Exact Representation of Any Polynomial Function by a Single Bézier Curve

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Abstract

A brief overview of Bézier curves and their properties is given. Then, the issue of accurately modeling a segment of a polynomial curve utilizing a single Bézier curve is considered. Through a recursive construction of the control points of a Bézier curve on the order of the curve, the existence of an exact representation of any interval of any polynomial curve is proven.

1 Introduction

Bézier curves are mathematical modeling tools with applications in computer-aided design, computer graphics, and HTML encoding. Consider Adobe PostScript, SWF, and SVG. These popular file formats all employ linear, quadratic, and cubic Bézier curves to display vector graphics (although linear Bézier curves are merely line segments). Moreover, other SVG elements, such as elliptical arcs, are represented by B-splines [Pro+01], which can be viewed as a series of connected Bézier curves. In this regard, the prevalence of Bézier curves in computer graphics entices further exploration of its mathematical basis.

Two areas that warrant further study are higher-order and non-approximating Bézier curves. Certainly, advancements in computational power and algorithms will greatly increase the viability of higher-order Bézier curves in vector graphics. For example, three algorithms for computing control points on Bézier curves proposed by Phien H.N. and Dejdumrong N. achieved more efficiency than the traditional de Casteljau's algorithm as the degree of the curve exceeds the cubic [PD00]. These higher-order curves circumvent the limitations associated with modifying quadratic and cubic Bézier curves when applications require large profiles or surfaces [Fit+14]. Furthermore, in related fields such as medical image processing, where highly accurate estimations are pertinent [SY16], higher-order curves could improve medical images. In the extreme case of non-approximating Bézier curves, the highest degree of accuracy can be achieved.

To examine the extent of the ability of Bézier curves to model different shapes accurately, we will compare it to a standard polynomial curve. In particular, we construct the control points of a Bézier curve equivalent to any polynomial curve in an attempt to synthesize mathematical and computer graphical standards.

2 Bernstein Polynomials and Bézier Curves

The Bernstein polynomial of order n for a function f, defined on the closed interval $[0, 1]$ is

$$B_n(x) = \sum_{v=0}^{n} f(\frac{v}{n}) \binom{n}{v} x^v (1-x)^{n-v} \tag{2.1}$$

where

$$b_{n,v}(x) = \binom{n}{v} x^v (1-x)^{n-v} \tag{2.2}$$

are the Bernstein basis polynomials as introduced by S. Bernstein (qtd. in Lorentz) [Lor53] to prove the Weierstrass Approximation Theorem.

The application of Bernstein polynomials to describing freeform shapes can be largely attributed to the work of Paul de Casteljau [Far93]. This innovative system, capable of manipulating and defining curves, relied on polynomial functions rather than the traditional harmonic functions because it more accurately described properties such as tangency, curvature, etc. As it was later discovered, these polynomial functions could be considered as sums of Bernstein's functions.

The resultant Bézier curve of order n, $P(t)$, is defined as

$$P(t) = \sum_{v=0}^{n} P_v b_{n,v}(t), \quad 0 \leq t \leq 1 \tag{2.3}$$

where $b_{n,v}(t)$ is defined in equation 2.2 and $P_0, P_1, ..., P_n$ are control points (coordinate pairs) that influence the shape of the resultant curve. Here, the Bernstein polynomials act as a blending function that blends the contributions of the different control points [Sal06].

3 Properties of Bézier Curves

A Bézier curve is a parametric curve that employs control points to create an approximating curve. This sets up the foundation for the first two important properties of Bézier curves:

1. Bézier curves are advantageous because they are not restricted by the surjectivity of an explicitly defined function. That is, Bézier curves can sketch curves that do not satisfy the vertical line test.

2. Bézier curves do not have to pass through every control point. As a result, they are more stable and convenient because individual control points can be moved to modify the curve.

