Introduction

> Thinking skills and problem-solving activities are indispensable to every area of our lives. To some extent, we are all problem solvers. The problem solver's work is mostly a tangle of guesswork, analogy, wishful thinking, observing patterns, and frustration. To become a master problem solver may be as inaccessible as acquiring the skills of a virtuoso, but everyone can become a better, more confident problem solver.[1]

In this introductory calculus course, you will learn to use functions to solve problems. This course covers approximately 34 sections of the textbook *Single Variable Calculus*, 5th ed, by James Stewart [Scarborough, ON : Nelson, 2002]. These sections are reproduced in a textbook customized for Athabasca University.

We do not cover all of the material presented in the original textbook because, as the author indicates, the textbook "contains more material than can be covered in any one course" (2006, p. 8). Furthermore, we do not follow the author's suggested order; instead, we ask you to use your intuition before we get into formal definitions. For instance, we do not require that you understand the formal definition of limit in order to evaluate limits, we first consider what a limit is in a visual, intuitive way, and once you have learned to apply theorems to evaluate limits, we then study the formal definition. We are very selective about the material we ask you to read and the exercises we recommend that you do from the textbook.

Note: You will learn the material of this course primarily through the *Study Guide*; the textbook will assist your learning. So pay attention and follow the indications as presented. Although you may think that we are jumping all over the textbook, the *Study Guide* will keep you on track, guiding you through trees and leaves so that at the end you can appreciate the forest. In other words, we ask you to put your trust in our approach.

Recommendations for Learning Mathematics

We have some recommendations for you to become a better, more confident problem solver.

- **Be prepared to practise.** To master mathematical concepts, all we need to do is practise, practise and practise again. You must do *at least* all of the exercises presented in this guide, and we also encourage you to try non-assigned exercises from your textbook. A steady pace, practice and patience will take you farther than any short cuts. You may have to change your study habits, because those that were sufficient for success in high school are not good enough for university.

  We provide you with answers to all of the assigned problems in the Appendix of this *Study Guide*, and with hints to the "Learning from Mistakes" exercises of each chapter. Do not look at our answers and hints before making at least one honest try to find the answer (if there is one). In other words, do not give up too easily. The process of learning from an exercise starts with asking the right questions: What do I want to know? What do I have to explain? What do I know about this? What is this exercise saying? What is this exercise testing?

- **Read your *Study Guide* and textbook.** The textbook is not a collection of solved problems; you are expected to understand the course material, and put it to use to solve the problems yourself. Hunting through a section to find a worked-out exercise that is similar to the assigned problem is totally inappropriate. Solving a mathematical problem requires an understanding of the concepts discussed.

  Never read mathematics without pencil and paper in hand. Read the corresponding assigned section before trying the exercises. You should treat the examples as exercises, and try to solve them without the author's help. Be prepared to check all the details as you read, and above all, read critically. Take nothing for granted.

  If you do not understand a statement, go back in the section, or to a previous section, to see if you missed or misunderstood something. If you still fail to understand a concept after puzzling over it for a while, mark the place, continue reading, and ask your tutor for help on that point.

  You will be asked to do exercises as concepts arise, do them at that precise moment, not later: you must make sure you understand what is presented before going on to the next concept.

  You must learn to make reference notes that are useful to you. Some reference tables are provided as examples to show you how information may be displayed for ease of use.

Once again, do not limit yourself to the assigned exercises. Instead, try to do as many exercises as you have time for, do not avoid those that appear to be challenging—the fun in learning is the satisfaction of being able to respond positively to a challenge. The time you spend in reading the *Study Guide* and the textbook will save you time in the long run. You should note any difficulty you have with the homework, and ask your tutor for help.

- **Raise your expectations.** In high school, students are accustomed to finding that the first few weeks of a new course are nothing more than a review of the last course, or even the last few courses. When a new idea is finally presented, the class spends several days working on it before moving ahead. Students quickly learn that there is no major crisis if they do not follow what is said the first time, or if they miss a class. The concept is repeated soon (and sometimes *ad nauseam*). In this course, however, the review of previous material is minimal; new concepts are introduced in the first few pages of Unit 2. Moreover, every subsequent concept assumes mastery of the previous ones, and involves the presentation of new ideas. This is why you must study steadily and constantly throughout each unit. Mathematics cannot be learned on a hurry.

  With the preliminaries presented in Unit 1, we intend to review or complement what you already know of algebra and trigonometry. After this first unit, we assume that you will be capable of working out the basics.

  For example, to find the solutions of the equation $5x^2 + 6 = 7$, you would expect your high school teacher to write

  $5x^2 + 6 = 7$,

  hence, $5x^2 = 7 - 6 = 1$,

  which implies $x^2 = \frac{1}{5}$,

  and we conclude that $x = \pm\sqrt{\frac{1}{5}}$.

  In this course, we would probably say, $5x^2 + 6 = 7$, so $x^2 = 1/5$, and $x = \pm\sqrt{1/5}$. We would give no further details. If you cannot work out the details, review your basics and make sure you understand the algebraic process. Ask your tutor for help if necessary. Do not leave it for another day. Lack of proficiency on the basics is the major cause of students' frustration in mathematics courses.

  Questions are posed throughout the *Study Guide*. These questions are not rhetorical; rather, they are designed to encourage you to think through and reflect on the concepts and processes to arrive at an answer. We expect you to answer the questions in order to get a deeper understanding of the mathematics involved in calculus.

- **Learn to evaluate your own work.** Solutions for all odd-numbered exercises are provided in the *Student Solution Manual*. Solutions of even-numbered exercises are not given, because you are expected to start relying more on your own knowledge and less on external verification to determine whether an answer is correct. The intention is that, eventually, you will be able to correct other people's work, but you must start with your own. To help you to achieve this goal, each unit ends with a section titled "Learning from Mistakes"—a series of questions with common errors that you are asked to identify and correct. Hints are found at the end of this guide. Complete solutions are provided in the *Student Manual*.

- **Ask for help.** You may have the opportunity of consulting other students or knowledgeable professionals, or you may even want to hire a private tutor. Discussing problems and course material with someone can help you to gain insight into difficult concepts. But take warning: others cannot understand the concept for you. It may be tempting to let others do your homework for you, but you will not learn if you do so. You *must* take the time to learn! It is unwise to pretend that you understand when you do not.

- **Use the evaluation mechanisms.** A test is a learning experience, not just a bureaucratic way to quantify your knowledge. Study your self-graded exercises and graded TME s carefully; make sure you understand why you made mistakes, and how you can avoid making them again. The best time to understand missing concepts is right away, when the ideas are still fresh in your mind. In a test or assignment, you must show that you have mastered the concepts and can solve problems. Rote memorization and regurgitation are not what we are looking for. Your grade reflects what you know and how well you know it, as demonstrated by your personal performance, not on the basis of a comparison of your performance with that of others. Do not rely on the good will of the marker to get credit for your work. The marker will grade what is on the paper, not what you might have intended to do.

- **Prepare for the examinations.** Sample examinations are provided to allow you to practise. But we do not want you to have false expectations because you did well in a practice examination. The best preparation for an examination is to try different exercises at random. At the end of each unit, we give you a list of exercises from the “Review” section of each chapter. Use this list to prepare for the examinations. Do the exercises in a random order.

  Making summaries of each unit is another excellent way to prepare yourself for an examination. And *never* cram for a mathematics examination.

Note: All questions in assignments and examinations are worth several points, because several steps are required to solve them. So please *show all your work* and *provide proper justification for all your answers*.

> *In mathematics, it pays to be persistent: people who succeed in mathematics are those who do not give up after the first attempt at solving a problem, but keep asking themselves, “What else can I try?”*

References in This Study Guide

References that provide page numbers may refer to the textbook, Readings from *Stewart: Single Variable Calculus, Fifth Edition* [Scarborough, ON : Thomson Nelson, 2006], or to this *Study Guide*. We have endeavoured to ensure that there is no ambiguity.

Note: This is a digital textbook (eTextbook). If you haven’t already done so, access or download it now through the link on the course home page.

Within the *Study Guide*, examples, theorems, definitions, etc., are numbered; for example, Theorem 3.23. Note that the first number points to the unit where the item is found.

Definitions marked with an asterisk (*) are not formal definitions, but they do indicate how you should think about the concept presented. Some formal definitions are provided in the last section of Unit 3.

Chapter Review

Your textbook provides a “Review” section at the end of each chapter. We recommend that you study the “Concept Check” and the “True-False Quiz” in each of these reviews.

When you come to the review, you should feel that you have learned enough to do the exercises presented. If you need assistance, consult your tutor.

Proofs

A proof is a logical explanation of why something is true. We present some proofs in this course. At a minimum, you should be able to understand a proof’s arguments, and to prove simple results independently. The section titled “Proofs of Theorems,” pages 373-378 of the textbook, provides proofs of several foundational theorems.

Other formal definitions and proofs can be found in advanced calculus textbooks. We recommend:

Apostol, Tom A. *Mathematical Analysis: A Modern Approach to Advanced Calculus*, 2nd ed. New York: Addison Wesley, 1975.

You may also want to look into the websites below.

Wikipedia: The Free Encyclopedia. *Mathematics*. Retrieved November 7, 2007, from: http://en.wikipedia.org/wiki/Mathematics

Wikipedia: The Free Encyclopedia. *Mathematical Proof*. Retrieved November 7, 2007, from: http://en.wikipedia.org/wiki/Mathematical_proof

It is fair to say that students are not penalized if they do not master formal definitions and proofs, but are rewarded if they do.

Technology

In this course, we do not use technological aids in our work. You are not required to have a particular calculator, or to use a computer algebra system or software. The textbook does have exercises that require a graphing calculator or computer—they are indicated with special icons. You are welcome to try them, if you have the appropriate technology and know how to use it. Read the introduction titled “To the Student” on page 7 of the textbook to understand the meaning of the various icons used.

In examinations, you are allowed to use a simple calculator, but *not* a graphing calculator, and *not* a programmable calculator.

NOTE: The PDF of the Study Guide does not include active hyperlinks to the tutorials. The links to all the tutorials for the course have been compiled in their respective order for the Study Guide units in a separate document, and found on the Moodle front page (http://science22.lms.athabascau.ca/course/view.php?id=186) directly under the “MATH 265 Study Guide” PDF link.

FOOTNOTE

[1] Blitzer, Robert. *Introductory Algebra for College Students*, 2nd ed., p. 123. Upper Saddle River, NJ: Prentice Hall, 1998.

Brief Review of Algebra and Trigonometry for Calculus

Objectives

When you have completed this unit, you should be able to

1. perform basic algebraic operations and factorizations.
2. define all trigonometric functions.
3. find exact values of trigonometric functions.

Learning calculus requires a knowledge of basic geometry, algebra and trigonometry. Throughout *Mathematics 265*, you will be informed about the background concepts needed in each section. However, to start the course, you must make sure that you have the minimal basic background to succeed. The time you spend in reviewing the prerequisite material is an investment. In most cases, frustrations and disappointments in learning calculus result from deficiencies in algebra and trigonometry.

In this unit, we help you by briefly reviewing some basic algebra and trigonometry. We present a series of exercises, and recommend that you submit the first assignment before starting Unit 2. The textbook presents some necessary material on geometry, algebra and trigonometry in the “Reference Pages” section (pp. 1-6) and on pages 336-367.

You may wish to consult one or more of the references listed below. The first three are available in the Athabasca University Library. See the “Library Services” section of the *Student Manual* for instructions for making a loan request.

Drooyan, Irving, and William Wooton. *Elementary Algebra with Geometry*, 2nd ed. New York: Wiley, 1984.

Steffensen, A., and M. L. Johnson. *Introductory Algebra*, 2nd ed. Glenview, IL: Scott, Foresman, 1981.

Swokowski, E. W. *A Precalculus Course in Algebra and Trigonometry*. Boston: Prindle, Weber & Schmidt, 1973.

Blitzer, R. F. *Precalculus Essentials*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 2004.

Connally, E., D. Hughes-Hallett, A. M. Gleason, P. Cheifetz, D. E. Flath, P. Frazer Lock, K. Rhea, C. Swenson, F. Avenoso, A. Davidian, B. Lahme, J. Morris, P. Shure, K. Yoshiwara, and E. J. Marks. *Functions Modeling Change: A Preparation for Calculus*, 2nd ed. New York: Wiley, 2004.

Slavin, S., and G. Crisonino, *Precalculus: A Self-teaching Guide*. Mississauga, ON: Wiley Canada, 2004.

In general, any precalculus textbook will do. If you are not sure about the quality of a precalculus textbook, consult your tutor.

You may also wish to consult the websites below.

College-level Geometry. Retrieved November 7, 2007, from website of The Math Forum@Drexel: http://mathforum.org/geometry/coll.geometry.html (http://mathforum.org/geometry/coll.geometry.html)

Kreider, Donald, and Dwight Lahr. *Principles of Calculus Modeling: An Interactive Approach*. Retrieved November 7, 2007, from http://www.math.dartmouth.edu/~calcsite/video1.html (http://www.math.dartmouth.edu/~calcsite/video1.html)

This site presents a series of videos with brief explanation of key concepts.

Stewart Calculus Home Page. Retrieved November 7, 2007, from http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

This website accompanies your textbook. Bookmark the page, and be prepared to refer to it often—it contains many useful links and other features. At this point, pay special attention to the PDF document titled “Review of Algebra,” which is available through the “Algebra Review” tab on the right-hand side of the Stewart Calculus home page.

Minimum Requirements

Before you begin this course, you must, at the least, have a firm grasp of the concepts listed below.

Geometry

- points and lines
- similar triangles
- Pythagoras' theorem
- perimeters, areas and volumes of regular geometric figures

Algebra

- the real line
- arithmetic operations: addition, subtraction, multiplication and division
- addition, subtraction, multiplication and division of algebraic expressions
- factoring
- distributive law, factoring polynomials
- completing a perfect square
- quadratic formula
- exponents
- radicals and rationalization

Trigonometry

- angles in radians
- trigonometric functions
- graphs of trigonometric functions
- common values of trigonometric functions
- trigonometric identities
- laws of sine and cosine

The material presented in this unit is only a brief review of algebra and trigonometry, it is by no means a comprehensive treatment of what is required in this course; in each unit, we continue working on these prerequisites.

Algebra

To keep the topic simple, in the following calculations, we neglect the question of limitations on the values of x for which the statements are valid.

Exponents

Pay attention to how the given expressions are simplified using the laws of exponents.

1. $\dfrac{2x^5+3x^7}{15x^6} = \dfrac{2x^5}{15x^6} + \dfrac{3x^7}{15x^6} = \dfrac{2}{15x} + \dfrac{x}{5}$

2. $\left(a^{-7}b^{-2}c^3\right)^{-2} = a^{14}b^4c^{-6} = \dfrac{a^{14}b^4}{c^6}$

3. $\left(\sqrt{a^3}\sqrt[3]{b^2}\right)^{-1/2} = \dfrac{1}{\sqrt{a^{3/2}b^{2/3}}}$ or

   $\left(\sqrt{a^3}\sqrt[3]{b^2}\right)^{-1/2} = a^{-3/4}b^{-1/3} = \dfrac{1}{a^{3/4}b^{1/3}}$

4. $\dfrac{x^2}{y^3}\left(\sqrt{x}+\sqrt{y^3}\right) = \dfrac{x^{5/2}}{y^3} + \dfrac{x^2}{y^{3/2}}$

Operations of Algebraic Expressions

Study the operations and simplification of the algebraic expressions given below.

1. $(2x+3)(5-2x)+x^2+3x\left(4-\dfrac{1}{x}\right)$

   $= 10x+15-4x^2-6x+x^2+12x-3$

   $= (-4x^2+x^2)+(10x-6x+12x)+(15-3)$

   $= -3x^2+16x+12$

2. $(4x^2+y^2z)\left(\dfrac{x}{z}+3z^2\right)-(x+y)(y^2-xz)$

   $= \dfrac{4x^3}{z}+12x^2z^2+xy^2+3y^2z^3-xy^2+x^2z-y^3+xyz$

   $= \dfrac{4x^3}{z}+12x^2z^2+3y^2z^3-y^3+x^2z+xyz$

3. $3x(4x-z)^2+\dfrac{8x^6y-x^5yz+3x^4yz}{-x^3y}$

   $= 48x^3-24x^2z+3xz^2-8x^3+x^2z-3xz$

   $= 40x^3-23x^2z+3xz^2-3xz$

4. $\dfrac{5}{x+3}+\dfrac{1}{x-9} = \dfrac{5(x-9)+(x+3)}{(x+3)(x-9)} = \dfrac{6x-42}{x^2-6x-27}$

Factorization

We will be factoring algebraic expressions continuously in this course, so pay attention to the following examples.

SIMILAR TERMS

$$\begin{aligned} &x^3 - 5x^2 - 4x + 20 + 10x && \text{do the operations of similar terms} \\ &\quad = (x^3 - 5x^2 + 6x) + 20 && \text{group similar terms} \\ &\quad = x(x^2 - 5x + 6) + 20 && \text{factor the common term with the lowest exponent} \end{aligned}$$

CONJUGATES

The binomial terms $(a + b)$ and $(a - b)$ are conjugates of each other. Observe that their product

$$(a - b)(a + b) = a^2 - b^2$$

is a difference of squares. So, for example,

$$\begin{aligned} 4x^2y^4 - 6z^6 &= \left(2xy^2\right)^2 - \left(\sqrt{6}z^3\right)^2 \\ &= (2xy^2 + \sqrt{6}z^3)(2xy^2 - \sqrt{6}z^3). \end{aligned}$$

QUADRATICS

A quadratic is an expression of the form $ax^2 + bx + c$ with b and c any real numbers and a a nonzero real number.

- A quadratic is a perfect square if $b = \pm 2\sqrt{ac}$. If so, then

  $$ax^2 + bx + c = \left(\sqrt{a}x \pm \sqrt{c}\right)^2.$$

  The quadratic $4x^2 - 20x + 25$ is a perfect square because

  $$a = 4, \quad c = 25 \quad \text{and} \quad b = -2\sqrt{4(25)} = -2\sqrt{100} = -20.$$

  Then,

  $$4x^2 - 20x + 25 = (2x - 5)^2.$$

  Similarly, $4t^2 - 12t + 9$ is a perfect square, and

  $$4t^2 - 12t + 9 = (2t - 3)^2.$$

- If we have a quadratic expression of the form $x^2 + bx + c$, and we can find two numbers k and m such that $c = km$ and $b = k + m$, then

  $$x^2 + bx + c = (x + k)(x + m).$$

  In the quadratic equation $x^2 - 2x - 35$, we see that $(-7)(5) = -35$ and $-7 + 5 = -2$; hence, $x^2 - 2x - 35 = (x - 7)(x + 5)$.

- The solutions of the quadratic equation $ax^2 + bx + c = 0$ is given by the “quadratic formula”:

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

  Using the quadratic formula, we find that the solutions of the equation $32y^2 + 4y - 6 = 0$ are

  $$y = \frac{-4 \pm \sqrt{16 + 768}}{64} = \frac{-4 \pm 28}{64};$$

  that is, $y = -\dfrac{1}{2}$ and $y = \dfrac{3}{8}$. Moreover,

$$\left(y+\frac{1}{2}\right)\left(y-\frac{3}{8}\right)=y^2+\frac{y}{8}-\frac{3}{16}$$

and we see that

$$32\left(y+\frac{1}{2}\right)\left(y-\frac{3}{8}\right)=32y^2+4y-6.$$

Verify the following factorization:

$$32y^2+4y-6=2\left(2\left(y+\frac{1}{2}\right)\right)\left(8\left(y-\frac{3}{8}\right)\right)=2(2y+1)(8y-3).$$

Let us try another example: the solutions of $-8x^2+10x+3=0$ are

$$x=\frac{-10\pm\sqrt{100+96}}{-16}=\frac{-10\pm 14}{-16};$$

hence, $x=\dfrac{3}{2}$ and $x=-\dfrac{1}{4}$.

Since

$$\left(x-\frac{3}{2}\right)\left(x+\frac{1}{4}\right)=x^2-\frac{5x}{4}-\frac{3}{8},$$

we have

$$\begin{aligned}-8x^2+10x+3 &= -8\left(x-\frac{3}{2}\right)-\left(x+\frac{1}{4}\right)\\ &= -\left(2\left(x-\frac{3}{2}\right)4\left(x+\frac{1}{4}\right)\right)\\ &= -(2x-3)(4x+1).\end{aligned}$$

OTHER FACTORIZATIONS

See the sections titled “Factoring Special Polynomials” and “Binomial Theorem” on page 1 of the “Reference Pages” at the beginning of the textbook. To apply these factorizations, identify the corresponding terms of x and y. For example, consider the following difference of cubes.

The expression $\left(8a^3-27z^6\right)$ is a difference of cubes where $x=2a$ and $y=3z^2$; hence,

$$\left(8a^3-27z^6\right)=\left(2a-3z^2\right)\left(4a^2+6az^2+9z^4\right).$$

RATIONAL EXPRESSIONS

To simplify rational expressions, we factor and cancel equal terms.

1. $\dfrac{5x+35}{20x}=\dfrac{5(x+7)}{5(4x)}=\dfrac{x+7}{4x}$
2. $\dfrac{x^3+x^2}{x+1}=\dfrac{x^2(x+1)}{x+1}=x^2$
3. $\dfrac{x^2+6x+5}{x^2-25}=\dfrac{(x+5)(x+1)}{(x+5)(x-5)}=\dfrac{x+1}{x-5}$
4. $\dfrac{x^3+5x^2+6x}{x^2-x-12}=\dfrac{x\left(x^2+5x+6\right)}{(x-4)(x+3)}=\dfrac{x(x+2)(x+3)}{(x-4)(x+3)}=\dfrac{x(x+2)}{x-4}$

We can also simplify rational expressions by long division.

