Athabasca University

MATH 265

# Assignment 3

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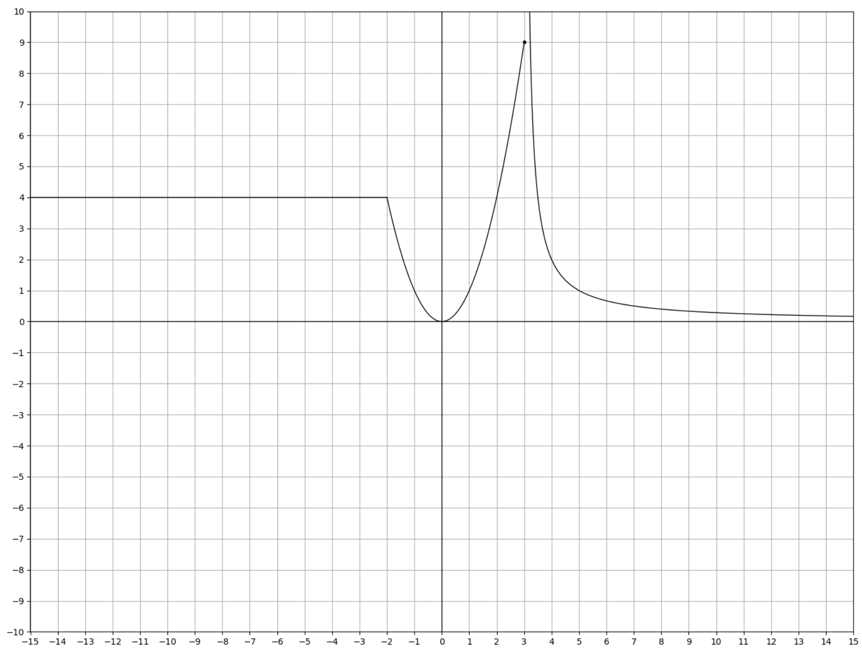
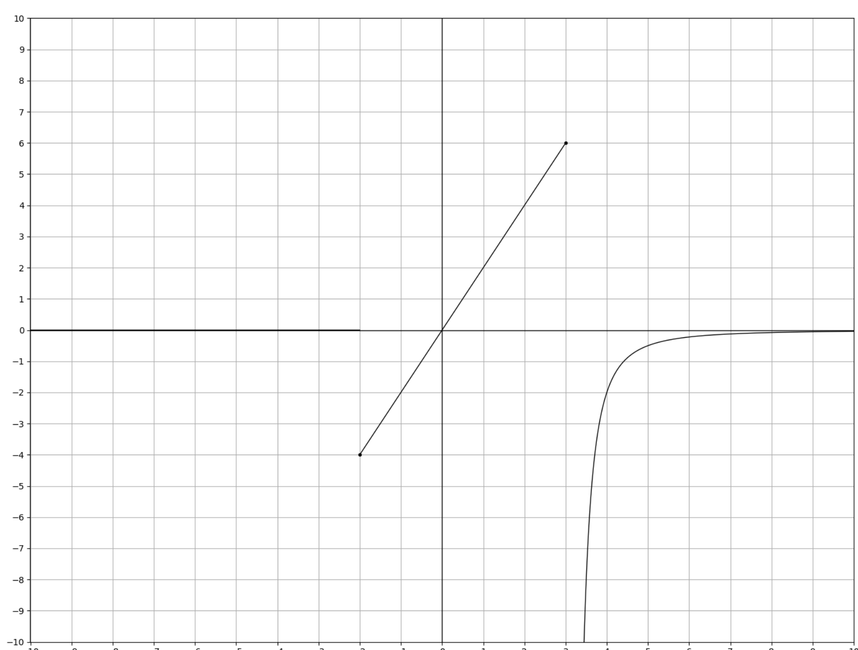
1. (a) Graph of $f(x)$:



Graph of $f'(x)$



2. (a)

$$\frac{d}{dx}x\cos\sqrt{x-3} = \cos(\sqrt{x-3}) - \frac{x\cdot\sin(\sqrt{x-3})}{2\cdot\sqrt{x-3}}$$

