Athabasca University

MATH 270

Assignment 4

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October 27, 2020

1. a) We can substitute our given functions into each axiom and check for contradictions. We are given $(\vec{f}+\vec{g}) = f(x)+g(x)$ and $f(x) = \vec{f}, g(x) = \vec{g}$. As a result, our first and second axiom, if $u+v = u+w, v = w$, holds. Next, we need to prove that there is a unique zero vector.

$$(f+0)(x) = 0, f(x)+0 = f(x)$$

As a result, our zero vector is 0. Next, we find the additive inverse.

$$\vec{f} = f(x), f(x)+g(x) = 0, g(x) = -f(x)$$

$-f(x)$ is the additive inverse of $f(x)$. Finally, we need to prove that the constant multiplication.

$$k(f+g)(x) = k(f(x)+g(x)) = c(f(x)+g(x)) = (cf+cg)x$$

b) For the first axiom, we have vector $\vec{w}$

$$(\vec{u}+\vec{v})+\vec{w} = (u_1, u_2, \ldots u_n) + (v_1, v_2, \ldots v_n) + (w_1, w_2, \ldots w_n)$$

$$= (u_1+v_1+w_1, u_2+v_2+w_2, \ldots u_n+v_n+w_n)$$
$$= \vec{u}+(\vec{v}+\vec{w})$$

Next, we need to find the zero vector, which is $\vec{0}$.

$$\vec{u}+\vec{0} = (u_1, u_2, \ldots u_n) + (0, 0, \ldots 0)$$

$$= (u_1+0, u_2+0, \ldots u_n+0)$$
$$= (u_1, u_2, \ldots u_n) = \vec{u}$$

Finally, we must find the additive inverse, $-\vec{u}$.

$$\vec{u}+(-\vec{u}) = (u_1, u_2, \ldots u_n) + (-1)(u_1, u_2, \ldots u_n)$$

$$= (u_1-u_1, u_2-u_2 \ldots u_n-u_n)$$
$$= (0, 0 \ldots 0) = \vec{0}$$

2. a) i) A linear combination is defined by this equation, for vectors w_1, w_2, w_3 and polynomials p_1, p_2, p_3.

$$p = w_1p_1 + w_2p_2 + w_3p_3$$

As such, we can begin to form this systems of linear equations.

$$-9-7x-15x^2 = w_1(2+x+4x^2) + w_2(1-x+3x^2) + w_3(3+2x+5x^2)$$

$$= (2w_1+w_2+3w_3) + x(w_1-w_2+2w_3) + w^2(4w_1+3w_2+5w_3)$$

Separating into a system of linear equations, we have

$$2w_1+w_2+3w_3 = -9$$

$$w_1-w_2+2w_3 = -7$$
$$4w_1+3w_2+5w_3 = -15$$

We can now put this into an augmented matrix to solve for w

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 5 & -15 \\ 0 & -\frac{7}{4} & \frac{3}{4} & -\frac{13}{4} \\ 0 & 0 & \frac{2}{7} & -\frac{4}{7} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore, we have coefficients $(w_1, w_2, w_3) = (-2, 1, -2)$. The entire equation becomes $-9 - 7x - 15x^2 = -2(2 + x + 4x^2) + (1 - x + 3x^2) - 2(3x + 2x + 5x^2)$

ii) We can follow similar steps to the above question.

