Functions

Objectives

When you have completed this unit, you should be able to

1. distinguish between a relation and a function.
2. apply the definition of a function to practical situations.
3. find the domain of a function.
4. apply basic transformations to sketch graphs of functions.

Relations and Functions

Prerequisites

To complete this section, you must know how to write inequalities in interval notation. See the section titled “Intervals,” pages 337-338 of the textbook.

Calculus is the area of mathematics that specializes in solving problems by establishing the function or functions that represent the quantities involved. In this introductory course, you will learn to apply different types of functions to solve problems, and it is our hope that, in the process, you will come to appreciate how powerful the concept of function is for problem solving. But before we discuss functions, we must consider variables and relations.

In mathematics, quantities are called “variables.” Quantities that do not change are called constant variables or constants; those that do change are called changing variables or simply variables.

Definition 2.1. A *variable* is a quantity that may or may not change (vary) according to what it represents.

You may think that this definition contains a contradiction, but what we intend is to give ourselves the flexibility of changing a constant into a variable if we need to. In this course, all variables are real numbers.

Example 2.1. Your age is a changing variable. It changes (varies) every year while you age, but it becomes a constant on your death.

Example 2.2. Income is a changing variable (increasing, one hopes). It changes according to cost of living adjustments, promotions, etc.

Example 2.3. The distance of a traveling object from its origin is a changing variable.

Example 2.4. The volume of a wooden box is a constant variable. [Why?]

Example 2.5. The volume of a melting ice cube is a changing variable. [Agree?]

In general, it is easy to decide when a variable is constant or changing. Consider the following exercises.

Exercises

1. Is the number of children in a family a changing or a constant variable?
2. Is the acceleration of Earth's gravity a constant variable?
3. Is the number of days in a year a constant or a changing variable?

Answers to Exercises (appendix-a.htm)

When working with variables, it is important to know which quantities are changing, and which are constant.

When we identify one or more variables, we name them for easy reference; therefore, different variables must have different names. Once a variable is named, we represent it as a symbol; this is the first step in abstract thinking—what mathematicians call "mathematical modeling." The type of symbol is not important, but to facilitate the manipulation of the variables, we tend to give them symbols that are related to what they represent, using as few letters or other characters as possible. Since each person could choose a different symbol for each variable, once we have assigned a symbol, we must let others know what the symbol represents. That is, we assign names and their corresponding symbols to the variables in order to define them.

Always define the variables you are using and give different names to different variables.

It is also convenient to reflect on the possible range of values of the variables we define.

Example 2.6. We could assign the symbols I and $\$$ to the variables that refer to two different types of income. We can define them by stating: "We denote by I the income earned by an employee before the year 2000, and by $\$$ the income earned by the same employee after the year 2000." What are the possible values for I and $\$$? We expect that I and $\$$ are positive nonzero real numbers; that is, $I > 0$ and $\$ > 0$.

Example 2.7. To the variable "age" of a person we could assign the letter A and state: "The age (in years) of a person is denoted by A." The variable A is positive or zero, and we are probably safe to assume that it is less than 120; hence, $0 \leq A < 120$; that is, the *range* of values for A is the interval [0,120).

Example 2.8. The variable "distance" can be represented by s, and we can say: " . . . the distance s of a traveling object . . . . The variable s is positive; that is, $s \geq 0$.

Exercises

4. What symbol would you give to the variable "volume of an ice cube"? How would you define it?
5. How would you define the variable "the body temperature of a living human being"? What is the possible range of values for this variable?

Answers to Exercises (appendix-a.htm)

Having defined the variables, we must try to understand the possible relationships among them.

Example 2.9. The "income" of an employee is related to the "number the employee's working hours." So we have a relation between the changing variables "income" and "number of worked hours."

Example 2.10. The cost of mailing a letter depends on its weight, so we have a relation between the changing variables "weight of a letter" and "cost of postage."

Example 2.11. The "area of any triangle" depends on the "length of its base and its height." So we have a relation between the variables "area of a triangle" and "length of base and height."

Example 2.12. The "distance" of a traveling object depends on the "time" traveled.

Observe that in the examples give above, we have related variables because we found a relation of dependency between them. But we can also relate variables based on other criteria.

