Partial Differential Equations
Problem sheet 2

Answers

1. (a) **[6 marks]** Compute the full Fourier series representation of

$$f(x) = e^{ax}, \qquad -\pi \le x < \pi. \tag{1}$$

when extended as a 2π-periodic function.

Solution 1(a)
Using the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

we have that

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\mathrm{d}x \\
&= \frac{1}{a\pi}\left[e^{ax}\right]_{-\pi}^{\pi} \\
&= \frac{1}{a\pi}\left(e^{a\pi} - e^{-a\pi}\right) \\
&= \frac{2}{a\pi}\sinh a\pi,
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos(nx)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^{x(a+n\mathrm{i})} + e^{x(a-n\mathrm{i})}\right)\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\frac{1}{a+n\mathrm{i}}e^{x(a+n\mathrm{i})} + \frac{1}{a-n\mathrm{i}}e^{x(a-n\mathrm{i})}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{2ae^{ax}\cos(nx) + 2\mathrm{i}ne^{ax}\sin(nx)}{a^2+n^2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi(a^2+n^2)}\left(ae^{a\pi}\cos(n\pi) - ae^{-a\pi}\cos(n\pi)\right) \\
&= 2\frac{a(-1)^n}{\pi(n^2+a^2)}\sinh(a\pi),
\end{aligned}$$

and finally

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\mathrm{d}x \\
&= \frac{1}{2\mathrm{i}\pi}\int_{-\pi}^{\pi}\left(e^{x(a+n\mathrm{i})} - e^{x(a-n\mathrm{i})}\right)\mathrm{d}x
\end{aligned}
$$

then, using the same ideas as above,

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[-\mathrm{i}\ \frac{2ne^{ax}\cos(nx) - 2ae^{ax}\sin(nx)}{a^2+n^2}\right]_{-\pi}^{\pi} \\
&= 2\frac{n(-1)^{n+1}}{\pi(n^2+a^2)}\sinh(a\pi).
\end{aligned}
$$

Putting this altogether we have that

$$
e^{ax} = \frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\left(a\cos(nx) - n\sin(nx)\right)\right\}. \tag{2}
$$

Note that an alternative would be to integrate by parts twice.

(b) **[4 marks]** By using the result of equation (1) or otherwise determine the full Fourier series expansion for the function

$$
g(x) = \sinh(x), \qquad -\pi \le x < \pi.
$$

when extended as a 2π-periodic function.

Solution 1(b)

We use the fact that

$$
\begin{aligned}
g(x) &= \sinh(x) \\
&= \frac{1}{2}\left(e^{x} - e^{-x}\right) \\
&= \frac{1}{2}\left([f(x)]_{a=1} - [f(x)]_{a=-1}\right).
\end{aligned}
$$

We then note that even parts will disappear to get

$$
= \frac{\sinh(\pi)}{\pi}\left\{2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(-n\sin(nx))\right\}.
$$

2. The vibrations $u(x,t)$ of air in an open pipe of unit length satisfy the wave equation

$$
\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad t > 0, \tag{3}
$$

with boundary conditions

$$
\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0, \quad \text{for all } t > 0. \tag{4}
$$

The air is initially at rest so that

$$
\frac{\partial u}{\partial t}(x,0) = 0, \quad \text{for all } 0 \le x \le 1. \tag{5}
$$

(a) **[13 marks]** Use separation of variables to show that the general solution of the wave equation (3) subject to the boundary conditions (4) and initial condition (5) is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)\cos(cn\pi t).$$

(b) **[7 marks]** Find the unique solution that in addition satisfies the initial condition

$$u(x,0) = \begin{cases} 1 & \text{for } 0 \le x \le \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < x \le 1 \end{cases}$$

Solution question 2 (both parts)
This is a standard separation of variables question. Assume that $u(t,x) = T(t)X(x)$. Substituting in to the wave equation gives

$$\frac{1}{c^2}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}X = \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}T$$

which can be rewritten as

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}.$$

As one side depends solely on t and one on x they must be separately constant (e.g., both equal λ). This gives the two ODEs

$$\begin{aligned} \ddot{T} - \lambda c^2 T &= 0, \\ X'' - \lambda X &= 0. \end{aligned}$$

First we note that the boundary conditions on u in space imply boundary conditions on X (or trivial solutions):

$$X'(0) = 0 = X'(1).$$

We can then look at the solution for X by looking at the three possible cases for λ:

(a) $\lambda = \mu^2 > 0$: In this case the general solution is

$$X(x) = Ae^{\mu x} + Be^{-\mu x}.$$

Combining this with the boundary conditions on X gives that the only solution is the trivial one $A = B = 0$.

