Athabasca University

MATH 270

Assignment 1

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1. (a) When we multiply the second equation by 2, we get the system

$$6x_1 + 2x_2 = -8$$
$$6x_1 + 2x_2 = -8$$

Equation 1 is equal to equation 2, and as a result, the two equations in this linear system are collinear and have an infinite amount of solutions.
To find the solutions, we can let $x_1 = t$. Then,

$$6t - 2x_2 = -8$$
$$x^2 = -3t - 4$$

Therefore, the solution set is $x_1 = t$, $x_2 = -3t - 4$

(b) We will attempt to make all of the equations collinear to equation 2. In order to make equation 1 equal equation 2, we can multiply equation 1 by 2. In order to make equation 3 collinear to equation 2, we can multiply equation 3 by $-\frac{3}{2}$. When these operations are performed, we get the system

$$6x - 3y + 6z = -12$$
$$6x - 3y + 6z = -12$$
$$6x - 3y + 6z = -12$$

Since all the equations are the same, the solution set is infinite. We can find the solutions by letting $y = s$, and $z = t$. Then,

$$6x - 3s + 6t = -12$$
$$x = \frac{1}{2}s - t - 2$$

Therefore, we have the solutions $x = \frac{1}{2}s - t - 2, y = t, z = s$

2. We can form an augmented matrix from the system of linear equations to use Gaussian and Gaussian-Jordan elimination.

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$

We can eliminate rows to form the matrix into row echelon form. Our first step is to get a leading 1, which can be done by dividing R_2 by 3.

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 1 & 2 & -1 & -\frac{2}{3} \\ 6 & 6 & 3 & 5 \end{bmatrix}$$

Then, we can eliminate the 6 in R_3 by multiplying R_2 by -6 and adding. In addition, we can swap R_1 and R_2 to have a leading 1.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix}$$

To get a leading 1 for R_2, we can divide by -2.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix}$$

Finally, we can multiply R_2 by 6 and add to R_3 to eliminate the -6, and our matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the matrix is in row echelon form, we can use Gaussian elimination and back-substitute. The corresponding linear equation is

$$\begin{aligned} a + 2b - c &= \frac{2}{3} \\ b - \frac{3}{2} &= -\frac{1}{2} \\ 0a + 0b + 0c &= 1 \end{aligned}$$

From the equation $0a + 0b + 0c = 1$, it is evident that this system is inconsistent, and therefore has no solutions. Another way we could find this solution is through Gauss-Jordan elimination. We can find this by further eliminating our row echelon form matrix. Our first step is to multiply R_2 by -2 and add it to R_1.

$$\begin{bmatrix} 1 & 0 & 2 & \frac{1}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Our final step to put this equation into reduced row echelon form is to multiply R_3 by $-\frac{1}{3}$ and add to R_1 to eliminate the $\frac{1}{3}$, and to multiply R_3 by $\frac{1}{2}$ and add to R_2 in order to eliminate the $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding system of linear equations is

$$\begin{aligned} a \qquad + 2c &= 0 \\ b - \frac{3}{2}c &= 0 \end{aligned}$$

$$0a + 0b + 0c = 1$$

Again, it is evident that the system of equations is inconsistent because of the equation $0a + 0b + 0c = 1$.

3. Since we are not allowed to introduce fractions, we can only start by adding and subtracting rows. We can start by subtracting R_1 by R_3 and replacing R_3.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix}$$

Next, we can multiply R_3 by -2 and add it to R_1 in order to eliminate the 2 and put the first row into reduced row echelon form. Since the first column is in reduced row echelon form, we can move the third row to the top.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & -2 & -29 \end{bmatrix}$$

Then, we can multiply R_3 by -2, add it to R_2, and multiply the new R_2 by -1 in order to make a 1 in the second column.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -57 \\ 0 & -2 & -29 \end{bmatrix}$$

We can then eliminate R_1 by multiplying R_2 by -3 and adding to R_1, and we can eliminate R_3 by multiplying R_2 by 2 and adding to R_3.

$$\begin{bmatrix} 1 & 0 & 173 \\ 0 & 1 & -57 \\ 0 & 0 & 143 \end{bmatrix}$$

Finally, we can multiply R_3 by $\frac{57}{143}$ and add to R_2 in order to eliminate it, and we can multiply R_3 by $-\frac{173}{143}$ and add to R_1 in order to eliminate. In addition, we can divide R_3 by 143 in order to make it zero. Our final answer for the reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. (a) We can start by splitting the addition into $2A^T$.