Example 2.13. We can relate the "Social Insurance Number" of a Canadian citizen with his or her "age in the year 1998."

Example 2.14. Your "age" can be related to your "weight."

Definition 2.2. A *relation* is an association between two variables or among several variables.

The criterion used to associate the variables must be established—that is, defined—and to establish it, we must introduce a mathematical model.

Example 2.15. In Example 2.9, we have the relation between the variable "income" and the variable "number of worked hours." We start by defining these variables. Let I be the income and w be the number of worked hours. There are two ways to define the relation between these two variables:

- by ordered pairs. The pair (w, I) indicates that I and w are two related variables. To establish the relationship between them, we say that the relation is the set of all pairs (w, I) where I is the income earned for w worked hours, and we write $\{(w, I) \mid I$ is the income earned for w worked hours $\}$. We can also give a name to this relation, say T, and we write

  $$T = \{(w, I) \mid I \text{ is the income for } w \text{ worked hours}\}.$$

  The brackets $\{\}$ are read as "the set of," and the symbol $\mid$ is read as "such that" or "where." So the expression is, "T is the set of all pairs of the form (w, I) such that I is the income for w worked hours."

- by explicit definition of the relation. We decide on the name of the relation first, say T and then we write wTI to indicate that w is related to I by T. We define this relation explicitly, as follows:

  wTI if and only if I is the income earned for w worked hours.

In calculus, we prefer the notation that uses ordered pairs.

Example 2.16. In Example 2.10, we have the variables cost of postage and weight, which we define as P and w, respectively. If we name the relation C, we either write

$$C = \{(w, P) | P \text{ is the cost of postage for a piece of mail of weight } w\}$$

to be read as "C is the set of all pairs (w, P) such that P is the cost of postage of a piece of mail of weight w," or we write wCP iff[1] P is the cost of posting a piece of mail of weight w.

We read "P is related to w by the relation C if and only if P is the cost of posting a piece of mail of weight w."

Example 2.17. In Example 2.11, we define A as the area of a triangle, and b and h as the base and height of the triangle, respectively. We name the relation T, and we define it as

$$T = \{((b, h), A) | A \text{ is the area of a triangle of base } b \text{ and height } h\}$$

or

$(b, h)TA$ iff A is the area of a triangle with base b and height h.

Example 2.18. Refer to Example 2.13. We will let SIN be the social insurance number of a person, and we let A be that person's age in the year 1998. We name the relation K, and we have

$K = (\text{SIN}, A) | A$ is the age, in the year 1998, of the person associated to SIN

or

SINK A iff A is the age in the year 1998 of the person associated to SIN.

Example 2.19. The relation S between a positive integer n and its square root q is written as

$S = \{(n, q) \mid q \text{ is the square root of the positive integer } n\}.$

Since we already have a symbol to denote the square root of a positive integer n, we can also write $S = \left\{(n, \sqrt{n}) \mid n \text{ is a positive integer}\right\}$.

Exercises

6. How would you state the relation in Example 2.12?
7. How would you read the relation below?

   $T = \{((b, h), A) \mid A \text{ is the area of a triangle of base } b \text{ and height } h\}$
8. How would you read the relation below?

   $(b, h)\, T A$ iff A is the area of a triangle with base b and height h
9. Define the variables “age” and “weight” and establish their relation as given in Example 2.14.
10. How would you read the relation below?

   $T = \{(A, y) \mid A \text{ is your age in the year } y\}$

Answers to Exercises (appendix-a.htm)

Once a relation is established (defined), we can decide which particular pairs belong to it.

Definition 2.3. We say that a pair (a, b) *belongs* to a relation T if a is related to b by T, and we write $(a, b) \in T$. If a is not related to b by T, then we write $(a, b) \notin T$.

A relation T is *well defined* if we can determine whether any given pair (a, b) belongs to the relation T or not.

From Example 2.15, we can see that if the pair $(12, 500) \in T$, then for 12 hours of work the income is \$500.00, we also see that the pair $(12, 0) \notin T$, since the income cannot be \$0.00 for 12 hours of work. It is also clear that $(-4, 100) \notin T$.

