Athabasca University

MATH 270

Assignment 4

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1. a) $\vec{v} + \vec{w} = (0+7, 1+4, -4+(-1), 1+(-2), 3) = (7, 5, -5, -1, 5)$

   b) $3(2\vec{u} - \vec{v}) = 3(2(1, 2, -3, 5, 0) - (0, 4, -1, 1, 2))$

   c) $(3\vec{u} - \vec{v}) - (2\vec{u} + 4\vec{w})$

   d) $\frac{1}{2}((7, 1, -4, -2, 3) - 5(0, 4, -1, 1, 2) + 2(1, 2, -3, 5, 0)) + (1, 2, -3, 5, 0)$

2. a i) The Euclidean distance can be found with the equation $||u-v|| = \sqrt{(u-v)\cdot(u-v)}$. Substituting our vectors, we have $u - v = (1, 2, -3, 0) - (5, 1, 2, -2) = (-4, 1, -1, 2)$. The Euclidean distance is

$$\sqrt{(-4, 1, -1, 2)(-4, 1, -1, 2)} = \sqrt{16 + 1 - 1 + 4} = \sqrt{2}0$$

To find the angle, we must find $||u||$ and $||v||$ by square rooting and squaring.

$$||u|| = \sqrt{u^2} = \sqrt{1 \cdot 1 + 2 \cdot 2 + (-3) \cdot (-3) + 0 \cdot 0} = \sqrt{14}$$

$$||v|| = \sqrt{v^2} = \sqrt{5 \cdot 5 + 1 \cdot 1 + 2 \cdot 2 + (-2) \cdot (-2)} = \sqrt{34}$$

We know that $\cos\theta = \frac{u \cdot v}{||u|| \cdot ||v||}$.

$$\frac{(1, 2, -3, 0)(5, 1, 2, -2)}{\sqrt{14} \cdot \sqrt{34}} = \frac{5 + 2 - 3 + 0}{\sqrt{476}} = \frac{1}{\sqrt{476}} = \frac{1}{2\sqrt{119}}$$

Since $\cos\theta > 0$, we know that the angle between the vectors is acute.

ii) We can use the same methodology as above.

$$u - v = (0, 1, 1, 1, 2) - (2, 1, 0, -1, 3) = (-2, 0, 1, 2, -1)$$

Our Euclidean distance is

$$\sqrt{(-2, 0, 1, 2, -1) \cdot (-2, 0, 1, 2, -1)} = \sqrt{(4 + 0 + 1 + 4 + 1)} = 10$$

We know that $\cos\theta = \frac{u \cdot v}{||u|| \cdot ||v||}$. Substituting in, we have

$$\begin{aligned}
\cos\theta &= \frac{(0, 1, 1, 1, 2) \cdot (2, 1, 0, -1, 3)}{\sqrt{(0, 1, 1, 1, 2) \cdot (0, 1, 1, 1, 2)} \cdot \sqrt{(2, 1, 0, -1, 3) \cdot (2, 1, 0, -1, 3)}} \\
&= \frac{0 + 1 + 0 + -1 + 6}{\sqrt{0 + 1 + 1 + 1 + 4} + \sqrt{4 + 1 + 0 + 1 + 9}} \\
&= \frac{6}{\sqrt{105}}
\end{aligned}$$

Again, since $\cos\theta > 0$, the angle θ is acute.

b i) We can find whether $|u \cdot v| < ||u||||v||$.

$$||u|| = \sqrt{4 \cdot 4 + 1 \cdot 1 + 1 \cdot 1} = \sqrt{18}$$

$$||v|| = \sqrt{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3} = \sqrt{14}$$

$$u \cdot v = 4 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 9$$

Then, $|u \cdot v| = 9 < \sqrt{252} = ||u||||v||$, so therefore, the Cauchy-Schwarz inequality holds.

ii) Again, we can calculate $||u||, ||v||$, and $|u \cdot v|$.

$$||u|| = \sqrt{1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3} = \sqrt{19}$$

$$||v|| = \sqrt{0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 + 5 \cdot 5 + (-2) \cdot (-2)} = \sqrt{31}$$

$$u \cdot v = \sqrt{1 \cdot 0 + 2 \cdot 1 + 1 \cdot 1 + 2 \cdot 5 + (-2) \cdot 3} = \sqrt{7}$$

Then, $|u \cdot v| = 7 < \sqrt{539} = ||u||||v||$, so therefore, the Cauchy-Schwarz inequality holds.

