Athabasca University

MATH 270

Assignment 2

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June 21, 2020

1. (a) i. We can start by multiplying A^2 then multiplying by A.

$$\begin{aligned}\begin{bmatrix}2 & 0\\4 & 1\end{bmatrix}^3 &= \begin{bmatrix}4 & 0\\12 & 1\end{bmatrix}\cdot\begin{bmatrix}2 & 0\\4 & 1\end{bmatrix}\\ &= \begin{bmatrix}8 & 0\\28 & 1\end{bmatrix}\end{aligned}$$

ii. We can invert the previous answer by using the following equation

$$\begin{aligned}\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} &= \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}\\ \begin{bmatrix}8 & 0\\28 & 1\end{bmatrix}^{-1} &= \frac{1}{8-0}\begin{bmatrix}1 & 0\\-28 & 8\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{8} & 0\\-\frac{7}{2} & 1\end{bmatrix}\end{aligned}$$

iii. We can start by finding A^2.

$$\begin{bmatrix}2 & 0\\4 & 1\end{bmatrix}^2 = \begin{bmatrix}4 & 0\\12 & 1\end{bmatrix}$$

Next, we can subtract $2A$ and add I

$$\begin{aligned}A^2-2A+I &= \begin{bmatrix}4 & 0\\12 & 1\end{bmatrix} - 2\cdot\begin{bmatrix}2 & 0\\4 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\\ &= \begin{bmatrix}-1 & 0\\4 & 0\end{bmatrix}\end{aligned}$$

(b) We can start by defining an arbitrary square matrix A. Let $A = \left[\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right]$. We can then find A^2-9 as well as $A+3$ and $A-3$.We know that in order to add or subtract a vector and a scalar, we need to multiply by the identity matrix. As such, we have

$$\begin{aligned}A^2-9 &= \begin{bmatrix}a & b\\c & d\end{bmatrix}^2 - 9\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\\ &= \begin{bmatrix}a^2+bc-9 & ab+bd\\ac+cd & bc+d^2-9\end{bmatrix} - \begin{bmatrix}9 & 0\\0 & 9\end{bmatrix}\end{aligned}$$

Next, we can do the same to $A+3$ and $A-3$.

$$\begin{aligned}A+3 &= \begin{bmatrix}a & b\\c & d\end{bmatrix} + 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\\ &= \begin{bmatrix}a+3 & b\\c & d+3\end{bmatrix}\\ A-3 &= \begin{bmatrix}a & b\\c & d\end{bmatrix} - 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\\ &= \begin{bmatrix}a-3 & b\\c & d-3\end{bmatrix}\end{aligned}$$

Then, we can multiply $(A+3)(A-3)$ to get A^2-9.

$$\begin{aligned}(A+3)(A-3) &= \begin{bmatrix} a+3 & b \\ c & d+3 \end{bmatrix} \cdot \begin{bmatrix} a-3 & b \\ c & d-3 \end{bmatrix} \\ &= \begin{bmatrix} a^2+bc-9 & ab+bd \\ ac+cd & bc+d^2-9 \end{bmatrix}\end{aligned}$$

Recall that $A^2-9 = \left[\begin{smallmatrix} a^2+bc-9 & ab+bd \\ ac+cd & bc+d^2-9 \end{smallmatrix}\right]$. Thus, we have proved that with an arbitrary matrix A, $A^2 = (A+3)(A-3)$ and that $p(A) = p_1(A)p_2(A)$ for any real values.

2. (a) In order to invert this augmented matrix, we can perform elementary row operations to manipulate it into an identity matrix, then perform those same elementary row operations on the identity matrix. We can start by subtracting R_1 from each other row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 3 & 5 & 7 \end{bmatrix}$$

Next, we can subtract R_2 from R_3 and R_4 and divide R_2 by 3.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 7 \end{bmatrix}$$

We can then subtract R_3 from R_4, divide R_3 by 5, and divide R_4 by 7. With these operations, we have an identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can now perform the elementary row operations above to the identity matrix to find the inverse of our initial matrix. Our first step was to subtract R_1 from all other rows and subtract R_2 from R_3 and R_4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Next, we divided R_2 by 3 and subtracted subtracted R_3 from R_4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Finally, we divided R_3 by 5 and R_4 by 7. Therefore, the inverse of the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & -\frac{1}{7} & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

