Limits

Objectives

When you have completed this unit, you should be able to

1. analyse graphs of functions to evaluate limits.
2. apply definitions and known results to evaluate limits.
3. use limits to find vertical and horizontal asymptotes.
4. prove results about limits.

The cornerstone concept of calculus is the "limit." The importance of understanding limits, and of being able to evaluate them, should not be underestimated. You must learn how to apply definitions, theorems, propositions and corollaries[1] to the evaluation of limits. The "guessing" strategy used in the textbook section titled "The Limit of a Function" is not appropriate, as Example 5 on page 52 clearly shows.

In this unit, we begin by presenting the concept of limits visually; that is, you will learn what a limit is by analysing graphs of functions. Next, we consider continuity, a topic that takes us to the application of theorems, propositions and corollaries that allow us to evaluate limits without having to look at the function's graph. We finish with the formal definition of a limit.

Note: We recommend that you make a list of definitions, propositions, theorems and corollaries, because we will refer to them constantly throughout this unit.

Evaluating Limits Visually

Prerequisites

To complete this section, you must be able to write inequalities in interval notation on the real line. See the section titled “Intervals” on pages 337-338 of the textbook.

Keep in mind that, in the ordered real line, the farther a positive number is from 0, the larger it is, and the farther a negative number is from 0, the smaller it is (see Figure 3.1, below). For instance -10^{10} is a very small negative number, smaller than -10^4, on the other hand -10^{-10} is a large negative number, bigger than -10^{-4}. Thus, when we say that a number is very small we do not mean a small positive number such as 10^{-10}, but a small negative number like -10^{10}. [Clearly, -10^{10} is interpreted as $-\left(10^{10}\right)$, not as $(-10)^{10}$.]

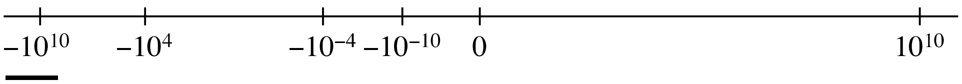

Figure 3.1. Real number line (not to scale)

As we have stated, the graph of a function is its visual representation. When you see a graph, you should get an immediate sense of the information that the graphs is providing. We pay particular attention to the points where the graph “breaks.” We want to understand what happens around the value of the function at the points where such breaks occur. Why does the graph break? What are the values of the function to the left and the right of this particular value? Answering these questions enables us to understand the concept of side limits, and once we know what side limits are, we will be able to understand visually what a limit is.

Given the graph of a function f, we can see whether the function is or is not defined at a number a. The function f is defined at a if the pair (a, b) is on the graph for some number b, and the value of the function f at a is b, that is $f(a) = b$. It may be difficult to know the exact value of $f(a)$ from its graph, but we can *see* its approximate value.

Example 3.1. Examine the graph of Exercise 4 on page 57 of the textbook.

We see that

$$f(0) = 3, \quad f(2) \approx 3 \quad \text{and} \quad f(3) = 3.$$

Example 3.2. Examine the graph of Exercise 5 on page 58 of the textbook.

We see that

$$f(2) = 2, \quad f(0) \approx \frac{5}{2}, \quad f(3) = 2, \quad \text{and} \quad f(5) \;\text{ is undefined.}$$

Also note that

$$f(0.9) \approx 2 \quad \text{and} \quad f(1.9) \approx 2.$$

Example 3.3. Examine the graph of Exercise 7 on page 58 of the textbook.

We see that

$$g(0) = -1, \quad g(2) = 1 \quad \text{and} \quad g(4) = 3.$$

Also note that

$$g(-0.001) \approx -1, \quad g(0.001) \approx -2, \quad g(1.9) \approx 2 \quad \text{and} \quad g(2.01) \approx 0.$$

Example 3.4. Examine the graph of Exercise 8 on page 58 of the textbook.

We see that

$$R(0) = 0 \quad \text{and that} \quad R(-3), R(2) \quad \text{and} \quad R(5) \quad \text{are undefined.}$$

Observe that $R(-2.9)$ is a very large positive number, and that $R(-3.01)$ is a very small negative number.

Similarly, both $R(1.9)$ and $R(2.01)$ are very small negative numbers.

What are the values of $R(4.9)$ and $R(5.1)$?

