PARTIAL DIFFERENTIAL EQUATIONS

Quiz 3, Time 2 hours

Instructor: James Vickers

Date: Friday 22nd November

---

1. This question carries [20 MARKS] in total.

Let $f(x)$ be the 2π-periodic function which is given by

$$f(x) = x, \qquad \text{for } -\pi < x \leqslant \pi.$$

(a) [3 MARKS] Sketch the graph of $f(x)$ between -3π and 3π.

(b) [5 MARKS] Calculate the Fourier series of $f(x)$.

(c) [3 MARKS] Let $g(x) = \int_{s=0}^{x} f(s)\mathrm{d}s$. Sketch the graph of $g(x)$.

(d) [4 MARKS] By integrating the Fourier series of $f(x)$ term by term calculate the Fourier series of $g(x)$.

(e) [3 MARKS] By evaluating the Fourier series of $g(x)$ at $x = 0$ show that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

(f) [2 MARKS] Is it possible to differentiate the Fourier series of $f(x)$ term by term to obtain the Fourier series of $f'(x)$? Give a brief reason for your answer.

See next page for question 2

2. This question carries [25 MARKS] in total.

The function $u(x,t)$ satisfies the wave equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where c is a constant.

It also satisfies the boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0\,, \text{ for all } t,$$

and the initial condition: $u(x,0) = 0\,$, for $0 < x < L$.

(a) [15 MARKS] Use the method of separation of variables to show that the general solution may be written as

$$u(x,t) = D_0 t + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right),$$

where D_n $(n = 0, 1, 2, \ldots)$ are arbitrary constants.

(b) [10 MARKS] Find the solution if at $t = 0$ the function also satisfies the initial velocity condition :

$$\frac{\partial u}{\partial t}(x,0) = L - x, \qquad 0 < x < L.$$

See next page for question 3

3. This question carries [25 MARKS] in total.

In plane polar coordinates Laplace's equation is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

We want to find a solution $u(r, \theta)$ on the unit disk that is 2π-periodic in θ and finite at $r = 0$.

(a) [15 MARKS] Use the method of separation of variables to show that the general solution is given by

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right].$$

(b) [10 MARKS] Find the solution u to part (a) that also satisfies the boundary condition

$$u(1, \theta) = |\theta|, \quad \text{for } -\pi \leqslant \theta < \pi$$

Hint: Is $u(1, \theta) = |\theta|$ an odd or even function?

See next page for bonus question and last page for Formulae for Fourier Series

This is a bonus question. You can obtain full marks for the quiz without attempting the bonus question. This question will only be marked for those students who obtain at least 60% in each of the previous three questions.

Bonus Question

The forced heat equation

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = F(x) \tag{1}$$

is defined on the domain $x \in [-\pi, \pi]$ and for $t > 0$ and is subject to the Neumann boundary conditions

$$\left.\frac{\partial y}{\partial x}\right|_{x=-\pi} = 0, \qquad \text{and} \qquad \left.\frac{\partial y}{\partial x}\right|_{x=\pi} = 0. \tag{2}$$

1. [15 MARKS] Assuming that the forcing term $F(x)$ is given by

$$F(x) = \begin{cases} \pi + x & x < 0, \\ \pi - x & x > 0 \end{cases}$$

   show that the general solution of the heat equation (1) is

$$y(x,t) = \frac{\pi t}{2} + C_0 + \sum_{n=1}^{\infty} \left[\frac{2}{\pi n^4} (1 - (-1)^n) + C_n e^{-n^2 t} \right] \cos(nx). \tag{3}$$

2. [10 MARKS] If in addition the initial conditions are

$$y(x, 0) = 0,$$

   show that the solution can be written as

$$y(x,t) = \frac{\pi t}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^4} \left(1 - e^{-(2k-1)^2 t}\right) \cos\left((2k-1)x\right).$$

See next page for Formulae for Fourier Series

Formulae for Fourier Series

A function $f(x)$ of period $2L$ has Fourier series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{4}$$

where the Fourier coeficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{5}$$

If the function $f(x)$ is **even** then it has a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{6}$$

where the Fourier coeficients are given by the *half-range* formula

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{7}$$

If the function $f(x)$ is **odd** then it has a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{8}$$

where the Fourier coeficients are given by the *half-range* formula

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{9}$$