(b) $\lambda = 0$: In this case the general solution is

$$X(x) = Ax + B.$$

Then $X'(x) = A$. Combining this with the boundary conditions on X' gives that $A = 0$ so that we have the non-trivial solution $X_0(x) = A_0 = const.$

(c) $\lambda = -\mu^2 < 0$: In this case the general solution is

$$X(x) = A\cos(\mu x) + B\sin(\mu x).$$

Hence
$$X'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x).$$
The boundary condition $X'(0) = 0$ implies that $B = 0$. The boundary condition at $X'(1) = 0$ implies that the only non-trivial solution occurs when

$$\sin(\mu) = 0$$

which implies that $\mu = n\pi$ for any integer n.

This implies that the only non-trivial solution for X is

$$X_n(x) = A_n \cos(n\pi x)$$

with eigenvalue $\lambda_n = -(n\pi)^2$.

We can now solve to find T. We first deal with the case $\lambda = 0$. The T equation is just $\ddot{T} = 0$ with solution $T_0(t) = C_0 + D_0 t$ The initial condition (5) gives $\dot{T}(0) = 0$ and hence $D_0 = 0$ and $T_0 = C_0 = const$.

We now solve the equation for $n = 1, 2, 3 \ldots$.

$$\ddot{T}_n + (n\pi c)^2 T_n = 0.$$

This has the general solution

$$T_n(t) = C_n \cos(n\pi ct) + D_n \sin(n\pi ct).$$

Again we can use the initial condition on u to impose that $\dot{T}(0) = 0$ which tells us that $D_n = 0$.

From this we can combine the non-trivial solutions for T_n and X_n to find that

$$u_n = A_n \cos(n\pi ct)\cos(n\pi x)$$

is a solution for any non-negative integer n (where this also works for $n = 0$ as $\cos(0) = 1$). As the equation is linear we can use the principle of superposition to write the general solution as

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi ct)\cos(n\pi x).$$

Solution 2(b) This is essentially a Fourier series question. We take the general solution, and set $t = 0$, which gives
$$p(x) := u(x,0) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)$$
So that $A_0 = a_0/2$ and $A_n = a_n$, $n = 1, 2, 3 \ldots$, where a_n and the Fourier cosine coefficients given by the half-range formula.

Using the standard half-range formula we have that

$$a_0 = 2\int_0^1 p(x)dx = 2\int_0^{1/2} 1dx = 1$$

$$
\begin{aligned}
a_n &= 2\int_0^1 p(x)\cos(n\pi x)dx \\
&= 2\int_0^{1/2} \cos(n\pi x)dx \\
&= 2\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^{1/2} \\
&= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

So that

$$
u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos(n\pi x)\cos(n\pi ct) \tag{6}
$$

Noting that $\sin\left(\frac{n\pi}{2}\right) = 0$ for n even we put $n = 2m-1$ then $\sin\left(\frac{(2m-1)\pi}{2}\right) = (-1)^{m+1}$ so that

$$
u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}\cos((2m-1)\pi x)\cos((2m-1)\pi ct) \tag{7}
$$

3. This question deals with the behaviour of the Fourier series near a discontinuity. The effect described below is often called the *Gibbs phenomenon* and is also discussed on many websites.