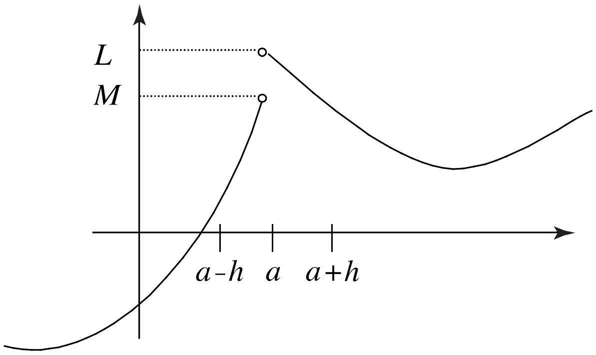

Figure 3.2. Function f, not defined at a

Example 3.5. In Figure 3.2, above, we can see that the function, let us call it f, is not defined at a, but we can also see what is happening with the function at values close to the number a. If h is a small positive number, then $a+h$ is a number on the right of a, so we say that $a+h$ is close to a from the right; and $a-h$ is a number on the left of a, so we say that $a-h$ is close to a from the left. We see from the graph that

$$f(a+h) \approx L \quad \text{and} \quad f(a-h) \approx M.$$

In conclusion, we see that

- for any x close to a from the right, $f(x) \approx L$, and
- for any x close to a from the left, $f(x) \approx M$.

Example 3.6. In Figure 3.3, below, we see that the function, say g, is defined at a and that $g(a) = L$. As in Example 3.5, for any x close to a from the right, $g(x) \approx L$, and for any x close to a from the left, $g(x) \approx M$.

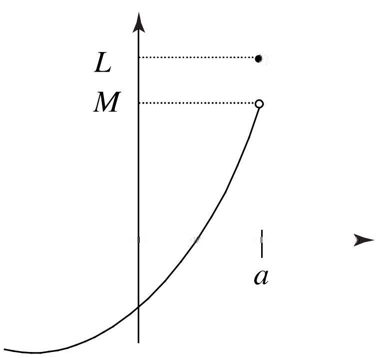

Figure 3.3. Function g, defined at a such that $g(a) = L$

Example 3.7. Figure 3.4 is the graph of a function we will call h. We see that $h(a) = L$, and that, for any x close to a from the right, $h(x) \approx N$; and for any x close to a from the left, $h(x) \approx M$.

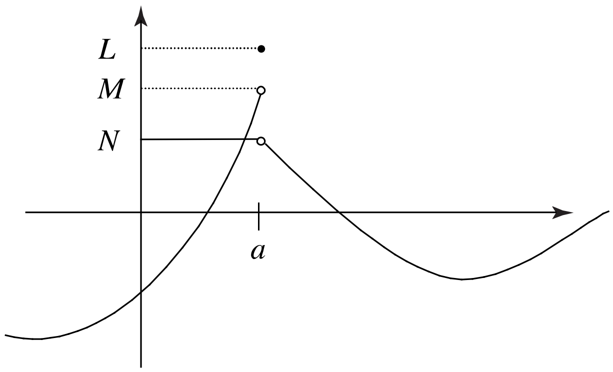

Figure 3.4. Function h, defined at a such that $h(a) = L$

In Examples 3.5 to 3.7, above, we pay attention to the point a because it is where the graph of the function "breaks." The function may or may not be defined at a: what determines the breaking of the graph are the values of the function on the left and right of a. That is, the graph of f breaks at a because

- $f(x) \approx L$ for any x close to a from the right,
- $f(x) \approx M$ for any x is close to a from the left, and
- $L \neq M$.

Example 3.8. In Figure 3.5, below, we see that the graph of the function g breaks, but not because the values of the function from the right and left are different. Rather, it breaks because $g(a)$ is undefined.

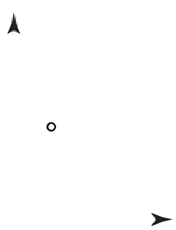

Figure 3.5. Function g, $g(a)$ undefined

In this case, we have

- $g(x) \approx L$ for any x close to a from the right or from the left, and
- $g(a)$ is undefined.

Example 3.9. In Figure 3.6, below, the graph of the function g breaks, even though $g(a) = M$ is defined, and the values of the function from the right and left are equal; that is $g(x) \approx L$ for any x close to a from the right or from the left. It breaks because $L \neq M$.

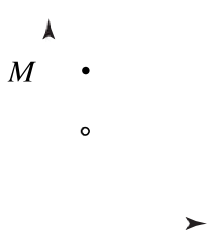

Figure 3.6. Function g, defined at a such that $g(a) = M$; $L \neq M$

All of these examples tell us that it is important to study the values of the function from the left and right at a given point.

