

AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES
(AIMS RWANDA, KIGALI)

Name: Akor stanley
Course: ODE

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Question 1

- (i) $\frac{dx}{dt} = |x|^{\frac{1}{2}}$
Case 1: $-x^{\frac{1}{2}}$
Case 2: $x^{\frac{1}{2}}$

$$\frac{dx}{dt} = -x^{\frac{1}{2}}$$

Applying separation of variables techniques,

$$\begin{aligned}\frac{dx}{-x^{\frac{1}{2}}} &= dt \\ \int -x^{\frac{1}{2}} dx &= dt \\ \int -x^{\frac{1}{2}} dx &= t + c \\ x &= \sqrt{\frac{-t}{2} + c}\end{aligned}$$

Taking similar steps for case 2, we shall obtain the second solution as; $x = \sqrt{\frac{t}{2} + c}$

- (ii)

$$\begin{aligned}x \frac{dx}{dt} &= t \\ x dx &= t dt \\ \int x dx &= \int t dt \\ \frac{x^2}{2} &= \frac{t^2}{2} + c \\ x &= \pm \sqrt{t^2 + c}\end{aligned}$$

- (2) (i) If a function $f(z)$ and its partial derivatives are continuous in domain of z , i.e the function is defined, then the hypothesis of Picard-Lindelof's theorem are satisfied

$$\frac{dx}{dt} = |x|^{\frac{1}{2}}$$

Taking partial derivative of this function with respect to x yields;

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{2\sqrt{x}}$$

The result is not continuous for $x = 0$, hence the hypothesis of Picard-Lindelof's theorem are not satisfied

(ii)

$$\begin{aligned} x \frac{dx}{dt} &= t \\ x' &= \frac{t}{x} \end{aligned}$$

The function is not continuous at $x = 0$, hence the hypothesis of the Picard-Lindelof theorem are not satisfied.

Question 2

$$\frac{du}{dt} = Au \quad u(0) = u_0$$

(i)

$$A = \begin{pmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} -5 - \lambda & -4 & 2 \\ -2 & -2 - \lambda & 2 \\ 4 & 2 & 2 - \lambda \end{pmatrix}$$

$$(-5 - \lambda) \begin{bmatrix} -2 - \lambda & 2 \\ 2 & 2 - \lambda \end{bmatrix} + 4 \begin{bmatrix} -2 & -4 \\ 4 & 2 \end{bmatrix} + 2 \begin{bmatrix} -2 & -2 - \lambda \\ 4 & 2 \end{bmatrix}$$

$$((-5 - \lambda) [(-2 - \lambda)(-2 - \lambda) - 4] + 4[-2(2) - (4)(-4)] + 2[-2(2) - (4)(-2 - \lambda)])$$

$$(-5 - \lambda)(\lambda^2 - 8) + 4(-4 + 16) + 2(-4 + 8 + 4\lambda)$$

$$-\lambda^3 - 5\lambda^2 + 24\lambda = 0$$

$$(\lambda^2 + 5\lambda - 24) - \lambda = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = -8 \quad \lambda_3 = 3$$

The eigen vectors attributed to $\lambda_1 = 0$ can be obtained as follows;

$$\begin{pmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-5x - 4y + 2z = 0 \quad (1)$$

$$-2x - 2y + 2z = 0 \quad (2)$$

$$4x + 2y + 2z = 0 \quad (3)$$

Resolving equations(1),(2)and (3) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_1 = 0 \quad u_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

Similarly we can obtain the eigen vectors for the eigen value $\lambda_2 = 3$.

$$\begin{pmatrix} -5 - (3) & -4 & 2 \\ -2 & -2 - (3) & 2 \\ 4 & 2 & 2 - (3) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -8 & -4 & 2 \\ -2 & -5 & 2 \\ 4 & -1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-8x - 4y + 2z = 0 \quad (4)$$

$$-2x - 5y + 2z = 0 \quad (5)$$

$$4x + 2y + -z = 0 \quad (6)$$

Resolving equations(4),(5)and (6) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_2 = 3 \quad u_2 = \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix}$$

Similarly we can obtain the eigen vectors for the eigen value $\lambda_3 = -8$.