For example, to simplify $\dfrac{7x-9-4x^2+4x^3}{2x-1}$, we divide:

$$2x-1\,\overline{)\,4x^3-4x^2+7x-9}$$

arrange terms in the dividend and divisor in decreasing order of their exponents

$$\begin{array}{r}2x^2\\ 2x-1\,\overline{)\,4x^3-4x^2+7x-9}\end{array}$$

divide $\dfrac{4x^3}{2x}=2x^2$

$$\begin{array}{r}2x^2\\ 2x-1\,\overline{)\,4x^3-4x^2+7x-9}\\ 4x^3-2x^2\end{array}$$

multiply $2x^2(2x-1)=4x^3-2x^2$

$$\begin{array}{r}2x^2\\ 2x-1\,\overline{)\,4x^3-4x^2+7x-9}\\ \underline{-4x^3+2x^2}\\ -2x^2+7x-9\end{array}$$

subtract $4x^3-4x^2-(4x^3-2x^2)$
$=4x^3-4x^2-4x^3+2x^2$
$=-2x^2$;
bring down $7x-9$;
the new dividend is $-2x^2+7x-9$

$$\begin{array}{r}2x^2-x\\ 2x-1\,\overline{)\,4x^3-4x^2+7x-9}\\ \underline{-4x^3+2x^2}\\ -2x^2+7x-9\\ -2x^2+x\end{array}$$

we repeat the process:
divide $\dfrac{-2x^2}{2x}=-x$
then multiply

$$\begin{array}{r}2x^2-x\\ 2x-1\,\overline{)\,4x^3-4x^2+7x-9}\\ \underline{-4x^3+2x^2}\\ -2x^2+7x-9\\ \underline{2x^2-x}\\ 6x-9\end{array}$$

subtract: note that
$-(-2x^2+x)=2x^2-x$;
the new dividend is $6x-9$

$$\begin{array}{r}2x^2-x+3\\ 2x-1\,\overline{)\,4x^3-4x^2+7x-9}\\ \underline{-4x^3+2x^2}\\ -2x^2+7x-9\\ \underline{2x^2-x}\\ 6x-9\\ \underline{-6x+3}\\ -6\end{array}$$

repeating the process

The remainder is -6; hence,

$$\frac{7x-9-4x^2+4x^3}{2x-1}=2x^2-x+3-\frac{6}{2x-1}.$$

Rationalization

To change quotient expressions with square roots, we rationalize; that is, we multiply by the conjugate to change the expression into a different, more manageable one.

For example, to simplify

$$\frac{x^2 - 81}{\sqrt{x} - 3},$$

it is helpful to realize that the conjugate of $\sqrt{x} - 3$ is $\sqrt{x} + 3$, and $\left(\sqrt{x} - 3\right)\left(\sqrt{x} + 3\right) = x - 9$. So,

$$\begin{aligned} \frac{x^2 - 81}{\sqrt{x} - 3} &= \frac{(x-9)(x+9)(\sqrt{x}+3)}{x-9} \\ &= (x+9)(\sqrt{x}+3) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\sqrt{h+1} - 1}{h} &= \frac{(\sqrt{h+1} - 1)(\sqrt{h+1} + 1)}{h(\sqrt{h+1} + 1)} \\ &= \frac{1 + h - 1}{h(\sqrt{1+h} + 1)} \\ &= \frac{1}{\sqrt{1+h} + 1} \end{aligned}$$

Trigonometry

Prerequisites

For this review you must be able to

1. locate points on the Cartesian plane (read "Coordinate Geometry and Lines," pages 344-345 of the textbook).
2. identify similar triangles.

Definitions of Trigonometric Functions

Consider the unit circle (centred at the origin 0 and with radius 1) on the Cartesian plane. Each point P on this circle has coordinates (x_θ, y_θ), where the θ refers to the angle from the point $(1, 0)$ to P. The angle θ is a positive angle if it is measured in the counterclockwise direction; it is negative if it is measured in the clockwise direction, as shown in Figure 1.1, below.

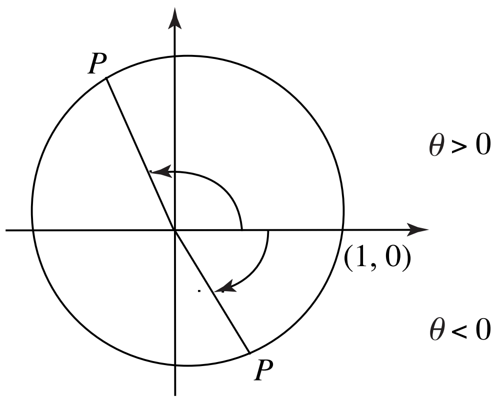


Figure 1.1. Direction of movement for angle θ

For each angle θ, consider the corresponding point $P = (x_\theta, y_\theta)$ on the unit circle, and define:

$\sin\theta = y_\theta$	second coordinate of the point P	$\cos\theta = x_\theta$	first coordinate of the point P
$\tan\theta = \dfrac{y_\theta}{x_\theta}$		$\cot\theta = \dfrac{x_\theta}{y_\theta}$	
$\sec\theta = \dfrac{1}{x_\theta}$		$\csc\theta = \dfrac{1}{y_\theta}$	

These trigonometric functions are defined for angles, not real numbers. In calculus, however, we work with these functions defined on real numbers; that is, in radians. To do so, we define a "winding function"—a correspondence between real numbers and angles—as follows.

Consider again the unit circle with centre 0, and let P_0 be the point $(1, 0)$. For each number a, we associate a point P_a on the unit circle using the rules given below.

- If $a > 0$, then we start at P_0 and move around the circle in the counterclockwise direction, until we have traced out a path whose length is a. The point where our path ends is P_a. See Figure 1.2, below.

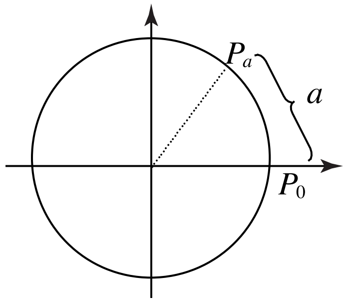


Figure 1.2. $P_a, \quad a > 0$

- If $a < 0$, then we start at P_0 and move around the circle in the clockwise direction, until we have traced out a path whose length is $|a|$. The point where our path ends is P_a. See Figure 1.3, below.

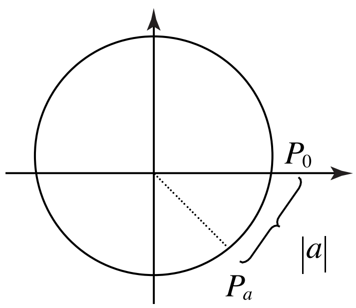


Figure 1.3. $P_a, \quad a < 0$

In either case, we associate the angle $\angle P_0 0 P_a$ to this number a.

REMARKS 1.1

a. The circumference of the unit circle is 2π, so, to this number, we associate the angle 360°. That is, the angle that corresponds to π (radians) is 180°.

b. The numbers a and $a + 2\pi$ correspond to the same angles, since P_a and $P_{a+2\pi}$ coincide on the unit circle. The same is true for $a + 2k\pi$, for any integer k.

c. When we define a trigonometric function on a real number a, we consider this association; that is, $\sin a = \sin \angle P_0 0 P_a$. For example, $\sin \pi = \sin 180^\circ$.

d. We always work with numbers (radians) in this course.

As you can see from this discussion, the coordinates of the point P are $(\cos \theta, \sin \theta)$. Furthermore, since $\theta = \theta + 2k\pi$ for any integer k, to show a trigonometric identity or property for any number θ, it is enough to show it for the corresponding number between 0 and 2π.

Therefore,

- If $\theta = 0$, the coordinates of P are $(1,0)$; hence,

  $$\cos 0 = 1 \quad \text{and} \quad \sin 0 = 0.$$

- If $\theta = \dfrac{\pi}{2}$, the coordinates of P are $(0,1)$; hence,

  $$\cos \frac{\pi}{2} = 0 \quad \text{and} \quad \sin \frac{\pi}{2} = 1.$$

- If $\theta = \pi$, the coordinates of P are $(-1,0)$; hence,

$$\cos \pi = -1 \quad \text{and} \quad \sin \pi = 0.$$

- If $\theta = \frac{3}{2}\pi$, the coordinates of P are $(0, -1)$; hence,

$$\cos \frac{3}{2}\pi = 0 \quad \text{and} \quad \sin \frac{3}{2}\pi = -1.$$

Consider the right-angle (rectangular) triangle T in Figure 1.4, below, with vertices ABC, legs a and b, and hypotenuse $c > 1$. Then, $\theta = BAC$ is the angle at the vertex A.

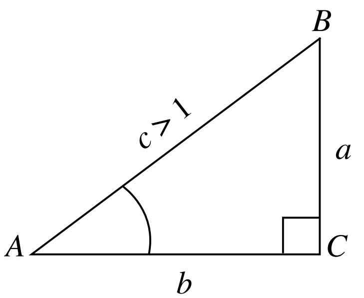


Figure 1.4. Right-angle triangle, T, with a hypotenuse $c > 1$

Next, consider the rectangular triangle R with sides x,y,z, hypotenuse $z = 1$ and the angle at the vertex Y equal to $\theta = XYZ$, as shown in Figure 1.5, below. The coordinates of the vertex Y are $(\cos \theta, \sin \theta)$; that is, $x = \cos \theta$ and $y = \sin \theta$.

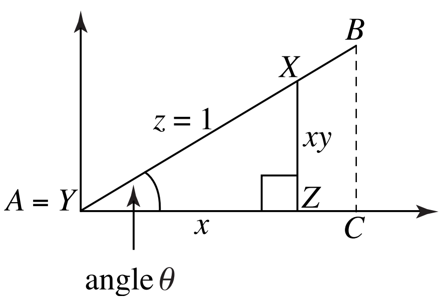


Figure 1.5. Right-angle triangle, R, with a hypotenuse $z = 1$

The triangles T (Figure 1.4) and R (Figure 1.5) are similar. [Why?]

So,

$$\frac{a}{c} = \frac{y}{z} = \frac{\sin \theta}{1} \quad \text{and} \quad \frac{b}{c} = \frac{x}{z} = \frac{\cos \theta}{1}.$$

We conclude that

$$\sin \theta = \frac{a}{c} \quad \text{and} \quad \cos \theta = \frac{b}{c}.$$

Therefore,

$$\tan \theta = \frac{a}{b}; \quad \cot \theta = \frac{b}{a}; \quad \sec \theta = \frac{c}{b}; \quad \csc \theta = \frac{c}{a}.$$

You should convince yourself that the same argument can be used to explain why the values of the trigonometric functions are the same, even if the triangle T has hypotenuse smaller than 1; that is, when the triangle is as shown in Figure 1.6, below.

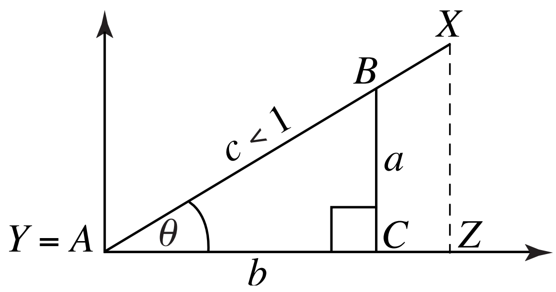


Figure 1.6. Right-angle triangle, T, with a hypotenuse $c < 1$

Exact Values of the Trigonometric Functions

See the table of exact values for sine and cosine on page 358 of the textbook.

The exact values for $\theta = 45^\circ$ can be deduced from the right-angle triangle shown in Figure 1.7, below. The acute angles are 45° each; hence,

$$\sin 45^\circ = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \text{and} \qquad \cos 45^\circ = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

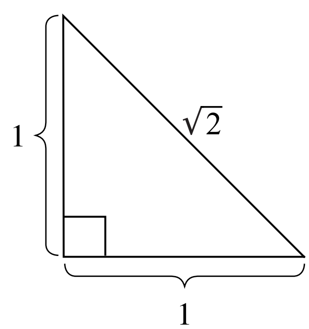


Figure 1.7. Right-angle triangle, with acute angles of 45°

The exact values for 60° and 30° are deduced from the right-angle triangle made up of half of an equilateral triangle of side 2 (see Figure 1.8), below.

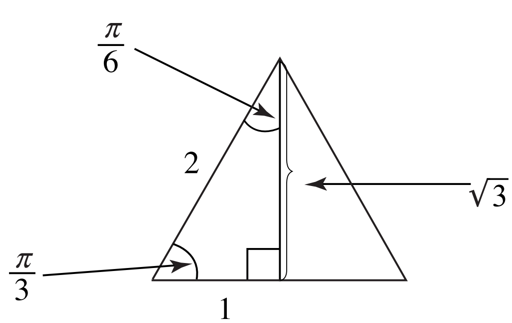


Figure 1.8. Right-angle triangle, half of an equilateral triangle of side 2

The height of this triangle is $\sqrt{3}$ and the acute angles are $30^\circ = \pi / 6$ and $60^\circ = \pi / 3$; hence,

$$\sin 60^\circ = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\sin 30^\circ = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos 30^\circ = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Study Figure 1.9, below.

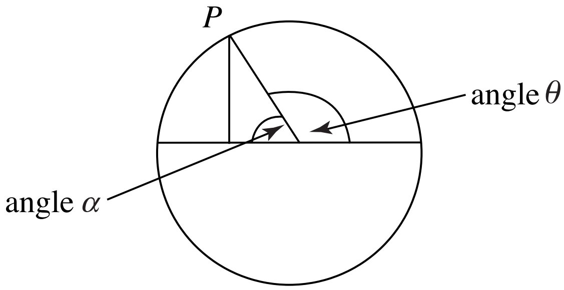


Figure 1.9. Point *P* at $(\cos\theta, \sin\theta)$

If the angle $\theta = \frac{2}{3}\pi$, then $\alpha = \frac{1}{3}\pi$.

The coordinates of P are $(\cos\theta, \sin\theta)$. On the other hand, from the rectangular triangle, we see that the legs of P are $\cos\alpha$ (horizontal) and $\sin\alpha$ (vertical). The first coordinate of P is negative and the second coordinate is positive. Therefore, $\cos\theta = -\cos\alpha$, and $\sin\theta = \sin\alpha$, and

$$\cos\frac{2}{3}\pi = -\frac{1}{2}, \quad \text{and} \quad \sin\frac{2}{3}\pi = \frac{\sqrt{3}}{2}.$$

Exercises

1. Read the section titled "Inequalities," pages 338-340 of the textbook.
2. Do the odd-numbered Exercises from 13 to 23 on page 343.
3. Read the section titled "Trigonometry," pages 358-365 of the textbook.
4. Memorize Identities 7, 8, 10, 12, 15 and 17, on pages 362-363.
5. Do Exercises 9, 11, 19, 21, 29, 31, 59, 61, 65, 67 and 73, on pages 366-367.

Finishing This Unit

1. Review the objectives of this unit and make sure you are able to meet all of them.
2. If there is a concept, definition, example or exercise that is not yet clear to you, go back and reread it, then contact your tutor for help.
3. Definitions are important, for each definition you should be able to give an example (something that satisfies the definition) and a counterexample (something does not satisfy the definition).
4. Do the exercises in "Learning from Mistakes" section for this unit.

Learning from Mistakes

There are mistakes in each of the following solutions. Identify the errors and give the correct answer.

1. If x is any number in the interval $[4, \infty)$, in which interval is $1 - \dfrac{x}{2}$?

   Erroneous Solution

   If x is in $[4, \infty)$, then $4 \leq x$, and $-4 \leq -x$, thus $-2 \leq \dfrac{-x}{2}$.

   Therefore, $-1 \leq 1 - \dfrac{x}{2}$, and we conclude that $1 - \dfrac{x}{2}$ is in the interval $[-1, \infty)$.

2. Simplify the following expressions.

   a. $(a+3b)^2 ab^{-1}$

   b. $\left(x^3 y^{-3}\right)^2$

   c. $\left(ab^2 - bc^3\right)abc$

   Erroneous Solutions

   a. $(a+3b)^2 ab^{-1} = \left(a^2 + 9b^2\right)ab^{-1} = a^3 b^{-1} + 9ab = \dfrac{a^3}{b} + 9ab$

   b. $\left(x^3 y^{-3}\right)^2 = x^5 y^{-1} = \dfrac{x^5}{y}$

   c. $\left(ab^2 - bc^3\right)abc = ab^2 - ab^2c^3$

3. Factor the following expressions.

   a. $2x^2 - 7x - 4$

   b. $3x^5 + 24x^2$

   c. $81x^4 y - y^5$

   Erroneous Solutions

   a. By the quadratic formula,

$$x = \frac{7 \pm \sqrt{49 - 4(2)(-4)}}{4} = \frac{7 \pm 9}{4};$$

hence, the solutions are

$$x = 4 \text{ and } x = \frac{-1}{2},$$

and the factors are

$$2x^2 - 7x - 4 = (x-4)\left(x + \frac{1}{2}\right).$$

b. $$\begin{aligned} 3x^5 + 24x^2 &= 3x^2(x^3+8) \\ &= 3x^2(x+2)(x^2+xy+4) \end{aligned}$$

c. $$\begin{aligned} 81x^4y - y^5 &= (9x^2\sqrt{y} - y^{5/2})(9x^2\sqrt{y} + y^{5/2}) \\ &= \sqrt{y}(9x^2+y^2) \\ &= \sqrt{y}(3x-y)(3x+y)(9x^2+y^2) \end{aligned}$$

4. a. Simplify the expression $\dfrac{x^2+9x+20}{x^2+5x}$.

b. Rationalize the expression $\dfrac{\sqrt{4+h}-2}{h}$.

Erroneous Solutions

a. $$\frac{x^2+9x+20}{x^2+5x} = \frac{9x+20}{5x}$$

b. $$\frac{\sqrt{4+h}-2}{h} = \frac{2+\sqrt{h}-2}{h} = \frac{\sqrt{h}}{h} + \frac{1}{\sqrt{h}}$$

5. Give the exact values of the following trigonometric functions

a. $\sec\left(\dfrac{7\pi}{6}\right)$

b. $\sin\left(\dfrac{7\pi}{12}\right)$

Erroneous Solutions

a. $$\sec\left(\frac{7\pi}{6}\right) = \frac{1}{\cos\left(\frac{7\pi}{6}\right)} = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = 2$$

b. $$\begin{aligned} \sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{5}}{2} \end{aligned}$$

Functions

Objectives

When you have completed this unit, you should be able to

1. distinguish between a relation and a function.
2. apply the definition of a function to practical situations.
3. find the domain of a function.
4. apply basic transformations to sketch graphs of functions.

Relations and Functions

Prerequisites

To complete this section, you must know how to write inequalities in interval notation. See the section titled "Intervals," pages 337-338 of the textbook.

Calculus is the area of mathematics that specializes in solving problems by establishing the function or functions that represent the quantities involved. In this introductory course, you will learn to apply different types of functions to solve problems, and it is our hope that, in the process, you will come to appreciate how powerful the concept of function is for problem solving. But before we discuss functions, we must consider variables and relations.

In mathematics, quantities are called "variables." Quantities that do not change are called constant variables or constants; those that do change are called changing variables or simply variables.

Definition 2.1. A *variable* is a quantity that may or may not change (vary) according to what it represents.

You may think that this definition contains a contradiction, but what we intend is to give ourselves the flexibility of changing a constant into a variable if we need to. In this course, all variables are real numbers.

Example 2.1. Your age is a changing variable. It changes (varies) every year while you age, but it becomes a constant on your death.

Example 2.2. Income is a changing variable (increasing, one hopes). It changes according to cost of living adjustments, promotions, etc.

Example 2.3. The distance of a traveling object from its origin is a changing variable.

Example 2.4. The volume of a wooden box is a constant variable. [Why?]

Example 2.5. The volume of a melting ice cube is a changing variable. [Agree?]

In general, it is easy to decide when a variable is constant or changing. Consider the following exercises.

Exercises

1. Is the number of children in a family a changing or a constant variable?
2. Is the acceleration of Earth's gravity a constant variable?
3. Is the number of days in a year a constant or a changing variable?

Answers to Exercises (appendix-a.htm)

When working with variables, it is important to know which quantities are changing, and which are constant.

When we identify one or more variables, we name them for easy reference; therefore, different variables must have different names. Once a variable is named, we represent it as a symbol; this is the first step in abstract thinking—what mathematicians call "mathematical modeling." The type of symbol is not important, but to facilitate the manipulation of the variables, we tend to give them symbols that are related to what they represent, using as few letters or other characters as possible. Since each person could choose a different symbol for each variable, once we have assigned a symbol, we must let others know what the symbol represents. That is, we assign names and their corresponding symbols to the variables in order to define them.

Always define the variables you are using and give different names to different variables.

It is also convenient to reflect on the possible range of values of the variables we define.

Example 2.6. We could assign the symbols I and $ to the variables that refer to two different types of income. We can define them by stating: "We denote by I the income earned by an employee before the year 2000, and by $ the income earned by the same employee after the year 2000." What are the possible values for I and $? We expect that I and $ are positive nonzero real numbers; that is, $I > 0$ and $\$ > 0$.

Example 2.7. To the variable "age" of a person we could assign the letter A and state: "The age (in years) of a person is denoted by A." The variable A is positive or zero, and we are probably safe to assume that it is less than 120; hence, $0 \leq A < 120$; that is, the *range* of values for A is the interval [0,120).

Example 2.8. The variable "distance" can be represented by s, and we can say: " . . . the distance s of a traveling object . . . . The variable s is positive; that is, $s \geq 0$.

Exercises

4. What symbol would you give to the variable "volume of an ice cube"? How would you define it?
5. How would you define the variable "the body temperature of a living human being"? What is the possible range of values for this variable?

Answers to Exercises (appendix-a.htm)

Having defined the variables, we must try to understand the possible relationships among them.

Example 2.9. The "income" of an employee is related to the "number the employee's working hours." So we have a relation between the changing variables "income" and "number of worked hours."

Example 2.10. The cost of mailing a letter depends on its weight, so we have a relation between the changing variables "weight of a letter" and "cost of postage."

