Proof of Green's theorem Math 131 Multivariate Calculus

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1 Introduction

Green's theorem is simply a relationship between the macroscopic circulation around the curve C and the sum of all the microscopic circulation that is inside C. If C is a simple closed curve in the plane, then it surrounds some region D (shown in red 1) in the plane. D is the "interior" of the curve C [1]. Green's

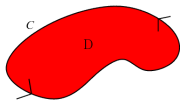

Figure 1: closed curve region

theorem says that if you add up all the microscopic circulation inside C (i.e., the microscopic circulation in D), then that total is exactly the same as the macroscopic circulation around C (as shown in 1).

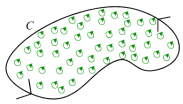

Figure 2: macroscopic-microscopic circulation

2 Theorem

Theorem 2.1 (Greens theorem)

$$\oint_{\delta D} M dx + N dy = \iint_D \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right) dA$$

Green's theorem can be interpreted as a planer case of Stokes' theorem2.1 [1]

$$\oint_{\delta D} F.ds = \iint_D (\bigtriangledown \times F).kdA$$

In words, that says the integral of the vector field F around the boundary δD equals the integral of the curl of F over the region D. In the next chapter we'll study Stokes' theorem in 3-space 2.1. Green's theorem implies the divergence theorem in the plane.

It says that the integral around the boundary δD of the the normal component of the vector field F equals the double integral over the region D of the divergence of F

proof 2.2 *We'll show whyGreen's theorem is true for elementary regions D These regions can be patched together to give more general regions [2].*

First, suppose that D is a region of that is, it can be described by inequalities $a \leq x \geq b$ and $\gamma(x) y \leq \delta(x)$ where γ and δ are C^1 where functions First, we'll show that

$$\iint_D -\frac{\partial M}{\partial y} dA = \oint_{\partial D} M(x, y) dx$$

We can directly integrate the left integral as a dou- ble integral

$$\iint_D -\frac{\partial M}{\partial y} dA$$

$$= \int_a^b \int_{\gamma(x)}^{\partial(y)} -\frac{\partial M}{\partial y} dydx$$

$$= \int_a^b -M(x,y)\,|_{y=\gamma(x)}^{\partial(x)} \quad dx$$

$$= \int_a^b \left(M(x,\gamma(x)) - M(x,\partial(t))\right) dx$$

$$= \int_a^b M(t,\gamma(t))\,dt - \int_a^b (t,\delta(t))\,dt$$

We're almost done. The first integral

$$\int_b^a M(t,\gamma(t))\,dt$$

is the path integral along $y = \gamma(x)$ from the left to right, that is, it is $\int_\gamma M(x, y(t))$ *likewize' the second integral $\int_a^b (t, \partial(t))\,dt$ is the parameterization along the curve $y = \partial(x)$ from left to right,but that portion of the boundary δD should go with from right to left, and the minus sign reverses the orientation. The two vertical sides $x = a$ and $x = b$ of D form the other two parts of δD. since $\frac{dx}{dt}$ on those vertical paths, therefore*

$$\int M(x,y)\,dx = \int M(x,y)\frac{dx}{dt}dt = \int 0dt = 0$$

along them. therefore,

$$\int_a^a M(t,\gamma(t))\,dt - \int_a^b (t,\partial(t))\,dt = \oint_{\delta D} M(x,y)\,dx$$

Likewise, if D is a region of "type 2," that is, one bounded between horizontal lines, then

$$\iint_D \frac{\partial D}{\partial N} dA = \oint_{\delta D} N(x,y)\,dy$$

where the minus sign is dropped because the symmetry exchanging y for x reverses orientation.Adding these two equations 2.2 and 2 gives Green's theorem for D

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_{\partial D} M(x,y)\,dx + N(x,y)\,dy$$

That takes care of the case when the region is of both type 1 and type 2. But regions that can be decomposed into a finite number of these can be patched together to take care of the general case. q.e.d.[2] There are other proofs that are more inclusive to show that some regions that are a union of an infinite number of these regions also satisfy Green's theorem.

References

[1] Nik Weaver. Lipschitz algebras and derivations ii. exterior differentiation. *Journal of Functional Analysis*, 178(1):64–112, 2000.

[2] Dongwoo Sheen. A generalized green's theorem. *Applied mathematics letters*, 5(4):95–98, 1992.