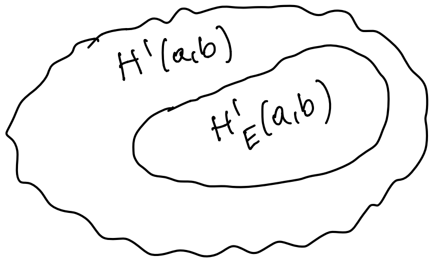
Finite element method - continuation

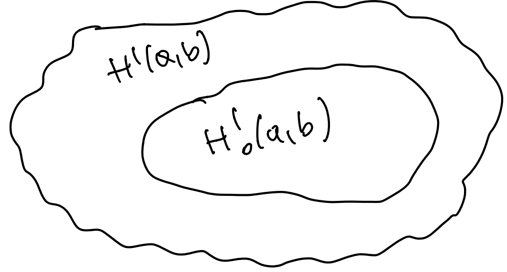
Boundary-value problem: $\begin{cases} -\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + r(t)u(t) = f(t), & t \in (a,b) \\ u(a) = A, & u(b) = B \end{cases}$

where $p \in C^1([a,b], \mathbb{R})$, $r \in C([a,b], \mathbb{R})$, $f \in L^2((a,b), \mathbb{R})$
and $\forall t \in [a,b]$ $\quad r(t) \geq 0$, $\quad p(t) \geq C_0 > 0$

Definition

$$H^1_E(a,b) = \{ v \in H^1(a,b) : v(a) = A, \; v(b) = B \}$$
$$H^1_0(a,b) = \{ v \in H^1(a,b) : v(a) = v(b) = 0 \}$$





Let $\mathcal{F} : H^1_E(a,b) \to \mathbb{R}$ be the following functional

$$\mathcal{F}(W) = \frac{1}{2}\int_a^b p(t)\left(W'(t)\right)^2 + r(t)\left(W(t)\right)^2 dt - \int_a^b f(t)W(t)\,dt, \text{ where}$$

$$W \in H^1_E(a,b) = \{ W : [a,b] \to \mathbb{R} \mid W \text{ is absolutely continuous on } [a,b], \; W' \in L^2(a,b), \; W(a) = A, \; W(b) = B \}.$$

Variational problem: find $u \in H^1_E(a,b)$ such that $\mathcal{F}(u) = \min_{W \in H^1_E(a,b)} \mathcal{F}(W)$

The last component in $\mathcal{F}(W)$ is the inner product

$$\langle f, W \rangle = \int_a^b f(t)W(t)\,dt \quad \text{in } L^2(a,b).$$

For the first component in $\mathcal{F}(W)$, we define

$$\mathcal{A}(W, v) \underset{\text{definition}}{=} \int_a^b p(t)W'(t)v'(t) + r(t)W(t)v(t)\,dt.$$

Then, $\mathcal{A}(w,w) = \int_a^b p(t)\left(w'(t)\right)^2 + r(t)\left(w(t)\right)^2 dt$ and

$\mathcal{F}(w) = \frac{1}{2}\mathcal{A}(w,w) - \langle f, w\rangle$.

<u>Lemma</u> The functional $\mathcal{A}: H^1_E(a,b) \times H^1_E(a,b) \to \mathbb{R}$ satisfies the properties:

$$\left.\begin{aligned}
&1^\circ\ \forall \lambda,\mu \in \mathbb{R}\ \ \forall w_1, w_2, v \in H^1_E(a,b) \\
&\quad \mathcal{A}(\lambda w_1 + \mu w_2, v) = \lambda \mathcal{A}(w_1,v) + \mu \mathcal{A}(w_2, v) \\
&2^\circ\ \forall \lambda,\mu \in \mathbb{R}\ \ \forall w_1, w_2, v \in H^1_E(a,b) \\
&\quad \mathcal{A}(v, \lambda w_1 + \mu w_2) = \lambda \mathcal{A}(v,w_1) + \mu \mathcal{A}(v_1, w_2)
\end{aligned}\right\}\ \mathcal{A} \text{ is bilinear}$$

<u>Lemma</u> The functional $\mathcal{A}: H^1_E(a,b) \times H^1_E(a,b) \to \mathbb{R}$ is symmetric, that is, $\forall w, v \in H^1_E(a,b)$ $\mathcal{A}(w,v) = \mathcal{A}(v,w)$

