Athabasca University

MATH 265

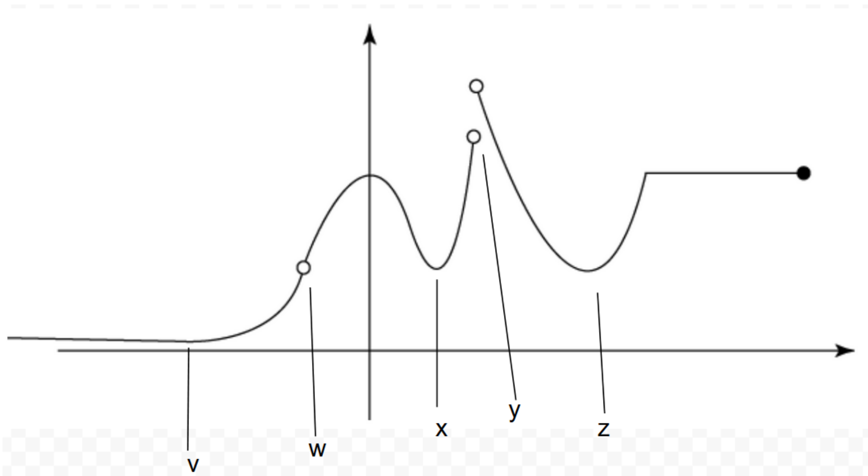
Assignment 4

Stanley Zheng

October 17, 2020

My sincere apologies if there are any typos or work is not shown fully. I wrote this assignment hastily, and the length made it difficult to check for typos. I hope you can understand. Thanks.

1. a)



The domain of the function g is $(-\infty, w) \cup (w, x) \cup (x, y]$

b) The function g is continuous from $(-\infty, w) \cup (w, x) \cup (x, y]$.

c) The local maximum ix at $x = 0$, since the derivative is 0.

d) The local minimum is at two points, x and z, where the derivative is zero.

e) There is no absolute maximum as the derivative at y is undefined, and y is not included in the domain.

f) The absolute minimum is at v. It appears as though the graph has negative slope as x decreases beyond v to infinity.

2. The first condition specifies that this is an odd function, so we must verify this by reflecting over the origin.

3. Our first step is to find the domain of the function. Setting the denominator to zero, we have $x^2 + 6 = 0$, so our domain is $x \neq \pm\sqrt{6}$

Next, we can find the intercepts. We can find the x intercept by letting $y = 0$. We have $x^2 = 4$, so our x-intercepts are $x = \pm 2$. Likewise, our y-intercepts are $-\frac{4}{6}$.

We can then test if the function is even or odd. We can test whether it is even or not by simplifying $f(-x)$. If $f(-x) = f(x)$, then our function is even.

$$f(-x) = \frac{(-x)^2 - 4}{(-x)^2 + 6}$$

Since $-x$ is squared, we know that $f(-x) = f(x)$, and therefore, the function is even.

Our function is not periodic, but it does have a horizontal asymptote. At extremely large values of x, our function will be approximately equal to $\frac{x^2}{x^2}$, so we have a horizontal asymptote at $x = 1$.

Next, we find the derivative of $f(x)$.

$$f'(x) = \frac{2x(x^2 + 6) - 2x(x^2 - 4)}{(x^2 + 6)^2}$$

$$f'(x) = \frac{20x}{(x^2+6)^2}$$

Since $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$, we know that the function is decreasing on $(-\infty, 0)$ and increasing on $0, \infty$.

Our only critical number is $x = 0$, where f' changes from positive to negative. We know it is a local minimum since the derivative to the right is positive, but negative to the left.

We can find concavity with the second derivative of the function.

$$f''(x) = -\frac{60x^2 - 120}{(x^2+6)^3}$$

From this, the curve is concave upward at $(-\sqrt{2}, \sqrt{2})$ and points of concavity are $\pm\sqrt{2}$.

Finally, we can draw the graph of the function.

4. First, we need to find relative distances for swimming and jogging. Let's let $\angle PWE$ be θ. We can create a triangle by connecting points PWE with straight lines. We know from Thale's theorem that an angle inscribed across the diameter of a circle is always $90°$, so this triangle is a right triangle. Then, the swimming distance, line PW, is $3\cos\theta$. The arc length for jogging is θr, and our diameter is 3km, so the jogging distance is $2 \cdot 1.5\theta$. We must multiply by 2 since the angle is not measured at the center of the circle, but rather subtends it.

   We know that critical points are found where the derivative of a function is 0 or undefined. To find the derivative, we can find an equation for the time of the journey, then differentiate.

$$\frac{1}{24} \cdot 3\theta + \frac{1}{3} \cdot 3\cos\theta$$

$$\frac{\theta}{8} + \cos\theta$$

   We can then differentiate and find zeros.

