_0, y_1, y_2, \dots, y_{k-1}$

Examples

$y_{n+1} = y_n + h f(t_n, y_n) \longrightarrow$ explicit Euler method

$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) \longrightarrow$ implicit Euler method

$$\left.\begin{aligned} y_{n+1} &= y_n + \tfrac{h}{2} f(t_{n+1}, y_{n+1}) + \tfrac{h}{2} f(t_n, y_n) \\ &= y_n + \tfrac{h}{2} f_{n+1} + \tfrac{h}{2} f_n \end{aligned}\right\} \text{trapezoidal rule}$$

$y_{n+4} = y_{n+3} + \frac{h}{24}\left(55 f_{n+3} - 59 f_{n+2} + 37 f_{n+1} - 9 f_n\right) \rightarrow$ explicite four-step method Adams-Bashforth

$y_{n+3} = y_{n+2} + \frac{h}{24}\left(9 f_{n+3} + 19 f_{n+2} - 5 f_{n+1} - 9 f_n\right) \rightarrow$ implicit three-step method Adams-Moulton