In Example 2.16, we write $(20, 0.50)$ to indicate that, for a piece of mail weighing 20 grams, the cost of postage is \$0.50.

In Example 2.17, we have $((4, 3), 6) \in T$ and $((5, 2), 7) \notin T$.

In Example 2.19, we have $(9, 3) \in S$, $(9, -3) \in S$ and $(9, 4) \notin S$. [Why?]

Observe that the order in which the pair is presented matters, the pairs $(12, 500)$ and $(500, 12)$ in the relation T of Example 2.15 are not the same: as we said $(12, 500)$ means that the income for 12 hours of work is \$500.00, the pair $(500, 12)$ says that for 500 hours of work the income is \$12.00.

Exercises

11. Consider the relation

   $M = \{(s, A) \mid A \text{ is the area of a square of side } s\}.$

   Identify a pair that is in M and a pair that is not in M.
12. Indicate whether the pair $(34, 2000)$ belongs to the relation

   $T = \{(A, y) \mid A \text{ is your age in the year } y\}.$

Answers to Exercises (appendix-a.htm)

We use a special notation when there is a relation of dependency between variables. In Example 2.15, the income I depends on the number of worked hours w, and we express this fact by writing $I(w)$. In Example 2.16, we write $P(w)$ to indicate that the cost P of posting a piece of mail depends on its weight w. In Example 2.17, we are told that $A(b, h)$—the area of a triangle depends on its base b and height h. In Example 2.12, the distance traveled s depends on the time traveled t; hence, $s(t)$. However, in Example 2.18, the variable SIN does not depend on the variable A. This is not a relation of dependency, so it is not correct to write A(SIN).

Definition 2.4. If a variable F depends on the variable m, we write $F(m)$. In this case, we refer to F as the *dependent variable*, and m as the *independent variable*.

For convenience, the relation of dependency takes the name of the dependent variable F. Moreover, the pair (a, b) belongs to the relation F iff $F(a) = b$.

Observe that if the pair (a, b) belongs to the relation F, then a is the independent variable and b is the dependent variable. When we write $F(a) = b$, we indicate two things: F depends on a, and the value F that corresponds to a is b. So we write

$$F = \{(a, b) \mid F(a) = b\}.$$

The relation in Example 2.15 is a relation of dependency; therefore, it is no longer referred to as T; instead, it is called I. Note that $I(12) = 500$ because $(12, 500)$ belongs to the relation. The statement $I(10) = 70$ indicates that for 10 worked hours, the income is \$70.00; that is, the pair $(10, 70)$ belongs to the relation I. We also recognize that $(-4, 100)$ does not belong to the relation; hence, $I(-4)$ is undefined.

In Example 2.16, $P(20) = 0.50$; in Example 2.17, $A(3, 4) = 6$; and in Example 2.19, $S(9) = 3$, $S(9) = -3$ and $S(9) \neq 4$.

Exercises

13. Is the relation in Example 2.11 a relation of dependency? How would you write it?
14. If s is the distance traveled (in metres) and t is the time (in seconds), then $s(t)$ is the relation of dependency between s and t. How would you write the statement, “the object travels 50 m in 30 seconds,” in symbolic form?
15. Establish the relation of dependency between the following pairs of variables.
    a. temperature T, and wind chill factor c
    b. cost of living L, and taxes t
    c. amount of interest paid r, and principal amount P

Answers to Exercises (appendix-a.htm)

The problems that calculus investigates are those that can be represented by a relation of dependency, where the dependent variable is uniquely determined by the value of the independent variable or variables. We call these special relations “functions.” Some functions have to do with variables that change with respect to time, such as distance, velocity, area, volume, population, etc. [Note that, for a given time, there is only one distance, one velocity, one area, one volume or one population.]

In this course, we consider only functions of one independent variable; functions of more than one independent variable, such as that shown in Example 2.17, are studied in more advanced courses.

Definition 2.5. A *function* is a relation of dependency F between two variables, such that if the pairs (a, b) and (a, c) belong to the relation [i.e., if $F(a) = b$ and $F(a) = c$], then $b = c$. That is, the dependent variable F associated to the independent variable a is uniquely determined.