Example 2.11. The "area of any triangle" depends on the "length of its base and its height." So we have a relation between the variables "area of a triangle" and "length of base and height."

Example 2.12. The "distance" of a traveling object depends on the "time" traveled.

Observe that in the examples give above, we have related variables because we found a relation of dependency between them. But we can also relate variables based on other criteria.

Example 2.13. We can relate the "Social Insurance Number" of a Canadian citizen with his or her "age in the year 1998."

Example 2.14. Your "age" can be related to your "weight."

Definition 2.2. A *relation* is an association between two variables or among several variables.

The criterion used to associate the variables must be established—that is, defined—and to establish it, we must introduce a mathematical model.

Example 2.15. In Example 2.9, we have the relation between the variable "income" and the variable "number of worked hours." We start by defining these variables. Let I be the income and w be the number of worked hours. There are two ways to define the relation between these two variables:

- by ordered pairs. The pair (w, I) indicates that I and w are two related variables. To establish the relationship between them, we say that the relation is the set of all pairs (w, I) where I is the income earned for w worked hours, and we write $\{(w, I) \mid I$ is the income earned for w worked hours $\}$. We can also give a name to this relation, say T, and we write

  $$T = \{(w, I) \mid I \text{ is the income for } w \text{ worked hours}\}.$$

  The brackets $\{\}$ are read as "the set of," and the symbol $\mid$ is read as "such that" or "where." So the expression is, "T is the set of all pairs of the form (w, I) such that I is the income for w worked hours."

- by explicit definition of the relation. We decide on the name of the relation first, say T and then we write wTI to indicate that w is related to I by T. We define this relation explicitly, as follows:

  wTI if and only if I is the income earned for w worked hours.

In calculus, we prefer the notation that uses ordered pairs.

Example 2.16. In Example 2.10, we have the variables cost of postage and weight, which we define as P and w, respectively. If we name the relation C, we either write

$$C = \{(w, P)|P \text{ is the cost of postage for a piece of mail of weight } w\}$$

to be read as "C is the set of all pairs (w, P) such that P is the cost of postage of a piece of mail of weight w," or we write wCP iff[1] P is the cost of posting a piece of mail of weight w.

We read "P is related to w by the relation C if and only if P is the cost of posting a piece of mail of weight w."

Example 2.17. In Example 2.11, we define A as the area of a triangle, and b and h as the base and height of the triangle, respectively. We name the relation T, and we define it as

$$T = \{((b, h), A)|A \text{ is the area of a triangle of base } b \text{ and height } h\}$$

or

$(b, h)TA$ iff A is the area of a triangle with base b and height h.

Example 2.18. Refer to Example 2.13. We will let SIN be the social insurance number of a person, and we let A be that person's age in the year 1998. We name the relation K, and we have

$$K = (\text{SIN}, A)|A \text{ is the age, in the year 1998, of the person associated to SIN}$$

or

SINK A iff A is the age in the year 1998 of the person associated to SIN.

Example 2.19. The relation S between a positive integer n and its square root q is written as

$$S = \{(n, q) \mid q \text{ is the square root of the positive integer } n\}.$$

Since we already have a symbol to denote the square root of a positive integer n, we can also write $S = \left\{(n, \sqrt{n}) \mid n \text{ is a positive integer}\right\}$.

Exercises

6. How would you state the relation in Example 2.12?
7. How would you read the relation below?

   $T = \{((b, h), A) \mid A \text{ is the area of a triangle of base } b \text{ and height } h\}$
8. How would you read the relation below?

   $(b, h)\,TA$ iff A is the area of a triangle with base b and height h

9. Define the variables "age" and "weight" and establish their relation as given in Example 2.14.
10. How would you read the relation below?

    $T = \{(A, y) \mid A \text{ is your age in the year } y\}$

Answers to Exercises (appendix-a.htm)

Once a relation is established (defined), we can decide which particular pairs belong to it.

Definition 2.3. We say that a pair (a, b) *belongs* to a relation T if a is related to b by T, and we write $(a, b) \in T$. If a is not related to b by T, then we write $(a, b) \notin T$.

A relation T is *well defined* if we can determine whether any given pair (a, b) belongs to the relation T or not.

From Example 2.15, we can see that if the pair $(12, 500) \in T$, then for 12 hours of work the income is \$500.00, we also see that the pair $(12, 0) \notin T$, since the income cannot be \$0.00 for 12 hours of work. It is also clear that $(-4, 100) \notin T$.

In Example 2.16, we write $(20, 0.50)$ to indicate that, for a piece of mail weighing 20 grams, the cost of postage is \$0.50.

In Example 2.17, we have $((4, 3), 6) \in T$ and $((5, 2), 7) \notin T$.

In Example 2.19, we have $(9, 3) \in S$, $(9, -3) \in S$ and $(9, 4) \notin S$. [Why?]

Observe that the order in which the pair is presented matters, the pairs $(12, 500)$ and $(500, 12)$ in the relation T of Example 2.15 are not the same: as we said $(12, 500)$ means that the income for 12 hours of work is \$500.00, the pair $(500, 12)$ says that for 500 hours of work the income is \$12.00.

Exercises

11. Consider the relation

    $M = \{(s, A) \mid A \text{ is the area of a square of side } s\}.$

    Identify a pair that is in M and a pair that is not in M.
12. Indicate whether the pair $(34, 2000)$ belongs to the relation

    $T = \{(A, y) \mid A \text{ is your age in the year } y\}.$

Answers to Exercises (appendix-a.htm)

We use a special notation when there is a relation of dependency between variables. In Example 2.15, the income I depends on the number of worked hours w, and we express this fact by writing $I(w)$. In Example 2.16, we write $P(w)$ to indicate that the cost P of posting a piece of mail depends on its weight w. In Example 2.17, we are told that $A(b, h)$—the area of a triangle depends on its base b and height h. In Example 2.12, the distance traveled s depends on the time traveled t; hence, $s(t)$. However, in Example 2.18, the variable SIN does not depend on the variable A. This is not a relation of dependency, so it is not correct to write A(SIN).

Definition 2.4. If a variable F depends on the variable m, we write $F(m)$. In this case, we refer to F as the *dependent variable*, and m as the *independent variable*.

For convenience, the relation of dependency takes the name of the dependent variable F. Moreover, the pair (a, b) belongs to the relation F iff $F(a) = b$.

Observe that if the pair (a, b) belongs to the relation F, then a is the independent variable and b is the dependent variable. When we write $F(a) = b$, we indicate two things: F depends on a, and the value F that corresponds to a is b. So we write

$$F = \{(a, b) \mid F(a) = b\}.$$

The relation in Example 2.15 is a relation of dependency; therefore, it is no longer referred to as T; instead, it is called I. Note that $I(12) = 500$ because $(12, 500)$ belongs to the relation. The statement $I(10) = 70$ indicates that for 10 worked hours, the income is $\$70.00$; that is, the pair $(10, 70)$ belongs to the relation I. We also recognize that $(-4, 100)$ does not belong to the relation; hence, $I(-4)$ is undefined.

In Example 2.16, $P(20) = 0.50$; in Example 2.17, $A(3, 4) = 6$; and in Example 2.19, $S(9) = 3$, $S(9) = -3$ and $S(9) \neq 4$.

Exercises

13. Is the relation in Example 2.11 a relation of dependency? How would you write it?
14. If s is the distance traveled (in metres) and t is the time (in seconds), then $s(t)$ is the relation of dependency between s and t. How would you write the statement, "the object travels 50 m in 30 seconds," in symbolic form?
15. Establish the relation of dependency between the following pairs of variables.
    a. temperature T, and wind chill factor c
    b. cost of living L, and taxes t
    c. amount of interest paid r, and principal amount P

Answers to Exercises (appendix-a.htm)

The problems that calculus investigates are those that can be represented by a relation of dependency, where the dependent variable is uniquely determined by the value of the independent variable or variables. We call these special relations "functions." Some functions have to do with variables that change with respect to time, such as distance, velocity, area, volume, population, etc. [Note that, for a given time, there is only one distance, one velocity, one area, one volume or one population.]

In this course, we consider only functions of one independent variable; functions of more than one independent variable, such as that shown in Example 2.17, are studied in more advanced courses.

Definition 2.5. A *function* is a relation of dependency F between two variables, such that if the pairs (a, b) and (a, c) belong to the relation [i.e., if $F(a) = b$ and $F(a) = c$], then $b = c$. That is, the dependent variable F associated to the independent variable a is uniquely determined.

If two pairs (a, b) and (a, c) belong to a relation and $b \neq c$, then the relation is not a function.

Example 2.20. In Example 2.19, the relation S is a relation of dependency, but it is not a function, because $S(9) = 3$ and $S(9) = -3$. That is, it is not a function because both of the pairs $(9, 3)$ and $(9, -3)$ belong to S.

Example 2.21. The relation between the area of a square and the length of its side is a function because each length of a square's side is associated with only one area.

Example 2.22. The GST we pay for a product of cost C is a relation of dependency, GST(C), and it is a function. [Why?]

The relations in Examples 2.15 and 2.16 are functions. [Why?]

Later, we will need to solve problems by establishing the function or functions that represent the problem. The key to doing so is understanding the relation of dependency between the variables of the problem.

We have been paying attention to the range of values of the dependent and independent variables. These ranges of values are important for identifying the variables we are working with. We have special names for them.

Definition 2.6. The *domain* D_F of a function $F(s)$ the largest set of acceptable values of the independent variable s.

The *range* or *rank* R_F of a function F is the largest set of values of the dependent variable F.

Thus a pair (s, F) belong to the function F, if s is in D_F and F is in R_F.

To find the domain of a function, ask yourself, "What are the possible values for the independent variable?"

If we have a function F with independent variable v—that is, $F(v)$—what we try to do next is to find a mathematical expression that gives the value of F for each value of v. This mathematical expression is the mathematical representation (model) of the function F. This model is also referred as a "formula" for F. In this course, the formula we want must involve constants and the variable v only. That is, we want to express F only in terms of the independent variable v.

Example 2.23. If $A(s)$ is the function that gives the area A of a square of side s, then the mathematical representation of A is s^2. [Why?] So, we write $A(s) = s^2$. In this case, A is expressed in terms of s. The domain and range of the function A is the interval $D_A = R_A = [0, \infty)$. [Why?]

The formula $A(s) = s^2$ can be used to solve any problem that involves the area of a square. Observe that $A(10) = 100$, this means that the area of a square of side 10 is 100. We say that "the value of A at 10 is 100," or "the area A is 100 if s is 10."

Example 2.24. If $\mathrm{GST}(C)$ is the function of Example 2.22, above, then $\mathrm{GST}(C) = 0.05C$. [Agree?] What is the formula for the total cost T (including the GST) that we would pay for a product of cost C? That is, what is the mathematical model (formula) in terms of C of T? What is the value of T for an item priced at \$165.45?

Example 2.25. If income paid I is at a rate of \$10.50 per hour, then the mathematical model of the function $I(w)$ of Example 2.15 is $I(w) = 10.50w$. The income for 10 worked hours is $I(10) = 105.00$. Although we can multiply 10.50 by a negative number, it does not make sense to say that the value of I at -2 is -21; that is, it is true that $I(-2) = 10.50(-2) = -21$, but the meanings of I and w are lost here. The domain and range of this function are the interval $D_I = R_I = [0, \infty)$, and we say that we cannot evaluate the function I at negative numbers, not because we cannot multiply 10.50 by a negative number, but because the variable w represents worked hours, and this variable takes only positive values.

We now have the concepts of *mathematical domain*, as defined in Definition 2.6, above, and of *physical domain*, as the set of acceptable values of the independent variable, according to what the independent and dependent variables involved represent.

Example 2.26. If A is the area of an equilateral triangle with side s, then the formula of the function $A(s)$ in terms of s is $(\sqrt{3}s^2)/4$; that is,

$$A(s) = \frac{\sqrt{3}s^2}{4}.$$

To arrive at this relation, we must study the area of equilateral triangles. The area of any triangle is half of the product of the base and the height. The base of the equilateral triangle is s. To find the height h (height is a variable) we use Pythagoras' Theorem.

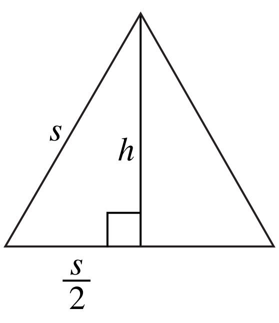


Figure 2.1. Equilateral triangle with side equal to s

According to Pythagoras' Theorem,

$$\left(\frac{s}{2}\right)^2 + h^2 = s^2,$$

so

$$h^2 = s^2 - \frac{s^2}{4} = \frac{3s^2}{4},$$

since h and s are positive. [Why?] Therefore,

$$h = \frac{\sqrt{3}s}{2},$$

and the area is as indicated.

An equilateral triangle with side 56 has an area of

$$A(56) = \frac{56^2\sqrt{3}}{4} \approx 1357.93.$$

In words, the area of an equilateral triangle with side 56 is approximately equal to 1357.93. What are the domain and range of A? What is the value of A at 9? What is the interpretation of $A(4) = 4\sqrt{3}$?

Note: We are ignoring the units for the time being. Be aware, however, that you will be expected to use the correct units in assignments or examinations.

Example 2.27. If the area of a rectangle is 16 m^2, what is the formula for $P(l)$, where P is the perimeter of the rectangle and l is the length of one of its sides? In other words, how can we express P in terms of l?

Step 1

We start with a picture of the problem:

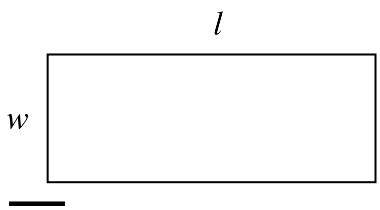


Figure 2.2. Rectangle with length equal to l, and width equal to w

Step 2

We define the variables:

l is the length of the base	P is the perimeter of the rectangle
w is the height of the rectangle	A is the area

Step 3

We relate the variables to what we already know about the perimeter and area of the rectangle; hence, $P = 2l + 2w$ and $A = lw = 16$.

Since we want to find a formula for the perimeter that depends only on l, we must substitute for the variable w an expression with only ls. Hence,

$$w = \frac{16}{l} \qquad \text{and} \qquad P = 2l + 2\frac{16}{l} = 2l + \frac{32}{l},$$

and we conclude that $P(l) = 2l + \dfrac{32}{l}$.

We have found the formula that gives the perimeter P of a rectangle of side l and area 16 m^2. We can find the value of P for any positive value of l—all we have to do is to apply the formula. For example,

$$P(6) = 2(6) + \frac{32}{6} = \frac{52}{3}.$$

We call this process "evaluating P at 6," and the answer is expressed as, "the value of P at 6 is $52/3$."

The domain and range of P are the interval $(0, \infty)$. [Agree?]

What is the perimeter of a rectangle of area 16 m^2 if one of its sides is 5 m?

Exercises

16. A rectangle has perimeter of 20 m. Express the area of the rectangle as a function of the length of one of its sides.
17. Express the surface area of a cube as a function of its volume.
18. An open rectangular box with volume 2 m^2 has a square base. Express the surface area of the box as a function of the length of a side of the base.
19. The cost of renting a car is \$50.00 plus 43 cents per kilometre traveled.
    a. Define the variables that correspond to this problem, and establish the relation of dependency between them.
    b. Find the formula that gives the rental cost in terms of the kilometres traveled.
    c. Give the domain and range of this function.
    d. What is the cost for renting a car in Edmonton to travel to Calgary? Include the appropriate taxes.

Answers to Exercises (appendix-a.htm)

The formulas or mathematical models of functions can take different forms. We cannot always give a single formula for a function. For instance, in Example 2.16, the cost of postage P is fixed depending on a certain range of values of w. That is, P is \$0.50 if $0 < w \leq 30$ and P is \$1.00 if $30 < w \leq 100$. In this case, we write

$$P(w) = \begin{cases} 0.50 & \text{if} \quad 0 < w \leq 30 \\ 1.00 & \text{if} \quad 30 < w \leq 100 \\ 1.70 & \text{if} \quad 100 < w \leq 200 \\ 2.45 & \text{if} \quad 200 < w \leq 500 \end{cases}$$

As you can see, $P(102) = 1.70$ and $P(100) = 1.00$. The range of P, written as a set, is $\{0.50, 1.00, 1.70, 2.45\}$, and the domain of P is the interval $(0, 500]$. See Example 2.10.

We may also be able to give different formulas for different ranges of the independent variable. Consider a function defined as follows:

$$g(s) = \begin{cases} 3s - 1 & \text{if} \quad 0 < s \leq 20 \\ s^2 + 4 & \text{if} \quad 20 < s < 50 \\ 12 & \text{if} \quad 50 < s \leq 60 \end{cases}$$

First, we notice that the values for s are in the intervals $(0, 50)$ and $(50, 60]$, that is $(0, 50) \cup (50, 60]$ is the domain of g. Then 50 and 0 are not in the domain, in other words there is no value for g at 0 or 50. And we say that g is undefined at 0 and 50. Observe also that g is not defined at any number bigger than 60 (e.g., 60.001 or 198), nor is it defined for negative numbers.

James Stewart, the author of your textbook, calls the functions defined in this fashion *piecewise* functions, and we will adopt this name. See Examples 7 and 8 on pages 14 and 15 of the textbook.

Exercises

20. For each of the piecewise functions (i)-(iv), below,
    i. $f(x) = \begin{cases} x & \text{if} \quad x \leq 0 \\ x + 1 & \text{if} \quad x > 0 \end{cases}$
    ii. $f(x) = \begin{cases} 2x + 3 & \text{if} \quad x < -1 \\ 3 - x & \text{if} \quad x \geq -1 \end{cases}$
    iii. $f(x) = \begin{cases} x + 2 & \text{if} \quad x \leq -1 \\ x^2 & \text{if} \quad x > -1 \end{cases}$

iv. $f(x) = \begin{cases} -1 & \text{if } \ x \leq -1 \\ 3x+2 & \text{if } \ |x| < 1 \\ 7-2x & \text{if } \ x \geq 1 \end{cases}$

a. give the domain of the function f.
Hint: For function (iv), see Box 6 on page 341 of the textbook.

b. give the values of $f(1)$ and $f(-1)$.

Answers to Exercises (appendix-a.htm)

Finally, we may not be able to find one given formula for a function, but we may be able to give a table of pairs that belong to the function (see Example 4 on page 12 of the textbook). This is the case when we make a finite number of observations, such as temperature, population, number of consumers of a product, etc.

Example 2.28. The table below shows the population of a certain city recorded every two years for 10 years.

Table showing the population, *P*, of a certain city recorded on a particular year, *Y*

P	***Y***
56000	1990
56800	1992
57000	1994
57500	1996
57850	1998
58000	2000

The variables are the population P and the year Y when the population was recorded. The function is $P(Y)$. This table indicates that the pairs

$$(1990, 56000), (1992, 56800), (1994, 57000), (1996, 57500), (1998, 57850), (2000, 58000)$$

are in the function P. So $P(1990) = 56000$, $P(1992) = 56800$, and so on.

If the data show a certain pattern or uniformity, one would hope to be able to associate a formula to the function. Different techniques are available.

Exercises

21. If a metal rod is being heated, its volume depends on the temperature. If T denotes the temperature (in °C) and V the volume (in m^3), then $V(T)$, and the domain of the function V is all possible values of the temperature T. We have the following recorded volumes:

Table showing how the volume of metal rod, *V*, varies with the temperature, *T*

V	***T***
15	70
18	75
21	80
20	78

18	75
16	72

a. What do you observe about the volume of the rod with respect to the temperature?
b. What is the value of V for $T = 80$?
c. Estimate the volume of the rod when the temperature is 73°C.

Answers to Exercises (appendix-a.htm)

So far we have been working with functions for which the variables have a very specific interpretation; however, to get to the problems we want to solve, we must learn to work with functions in a general sense. That is, we must learn to work with functions and their corresponding formulas when the variables do not have any specific meaning. We must learn about the general properties of functions, and we must learn to do different operations with them. Once we can perform these operations, we will be able to solve problems using functions. This learning approach is the same as the one you used when learning to read: you learned the basic alphabet first.

For example, in general terms, we defined the domain of a function as the set of possible values of the independent variable. In Example 2.23, the variables of the function $A(s)$ have specific meaning. Therefore, the domain of the function $A(s)$ is the interval $[0, \infty)$ because s represents a positive quantity: the length of one side of a square. However, if we ignore the meaning of s as a length, and consider the function $A(s) = s^2$ in general terms, then we see that we can evaluate the function A at any value of s, including negative numbers:

$$A(-3) = (-3)^2 = 9.$$

Hence, the domain of the function $A(s) = s^2$ in general is all of the real numbers.

Working in general terms has its advantages, because different practical problems can be solved with functions that share the same formula.

Example 2.29. If we consider the function

$$T(t) = \frac{t^2 + 5t - 1}{2t - 3},$$

in general terms, then we can evaluate the function T for any value of t except $t = 3 / 2$. For this value of t, the denominator of this expression is zero; that is, the expression is undefined. So, for any value of t other than $t = 3 / 2$, it is possible to apply the formula of T.

You can check that $T(3 / 4) = -2.08333$, and that $T(-5.67) = -0.19518$.

Since $3 / 2$ is the only value for which T is not defined, the domain of T is all real numbers except $3 / 2$. In interval notation, we would write

$$\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right);$$

in set notation,

$$\mathbb{R} - \left\{\frac{3}{2}\right\}.$$

If T represents the temperature of an object at time t, then the possible values of t are all positive real numbers except $t = 3 / 2$. In interval notation, the domain is

$$\left[0, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right).$$

Example 2.30. The domain of the function

$$F(c) = \sqrt{c-6}$$

consists of all numbers such that $c-6 \geq 0$ (since only positive numbers have real square roots). So, the domain of F is all real numbers greater than or equal to 6; that is, $c \geq 6$, or in interval notation, $[6, \infty)$.