Some other important properties related to the geometric interpretation of Bézier curves include:

3. Bézier curves always pass through their endpoints P_0 and P_n [Sal06]. This is easy to show as it is convention to let $0^0 = 1$,

$$P(0) = \sum_{v=0}^{n} P_v b_{n,v}(0) = \sum_{v=0}^{n} P_v \binom{n}{v} 0^v (1-0)^{n-v} = P_0 \tag{3.1}$$

$$P(1) = \sum_{v=0}^{n} P_v b_{n,v}(1) = \sum_{v=0}^{n} P_v \binom{n}{v} 1^v (1-1)^{n-v} = P_n \tag{3.2}$$

Hence, a degree n Bézier curve begins at P_0 and ends at P_n as t ranges from 0 to 1. (Note: Here $b_{n,v}(0)$ and $b_{n,v}(1)$ function as $\delta_{v,0}$ and $\delta_{v,n}$ respectively where $\delta_{i,j}$ is the Kronecker delta function which equals 1 when $i = j$ and 0 otherwise [Far93]).

4. The first derivative or tangent vector of a Bézier curve is as follows [Sal06]:

$$P'(t) = n \sum_{v=0}^{n-1} \Delta P_v b_{n-1,v}(t) \tag{3.3}$$

where

$$\Delta P_v = P_{v+1} - P_v \tag{3.4}$$

This implies that $\overline{P_0 P_1}$ is tangent to $P(t)$ at $t = 0$ and $\overline{P_{n-1} P_n}$ is tangent to $P(t)$ at $t = 1$ since

$$P'(0) = n(P_1 - P_0), P'(1) = n(P_n - P_{n-1}) \tag{3.5}$$

from Equation 3.3

5. A control point P_v exerts maximum influence on the Bézier curve at $t = \frac{v}{n}$. Consider

$$\begin{aligned} b'_{n,v}(t) &= \binom{n}{v} [vt^{v-1}(1-t)^{n-v} + t^v(n-v)(1-t)^{n-v-1}(-1)] \\ &= \binom{n}{v} t^{v-1}(1-t)^{n-v-1}(v(1-t) - (n-v)t) \\ &= \binom{n}{v} t^{v-1}(1-t)^{n-v-1}(v - nt) \end{aligned} \tag{3.6}$$

Since $b'_{n,v}(t) = 0$ when $t = \frac{v}{n}$ and $b_{n,v}(\frac{v}{n}) > b_{n,v}(0) = b_{n,v}(1) = 0$ when $0 < v < n$, $b_{n,v}(t)$ has a maximum at $t = \frac{v}{n}$. For $v = 0$ and $v = n$, it has been shown that $P(0)$ and $P(1)$ pass through P_0 and P_n respectively. Therefore, $P(\frac{v}{n})$ is most impacted by P_v.

4 Polynomials and Bézier Curves

As Bézier curves are an approximating curve, many curves cannot be exactly represented by a Bézier curve or even a piecewise Bézier curve. Consider circular arcs, which can be represented by rational Bézier curves but not polynomial Bézier curves. Although much research has gone into improving the approximation of such curves, the same cannot be said about curves that can be exactly represented by a polynomial Bézier curve [Rab16].

Intuition suggests that the graph of any segment of every nth degree polynomial can be exactly represented by a Bézier curve of order n. As a first step to proving this, we consider small cases of $n = 1, 2$.