If two pairs (a, b) and (a, c) belong to a relation and $b \neq c$, then the relation is not a function.

Example 2.20. In Example 2.19, the relation S is a relation of dependency, but it is not a function, because $S(9) = 3$ and $S(9) = -3$. That is, it is not a function because both of the pairs $(9, 3)$ and $(9, -3)$ belong to S.

Example 2.21. The relation between the area of a square and the length of its side is a function because each length of a square's side is associated with only one area.

Example 2.22. The GST we pay for a product of cost C is a relation of dependency, GST(C), and it is a function. [Why?]

The relations in Examples 2.15 and 2.16 are functions. [Why?]

Later, we will need to solve problems by establishing the function or functions that represent the problem. The key to doing so is understanding the relation of dependency between the variables of the problem.

We have been paying attention to the range of values of the dependent and independent variables. These ranges of values are important for identifying the variables we are working with. We have special names for them.

Definition 2.6. The *domain* D_F of a function $F(s)$ the largest set of acceptable values of the independent variable s.

The *range* or *rank* R_F of a function F is the largest set of values of the dependent variable F.

Thus a pair (s, F) belong to the function F, if s is in D_F and F is in R_F.

To find the domain of a function, ask yourself, "What are the possible values for the independent variable?"

If we have a function F with independent variable v—that is, $F(v)$—what we try to do next is to find a mathematical expression that gives the value of F for each value of v. This mathematical expression is the mathematical representation (model) of the function F. This model is also referred as a "formula" for F. In this course, the formula we want must involve constants and the variable v only. That is, we want to express F only in terms of the independent variable v.

Example 2.23. If $A(s)$ is the function that gives the area A of a square of side s, then the mathematical representation of A is s^2. [Why?] So, we write $A(s) = s^2$. In this case, A is expressed in terms of s. The domain and range of the function A is the interval $D_A = R_A = [0, \infty)$. [Why?]

The formula $A(s) = s^2$ can be used to solve any problem that involves the area of a square. Observe that $A(10) = 100$, this means that the area of a square of side 10 is 100. We say that "the value of A at 10 is 100," or "the area A is 100 if s is 10."

Example 2.24. If $\mathrm{GST}(C)$ is the function of Example 2.22, above, then $\mathrm{GST}(C) = 0.05C$. [Agree?] What is the formula for the total cost T (including the GST) that we would pay for a product of cost C? That is, what is the mathematical model (formula) in terms of C of T? What is the value of T for an item priced at $\$165.45$?

Example 2.25. If income paid I is at a rate of $\$10.50$ per hour, then the mathematical model of the function $I(w)$ of Example 2.15 is $I(w) = 10.50w$. The income for 10 worked hours is $I(10) = 105.00$. Although we can multiply 10.50 by a negative number, it does not make sense to say that the value of I at -2 is -21; that is, it is true that $I(-2) = 10.50(-2) = -21$, but the meanings of I and w are lost here. The domain and range of this function are the interval $D_I = R_I = [0, \infty)$, and we say that we cannot evaluate the function I at negative numbers, not because we cannot multiply 10.50 by a negative number, but because the variable w represents worked hours, and this variable takes only positive values.

We now have the concepts of *mathematical domain*, as defined in Definition 2.6, above, and of *physical domain*, as the set of acceptable values of the independent variable, according to what the independent and dependent variables involved represent.

Example 2.26. If A is the area of an equilateral triangle with side s, then the formula of the function $A(s)$ in terms of s is $(\sqrt{3}s^2)/4$; that is,

$$A(s) = \frac{\sqrt{3}s^2}{4}.$$

To arrive at this relation, we must study the area of equilateral triangles. The area of any triangle is half of the product of the base and the height. The base of the equilateral triangle is s. To find the height h (height is a variable) we use Pythagoras' Theorem.

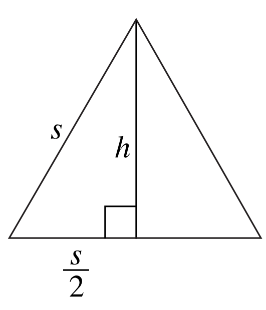

Figure 2.1. Equilateral triangle with side equal to s

According to Pythagoras' Theorem,

$$\left(\frac{s}{2}\right)^2 + h^2 = s^2,$$

so

$$h^2 = s^2 - \frac{s^2}{4} = \frac{3s^2}{4},$$

since h and s are positive. [Why?] Therefore,

$$h = \frac{\sqrt{3}s}{2},$$

and the area is as indicated.