We can also write, $F(c) = \sqrt{c-6}$ only for $c \geq 6$.

Example 2.31. What is the domain of

$$H(a) = \frac{a-6}{\sqrt{2a-8}}?$$

To find the domain, we look for those values a for which we may have problems evaluating $H(a)$, either because the denominator is 0 for this particular value, or because the square root is negative. We see that what we need is $2a-8 > 0$, so $a > 4$. So the domain of H is all values a that are strictly greater than 4—in interval notation $(4, \infty)$.

Example 2.32. For the function

$$h(s) = \frac{\sqrt{s-1}}{2\ \cos\ s-1},$$

we see that $s-1 \geq 0$ (the square root must be non-negative), and the divisor must be nonzero.

If $2\ \cos\ s-1=0$ then $\cos\ s = 1\ /\ 2,$ and therefore,

$$s = \frac{\pi}{3} + 2k\pi \text{ and } s = -\frac{\pi}{3} + 2k\pi, \text{ for any integer } k.$$

Since $s \geq 1$, the function is undefined for

$$s = \frac{\pi}{3}, \text{ and for all } s = \frac{(6k \pm 1)\pi}{3}, \text{ for any } k > 0.$$

We conclude that the domain is all numbers $s \geq 1$, except

$$s = \frac{\pi}{3}, \text{ and } s = \frac{(6k \pm 1)\pi}{3}, \text{ for any integer } k > 1.$$

If we have a function with its corresponding formula, we need to understand what we mean by "evaluating the function."

To evaluate the function

$$H(a) = \frac{a-6}{\sqrt{2a-8}}$$

at 9, we replace the independent variable a by 9 in the formula:

$$H(9) = \frac{9-6}{\sqrt{2(9)-8}} = \frac{3}{\sqrt{10}}.$$

We can also evaluate the function at any other expression in the same way. For example, we can evaluate this function at $x+h$ by replacing the independent variable a by $x+h$:

$$\begin{aligned} H(x+h) &= \frac{x+h-6}{\sqrt{2(x+h)-8}} \\ &= \frac{x+h-6}{\sqrt{2x+2h-8}}. \end{aligned}$$

Exercises

22. Find the domain of each of the functions below.

a. $f(x) = \dfrac{x}{3x - 1}$

b. $f(x) = \dfrac{5x + 4}{x^2 - 3x + 2}$

c. $f(t) = \sqrt{t} + \sqrt[3]{t}$

d. $g(u) = \sqrt{u} + \sqrt{4 - u}$

e. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

23. Express the domain of each of the functions below in interval notation.

a. $R(u) = \dfrac{u^2 - 9}{u^2 + 3u}$

b. $F(u) = \dfrac{4}{\sqrt{3u^2 + 4}}$

24. Evaluate the functions in Exercise 23, above, at $3x + 1$.

25. Give an example of a function such that 5 and 2 are *not* in its domain.

Answers to Exercises (appendix-a.htm)

FOOTNOTE

[1] It is customary to write iff instead of “if and only if”.

The Graph of a Function

Prerequisites

To complete this section, you must be able to locate points on the Cartesian plane (read pages 344-345 of the textbook, and do Exercises 1-6, 11, 13 and 15 on page 349).

To aid our understanding of a function, we resort to its graph whenever possible. The graph of a function is its visual representation. We must learn how to obtain information about a function from its graph; that is, *visually*.

As we stated above, a function is a set of pairs

$$\{(a, F(a)) \mid a \text{ is in the domain of } F\}.$$

We can plot the pairs $(a, F(a))$ on the Cartesian plane, with a as the x-coordinate and $F(a)$ as the y-coordinate. If we could do this for every number a in the domain of the function F, we would end up with a set of points on the Cartesian plane—as many points as the number of elements in the domain. Taken together, all of these points make up the graph of the function F.

> **Definition 2.7.** The *graph of a function* F, is the set of all points $(a, F(a))$ on the Cartesian plane, where a is in the domain of F.

You can see that if the domain of F is an infinite set, we cannot possibly plot all the pairs $(a, F(a))$ to obtain the graph of the function F. We will learn later how to overcome this problem. For now, we want to learn to read a function's graph; that is, to obtain information about the function from its graph. Later, we will also learn to graph functions.

If a pair (a, b) is on the graph of a function F, then a is in the domain of F and $F(a) = b$. Observe that the domain of the function F is all numbers a on the x-axis, such that (a, b) is on the graph of F for some b on the y-axis.

Example 2.33. See the graph below.

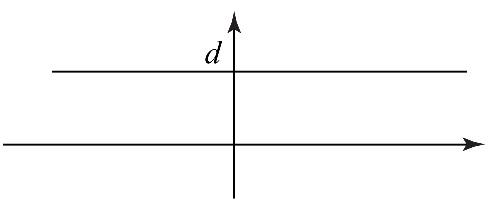


Figure 2.3. Constant function d

Figure 2.3 is the graph of a function, say C, represented by an infinite horizontal line passing through the point $(0, d)$. [Why?] All of the points on this line are of the form (a, d), with d fixed and a any real number. Since any of these pairs is on the graph of the function C, we understand two things: the domain of C is all real numbers, and $C(a) = d$ for any real number a. This function is called the *constant* function d, because the value of the function is d for any real number.

Example 2.34. Compare the graph in Figure 2.3, above, with the following graph:

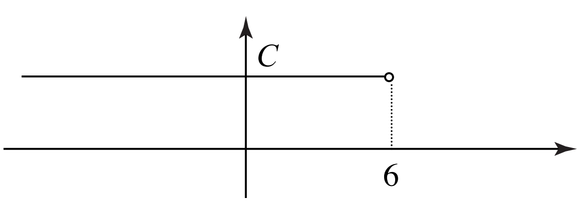


Figure 2.4. Infinite horizontal line with domain $a < 6$

This function, say P, also describes an infinite horizontal line, but the line stops at the point $(6, C)$. Any point on this line is of the form (a, C), but $a < 6$ (the "open" circle indicates that there is no number on the y-axis associated to 6). That is, the pair $(6, b)$ is not in the graph of P for any b, and we say that P is *not defined* at 6. Hence, it is not true that $P(6) = C$. From this graph, we can *see* that the domain of P is all $a < 6$ [in interval notation $(-\infty, 6)$], and that $P(a) = C$ for any $a < 6$.

Hence,

$$P\left(-\frac{3}{4}\right) = C, \ \ P(3) = C, \ \ \text{and} \ \ P(9) \ \ \text{is undefined.}$$

Example 2.35. Let Q be the function whose graph is shown in Figure 2.5, below.

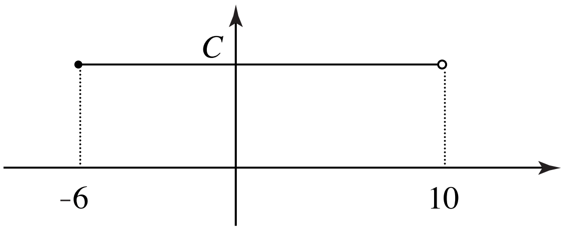


Figure 2.5. Horizontal line with domain $-6 \leq a < 10$

The graph of Q is a horizontal line from the point $(-6, C)$ to the point $(10, C)$. The graph of Q has infinitely many points, and the "closed" circle indicates that the pair $(-6, C)$ is on the graph of Q. As you have probably concluded from the open circle, the pair $(10, C)$ is not on the graph of Q. The pairs on this line are of the form (a, C) with $-6 \leq a < 10$. Hence, the domain of Q is the interval $[-6, 10)$. You can see that $Q(-6) = C$, $Q(0) = C$, and $Q(10)$ is undefined.

Example 2.36. The graph of the function I is as follows:

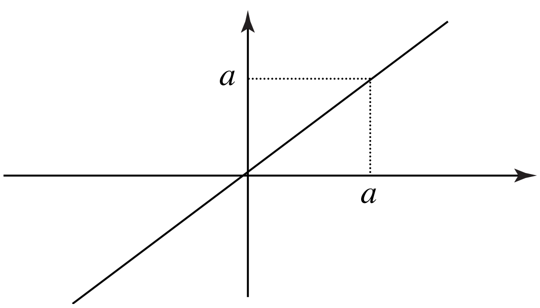


Figure 2.6. Identity function I

This line makes an angle of 45° with the x-axis. Any pair on this line has coordinates (a, a). Hence, the domain is all real numbers and $I(a) = a$ for any real number a. For example, $I(4 / 7) = 4 / 7$, and $I(\pi) = \pi$.

This function is called the *identity* function.

Example 2.37. Let us study the graph of the function f shown in Figure 2.7, below.

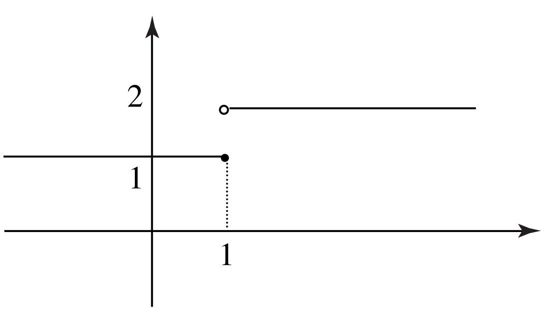


Figure 2.7. Function f, Example 2.37

First, let us see which pairs are on the graph and which are not. Are the pairs $(0, 2)$ and $(1, 2)$ on the graph? What is the domain of the function graphed? What is the value of $f(1)$?

The function has the constant value 2 for $x > 1$, that is $f(x) = 2$ for $x > 1$. The pair $(0, 1)$ is on the graph, but $(0, 2)$ is not; hence, $f(0) = 1$. The pair $(1, 1)$ is on the graph, but $(1, 2)$ is not; hence, $f(1) = 1$. For any x-coordinate there is a y-coordinate, such that (x, y) is on the graph. To see that the domain is all of the real numbers, take any number x and draw a vertical line through the point $(x, 0)$, you see that this line cuts the graph of the function at one point, (x, y). This point of intersection is on the graph, and the value of f at x is y; that is $f(x) = y$. See Figure 2.8, below.

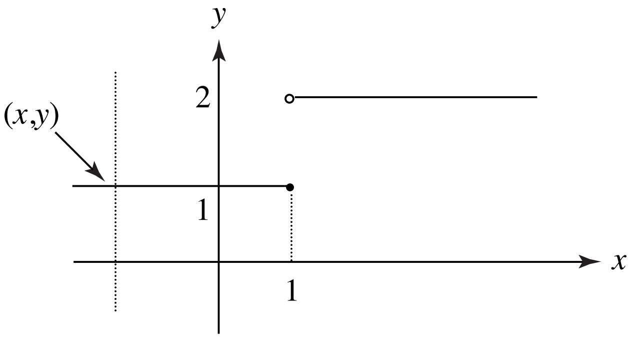


Figure 2.8. Function f, showing intersection with the vertical line through x

Example 2.38. Study the following graph of the function G, and check that the information given is correct.

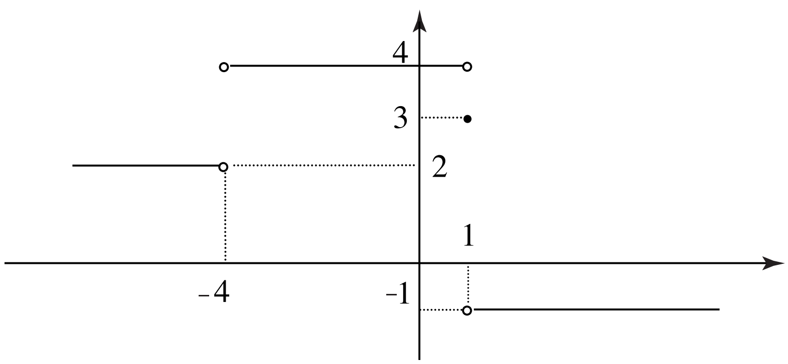


Figure 2.9. Function G, Example 2.38

The pairs $(-4, 2)$, $(-4, 4)$ and $(1, 4)$ are not on the graph, but the pair $(1, 3)$ is on the graph. The domain is all real numbers except -4.

$$G(a) = 2 \text{ for } a < -4, \quad G(a) = 4 \text{ for } -4 < a < 1,$$

$$G(a) = -1 \text{ for } a > 1, \quad \text{and} \quad G(1) = 3.$$

Example 2.39. Consider the following graph of a function f:

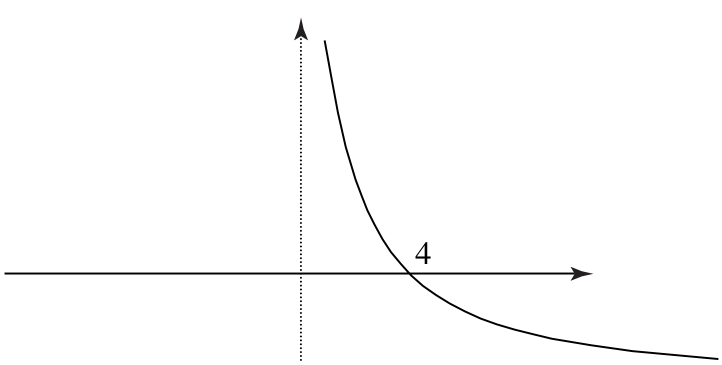


Figure 2.10. Function f, Example 2.39

The domain of the function is all positive numbers; that is the interval $(0, \infty)$, the value of the function at 4 is 0.

Exercises

26. Graph the identity function with the domain being the interval $[5, 9)$.
27. See the graph of Problem 4 on page 57 of the textbook.
    a. What is the domain of the function?
    b. What is $f(3)$?
28. See the graph of the function g in Problem 6 on page 58 of the textbook.
    a. Is 0 in the domain of g?
    b. What is the value of g at -2?
    c. What is the value of g at 2?
    d. What is the domain of g?
29. See the graph R in Problem 8 on page 58 of the textbook. Explain why the function is not defined at -3, 2 and 5.
30. Sketch the graph of one function f that has all of the properties listed below.
    a. domain is the interval $(-4, 12]$
    b. $f(u) = 1$ on $(-4, 0)$
    c. $f(0) = -1$
    d. $f(u) = -3$ on $(0, 12)$
    e. $f(12) = 0$

Answers to Exercises (appendix-a.htm)

In a function, a pair (a, b) is a point on the Cartesian plane. Observe that pairs of the form (a, b) and (a, c) are on the vertical line $x = a$, as shown in Figure 2.11, below.

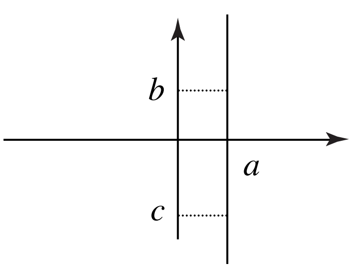


Figure 2.11. Multiple points on the vertical line $x = a$

A function does not include pairs like (a, b) and (a, c), so the graph of a function does not include such pairs either.

> **Definition 2.8.** A curve on a Cartesian plane is the graph of a function only if any vertical line cuts the curve at only one point. This criterion is known as *the vertical line test* (see page 13 of the textbook).

Example 2.40. The following curves are not graphs of functions, because all of them fail the vertical line test.

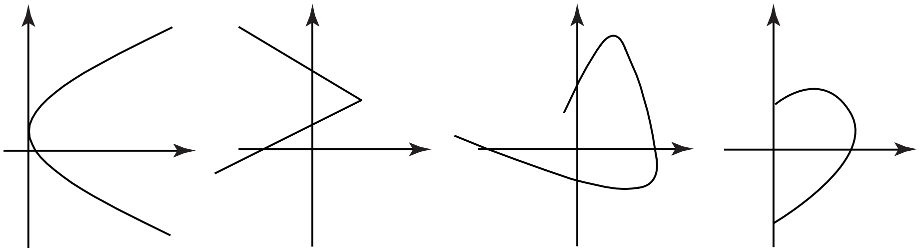


Figure 2.12. Curves that do not meet the vertical line test

Example 2.41. The following curves are graphs of functions. You can see that no vertical line intersects a curve at more than one point.

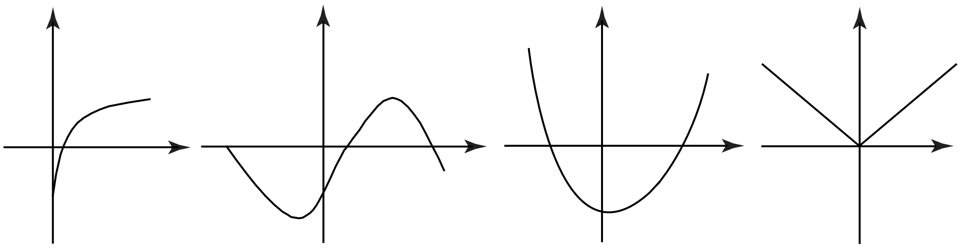
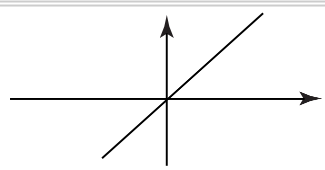
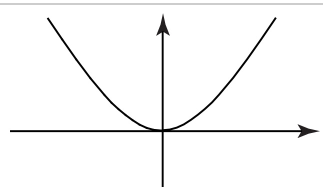
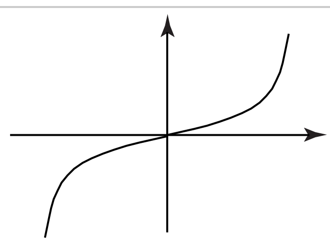
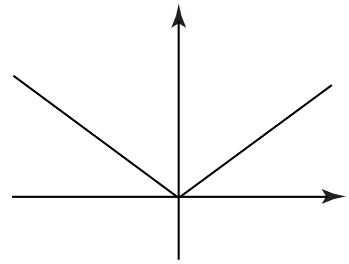
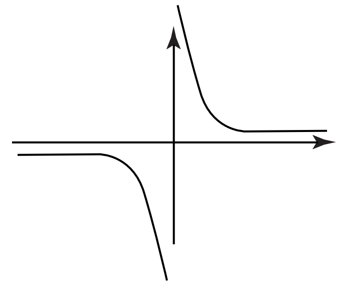
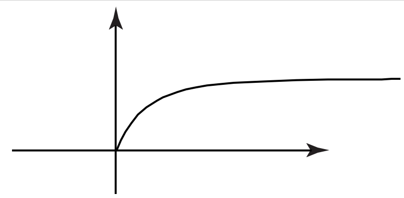


Figure 2.13. Curves that meet the vertical line test

To continue our study of functions, we must memorize a few basic graphs. For now, we will accept them as given; later, we will understand why the graphs are the way they are.

Note: In addition to the graphs shown in Table 2.1, below, consider the graphs of the trigonometric functions (shown on page 2 of the "Reference Pages" at the front of the textbook). Study each of these twelve graphs carefully: look for the domain of each graph, and identify some of its points.

Function	Graph
$F(x) = x$ Identity function	
$F(x) = x^2$ Basic quadratic function †	
$F(x) = x^3$ Basic cubic function	

$F(x) = \lvert x \rvert$ Absolute value function	
$F(x) = \dfrac{1}{x}$	
$F(x) = \sqrt{x}$ Basic square root function	

† We refer to the function $F(x) = x^2$ as the "basic" quadratic function to distinguish it from other quadratic functions, such as $G(x) = x^2 + 4$. We do the same for the cubic and square root functions.

Table 2.1. Graphs of basic functions

Transformations

In this course, we study four basic transformations:

- shifts (upward, downward, to the left or to the right).
- stretches (vertical or horizontal).
- compressions (vertical or horizontal).
- reflections (with respect to the x-axis or the y-axis).

Shifts

Shifts may be vertical or horizontal. Let us consider vertical shifts first.

A vertical shift moves the graph of a function up or down without modifying the graph, as in Figures 2.14 and 2.15, and 2.16 and 2.17, below.

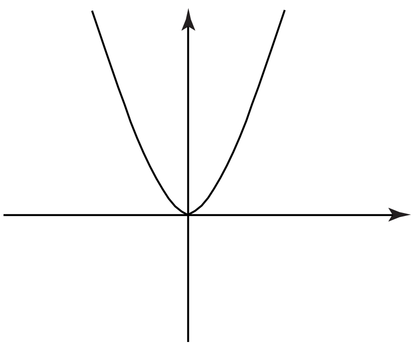


Figure 2.14. Basic graph 1

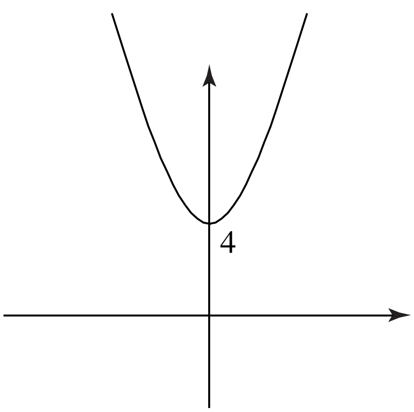


Figure 2.15. Basic graph 1 shifted 4 units upward

The graph in Figure 2.15 is the shift 4 units upward of the graph in Figure 2.14.

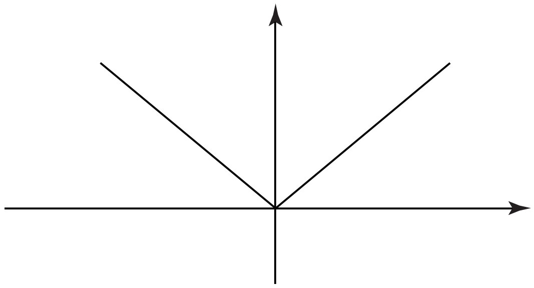


Figure 2.16. Basic graph 2

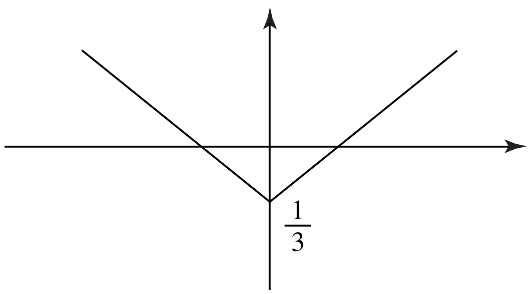


Figure 2.17. Basic graph 2 shifted 1/3 units downward

The graph in Figure 2.17 is the shift $1/3$ units downward of the graph in Figure 2.16.

