Athabasca University

MATH 265

Assignment 2

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1. (a) $x = 30\tan(\theta)$
   (b) $\theta \in \mathbb{R}, [0, \frac{\pi}{2})$
   (c) We have $\theta = \frac{\pi}{3}$. Plugging in, we get $x = 30\tan(\frac{\pi}{3}) = \boxed{30\sqrt{3}m}$

2. (a)

x	-2	-1	0	1	2	3	4	5
$f(x)$	0	-0.25	-0.5	-2	-1	0	1	2
$g(x)$	2	2	2	1	0	-1	-2	

   (b) i. $g \circ f(-2) = g(f(-2)) = g(0) = 2$
   ii. $g \circ f(1) = g(f(1)) = g(-2) = 2$
   iii. $g \circ f(4) = g(f(4)) = g(1) = 1$
   iv. $f \circ g(0) = f(g(0)) = f(2) = -1$
   v. $f \circ g(4) = f(g(4)) = f(-2) = 0$
   vi. $f \circ g(-1) = f(g(-1)) = f(2) = -1$

3. (a) Since $\frac{f}{g} = fg^{-1}$, we have

$$\begin{aligned} fg^{-1} &= (\sqrt{1-2x})\left(\frac{x}{x^2-1}\right)^{-1} \\ &= \frac{\sqrt{1-2x}(x^2-1)}{x} \end{aligned}$$

Therefore our domain is $(-\infty, 0) \cup (0, \frac{1}{2}], x \in \mathbb{R}$

   (b)

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= \frac{\sqrt{1-2x}}{(\sqrt{1-2x})^2 - 1} \\ &= \frac{\sqrt{1-2x}}{-2x} \end{aligned}$$

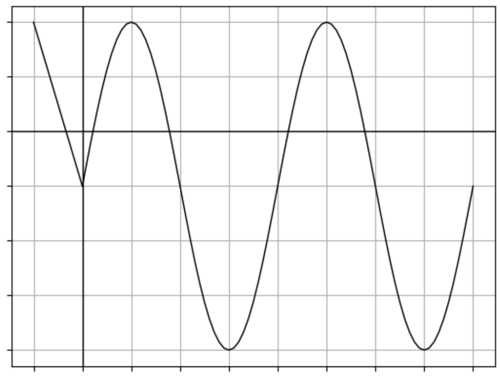
The domain of this function is still $(-\infty, 0) \cup (0, \frac{1}{2}], x \in \mathbb{R}$

   (c)

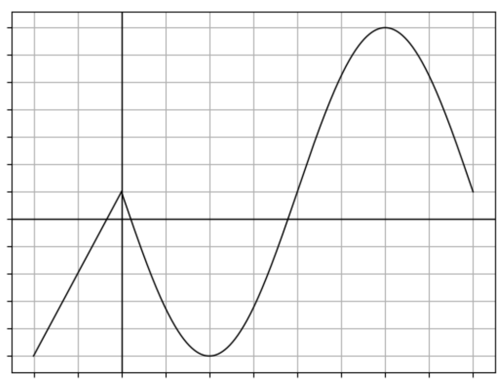
$$\begin{aligned} gf^2 &= g(x) \cdot f(x)^2 \\ &= \left(\frac{x}{x^2-1}\right)(\sqrt{1-2x})^2 \\ &= \frac{x-2x^2}{x^2-1} \\ &= \frac{x-2x^2}{(x-1)(x+1)} \end{aligned}$$

The domain for gf^2 is $(-\infty, -1) \cup (-1, \frac{1}{2}]$

4. (a) We begin our manipulations by horizontally stretching our graph by a factor of $\frac{1}{2}$. Next, we translate our graph down one unit. Our final transformed graph is as follows:



(b) We first reflect our graph accross the x axis, then vertically stretch it by a factor of 2. Next, we translate one unit up. Our final transformed graph is as follows:



5. (a) The limit is well defined. As x approaches 2 frome either the right or left side, y approaches 0.

(b) The limit is not defined. First calculating $\lim_{x\to\infty} \frac{x^2+1}{x+3}$, we can divide each side by the highest denominator power, x. $\frac{x+\frac{1}{x}}{1+\frac{3}{x}}$. The numerator is ∞, and the denominator is 1. Since $\cot(\infty)$ is not defined, our limit is not defined.

6. (a) Step 2 is incorrect. We have $\lim_{x\to-\infty} \frac{\sqrt{x^2-x^3}}{x}$, and we can factor the numerator into $\lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{1-x}}{x}$. Then, simplifying, we have $\lim_{x\to-\infty} -\sqrt{1-x}$.