Definition 3.1. We say that a function f is *defined at the right* of a, if f is defined on an interval (a, b) for some b; that is,

$f(x)$ is defined for any x such that $a < x < b$ for some b.

Similarly, we say that a function f is *defined at the left* of a, if f is defined on an interval (c, a) for some c; that is,

$f(x)$ is defined for any x such that $c < x < a$ for some c.

A function is *defined around or near a* if it is defined at the right and left of a.

Let us now look at the values of the function in a dynamic way. Calculus is the subject that studies motions and changes. We want to know how the values of the function *change*.

In Figure 3.7, below, we start with an x close to a from the right, and we consider the corresponding point $(x, f(x))$ on the graph. Next we move x towards a, and we continue looking at the corresponding points on the graph, this strategy gives us a succession of points, as indicated.

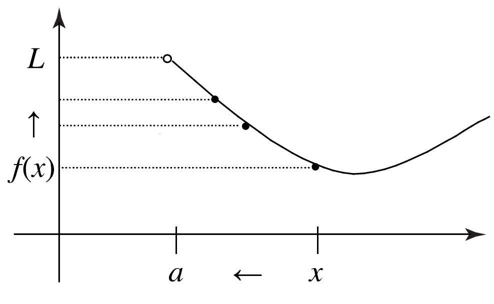

Figure 3.7. Function f, with points x close to a from the right

What happens to the corresponding values of $f(x)$ (on the y-axis) as x moves towards a? They approach L, and the closer x is to a, the closer $f(x)$ is to L.

Another way of looking at this is to recognize that there is always an x that is closer to a from the right, so that $f(x)$ is as close as L as we want.

Therefore, we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficiently close to a from the right, but not equal to a.

Similarly in Figure 3.8, below, we start with an x close to a from the left, and we consider the corresponding point $(x, f(x))$ on the graph. When we move x towards a and look at the corresponding points of the graph, we again obtain a succession of points.

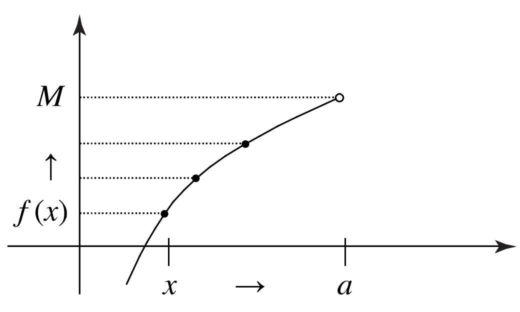

Figure 3.8. Function f, with points x close to a from the left

The values $f(x)$ (on the y-axis) approach M, and the closer x is to a, the closer $f(x)$ is to M. As before, there is always an x closer to a from the left, so that $f(x)$ is as close as M as we want.

Therefore, we can make the values of $f(x)$ arbitrarily close to M (as close as we like) by taking x to be sufficiently close to a from the left, but not equal to a.

We write $x \to a^+$ to indicate that x approaches a from the right, and we write $x \to a^-$ to indicate that x approaches a from the left.

To indicate that the values of the function $f(x)$ approach L, we write $f(x) \to L$.

Hence, “if $x \to a^+$, then $f(x) \to L$” is read in two ways:

“if x approaches a from the right, then the values of the function $f(x)$ approach L,”

or

“$f(x)$ approaches L as x approaches a from the right.”

We have the corresponding notation for the left:

“If $x \to a^-$, then $f(x) \to M$,” is read as

“if x approaches a from the left, then the values of the function $f(x)$ approach M,”

or

“$f(x)$ approaches M as x approaches a from the left.”

The behaviour of the values of the function as they get arbitrarily close to a number is what we call the *limit* of the function.

Definition* 3.2. Let f be a function defined on the right of a.

Then, if we can make the values of $f(x)$ arbitrarily close to L (as close as we like), by taking x to be sufficiently close to a from the right (but not equal to a), we say that

the *limit* of $f(x)$, as x approaches a from the right, is L,

and we write

$$\lim_{x \to a^+} f(x) = L.$$

Another notation for this limit is

$$f(x) \to L \quad \text{as} \quad x \to a^+.$$

Definition* 3.3. Let f be a function defined on the left of a.