$$6 + 11x + 6x^2 = w_1(2 + x + 4x^2) + w_2(1 - x + 3x^2) + w_3(3 + 2x + 5x^2)$$

$$= (2w_1 + w_2 + 3w_3) + x(w_1 - w_2 + 2w_3) + w^2(4w_1 + 3w_2 + 5w_3)$$

Separating into a system of linear equations, we have

$$2w_1 + w_2 + 3w_3 = 6$$

$$w_1 - w_2 + 2w_3 = 11$$

$$4w_1 + 3w_2 + 5w_3 = 6$$

Then rewriting as an augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 21 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 1 & -1 & -6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, we have $(w_1, w_2, w_3) = (4, -5, 1)$, and our entire equation is

$$6 + 11x + 6x^2 = 4(2 + x + 4x^2) - 5(1 - x + 3x^2) + 1(3x + 2x + 5x^2)$$

iii) We can skip straight to the augmented matrix, putting in our coefficients and then using gaussian-jordan elimination.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & -1 \quad 0 & \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 21 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, we have $(w_1, w_2, w_3) = (0, 0, 0)$ and equation $0 = 0$

iv) Following similar steps as above, we can begin by substituting our coefficients into the augmented matrix and solving.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 7 \\ 1 & -1 & 2 & 8 \\ 4 & 3 & 5 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{7}{2} \\ 0 & -\frac{3}{2} & \frac{1}{2} & \frac{9}{2} \\ 0 & 1 & -1 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 5 \\ 0 & 1 & -\frac{1}{3} & -3 \\ - & - & -\frac{2}{3} \quad -2 & \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore, we have $(w_1, w_2, w_3) = (0, -2, 3)$, and our entire equation is

$$7 + 8x + 9^2 = 0(2 + x + 4x^2) - 2(1 - x + 3x^2) + 3(3x + 2x + 5x^2)$$

b) A vector in span $\{v_1, v_2, v_3\}$ means that there is some linear combination of w_1, w_2, w_3 in which

$$p = w_1(2, 1, 0, 3) + w_2(3, -1, 5, 2) + w_3(-1, 0, 2, 1)$$

We can rewrite this using parametric equations to form a system. Our first value p is $2, 3, -7, 3$. We can form a linear system

$$2w_1 + 3w_2 - w_3 = 2$$

$$w_1 - w_2 = 3$$

$$5w_2 + 2w_3 = -7$$

$$3w_1 + 2w_2 + w_3 = 3$$

First rearrange the second equation to $w_1 = 3 + w_2$ Adding the first equation and last equation, we have

$$5w_1 + 5w_2 = 5$$

Then, substituting this in, $5(3 + w_2) + 5w_2 = 5$, $w_2 = -1$. Substituting this back into $w_1 = 3 + w_2$, we have $w_1 = 2$. Finally, we can substitute this into the third equation

$$5(-1) + 2(w_3) = -7$$

$$w_3 = -1$$

Therefore, our first vector, (2, 3, -7, 3), is in span $\{v_1, v_2, v_3\}$ with linear combination with coefficients (2, -1, -1).

Our next vector is (0,0,0,0). We can immediately see that this is a linear combination, since

$$(0, 0, 0, 0) = 0(2, 1, 0, 3) + 0(3, -1, 5, 2) + 0(-1, 0, 2, 1)$$

As such, (0,0,0,0) is in span $\{v_1, v_2, v_3\}$ with linear combination of coefficients (0,0,0).

Our next vector is (1,1,1,1). Again, we can form a linear system with the same coefficients for w but updated constants.

$$2w_1 + 3w_2 - w_3 = 1$$

$$w_1 - w_2 = 1$$

$$5w_2 + 2w_3 = 1$$

$$3w_1 + 2w_2 + w_3 = 1$$

First rearrange the second equation to $w_1 = 1 + w_2$ Adding the first equation and last equation, we have

$$5w_1 + 5w_2 = 2$$

Then, substituting this in, $5(3 + w_2) + 5w_2 = 2$, $w_2 = -\frac{1}{15}$. Substituting this back into $w_1 = 1 + w_2$, we have $w_1 = \frac{14}{15}$. Finally, we can substitute this into the third equation

$$5(-\frac{1}{15}) + 2(w_3) = 1$$

$$w_3 = \frac{2}{3}$$

As such, (1,1,1,1) is in span $\{v_1, v_2, v_3\}$ with linear combination of coefficients $\left(\frac{14}{15}, -\frac{1}{15}, \frac{2}{3}\right)$.