An equilateral triangle with side 56 has an area of

$$A(56) = \frac{56^2\sqrt{3}}{4} \approx 1357.93.$$

In words, the area of an equilateral triangle with side 56 is approximately equal to 1357.93. What are the domain and range of A? What is the value of A at 9? What is the interpretation of $A(4) = 4\sqrt{3}$?

> **Note:** We are ignoring the units for the time being. Be aware, however, that you will be expected to use the correct units in assignments or examinations.

Example 2.27. If the area of a rectangle is 16 m^2, what is the formula for $P(l)$, where P is the perimeter of the rectangle and l is the length of one of its sides? In other words, how can we express P in terms of l?

Step 1

We start with a picture of the problem:

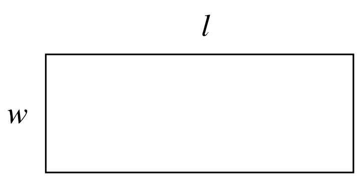

Figure 2.2. Rectangle with length equal to l, and width equal to w

Step 2

We define the variables:

l is the length of the base	P is the perimeter of the rectangle
w is the height of the rectangle	A is the area

Step 3

We relate the variables to what we already know about the perimeter and area of the rectangle; hence, $P = 2l + 2w$ and $A = lw = 16$.

Since we want to find a formula for the perimeter that depends only on l, we must substitute for the variable w an expression with only ls. Hence,

$$w = \frac{16}{l} \qquad \text{and} \qquad P = 2l + 2\frac{16}{l} = 2l + \frac{32}{l},$$

and we conclude that $P(l) = 2l + \dfrac{32}{l}$.

We have found the formula that gives the perimeter P of a rectangle of side l and area 16 m^2. We can find the value of P for any positive value of l—all we have to do is to apply the formula. For example,

$$P(6) = 2(6) + \frac{32}{6} = \frac{52}{3}.$$

We call this process "evaluating P at 6," and the answer is expressed as, "the value of P at 6 is $52/3$."

The domain and range of P are the interval $(0, \infty)$. [Agree?]

What is the perimeter of a rectangle of area 16 m^2 if one of its sides is 5 m?

Exercises

16. A rectangle has perimeter of 20 m. Express the area of the rectangle as a function of the length of one of its sides.
17. Express the surface area of a cube as a function of its volume.
18. An open rectangular box with volume 2 m^2 has a square base. Express the surface area of the box as a function of the length of a side of the base.
19. The cost of renting a car is $50.00 plus 43 cents per kilometre traveled.
    a. Define the variables that correspond to this problem, and establish the relation of dependency between them.
    b. Find the formula that gives the rental cost in terms of the kilometres traveled.
    c. Give the domain and range of this function.
    d. What is the cost for renting a car in Edmonton to travel to Calgary? Include the appropriate taxes.

Answers to Exercises (appendix-a.htm)

The formulas or mathematical models of functions can take different forms. We cannot always give a single formula for a function. For instance, in Example 2.16, the cost of postage P is fixed depending on a certain range of values of w. That is, P is $\$0.50$ if $0 < w \leq 30$ and P is $\$1.00$ if $30 < w \leq 100$. In this case, we write

$$P(w) = \begin{cases} 0.50 & \text{if} \quad 0 < w \leq 30 \\ 1.00 & \text{if} \quad 30 < w \leq 100 \\ 1.70 & \text{if} \quad 100 < w \leq 200 \\ 2.45 & \text{if} \quad 200 < w \leq 500 \end{cases}$$

As you can see, $P(102) = 1.70$ and $P(100) = 1.00$. The range of P, written as a set, is $\{0.50, 1.00, 1.70, 2.45\}$, and the domain of P is the interval $(0, 500]$. See Example 2.10.