   (a) **[4 marks]** The function $f(x)$ is defined as

   $$
   f(x) = 1 \qquad 0 < x < \pi.
   $$

   Sketch the odd extension and show that the Fourier sine series expansion is

   $$
   f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1}. \tag{8}
   $$

   Solution 3(a)
   The extension is as given in figures 1, 2. As it is an odd extension we need only compute

   $$
   \begin{aligned}
   b_m &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\mathrm{d}x \\
   &= \frac{2}{\pi}\int_0^{\pi}\sin(mx)\mathrm{d}x \\
   &= -\frac{2}{m\pi}\left[\cos(mx)\right]_0^{\pi} \\
   &= \begin{cases} 0 & \text{if } m \text{ is even} \\ \frac{4}{m\pi} & \text{if } m \text{ is odd.} \end{cases}
   \end{aligned}
   $$

   Therefore the Fourier series is given by

   $$
   \begin{aligned}
   f(x) &= \sum_{k \text{ odd}}\frac{4}{k\pi}\sin(kx) \\
   &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1}.
   \end{aligned}
   $$

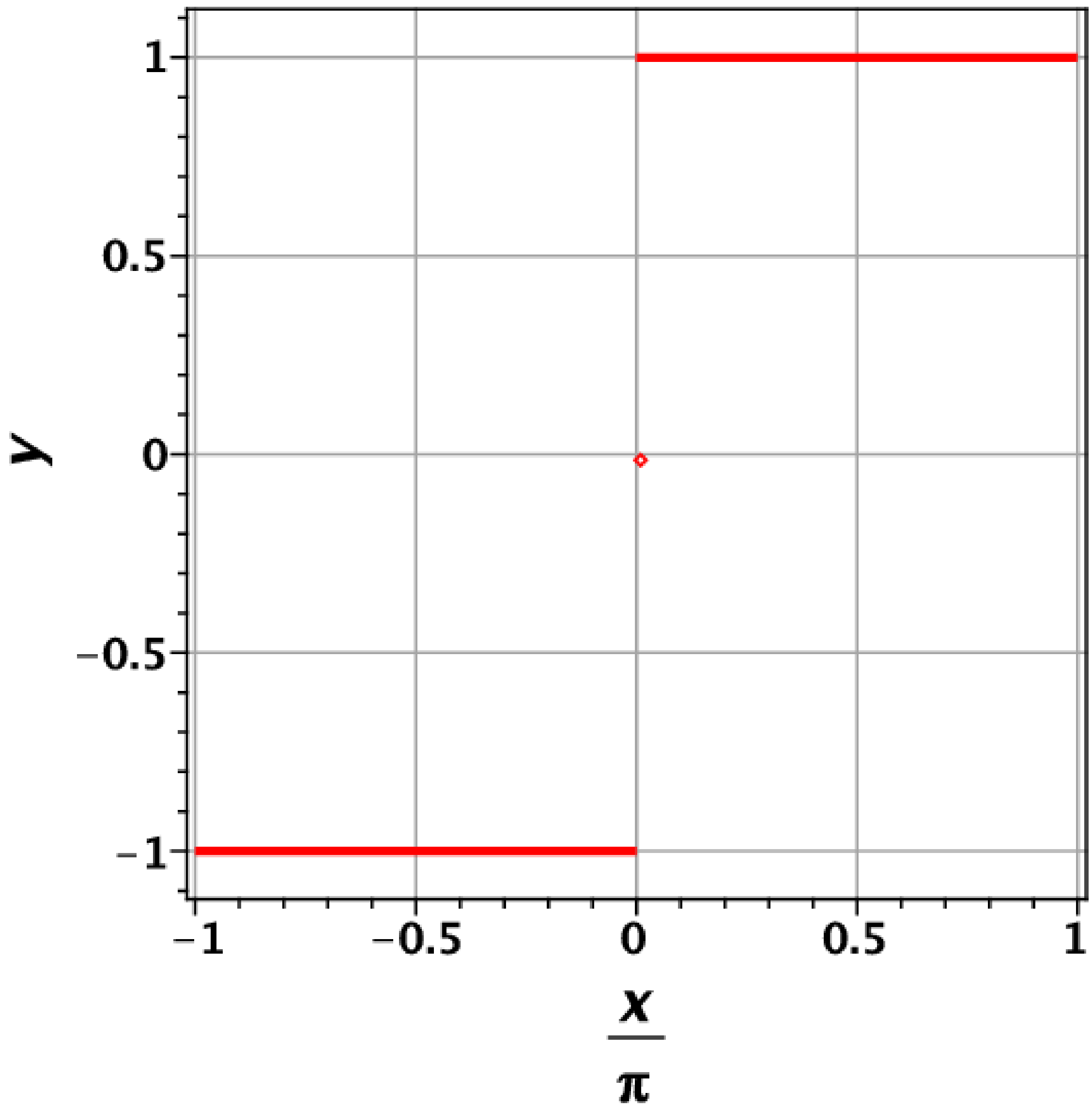

Figure 1: Odd extension of $f = 1$ over the standard range.

