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Question 1

The gradient of the tangent line is the derivative of the equation of the line; this derivative is given below:

$$\frac{dy}{dx} = 6\sqrt{y} + 5x^3$$

Now substituting the values of $x = -1$ and $y = 4$ we shall obtain:

$$\begin{aligned}\frac{dy}{dx} &= 6\sqrt{4} + 5\left(-1\right)^3 \\ &= 7.\end{aligned}$$

Therefore the gradient of the tangent line is 7.

Question 2

(2a) The solution of the curve for the initial value problem can be obtained as follows;

$$\frac{dy}{dx} = \frac{-x}{y} \qquad\qquad y\left(4\right) = -3$$

We apply seperation of variables method to solve the O.D.E :

$$ydy = -xdx$$

Since all the variable have been seperated accordingly, we can now integrate.

$$\begin{aligned}\int ydy &= -\int xdx \\ \frac{y^2}{2} &= -\frac{x^2}{2} + c\end{aligned}$$

$$y^2 = -x^2 + c \tag{1}$$

Now applying the boundary conditions $y(4) = -3$.

$$\begin{aligned} c &= 4^2 + (-3)^2 \\ &= 25 \end{aligned}$$

Substituting this value for c in equation[1] above, we shall obtain:

$$y = \sqrt{25 - x^2} \qquad \forall x \in (-5, 5)$$

(2b) .

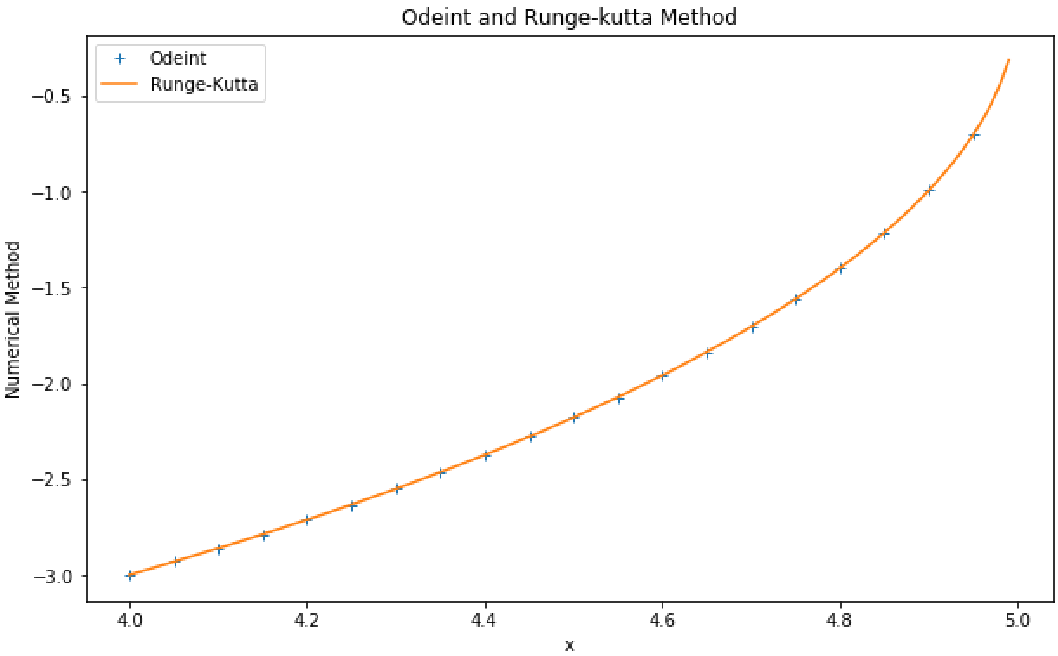

Figure 1: A graph of Runge-Kutta and Odeint Against x

The figure above shows the solution of the given differential equation using the Odeint and Runge-Kutta methods. The Runge-kutta solution is depicted by the orange curve while the Odeint solution is depicted by the little blue cross curve '+', we cannot draw any conclusion as to which method is more accurate since both curves are almost the same, in order to find the more accurate one, we have to plot the graph of their absolute errors.

(2c) .