We may also be able to give different formulas for different ranges of the independent variable. Consider a function defined as follows:

$$g(s) = \begin{cases} 3s - 1 & \text{if} \quad 0 < s \leq 20 \\ s^2 + 4 & \text{if} \quad 20 < s < 50 \\ 12 & \text{if} \quad 50 < s \leq 60 \end{cases}$$

First, we notice that the values for s are in the intervals $(0, 50)$ and $(50, 60]$, that is $(0, 50) \cup (50, 60]$ is the domain of g. Then 50 and 0 are not in the domain, in other words there is no value for g at 0 or 50. And we say that g is undefined at 0 and 50. Observe also that g is not defined at any number bigger than 60 (e.g., 60.001 or 198), nor is it defined for negative numbers.

James Stewart, the author of your textbook, calls the functions defined in this fashion *piecewise* functions, and we will adopt this name. See Examples 7 and 8 on pages 14 and 15 of the textbook.

Exercises

20. For each of the piecewise functions (i)-(iv), below,
    i. $f(x) = \begin{cases} x & \text{if} \quad x \leq 0 \\ x + 1 & \text{if} \quad x > 0 \end{cases}$

ii. $f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ 3-x & \text{if } x \geq -1 \end{cases}$

iii. $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

iv. $f(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ 3x+2 & \text{if } |x| < 1 \\ 7-2x & \text{if } x \geq 1 \end{cases}$

a. give the domain of the function f.
Hint: For function (iv), see Box 6 on page 341 of the textbook.

b. give the values of $f(1)$ and $f(-1)$.

Answers to Exercises (appendix-a.htm)

Finally, we may not be able to find one given formula for a function, but we may be able to give a table of pairs that belong to the function (see Example 4 on page 12 of the textbook). This is the case when we make a finite number of observations, such as temperature, population, number of consumers of a product, etc.

Example 2.28. The table below shows the population of a certain city recorded every two years for 10 years.

Table showing the population, *P*, of a certain city recorded on a particular year, *Y*

P	***Y***
56000	1990
56800	1992
57000	1994
57500	1996
57850	1998
58000	2000

The variables are the population P and the year Y when the population was recorded. The function is $P(Y)$. This table indicates that the pairs

$$(1990, 56000), (1992, 56800), (1994, 57000), (1996, 57500), (1998, 57850), (2000, 58000)$$

are in the function P. So $P(1990) = 56000$, $P(1992) = 56800$, and so on.

If the data show a certain pattern or uniformity, one would hope to be able to associate a formula to the function. Different techniques are available.

Exercises

21. If a metal rod is being heated, its volume depends on the temperature. If T denotes the temperature (in °C) and V the volume (in m^3), then $V(T)$, and the domain of the function V is all possible values of the temperature T. We have the following recorded volumes:

Table showing how the volume of metal rod, *V*, varies with the temperature, *T*

V	***T***
15	70
18	75

21	80
20	78
18	75
16	72

a. What do you observe about the volume of the rod with respect to the temperature?

b. What is the value of V for $T = 80$?

c. Estimate the volume of the rod when the temperature is 73°C.

Answers to Exercises (appendix-a.htm)

So far we have been working with functions for which the variables have a very specific interpretation; however, to get to the problems we want to solve, we must learn to work with functions in a general sense. That is, we must learn to work with functions and their corresponding formulas when the variables do not have any specific meaning. We must learn about the general properties of functions, and we must learn to do different operations with them. Once we can perform these operations, we will be able to solve problems using functions. This learning approach is the same as the one you used when learning to read: you learned the basic alphabet first.

For example, in general terms, we defined the domain of a function as the set of possible values of the independent variable. In Example 2.23, the variables of the function $A(s)$ have specific meaning. Therefore, the domain of the function $A(s)$ is the interval $[0, \infty)$ because s represents a positive quantity: the length of one side of a square. However, if we ignore the meaning of s as a length, and consider the function $A(s) = s^2$ in general terms, then we see that we can evaluate the function A at any value of s, including negative numbers:

$$A(-3) = (-3)^2 = 9.$$

Hence, the domain of the function $A(s) = s^2$ in general is all of the real numbers.

Working in general terms has its advantages, because different practical problems can be solved with functions that share the same formula.