(b)

$$\begin{aligned}\frac{d^2}{dx^2}x^2\tan x|_{x=\pi} &= \frac{d}{dx}2x\tan(x) + x^2\sec^2(x)|_{x=\pi}\\ &= 2\tan(x) + 2x\sec^2(x) + 2x\sec^2(x) + 2x^2\sec^2(x)\tan(x)|_{x=\pi}\\ &= 2(0) + 4\pi + 2\pi^2(0) = \boxed{4\pi}\end{aligned}$$

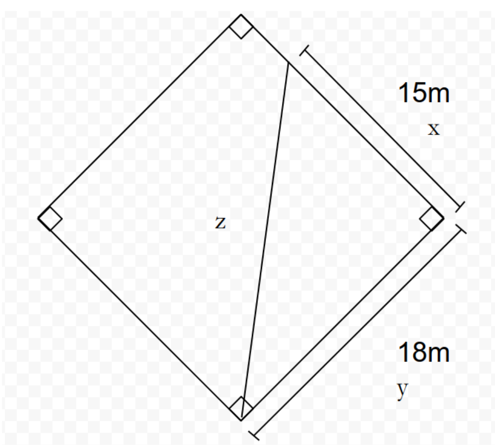
(c)

$$\frac{d}{dx}\frac{\sin(x)-\cos(x)}{x^3} = \frac{x(\cos(x)+\sin(x)) - 3(\sin(x)-\cos(x))}{x^4}$$

(d)

$$\begin{aligned}\frac{d}{dx}\frac{f(x)-x^2g(x)}{f(x)+g(x)}|_{x=0} &= \frac{(f(x)-x^2g(x))'(f(x)+g(x)) - (f(x)-x^2g(x))(f(x)+g(x))'}{(f(x)+g(x))^2}\\ &= \frac{(f'(x)-x^2g'(x)+2xg(x))(1) - (1-(0)^2(0))(f'(x)+g'(x))}{(1+0))^2}\\ &= \frac{(2-(0^2)(-1)+2(0)(0)) - (1)(2+(-1))}{1}\\ &= 1\end{aligned}$$

3. Let the distance from home base to first base be $y$, the distance from first base to the runner be $x$, and the distance from home plate to the runner be $z$.




We have $\frac{dt}{dx} = 7.5\text{m/s}$. We can also find $z$ though the pythagorean theorem. $z^2 = 18^2 + x^2$. Next we differentiate over $t$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dz}{dt} = \frac{x}{z}\cdot\frac{dx}{dt}$$

We are given $x = 18 - 3 = 15$. Substitute this into the pythagorean theorem to find $z$

$$z = \sqrt{18^2 + 15^2} = 3\sqrt{61}.$$

Substituting this back into the derivative, we have

$$\frac{dz}{dt} = \frac{15}{3\sqrt{61}} \cdot 7.5 = \boxed{4.8m/s}$$

4. (a) We can first find the derivative of function. $f(x) = \frac{x-1}{\sqrt{x}}$

$$f'(x) = \frac{(x-1)'\sqrt{x} - (\sqrt{x}'(x-1))}{(\sqrt{x})^2}$$
$$= \frac{1}{\sqrt{x}} - \frac{x-1}{2x^{\frac{3}{2}}}$$
$$= \frac{2x}{2x^{\frac{3}{2}}} - \frac{x-1}{2x^{\frac{3}{2}}} = \frac{x+1}{2x^{\frac{3}{2}}}$$

Plugging in $x = 4$, we have

$$\frac{4+1}{2(4)^{\frac{3}{2}}} = \frac{5}{16}.$$

The $y$ coordinate is $f(4) = \frac{3}{2}$. In order for a line with slope $\frac{5}{16}$ to pass through this point, the $y$ intercept must be $\frac{1}{4}$.

Therefore, the equation of the tangent line at $(4, f(4))$ is $\boxed{y = \frac{5}{16}x + \frac{1}{4}}$.