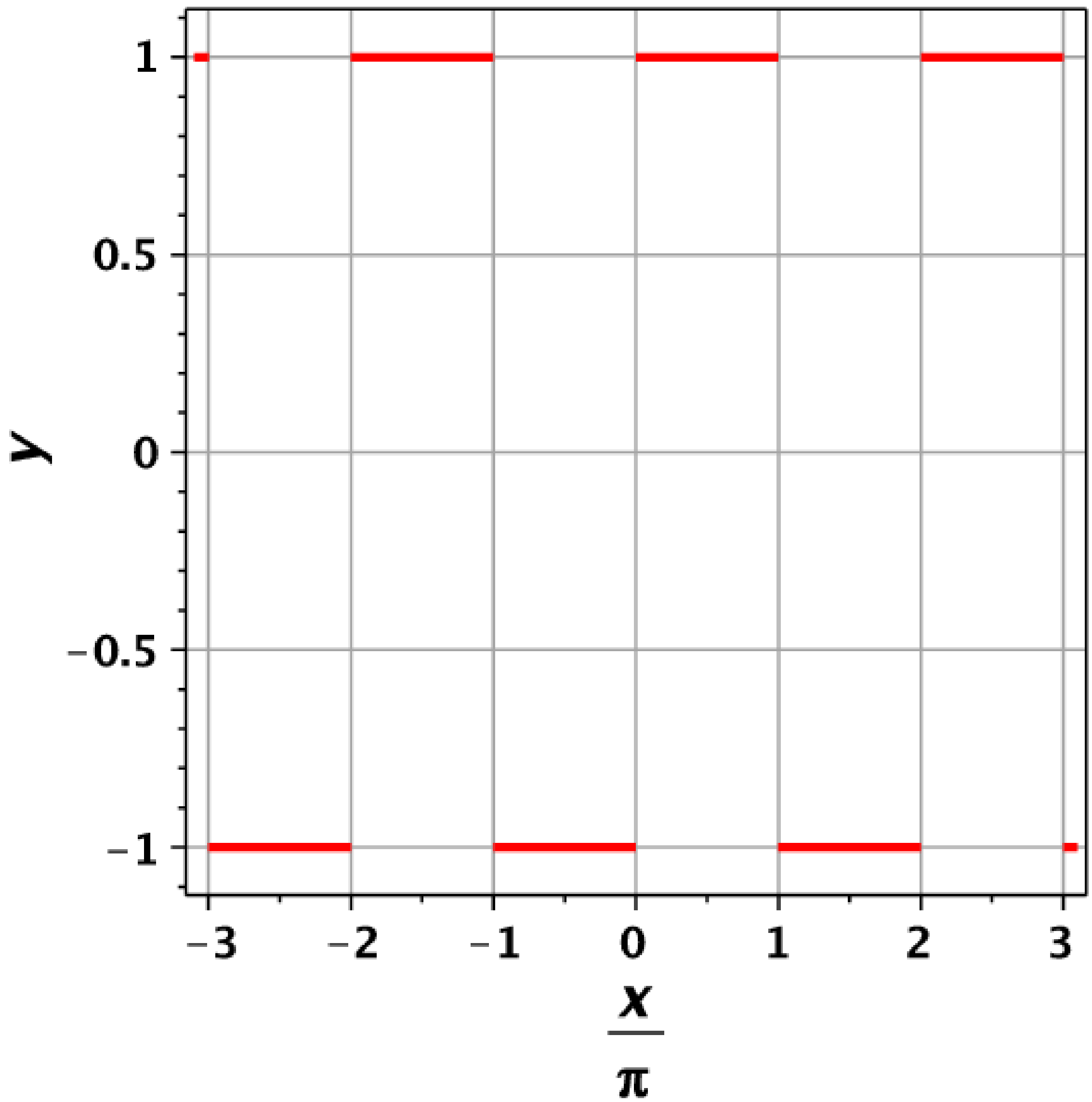

Figure 2: Odd extension of $f = 1$ over three standard ranges.