(b) We can start by manipulating the augmented matrix into reduced row echlon form by elementary row operations. However, since we are working with a variable, c, we must look out for non-permissible values. We can start by dividing R_1 by c, then adding $-1 \times R_1$ to R_2. However, since we divided by c, we need to add 0 to our list of non-permissible values.

$$\begin{aligned}\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix} &= \begin{bmatrix} 1 & \frac{1}{c} & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{c} & 0 \\ 0 & c - \frac{1}{c} & 1 \\ 0 & 1 & c \end{bmatrix}\end{aligned}$$

Next, we can add $-\frac{1}{c}R_3$ to R_1 and $(\frac{1}{c} - c$ to R_2. In addition, we can swap R_2 and R_3.

$$\begin{aligned}\begin{bmatrix} 1 & \frac{1}{c} & 0 \\ 0 & c - \frac{1}{c} & 1 \\ 0 & 1 & c \end{bmatrix} &= \begin{bmatrix} 1 & \frac{1}{c} - \frac{1}{c} & -\frac{1}{c} \cdot c \\ 0 & 1 & c \\ 0 & 0 & 1 - c^2 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & c \\ 0 & 0 & -c^2 + 2 \end{bmatrix}\end{aligned}$$

Finally, we can add $\frac{1}{-c^2+2}R_3$ to R_1 and add $\frac{c}{-c^2+2}R_3$ to R_2. Again, since we are dividing a variable, we need to account for non-permissible values. The solutions to $-c^2+2=0$ are $\pm\sqrt{2}$, so these are our non-permissible values along with 0 from our previous operations.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since we found non-permissible values of $0, \pm\sqrt{2}$ through division of varibles, these are the values of c such that the matrix cannot be inverted. Therefore, the matrix is invertible for all real values $\boxed{c \neq 0, \pm\sqrt{2}}$

(c) We can find C by finding the elementary row operations that manipulate A into B then performing identical operations to an identity matrix. Our first step to manipulate A into B is to add $-2 \times R_1$ to R_3 and R_2

$$\begin{bmatrix} 2 & 1 & 0 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Finally, to manipulate A into B, we can add $-4 \times R_3$ to R_1

$$\begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

We can now perform these operations to the identity matrix. First, we add $-2 \times R_1$ to R_3 and R_2.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Next, we add $-4 \times R_3$ to R_1. This is the value C such that $CA = B$

$$\begin{bmatrix} 9 & 0 & -4 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

3. (a) We can begin by forming an augmented matrix with the provided system of equations.

$$\begin{bmatrix} 0 & -1 & -2 & -3 & 0 \\ 1 & 1 & 4 & 4 & 7 \\ 1 & 3 & 7 & 9 & 4 \\ -1 & -2 & -4 & -6 & 6 \end{bmatrix}$$

While it is impossible to invert a nonsquare matrix, we know that for an invertible $m \times n$ matrix A, each $n \times 1$ matrix b has one solution in the system $Ax = b$, which is $x = A^{-1}b$. As a result, we can separate our 5×4 matrix into three matricies.

$$\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 1 & 4 & 4 \\ 1 & 3 & 7 & 9 \\ -1 & -2 & -4 & -6 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix}$$

We can now invert the 4×4 matrix A by using elementary row operations to reduce it into an identity matrix, then performing the same set of operations on a 4×4 identity matrix. We can start off by adding $-1 \times R_2$ to R_3 and adding R_2 to R_4

$$\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 1 & 4 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & -1 & 0 & -2 \end{bmatrix}$$

Next, we can swap R_1 and R_2 since R_1 has a leading 1. We can also add $-1 \times R_1$ to R_4, $2 \times R_1$ to R_3, and R_1 to R_2. Finally, we can multiply R_1 by -1 to get a leading 1.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Then, we can add $2 \times R_3$ to all other rows and divide R_3 by -1.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Finally, we can add R_4 to R_2 and R_3, and add $-1 \times R_4$ to R_1. To complete our identity matrix, we multiply R_4 by -1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can now reverse all of these steps, but starting with an identity matrix instead of the matrix A. Our first step was to add $-1 \times R_2$ to R_3 and add R_2 to R_4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Our second step was to swap R_1 and R_2 and add $-1 \times R_1$ to R_4, $2 \times R_1$ to R_3, and R_1 to R_2. Finally, we multiplied R_1 by -1.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Our third step was to add $2 \times R_3$ to all other rows and divide R_3 by -1.

$$\begin{bmatrix} 5 & -1 & 2 & 0 \\ 3 & -2 & 2 & 0 \\ -2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 1 \end{bmatrix}$$