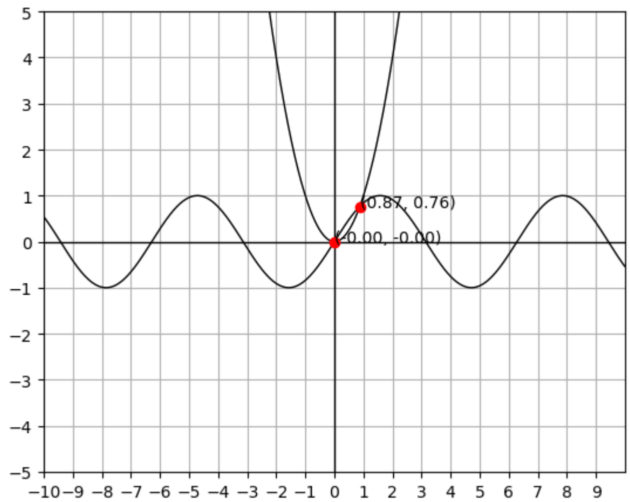
$$0 = \frac{1}{8} - \sin\theta$$

$$\sin\theta = \frac{1}{8}$$

$$\sin^{-1}\left(\frac{1}{8}\right) = \theta$$

   We have $\theta \approx 0.125$. To find the distance to jog, we multiply by 3. The final distance Peter should jog to arrive at point W in the least amount of time is $\boxed{0.376\text{km}}$

5. a) Graph is as follows:



b) We can evaluate the function at various reference angles, since we know that x^2 will always be positive and $\sin x$ has a maximum value of 1.
When $x = 0$, we have $\sin 0 - (0^2) = 0$.
When $x = \frac{\pi}{2}$, we have $\sin\frac{\pi}{2} - (\frac{\pi}{2})^2$, which is positive.
When $x = \frac{\pi}{3}$, we have $\sin\frac{\pi}{3} - (\frac{\pi}{3})^2$, which is negative.
When $x = \frac{\pi}{4}$, we have $\sin\frac{\pi}{4} - (\frac{\pi}{4})^2$, which is positive.
Then, we know that the interval in which x is nonzero is $[\frac{\pi}{4}, \frac{\pi}{3}]$

c) In order to use Newton's method, we must have the function in the form $f(x) = 0$, so our function is $\sin x - x^2 = 0$.
We can now write down the general formula for Newton's method.

$$x_{n+1} = x_n - \frac{\sin x_n - x_n^2}{\cos x_n - 2x_n}$$

Next, we can get our first approximation by substituting our lower bound, $\frac{\pi}{3}$.

$$x_1 = \frac{\pi}{3} - \frac{\sin(\frac{\pi}{3}) - \left(\frac{\pi}{3}\right)^2}{\cos(\frac{\pi}{3}) - 2\left(\frac{\pi}{3}\right)} \approx 0.902567586$$

We must do this again, but substituting x_n with 0.902567586.

$$x_2 = 0.902567586 - \frac{\sin 0.902567586 - (0.902567586)^2}{\cos 0.902567586 - 2\,(0.902567586)} \approx 0.8775090289$$

Again, we can substitute x_n with 0.8775090289.

$$x_3 = 0.8775090289 - \frac{\sin 0.8775090289 - (0.8775090289)^2}{\cos 0.8775090289 - 2\,(0.8775090289)} \approx 0.8767269756$$

After one more repetition, we get $x_4 = 0.876726215396$, which has 6 digits consistent with x_3. Therefore, x has a nonzero solution of $\boxed{x = 0.876726}$

6. We can begin by finding the integral of the first function to see if the ant walks 4cm in the first second.

$$\int_0^1 (5t)dt = \frac{5t^2}{2} = \frac{5}{2}$$

The ant can only walk 2.5cm in the first second, therefore, we need to find the integral of the second function.

$$\int_1^t (6\sqrt{t} - \frac{1}{t})dt = 4t^{\frac{3}{2}} - \ln t$$

We can then add our first integral to the second integral and equate it to 4cm.

$$4 = \frac{5}{2} + 4t^{\frac{3}{2}} - \ln t$$

We have $t \approx 0.266$ for our second function. Adding 1 second from our first function, the ant travelled for $\boxed{1.266s}$.

7. a) We can use u substitution. First, we substitute $u = x^2 - 6x + 1$, the polynomial under the square root.

$$u' = 2x - 6dx$$

$$\frac{u'}{2} = x - 3dx$$

Conveniently, we see that the numerator has a factor of this.

$$3\int \frac{\frac{u'}{2}}{\sqrt{u}}$$
$$= 3u^{0.5} + c$$
$$= \boxed{3\sqrt{(x^2 - 6x + 1)} + C}$$

b) Again, we can use u substitution, this time, we can substitute $u = 3 - \tan\theta$, the numerator of the integral.

The derivative is

$$u' = -\sec^2\theta d\theta$$

Simplifying to find $\cos^2\theta$, we have $-u' = \frac{d\theta}{cos^2\theta}$

Our integral becomes $\int u(-du)$.

