Athabasca University

MATH 270

Assignment 3

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1. (a) We will cofactor expand along row 1.

$$\det\begin{bmatrix}3 & 3 & 1\\ 1 & 0 & -4\\ 1 & -3 & 5\end{bmatrix} = 3\cdot\det\begin{bmatrix}0 & -4\\ -3 & 5\end{bmatrix} - 3\cdot\det\begin{bmatrix}1 & -4\\ 1 & 5\end{bmatrix} + 1\det\begin{bmatrix}1 & 0\\ 1 & -3\end{bmatrix}$$
$$= 3(\cdot 5 - (-4)\cdot(-3)) - 3(5\cdot 1 - (-4)\cdot 1) + (1\cdot(-3) - 0\cdot 1)$$
$$= 3\cdot -12 - 9\cdot 3 - 3$$
$$= \boxed{-66}$$

(b) Again, we will cofactor expand along row 1.

$$\det\begin{bmatrix}k+1 & k-1 & 7\\ 2 & k-3 & 4\\ 5 & k+1 & k\end{bmatrix} = (k+1)\cdot\det\begin{bmatrix}k-3 & 4\\ k+1 & k\end{bmatrix} - (k-1)\cdot\det\begin{bmatrix}2 & 4\\ 5 & k\end{bmatrix} + 7\cdot\det\begin{bmatrix}2 & k-3\\ 5 & k+1\end{bmatrix}$$
$$= (k+1)(k(k-3) - 4(k+1)) - (k-1)(2k-20) + 7(2(k+1) - 5(k-3))$$
$$= (k^3 - 6k^2 - 11k - 4) - (2k^2 - 22k + 20) + (-21k + 119)$$
$$= \boxed{k^3 - 8k^2 - 10k + 95}$$

2. (a) We can start by manipulating this matrix into row echelon form by dividing R_1 by 3, adding $2R_1$ to R_3, and swapping R_2 and R_3. Finally, we can divide R_2 by 5 and R_3 by -2.

$$\det A = \begin{vmatrix}3 & 6 & -9\\ 0 & 0 & -2\\ -2 & 1 & 5\end{vmatrix}$$
$$\frac{\det A}{3} = \begin{vmatrix}1 & 2 & -3\\ 0 & 5 & -1\\ 0 & 0 & -2\end{vmatrix}$$
$$\frac{\det A}{30} = \begin{vmatrix}1 & 2 & -3\\ 0 & 1 & -\frac{1}{5}\\ 0 & 0 & 1\end{vmatrix}$$

Finally, we can find the determinant of our row echelon matrix by cofactor expanding on R_3.

$$\frac{\det A}{30} = 1\det\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = 1$$
$$\det A = 30$$

(b) We can simply manipulate the second matrix using elementary row operations until it is equal to the first, then observe the result. Let $A = \left[\begin{smallmatrix}a & b & c\\ d & e & f\\ g & h & i\end{smallmatrix}\right]$. Then, we have $\det A = -6$. We can then set this equal to A, then manipulate A into the matrix we are trying to find the determinant of. To do this, we first subtract R_2 from R_1, then multiply R_2 by -1.

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\det A = \begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$-\det A = \begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$$

Therefore, we have $\begin{vmatrix} a+d & b+c & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix} = \boxed{6}$

(c) We can use elementary column operations on the first matrix to make it equal to the second matrix since elementary row operations do not change the discriminant of a matrix. First, we can divide C_2 by $t \cdot C_1$ and C_3 by $s \cdot C_1$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 + rb_1 \\ a_2 & b_2 & c_2 + rb_2 \\ a_3 & b_3 & c_3 + rb_3 \end{vmatrix}$$

Next, we divide C_3 by $r \cdot C_2$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

3. (a) We can begin by finding the determinant of A using cofactor expansion.

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & k \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} k & k \\ 0 & 1 \end{vmatrix} \\ &= 1 - 2k - 2k \\ &= 1 - 4k \end{aligned}$$

Given that a matrix is invertible when its determinant is nonzero and real, we know that the matrix is invertible when $\boxed{k \neq \frac{1}{4}}$

(b) i. Recall that $\det(AB) = \det A \cdot \det B$.
We have $\det(-I \cdot A)$, and we know that $\det I = 1$, so $\det(-1) = -\det(A) = \boxed{2}$

ii. We know that $\det(AB) = \det A \cdot \det B$ and $A \cdot A^{-1} = I$.
Rearranging this, we get $\det(A \cdot A^{-1}) = \det(I) = 1$.
Finally, we have $\det(A^{-1}) = \frac{1}{\det(A)}$, so $\det(A^{-1}) = \boxed{-\frac{1}{2}}$

iii. If we cofactor expand A using the first row before A is transposed, it is equivalent to cofactor expanding along the first column after A is transposed. Therefore, we

know that $\det A^T = \det A$.
We can use the property $\det(AB) = \det A \cdot \det B$ to break up the two parts. $\det 2I \cdot \det A = 2 \det A = \boxed{-4}$

iv. $\det(A^3)$ Similarly to the previous questions, we can use the property $\det(AB) = \det A \cdot \det B$. We have $\det(A^3) = \det A \cdot \det A \cdot \det A = (-2)^3 = \boxed{-8}$

4. (a) We can find if the matrix is invertible by finding its discriminant. We can use cofactor expansion along R_1

$$\det A = 2 \begin{vmatrix} 1 & 0 \\ 3 & 6 \end{vmatrix} = 12$$

The matrix is invertible since its determinant is nonzero. We then need to find the adjugate matrix of A by cofactoring the matrix, then we can multiply by the discriminant to find the inverse.

$$\begin{aligned}
A^{-1} &= \frac{1}{12} adj \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix} \\
&= \frac{1}{12} \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 8 & 0 \\ -5 & 6 \end{vmatrix} & \begin{vmatrix} 8 & 1 \\ -5 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ -5 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 8 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 8 & 1 \end{vmatrix} \end{bmatrix}^T \\
&= \frac{1}{12} \begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}^T \\
&= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}
\end{aligned}$$

(b) We can first rewrite this system of equations as three matrices, one for each variable.

$$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = 1 \cdot (-1) - (-4) \cdot (-16) + 1 \cdot 10 = -55$$

$$D_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 6 \cdot (-1) - (-4) \cdot (43) + 1 \cdot (-22) = 144$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 1 \cdot (43) - 6 \cdot (-16) + 1 \cdot (-78) = 61$$

$$D_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = 1 \cdot (22) - (-4) \cdot (-78) + 6 \cdot (10) = -230$$

Cramer's rule states that $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, and $z = \frac{D_z}{D}$. Therefore, we have

$$x = \frac{144}{-55} = -\frac{144}{55}$$
$$y = \frac{61}{-55} = -\frac{61}{55}$$
$$z = \frac{-230}{-55} = \frac{46}{11}$$

5. (a)