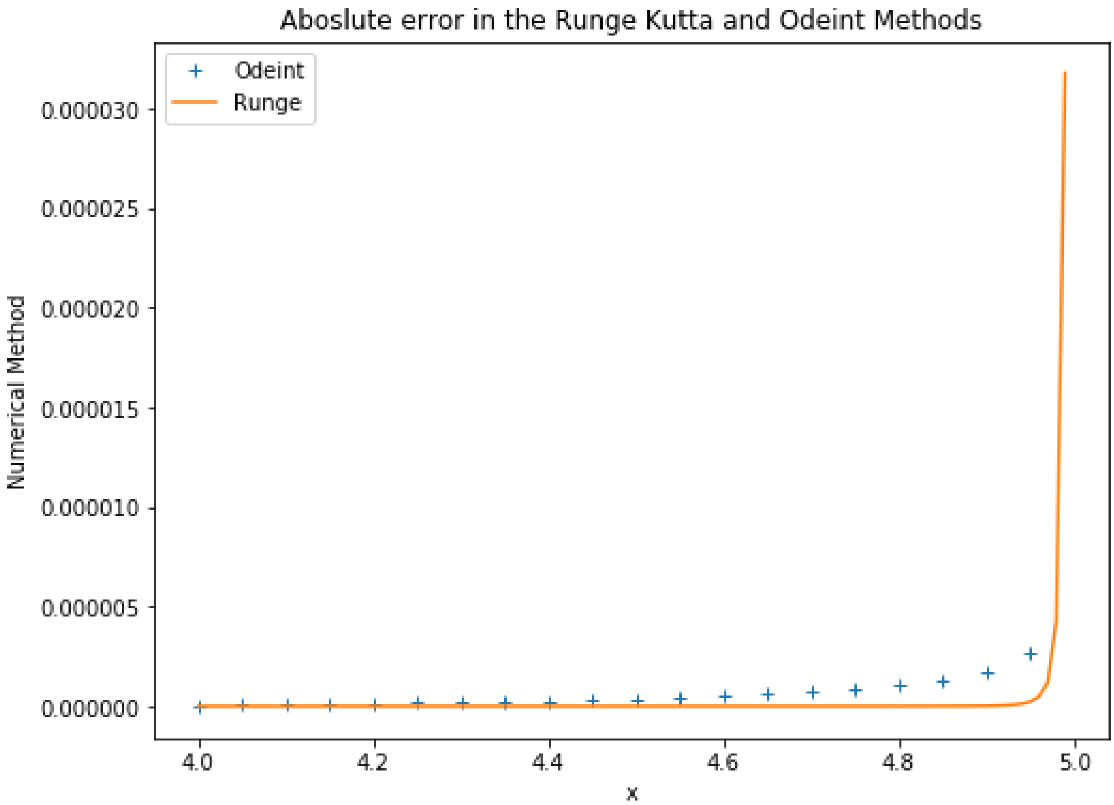

Figure 2: A graph of the Absolute error in the Runge-Kutta and Odeint Method againt x

The figure above shows the absolute error in the Runge-Kutta and Odeint Methods , for this paricular problem, the Runge-kutta method gives the closest value to what can be obtained from the exact solution, this can be deduced from the orange curve in the figure which is at zero between $4 \leq x < 5$, the Odeint curve is the blue little cross curve '+' on the graph which is slightly above the orange curve.

Question 3

(3a) $U = Uninfected \quad Number$
$I = Infected \quad Number$
$N = Population \quad Size$

$x = \frac{I}{N}$
$y = \frac{U}{N} \qquad\qquad x, y \in [0, 1]$

$$x + y = 1 \tag{2}$$
$$\frac{dx}{dt} = \beta xy \tag{3}$$

Substituting $y = 1 - x$ for y in equation[3],we shall obtain;

$$\frac{dx}{dt} = \beta x (1 - x) \tag{4}$$

We can now solve equation[4] by seperation of variables.

$$\frac{dx}{x(1-x)} = \beta dt \tag{5}$$

We shall decompose $\frac{1}{x(1-x)}$ into partial fractions and substitute it back into equation[5].

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

Now we shall substitute this result into equation[5], which then yields:

$$\int \frac{1}{x} dx + \int \frac{1}{1-x} dx = \int \beta dt$$
$$\ln|x| - \ln|1-x| = \beta t + c$$
$$\ln \left| \frac{x}{1-x} \right| = \beta t + c$$

$$\frac{x}{1-x} = Ae^{\beta t} \tag{6}$$

Now applying the initial value condition of $x(0) = x_0$, we shall obtain A as;

$$A = \frac{x_0}{1-x_0}$$

Substituting the value of A into equation[6] we shall obtain:

$$\frac{x}{1-x} = \frac{x_0}{1-x_0} e^{\beta t} \tag{7}$$

Simplyfying equation[7] further;

$$
\begin{aligned}
x\left(1-x_0\right) &= x_0\left(1-x\right)e^{\beta t} \\
x\left(1+x_0e^{\beta t}-x_0\right) &= x_0e^{\beta t} \\
x &= \frac{x_0e^{\beta t}}{1+x_0e^{\beta t}-x_0} \\
&= \frac{x_0}{\left(1+x_0e^{\beta t}-x_0\right)\times e^{-\beta t}} \\
&= \frac{x_0}{x_0+\left(1-x_0\right)e^{-\beta t}}
\end{aligned}
$$