Then, if we can make the values of $f(x)$ arbitrarily close to M (as close as we like), by taking x to be sufficiently close to a from the left (but not equal to a), we say that

the *limit* of $f(x)$, as x approaches a from the left, is M,

and we write

$$\lim_{x \to a^-} f(x) = M.$$

Another notation for this limit is

$$f(x) \to M \quad \text{as} \quad x \to a^-.$$

Example 3.10. In Figure 3.2, we see that

$$\lim_{x \to a^+} f(x) = L, \quad \lim_{x \to a^-} f(x) = M, \quad \text{and} \quad f(a) \text{is undefined.}$$

Example 3.11. In Figure 3.3, we see that

$$\lim_{x \to a^+} g(x) = L, \quad \lim_{x \to a^-} g(x) = M, \quad \text{and} \quad g(a) = L.$$

Example 3.12. In Figure 3.4, we see that

$$\lim_{x \to a^+} h(x) = N, \quad \lim_{x \to a^-} h(x) = M, \quad \text{and} \quad h(a) = L.$$

Example 3.13. In Figure 3.5, we see that

$$\lim_{x \to a^+} g(x) = L, \quad \lim_{x \to a^-} g(x) = L, \quad \text{and} \quad g(a) \text{is undefined.}$$

Example 3.14. In Figure 3.6, we see that

$$\lim_{x \to a^+} g(x) = L, \quad \lim_{x \to a^-} g(x) = L, \quad \text{and} \quad g(a) = M.$$

If we analyse the graphs in Figures 3.2 to 3.6, we see that the break on the graph is "bad" in Figures 3.2 and 3.4, since the "gap" is large. However, in Figures 3.6 and 3.7, the break is only one point. The situation is considered to be "better" when the limits from the left and the right are equal. In this case, we say that the limit of the function is the common value of the limits from both sides.

Definition* 3.4. Let f be a function defined around a.

If we can make the values of $f(x)$ arbitrarily close to L (as close as we like), by taking x to be sufficiently close to a from the right and from the left (but not equal to a), we say that

the *limit* of $f(x)$, as x approaches a is L,

and we write

$$\lim_{x \to a} f(x) = L.$$

Definition 3.4 says that

$$\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L.$$

We write $x \to a$ to indicate both $x \to a^+$ and $x \to a^-$.

Hence, another notation for this limit is

$$f(x) \to L \quad \text{as} \quad x \to a.$$

So,

$$f(x) \to L \text{ as } x \to a \quad \text{iff} \quad f(x) \to L \text{ as } x \to a^+ \text{ and } x \to a^-.$$

Definition 3.5. If

$$\lim_{x \to a^+} f(x) = L, \quad \lim_{x \to a^-} f(x) = M, \quad \text{and} \quad L \neq M,$$

then $\lim_{x \to a} f(x)$ does not exist.

Example 3.15. In Figures 3.2 to 3.4, the limits of the functions at a do not exist.

Example 3.16. In Figures 3.5 and 3.6, the limit of the function at a exists, and it is L.

Example 3.17. To evaluate the limit of the function $f(x) = \sqrt{x}$ at 0 from the right, we examine its graph, noting that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Since this function is not defined on the left of 0, we say that

$\lim_{x \to 0^-} \sqrt{x}$ is undefined.

Exercises

1. Do Exercises 4, 5, 6 and 7 on pages 57 and 58 of the textbook.
2. Sketch the graph of *a single* function f that satisfies *all* of the conditions given below.
    a. domain is the interval $(3, \infty)$
    b. $\lim_{x \to 3^+} f(x) = 0$
    c. $\lim_{x \to 5^-} f(x) = 6$
    d. $\lim_{x \to 5^+} f(x) = 6$
    e. $\lim_{x \to 6^-} f(x) = 6$
    f. $\lim_{x \to 6^+} f(x) = -1$

g. $f(5) = 0$

h. $f(6) = -1$

Answers to Exercises (appendix-a.htm)

Let us look at the graph in Figure 3.9, below, in a dynamic way.

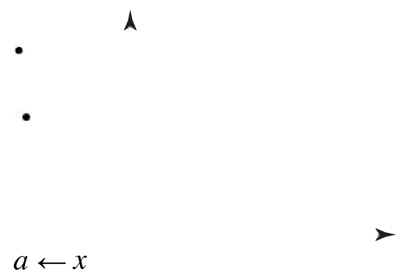

Figure 3.9. Function f, with points x close to a from the right; $f(x)$ increasing

In Figure 3.9, we also start with an x close to a from the right, and we consider the corresponding point $(x,\ f(x))$ on the graph. Next, we move x towards a as we consider the points on the graph, and we obtain a succession of points, as indicated.