(b) Using the equation of the tangent line we found in the previous question, we have

$$f(x) = \frac{5}{16}x + \frac{1}{4}$$

$$f(4.02) = \frac{5}{16}(4.02) + \frac{1}{4} = \boxed{1.50625}$$

5. Using implicit derivation, we can start by finding the derivative over $x$.

$$y^2 + x^2 = (2x^2 + 2y^2 - x)^2$$

$$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x)(4x + 4y\frac{dy}{dx} - 1)$$

Now, we need to isolate $y'$. If the denominator is 0, then there is a point in which the tangent line is vertical.

$$\frac{dy}{dx} = \frac{2(2x^2 + 2y - x)(4x + 4y\frac{dy}{dx} - 1) - 2x}{y^2}$$

$$\frac{dy}{dx} = \frac{2(3-4x)x^2 + (2-8x)y^2}{y(8x^2 - 4x + 8y^2 - 1)}$$

The term in the denominator is $y(8x - 4x + 8y^2 - 1)$, so the tangent is vertical when $y = 0$ since the slope will be undefined.

Then, we can substitute $y = 0$ back into the original equation in order to find $x$.

$$0^2 + x^2 = (2x^2 + 2(0)^2 - x)^2$$

We have $x = \pm 1$. Therefore, there is a vertical tangent line at (1,0) and (-1, 0).

6. (a) The equation for the hypotenuse is $25\csc(x)$, and the equation for the adjacent side is $25\cot(x)$.

   To find the error for the hypotenuse, we can find the derivative.

$$\frac{d}{dx} 25\csc(x) = 25(-\csc(x))\cot(x).$$

   Next, to find the error of the hypotenuse, we substitute $x = 60° = \frac{\pi}{3}$ as well as multiplying by the error of $\pm\frac{\pi}{360}$.

$$25(-\csc\left(\frac{\pi}{3}\right))\cot\left(\frac{\pi}{3}\right)\left(\pm\frac{\pi}{360}\right) \approx \pm 0.145cm$$

   Similarly, to find the error for the adjacent side, we can find the derivative of the equation for the adjacent side.

$$\frac{d}{dx} 25\cot(x) = -25\csc^2(x)$$

   Again, we substitute $\frac{\pi}{3}$ and multiply by our error of $\pm\frac{\pi}{360}$

$$-25\csc^2\left(\frac{\pi}{3}\right)\left(\pm\frac{\pi}{360}\right) \approx \pm 0.290cm$$

   (b) We can begin by calculating the error for the hypotenuse

$$\frac{\pm 0.145}{25\csc\left(\frac{\pi}{3}\right)} \approx \pm 0.005 = \pm 0.5\%.$$

   Next, we can calculate the error of the adjacent side

$$\frac{\pm 0.29}{25\cot\left(\frac{\pi}{3}\right)} \approx \pm 0.02 = \pm 2\%.$$

7. (a) We can use trigonometry and rearrange our equation. We have

$$\sin(\theta) = \frac{r}{r+h}$$

   Then, we can replace sin with reciprocal trig functions and simplify.

$$\frac{1}{\csc(\theta)} = \frac{r}{r+h}$$

$$r + h = r\csc(\theta)$$

$$h = r\csc(\theta) - r$$

   Finally, common factoring out $r$, we have

$$h = r(\csc(\theta) - 1)$$

(b) Average rate of change can be found with the formula $\frac{f(x)-f(y)}{x-y}$. We have $f(x) = 6738(\csc(\theta) - 1)$, and our values are $\frac{\pi}{3}$ and $\frac{\pi}{3}$.

$$\frac{6738(\csc(\frac{\pi}{3}) - 1) - 6738(\csc(\frac{\pi}{4}) - 1)}{\frac{\pi}{4} - \frac{\pi}{3}} \approx -6679.155\text{km per rad}$$

(c) We found the equation for $h$ in part (a), and the derivative in part (b). Substituting $r = 6378$km and $\theta = \frac{\pi}{6}$, we have

$$h = 6738(\csc\left(\frac{\pi}{6}\right) + 1)$$

$$\Delta h = 6738(-\csc\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)) \approx -23341\text{km per rad}$$