We know that $\lim_{x\to-\infty} \sqrt{1-x}$ is infinite, and therefore, $\lim_{x\to-\infty} \sqrt{1-x}$ is infinite, therefore, $\lim_{x\to-\infty} \frac{\sqrt{x^2-x^3}}{x} = \boxed{-\infty}$

(b) The fourth step is incorrect. The first two limits are correct, however, a mistake was made for $\lim_{x\to 0} 1+\cos(3x)$. Since $\cos(3x) = 1$, we know that $\lim_{x\to 0} 1+\cos(3x) = 2$. Therefore, the correct answer is $\boxed{-\frac{2}{9}}$

7. (a) We can simply plug in $x = 0$. $\lim_{x\to 0} \frac{\sin(\pi - x)}{\sqrt{x^2 - x + 1}} = \frac{sin(\pi - 0)}{\sqrt{0^2 - 0 + 1}} = \sin\pi = \boxed{0}$

(b) We can begin by simplifying the fraction.

$$\lim_{x\to 3} \frac{x^3 - 3x^2 + 4x - 12}{3x^2 - 7x - 6} = \lim_{x\to 3} \frac{(x-3)(x^2+4)}{(x-3)(3x+2)}$$

Finally, we can plug in $x = 3$ and solve.

$$\lim_{x\to 3} = \frac{x^2+4}{3x+2} = \frac{3^2+4}{3\cdot 3+2} = \boxed{\frac{13}{11}}$$

(c) We can start by expanding, then dividing the fraction by the highest denominator power, which is x^3.

$$\begin{aligned}\lim_{x\to\infty} \cos\left(\frac{(\pi x+1)(3-x^2)}{x^3-\pi}\right) &= \lim_{x\to\infty} \cos\left(\frac{-\pi x^3 - x^2 + 3\pi x + 3}{x^3 - \pi}\right) \\ &= \lim_{x\to\infty} \cos\left(\frac{-\pi - \frac{1}{x} + \frac{3\pi}{x^2} + \frac{3}{x^3}}{1 - \frac{\pi}{x^3}}\right) \\ &= \cos -\pi \\ &= \boxed{-1}\end{aligned}$$

(d) Again, we can find the limit by dividing by the highest denominator power, x^2.

$$\lim_{x\to\infty} \frac{x\sin(3x)}{x^2+1} = \frac{\frac{\sin(3x)}{x}}{1+\frac{1}{x^2}}$$

We can now evaluate the limit of the numerator and denominator separately. The denominator is $\lim_{x\to\infty} 1+\frac{1}{x^2} = 1+0 = 1$. Since $\sin 3x$ is between 1 and -1, and by sandwich theorem, $-\frac{1}{x} \le \frac{\sin 3x}{x} \le \frac{1}{x}$, the numerator is 0. Therefore, we have $\lim_{x\to\infty} \frac{x\sin(3x)}{x^2+1} = \boxed{0}$.

(e) We can begin by rationalizing the numerator.

$$\lim_{x\to 1^+} \sin\left(\frac{\sqrt{x+1}}{x^2-1}\right) = \lim_{x\to 1^+} \sin\left(\frac{1}{(x-1)\sqrt{x+1}}\right)$$

The denominator and numerator are positive and the denominator approaches 0 as x approaches 1 from the right. As such, the fraction approaches ∞. Then, $\lim_{x\to 1^+} \sin\left(\frac{\sqrt{x+1}}{x^2-1}\right)$ is convergent and undefined since sine is a convergent function and $\lim_{\theta\to\infty} \sin\theta$ is undefined.

(f) We can start by dividing the fraction by the highest denominator power, $\sqrt{x+1}$.

$$\lim_{x\to 0^+} \frac{\sqrt{x^3+x^2}}{\sqrt{x+1}-1} = \frac{x}{1-\frac{1}{\sqrt{x+1}}}$$

Next, we can rationalize

$$\lim_{x\to 0^+} \left(\frac{x}{1-\frac{1}{\sqrt{x+1}}}\right)\left(\frac{1+\frac{1}{\sqrt{x+1}}}{1+\frac{1}{\sqrt{x+1}}}\right) = \lim_{x\to 0^+} x + \sqrt{x+1} + 1$$

Finally, we can plug in 0 to $x + \sqrt{x+1} + 1$. $0 + \sqrt{1+0} + 1 = \boxed{2}$

(g) We know that $\cos(2x) = \cos^2(x) - \sin^2(x)$ and that $\cos^2(x) = 1 - \sin^2(x)$. Substituting this into the equation, we have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin^2(3x)}{1-\cos(2x)} &= \lim_{x\to 0} \frac{\sin^2(3x)}{1-(1-2\sin^2(2x))} \\
&= \lim_{x\to 0} \frac{\sin^2(3x)}{2\sin^2 x} \\
&= \frac{1}{2} \lim_{x\to 0} \left(\frac{\sin(3x)}{\sin x}\right)^2
\end{aligned}$$

We know that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, so we will try to manipulate our limit into it. To do this, we divide the numerator and denominator of the fraction by x.

$$\frac{1}{2} \lim_{x\to 0} \left(\frac{\frac{\sin(3x)}{x}}{\frac{\sin(x)}{x}}\right)^2$$

Next, we multiply the numerator by 3 and divide it by 3.

$$\frac{1}{2} \lim_{x\to 0} \left(\frac{3\frac{\sin(3x)}{3x}}{\frac{\sin(x)}{x}}\right)^2$$

We can then move the 3 out of the brackets and find the limit of the numerator and denominator individually.

$$\frac{9}{2} \left(\frac{\lim_{x\to 0} \frac{\sin(3x)}{3x}}{\lim_{x\to 0} \frac{\sin(x)}{x}}\right)^2$$

$$\frac{9}{2} \left(\frac{1}{1}\right)^2 = \boxed{4.5}$$

8. (a) $\lim_{x\to 0^-} f(x)$ approaches from the left side of 0, so therefore, $f(x) = \frac{2}{x}$. Then, we have $\lim_{x\to 0^-} \frac{2}{x}$, which is $\boxed{\infty}$.