(b) **[4 marks]** Using Python plot the partial sum (6)

$$f_N(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin\left((2n-1)x\right)}{2n-1}. \tag{9}$$

for the cases when $N = 5, 10, 20$, over the range $0 < x < \pi$. In addition plot the series with $N = 20, 50, 500$ terms over the range $0 < x < 0.1$.

Solution 3(b)

The figures are as in figures 3, 4, 5.

(c) **[3 marks]** Show that the partial sum in equation (9) may be written as

$$f_N(x) = \frac{2}{\pi}\int_0^x \frac{\sin\left(2Nt\right)}{\sin\left(t\right)}\mathrm{d}t.$$

Hint: Consider the derivative of f_N and show that

$$\sum_{n=1}^{N}\sin\left(x\right)\cos\left((2n-1)x\right) = \tfrac{1}{2}\sin\left(2Nx\right).$$

Solution 3(c)

Given the definition of the partial sum the derivative is

$$f_N'(x) = \frac{4}{\pi}\sum_{n=1}^{N}\cos\left((2n-1)x\right).$$

Multiplying both sides by $\sin\left(x\right)$ and using the hint we have that

$$f_N'(x)\sin\left(x\right) = \frac{2}{\pi}\sin\left(2Nx\right).$$

Hence

$$f_N'(x) = \frac{2}{\pi}\frac{\sin\left(2Nx\right)}{\sin x}.$$

Integrating both sides we may write the partial sum as

$$f_N(x) = \frac{2}{\pi}\int_0^x \frac{\sin\left(2Nt\right)}{\sin\left(t\right)}\mathrm{d}t,$$

Where there is no need for a constant of integration since both sides vanish at $x = 0$.

To prove the identity given in the hint we use the standard multiple angle formulae to get

$$\sin\left(a\right)\cos\left(b\right) = \tfrac{1}{2}\left(\sin\left(b+a\right) - \sin\left(b-a\right)\right).$$

Hence the sum is

$$\sum_{n=1}^{N}\sin\left(x\right)\cos\left(2n-1\right)x = \sum_{n=1}^{N}\tfrac{1}{2}\left(\sin\left(2nx\right) - \sin\left(2(n-1)x\right)\right)$$