Again, we can simplify, and our final answer is $\boxed{\frac{-(3 - \tan\theta)^2}{2} + C}$

c) We can begin by expanding the numerator.

$$\int \frac{(2-x+x^2)^2}{\sqrt{x}}$$
$$= \int \frac{x^4-2x^3+5x^2-4x+4}{\sqrt{x}}$$
$$= \int x^{\frac{7}{2}} - 2x^{\frac{5}{2}} + 5x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}$$

From here, we can simply find the anti-derivative with the equation $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

$$\boxed{\frac{2x^{\frac{9}{2}}}{9} - \frac{4x^{\frac{7}{2}}}{7} + 2x^{\frac{5}{2}} - \frac{8x^{\frac{3}{2}}}{3} + 8\sqrt{x} + c}$$

d) We can begin by using trigonometric identities to simplify the integral.

$$\int \frac{1+\cos(6x)}{2} dx$$

Then, we can remove the constant and apply the sum rule

$$\frac{1}{2}\int 1dx + \int \cos(6x)dx$$

Finally, evaluating the integrals, we have

$$\boxed{\frac{x}{2} + \frac{\sin(6x)}{12} + C}$$

8. Mean value theorem states that $f(x)$ is defined and continuous on the interval $[a,b]$ and is differentiable on interval (a,b), then there is at least one number c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
For this question, we can let $f(x) = \cos(x)$. Plugging into our equation, we have

$$f'(c) = \frac{\cos(a)-\cos(b)}{b-a}$$

We can put absolute value to the left-hand side as well as the denominator and numerator.

$$|f'(c)| - \frac{|\cos(a)-\cos(b)|}{|b-a|}$$

We know that $f(c) = \cos(c)$, so $f'(c) = -\sin(c)$, and the domain is $[-1,1]$. Therefore, we have

$$|a-b| \geq |cos\prime(c)||a-b|$$

This is since the largest possible value of of $\cos'(c)$ and $|a-b|$ is 1.

9. a) We can use the intermediate value theorem to find possible intervals of a root. To this, we can evaluate the function at a few points. Looking at the equation, it is not defined when $x < 0$. We can start with points $x = 0, 1, 2$.
When $x = 0$: $f(0) = 3 \cdot 0^3 + \sqrt{0} - 2 = -2$
When $x = 0.25$: $f(0.25) = 3 \cdot 0^3 + \sqrt{0.25} - 2 = -1.3125$
When $x = 1$: $f(1) = 3 \cdot 1^3 + \sqrt{(1)} - 2 = 2$

b) We can use Rolle's theorem and argue by contradiction. It states that for a function with two real roots, a and b, we would have $f(a) = f(b) = 0$.
Since f is a polynomial, it must be differentiable on (a, b) and continuous on $[a, b]$.
Then, we need to find a number c between a and b such that $f'(c) = 0$.
The derivative of our equation is $9x^2 + \frac{1}{2\sqrt{x}}$
Given our bound $(0.25, 1)$, this is impossible since $9x^2 \geq 0$ and $x > 0.25$ so also $\frac{1}{2\sqrt{x}} > 0$.
Thus, we have a contradiction and our function f has exactly one root.

10. We can first split the integral into parts to use the trigonometric identity.

$$\int \cos^2(x)\cos(x)\sin^2(x)dx$$

Next, we will substitute $\cos^2(x) = 1 - \sin^2(x)$

$$\int (1 - \sin^2(x))\cos(x)\sin^2(x)dx$$

We can then use u substitution, with $u = \sin(x)$.

$$\int u^2(1 - u^2)du$$

$$\int u^2 du = \frac{u^3}{3}$$

$$\frac{u^3}{3} - \frac{u^5}{5}$$

Therefore, after substituting $u = \sin(x)$ back in, our solution is

$$\frac{sin^3(x)}{3} - \frac{sin^5(x)}{x} + C$$

11.

12. Let a be a constant in the interval $(-x, x^2)$.

From the fundamental theorem of calculus, we can split the integral into two parts. First, we have

$$\frac{d}{dx}\int_x^a f(t)dt + \int_a^{x^2} f(t)dt$$

$$= \frac{d}{dx} - \int_{-a}^{-x} f(t)dt \int_{a}^{x^2} f(t)dt$$

Let $g(x)$ be the anti-derivative of $\tan(3x)$. Then, $g'(x) = \tan(3x)$.

Then, we can rewrite the integral above as the following.

$$\frac{d}{dx} g(x^2) - g(-x)$$

We can then differentiate this using the chain rule

$$2x \cdot g'(x^2) + g'(-x)$$

Recalling $g' = \tan(x) = f(x)$

We have $2x \cdot f(x^2) + f(-x)$. Substituting $f(x) = \tan(3x)$, we have $\boxed{2x \cdot \tan(3x^2) + \tan(-3x)}$