$$\begin{pmatrix} -5 - (-8) & -4 & 2 \\ -2 & -2 - (-8) & 2 \\ 4 & 2 & 2 - (-8) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -4 & 2 \\ -2 & 6 & 2 \\ 4 & 2 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x - 4y + 2z = 0 \quad (7)$$

$$-2x + 6y + 2z = 0 \quad (8)$$

$$4x + 2y + 10z = 0 \quad (9)$$

Resolving equations(4),(5)and (6) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_3 = -8 \quad u_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

From the given equation;

$$\begin{aligned} \frac{du}{dt} &= Au \\ \frac{du}{u} &= A dt \\ \int \frac{du}{u} &= \int A dt \\ \ln |u| &= At + c \\ u &= ce^{tA} \end{aligned}$$

Applying the initial value condition $u(0) = u_0$ we shall obtain;

$$\begin{aligned} u_0 &= ce^0 \\ c &= u_0 \end{aligned}$$

$$\begin{aligned} u_1(t) &= e^{\lambda_1 t} u_1 = e^0 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \\ u_2(t) &= e^{\lambda_2 t} u_1 = e^{3t} \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix} \\ u_3(t) &= e^{\lambda_3 t} u_1 = e^{-8t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \\ u(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} \end{aligned}$$

$$u(t) = c_1 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix} + c_3 e^{-8t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{At } t = 0 \quad u(0) = u_0$$

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & -2 \\ 3 & 6 & -1 \\ 1 & 16 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad (10)$$

$$-2c_1 + -c_2 + -2c_3 = 2 \quad (11)$$

$$3c_1 + 6c_2 + -c_3 = 1 \quad (12)$$

$$c_1 + 16c_2 + c_3 = 2 \quad (13)$$

Resolving equations (11),(12)and (13) simultaneously, we shall obtain the value of the constants as;

$$c_1 = \frac{-1}{4} \quad c_2 = \frac{2}{11} \quad c_3 = \frac{-29}{44}$$

The general solution is given as;

$$u(t) = \frac{-1}{4} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \frac{2}{11} e^{3t} \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix} + \frac{-29}{44} e^{-8t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

(ii)

$$A = \begin{pmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \\ 5 & 10 & 10 \end{pmatrix}$$

We want to determine the eigen values of the matrix A .

$$\begin{pmatrix} 4 - \lambda & -3 & 0 \\ 3 & 4 - \lambda & 0 \\ 5 & 10 & 10 - \lambda \end{pmatrix}$$

We want to obtain the determinant the of the matrix A .

$$(4 - \lambda) ((4 - \lambda) (10 - \lambda)) + 3 (3 (10 - \lambda)) = 0$$

$$-\lambda^3 + 18\lambda^2 - 105\lambda + 250 = 0$$

$$(\lambda_1 - 10) (\lambda_2 - 4 + 3i) (\lambda_3 - 4 - 3i)$$

$$\lambda_1 = 10 \quad \lambda_2 = 4 - 3i \quad \lambda_3 = 4 + 3i$$

Now we want to determine the eigen vectors associated with the eigen values;

$$\lambda_1 = 10$$

$$\begin{pmatrix} 4 - (10) & -3 & 0 \\ 3 & 4 - (10) & 0 \\ 5 & 10 & 10 - (10) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -3 & 0 \\ 3 & -6 & 0 \\ 5 & 10 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-6x - 3y + 0 = 0 \quad (14)$$

$$3x + 6y + 0 = 0 \quad (15)$$

$$5x + 10y + 0 = 0 \quad (16)$$

Resolving equations(14),(15)and (16) simultaneously, we shall obtain the associated eigen vectors.

$$\lambda_1 = 10 \quad u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Similarly for $\lambda_2 = 4 + 3i$

$$\begin{pmatrix} 4 - (4 + 3i) & -3 & 0 \\ 3 & 4 - (4 + 3i) & 0 \\ 5 & 10 & 10 - (4 + 3i) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By resolving the above matrix, we shall obtain;

$$\lambda_2 = 4 + 3i \quad u_1 = \begin{pmatrix} 12 - 9i \\ -9 + 12i \\ 25 \end{pmatrix}$$