The values of $f(x)$ (on the y-axis) increase to a very large number as x gets closer to a. We see that there is always an x closer to a from the right, so that $f(x)$ is as large as we want.

> *Therefore, we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the right, but not equal to a.*

Furthermore, we see that the values $f(x)$ are infinitely large for x very close to a from the right. In this case, we say that "*$f(x)$ increases without bounds*, as $x \to a^+$."

Figure 3.10. Function f, with points x close to a from the right; $f(x)$ decreasing

In Figure 3.10, above, we see that as x approaches a from the right, the values of $f(x)$ (on the y-axis) decrease to a very small negative number. There is always a number x on the right of a, so that $f(x)$ can be as (negatively) small as we want.

> *Therefore, we can make the values of $f(x)$ arbitrarily small negatively (as small as we like) by taking x to be sufficiently close to a from the right, but not equal to a.*

Since the values of $f(x)$ are infinitely small (negatively) for x very close to a from the right, we say that "*$f(x)$ decreases without bounds*, as $x \to a^+$."

We write $f(x) \to \infty$ to indicate that the values of $f(x)$ increase without bounds, and $f(x) \to -\infty$ to indicate that the values of $f(x)$ decrease without bounds.

When the values of the function increase or decrease without bounds, we say that the limit of the function is infinity or negative infinity.

Definition* 3.6. Let f be a function defined on the right of a.

a. If we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the right (but not equal to a), we say that

   the limit of $f(x)$, as x approaches a from the right, is infinity,

   and we write

   $$\lim_{x \to a^+} f(x) = \infty.$$

   Another notation for this limit is

   $$f(x) \to \infty \quad \text{as} \quad x \to a^+.$$

b. If we can make the values of $f(x)$ arbitrarily (negatively) small (as small as we like) by taking x to be sufficiently close to a from the right (but not equal to a), we say that

   the limit of $f(x)$, as x approaches a from the right, is negative infinity,

   and we write

   $$\lim_{x \to a^+} f(x) = -\infty.$$

   Another notation for this limit is

   $$f(x) \to \ -\infty \quad \text{as} \quad x \to a^+.$$

The limits from the left are similar. The graphs of the functions in Figures 3.11 and 3.12, below, show that the limits are infinity and negative infinity as $x \to a^-$.

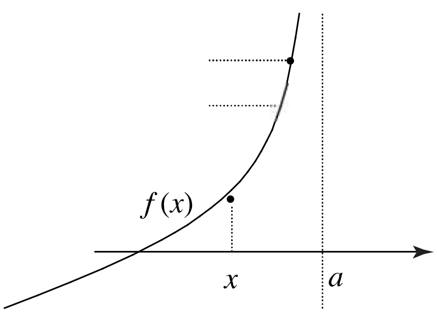

Figure 3.11. Function f, with points x close to a from the left; $f(x)$ increasing

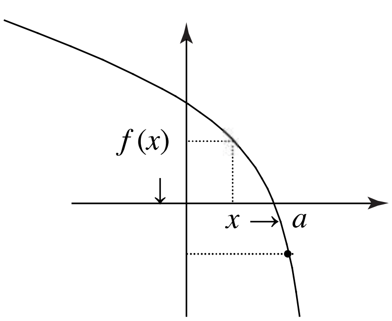

Figure 3.12. Function f, with points x close to a from the left; $f(x)$ decreasing

In Figure 3.11, we see that as x approaches a from the left, the values of $f(x)$ (on the y-axis) increase to a very large number. That is, there is always a number x on the left of a, so that $f(x)$ is as large as we want.

Therefore, we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the left but not equal to a.

In Figure 3.12, we see that as x approaches a from the left, the values of $f(x)$ (on the y-axis) decrease to a very small (negative) number. Hence, there is always a number x on the left of a, so that $f(x)$ is as negatively small as we want.

Therefore, we can make the values of $f(x)$ arbitrarily small negatively (as small as we like) by taking x to be sufficiently close to a from the left, but not equal to a.