Our final step was to add R_4 to R_2 and R_3, add $-1 \times R_4$ to R_1, and multiply R_4 by -1. Therefore, we have

$$A^{-1} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

Plugging back into the equation $x = A^{-1}b$, we have

$$\begin{aligned} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ -3 \cdot 7 + 4 \cdot 4 + 6 \cdot 1 \\ 4 \cdot 1 + 6 \cdot 1 \\ 7 \cdot 1 - 2 \cdot 4 - 6 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ 1 \\ 10 \\ -7 \end{bmatrix} \end{aligned}$$

(b) We can start by making an augmented matrix to represent the systems of equations.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right]$$

Next, we can manipulate this matrix into reduced row echlon and observe any restrictions. We can start by adding $4 \times R_1$ to R_2 and R_3

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & 4b_1 + b_2 \\ 0 & -1 & 0 & 4b_1 + b_3 \end{array}\right]$$

We can then multiply R_3 by -1 and swap R_2 and R_3

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & -3 & -2 & 4b_1 + b_2 \end{array}\right]$$

Next, we add $2 \times R_2$ to R_1 and $3 \times R_2$ to R_3

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -7b_1 - 2b_3 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & 0 & -2 & -8b_1 + b_2 - 3b_3 \end{array}\right]$$

Finally, we divide R_3 by -2 and add to R_1.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3b_1 - \frac{1}{2}b_2 + \frac{1}{2}b_3 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & 0 & 1 & 4b_1 - \frac{1}{2}b_2 + \frac{1}{2}b_3 \end{array}\right]$$

Therefore, the system is consistent for all real values b_1, b_2, and b_3.

4. (a) We can simply multiply each of the diagonal entries together. We will replace unknown values with $-$.

$$\begin{aligned} AB &= \begin{bmatrix} 4 \times 6 & 0 & 0 \\ - & 0 & 0 \\ - & - & 7 \times 6 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 0 & 0 \\ - & 0 & 0 \\ - & - & 42 \end{bmatrix} \end{aligned}$$

(b) Since the matrix is a diagonal matrix, in order to root and invert the matrix, we can simply give every diagonal entry a power of $-\frac{1}{2}$.

$$\begin{aligned} A &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 & 0 \\ 0 & 4^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1^{-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

5. The consumption matrix represents the number of inputs needed to create one unit of product. For example, sector 1 requires 0.5 units from sector 1, sector 2, and sector 3. Sector 2 requires 0.25, 0.125, and 0.25 units from sector 1, 2, and 3 respectively, and sector 3 requires 0.25,

0.25, and 0.125 units of input from the other sectors. The sector that requires the largest amount of inputs from the other sector will have the greatest dollar value. Sector 1 requires the maximum number of units of input from the other sectors, and therefore, Sector 1 has the greatest dollar value.

We know that an economy is productive if $(I-C)$ is invertible and $(I-C)^{-1} \geq 0$. Therefore, we can subtract C from the identity matrix and then invert.

$$
\begin{aligned}
(I-C)^{-1} &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{7}{8} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{7}{8} \end{bmatrix}^{-1}
\end{aligned}
$$

To invert $(I-C)$, we can manipulate it into an identiy matrix and perform the same operations on an identity matrix. We can start by adding R_1 to R_2 and R_3 then multiplying R_1 by 2

$$
\begin{aligned}
\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{7}{8} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{7}{8} \end{bmatrix}^{-1} &= \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{5}{8} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{5}{8} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{4}{5} \\ 0 & -\frac{1}{2} & \frac{5}{8} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \frac{9}{10} \\ 0 & 1 & -\frac{4}{5} \\ 0 & 0 & \frac{9}{40} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Finally, reversing these operations on an identity matrix,

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{14}{5} & \frac{4}{5} & 0 \\ \frac{8}{5} & \frac{8}{5} & 0 \\ \frac{9}{5} & \frac{4}{5} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{14}{5} & \frac{4}{5} & 0 \\ \frac{8}{5} & \frac{8}{5} & 0 \\ \frac{72}{9} & \frac{32}{9} & \frac{40}{9} \end{bmatrix} \\
&= \begin{bmatrix} 10 & 4 & 4 \\ 8 & \frac{40}{9} & \frac{32}{9} \\ 8 & \frac{32}{9} & \frac{40}{9} \end{bmatrix}
\end{aligned}
$$

Since $(I-C)^{-1} > 0$, our open economy is productive.