Finite element method

reminder : $a_{ij} = A(\varphi_i, \varphi_j) = \int_a^b p(t)\varphi_i'(t)\varphi_j'(t)dt + \int_a^b r(t)\varphi_i(t)\varphi_j(t)dt$

last time we determined $a_{ij} = 0$, for $j \neq i, i-1, i+1$, $i = 1, 2, \cdots, n-1$.

Notice that

$$\int_{t_{i-1}}^{t_i} r(t)\varphi_{i-1}(t)\varphi_i(t)\,dt \underset{\text{trapezoidal rule}}{\approx} (t_i - t_{i-1})\frac{r(t_{i-1})\overbrace{\varphi_{i-1}(t_{i-1})}^{=1}\overbrace{\varphi_i(t_{i-1})}^{=0} + r(t_i)\overbrace{\varphi_{i-1}(t_i)}^{=0}\overbrace{\varphi_i(t_i)}^{=1}}{2} = 0$$

and

$$\int_a^b r(t)\varphi_{i-1}(t)\varphi_i(t)\,dt \approx 0$$

So, $a_{i,i-1} = a_{i-1,i} \approx \underbrace{-\frac{1}{h}p(t_{i-1/2})}_{\text{last time}} + 0 = -\frac{1}{h}p_{i-1/2}$

Similarly,

$$a_{i,i+1} = a_{i+1,i} \approx -\frac{1}{h}p(t_{i+1/2}) = -\frac{1}{h}p_{i+1/2}$$

We now consider

$$\int_{t_{i-1}}^{t_i} r(t)(\varphi_i(t))^2\,dt \underset{\text{trapezoidal rule}}{\approx} (\underbrace{t_i - t_{i-1}}_{=h})\frac{r(t_{i-1})(\overbrace{\varphi_i(t_{i-1})}^{=0})^2 + r(t_i)(\overbrace{\varphi_i(t_i)}^{=1})^2}{2} = \frac{1}{2}hr_i$$

$$\int_{t_i}^{t_{i+1}} r(t)(\varphi_i(t))^2\,dt \approx (\underbrace{t_{i+1} - t_i}_{=h})\frac{r(t_i)(\overbrace{\varphi_i(t_i)}^{=1})^2 + r(t_{i+1})(\overbrace{\varphi_i(t_{i+1})}^{=0})^2}{2} = \frac{1}{2}hr_i$$

and get

$$\int_a^b r(t)(\varphi_i(t))^2\,dt \approx hr_i$$

Also,

$$\int_a^b p(t)(\varphi_i'(t))^2\,dt = \int_{t_{i-1}}^{t_i} p(t)\left(\frac{1}{h}\right)^2 dt + \int_{t_i}^{t_{i+1}} p(t)\left(-\frac{1}{h}\right)^2 dt \underset{\text{midpoint rule}}{\approx} \frac{1}{h}\underbrace{p(t_{i-1/2})}_{=p_{i-1/2}} + \frac{1}{h}\underbrace{p(t_{i+1/2})}_{p_{i+1/2}}$$

and

$$a_{ii} = \underbrace{\int_a^b p(t)(\varphi_i'(t))^2\,dt}_{\approx \frac{1}{h}(p_{i-1/2}+p_{i+1/2})} + \underbrace{\int_a^b r(t)(\varphi_i(t))^2\,dt}_{\approx hr_i} \approx \frac{1}{h}(p_{i-1/2}+p_{i+1/2}) + hr_i$$