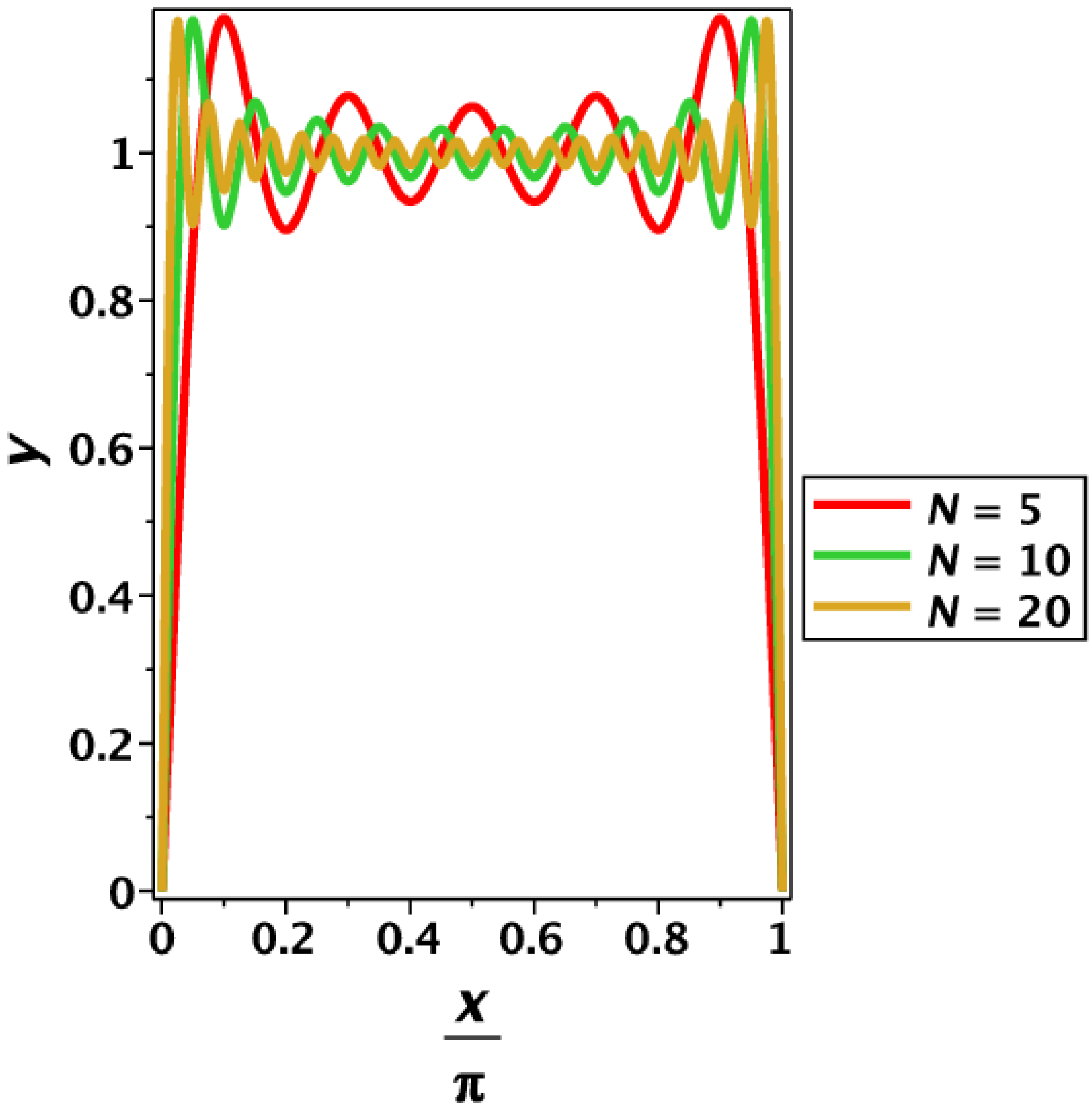

Figure 3: The partial sums of the Fourier Series approximation to this discontinuous function. Note the behaviour near $x = 0$.

