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Question 1

$$\begin{cases} v'''(t) = v(t)v' - s(t)(v''(t))^2, \quad \forall t > 0 \\ v(0) = 1, \quad v'(0) = 0; \quad v''(0) = 2 \end{cases}$$

Where $s(t)$ is a given function.

(1) We start by re-writing the given equation as a system of first order ordinary differential equations

$$\begin{cases} v'(t) = x(t) \\ x'(t) = y(t) \\ y'(t) = v(t)x(t) - s(t)(y(t))^2 \end{cases}$$

Let;

$$\mathbf{Y}(t) = \begin{pmatrix} v \\ x \\ y \end{pmatrix} \implies \mathbf{Y}'(t) = \begin{pmatrix} v' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \\ vx - sy^2 \end{pmatrix} \tag{1}$$

Thus;

$$F(t, \mathbf{Y}) = \begin{pmatrix} x \\ y \\ vx - sy^2 \end{pmatrix}$$

Where

$$\mathbf{Y(0)} = \mathbf{Y}_\circ = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Therefore;

$$\mathbf{Y}'(t) = F(t, \mathbf{Y})$$

(2) The associated midpoint is given by;
$v(0) = v_\circ$
$x(0) = x_\circ$
$y(0) = y_\circ$
$\Delta t_n = \Delta t$
$t_n = t_\circ + n\Delta t$
$t_{n+\frac{1}{2}} = t_n + \frac{\Delta t}{2}$
$\mathbf{Y}_{n+\frac{1}{2}} = \mathbf{Y}_n + \frac{\Delta t}{2} F(t_n, \mathbf{Y}_n)$
$\mathbf{Y}_{n+1} = \mathbf{Y}_n + \Delta t F(t_{n+\frac{1}{2}}, \mathbf{Y}_{n+\frac{1}{2}})$

(3) Kindly note that I made a change of variable from the initial vector $\mathbf{Y}_n$ in equation1 to $\mathbf{V}_n$ in order to be consistent with the requirements of the given problem.

$$\begin{cases} t_\circ = 0 \\ \Delta t = 0.5 \\ s = 1 + t^2 \end{cases}$$

For $n = 0$;

$$\begin{aligned} t_{n+\frac{1}{2}} &= t_n + \frac{\Delta t}{2} \\ t_{\frac{1}{2}} &= t_\circ + \frac{0.5}{2} = 0.25 \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{\frac{1}{2}} &= \mathbf{V}_\circ + \frac{\Delta t}{2} F(t_\circ, \mathbf{V}_\circ) \\ \mathbf{V}_{\frac{1}{2}} &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0.25 \begin{pmatrix} x_\circ \\ y_\circ \\ v_\circ x_\circ - (1 + t_\circ^2) y_\circ^2 \end{pmatrix} \\ \mathbf{V}_{\frac{1}{2}} &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0.25 \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} \\ \mathbf{V}_{\frac{1}{2}} &= \begin{pmatrix} 1 \\ 0.5 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{n+1} &= \mathbf{v}_n + \Delta F(t_{n+\frac{1}{2}}, \mathbf{V}_{n+\frac{1}{2}}) \\ \mathbf{V}_1 &= \mathbf{v}_\circ + \Delta F(t_{\frac{1}{2}}, \mathbf{V}_{\frac{1}{2}}) \\ \mathbf{V}_1 &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0.5 \begin{pmatrix} 0.5 \\ 1 \\ 0.5 \times 1 - (1 + 0.25^2) \times 1 \end{pmatrix} \\ \mathbf{V}_1 &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.25 \\ 0.5 \\ -0.28125 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.5 \\ 1.71875 \end{pmatrix} \end{aligned}$$