For the right-hand side of $(*)$, we get

$$b_i = \underbrace{\langle f, \varphi_i \rangle}_{= \int_a^b f(t)\varphi_i(t)\,dt} - \underbrace{A(\psi, \varphi_i)}_{\int_a^b p(t)\psi'(t)\varphi_i'(t)\,dt + \int_a^b r(t)\psi(t)\varphi_i(t)\,dt} \qquad \psi(t) = \frac{B-A}{b-a}(t-a) + A$$

$$\int_a^b f(t)\varphi_i(t)\,dt = \int_{t_{i-1}}^{t_i} f(t)\underbrace{\varphi_i(t)}_{= \frac{t-t_{i-1}}{h}}\,dt + \int_{t_i}^{t_{i+1}} f(t)\underbrace{\varphi_i(t)}_{\frac{t_{i+1}-t}{h}}\,dt \approx$$

trapezoidal rule

$$\approx \underbrace{(t_i - t_{i-1})}_{=h} \frac{f(t_{i-1})\overbrace{\varphi_i(t_{i-1})}^{=0} + f(t_i)\overbrace{\varphi_i(t_i)}^{=1}}{2} + \underbrace{(t_{i+1} - t_i)}_{=h} \frac{f(t_i)\overbrace{\varphi_i(t_i)}^{=1} + f(t_{i+1})\overbrace{\varphi_i(t_{i+1})}^{=0}}{2}$$

$$= \frac{1}{2}hf_i + \frac{1}{2}hf_i = hf_i$$

and

$$\int_a^b p(t)\psi'(t)\varphi_i'(t)\,dt + \int_a^b r(t)\psi(t)\varphi_i(t)\,dt = \overbrace{\int_{t_{i-1}}^{t_i} p(t)\frac{B-A}{b-a}\cdot\frac{1}{h}\,dt}^{\approx \frac{B-A}{b-a}\cdot\frac{1}{h}\cdot\frac{h}{2}(p_i + p_{i-1})} + \overbrace{\int_{t_i}^{t_{i+1}} p(t)\frac{B-A}{b-a}\left(-\frac{1}{h}\right)dt}^{\approx \frac{B-A}{b-a}\left(-\frac{1}{h}\right)\frac{h}{2}(p_{i+1} + p_i)}$$

$$+ \underbrace{\int_{t_{i-1}}^{t_i} r(t)\psi(t)\varphi_i(t)\,dt}_{\approx \frac{t_i - t_{i-1}}{2}\left(r_i\psi_i\underbrace{\varphi_i(t_i)}_{=1} + r_{i-1}\psi_{i-1}\underbrace{\varphi_i(t_{i-1})}_{=0}\right)} + \underbrace{\int_{t_i}^{t_{i+1}} r(t)\psi(t)\varphi_i(t)\,dt}_{\approx \frac{t_{i+1}-t_i}{2}\left(r_i\psi_i\underbrace{\varphi_i(t_i)}_{=1} + r_{i+1}\psi_{i+1}\underbrace{\varphi_i(t_{i+1})}_{=0}\right)}$$

$$\approx \frac{1}{2}\frac{B-A}{b-a}(p_{i-1} - p_{i+1}) + hr_i\psi_i$$

So, we get

$$b_i \approx hf_i + \frac{1}{2}\frac{B-A}{b-a}(p_{i-1} - p_{i+1}) + hr_i\psi_i$$

Therefore, the i-th equation of system $(*)$ is written in the form

$$\underbrace{a_{i,i-1}}_{\approx -\frac{1}{h}p_{i-1/2}} c_{i-1} + \underbrace{a_{ii}}_{\approx \frac{1}{h}(p_{i-1/2} + p_{i+1/2}) + hr_i} c_i + \overbrace{a_{i,i+1}}^{\approx -\frac{1}{h}p_{i+1/2}} c_{i+1} = b_i, \quad \text{where } i = 1, 2, \dots, n-1, \quad a_{1,0} = 0, \; a_{n,n+1} = 0$$

So,

$$-\frac{1}{h} p_{i-1/2} c_{i-1} + \frac{p_{i-1/2}+p_{i+1/2}}{h} c_i - \frac{1}{h} p_{i+1/2} c_{i+1} + h r_i c_i = b_i$$

<u>Remark</u> : using different basis functions $\{\varphi_i\}_{i=1}^{n-1}$, we get different schemes and systems.