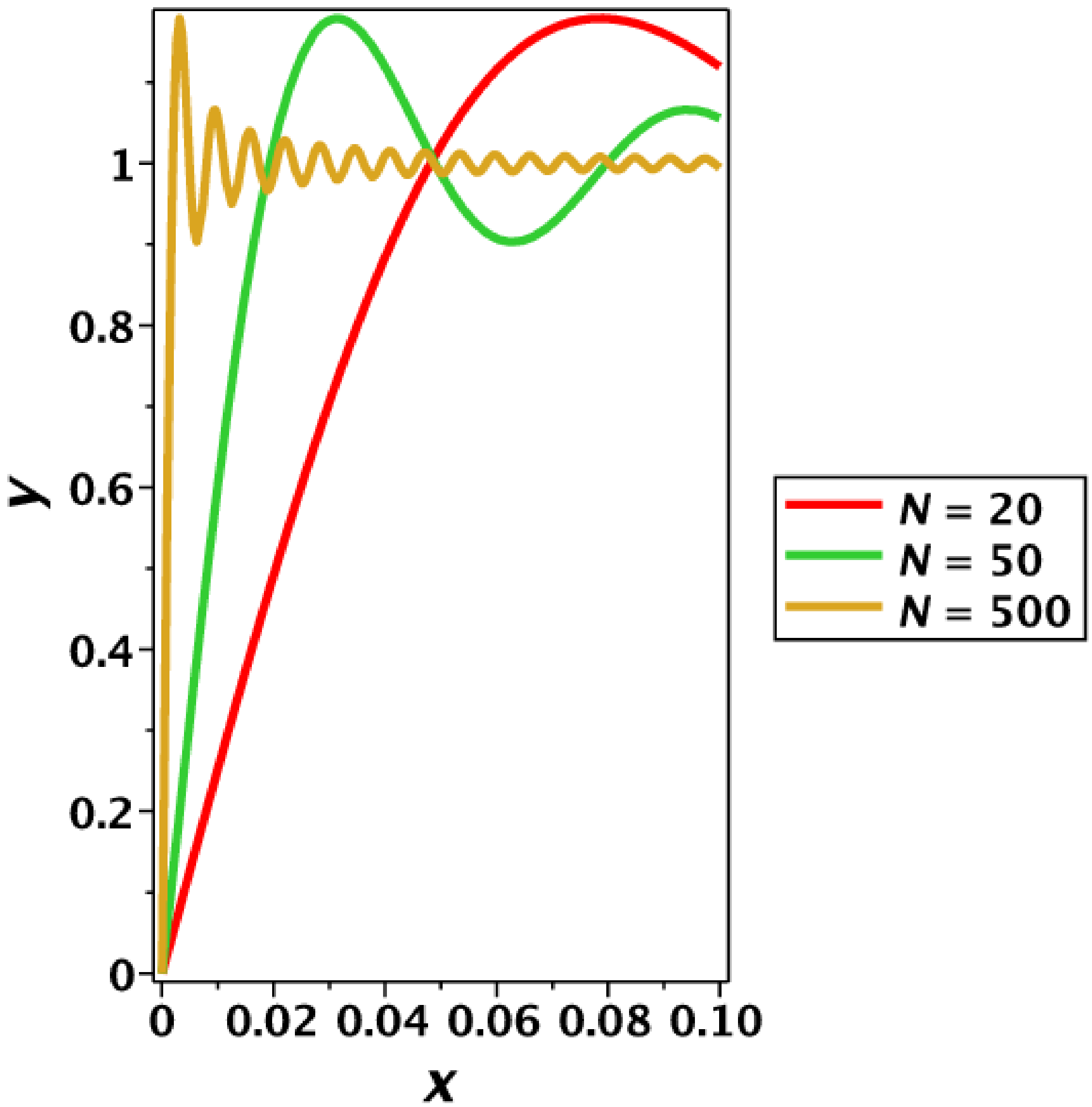

Figure 4: The partial sums of the Fourier Series approximation to this discontinuous function. Note the behaviour near $x = 0$.