For $n = 1$, on the interval $t \in [0, 1]$

$$\begin{aligned} P(t) &= \sum_{v=0}^{n} P_v b_{n,v}(t) \\ &= P_0 b_{1,0}(t) + P_1 b_{1,1}(t) \\ &= (1-t)P_0 + tP_1 \end{aligned} \tag{4.1}$$

Therefore, if $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, the coordinate parameterizations of $x(t)$ and $y(t)$ are

$$\begin{aligned} x(t) &= (1-t)x_0 + tx_1 = (x_1 - x_0)t + x_0 \\ y(t) &= (1-t)y_0 + ty_1 = (y_1 - y_0)t + y_0 \end{aligned} \tag{4.2}$$

Rewriting y in terms of x yields

$$y - y_0 = \frac{(y_1 - y_0)}{(x_1 - x_0)}(x - x_0), \quad x \in [x_0, x_1] \tag{4.3}$$

which is merely the point-slope form for a line with slope $\frac{y_1 - y_0}{x_1 - x_0}$ passing through (x_0, y_0) or the line segment connecting (x_0, y_0) and (x_1, y_1). Hence, any interval $x \in [m, n]$ of every degree 1 polynomial of the form $ax + b$ can be exactly represented by a Bézier curve $P(t) = (1-t)P_0 + tP_1,\ t \in [0, 1]$ where $P(0) = (m, am + b)$ and $P(1) = (n, an + b)$.

For $n = 2$, a construction for a second order Bézier curve for any interval $x \in [m, n]$ of a degree 2 polynomial of the form $ax^2 + bx + c$ can be identified geometrically.
By property 3 of Bézier curves,

$$\begin{aligned} P_0 &= (m, am^2 + bm + c) \\ P_2 &= (n, an^2 + bn + c) \end{aligned} \tag{4.4}$$

as they are the endpoints of a segment of the curve of the polynomial $f(x) = ax^2 + bx + c$.

By property 4 of Bézier curves, $\overline{P_0P_1}$ must be tangent to $f(x)$ at $x = m$ and $\overline{P_1P_2}$ must be tangent to $f(x)$ at $x = n$. Therefore, P_1 must be the intersection of the tangents to $f(x)$ at P_0 and P_2. Consider the equations for these 2 tangent lines

$$\begin{aligned} y - (am^2 + bm + c) &= f'(m)(x - m) \\ y - (am^2 + bm + c) &= (2am + b)(x - m) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} y - (an^2 + bn + c) &= f'(n)(x - n) \\ y - (an^2 + bn + c) &= (2an + b)(x - n) \end{aligned} \tag{4.6}$$

Therefore, the intersection of these two lines $(\frac{m+n}{2}, amn + \frac{b(m+n)}{2} + c)$ is P_1. This geometric construction can be verified algebraically. For a second order Bézier curve with $t \in [0, 1]$,

$$\begin{aligned} P(t) &= \sum_{v=0}^{n} P_v b_{n,v}(t) \\ &= P_0 b_{2,0}(t) + P_1 b_{2,1}(t) + P_2 b_{2,2}(t) \\ &= (1 - t)^2 P_0 + 2t(1 - t)P_1 + t^2 P_2 \end{aligned} \tag{4.7}$$

Hence, for P_0,P_1, and P_2 as calculated as above

$$\begin{aligned} x(t) &= (1 - t)^2(m) + 2t(1 - t)(\frac{m + n}{2}) + t^2(n) \\ &= (n - m)t + m \\ y(t) &= (1 - t)^2(am^2 + bm + c) + 2t(1 - t)(amn + \frac{b(m + n)}{2} + c) \\ &\quad + t^2(an^2 + bn + c) \\ &= a(m^2 + 2mn + n^2)t^2 + (2amn - 2am^2 + 2bn - 2bm)t \\ &\quad + am^2 + bm + c \end{aligned} \tag{4.8}$$

Rewriting y in terms of x yields

$$y = ax^2 + bx + c, \quad x \in [m, n] \tag{4.9}$$

which verifies the construction of a second order Bézier curve $P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$ where $P_0 = (m, am^2 + bm + c)$, $P_1 = (\frac{m+n}{2}, amn + \frac{b(m+n)}{2} + c)$, and $P_2 = (n, an^2 + bn + c)$ that exactly represents any segment $x \in [m, n]$ of every degree 2 polynomial of the form $ax^2 + bx + c$.