$v_1 = 1.25$

For $n = 1$

$$
\begin{aligned}
t_{\frac{3}{2}} &= t_1 + \frac{\Delta t}{2} = 0.5 + 0.25 = 0.75 \\
\mathbf{V}_{\frac{3}{2}} &= \mathbf{V}_1 + \frac{\Delta t}{2} F(t_1, \mathbf{V}_1) \\
&= \begin{pmatrix} 1.25 \\ 0.5 \\ 1.71875 \end{pmatrix} + 0.25 \begin{pmatrix} 0.5 \\ 1.71875 \\ 1.25 \times 0.5 - (1 + 0.5^2) 1.71875^2 \end{pmatrix} \\
&= \begin{pmatrix} 1.25 \\ 0.5 \\ 1.71875 \end{pmatrix} + \begin{pmatrix} 0.125 \\ 0.42968 \\ -0.7669 \end{pmatrix} \\
\mathbf{V}_{\frac{3}{2}} &= \begin{pmatrix} 1.375 \\ 0.92968 \\ 0.9518 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}_2 &= \mathbf{v}_1 + \Delta t F(t_{\frac{3}{2}}, \mathbf{V}_{\frac{3}{2}}) \\
\mathbf{V}_2 &= \begin{pmatrix} 1.25 \\ 0.5 \\ 1.71875 \end{pmatrix} + 0.5 \begin{pmatrix} 0.92968 \\ 0.9518 \\ 1.375 \times 0.92968 - (1 + 0.75^2) 0.9518^2 \end{pmatrix} \\
&= \begin{pmatrix} 1.25 \\ 0.5 \\ 1.71875 \end{pmatrix} + \begin{pmatrix} 0.46484 \\ 0.4759 \\ -0.06859 \end{pmatrix} \\
\mathbf{V}_2 &= \begin{pmatrix} 1.7148 \\ 0.9759 \\ 1.65016 \end{pmatrix}
\end{aligned}
$$

$v_2 = 1.7148$

Question 2

Given;

$$C\frac{dT}{dt} = \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma T^4 \tag{2}$$

Where: $C = 85, \quad \alpha = 0.3, \quad S_\circ = 1367, \epsilon = 0.6, \sigma = 5.67 \times 10^{-8}$ and $T = T(t)$ is the globally averaged surface temperature.

(1) At equilibruim temperature T_{eq}, $\frac{dT}{dt} = 0$, thus equation2 becomes;

$$\begin{aligned}
C \times 0 &= \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma T_{eq}^4 \\
T_{eq}^4 &= \frac{(1-\alpha)S_\circ}{4\epsilon\sigma} \\
T_{eq} &= \sqrt[4]{\frac{(1-\alpha)S_\circ}{4\epsilon\sigma}} \\
&= \sqrt[4]{\frac{(1-0.3)1367}{4 \times 0.6 \times 5.67 \times 10^{-8}}} \\
&= 289.57K
\end{aligned}$$

Therefore, $T_{eq} = 289.57K$

(2) We want to show that;

$$C\frac{dT}{dt} = \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma T^4 \tag{3}$$

Where; $T(t) = T_{eq} + \tilde{T}(t)$, by substituting $T(t)$ in equation3, we shall have;

$$\begin{aligned}
C\frac{d(T_{eq} + \tilde{T}(t))}{dt} &= \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma(\tilde{T} + T_{eq})^4 \\
C\frac{dT_{eq}}{dt} + C\frac{d\tilde{T}}{dt} &= \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma(\tilde{T} + T_{eq})^4
\end{aligned}$$

At equilibrium, $\frac{dT_{eq}}{dt} = 0$, thus;

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma(\tilde{T} + T_{eq})^4 \tag{4}$$

Hence proved!

(3) Assume that

$$\left(1 + \frac{\tilde{T}}{T_{eq}}\right)^4 = 1 + 4\frac{\tilde{T}}{T_{eq}} \tag{5}$$

we are required to prove

$$\frac{d\tilde{T}}{dt} = -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\tilde{T}$$

From equation4, we showed that

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma(\tilde{T}+T_{eq})^4$$

we can further simplify it as;

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma\left(T_{eq}\left(1+\frac{\tilde{T}}{T_{eq}}\right)\right)^4$$
$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma T_{eq}^4\left(1+\frac{\tilde{T}}{T_{eq}}\right)^4$$

Using the relation in equation5, we get;

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_\circ}{4} - \epsilon\sigma T_{eq}^4\left(1+4\frac{\tilde{T}}{T_{eq}}\right)$$
$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_\circ}{4} - 4\epsilon\sigma T_{eq}^3\tilde{T} - \epsilon\sigma T_{eq}^4$$

But at equilibrium,

$$\epsilon\sigma T_{eq}^4 = \frac{(1-\alpha)S_\circ}{4}$$

by substitution, we then have,

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_\circ}{4} - 4\epsilon\sigma T_{eq}^3\tilde{T} - \frac{(1-\alpha)S_\circ}{4}$$
$$C\frac{d\tilde{T}}{dt} = -4\epsilon\sigma T_{eq}^3\tilde{T}$$
$$\frac{d\tilde{T}}{dt} = -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\tilde{T} \qquad 2(a)$$

Hence shown!