3. a) We can begin by finding a point on either plane. For the plane $2x - y + z = 1$, we can let $x, z = 0$ so we find point $(0, -1, 0)$.

Next, we need to find the distance between this point and the second plane, $2x - y + z = -1$. We can an equation to find the distance between this point and the plane.

$$\frac{|(2 \cdot 0 + (-1) \cdot (-1) + 1 \cdot 0 + 1)}{\sqrt{2^2 + (-1)^2 + 1^2}} = \boxed{\frac{2}{\sqrt{6}}}$$

b) Two vectors are orthogonal if $|a \cdot b| = 0$.

$$|a \cdot b| = -ab + ab = 0$$

Therefore, these two vectors are orthogonal.

We can set the dot product of $\vec{v}$ and a second vector, $\vec{u} = (a, b)$ to 0.

$$|(a, b) \cdot (2, -3)| = 0$$

$$2a - 3b = 0$$

A few possible values include $(a, b) = (3, 2)$ or $(a, b) = (6, 4)$

ii) We begin by finding orthogonal vectors to $(-3, 4)$ like the previous question.

$$-3a + 4b = 0$$

One possible solution is $(a, b) = (4, 3)$ or $(a, b) = (8, 6)$. To find the unit vector, we multiply a vector by the reciprocal of its magnitude.

$$\frac{1}{||(4,3)||}(4,3) = \frac{1}{\sqrt{4 \cdot 4 + 3 \cdot 3}}(4,3) = \frac{1}{5}(4,3) = \left(\frac{4}{5}, \frac{3}{5}\right)$$

Since the original vector, (-3, 4) is of dimension 2, we can make our unit vector negative and it will still be orthogonal. As such, our two unit vectors are $\boxed{\left(\frac{4}{5}, \frac{3}{5}\right), \left(-\frac{4}{5}, -\frac{3}{5}\right)}$

4. a) We can begin by manipulating this system into an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & 1 & -7 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 17 & 0 \\ 0 & 1 & -7 & 0 \end{array}\right]$$

Therefore, our solutions are $x_1 = -17x_3$ and $x_2 = 7x_3$. We can form a vector: $(x_1, x_2, x_3) = (-17x_3, 7x_3, a)$. Finally, we can check whether this vector is orthogonal to the coefficient vectors. Note that $a = x_3$

$$1(-17x_3) + 3(7x_3) - 4(a) = -17x_3 - 4a + 21x_3 = 0$$

$$1(-17x_3) + 2(7x_3) + 3(a) = -17x_3 + 14x_3 + 3a = 0$$

b) i) We can simply let the vectors be the coefficients for a homogenous system of linear equations.

$$-3x_1 + 2x_2 - 1x_3 = 0$$

$$-2x_2 - 2x_3 = 0$$

ii) Each equation is a plane, and the intersections of the equations are the solutions. We have 2 planes, so their intersection will always be a line.

5. a) First, we calculate the vectors by the initial and terminal points. We are given points $P(1, -1, 2), Q(0, 3, 4), R(6, 1, 8)$.

$$\vec{PQ} = 0 - 1; 3 - (-1); 4 - 2 = (-1, 4, 2)$$

$$\vec{PR} = 6 - 1; 1 - (-1); 8 - 2 = (5, 2, 6)$$

$$P = \frac{1}{2}|\vec{PQ} \cdot \vec{PR}|$$

Next, we calculate the cross product of our vectors $\vec{PQ}, \vec{PR}$

$$\vec{r} = \vec{PQ} \times \vec{PR}$$

$$= (4 \cdot 6 - 2 \cdot 2, 2 \cdot 5 - (-1 \cdot 6), -1 \cdot 2 - 4 \cdot 5) = (20, 16, -22)$$

Finally, we calculate the magnitude of the vector and divide by two..

$$\vec{r} = \frac{\sqrt{20^2 + 16^2 + (-22)^2}}{2} = \frac{\sqrt{1140}}{2} = \boxed{\sqrt{285}}$$

b) We have $\vec{u} = (-1, 2, 4), \vec{v} = (3, 4, -2), \vec{w} = (-1, 2, 5)$. We know that

$$u \cdot (v \times w) = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 4 & -2 \\ -1 & 2 & 5 \end{bmatrix}$$

With cofactor expansion, we have $-1(25+4) - 2(15-2) + 4(6+4) = -29 - 26 + 40 = -15$