Partial functional differential equations

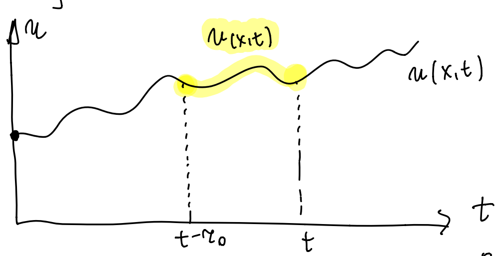
We consider the initial-boundary value problem

$$(1) \quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = a(t)\dfrac{\partial^2 u}{\partial x^2}(x,t) + b(t)\, f(\underbrace{u_{(x,t)}}_{\text{function}}) + g(x,t) \\ u(x,t) = \phi(x,t), \quad x \in [-L,L], \\ \qquad\qquad\qquad\quad t \in [-r_0, 0] \\ u(L,t) = \psi_+(t), \quad u(-L,t) = \psi_-(t), \quad t \in [0,T], \end{cases}$$

where $a: [0,T] \to \mathbb{R}_+$, $b: [0,T] \to \mathbb{R}$, $f: C([-r_0,0],\mathbb{R}) \to \mathbb{R}$
$g: [-L,L] \times [0,T]$ are continuous functions, $r_0 \geq 0$, $T > 0$, and
$u_{(x,t)} \in C([-r_0,0],\mathbb{R})$ is a functional argument defined by

$$u_{(x,t)}(r) = u(x, t+r), \quad r \in [-r_0, 0],$$

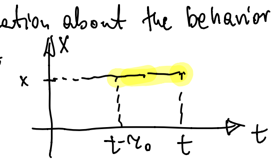
for any $(x,t) \in [-L,L] \times [0,T]$.



x - fixed

Example If we define $f(w) = \int_{-r_0}^{0} w(r)\,dr$, where $w \in C([-r_0,0],\mathbb{R})$,
then the partial functional differential equation in (1) is written in the form

$$(2) \quad \frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t)\int_{-r_0}^{0} \underbrace{u(x,t+r)\,dr}_{\substack{\text{information about the behavior of } u \text{ over} \\ \text{the set}}} + g(x,t)$$



__Example__ If we define $f(w) = w(-r_0)$, then the partial functional differential equation in (1) is written in the form

$$(3) \qquad \frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t)u(x,t-r_0) + g(x,t)$$

__Spatial discretization__

Finite difference scheme for (1):

$$(4) \quad \frac{d}{dt}u_i(t) = \frac{a(t)}{(\Delta x)^2}\Big(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\Big) + b(t)f\big((u_i)_t\big) + g(i\Delta x, t),$$

where $(u_i)_t \in C([-r_0,0],\mathbb{R})$, $(u_i)_t(\tau) = u_i(t+\tau)$, $\tau \in [-r_0,0]$.

Finite difference scheme for (2):

$$\frac{d}{dt}u_i(t) = \frac{a(t)}{(\Delta x)^2}\Big(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\Big) + b(t)\int_{-r_0}^{0} u_i(t+\tau)\,d\tau + g(i\Delta x, t)$$

Finite difference scheme for (3):

$$\frac{d}{dt}u_i(t) = \frac{a(t)}{(\Delta x)^2}\Big(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\Big) + b(t)u_i(t-r_0) + g(i\Delta x, t).$$

(1), (2), (3) are supplemented by the initial and boundary conditions

$u_i(t) = \phi(i\Delta x, t)$, $\quad t \in [-r_0, 0]$, $\quad i = -M+1, \dots, M-1$

$u_{-M}(t) = \psi_-(t)$, $\quad u_M(t) = \psi_+(t)$, $\quad t \in [0,T]$,

where $M \in \mathbb{N}$ is such that $M\Delta x = L$.