Last time

$$(1) \quad \frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t) f\left(u_{(x,t)}\right) + g(x,t)$$

$$(4) \quad \frac{du_i}{dt}(t) = \frac{a(t)}{(\Delta x)^2}\left[u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\right] + b(t) f\left((u_i)_t\right) + g(i\Delta x, t)$$

Finite difference approximation

$$(7) \quad \frac{dv_i}{dt}(t) = a(t)\sum_{j=1}^{N-1}{}' d_{ij} v_j(t) + b(t) f\left((v_i)_t\right) + g_i(t)$$

Error bounds

Systems (4) and (7) can be written in the form

$$(10) \quad \frac{dy}{dt} = a(t)(A+B)y(t) + b(t)F(y_t) + G(t),$$

where $y_t : [-r_0, 0] \to \mathbb{R}^n$, $y_t(\tau) = y(t+\tau)$, $\tau \in [-r_0, 0]$,

$F : C([-r_0,0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$, $F_i(w) = f(w_i)$, $w \in C([-r_0,0], \mathbb{R}^n)$,

and A, B are n by n square matrices. Here, $n = 2M+1$ for system (4), and $n = N+1$, for system (7).

Systems (5), (6), (8), (9) can be written in the form

$$(11) \quad \frac{dy^{(k+1)}}{dt}(t) = a(t)Ay^{(k+1)}(t) + a(t)By^{(k)}(t) + b(t)F\left(y_t^{(k)}\right) + G(t)$$

Error : $e^{(k)}(t) = y^{(k)}(t) - y(t)$

If F is additive (that is, $F(x+\tilde{x}) = F(x) + F(\tilde{x})$) as in examples (2) and (3), then subtracting (10) from (11), we get

$$\frac{de^{(k+1)}}{dt}(t) = a(t)Ae^{(k+1)}(t) + a(t)Be^{(k)}(t) + \underbrace{b(t)F(y_t^{(k)}) - b(t)F(y_t)}$$

$$= b(t)F(y_t^{(k)} - y_t)$$

$$= b(t)F(e_t^{(k)})$$

So,

$$\frac{de^{(k+1)}(t)}{dt} = a(t)Ae^{(k+1)}(t) + a(t)Be^{(k)}(t) + b(t)F(e_t^{(k)})$$

Let $\|\cdot\|$ be an arbitrary vector norm or the induced matrix norm.

Then,

$$\|e^{(k+1)}(t+h)\| = \|e^{(k+1)}(t) + h\frac{de^{(k+1)}}{dt}(t)\| + O(h^2)$$

$$= \|\underbrace{e^{(k+1)}(t)}_{I e^{(k+1)}(t)} + h a(t) A e^{(k+1)}(t) + h a(t) B e^{(k)}(t) + h b(t) F(e_t^{(k)})\| + O(h^2)$$

identity matrix

triangle inequality

$$\leq \|(I + h a(t) A) e^{(k+1)}(t)\| + \|\overset{\text{positive}}{h a(t)} B e^{(k)}(t)\| + \|\overset{\text{positive}}{h} b(t) F(e_t^{(k)})\| + O(h^2)$$

$$\leq \|I + h a(t) A\| \cdot \|e^{(k+1)}(t)\| + h a(t) \|B\| \cdot \|e^{(k)}(t)\| + h \|b(t) F(e_t^{(k)})\| + O(h^2)$$

We define $\|F\|_0 = \inf\{ r > 0 : \|F(W)\| \leq r \|W\|_0,\ W \in C([-r_0, 0], \mathbb{R}^n)\}$

where $\|W\|_0 = \max\{\|W(\tau)\| : \tau \in [-r_0, 0]\}$. Then,

$$\|e^{(k+1)}(t+h)\| \leq \|I + h a(t) A\| \cdot \|e^{(k+1)}(t)\| + h a(t) \|B\| \cdot \|e^{(k)}(t)\| + $$
$$+ h |b(t)| \cdot \|F\|_0 \cdot \|e_t^{(k)}\|_0 + O(h^2)$$

So, since $h > 0$, we get

$$(12) \begin{cases} \dfrac{\|e^{(k+1)}(t+h)\| - \|e^{(k+1)}(t)\|}{h} \leq \dfrac{\|I + h a(t) A\| - 1}{h} \|e^{(k+1)}(t)\| + \\ + a(t) \|B\| \cdot \|e^{(k)}(t)\| + |b(t)| \cdot \|F\|_0 \cdot \|e_t^{(k)}\|_0 + O(h) \end{cases}$$

We now define the logarithmic norm

$$\mu(C) = \lim_{\varepsilon \to 0+} \frac{\|I + \varepsilon C\| - 1}{\varepsilon}, \text{ where } C \text{ is any square matrix.}$$

Properties of the logarithmic norm:

$\mu(C) \leq \|C\|$

$\mu(\alpha C) = \alpha \mu(C)$, for any positive scalar α

$\mu(C_1 + C_2) \leq \mu(C_1) + \mu(C_2)$

$\|e^{tC}\| \leq e^{t\mu(C)}$, for any $t \geq 0$

We now take $h \to 0+$ in (12) and get

$$D_+ \|e^{(k+1)}(t)\| \leq \mu(a(t)A)\|e^{(k+1)}(t)\| + a(t)\|B\|\,\|e^{(k)}(t)\| + |b(t)|\cdot\|F\|_0\cdot\|e_t^{(k)}\|_0,$$

where D_+ is the right-hand side derivative.