Definition* 3.7. Let f be a function defined on the left of a.

a. If we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the left, but not equal to a, we say that

   the limit of $f(x)$, as x approaches a from the left, is infinity,

   and we write

   $$\lim_{x \to a^-} f(x) = \infty.$$

   Another notation for this limit is

   $$f(x) \to \infty \quad \text{as} \quad x \to a^-.$$

b. If we can make the values of $f(x)$ arbitrarily (negatively) small (as small as we like) by taking x to be sufficiently close to a from the left, but not equal to a, we say that

   the limit of $f(x)$, as x approaches a from the left, is negative infinity,

   and we write

   $$\lim_{x \to a^-} f(x) = -\infty.$$

   Another notation for this limit is

   $$f(x) \to -\infty \quad \text{as} \quad x \to a^-.$$

Example 3.18. Looking at the graph of selected trigonometric functions (see Figure 14 on page 365 of the textbook), we see that

a. $\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$

b. $\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$

c. $\lim_{x \to 0^+} \cot x = \infty$

d. $\lim_{x \to \frac{\pi}{2}^+} \sec x = -\infty$

Definition* 3.8. Let f be a function defined around a.

a. We write

   $$\lim_{x \to a} f(x) = \infty,$$

   and say

   the limit of $f(x)$, as x approaches a, is infinity,

   if we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the right and from the left, but not equal to a.

b. We write

   $$\lim_{x \to a} f(x) = -\infty,$$

and say

the limit of $f(x)$, as x approaches a, is negative infinity,

if we can make the values of $f(x)$ arbitrarily (negatively) small (as small as we like) by taking x to be sufficiently close to a from the right and from the left, but not equal to a.

Definition 3.8 state that

$$\begin{array}{lllll} \lim_{x\to a} f(x)=\infty & \text{iff} & \lim_{x\to a^+} f(x)=\infty & \text{and} & \lim_{x\to a^-} f(x)=\infty \\ \lim_{x\to a} f(x)=-\infty & \text{iff} & \lim_{x\to a^+} f(x)=-\infty & \text{and} & \lim_{x\to a^-} f(x)=-\infty. \end{array}$$

Another notation for $\lim_{x\to a} f(x) = \pm\infty$ is $f(x)\to \pm\infty$ as $x\to a$. So,

$$\begin{array}{lllll} f(x)\to\infty \text{ as } x\to a & \text{iff} & f(x)\to\infty \text{ as } x\to a^+ & \text{and} & \text{as } x\to a^- \\ f(x)\to -\infty \text{ as } x\to a & \text{iff} & f(x)\to -\infty \text{ as } x\to a^+ & \text{and} & \text{as } x\to a^-. \end{array}$$

Definition 3.9. The limit $\lim_{x\to a} f(x)$ does *not* exist if and only if

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x).$$

REMARKS 3.1

a. Definition 3.9 includes Definition 3.5 because, by definition, a limit does *not* exist if the limit from the right and the limit from the left are unequal, regardless of whether they are finite or infinite.

b. The limits from the right or from the left at a number a always exist if the function is defined on the right or on the left of a.

c. Although it is customary to say that a limit does not exist if it is not finite, it is not convenient to give the same designation to two completely different facts. The case where a function increases or decreases without bounds, as in Figures 3.11 and 3.12, above, is not the same as the case where the limit from the right is not equal to the limit from the left, as in Figure 3.3. Thus, we must learn to distinguish between these two entirely different situations.

d. To show that a limit does not exist at a number a, it is necessary and sufficient to show that the limits at a, from the right and from the left, are not equal.

Observe, and this is a very important observation, that when we look for (evaluate) the limit of a function at some number a, we do not pay attention to the value of the function at a; instead, we look at the values of the function *around* the number a. Whether the function is or is not defined at a is irrelevant to the process of finding the limit at a.

Exercises

3. Do Exercises 8, 9 and 10 on page 58 of the textbook.
4. Explain why each of the limits listed below does *not* exist.
   a. $\lim\limits_{x \to \frac{\pi}{2}} \tan x$
   b. $\lim\limits_{x \to 0} \cot x$
   c. $\lim\limits_{x \to 0} \csc x$
5. Sketch the graph of a *single* function g that satisfies all of the conditions listed below.
   a. $\lim\limits_{x \to 0} g(x) = \infty$
   b. $\lim\limits_{x \to 3^+} g(x) = \infty$
   c. $\lim\limits_{x \to 3^-} g(x) = -\infty$

Answers to Exercises (appendix-a.htm)

FOOTNOTE

[1] A *theorem* is an proved result with important applications; a *proposition* is also a proved result, but one with applications that are less important than those of a theorem; a *corollary* is a direct consequence of a theorem or a proposition.