(3c) $\lim_{x(t)\longrightarrow\infty}$ when $x_0 > 0$.

$$
\begin{aligned}
x &= \frac{x_0}{x_0+\left(1-x_0\right)e^{-\beta t}} \\
&= \frac{x_0}{x_0+\left(1-x_0\right)\times 0} \\
&= \frac{x_0}{x_0} \\
&= 1
\end{aligned}
$$

Since $x = \frac{I}{N} = 1$, this model predicts that as time increases the infection rate will increase and at some very large time, every member of the population will be infected by the disease.

(3d) The epidemic model is not feasible, it assumes that the population will remain constant throughout, thereby neglecting maternity, mortality, emmigration and immigration rates, also, it neglected the fact that some individuals will possess resistance due to their strong immune system and healthy life style.

Question 4

$$
\left(e^y \sin 2x\right)dx + \left(y-e^{2y}\right)\cos x dy = 0 \qquad\qquad y\left(0\right)=0
$$

From trigonometric indentity $\sin 2x = 2\sin x\cos x$.

$$
\cos x\left(2e^y\sin x\right)dx + \left(y-e^{2y}\right)\cos x dy
$$

We can divide through by $\cos x$ and obtain a simpler equation.

$$
\begin{aligned}
\left(2e^y\sin x\right)dx + \left(y-e^{2y}\right)dy = 0 \\
\left(2e^y\sin x\right)dx = \left(y-e^{2y}\right)dy
\end{aligned}
$$

Dividing through by e^y we shall obtain;

$$
\begin{aligned}
(2\sin x)\,dx &= -\left(ye^{-y} - e^{y}\right)dy \\
\int (2\sin x)\,dx &= -\int \left(ye^{-y} - e^{y}\right)dy
\end{aligned}
$$

$$
\begin{aligned}
-2\cos x + c &= ye^{-y} + e^{-y} + e^{y} && (8)\\
c &= 2\cos x + ye^{-y} + e^{-y} + e^{y} && (9)
\end{aligned}
$$

Applying the initial value conditions of $y\,(0) = 0$

$$
\begin{aligned}
c &= 2\cos 0 + ye^{0} + e^{0} + e^{0} \\
&= 4
\end{aligned}
$$

Substituting the value of $c = 4$ into equation [8], we shall obtain;

$$-2\cos x + 4 = ye^{-y} + e^{-y} + e^{y} \tag{10}$$

Question 5

$$y^{(5)} - 9y^{(4)} + 28y^{(3)} - 36y^{(2)} + 27y^{(1)} - 27y$$

We assume that $y = e^{mx}$ is a solution to the above equation aand then we re-write the equation as;

$$m^{5} - 9m^{4} + 28m^{3} - 36m^{2} + 27m^{1} - 27y$$

We can obtain the roots of the characteristic by obtaining the first zeros of the characteristic equation and subsequently we use polynomial division to obtain the other roots.

$$(m-3)\,(m-3)\,(m-3)\left(m^{2}+1\right)$$

$$(m-3)\,(m-3)\,(m-3)\left(m-\sqrt{-1}\right)\left(m+\sqrt{-1}\right)$$

We obtain a root multiplicity of three *i.e* $m_1 = m_2 = m_3 = 3$ and a complex root with its conjugate *i.e* $m_4 = -\sqrt{-1}, \quad m_5 = \sqrt{-1}$

The solution of the equation is given as;

$$y = A_1e^{3x} + A_2xe^{3x}A_3x^{2}e^{3x} + C_1\cos\sqrt{-1} + C_1 i\sin\sqrt{-1} + C_2\cos\sqrt{-1} - C_2 i\sin\sqrt{-1} \tag{11}$$

let $i = \sqrt{-1}$, we can then simplify equation(11) to obtain;

$$y = A_1e^{3x} + A_2xe^{3x}A_3x^{2}e^{3x} + C_1\cos ix + C_1 i\sin ix + C_2\cos ix - C_2 i\sin ix \tag{12}$$