The eigen vectors associated with the eigen value $\lambda_3 = 4 - 3i$ is the conjugate of the eigen vectors of $\lambda_2 = 4 + 3i$

$$\lambda_3 = 4 - 3i \quad u_1 = \begin{pmatrix} 12 + 9i \\ -9 - 12i \\ 25 \end{pmatrix}$$

$$u_1(t) = e^{\lambda_1 t} u_1 = e^{10t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_2(t) = e^{\lambda_2 t} u_1 = e^{4t} \left(\cos(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} - \sin(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} \right)$$

$$u_3(t) = e^{\lambda_3 t} u_1 = e^{4t} \left(\cos(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} + \sin(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} \right)$$

$$\begin{aligned}
u(t) &= c_1 e^{\lambda_1 t} u_1 = e^{10t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{4t} \left(\cos(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} - \sin(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} \right) \\
&+ c_3 e^{4t} \left(\cos(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} + \sin(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} \right) \\
\text{At } t = 0 \quad u(0) &= u_0
\end{aligned}$$

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -12 & -9 \\ 0 & -9 & 12 \\ 1 & 25 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad (17)$$

$$-12c_2 + -9c_3 = 2 \quad (18)$$

$$-9c_2 + 12c_3 = 1 \quad (19)$$

$$c_1 + 25c_2 = 2 \quad (20)$$

Resolving equations (18),(19)and (20) simultaneously, we shall obtain the value of the constants as;

$$c_1 = \frac{17}{3} \quad c_2 = \frac{-11}{75} \quad c_3 = \frac{-2}{75}$$

Our general solution now becomes;

$$\begin{aligned}
u(t) &= \frac{17}{3} e^{\lambda_1 t} u_1 = e^{10t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{-11}{75} e^{4t} \left(\cos(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} - \sin(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} \right) \\
&+ \frac{-2}{75} e^{4t} \left(\cos(3t) \begin{pmatrix} -9 \\ 12 \\ 0 \end{pmatrix} + \sin(3t) \begin{pmatrix} -12 \\ -9 \\ 25 \end{pmatrix} \right)
\end{aligned}$$

Question 3

(i)

$$\begin{aligned}
 \frac{du}{dt} &= Au + F(u, t) \quad \text{on } [a, b] \\
 e^{-At} \frac{du}{dt} &= e^{-At} (Au + F(u, t)) \\
 \frac{du}{dt} (ue^{At}) &= e^{-At} F(u, t) \\
 ue^{-At} \Big|_{t_n}^{t_{n+1}} &= \int_{t_n}^{t_{n+1}} e^{-At} F(u, t) \\
 u(t_{n+1})e^{At_{n+1}} - u(t_n)e^{At_n} &= \int_{t_n}^{t_{n+1}} e^{-At} F(u, t) \\
 u(t_{n+1})e^{At_{n+1}} &= u(t_n)e^{At_n} + \int_{t_n}^{t_{n+1}} e^{-At} F(u, t)
 \end{aligned}$$

Dividing through by $e^{A(t_{n+1})}$

$$\begin{aligned}
 u(t_{n+1}) &= u(t_n)e^{At_n}e^{-At_{n+1}} + \int_{t_n}^{t_{n+1}} e^{At_{n+1}}e^{-At} F(u, t) \\
 u(t_{n+1}) &= u(t_n)e^{A(-t_n+t_{n+1})} + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-t_n)} F(u, t)
 \end{aligned}$$

Setting $h = t_{n+1} - t_n$, $t = t_n + \tau$ and $\tau = t - t_n$ we shall obtain the desired solution.

$$u(t_{n+1}) = u(t_n)e^{A(-t_n+t_{n+1})} + \int_0^h e^{-(\tau-h)A} F(u(t_n + \tau), t_n + \tau) d\tau$$

$$(ii) \quad F(u_n, t_n) = F_n$$

$$\begin{aligned}
u(t_{n+1}) &= u(t_n)e^{A(-t_n+t_{n+1})} + \int_0^h e^{-(\tau-h)A} F(u(t_n+\tau), t_n+\tau) d\tau \\
&= e^{hA}u_n + \int_0^h e^{-(\tau-h)A} F(u(t_n+\tau), t_n+\tau) d\tau \\
&= u(t_n)e^{hA} + \int_0^h e^{-(\tau-h)A} F_n d\tau \\
&= u(t_n)e^{hA} - A^{-1}F_n e^{-(\tau-h)A} \Big|_0^h \\
&= u(t_n)e^{hA} - A^{-1}F_n (I - e^{hA}) \\
&= e^{hA}u(t_n) + A^{-1} (e^{hA} - I) F_n
\end{aligned}$$