<u>Proof</u>

$LHS = \mathcal{A}(w,v) = \int_a^b p(t)\, w'(t)\, v'(t) + r(t) w(t) v(t)\, dt = \mathcal{A}(v,w) = RHS$

<u>Theorem</u>

$$\underbrace{\mathcal{F}(u) = \min_{w \in H^1_E(a,b)} \mathcal{F}(w)}_{\text{Rayleigh-Ritz method}} \iff \forall v \in H^1_0(a,b)\ \ \underbrace{\mathcal{A}(u,v) = \langle f, v\rangle}_{\text{Galerkin method}}$$

<u>Proof</u> $\boxed{\Rightarrow}$ Suppose $u \in H^1_E(a,b)$ is such that $\mathcal{F}(u) = \min_{w \in H^1_E(a,b)} \mathcal{F}(w)$.

Then, $\forall w \in H^1_E(a,b)$ $\mathcal{F}(w) \geq \mathcal{F}(u)$.

Let $\lambda \in \mathbb{R}$ and $v \in H^1_0(a,b)$ be arbitrary. Then, $u + \lambda v \in H^1_E(a,b)$ and

$\mathcal{F}(u) \leq \mathcal{F}(u+\lambda v) = \frac{1}{2}\mathcal{A}(u+\lambda v, u+\lambda v) - \langle f, u+\lambda v\rangle =$

$= \frac{1}{2}\left[\mathcal{A}(u, u+\lambda v) + \lambda \mathcal{A}(v, u+\lambda v)\right] - \langle f, u+\lambda v\rangle =$

$= \frac{1}{2}\left[\mathcal{A}(u,u) + \lambda\mathcal{A}(u,v) + \lambda \underbrace{\mathcal{A}(v,u)}_{=\mathcal{A}(u,v)} + \lambda^2 \mathcal{A}(v,v)\right] - \langle f, u\rangle - \lambda\langle f, v\rangle$

$$(*) = \underbrace{\tfrac{1}{2}\mathcal{A}(u,u) - \langle f,u\rangle}_{=\mathcal{F}(u)} + \lambda \mathcal{A}(u,v) + \tfrac{1}{2}\lambda^2 \mathcal{A}(v,v) - \lambda\langle f,v\rangle$$

Therefore,

$$-\tfrac{1}{2}\lambda^2 \mathcal{A}(v,v) \le \lambda\left[\mathcal{A}(u,v) - \langle f,v\rangle\right].$$

We now take $\lambda > 0$ and get

$$(1) \quad -\tfrac{1}{2}\lambda \mathcal{A}(v,v) \le \mathcal{A}(u,v) - \langle f,v\rangle.$$

We now take $\lambda \to 0^+$,

$$(2) \quad 0 \le \mathcal{A}(u,v) - \langle f,v\rangle.$$

We now replace v by $-v$ and, from (1), we get

$$-\tfrac{1}{2}\lambda \underbrace{\mathcal{A}(-v,-v)}_{=\mathcal{A}(v,v)} \le \underbrace{\mathcal{A}(u,-v)}_{\substack{=-\mathcal{A}(u,v) \\ \text{from the definition} \\ \text{of } \mathcal{A}}} - \underbrace{\langle f,-v\rangle}_{=-\langle f,v\rangle}$$

$$-\tfrac{1}{2}\lambda \mathcal{A}(v,v) \le -\mathcal{A}(u,v) + \langle f,v\rangle$$

$$(3) \quad \tfrac{1}{2}\lambda \mathcal{A}(v,v) \ge \mathcal{A}(u,v) - \langle f,v\rangle$$

Taking $\lambda \to 0^+$ in (3), we get

$$(4) \quad 0 \ge \mathcal{A}(u,v) - \langle f,v\rangle.$$

From (2) and (4), we get $0 = \mathcal{A}(u,v) - \langle f,v\rangle$, and since $v \in H_0^1(a,b)$ was arbitrary, we conclude that

$$\forall v \in H_0^1(a,b) \quad \mathcal{A}(u,v) = \langle f,v\rangle,$$

which finishes the first part of the proof.