Goal: apply Theorem 1 (proved last time) to prove the convergence of Euler's method

Theorem 2 Suppose

1° $f(\cdot,\cdot)$ is continuous

2° f satisfies a Lipschitz condition with respect to the second argument on $[a,b]\times[y_0-C, y_0+C]$

$\forall t\in[a,b]\ \forall x, \tilde{x}\in[y_0-C, y_0+C]$

$|f(t,x)-f(t,\tilde{x})|\leq L|x-\tilde{x}|$

3° $\forall n\in\{0,1,\dots,N-1\}\quad y_{n+1}=y_n+hf(t_n,y_n)\in[y_0-C, y_0+C]$

Then,

$$\forall n\in\{1,2,\dots,N\}\quad |e_n|\leq\frac{h}{2L}\Big(\exp(L(t_n-a))-1\Big)\cdot\max_{\xi\in[a,b]}|y''(\xi)|,$$

where y is the exact solution to the initial value problem.

Proof We apply Theorem 1 where

$$T_n=\frac{y(t_{n+1})-y(t_n)}{h}-f(t_n,y(t_n))=\frac{y''(t_n)}{2!}h+\frac{y'''(t_n)}{3!}h^2+\dots$$

$$=\frac{y''(\xi_n)}{2}h,\quad\text{where } \xi_n \text{ is between } t_{n+1} \text{ and } t_n$$

Then,

$$|T_n|\leq\frac{1}{2}h\max_{\xi\in[a,b]}|y''(\xi)|$$

By Theorem 1,

$$|e_n|\leq\frac{1}{L}\Big(\exp(L(t_n-a))-1\Big)\cdot\frac{1}{2}h\cdot\max_{\xi\in[a,b]}|y''(\xi)|$$

which finishes the proof.