(b) $\lim_{x\to 0^+} f(x)$ approaches from the right side of 0, so therefore, we have $\lim_{x\to 0} \sqrt{x} + \cos(\pi x)$. Plugging in $x = 0$, we have $\sqrt{0} + \cos(\pi \cdot 0) = \cos(0) = \boxed{1}$

(c) $\lim_{x\to 5^-} f(x)$ approaches from the left side of 5, so we have $\lim_{x\to 5} \sqrt{x} + \cos(\pi x)$. Simply plugging in $x = 5$, we have $\sqrt{5} + \cos(\pi \cdot 5) = \boxed{\sqrt{5} - 1}$

(d) $\lim_{x\to 5^+} f(x)$ approaches from the right side of 5, so we have $\lim_{x\to 5^+} \frac{x}{x-5}$. To solve this limit, we first divide by the numerator's largest power, x.

$$\lim_{x\to 5^+} \frac{1}{x - \frac{5}{x}}$$

The denominator approaches 0 as x approaches 5, so we know that this limit approaches infinity.

(e) This limit does not exist. Depending on which side the limit is approached from, the limit approaches different values. Specifically, we found $\lim_{x\to 5^+} f(x) = \sqrt{5} - 1$, but $\lim_{x\to 5^-} f(x) = \infty$.

(f) According to the piecewise function, our limit is equivalent to $\lim_{x\to\infty} \frac{x}{x-5}$. To solve this limit, we can divide by the largest denominator fraction, x.

$$\lim_{x\to\infty} \frac{1}{1-\frac{5}{x}}$$

We can now see that the denominator approaches 1 as x approaches infintiy. As a result, our limit is equal to 1.

(g) From the piecewise function, our limit is equivalent to $\lim_{x\to-\infty} \frac{2}{x}$. Since we have infinity as a denominator, our limit aproaches 0.

This function is not continuous at $x = 0, 5$. This can be seen in the limits we calculated in this question, where $\lim_{x\to0^-}$ and $\lim_{x\to0^+}$ as well as $\lim_{x\to5^-}$ and $\lim_{x\to5^+}$ had different values.

9. In order to make a single function that fulfills these conditions, we need to create a piecewise function. We can arrive at a few conclusions from the provided conditions. Firstly, our graph needs to pass through points (5, 6) and (-5, 6). Next, we need a positive asymptote on the left side of 0, and a graph with a negative slope after point (5,6) on the right side.

$$f(x) = \begin{cases} -\frac{1}{x}+5 & -5 < x < 0 \\ \frac{6}{5}x & 0 < x < 5 \\ -x & 5 \leq x \end{cases}$$

The graph of this function which satisfies all of these conditions needs to have a zero, where $f(x) = 0$, since condition f specifies that $\lim_{x\to0} f(x) = 0$, condition c specifies $f(-5) = f(5)$, and condition d specifies $\lim_{x\to5} f(x) = 6$. Finally, condition g specifies that $\lim_{x\to\infty} f(x) = -\infty$. This means that the function must near the origin, have a positive slope going up to point (5,6), then have a negative slope after this point going down to negative infinity. As such, there must be a zero somewhere between (5,6) and negative infinity to fulfill all of the conditions.

10. (a) For $\varepsilon > 0, \delta > 0, 0 < |x-a| < \delta$,

$$|f(x) - L| < \varepsilon$$

$$c|f(x) - L| < c\varepsilon$$

$$|cf(x) - cL| < \varepsilon', \varepsilon' = c\varepsilon > 0$$

$$\lim_{x \to a} f(x) = cL, (= L', c' = cL)$$

Now, $\lim_{x\to\infty} f(x) = -\infty$
$\therefore \lim_{x\to\infty} f(x) = c(-\infty) = -\infty$

(b) For $\varepsilon > 0$, whenever $\delta > 0$, $0 < |x-a| < \delta$, and $|g(x) - L| < \varepsilon$

$$-\varepsilon < g(x) - L < \varepsilon$$

$$g(x) - L < \varepsilon$$

As $x \to A$, $f(x) \geq g(x)$

$$\therefore f(x) - L < \varepsilon', \varepsilon' > \varepsilon > 0$$

We also have $f(x) \geq g(x)$. Since $g(x) - L > -\varepsilon$,

$$f(x) - L \geq g(x) - L > -\varepsilon$$

Since ε', we have $\varepsilon > -\varepsilon'$

$$\therefore f(x) - L > -\varepsilon'$$