Our final vector is $-4, 6, -13, 4$. Once more, we can form the system of equations after updating the constants.

$$2w_1 + 3w_2 - w_3 = -4$$

$$w_1 - w_2 = 6$$

$$5w_2 + 2w_3 = -13$$

$$3w_1 + 2w_2 + w_3 = 4$$

First rearrange the second equation to $w_1 = 6 + w_2$ Adding the first equation and last equation, we have

$$5w_1 + 5w_2 = -17$$

Then, substituting this in, $5(3 + w_2) + 5w_2 = -17$, $w_2 = -3$. Substituting this back into $w_1 = 6 + w_2$, we have $w_1 = 3$. Finally, we can substitute this into the third equation

$$5(-3) + 2(w_3) = -13$$

$$w_3 = 2$$

As such, (-4, 6, -13, 4) is in span $\{v_1, v_2, v_3\}$ with linear combination of coefficients $(3, -3, 1)$.

3. a) i) Vectors are linearly dependent if and only if there is only one trivial solution, $w_1v_1 + w_2v_2 + w_3v_3 = \vec{0}$.
Similarly to the last question, we can find values w_1, w_2, w_3 to fulfill this. First, we form an equation.

$$(0, 0, 0, 0) = w_1(1, 2, 3, 4) + w_2(0, 1, 0, -1) + w_3(1, 3, 3, 3)$$

Splitting into a system of equations, we have

$$w_1 + w_3 = 0$$

$$2w_1 + w_2 + 3w_3 = 0$$

$$3w_1 + 3w_3 = 0$$

$$4w_1 - w_2 + 3w_3 = 0$$

To solve this, we can rearrange the first equation to find $w_1 = -w_3$. Substituting this into the last equation, we have

$$w_3 - w_2 = 0$$

$$w_3 = w_2$$

Therefore, the vectors are linearly dependent with coefficients $(t, t, -t)$

ii) Beginning with $\vec{v}_1$, we have

$$(1, 2, 3, 4) = w_1(0, 1, 0, -1) + w_2(1, 3, 3, 3)$$

Forming systems, we have

$$w_2 = 1$$

$$w_1 + 3w_2 = 2$$

$$3w_2 = 3$$

$$-w_1 + 3w_2 = 4$$

We have $w_2 = 1$. Substituting into the second equation, we have $w_1 = -1$. As a result, $\vec{v}_1$ is a linear combination of $\vec{v}_2, \vec{v}_3$ with a coefficient of (-1, 1).
Next, we can find $\vec{v}_2$ as a linear combination of the others.

$$(0, 1, 0, -1) = w_1(1, 2, 3, 4) + w_2(1, 3, 3, 3)$$

Splitting into systems, we have

$$w_1 + w_2 = 0$$

$$2w_1 + 3w_2 = 1$$

$$3w_1 + 3w_2 = 0$$

$$4w_1 + 3w_2 = -1$$

We have $w_1 = 0$, and substituting into the last equation, we have $w_1 = 1$. As a result, $\vec{v}_2$ is a linear combination of $\vec{v}_1, \vec{v}_3$ with a coefficient of (1, 0).
Next, we can find $\vec{v}_3$ as a linear combination of the others.

$$(0, 1, 0, -1) = w_1(1, 2, 3, 4) + w_2(0, 1, 0, -1)$$

We can then split into a system

$$0 = w_1$$

$$1 = 2w_1 + w_2$$

$$0 = 3w_2$$

$$-1 = 4w_1 - w_2$$

We have $w_1 = 0$, and we can substitute this into the second equation. $w_2 = 1$. As a result, $\vec{v}_3$ is a linear combination of $\vec{v}_1, \vec{v}_3$ with a coefficient of (0, 1).

b) We know that $\{v_1, v_2, v_3\}$ are linearly dependent if $w_1v_1 + w_2v_2 + w_3v_3 = 0$, or the only solution is the trivial solution. Then, we can split this equation into parts.

$$w_1v_1 + w_2v_2 = 0$$

$$w_nv_n + w_fv_f = 0$$

$$w_nv_n = 0$$

This implies that any two sets, $\{v_n, v_f\}$ are linearly dependent.

