Last time : $u^h(t) = \underbrace{\psi(t)}_{\psi \in H^1_E(a,b)} + \sum_{i=1}^{n-1} c_i \underbrace{\varphi_i(t)}_{\varphi_i \in H^1_0(a,b)}$ (the Galerkin approximations)

$$(*) \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & a_{22} & \cdots & a_{2n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}}_{=c} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}}_{=b}$$

A is symmetric

where $a_{ij} = \mathcal{A}(\varphi_i, \varphi_j) = \mathcal{A}(\varphi_j, \varphi_i)$

$b_i = \langle f, \varphi_i \rangle - \mathcal{A}(\psi, \varphi_i)$

$= A$ → symmetric matrix

We now verify whether A is positive definite, that is,

$$\forall v = (v_1, v_2, \ldots, v_{n-1})^T \in \mathbb{R}^{n-1} \qquad v^T A v > 0$$

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & a_{22} & \cdots & a_{2n-1} \\ \vdots & & & \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n-1} a_{1j} v_j \\ \sum_{j=1}^{n-1} a_{2j} v_j \\ \cdots \\ \sum_{j=1}^{n-1} a_{n-1j} v_j \end{bmatrix} =$$

$$= v_1 \sum_{j=1}^{n-1} a_{1j} v_j + v_2 \sum_{j=1}^{n-1} a_{2j} v_j + \ldots + v_{n-1} \sum_{j=1}^{n-1} a_{n-1j} v_j = \sum_{i=1}^{n-1} v_i \sum_{j=1}^{n-1} a_{ij} v_j =$$

$$= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{ij} v_j v_i = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \mathcal{A}(\varphi_i, \varphi_j) v_i v_j = \sum_{i=1}^{n-1} \mathcal{A}\Big(\varphi_i, \sum_{j=1}^{n-1} v_j \varphi_j\Big) v_i =$$

$$= \mathcal{A}\Big(\sum_{i=1}^{n-1} v_i \varphi_i, \sum_{j=1}^{n-1} v_j \varphi_j\Big) = \mathcal{A}\Big(\underbrace{\sum_{i=1}^{n-1} v_i \varphi_i}_{\in S^h_0}, \sum_{i=1}^{n-1} v_i \varphi_i\Big) \underset{\substack{\text{definition of the} \\ \text{functional } \mathcal{A}}}{=}$$

$$= \int_a^b \underbrace{p(t)}_{\geq c_0 > 0} \underbrace{\left[\frac{d}{dt} \sum_{i=1}^{n-1} v_i \varphi_i(t)\right]^2}_{\geq 0} + \underbrace{r(t)}_{\geq 0} \underbrace{\left[\sum_{i=1}^{n-1} v_i \varphi_i(t)\right]^2}_{\geq 0} dt > 0$$

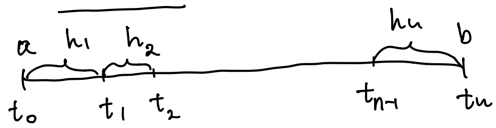
Since A is positive definite, it is invertable.

Once the solution $(c_1, c_2, \dots, c_{n-1}) \in \mathbb{R}^{n-1}$ of system $(*)$ is determined, it is used to get the approximations $u^h \in S^h_E$:

$$u^h(t) = \psi(t) + \sum_{i=1}^{n-1} c_i \varphi_i(t).$$

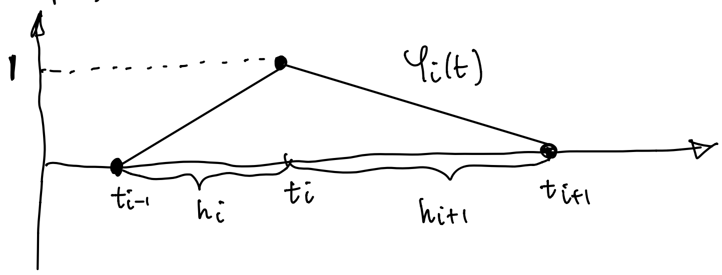
Note that u^h depends on the choice of $\varphi_1, \varphi_2, \dots, \varphi_{n-1} \in H^1_0(a,b)$ and $n \in \mathbb{N}$. The basis functions φ_i are constructed on the intervals $[t_i, t_{i+1}]$, $i = 0, 1, \dots, n-1$, called <u>elements</u>. The mesh points are

$a = t_0 < t_1 < \dots < t_n = b$



$h_i = t_i - t_{i-1}$, $t_i = t_{i-1} + h_i$

For example, we consider the hat functions $\varphi_1, \varphi_2, \dots, \varphi_{n-1}$



$i = 1, 2, \dots, n-1$

defined by

$$\varphi_i(t) = \begin{cases} 0 & \text{if } t \le t_{i-1} \\ (t - t_{i-1})/h_i & \text{if } t_{i-1} \le t \le t_i \\ (t_{i+1} - t)/h_{i+1} & \text{if } t_i \le t \le t_{i+1} \\ 0 & \text{if } t \ge t_{i+1} \end{cases}$$

Therefore,

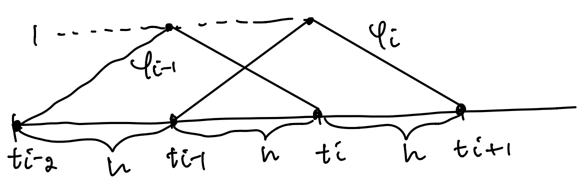
definition; piecewise constant functions; piecewise linear functions

$$a_{ij} = \mathcal{A}(\varphi_i, \varphi_j) = \int_a^b p(t)\varphi_i'(t)\varphi_j'(t) + r(t)\varphi_i(t)\varphi_j(t)\,dt$$

For equally spaced mesh points $t_i = t_0 + ih$, $i = 0, 1, \dots, n$, $h = \frac{b-a}{n}$, we get

$$\int_{t_{i-1}}^{t_i} p(t)\varphi_{i-1}'(t)\varphi_i'(t)\,dt = \int_{t_{i-1}}^{t_i} p(t)\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dt = -\frac{1}{h^2}\int_{t_{i-1}}^{t_i} p(t)\,dt \approx -\frac{1}{h^2}p(t_{i-1/2})\cdot h$$

$$= -\frac{p(t_{i-1/2})}{h}$$



only φ_{i-1} and φ_i are simultaneously nonzero over $[t_{i-1}, t_i]$

the integrand is zero outside $[t_{i-1}, t_i]$ and

$$\int_a^b p(t)\varphi_{i-1}'(t)\varphi_i'(t)\,dt = \int_{t_{i-1}}^{t_i} p(t)\varphi_{i-1}'(t)\varphi_i'(t)\,dt \approx -\frac{1}{h}p(t_{i-1/2})$$

Since only φ_{i-1} and φ_i are simultanously nonzero on $[t_{i-1}, t_i]$, we conclude that

$$\int_a^b p(t)\varphi_i'(t)\varphi_j'(t)\,dt = 0, \qquad \text{for } j \neq i, i-1, i+1, \text{ and}$$

$$\int_a^b r(t)\varphi_i(t)\varphi_j(t)\,dt = 0, \qquad \text{for } j \neq i, i-1, i+1.$$

Therefore, $a_{ij} = \int_a^b p(t)\varphi_i'(t)\varphi_j'(t)\,dt + \int_a^b r(t)\varphi_i(t)\varphi_j(t) = 0$, for $j \neq i, i-1, i+1$, and $i = 1, 2, \dots, n-1$.

So, the matrix A is tridiagonal.