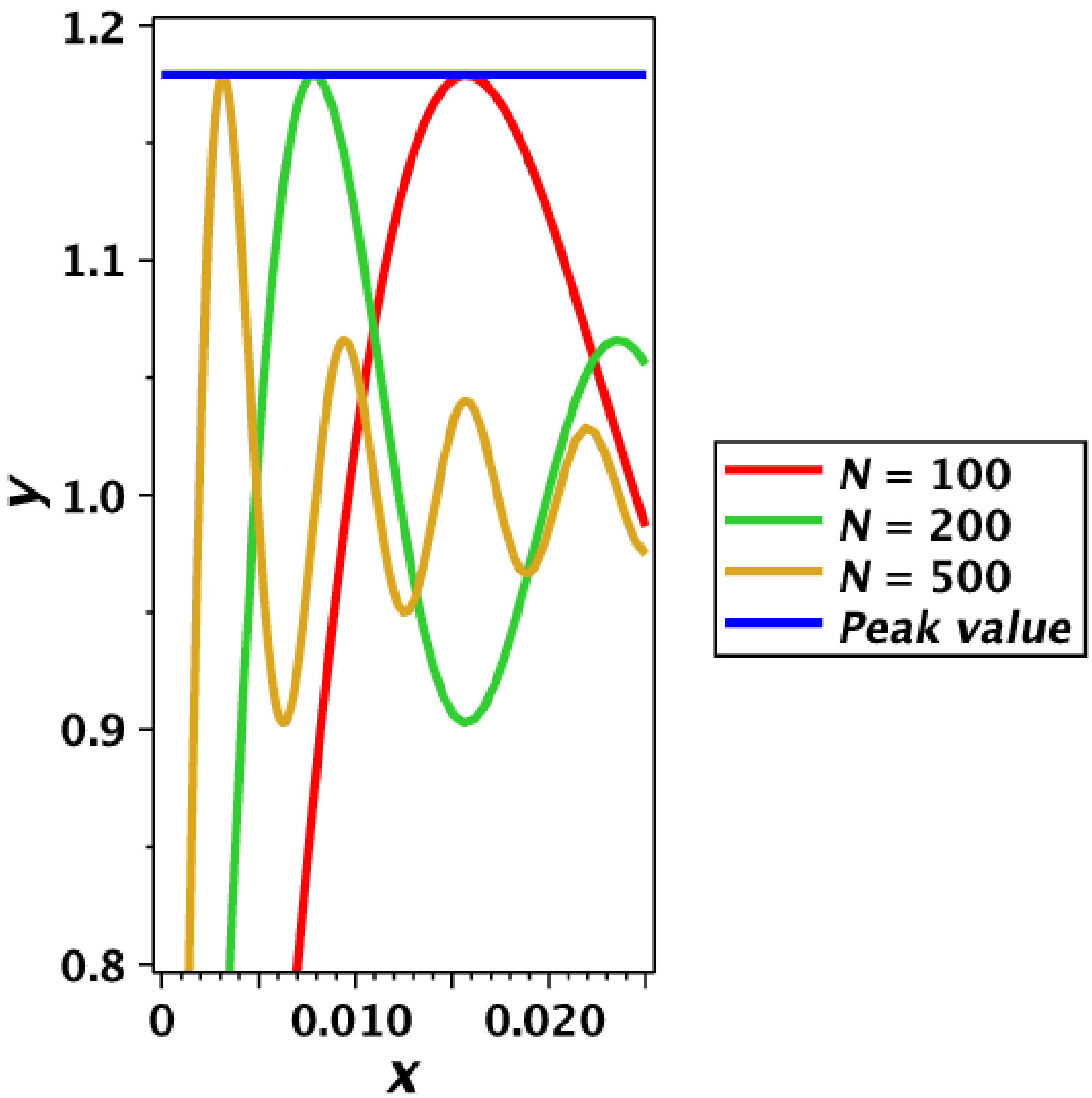

Figure 5: The partial sums of the Fourier Series approximation to this discontinuous function. Note the behaviour near $x = 0$. This illustrates the behaviour of the first peak, and the point of these questions.

which is a *telescoping* sum, i.e. most of the terms cancel:

$$\begin{aligned}
&= \sin(2Nx) - \sin(2(N-1)x) \\
&\quad + \sin(2(N-1)x) - \sin(2(N-2)x) \\
&\quad + \sin(2(N-2)x) - \cdots - \sin(2x) \\
&\quad + \sin(2x) - \sin(0x) \\
&= \sin(2Nx).
\end{aligned}$$

(An alternative method is to regard $\cos((2n-1)x)$ as the real part of $e^{i(2n-1)x}$, sum the geometric progression and then take the real part).

(d) **[4 marks]** One can show that the partial sum f_N has extrema at $x = \frac{m\pi}{2N}$.One can also show that the first extrema to the right of the discontinuity is located at $x = \frac{\pi}{2N}$ and has value

$$f_N\left(\frac{\pi}{2N}\right) \approx \frac{2}{\pi}\int_0^{\pi} \frac{\sin(u)}{u}\mathrm{d}u, \tag{10}$$

You do not have to prove this!. By using Python to evaluate (10) numerically show that

$$f_N\left(\frac{\pi}{2N}\right) \approx 1.1790$$

independently of N.

Hence comment on the accuracy of Fourier series at discontinuities (also known as *Gibbs phenomenon*). Given that the error at $\pi/(2N)$ is nearly constant explain why the Fourier Convergence theorem is, or is not, valid for this problem?

Solution 3(d)

The exact value of this integral requires numerical computation. My calculation gives 1.178979744. This should be compared against the earlier plots.

This result shows that the maximum error in the Fourier series approximation does not vanish with increasing N. However, the location of the extrema becomes steadily closer to the discontinuity where the value of the function is not defined. Therefore we do not violate the convergence theorem, as the convergence theorem works by considering a *fixed* point x whilst letting $N \to \infty$, and here we have a point whose location depends on N; in the limit the series converges everywhere except at the discontinuity to the correct value, and at the discontinuity at $x = 0$ the Fourier will always be zero, exactly half the value of the left and right limits. However, for any finite number of terms the maximum error will be non-zero and will not converge. This illustrates the somewhat tricky point about *what* actually converges when we look at Fourier Series.

For the Fourier Convergence theorem the measure of error is an "average". Hence although the error at a particular point $x = \pi/2N$ stays fixed the integral of the square of the error goes to zero because the region in which the error stays fixed goes to zero. Hence the Fourier Convergence Theorem **does hold** in this case.