4. a) i) Our standard contraction matrix in R^2 is k multiplied by the identity matrix. With point x, y and factor $k = \frac{1}{\alpha}$, we have

$$k\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\frac{1}{\alpha} & 0\\0 & \frac{1}{\alpha}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\frac{1}{\alpha}\\b\frac{1}{\alpha}\end{bmatrix}$$

ii) With $k = \alpha$ and $\alpha > 1$, our standard matrix for dilation is k times the identity matrix.

$$k\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\alpha a\\\alpha b\end{bmatrix}$$

b) I am not sure what it means by "the line that makes an angle of $\pi/4 (= 45°)$ with positive x axis.

5. a) We can begin by analyzing where we want our points to land. We already have two points lying on the x axis. This means the last point must have the same x values as one of the other points, specifically $x = 0$ since the right angle must remain at the origin. We also know that point (0, 0) must be static.

   Next, we know that the standard shear in the x direction is $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ We want to map point $(2, 1)$ to $0, y$ so we can multiply.

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+k \\ 1 \end{bmatrix}$$

   Since we must have the point be $0, y$, we have $2 + k = 0$ and $y = 1$.

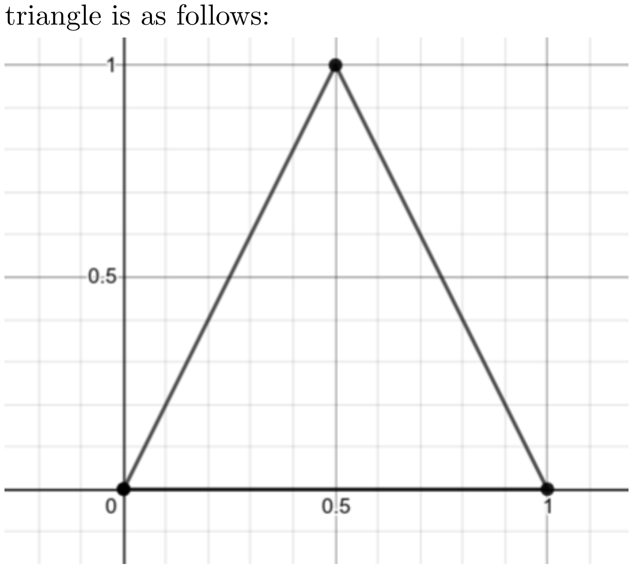
   As such, the shear to fulfill these requirements is $\boxed{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}$

   b) The standard matrix for a shear in the x direction with a factor of k is $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, so for our shear of factor 2, we have a standard matrix of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ We can then find the new points after the shear. Our point (0,0) will not move.

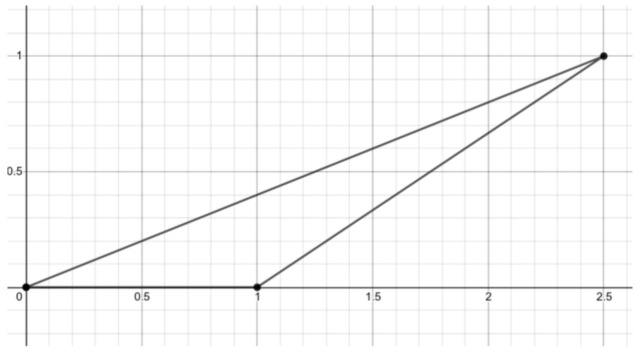
$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

   As such, our transformed triangle has points (0,0), (1,0), (2.5, 1). The un-sheared triangle is as follows:



   The sheared triangle is as follows:


1
0.5
0
0.5
1
1.5
2
2.5