We can recognize a shift from the formula of a function $f(x)$ by the constant that is added to it (see the definition below).

Definition 2.9. If c is a positive real number and $f(x)$ is a function, then

a. the graph of the function

$$F(x) = f(x) + c$$

is a *shift c units upwards* of the graph of the function $f(x)$.

b. the graph of the function

$$F(x) = f(x) - c$$

is a *shift c units downwards* of the graph of the function $f(x)$.

Example 2.42. The graph of $f(x) = x^2$ is shown in Figure 2.14; the graph in Figure 2.15 is the graph of the function $F(x) = x^2 + 4$, which is the shift 4 units up of the graph of $f(x)$. You can see that $F(x) = f(x) + 4$.

Example 2.43. The graph of $f(x) = |x|$ is shown in Figure 2.16 above. The graph in Figure 2.17 is the graph of the function

$$F(x) = f(x) - \frac{1}{3} = \left|x\right| - \frac{1}{3},$$

which is the shift $1/3$ units downward of the graph of $f(x)$.

Example 2.44. If we analyse the function $F(x) = \cos x + 2$, we see that it is the sum of the cosine function and the constant 2. Hence, the graph of $F(x)$ is the shift 2 units up of the graph of the cosine function (see Figure 2.18).

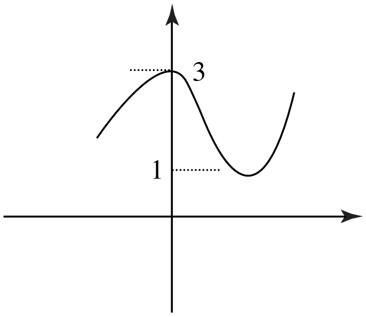


Figure 2.18. Graph of the cosine function shifted 2 units upward

Shifts to the left or right are obtained by adding a constant to the independent variable.

Definition 2.10. If c is a positive real number, and $f(x)$ is a function, then

a. the graph of the function

$$F(x) = f(x + c)$$

is a *shift* c *units to the left* of the graph of the function $f(x)$.

b. the graph of the function

$$F(x) = f(x - c)$$

is a *shift* c *units to the right* of the graph of the function $f(x)$.

Example 2.45. The graph of $f(x) = x^2$ is shown in Figure 2.14, and the graph of the function $F(x) = (x + 4)^2$, which is the shift 4 units to the left of the graph of $f(x)$, is shown in Figure 2.19, below. As you can see, $F(x) = f(x + 4)$; hence, $F(-4) = f(-4 + 4) = f(0) = 0$.

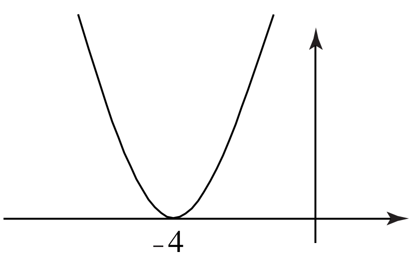


Figure 2.19. Graph of $F(x) = (x + 4)^2$; that is, $f(x) = x^2$ shifted 4 units to the left

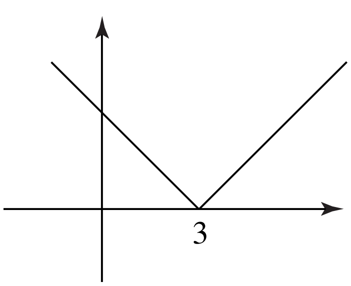


Figure 2.20. Graph of $F(x) = |x - 3|$; that is, $f(x) = |x|$ shifted 3 units to the right

Example 2.46. The graph of $f(x) = |x|$ is shown in Figure 2.16. The graph in Figure 2.20, above, is the graph of the function $F(x) = |x - 3|$, which is the shift to the right of the graph of $f(x)$. We also have $F(x) = f(x - 3)$. As you can see, $F(3) = f(3 - 3) = f(0) = 0$.

Exercises

31. Sketch the graph of the functions listed below.
    a. $F(x) = \sqrt{x} - 6$
    b. $F(x) = \tan x + 2$
    c. $F(x) = \dfrac{1}{x} - 1$
    d. $F(x) = \sqrt{x+5}$

Answers to Exercises (appendix-a.htm)

Stretches and Compressions

As can shifts, stretches and compressions can be vertical or horizontal. Figures 2.21 to 2.24, below, show vertical stretches and compressions.

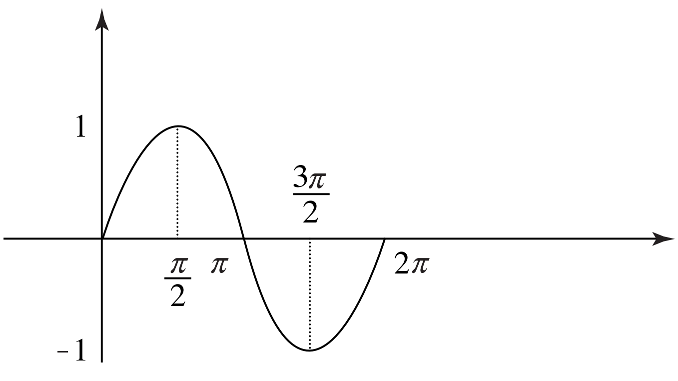


Figure 2.21. Basic graph

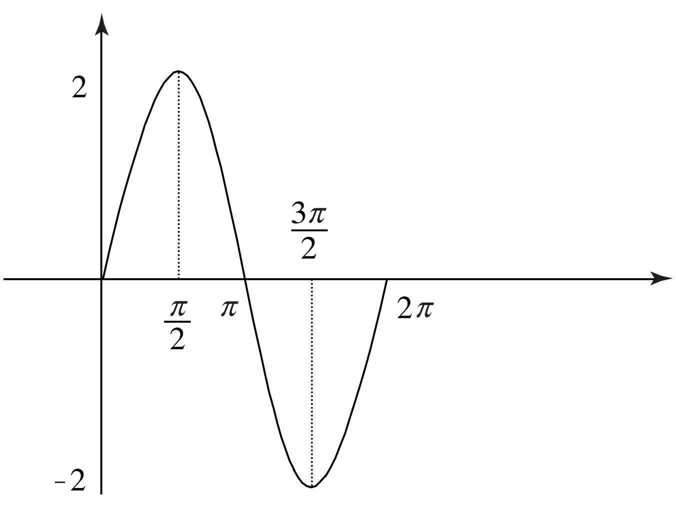


Figure 2.22. Basic graph stretched vertically by 2 units

The graph in Figure 2.22 is a stretch 2 units vertically of the graph in Figure 2.21. Observe that the points of intersection in the x-axis are the same in both graphs.

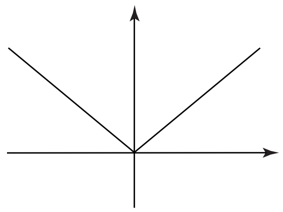


Figure 2.23. Basic graph

The graph in Figure 2.24, below, is a compression $1 \ / \ 2$ units vertically of the graph in Figure 2.23.

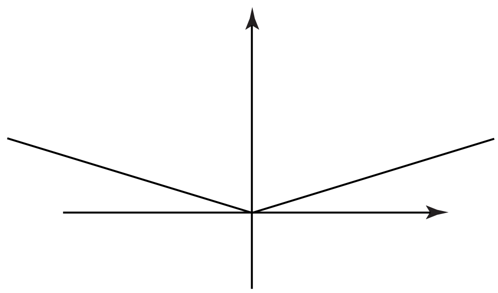


Figure 2.24. Basic graph compressed vertically by $\frac{1}{2}$ units

To stretch the graph of a function vertically, we multiply the function by a constant greater than 1; for a vertical compression, we multiply the function by a number smaller than 1.

Definition 2.11. If $c > 1$ is a real number and $f(x)$ is a function, then

a. the graph of the function

$$F(x) = cf(x)$$

represents a *vertical stretch* of the graph of the function $f(x)$ by a factor of *c*.

b. the graph of the function

$$F(x) = \frac{1}{c} f(x)$$

represents a *vertical compression* of the graph of the function $f(x)$ by a factor of *c*.

Example 2.47. The graph of $f(x) = \sin x$ is shown in Figure 2.21. The graph in Figure 2.22 is the graph of the function $F(x) = 2 \sin x$, which is the vertical stretch of the graph of $f(x)$ by a factor of 2. As you can see, $F(x) = 2f(x)$.

Example 2.48. The graph of $f(x) = |x|$ is shown in Figure 2.23. The graph in Figure 2.24 is the graph of the function

$$F(x) = \frac{1}{2}|x| = \frac{1}{2}f(x),$$

which is the vertical compression of the graph of $f(x)$ by a factor of 2.

Example 2.49. The graph of

$$F(x) = \frac{x^2}{3}$$

is a vertical compression of the function $f(x) = x^2$ by a factor of 3:

$$F(x) = \frac{x^2}{3} = \frac{1}{3}\, x^2 = \frac{1}{3}f(x)$$

(see Figure 2.25, below).

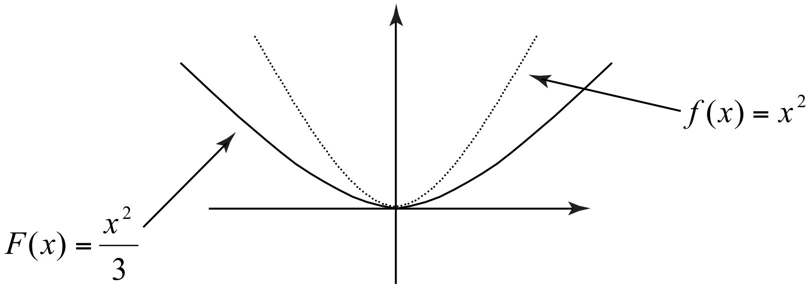


Figure 2.25. Graph of $f(x) = x^2$, and graph of $f(x) = x^2$ compressed vertically by a factor of 3

To compress the graph of a function horizontally, we multiply the independent variable of the function by a constant greater than 1; to stretch it horizontally, we multiply it by a number smaller than 1.

Definition 2.12. If $c > 1$ is a real number and $f(x)$ is a function, then

a. the graph of the function

$$F(x) = f(cx)$$

is a *horizontal compression* of the graph of the function $f(x)$ by a factor of c.

b. the graph of the function

$$F(x) = f\left(\frac{x}{c}\right)$$

is a *horizontal stretch* of the graph of the function $f(x)$ by a factor of c.

Example 2.50. The graph of $f(x) = \sin x$ is shown in Figure 2.26, below.

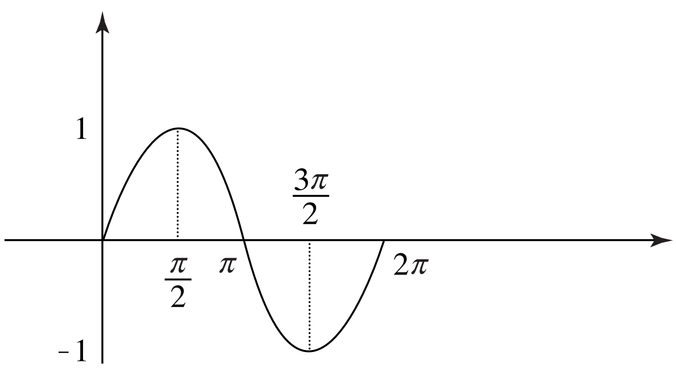


Figure 2.26. Graph of $f(x) = \sin x$

The graph in Figure 2.27 is the graph of the function $F(x) = \sin(2x)$, which is the horizontal compression of the graph of $f(x)$ by a factor of 2. As you can see, $F(x) = f(2x)$. Observe that the range values of the dependent variable F are the same, from -1 to 1.

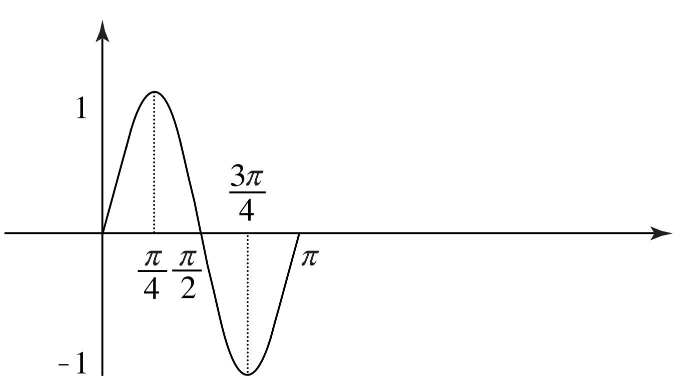


Figure 2.27. Graph of $F(x) = \sin(2x)$

The points of intersection with the x-axis of $\sin x$ are

$$0, \quad \pi, \quad 2\pi;$$

and the points of intersection with the x-axis of $\sin(2x)$ are

$$0 = \left(\frac{1}{2}\right)0, \quad \frac{\pi}{2} = \left(\frac{1}{2}\right)\pi, \quad \pi = \left(\frac{1}{2}\right)2\pi.$$

Example 2.51. The graph of $f(x) = \sin x$ is shown in Figure 2.26. The graph in Figure 2.28, below, is the graph of the function

$$F(x) = \sin\left(\frac{x}{2}\right),$$

which is the horizontal stretch of the graph of $f(x)$ by a factor of 2. As you can see,

$$F(x) = f\left(\frac{x}{2}\right).$$

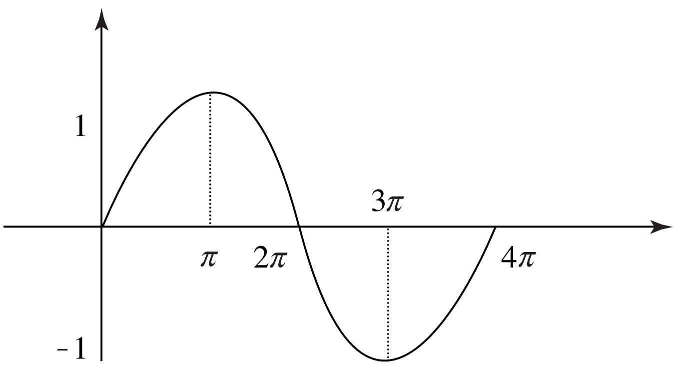


Figure 2.28. Graph of $F(x) = \sin\left(\frac{x}{2}\right)$

The points of intersection with the x-axis of $\sin x$ are

$$0, \quad \pi, \quad 2\pi;$$

and those of $\sin\left(\frac{x}{2}\right)$ are

$$0 = 2(0), \quad 2\pi = 2(\pi), \quad 4\pi = 2(2\pi).$$

Reflections

To reflect a graph with respect to the x-axis—that is, to have the x-axis act as a mirror—we multiply the function by -1; to reflect the function with respect to the y-axis, we multiply the independent variable by -1.

Definition 2.13.

a. The graph of the function
$$F(x) = -f(x)$$
is the *reflection with respect to the x-axis* of the function $f(x)$.

b. The graph of the function
$$F(x) = f(-x)$$
is the reflection with respect to the y-axis of the function $f(x)$.

Example 2.52. The reflection of the graph of the sine function with respect to the x-axis corresponds to the function $-\sin x$, as is shown in Figure 2.29, below.

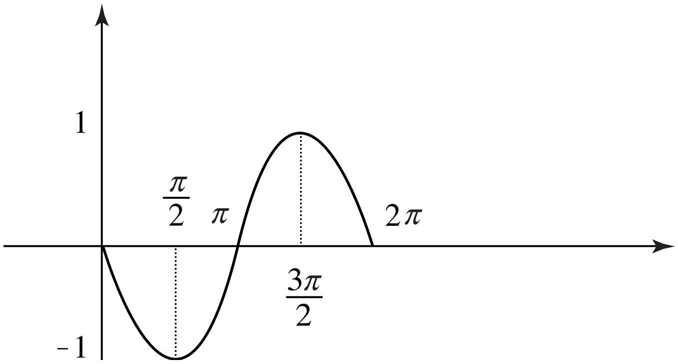


Figure 2.29. Graph of $f(x) = -\sin x$

Example 2.53. The graph the function $f(x) = \sqrt{x}$ is shown in Figure 2.30, below.

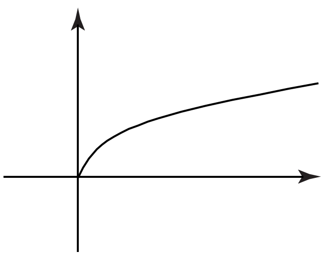


Figure 2.30. Graph of $f(x) = \sqrt{x}$

The reflection of $f(x) = \sqrt{x}$ with respect to the y-axis corresponds to the function $F(x) = \sqrt{-x}$, as is shown in Figure 2.31, below. Observe that the domain of the function f is $(0, \infty)$ and the domain of F is $(-\infty, 0)$.

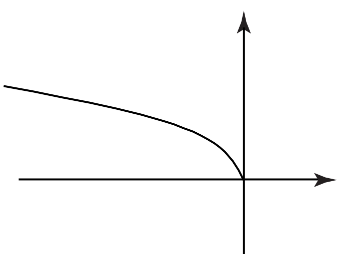


Figure 2.31. Graph of $F(x) = \sqrt{-x}$

Exercises

32. For each of the functions below, identify the basic function f and its transformation, and sketch the graph of the function F.
    a. $F(x) = x - 8$
    b. $F(x) = (x + 4)^2$
    c. $F(x) = \frac{1}{3} \cos x$
    d. $F(x) = 5C(x)$, where $C(x) = 4$ is the constant function 4.

Answers to Exercises (appendix-a.htm)

Multiple Transformations

We can apply several transformations to one basic graph.

Example 2.54. If the basic function is f, then a vertical stretch by a factor of 3 is the function $3f(x)$. Continuing with a shift up by a factor of 2 gives the function $3f(x) + 2$.

Let the basic function be $f(x) = x^2$, then the graph of $F(x) = 3x^2 + 2$ is obtained after two transformations: a vertical stretch first, and a shift second, in that order (see Figure 2.32, below).

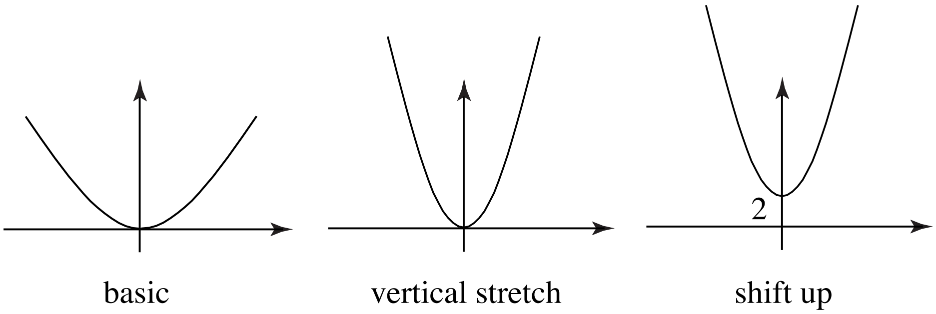


Figure 2.32. Deriving the graph of $F(x) = 3x^2 + 2$ from that of $f(x) = x^2$

The challenge in sketching a new graph from a basic one is to identify the transformations in the order in which they are applied. Observe that the order matters: if we shift by 2 units up and then stretch vertically by 3 units, we go to $f(x) + 2$ first, then to $3(f(x) + 2) = 3f(x) + 6$. If the basic function is $f(x) = x^2$, under these transformations in this order, $F(x) = 3f(x) + 6$, and the graph of this function, is as shown in Figure 2.33, below. [What would the function be if we applied the vertical stretch first? What would the graph look like?]

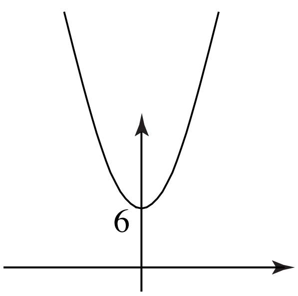


Figure 2.33. Graph of $F(x) = 3f(x) + 6$

Example 2.55. Consider the function $F(x) = 3(x + 5)^2 - 2$. If we were to evaluate this function at a number x, we would proceed as follows.

$x + 5$	add 5 to the independent variable	shift 5 units to the left
$(x + 5)^2$	evaluate $f(x) = x^2$ at $x + 5$	basic function $f(x) = x^2$
$3(x + 5)^2$	multiply the function by 3	vertical stretch by 3
$3(x + 5)^2 - 2$	add -2 to the function	shift 2 units down

Conclusion: The function $F(x) = 3(x + 5)^2 - 2$ is obtained from the basic function $f(x) = x^2$ with the three transformations listed in the third column, in the order given. Figures 2.34 and 2.35, below, show the first two stages of these transformations.

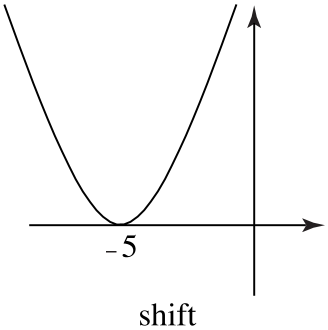


Figure 2.34. Stage 1 of the transformation of $f(x) = x^2$ to $F(x) = 3(x + 5)^2 - 2$

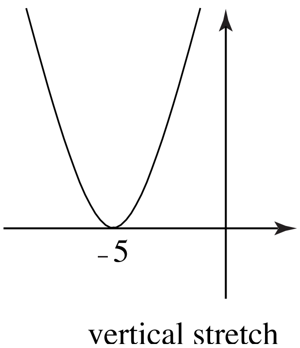


Figure 2.35. Stage 2 of the transformation of $f(x) = x^2$ to $F(x) = 3(x + 5)^2 - 2$

The final graph of the function is shown in Figure 2.36:

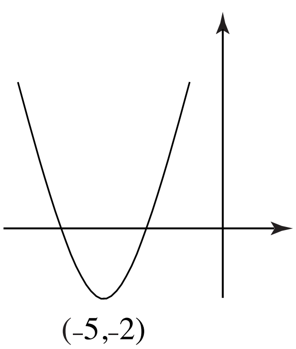


Figure 2.36. Graph of $F(x) = 3(x + 5)^2 - 2$

Example 2.56. To sketch the graph of

$$F(x) = 2\cos\left(\frac{x}{2}\right),$$

we make the following analysis:

$\frac{x}{2}$	horizontal stretch by a factor of 2	[Why?]
$\cos\left(\frac{x}{2}\right)$	evaluate $\cos x$ at $\frac{x}{2}$	[Why?]
$2\cos\left(\frac{x}{2}\right)$	vertical stretch by a factor of 2	[Why?]