(4) Given $\tilde{T}(0) = 0$, we want to find the exact solution of (2a)

$$\frac{d\tilde{T}}{dt} = -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\tilde{T}$$
$$\frac{d\tilde{T}}{\tilde{T}} = -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)dt$$
$$\int\frac{d\tilde{T}}{\tilde{T}} = -\int\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)dt$$
$$\tilde{T} = \exp\left[\frac{-4\epsilon\sigma T_{eq}^3 t}{C}\right] + B$$
$$\tilde{T} = A\exp\left[\frac{-4\epsilon\sigma T_{eq}^3 t}{C}\right]$$

$\tilde{T}(0) = 10$, thus

$$10 = A\exp(0)$$
$$A = 10$$

The exact solution is given as;

$$\tilde{T}(t) = 10\exp\left[\frac{-4\epsilon\sigma T_{eq}^3 t}{C}\right]$$

By substituting the values of C, σ, ϵ, T_{eq}, we shall obtain;

$$\tilde{T}(t) = 10\exp[-0.0388t]$$

(5a)

$$(P)\begin{cases} y_\circ & = y_\circ \\ K_1 & = f(t_n, y_n) \\ t_{n+\frac{3}{4}} & = t_n + \frac{3}{4}\Delta t \\ y_{n+\frac{3}{4}} & = y_n + \frac{3}{4}\Delta t K_1 \\ K_2 & = f(t_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}) \\ y_{n+1} & = y_n + \frac{1}{3}\Delta t(K_1 + 2K_2) \end{cases}$$

For a one step method;

$$y_{n+1} = y_n + \Delta t_n \phi(t_n, y_n, \Delta t_n) \tag{6}$$

From the given scheme

$$y_{n+1} = y_n + \frac{1}{3}\Delta t(K_1 + 2K_2)$$

By substituting for K_1 and K_2

$$y_{n+1} = y_n + \frac{1}{3}\Delta t\left(f(t_n, y_n) + 2f(t_{n+\frac{3}{4}}, y_{n+\frac{3}{4}})\right)$$

By substituting for $t_{n+\frac{3}{4}}$ and $y_{n+\frac{3}{4}}$, we obtain;

$$y_{n+1} = y_n + \frac{1}{3}\Delta t\left(f(t_n, y_n) + 2\left(t_n + \frac{3}{4}\Delta t, y_n + \frac{3}{4}\Delta t K_1\right)\right)$$

Now substituting for K_1, we get

$$y_{n+1} = y_n + \frac{1}{3}\Delta t\left(f(t_n, y_n) + 2\left(t_n + \frac{3}{4}\Delta t, y_n + \frac{3}{4}\Delta t \,.\, f(t_n, y_n)\right)\right) \tag{7}$$

From equation5 and equation7, we obtain ϕ

$$\phi(t_n, y_n, \Delta t_n) = \frac{1}{3}\left(f(t_n, y_n) + 2\left(t_n + \frac{3}{4}\Delta t, y_n + \frac{3}{4}\Delta t \,.\, f(t_n, y_n)\right)\right)$$

Thus for u, v, w

$$\phi(u, v, w) = \frac{1}{3}\left(f(u, v) + 2\left(u + \frac{3}{4}w, v + \frac{3}{4}w \,.\, f(u, v)\right)\right)$$

Where ϕ is a continuous function.

(5b) .

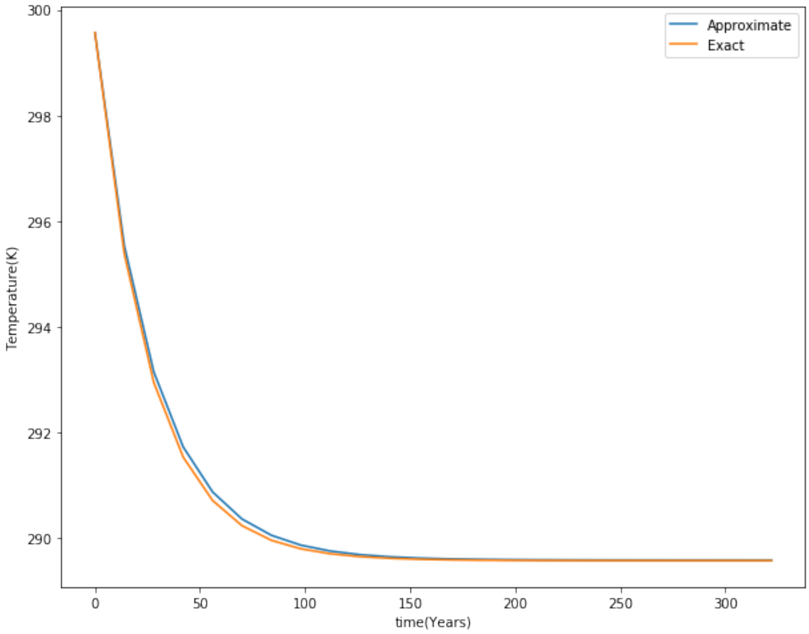

Figure 1: Exact and Approximate solution