The simultaneous considerations of the algebraic and geometric properties of Bézier curves in the base cases establishes the foundation for proving a generalization to all n. We claim that an order n Bézier curve with n control points of the form $P_i = (x_i, y_i)$ where $x_i = \frac{(l-k)i}{n} + k$ and i ranges from 0 to n can be constructed for any interval $x \in [k, l]$ of every degree n polynomial. In order to show the existence of such a curve, we will employ the following lemmas.

Lemma 1: *Given a polynomial $f(x)$ that can be represented by a Bézier curve of order n with control points $P_0 = (x_0, a_0)$, $P_1 = (x_1, a_1)$, ... $P_n = (x_n, a_n)$ and a polynomial $g(x)$ that can be represented by a Bézier curve of order n with control points $Q_0 = (x_0, b_0)$, $Q_1 = (x_1, b_1)$, ... $Q_n = (x_n, b_n)$. If $x_i = \frac{(l-k)i}{n} + k$ for $i \in \{0, 1, ...n\}$, the polynomial $(f + g)(x)$ can be represented by an order n Bézier curve with control points $R_0 = (x_0, a_0 + b_0)$, $R_1 = (x_1, a_1 + b_1)$, ... $R_n = (x_n, a_n + b_n)$.*

Proof: The coordinate parameterizations for $f(x)$ can be written as follows:

$$\begin{aligned}
x_1(t) &= \sum_{i=0}^{n} x_i b_{n,i}(t) \\
&= \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} (\frac{(l-k)i}{n} + k) \\
&= (l-k) \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} (\frac{i}{n}) + k \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \\
&= (l-k) \sum_{i=0}^{n} \binom{n-1}{i-1} t^i (1-t)^{n-i} + k(t + (1-t))^n \\
&= (l-k) \sum_{i=1}^{n} \binom{n-1}{i-1} t^i (1-t)^{n-i} + k \\
&= (l-k)t \sum_{i=1}^{n} \binom{n-1}{i-1} t^{i-1} (1-t)^{(n-1)-(i-1)} + k \text{ (Note:} \binom{n-1}{-1} = 0)
\end{aligned}$$

if we let $j = i - 1$,

$$\begin{aligned}
x_1(t) &= (l-k)t \sum_{j=0}^{n-1} \binom{n-1}{j} t^j (1-t)^{n-1-j} + k \\
&= ((l-k)t)(t + (1-t))^{n-1} + k \\
&= (l-k)t + k \\
y_1(t) &= \sum_{i=0}^{n} a_i b_{n,i}(t)
\end{aligned} \tag{4.10}$$

Appropriately, the coordinate parameterizations for $g(x)$ are:

$$\begin{aligned} x_2(t) &= \sum_{i=0}^{n} x_i b_{n,i}(t) \\ &= (l-k)t + k \\ y_2(t) &= \sum_{i=0}^{n} b_i b_{n,i}(t) \end{aligned} \tag{4.11}$$

As $x_1(t)$ and $x_2(t)$ are both linear, and therefore monotonic, both functions have an inverse. This allows us to substitute $t = \frac{1}{l-k}x - \frac{k}{l-k}$ and rewrite $f(x)$ and $g(x)$ as

$$\begin{aligned} f(x) &= y_1(\frac{1}{l-k}x - \frac{k}{l-k}) \\ &= \sum_{i=0}^{n} a_i b_{n,i}(\frac{1}{l-k}x - \frac{k}{l-k}) \\ g(x) &= y_2(\frac{1}{l-k}x - \frac{k}{l-k}) \\ &= \sum_{i=0}^{n} b_i b_{n,i}(\frac{1}{l-k}x - \frac{k}{l-k}) \end{aligned} \tag{4.12}$$

Accordingly,

$$(f+g)(x) = \sum_{i=0}^{n} (a_i + b_i) b_{n,i}(\frac{1}{l-k}x - \frac{k}{l-k}) \tag{4.13}$$