The graph is shown in Figure 2.37, below.

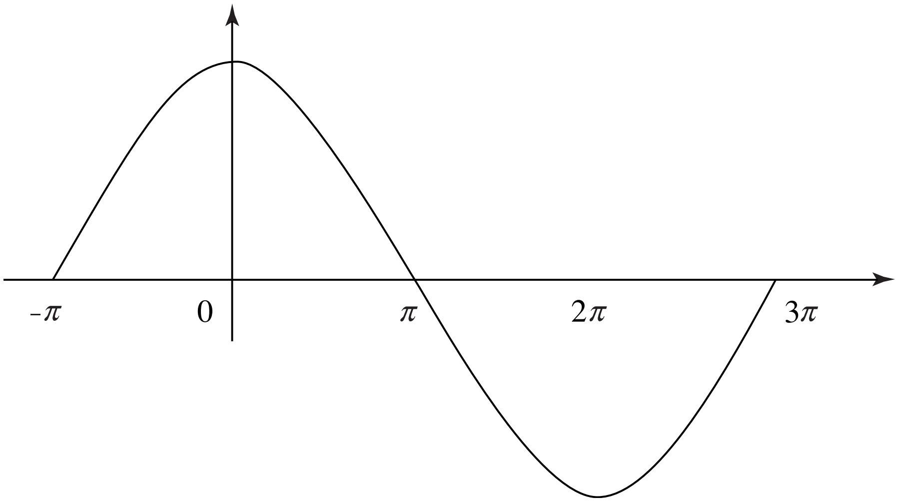


Figure 2.37. Graph of $F(x) = 2\ \cos\ \left(\frac{x}{2}\right)$

Exercises

33. Read the section titled “Transformations of Functions,” on pages 30-33 of the textbook.
34. Do Exercises 11, 14, 17, 18 and 20 on page 38.
35. Do Exercises 77, 79, 80 and 82 on page 367 of the textbook.

Answers to Exercises (appendix-a.htm)

Finishing This Unit

1. Review the objectives of this unit and make sure you are able to meet all of them.
2. If there is a concept, definition, example or exercise that is not yet clear to you, go back and reread it, then contact your tutor for help.
3. Definitions are important, for each definition you should be able to give an example (something that is covered by the definition) and a counterexample something that is not so covered.
4. Do the exercises in the "Learning from Mistakes" section for this unit.
5. You may want to do Exercises 1(*a*)-(*d*), 2, 5 (domain only), 6 (domain only), 9(*a*)-(*d*), 10 and 11-15 from the "Review" (page 42 of the textbook).

Learning from Mistakes

There are mistakes in each of the following solutions. Identify the errors, and give the correct answer.

1. Define each of the functions below as a set of pairs.

   a. The velocity v depends on the time t.
   b. The bacteria population B depends on the amount of oxygen o.

   Erroneous Solutions

   a. $\{(v, t) | v \text{ is the velocity at time } t\}$
   b. $\{(o, B) | o \text{ is the amount of oxygen for the population } B\}$

2. Let $S(t) = 3.75t^2 + 500$ be a function, where S is the salary for the number of units sold t. Find the value of S at $t = 4 - \sqrt{14}$.

   Erroneous Solution

$$\begin{aligned} S(4-\sqrt{14}) &= 3.75(4-\sqrt{14})^2 + 500 \\ &= 3.75(16-14) + 500 \\ &= 507.50 \end{aligned}$$

3. Find the domain of each of the following functions.

   a. $g(t) = \dfrac{4-t}{6-\sqrt{t^2-9}}$
   b. $h(x) = \tan(2x)$.

   Erroneous Solutions

   a. We need $t^2 - 9 > 0$; $t^2 > 9$; so $t > 3$ and $t < -3$. We also want $4 - t \neq 0$, so $t \neq 4$; therefore, the domain in interval notation is the union of the intervals, $(-\infty, -3)$, $(3, 4)$ and $(4, \infty)$.
   b. Since $\tan(u)$ is undefined for

$$u = \frac{k\pi}{2},$$

   we need

$$2x \neq \frac{k\pi}{2}, \text{ so } x \neq \frac{k\pi}{4},$$

the domain is all numbers except $\frac{k\pi}{4}$ for any integer k.

4. Sketch the graph of a single function that satisfies all of the conditions listed below.

- Domain is $[3, 12]$.
- The pairs $(4, 5)$ and $(5, 4)$ are in the function.
- The function is constant and equal to $(8, 12]$.

Erroneous Solution

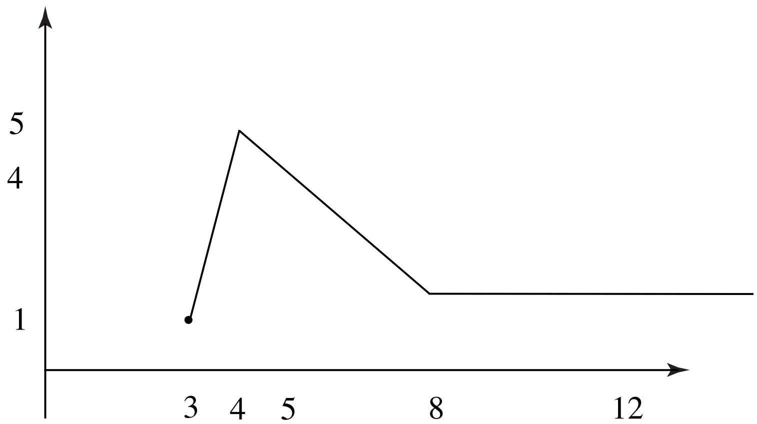


Figure 2.38. Erroneous solution to "Learning from Mistakes" Question 4.

5. If the graph of the function f is shown below, give a sketch of the following functions.

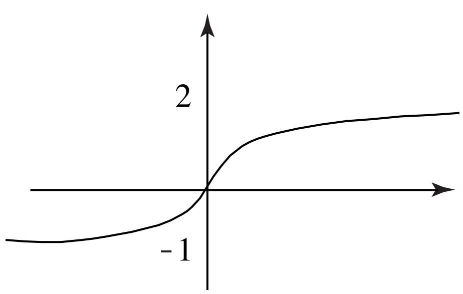


Figure 2.39. Graph for "Learning from Mistakes" Question 5.

a. $F(x) = 2f(x) - 1$
b. $F(x) = f(x+2) + 3$.

Erroneous Solutions

a. The function F is a vertical stretch by a factor of 2 and a shift down of one unit.

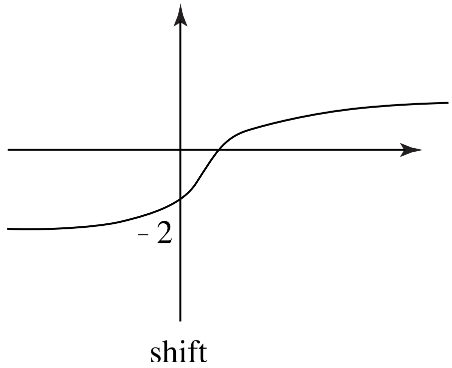


Figure 2.40. Erroneous solution to “Learning from Mistakes” Question 5(*a*).

b. The function F is shift to the right by 2 units and a shift up by 3 units.

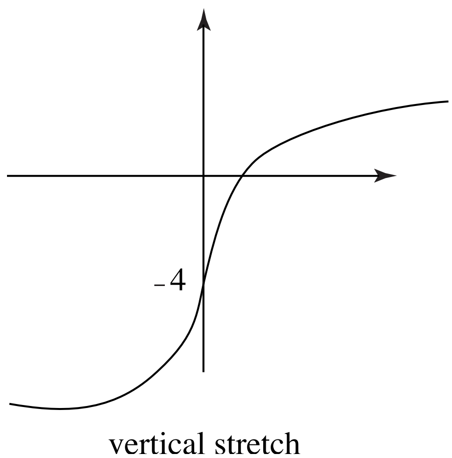


Figure 2.41. Erroneous solution to “Learning from Mistakes” Question 5(*b*).

Limits

Objectives

When you have completed this unit, you should be able to

1. analyse graphs of functions to evaluate limits.
2. apply definitions and known results to evaluate limits.
3. use limits to find vertical and horizontal asymptotes.
4. prove results about limits.

The cornerstone concept of calculus is the "limit." The importance of understanding limits, and of being able to evaluate them, should not be underestimated. You must learn how to apply definitions, theorems, propositions and corollaries[1] to the evaluation of limits. The "guessing" strategy used in the textbook section titled "The Limit of a Function" is not appropriate, as Example 5 on page 52 clearly shows.

In this unit, we begin by presenting the concept of limits visually; that is, you will learn what a limit is by analysing graphs of functions. Next, we consider continuity, a topic that takes us to the application of theorems, propositions and corollaries that allow us to evaluate limits without having to look at the function's graph. We finish with the formal definition of a limit.

> **Note:** We recommend that you make a list of definitions, propositions, theorems and corollaries, because we will refer to them constantly throughout this unit.

Evaluating Limits Visually

Prerequisites

To complete this section, you must be able to write inequalities in interval notation on the real line. See the section titled "Intervals" on pages 337-338 of the textbook.

Keep in mind that, in the ordered real line, the farther a positive number is from 0, the larger it is, and the farther a negative number is from 0, the smaller it is (see Figure 3.1, below). For instance -10^{10} is a very small negative number, smaller than -10^4, on the other hand -10^{-10} is a large negative number, bigger than -10^{-4}. Thus, when we say that a number is very small we do not mean a small positive number such as 10^{-10}, but a small negative number like -10^{10}. [Clearly, -10^{10} is interpreted as $-\left(10^{10}\right)$, not as $(-10)^{10}$.]

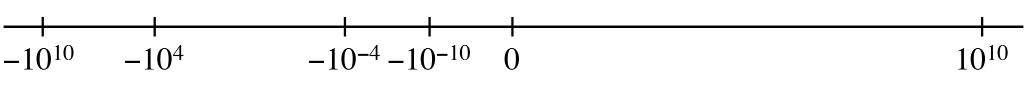


Figure 3.1. Real number line (not to scale)

As we have stated, the graph of a function is its visual representation. When you see a graph, you should get an immediate sense of the information that the graphs is providing. We pay particular attention to the points where the graph "breaks." We want to understand what happens around the value of the function at the points where such breaks occur. Why does the graph break? What are the values of the function to the left and the right of this particular value? Answering these questions enables us to understand the concept of side limits, and once we know what side limits are, we will be able to understand visually what a limit is.

Given the graph of a function f, we can see whether the function is or is not defined at a number a. The function f is defined at a if the pair (a, b) is on the graph for some number b, and the value of the function f at a is b, that is $f(a) = b$. It may be difficult to know the exact value of $f(a)$ from its graph, but we can *see* its approximate value.

Example 3.1. Examine the graph of Exercise 4 on page 57 of the textbook.

We see that

$$f(0) = 3, \quad f(2) \approx 3 \quad \text{and} \quad f(3) = 3.$$

Example 3.2. Examine the graph of Exercise 5 on page 58 of the textbook.

We see that

$$f(2) = 2, \quad f(0) \approx \frac{5}{2}, \quad f(3) = 2, \quad \text{and} \quad f(5) \ \text{is undefined.}$$

Also note that

$$f(0.9) \approx 2 \quad \text{and} \quad f(1.9) \approx 2.$$

Example 3.3. Examine the graph of Exercise 7 on page 58 of the textbook.

We see that

$$g(0) = -1, \quad g(2) = 1 \quad \text{and} \quad g(4) = 3.$$

Also note that

$$g(-0.001) \approx -1, \quad g(0.001) \approx -2, \quad g(1.9) \approx 2 \quad \text{and} \quad g(2.01) \approx 0.$$

Example 3.4. Examine the graph of Exercise 8 on page 58 of the textbook.

We see that

$$R(0) = 0 \quad \text{and that} \quad R(-3), R(2) \quad \text{and} \quad R(5) \quad \text{are undefined.}$$

Observe that $R(-2.9)$ is a very large positive number, and that $R(-3.01)$ is a very small negative number.

Similarly, both $R(1.9)$ and $R(2.01)$ are very small negative numbers.

What are the values of $R(4.9)$ and $R(5.1)$?

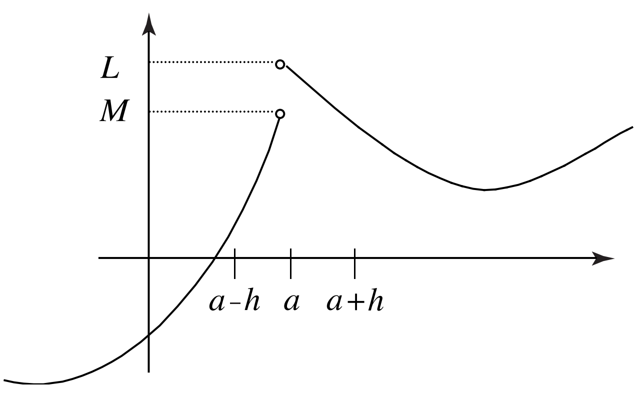


Figure 3.2. Function f, not defined at a

Example 3.5. In Figure 3.2, above, we can see that the function, let us call it f, is not defined at a, but we can also see what is happening with the function at values close to the number a. If h is a small positive number, then $a + h$ is a number on the right of a, so we say that $a + h$ is close to a from the right; and $a - h$ is a number on the left of a, so we say that $a - h$ is close to a from the left. We see from the graph that

$$f(a + h) \approx L \quad \text{and} \quad f(a - h) \approx M.$$

In conclusion, we see that

- for any x close to a from the right, $f(x) \approx L$, and
- for any x close to a from the left, $f(x) \approx M$.

Example 3.6. In Figure 3.3, below, we see that the function, say g, is defined at a and that $g(a) = L$. As in Example 3.5, for any x close to a from the right, $g(x) \approx L$, and for any x close to a from the left, $g(x) \approx M$.

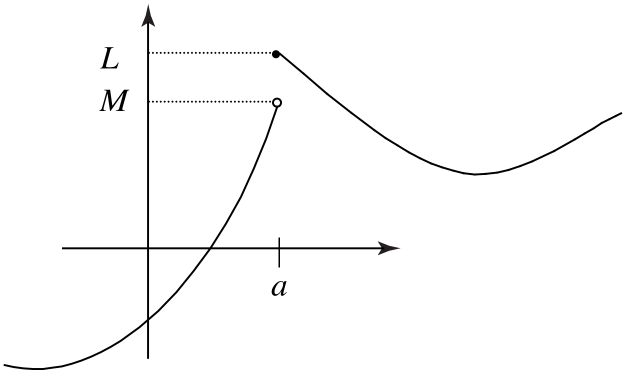


Figure 3.3. Function g, defined at a such that $g(a) = L$

Example 3.7. Figure 3.4 is the graph of a function we will call h. We see that $h(a) = L$, and that, for any x close to a from the right, $h(x) \approx N$; and for any x close to a from the left, $h(x) \approx M$.

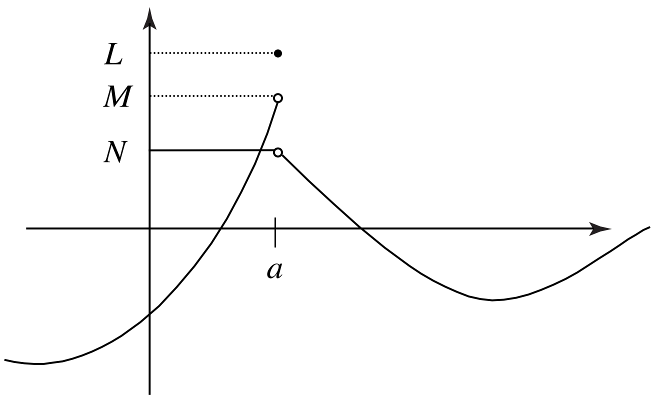


Figure 3.4. Function h, defined at a such that $h(a) = L$

In Examples 3.5 to 3.7, above, we pay attention to the point a because it is where the graph of the function "breaks." The function may or may not be defined at a: what determines the breaking of the graph are the values of the function on the left and right of a. That is, the graph of f breaks at a because

- $f(x) \approx L$ for any x close to a from the right,
- $f(x) \approx M$ for any x is close to a from the left, and
- $L \neq M$.

Example 3.8. In Figure 3.5, below, we see that the graph of the function g breaks, but not because the values of the function from the right and left are different. Rather, it breaks because $g(a)$ is undefined.

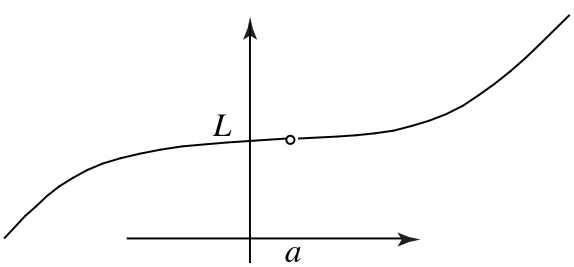


Figure 3.5. Function g, $g(a)$ undefined

In this case, we have

- $g(x) \approx L$ for any x close to a from the right or from the left, and
- $g(a)$ is undefined.

Example 3.9. In Figure 3.6, below, the graph of the function g breaks, even though $g(a) = M$ is defined, and the values of the function from the right and left are equal; that is $g(x) \approx L$ for any x close to a from the right or from the left. It breaks because $L \neq M$.

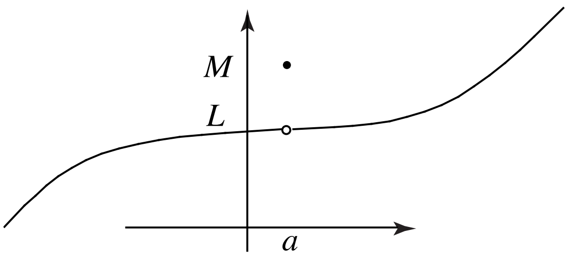


Figure 3.6. Function g, defined at a such that $g(a) = M$; $L \neq M$

All of these examples tell us that it is important to study the values of the function from the left and right at a given point.

Definition 3.1. We say that a function f is *defined at the right* of a, if f is defined on an interval (a, b) for some b; that is,

$f(x)$ is defined for any x such that $a < x < b$ for some b.

Similarly, we say that a function f is *defined at the left* of a, if f is defined on an interval (c, a) for some c; that is,

$f(x)$ is defined for any x such that $c < x < a$ for some c.

A function is *defined around or near a* if it is defined at the right and left of a.

Let us now look at the values of the function in a dynamic way. Calculus is the subject that studies motions and changes. We want to know how the values of the function *change*.

In Figure 3.7, below, we start with an x close to a from the right, and we consider the corresponding point $(x, f(x))$ on the graph. Next we move x towards a, and we continue looking at the corresponding points on the graph, this strategy gives us a succession of points, as indicated.

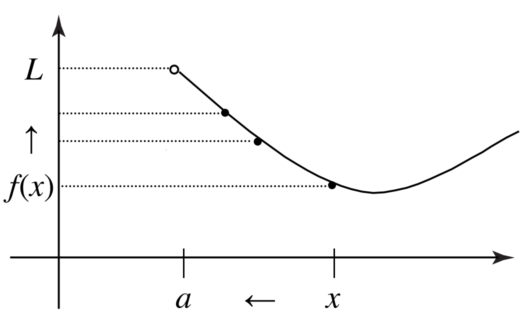


Figure 3.7. Function f, with points x close to a from the right

What happens to the corresponding values of $f(x)$ (on the y-axis) as x moves towards a? They approach L, and the closer x is to a, the closer $f(x)$ is to L.

Another way of looking at this is to recognize that there is always an x that is closer to a from the right, so that $f(x)$ is as close as L as we want.

Therefore, we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficiently close to a from the right, but not equal to a.

Similarly in Figure 3.8, below, we start with an x close to a from the left, and we consider the corresponding point $(x, f(x))$ on the graph. When we move x towards a and look at the corresponding points of the graph, we again obtain a succession of points.

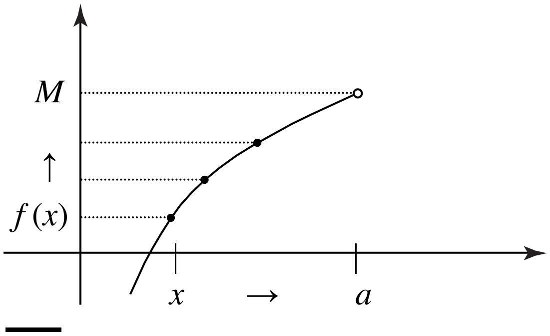


Figure 3.8. Function f, with points x close to a from the left

The values $f(x)$ (on the y-axis) approach M, and the closer x is to a, the closer $f(x)$ is to M. As before, there is always an x closer to a from the left, so that $f(x)$ is as close as M as we want.

> *Therefore, we can make the values of $f(x)$ arbitrarily close to M (as close as we like) by taking x to be sufficiently close to a from the left, but not equal to a.*

We write $x \to a^+$ to indicate that x approaches a from the right, and we write $x \to a^-$ to indicate that x approaches a from the left.