   $A^T = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$, so $2A^T = \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix}$

   Then, $2A^T + C = \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(b) Again, we can start by individually computing D^T and E^T.

$D^T \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$ and $E^T = \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

Then, $D^t - E^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(c) We can begin by computing the subtraction inside of the parenthesis.

$$D - E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Finally, transposing $(D - E)$, we get $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(d) Start by calculating B^T and $5C^T$.

We have $B^T = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$ and $5C^T = \begin{bmatrix} 5 & 15 \\ 20 & 1 \\ 10 & 25 \end{bmatrix}$

However, we cannot add B^T and $5C^T$ since they have different dimensions; matrices must have identical dimensions in order to add or subtract. Therefore, $B^T + 5C^T$ is undefined.

(e) Again, we can start by splitting the subtraction into $\frac{1}{2}C^T$ and $\frac{1}{4}A$.

$\frac{1}{2}C^T = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 2 & \frac{1}{2} \\ 1 & \frac{5}{2} \end{bmatrix}$ and $\frac{1}{4}A = \begin{bmatrix} \frac{3}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

$$\frac{1}{2}C^T - \frac{1}{4}A = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 2 & \frac{1}{2} \\ 1 & \frac{5}{2} \end{bmatrix} - \begin{bmatrix} \frac{3}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{9}{4} & 0 \\ \frac{3}{4} & \frac{9}{4} \end{bmatrix}$$

(f) We found B^T in a previous question, so we can subtract B^T from B.

$$B - B^T = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(g) Similarly to previous questions, we can split the subtraction into $2E^T$ and $3D^T$

$2E^T = \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix}$, and $3D^T = \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix}$

We have $2E^T - 3D^T = \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} = \begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$

(h) Since we calculated $(2E^T - 3D^T)$ in the previous question, we could transpose our previous answer.

$$(2E^T - 3D^T)^T = \begin{bmatrix} 9 & -13 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -6 \end{bmatrix}$$

(i) Let's start with the parenthesis, $B \cdot A$.

$$B \cdot A = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

However, this multiplication is not possible. In order to multiply two matrices $x \cdot y$, x must have the same number of columns as the rows in y. In this case, B has 2 columns, but A has 3 rows. Therefore, $C(BA)$ is undefined.

(j) We can start by finding E^T, then multiplying $D \cdot E^T$.

$$E^T = \begin{bmatrix} 6 & 1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} D \cdot E^T &= \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 6 & 1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 6 + 5 \cdot 1 + 2 \cdot 3 & 1 \cdot 1 + 5 \cdot 1 + 2 \cdot 2 & 1 \cdot 4 + 5 \cdot 3 + 2 \cdot 3 \\ (-1) \cdot 6 + 0 \cdot 1 + 1 \cdot 3 & (-1) \cdot 1 + 0 \cdot 1 + 2 \cdot 2 & (-1) \cdot 4 + 0 \cdot 1 + 4 \cdot 3 \\ 3 \cdot 6 + 2 \cdot 1 + 4 \cdot 3 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 2 & 3 \cdot 4 + 2 \cdot 1 + 4 \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} 17 & 8 & 15 \\ -3 & 3 & -1 \\ 32 & 7 & 26 \end{bmatrix} \end{aligned}$$

The trace of a matrix is the sum of the entries in the main diagonal, which in this case is $17 + 3 + 26 = \boxed{46}$

5. (a) We can start by making the systems of linear equations homogeneous by moving all the variables to one side.