It follows from Equations 4.10 and 4.12 that the coordinate parameterizations for $(f+g)(x)$ can be written as

$$\begin{aligned} x_3(t) &= (l-k)t + k \\ &= \sum_{i=0}^{n} x_i b_{n,i}(t) \\ y_3(t) &= \sum_{i=0}^{n} (a_i + b_i) b_{n,i}(t) \end{aligned} \tag{4.14}$$

As desired, this describes an order n Bézier curve with control points $R_0 = (x_0, a_0 + b_0)$, $R_1 = (x_1, a_1 + b_1)$, ... $R_n = (x_n, a_n + b_n)$.

Lemma 2: *A Bézier curve $P(t)$ with n control points $P_i = (x_{P_i}, y_{P_i})$ where $x_{P_i} = \frac{(l-k)i}{n} + k$, $i \in \{0, 1, ..., n\}$ has an equivalent Bézier curve $Q(t)$ with $n+1$ control points $Q_j = (x_{Q_j}, y_{Q_j})$ such that $x_{Q_j} = \frac{(l-k)j}{n+1} + k$, $j \in \{0, 1, ..., n, n+1\}$ and Q_j is on $\overline{P_{j-1}P_j}$ of the order n Bézier curve.*

Proof: If $P(t)$ is defined by

$$\begin{cases} x_P(t) = \sum\limits_{i=0}^{n} x_{P_i} b_{n,i}(t) \\ y_P(t) = \sum\limits_{i=0}^{n} y_{P_i} b_{n,i}(t) \end{cases} \quad t \in [0,1]$$

and $Q(t)$ is defined by

$$\begin{cases} x_Q(t) = \sum\limits_{j=0}^{n+1} x_{Q_j} b_{n+1,j}(t) \\ y_Q(t) = \sum\limits_{j=0}^{n+1} y_{Q_j} b_{n+1,j}(t) \end{cases} \quad t \in [0,1]$$

From Equation 4.10, the x-coordinate parameterizations for these two Bézier curves are both

$$x_P(t) = x_Q(t) = (l-k)t + k \tag{4.15}$$

Hence, it suffices to prove that $y_P(t) = y_Q(t)$.
Since P and Q are equally spaced on the interval $[k,l]$ and Q_j lies on $\overline{P_{j-1}P_j}$ for $j \in \{1, 2, ..., n, n+1\}$,

$$\begin{aligned} y_{Q_j} &= \frac{j}{n+1} y_{P_{j-1}} + \frac{(n+1-j)}{n+1} y_{P_j} \\ y_{Q_0} &= y_{P_0} \end{aligned} \tag{4.16}$$

We can then rewrite $y_Q(t)$ as,

$$\begin{aligned} y_Q(t) &= \sum_{j=0}^{n+1} y_{Q_j} b_{n+1,j}(t) \\ &= y_{Q_0} b_{n+1,0}(t) + \sum_{j=1}^{n+1} y_{Q_j} b_{n+1,j}(t) \\ &= y_{P_0} b_{n+1,0}(t) + \sum_{j=1}^{n+1} \left(\frac{j}{n+1} y_{P_{j-1}} b_{n+1,j}(t) + \frac{n+1-j}{n+1} y_{P_j} b_{n+1,j}(t) \right) \\ &= \sum_{j=0}^{n} \left(\frac{j+1}{n+1} y_{P_j} b_{n+1,j+1}(t) + \frac{n+1-j}{n+1} y_{P_j} b_{n+1,j}(t) \right) \end{aligned} \tag{4.17}$$

since $\frac{n+1-j}{n+1} y_{P_j} b_{n+1,j}(t) = 0$ at $j = n+1$.