To indicate that the values of the function $f(x)$ approach L, we write $f(x) \to L$.

Hence, “if $x \to a^+$, then $f(x) \to L$” is read in two ways:

> “if x approaches a from the right, then the values of the function $f(x)$ approach L,”

or

> “$f(x)$ approaches L as x approaches a from the right.”

We have the corresponding notation for the left:

“If $x \to a^-$, then $f(x) \to M$,” is read as

> “if x approaches a from the left, then the values of the function $f(x)$ approach M,”

or

> “$f(x)$ approaches M as x approaches a from the left.”

The behaviour of the values of the function as they get arbitrarily close to a number is what we call the *limit* of the function.

Definition* 3.2. Let f be a function defined on the right of a.

Then, if we can make the values of $f(x)$ arbitrarily close to L (as close as we like), by taking x to be sufficiently close to a from the right (but not equal to a), we say that

the *limit* of $f(x)$, as x approaches a from the right, is L,

and we write

$$\lim_{x \to a^+} f(x) = L.$$

Another notation for this limit is

$$f(x) \to L \quad \text{as} \quad x \to a^+.$$

Definition* 3.3. Let f be a function defined on the left of a.

Then, if we can make the values of $f(x)$ arbitrarily close to M (as close as we like), by taking x to be sufficiently close to a from the left (but not equal to a), we say that

the *limit* of $f(x)$, as x approaches a from the left, is M,

and we write

$$\lim_{x \to a^-} f(x) = M.$$

Another notation for this limit is

$$f(x) \to M \quad \text{as} \quad x \to a^-.$$

Example 3.10. In Figure 3.2, we see that

$$\lim_{x \to a^+} f(x) = L, \quad \lim_{x \to a^-} f(x) = M, \text{ and } f(a) \text{is undefined.}$$

Example 3.11. In Figure 3.3, we see that

$$\lim_{x \to a^+} g(x) = L, \quad \lim_{x \to a^-} g(x) = M, \quad \text{and } g(a) = L.$$

Example 3.12. In Figure 3.4, we see that

$$\lim_{x \to a^+} h(x) = N, \quad \lim_{x \to a^-} h(x) = M, \quad \text{and } h(a) = L.$$

Example 3.13. In Figure 3.5, we see that

$$\lim_{x \to a^+} g(x) = L, \quad \lim_{x \to a^-} g(x) = L, \quad \text{and } g(a) \text{is undefined.}$$

Example 3.14. In Figure 3.6, we see that

$$\lim_{x \to a^+} g(x) = L, \quad \lim_{x \to a^-} g(x) = L, \quad \text{and } g(a) = M.$$

If we analyse the graphs in Figures 3.2 to 3.6, we see that the break on the graph is "bad" in Figures 3.2 and 3.4, since the "gap" is large. However, in Figures 3.6 and 3.7, the break is only one point. The situation is considered to be "better" when the limits from the left and the right are equal. In this case, we say that the limit of the function is the common value of the limits from both sides.

Definition* 3.4. Let f be a function defined around a.

If we can make the values of $f(x)$ arbitrarily close to L (as close as we like), by taking x to be sufficiently close to a from the right and from the left (but not equal to a), we say that

the *limit* of $f(x)$, as x approaches a is L,

and we write

$$\lim_{x \to a} f(x) = L.$$

Definition 3.4 says that

$$\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L.$$

We write $x \to a$ to indicate both $x \to a^+$ and $x \to a^-$.

Hence, another notation for this limit is

$$f(x) \to L \quad \text{as} \quad x \to a.$$

So,

$$f(x) \to L \text{ as } x \to a \quad \text{iff} \quad f(x) \to L \text{ as } x \to a^+ \text{ and } x \to a^-.$$

Definition 3.5. If

$$\lim_{x \to a^+} f(x) = L, \quad \lim_{x \to a^-} f(x) = M, \quad \text{and} \quad L \neq M,$$

then $\lim_{x \to a} f(x)$ does not exist.

Example 3.15. In Figures 3.2 to 3.4, the limits of the functions at a do not exist.

Example 3.16. In Figures 3.5 and 3.6, the limit of the function at a exists, and it is L.

Example 3.17. To evaluate the limit of the function $f(x) = \sqrt{x}$ at 0 from the right, we examine its graph, noting that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Since this function is not defined on the left of 0, we say that

$\lim_{x \to 0^-} \sqrt{x}$ is undefined.

Exercises

1. Do Exercises 4, 5, 6 and 7 on pages 57 and 58 of the textbook.
2. Sketch the graph of *a single* function f that satisfies *all* of the conditions given below.
    a. domain is the interval $(3, \infty)$
    b. $\lim_{x \to 3^+} f(x) = 0$
    c. $\lim_{x \to 5^-} f(x) = 6$
    d. $\lim_{x \to 5^+} f(x) = 6$
    e. $\lim_{x \to 6^-} f(x) = 6$
    f. $\lim_{x \to 6^+} f(x) = -1$
    g. $f(5) = 0$
    h. $f(6) = -1$

Answers to Exercises (appendix-a.htm)

Let us look at the graph in Figure 3.9, below, in a dynamic way.

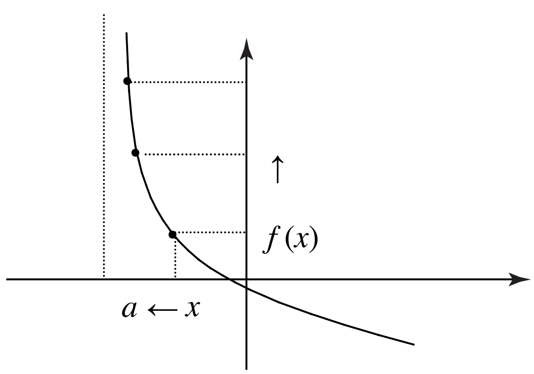


Figure 3.9. Function f, with points x close to a from the right; $f(x)$ increasing

In Figure 3.9, we also start with an x close to a from the right, and we consider the corresponding point $(x, f(x))$ on the graph. Next, we move x towards a as we consider the points on the graph, and we obtain a succession of points, as indicated.

The values of $f(x)$ (on the y-axis) increase to a very large number as x gets closer to a. We see that there is always an x closer to a from the right, so that $f(x)$ is as large as we want.

> *Therefore, we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the right, but not equal to a.*

Furthermore, we see that the values $f(x)$ are infinitely large for x very close to a from the right. In this case, we say that "$f(x)$ *increases without bounds*, as $x \to a^+$."

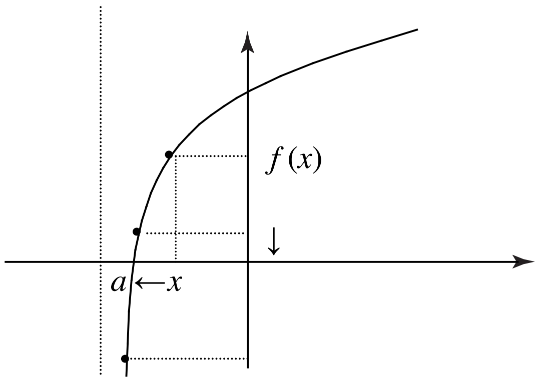


Figure 3.10. Function f, with points x close to a from the right; $f(x)$ decreasing

In Figure 3.10, above, we see that as x approaches a from the right, the values of $f(x)$ (on the y-axis) decrease to a very small negative number. There is always a number x on the right of a, so that $f(x)$ can be as (negatively) small as we want.

> *Therefore, we can make the values of $f(x)$ arbitrarily small negatively (as small as we like) by taking x to be sufficiently close to a from the right, but not equal to a.*

Since the values of $f(x)$ are infinitely small (negatively) for x very close to a from the right, we say that "$f(x)$ *decreases without bounds*, as $x \to a^+$."

We write $f(x) \to \infty$ to indicate that the values of $f(x)$ increase without bounds, and $f(x) \to -\infty$ to indicate that the values of $f(x)$ decrease without bounds.

When the values of the function increase or decrease without bounds, we say that the limit of the function is infinity or negative infinity.

Definition* 3.6. Let f be a function defined on the right of a.

a. If we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the right (but not equal to a), we say that

   the limit of $f(x)$, as x approaches a from the right, is infinity,

   and we write

$$\lim_{x \to a^+} f(x) = \infty.$$

   Another notation for this limit is

$$f(x) \to \infty \quad \text{as} \quad x \to a^+.$$

b. If we can make the values of $f(x)$ arbitrarily (negatively) small (as small as we like) by taking x to be sufficiently close to a from the right (but not equal to a), we say that

   the limit of $f(x)$, as x approaches a from the right, is negative infinity,

   and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

Another notation for this limit is

$$f(x) \to -\infty \quad \text{as} \quad x \to a^{+}.$$

The limits from the left are similar. The graphs of the functions in Figures 3.11 and 3.12, below, show that the limits are infinity and negative infinity as $x \to a^{-}$.

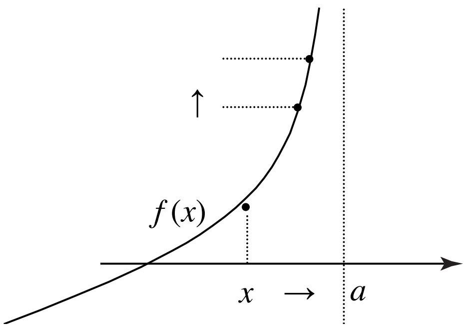


Figure 3.11. Function f, with points x close to a from the left; $f(x)$ increasing

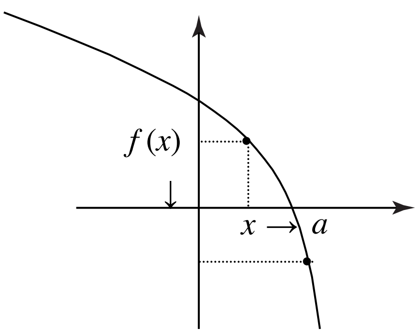


Figure 3.12. Function f, with points x close to a from the left; $f(x)$ decreasing

In Figure 3.11, we see that as x approaches a from the left, the values of $f(x)$ (on the y-axis) increase to a very large number. That is, there is always a number x on the left of a, so that $f(x)$ is as large as we want.

Therefore, we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the left but not equal to a.

In Figure 3.12, we see that as x approaches a from the left, the values of $f(x)$ (on the y-axis) decrease to a very small (negative) number. Hence, there is always a number x on the left of a, so that $f(x)$ is as negatively small as we want.

Therefore, we can make the values of $f(x)$ arbitrarily small negatively (as small as we like) by taking x to be sufficiently close to a from the left, but not equal to a.

Definition* 3.7. Let f be a function defined on the left of a.

a. If we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the left, but not equal to a, we say that

the limit of $f(x)$, as x approaches a from the left, is infinity,

and we write

$$\lim_{x \to a^{-}} f(x) = \infty.$$

Another notation for this limit is

$$f(x) \to \infty \quad \text{as} \quad x \to a^{-}.$$

b. If we can make the values of $f(x)$ arbitrarily (negatively) small (as small as we like) by taking x to be sufficiently close to a from the left, but not equal to a, we say that

the limit of $f(x)$, as x approaches a from the left, is negative infinity,

and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Another notation for this limit is

$$f(x) \to -\infty \quad \text{as} \quad x \to a^-.$$

Example 3.18. Looking at the graph of selected trigonometric functions (see Figure 14 on page 365 of the textbook), we see that

a. $\lim\limits_{x \to \frac{\pi}{2}^+} \tan x = -\infty$

b. $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x = \infty$

c. $\lim\limits_{x \to 0^+} \cot x = \infty$

d. $\lim\limits_{x \to \frac{\pi}{2}^+} \sec x = -\infty$

Definition* 3.8. Let f be a function defined around a.

a. We write

$$\lim_{x \to a} f(x) = \infty,$$

and say

the limit of$f(x)$, as x approaches a, is infinity,

if we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the right and from the left, but not equal to a.

b. We write

$$\lim_{x \to a} f(x) = -\infty,$$

and say

the limit of$f(x)$, as x approaches a, is negative infinity,

if we can make the values of $f(x)$ arbitrarily (negatively) small (as small as we like) by taking x to be sufficiently close to a from the right and from the left, but not equal to a.

Definition 3.8 state that

$$\lim_{x \to a} f(x) = \infty \quad \text{iff} \quad \lim_{x \to a^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to a^-} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \quad \text{iff} \quad \lim_{x \to a^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^-} f(x) = -\infty.$$

Another notation for $\lim\limits_{x \to a} f(x) = \pm\infty$ is $f(x) \to \pm\infty$ as $x \to a$. So,

$$f(x) \to \infty \text{ as } x \to a \quad \text{iff} \quad f(x) \to \infty \text{ as } x \to a^+ \quad \text{and} \quad \text{as } x \to a^-$$
$$f(x) \to -\infty \text{ as } x \to a \quad \text{iff} \quad f(x) \to -\infty \text{ as } x \to a^+ \quad \text{and} \quad \text{as } x \to a^-.$$

Definition 3.9. The limit $\lim\limits_{x \to a} f(x)$ does *not* exist if and only if

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x).$$

REMARKS 3.1

a. Definition 3.9 includes Definition 3.5 because, by definition, a limit does *not* exist if the limit from the right and the limit from the left are unequal, regardless of whether they are finite or infinite.

b. The limits from the right or from the left at a number a always exist if the function is defined on the right or on the left of a.

c. Although it is customary to say that a limit does not exist if it is not finite, it is not convenient to give the same designation to two completely different facts. The case where a function increases or decreases without bounds, as in Figures 3.11 and 3.12, above, is not the same as the case where the limit from the right is not equal to the limit from the left, as in Figure 3.3. Thus, we must learn to distinguish between these two entirely different situations.

d. To show that a limit does not exist at a number a, it is necessary and sufficient to show that the limits at a, from the right and from the left, are not equal.

Observe, and this is a very important observation, that when we look for (evaluate) the limit of a function at some number a, we do not pay attention to the value of the function at a; instead, we look at the values of the function *around* the number a. Whether the function is or is not defined at a is irrelevant to the process of finding the limit at a.

Exercises

3. Do Exercises 8, 9 and 10 on page 58 of the textbook.
4. Explain why each of the limits listed below does *not* exist.
   a. $\lim_{x \to \frac{\pi}{2}} \tan x$
   b. $\lim_{x \to 0} \cot x$
   c. $\lim_{x \to 0} \csc x$
5. Sketch the graph of a *single* function g that satisfies all of the conditions listed below.
   a. $\lim_{x \to 0} g(x) = \infty$
   b. $\lim_{x \to 3^+} g(x) = \infty$
   c. $\lim_{x \to 3^-} g(x) = -\infty$

Answers to Exercises (appendix-a.htm)

FOOTNOTE

[1] A *theorem* is an proved result with important applications; a *proposition* is also a proved result, but one with applications that are less important than those of a theorem; a *corollary* is a direct consequence of a theorem or a proposition.

Continuous Functions

Prerequisites

To complete this section, you must be able to

1. identify polynomial, power, rational, trigonometric and exponential functions. Read pages 21-26 of the textbook, and do Exercises 1 and 2 on page 27.
2. do algebraic operations with functions, as explained in the section titled "Combination of functions," on pages 34-35 of the textbook. Do Exercises 31 and 32 on page 39.
3. compose functions, as explained in the section titled "Composition of Functions," on pages 35-37 of the textbook. Do Exercises 35-40 on page 39.

   Note: In a composition $(f \circ g)(x) = f(g(x))$, the function $g(x)$ takes the place of the independent variable in the function f.

We can evaluate (find) limits by analysing the graphs of a function, but graphs are not always available or easy to obtain. So, instead, we must use the properties of the function, one of which is continuity.

In very general terms, we say that a function is continuous on an interval if the graph of the function on this interval does not break. From our basic graphs, we see that the graph of the function $f(x) = \sqrt{x}$ does not break on $[0, \infty)$, so this function is continuous on this interval. Similarly, the graphs of the trigonometric functions tell us that tangent is continuous on the interval $(-\pi / 2, \pi / 2)$, and many others; cosecant is continuous on $(0, \pi)$; and sine and cosine are continuous everywhere; that is, on $\mathbb{R} = (-\infty, \infty)$. The limit of a continuous function is easy to evaluate.

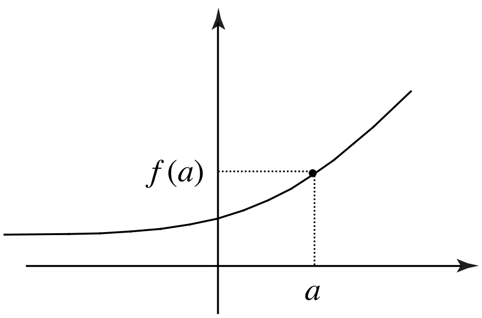


Figure 3.13. Function f, continuous at a

Figure 3.13, above, shows the graph of a function f that is continuous at a. We have

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a^-} f(x) = f(a);$$

hence,

$$\lim_{x \to a} f(x) = f(a).$$

It could not be easier: once we know that the function is continuous at a, its limit at a is $f(a)$. Thus, to evaluate limits, it helps to know which functions are continuous, and where.

All the function graphs in Figures 3.2 to 3.6 (unit03.htm#figure3-2) break in different ways.

In Figures 3.2 to 3.4 (unit03.htm#figure3-2), we see that the graphs break because

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x).$$

In Figures 3.5 and 3.6 (unit03.htm#figure3-5), $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, but the graph breaks because

$$\lim_{x \to a} f(x) \neq f(a).$$

So we can see that for a function to be continuous at a, the following three conditions must be met:

1. $\lim_{x \to a} f(x)$ exist, that is $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.
2. $f(a)$ is defined.
3. $\lim_{x \to a} f(x) = f(a)$.

Convince yourself that Condition 1 prevents the graph from breaking as in Figures 3.2 to 3.4 (unit03.htm#figure3-2), Condition 2 prevents a break as in Figure 3.5 (unit03.htm#figure3-5), and Condition 3 prevents a break as in Figure 3.6 (unit03.htm#figure3-6).

We have found the conditions for a function to be continuous at a number. We take these conditions as sufficient.

Definition* 3.10. A function f defined around a is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

All three conditions are included in Definition 3.10, because when we say that the limit of f at a is equal to $f(a)$ we understand that $\lim_{x \to a} f(x)$ exist, and that $f(a)$ is defined and is equal to the limit.

When a function is continuous at every number (i.e., on $\mathbb{R}$), we say that the function is *continuous everywhere*.

As we mentioned earlier, sine and cosine are continuous everywhere, and from our basic graphs, we see that the identity function and the basic quadratic and cubic functions are also continuous everywhere. In fact, polynomial functions are continuous everywhere.

Example 3.19. We can apply Definition 3.10 to evaluate limits of continuous functions as follows.

a. $\lim_{x \to \sqrt{3}} 6x^3 + 2x^2 - \sqrt{6} = 6\left(\sqrt{3}\right)^3 + 2\left(\sqrt{3}\right)^2 - \sqrt{6} = 18\sqrt{3} + 6 - \sqrt{6}$.

   This is so because $P(x) = 6x^3 + 2x^2 - \sqrt{6}$ is a polynomial function, and polynomials are continuous everywhere. In particular for our question, it is continuous at $\sqrt{3}$; hence, the limit is equal to $P\left(\sqrt{3}\right)$.

b. $\lim_{x \to \frac{5\pi}{3}} \sin x = \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

   Again, this is so because sine is continuous everywhere, so by Definition 3.10 the limit is equal to the value of sine at $5\pi / 3$.

Rational functions are continuous on their domain. Hence, to evaluate the limit of a rational function at a number a, we need to know if the number a is in the domain of the function, and then apply Definition 3.10.

Example 3.20. Let us evaluate the limit

$$\lim_{x \to 3} \frac{4x^2 - 3}{4x + 6}.$$

We see that the function

$$g(x) = \frac{4x^2 - 3}{4x + 6}$$

is rational, and that the number 3 is in its domain. [Why?] Hence, by Definition 3.10, the limit is equal to $g(3)$; that is,

$$\lim_{x \to 3} \frac{4x^2 - 3}{4x + 6} = \frac{4(3)^2 - 3}{4(3) + 6} = \frac{11}{6}.$$

Example 3.21. The limit

$$\lim_{x \to 3} \frac{4x^2 - 9}{x^2 - 9}$$

is also the limit of a rational function

$$R(x) = \frac{4x^2 - 9}{x^2 - 9},$$

but in this case, 3 is not in the domain of this function. Hence, we cannot apply Definition 3.10. We will learn later how such limits are evaluated.

Examples 3.19 to 3.21 show that, in order to apply Definition 3.10, we must know where the functions are continuous. The answer is given by the theorem below.

Theorem 3.11. The types of functions listed below are continuous at every number of their domains.

a. polynomials
b. rational functions
c. root functions
d. trigonometric functions

In the last section of this unit, we attempt to explain why this theorem is true. For now, we simply apply this theorem, together with Definition 3.10, to evaluate limits.

Example 3.22. Let us evaluate the limit $\lim_{x \to \frac{5\pi}{4}} \tan x$.

First, we identify the trigonometric function $\tan x$. We know from the graph of this function that $5\pi / 4$ is in the domain of $\tan x$; hence, $\tan x$ is continuous at $5\pi / 4$ by Theorem 3.11, and, by Definition 3.10, the limit is equal to $\tan (5\pi / 4)$. That is,

$$\lim_{x \to \frac{5\pi}{4}} \tan x = \tan\left(\frac{5\pi}{4}\right) = 1.$$

Example 3.23. The function in the limit $\lim_{x \to -5} \sqrt[3]{x}$ is a root function, and -5 is in its domain (i.e., $x^{1/3}$ is defined for any number x, positive or negative).