Example 2.29. If we consider the function

$$T(t) = \frac{t^2 + 5t - 1}{2t - 3},$$

in general terms, then we can evaluate the function T for any value of t except $t = 3 / 2$. For this value of t, the denominator of this expression is zero; that is, the expression is undefined. So, for any value of t other than $t = 3 / 2$, it is possible to apply the formula of T.

You can check that $T(3 / 4) = -2.08333$, and that $T(-5.67) = -0.19518$.

Since $3 / 2$ is the only value for which T is not defined, the domain of T is all real numbers except $3 / 2$. In interval notation, we would write

$$\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right);$$

in set notation,

$$\mathbb{R} - \left\{\frac{3}{2}\right\}.$$

If T represents the temperature of an object at time t, then the possible values of t are all positive real numbers except $t = 3 / 2$. In interval notation, the domain is

$$\left[0, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right).$$

Example 2.30. The domain of the function

$$F(c) = \sqrt{c-6}$$

consists of all numbers such that $c - 6 \geq 0$ (since only positive numbers have real square roots). So, the domain of F is all real numbers greater than or equal to 6; that is, $c \geq 6$, or in interval notation, $[6, \infty)$.

We can also write, $F(c) = \sqrt{c-6}$ only for $c \geq 6$.

Example 2.31. What is the domain of

$$H(a) = \frac{a-6}{\sqrt{2a-8}}\,?$$

To find the domain, we look for those values a for which we may have problems evaluating $H(a)$, either because the denominator is 0 for this particular value, or because the square root is negative. We see that what we need is $2a - 8 > 0$, so $a > 4$. So the domain of H is all values a that are strictly greater than 4—in interval notation $(4, \infty)$.

Example 2.32. For the function

$$h(s) = \frac{\sqrt{s-1}}{2\ \cos\ s-1},$$

we see that $s - 1 \geq 0$ (the square root must be non-negative), and the divisor must be nonzero.

If $2\ \cos\ s-1 = 0$ then $\cos\ s = 1\,/\,2,$ and therefore,

$$s = \frac{\pi}{3} + 2k\pi \ \text{ and } \ s = -\frac{\pi}{3} + 2k\pi, \text{for any integer } k.$$

Since $s \geq 1$, the function is undefined for

$$s = \frac{\pi}{3}, \ \text{ and for all } \ s = \frac{(6k \pm 1)\pi}{3}, \text{for any } k > 0.$$

We conclude that the domain is all numbers $s \geq 1$, except

$$s = \frac{\pi}{3}, \ \text{ and } \ s = \frac{(6k \pm 1)\pi}{3}, \text{for any integer } k > 1.$$

If we have a function with its corresponding formula, we need to understand what we mean by “evaluating the function.”

To evaluate the function

$$H(a) = \frac{a-6}{\sqrt{2a-8}}$$

at 9, we replace the independent variable a by 9 in the formula:

$$H(9) = \frac{9-6}{\sqrt{2(9)-8}} = \frac{3}{\sqrt{10}}.$$

We can also evaluate the function at any other expression in the same way. For example, we can evaluate this function at $x + h$ by replacing the independent variable a by $x + h$:

$$H(x+h) = \frac{x+h-6}{\sqrt{2(x+h)-8}}$$

$$= \frac{x+h-6}{\sqrt{2x+2h-8}}.$$

Exercises

22. Find the domain of each of the functions below.

a. $f(x) = \dfrac{x}{3x-1}$

b. $f(x) = \dfrac{5x+4}{x^2-3x+2}$

c. $f(t) = \sqrt{t} + \sqrt[3]{t}$

d. $g(u) = \sqrt{u} + \sqrt{4-u}$

e. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

23. Express the domain of each of the functions below in interval notation.

a. $R(u) = \dfrac{u^2 - 9}{u^2 + 3u}$

b. $F(u) = \dfrac{4}{\sqrt{3u^2 + 4}}$

24. Evaluate the functions in Exercise 23, above, at $3x + 1$.

25. Give an example of a function such that 5 and 2 are *not* in its domain.

Answers to Exercises (appendix-a.htm)

FOOTNOTE

[1] It is customary to write iff instead of “if and only if”.