Therefore, to prove $y_Q(t)$ and $y_P(t)$ equivalent, it suffices to show that the coefficients of y_{P_j} are equivalent for $j \in \{0, 1, ..., n\}$.

$$\begin{aligned}
&\frac{j+1}{n+1} b_{n+1,j+1}(t) + \frac{n+1-j}{n+1} b_{n+1,j}(t) \\
&= \frac{j+1}{n+1} \binom{n+1}{j+1} t^{j+1}(1-t)^{n-j} + \frac{n+1-j}{n+1} \binom{n+1}{j} t^j (1-t)^{n+1-j} \\
&= \binom{n}{j} t^{j+1}(1-t)^{n-j} + \binom{n}{j} t^j (1-t)^{n+1-j} \\
&= \binom{n}{j} t^j (1-t)^{n-j} \\
&= b_{n,i}(t)
\end{aligned} \tag{4.18}$$

As expected, $y_Q(t) = y_P(t)$ and $x_Q(t) = x_P(t)$. Hence, $Q(t)$ is an order $n+1$ Bézier curve that is equivalent to $P(t)$, an order n Bézier curve.

Lemma 3: *Any segment $x \in [k, l]$ of a degree n polynomial of the form $f(x) = ax^n$ can be represented by an order n Bézier curve*

Proof: We claim that the interval $[k, l]$ of $f(x) = ax^n$ can be represented by a Bézier curve $P(t)$ with n control points $P_0 = (x_0, y_0), P_1 = (x_1, y_1), ..., P_n = (x_n, y_n)$ where $x_i = \frac{(l-k)i}{n} + k$ and $y_i = ak^{n-i}l^i$ for $i \in \{0, 1, ...n\}$.
As $P(t)$ is defined by

$$\begin{cases} x_P(t) = \sum_{i=0}^{n} x_i b_{n,i}(t) \\ y_P(t) = \sum_{i=0}^{n} y_i b_{n,i}(t) \end{cases} \quad t \in [0, 1]$$

We have,

$$x_P(t) = (l-k)t + k \tag{4.19}$$

from Equation 4.10 and

$$\begin{aligned}
y_P(t) &= \sum_{i=0}^{n} ak^{n-i}l^i b_{n,i}(t) \\
&= \sum_{i=0}^{n} ak^{n-i}l^i \binom{n}{i} t^i (1-t)^{n-i} \\
&= a \sum_{i=0}^{n} \binom{n}{i} (lt)^i (k(1-t))^{n-i} \\
&= a(lt + (k(1-t)))^n \\
&= a((l-k)t + k)^n
\end{aligned} \tag{4.20}$$

by the binomial theorem.

Therefore, rewriting y as a function of x yields $f(x) = y = ax^n$ as expected.

Theorem 1: *Any segment of any nth degree polynomial curve can be exactly represented by a Bézier curve of order n.*

Proof: We proceed by induction.

Base cases ($n = 1$ and $n = 2$): Shown by Equation 4.3 and Equation 4.9.

Inductive step: Assume that there exists a Bézier curve of order $n - 1$ that can exactly represent any segment of any $n - 1$ degree polynomial curve. It is enough to prove that there exists a Bézier curve of order n that can exactly represent any segment of any n degree polynomial curve.

If there exists a Bézier curve $P(t)$ of order $n-1$ that exactly represents any segment $x \in [k, l]$ of a function $f(x) = a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$, from Lemma 2, there exists a Bézier curve $Q(t)$ of order n that exactly represents any segment $[k, l]$ of $f(x)$.

From Lemma 3, any segment $x \in [k, l]$ of a function $g(x) = a_0x^n$ can be represented by an order n Bézier curve $R(t)$.

As every degree n polynomial can be written in the form $h(x) = a_0x^n + (a_1x^{n-1} + a_2x^{n-2} + ...a_{n-1}x + a_n)$. From Lemma 1, since $h(x) = (f + g)(x)$ and $f(x)$ and $g(x)$ can be represented by an order n Bézier with control points that are evenly spaced horizontally between $x \in [k, l]$, $h(x)$ can be represented by an order n Bézier curve with control points that are evenly spaced horizontally between $x \in [k, l]$.