$$5x + y + z - K = 0$$
$$x + 7y + z - K = 0$$
$$x + y + 8z - K = 0$$

We can then form an augmented matrix

$$\begin{bmatrix} 5 & 1 & 1 & -1 & 0 \\ 1 & 7 & 1 & -1 & 0 \\ 1 & 1 & 8 & -1 & 0 \end{bmatrix}$$

Next, we can start to manipulate the augmented matrix into reduced row echelon form. Multiply R_3 by -1 and add to R_2 to eliminate the 1, and multiply R_3 by -5 and add to R_1 to eliminate the 5.

$$\begin{bmatrix} 0 & -4 & -39 & 4 & 0 \\ 0 & 6 & -7 & 0 & 0 \\ 1 & 1 & 8 & -1 & 0 \end{bmatrix}$$

Since R_3 has a leading 1, we can swap it with R_1. We can also divide R_2 by 6 in order to make a leading 1.

$$\begin{bmatrix} 1 & 1 & 8 & -1 & 0 \\ 0 & 1 & -\frac{7}{6} & 0 & 0 \\ 0 & -4 & -39 & 4 & 0 \end{bmatrix}$$

We can use the leading 1 in R_2 to eliminate the other rows. We multiply R_2 by 4 and add to R_3 to eliminate the -4, and multiply R_2 by -1 and add to R_1 to eliminate the -1.

$$\begin{bmatrix} 1 & 0 & \frac{55}{6} & -1 & 0 \\ 0 & 1 & -\frac{7}{6} & 0 & 0 \\ 0 & 0 & -\frac{131}{3} & 4 & 0 \end{bmatrix}$$

We need to make another leading 1, so we can divide R_3 by $-\frac{131}{3}$

$$\begin{bmatrix} 1 & 0 & \frac{55}{6} & -1 & 0 \\ 0 & 1 & -\frac{7}{6} & 0 & 0 \\ 0 & 0 & 1 & -\frac{12}{131} & 0 \end{bmatrix}$$

We can multiply R_3 by $\frac{7}{6}$ and add to R_2 to eliminate the $-\frac{7}{6}$, and multiply R_3 by $-\frac{55}{6}$ to eliminate the $\frac{55}{6}$. After these operations, our matrix is in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{21}{131} & 0 \\ 0 & 1 & 0 & -\frac{14}{131} & 0 \\ 0 & 0 & 1 & -\frac{12}{131} & 0 \end{bmatrix}$$

This corresponds to the homogeneous system of linear equations

$$\begin{aligned} x - \frac{21}{131}k &= 0 \\ y - \frac{14}{131}k &= 0 \\ z - \frac{12}{131}k &= 0 \end{aligned}$$

From this, it is evident that any one arbitrary parameter, whether it be x, y, z, or k, would provide enough information to calculate the other 3 variables.

(b) Taking the systems of linear equations from the question above, we can solve for the smallest positive integer solution set

$$\begin{aligned} x - \frac{21}{131}k &= 0 \\ y - \frac{14}{131}k &= 0 \\ z - \frac{12}{131}k &= 0 \end{aligned}$$

If we let k equal 131 to eliminate the fractions, we have $x = 21, y = 14, z = 12$. We can attempt to find a lowest common divisor between these positive integers to see if a lower solution exists. 21 prime factors as $3 \cdot 7$, 14 has the prime factors $2 \cdot 7$, and 12 has the prime factors $2 \cdot 3^2$. Since there are no shared common factors, the solution set $\boxed{k = 131, x = 21, y = 14, z = 12}$ is the smallest positive integer solution set.

(c) We can plug the solutions provided in the question into our system of linear equations and check if the equations are defined. The solution set is $k = 262, x = 42, y = 28, z = 24$.

$$
\begin{aligned}
42 - \frac{21}{131} \cdot 262 = 0 \\
28 - \frac{14}{131} \cdot 262 = 0 \\
24 - \frac{12}{131} \cdot 262 = 0
\end{aligned}
$$

Simplifying, we end up with

$$
\begin{aligned}
42 = 42 \\
28 = 28 \\
24 = 24
\end{aligned}
$$

Seeing as the equations are all true and defined, the solution set provided in the question is included among the solutions of our system of equations.