5 Conclusion

We conclude by defining a recursive method for constructing the $n + 1$ control points for an nth degree polynomial.

$$\begin{aligned}
&P_{1,0} = (k, a_1k + a_2)\\
&P_{1,1} = (l, a_1l + a_2)\\
&P_{n+1,i} = (\frac{(l-k)i}{n} + k, \frac{i}{n+1}y_{P_{n,i-1}} + \frac{n+1-i}{n}y_{P_{n,i}}), i \in \{0, 1, ...n + 1\}
\end{aligned} \tag{5.1}$$

where $P_{n,i}$ is the ith control point for the Bézier curve representation of an nth degree polynomial and $y_{P_{n,i}}$ is the y-coordinate of $P_{n,i}$. Observe that although $P_{n,i-1}$ is undefined for $i = 0$ and $P_{n,i}$ is undefined for $i = n+1$, $\frac{i}{n+1}$ and $\frac{n+1-i}{n}$ evaluate to 0 at these two points.

Utilizing our recursive definition, we can construct the control points for any

$x \in [k, l]$ of any third degree polynomial $f(x) = a_1x^3 + a_2x^2 + a_3x + a_4$ as

$$
\begin{aligned}
P_{3,0} &= (k, a_1k^3 + a_2k^2 + a_3k + a_4) \\
P_{3,1} &= (\frac{2}{3}k + \frac{1}{3}l, a_1k^2l + a_2(\frac{1}{3}k^2 + \frac{2}{3}kl) + a_3(\frac{2}{3}k + \frac{1}{3}l) + a_4) \\
P_{3,2} &= (\frac{1}{3}k + \frac{2}{3}l, a_1kl^2 + a_2(\frac{2}{3}kl + \frac{1}{3}l^2) + a_3(\frac{1}{3}k + \frac{2}{3}l) + a_4) \\
P_{3,3} &= (l, a_1l^3 + a_2l^2 + a_3l + a_4)
\end{aligned}
\tag{5.2}
$$

Similarly, the control points for any $x \in [k, l]$ of any fourth degree polynomial $f(x) = a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$ are

$$
\begin{aligned}
P_{4,0} =&(k, a_1k^4 + a_2k^3 + a_3k^2 + a_4k + a_5) \\
P_{4,1} =&(\frac{3}{4}k + \frac{1}{4}l, a_1k^3l + a_2(\frac{1}{4}k^3 + \frac{3}{4}k^2l) + a_3(\frac{1}{2}k^2 + \frac{1}{2}kl) \\
&+ a_4(\frac{3}{4}k + \frac{1}{4}l) + a_5) \\
P_{4,2} =&(\frac{1}{2}k + \frac{1}{2}l, a_1k^2l^2 + a_2(\frac{1}{2}k^2l + \frac{1}{2}kl^2) + a_3(\frac{1}{6}k^2 + \frac{2}{3}kl + \frac{1}{6}l^2) \\
&+ a_4(\frac{1}{2}k + \frac{1}{2}l) + a_5) \\
P_{4,3} =&(\frac{1}{4}k + \frac{3}{4}l, a_1kl^3 + a_2(\frac{3}{4}kl^2 + \frac{1}{4}l^3) + a_3(\frac{1}{2}kl + \frac{1}{2}l^2) \\
&+ a_4(\frac{1}{4}k + \frac{3}{4}l) + a_5) \\
P_{4,4} =&(l, a_1l^4 + a_2l^3 + a_3l^2 + a_4l + a_5)
\end{aligned}
\tag{5.3}
$$

These all-encompassing and generalized formulas for converting a segment of a polynomial curve to a set of control points may have potential ramifications in computer memory storage and vector graphics. However, a recursive formula may hinder the construction of control points for higher-order curves. Hence, the development of an explicit formula in terms of the degree of the polynomial curve could be an avenue for further research.

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