By Theorem 3.11 and Definition 3.10, the limit is equal to $(-5)^{1/3}$. That is,

$$\lim_{x \to -5} \sqrt[3]{x} = \sqrt[3]{-5} = (-5)^{1/3}.$$

Example 3.24. $\lim_{x \to \frac{3\pi}{4}} \csc x = \csc\left(\frac{3\pi}{4}\right) = \sqrt{2}$

This statement is true because $\csc x$ is continuous in its domain by Theorem 3.11, and $3\pi / 4$ is in its domain. Thus by Definition 3.10, the limit is equal to $\csc (3\pi / 4)$.

Example 3.25. The domain of the root function in the limit $\lim_{x \to -5} \sqrt[4]{x}$ is $[0, \infty)$. [Why?] So, the limit is undefined, since the function is not defined around -5.

The next two theorems extend the types of functions that are continuous on their domains.

Theorem 3.12. If f and g are continuous at a, and c is a constant, then all the functions listed below are also continuous at a.

a. $f + g$
b. $f - g$
c. cf
d. fg
e. $\frac{f}{g}$ if $g(a) \neq 0$

Functions $a,\ b,\ d$ and e refer to the algebra of functions, number c is the constant product of a function, which is defined as $cf(x) = c(f(x))$. For example, if $f(x) = \sin x + 6x^2$, then $4f(x) = 4(\sin x + 6x^2) = 4\sin x + 24x^2$.

For the composition of functions, we have the following theorem.

Theorem 3.13. If the function g is continuous at a, and the function f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Let us see how we use Theorems 3.11 to 3.13 to decide if a function is continuous on its domain.

Example 3.26. Explain why each of the functions listed below is continuous on its domain.

a. $F(x) = \sqrt{x^2+1}$
b. $G(x) = \sqrt{x^2+1} + \sin(x^2)$
c. $H(x) = \sqrt{x^2+1} + \sin(x^2) + \dfrac{x^3+6}{\sqrt{x^2+1}}$

Solutions

a. The function F is the composition of the functions $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$, as follows: $F(x) = (f \circ g)(x)$. The function g is a polynomial; hence, it is continuous everywhere. The function f is continuous on $[0, \infty)$, and $x^2 + 1 \geq 0$ for any x, so f is continuous at $g(x)$ for any x. By Theorem 3.13, above, F is continuous for all x; that is, everywhere.

b. By Theorem 3.12, above, the function G is continuous everywhere, because it is the sum of two functions, both of which are continuous everywhere; that is, F (as in (b), above, and $\sin\left(x^2\right)$. Indeed the function $\sin\left(x^2\right)$ is continuous everywhere by Theorem 3.13, because it is the composition of $\sin t$ and x^2, both of which are continuous everywhere by Theorem 3.12.

c. The function $g(x) = \dfrac{x^3+6}{\sqrt{x^2+1}}$ is the quotient of a polynomial and the function F, both continuous everywhere, and $F(x) \geq 0$ for all x. So, by Theorem 3.12, it is true that g is continuous everywhere, and the function H is the sum of two everywhere continuous functions. [Which ones?] Therefore, by Theorem 3.12, it is continuous everywhere.

If you think it through carefully, you will see that the theorem below summarizes Definition 3.10 and Theorems 3.11 to 3.13. This theorem is the key to evaluating limits for continuous functions.

Theorem 3.14. If f is any of the following functions

a. an algebraic function,
b. a trigonometric function,
c. the sum, difference, product or quotient of algebraic and trigonometric functions, or
d. a composition of any of the three previous types of functions,

then the function is continuous in its domain and

$$\lim_{x \to a} f(x) = f(a) \quad \text{for every } a \text{ in the domain of } f$$

Example 3.27. By Theorem 3.14, all of the following functions are continuous on their domains. To find the domain, look for the variables for which the function is not defined, and eliminate them from the set of real numbers; what is left is the domain of the function. Verify that the domains listed below are correct.

$F(x) = x^2 \cos(x+1) - \sqrt{x}$	$D_F = [0, \infty)$
$g(t) = \dfrac{t^2(6-\cos t)^2}{3t - \sqrt{1-t}}$	the domain is all $t \leq 1$ except $t = \dfrac{-1 \pm \sqrt{37}}{18}$
$h(x) = \sqrt{x-\pi} \tan x$	the domain is all intervals of the form $\left[\pi, \dfrac{3\pi}{2}\right), \left(\dfrac{2k+1}{2}\pi, \dfrac{2k+3}{2}\pi\right)$, for any integer $k > 0$

Example 3.28. Use Theorem 3.14 to determine whether each of the functions below is continuous at the given number.

a. $G(x) = \sec(2x - \pi) + \dfrac{3}{x + \pi}$ $\qquad x = \dfrac{\pi}{2}$

b. $h(t) = \dfrac{6t\sin(t - 1)}{t^2 - 1}$ $\qquad t = 1$

c. $u(s) = \csc(2s) - \sqrt{5s + 1}$ $\qquad s = 0$

Solutions

First observe that all of these functions are of the type listed in Theorem 3.14.

a. The function G is defined at $\pi / 2$; hence, $\pi / 2$ is in the domain of G, and by Theorem 3.14, G is continuous at $\pi / 2$.
b. The function h is not defined at 1 ($t^2 - 1 = 0$ for $t = 1$); hence, $t = 1$ is not in the domain of h, and the function is not continuous at 1.
c. The function u is not defined at 0 because $\csc(0)$ is not defined, so 0 is not in the domain of u, and by Theorem 3.14, u is not continuous at 0.

Example 3.29. Finally, we will use Theorem 3.14 to evaluate the limits given below.

a. $$\lim_{t \to -3} \frac{t^2(6 - \cos t)^2}{3t - \sqrt{1 - t}} = -\frac{9(6 - \cos 3)^2}{11},$$

because the function

$$g(t) = \frac{t^2(6 - \cos t)^2}{3t - \sqrt{1 - t}}$$

is defined at -3; hence, -3 is in its domain, and by Theorem 3.14, the function is continuous at -3 and the limit is equal to $g(-3)$.

b. We leave the explanation of the second result to you as an exercise:

$$\lim_{x \to \frac{\pi}{2}} \sec(2x - \pi) + \frac{3}{x + \pi} = \sec(0) + \frac{3}{(3 / 2)\pi} = 1 + \frac{2}{\pi}$$

Exercises

6. Use Theorem 3.14 to explain why the functions in Exercises 21-28 on page 69 of the textbook are continuous on their domains.
7. Use Theorem 3.14 to evaluate the limits in Exercises 3-9 on page 80 of the textbook.

Answers to Exercises (appendix-a.htm)

Pay attention to Theorem 3.14. Most of the functions we study in this course are of the type listed in the theorem, so to evaluate a limit

$$\lim_{x \to a} f(x),$$

the first thing we do is to see if we can apply Theorem 3.14. That is, we ask, “Is f continuous at a?” or, in other words, “Is $f(a)$ defined?” If the answer is “Yes,” then we conclude that

$$\lim_{x \to a} f(x) = f(a).$$

Remember, however, when solving problems, that you must be careful in applying Theorem 3.14.

We also have the concept of continuity on the right and left. The graph of a function that is continuous on the right at a looks like that shown in Figure 3.14, below, and the graph of a function that is continuous on the left of a looks like that shown in Figure 3.15.

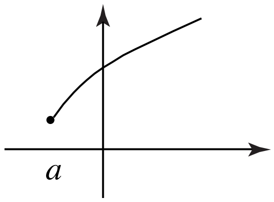


Figure 3.14. Function f, continuous on the right at a

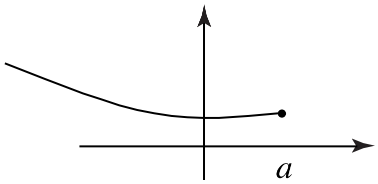


Figure 3.15. Function f, continuous on the left at a

If we have understood the definition of continuity, we can see that the following definitions correspond to the graphs in Figures 3.14 and 3.15.

Definition* 3.15.

a. A function f defined on the right of a is *continuous at the right* of a if

$$\lim_{x \to a^+} f(x) = f(a).$$

b. A function f defined on the left of a is *continuous at the left* of a if

$$\lim_{x \to a^-} f(x) = f(a).$$

From Definition 3.15, we understand that when a function is continuous on the right at a, then $f(a)$ is defined at a, and $\lim_{x \to a^+} f(x) = f(a)$. Similarly, when a function is continuous on the left at a, then $f(a)$ is defined, and $\lim_{x \to a^-} f(x) = f(a)$.

Example 3.30. Look carefully at the graph of Exercise 3 on page 68 of the textbook. The graph shows that this function is continuous on the left at -2, and on the right at 2 and 4. It is not continuous either to the left or the right at -4.

Example 3.31. Look carefully at the graph of Exercise 4 on page 68 of the textbook. The graph shows that this function is continuous on the right at -4 and 2. It is not continuous either to the left or the right at -2, 4, 6 or 8.

Example 3.32. Consider the following function:

$$g(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 3 \\ 3 - x & \text{if } 3 < x < 6 \\ \frac{1}{x+5} & \text{if } 6 < x \end{cases}$$

The polynomial function $1 - x^2$ is continuous everywhere; in particular for our question, in the interval $(-\infty, 3)$.

Similarly $3 - x$ is continuous on $(3, 6)$.

$\dfrac{1}{x+5}$ is continuous at all numbers but -5; hence, it is continuous on $(6, \infty)$.

Therefore, g is continuous on the intervals $(-\infty, 3)$, $(3, 6)$, and $(6, \infty)$.

We also see that $g(3) = 1 - (3)^2 = -8$, and $g(6)$ is undefined. Hence, g is not continuous at 6, and it may be continuous on the right or left at 3. To see if this is the case, we must evaluate the limits on the left and right of 3.

Observe that $x > 3$ if $x \to 3^+$; hence, for this x, we have $g(x) = 3 - x$. Thus,

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} 3 - x = 3 - 3 = 0 \qquad \text{(by Theorem 3.14).}$$

If $x \to 3^-$, then $x < 3$, and for this x, we have $g(x) = 1 - x^2$. Thus,

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} 1 - x^2 = -8 \qquad \text{(by Theorem 3.14).}$$

Since

$$\lim_{x\to 3^+} g(x) = 0 \neq g(3) = \lim_{x\to 3^-} g(x),$$

we conclude that the function is continuous on the left at 3, and that it is not continuous at 3.

Example 3.33. In this example, we will see that the piecewise function given below is continuous at 0, discontinuous at 1, and continuous on the right at 1.

$$h(x) = \begin{cases} x^3 - x^2 + \sqrt{6} & \text{if} \quad x \leq 0 \\ \sqrt{6+x} & \text{if} \quad 0 \leq x < 1 \\ x + 3 & \text{if} \quad 1 \leq x \end{cases}$$

First, we see that h is defined at 0 and 1, as follows: $h(0) = \sqrt{6}$ and $h(1) = 4$.

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \sqrt{x+6}$$

and

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} x^3 - x^2 + \sqrt{6} = \sqrt{6}.$$

Therefore, h is continuous at 0, since $\lim\limits_{x\to 0} h(x) = \sqrt{6} = h(0)$.

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} \sqrt{x+6} = \sqrt{7} \quad \text{and} \quad \lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} x + 3 = 4$$

Hence, the function is not continuous at 1, but is continuous on the right at 1. [Why?]

Exercises

8. Study Examples 4 and 9 on pages 75 and 77 of the textbook.
9. Do Exercises 17, 19, 33, 35, 36 and 37 on page 69 of the textbook.
10. Do Exercises 46 and 48 on page 81 of the textbook.

Answers to Exercises (appendix-a.htm)

Using Algebra to Evaluate Limits

Prerequisites

To complete this section, you must be able to

1. simplify algebraic expressions. See Unit 1 in this *Study Guide* and the PDF document titled "Review of Algebra," available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

2. factor special polynomials. See page 1 of your textbook.

In the next three sections, we consider evaluating limits at a point a to which Theorem 3.14 (unit03-01.htm#theorem3-14), does not apply; that is, limits of functions that are not continuous at a.

What happens when we simplify algebraic expressions? For example, to simplify

$$\frac{x^2-4}{x-2},$$

we would follow this procedure:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2; \quad \text{thus} \quad \frac{x^2-4}{x-2} = x+2.$$

When we are evaluating functions, this answer is not quite correct. What is correct in this equality is that

$$\frac{x^2-4}{x-2} = x+2 \quad \text{is true (holds) for all } x \text{ except } 2.$$

The function $f(x) = \dfrac{x^2-4}{x-2}$ is not defined at 2, whereas the function $g(x) = x+2$ is defined for all x.

These two functions are not equal, because two functions f and g are equal only if they have the same domain and if $f(x) = g(x)$ for all x in their domains.

The functions $f(x) = \dfrac{x^2-4}{x-2}$ and $g(x) = x+2$ do not have the same domain.

Figures 3.16 and 3.17, below, show the graphs of $f(x) = \dfrac{x^2-4}{x-2}$ and $g(x) = x+2$, respectively. As you can see, while the graph of f has a hole because it is not defined at 2, the graph of g is continuous.

To evaluate the limit, we must consider only values of $f(x)$ for x around 2. For these xs, we have $f(x) = g(x)$; that is, $f(x) = g(x)$ for $x \to a$. Moreover, from the graphs of these functions, we see that indeed

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x).$$

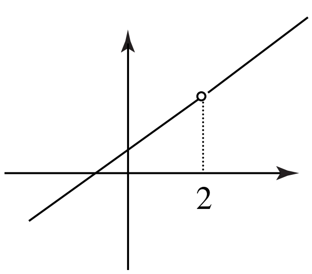


Figure 3.16. Function $f(x) = \dfrac{x^2-4}{x-2}$

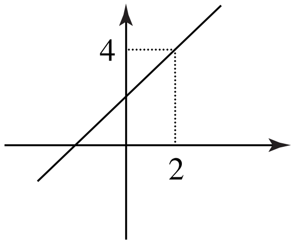


Figure 3.17. Function $g(x) = x + 2$

Theorem 3.14 (unit03-01.htm#theorem3-14) does not apply when we want to evaluate the limit $\lim_{x \to 2} f(x)$, because f is not continuous at 2, but it does apply to the limit $\lim_{x \to 2} g(x)$. [Why?] Hence, we conclude that

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} x + 2 = 4.$$

In summary, we may be able to evaluate the limit $\lim_{x \to a} f(x)$ of a function f which is not continuous at a, if we can obtain, through an algebraic manipulation of f, a function g, such that

$$f(x) = g(x) \quad \text{for all} \quad x \quad \text{around} \quad a, \quad \text{and} \quad \lim_{x \to a} g(x) \quad \text{exists.}$$

Thus

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

This conclusion is possible because of the following proposition.

Proposition 3.16.

a. If $f(x) = g(x)$ for all x close to a from the right, but not at a, then

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x).$$

b. If $f(x) = g(x)$ for all x close to a from the left, but not at a, then

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} g(x).$$

c. If $f(x) = g(x)$ for all x close to a, but not at a, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

To use this approach in the evaluation of limits, we must learn which algebraic manipulations we can try for which functions. Here is where our algebraic skills will pay off. In the following examples, we try to cover the most common cases; you may be able to find other examples on your own.

Example 3.34. Rational functions.

Consider the limit

$$\lim_{x \to 3} \frac{x^2 + x - 12}{3x^2 - 7x - 6}.$$

The rational function

$$f(x) = \frac{x^2 + x - 12}{3x^2 - 7x - 6}$$

is not continuous at 3.

If we evaluate the polynomials $p(x) = x^2 + x - 12$ and $q(x) = 3x^2 - 7x - 6$ at 3, we get 0 (try it).

There is a theorem that says that if $p(3) = 0$, then p can be factored as a product of the form $(x-3)r(x)$, where $r(x)$ is a polynomial. The same is true for $q(x)$: there is a polynomial $s(x)$ such that $q(x) = (x-3)s(x)$. As we indicated in Unit 1,

$$p(x) = (x-3)(x+4) \quad \text{and} \quad q(x) = (x-3)(3x+2).$$

Hence,

$$\frac{x^2+x-12}{3x^2-7x-6} = \frac{(x-3)(x+4)}{(x-3)(3x+2)} = \frac{x+4}{3x+2}$$

for all x around 3.

The function

$$g(x) = \frac{x+4}{3x+2}$$

is continuous at 3, and we conclude that

$$\lim_{x\to 3} \frac{x^2+x-12}{3x^2-7x-6} = \lim_{x\to 3} \frac{x+4}{3x+2} = \frac{7}{11}.$$

> **Note:** If you do not remember how to factor a polynomial, but you know one of the factors, you can always use long division to find the other factor, as is shown below (see also Unit 1).

$$\begin{array}{r|l} & \quad\;\; 3x+2 \\ \hline x-3 & 3x^2-7x-6 \\ & \underline{-3x^2+9x} \\ & \qquad\;\; 2x-6 \\ & \qquad \underline{-2x+6} \\ & \qquad\qquad 0 \end{array}$$

Hence, $3x^2-7x-6 = (x-3)(3x+2)$.

WARNING: It is incorrect to write

$$\lim_{x\to 3} \frac{x^2+x-12}{3x^2-7x-6} = \frac{x+4}{3x+2} = \frac{7}{11}.$$

This simply does not make sense: it is not true that

$$\lim_{x\to 3} \frac{x^2+x-12}{3x^2-7x-6} = \frac{x+4}{3x+2}.$$

A limit is not equal to an algebraic expression, and the equality

$$\frac{x+4}{3x+2} = \frac{7}{11}$$

is even worse: it is an equation and is not what we mean or want to mean.

Example 3.35. Algebraic operations.

The function in the limit

$$\lim_{h\to 0} \frac{(4+h)^2-16}{h^2+h}$$

is also rational and discontinuous at 0. To simplify it, we do the algebraic operations and factorization as follows:

$$\frac{(4+h)^2-16}{h^2+h} = \frac{16+8h+h^2-16}{h(h+1)} = \frac{8h+h^2}{h(h+1)} = \frac{8+h}{h+1}.$$

This is true for all $h \neq 0$. We arrive at a continuous function at 0, namely

$$g(h) = \frac{8+h}{h+1}.$$

Hence,

$$\lim_{h\to 0} \frac{(4+h)^2 - 16}{h^2 + h} = \lim_{h\to 0} \frac{8+h}{h+1} = 8.$$

Example 3.36. Fractions.

To simplify the function in the limit

$$\lim_{t\to 0} \frac{1}{t} - \frac{1}{t^2+t},$$

we must perform the operations of the indicated fractions.

$$\frac{1}{t} - \frac{1}{t^2+t} = \frac{t^2+t-t}{t(t^2+t)} = \frac{t^2}{t^2(t+1)} = \frac{1}{t+1} \quad \text{for } t \neq 0.$$

We again arrive at a continuous function at 0.

$$\lim_{t\to 0} \frac{1}{t} - \frac{1}{t^2+t} = \lim_{t\to 0} \frac{1}{t+1} = 1.$$

Example 3.37. Rationalization.

The function in the limit

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}$$

is not continuous at 0 and it involves square roots. So, we rationalize the function as follows:

$$\left(\frac{\sqrt{9+h}-3}{h}\right)\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right) = \frac{1}{\sqrt{9+h}+3} \quad \text{for } h \neq 0.$$

We can apply Theorem 3.14 (unit03-01.htm#theorem3-14) to conclude that

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6}.$$

Example 3.38. Special Polynomials.

For the limit

$$\lim_{x\to 8} \frac{x^{1/3}-2}{x-8},$$

we use the factorization of the difference of cubes.

$$\begin{aligned} \frac{x^{1/3}-2}{x-8} &= \frac{x^{1/3}-2}{\left(x^{1/3}\right)^3 - 2^3} \\ &= \frac{x^{1/3}-2}{\left(x^{1/3}-2\right)\left(x^{2/3}+2x^{1/3}+2^2\right)} \\ &= \frac{1}{x^{2/3}+2x^{1/3}+2^2} \quad \text{for } x \neq 8 \end{aligned}$$

Hence, by Proposition 3.16 and Theorem 3.14 (unit03-01.htm#theorem3-14),

$$\lim_{x\to 8} \frac{x^{1/3}-2}{x-8} = \lim_{x\to 8} \frac{1}{x^{2/3}+2x^{1/3}+2^2} = \frac{1}{12}.$$

Tutorial 1: Evaluation of a limit of a discontinuous function at 3, by algebraic manipulation. (../multimedia/tutorials/265Limit01/index.html)

Exercises

11. Study Examples 3, 5, 6 and 7 on pages 75-77 of the textbook.
12. Do at least 10 exercises from numbers 11 to 30 on page 80 of the textbook.

Answers to Exercises (appendix-a.htm)

Infinite Limits: Vertical Asymptotes

Prerequisites

To complete this section, you must be able to

1. draw the graphs of basic functions. See Unit 2 in this *Study Guide*, and pages 30-37 of the textbook.
2. sketch transformations of functions. See Unit 2 in this *Study Guide*, and pages 30-37 of the textbook.

In this section, we discuss evaluating limits $\lim_{x \to a} f(x)$, where the function f is neither continuous at a nor capable of being simplified into another function (as suggested in the previous section). For such limits, we must analyse the function.

We start with the problem of sketching the graph of the *reciprocal function*:

$$f(x) = \frac{1}{g(x)}$$

if the graph of $g(x)$ is known.

Study the following statements about the reciprocal function. You must understand and agree with them, not just believe or accept them.

- If $g(a) = 0$, then

  $$f(x) = \frac{1}{g(x)}$$

  is not defined at a.

- If $g(x) > 0$ on an interval J, then

  $$f(x) = \frac{1}{g(x)} > 0$$

  on the same interval J.

- If $g(x) < 0$ on an interval J, then

  $$f(x) = \frac{1}{g(x)} < 0$$

  on the same interval J.

- If $g(a)$ is positive and near 0, then

  $$f(x) = \frac{1}{g(a)}$$

  is positive and very large (far from 0).

- If $g(a)$ is negative and near 0, then

  $$f(x) = \frac{1}{g(a)}$$

  is negative and very small (far from 0).

What is $f(x) = \dfrac{1}{g(x)}$ like if $g(x)$ is positive and very large (far from 0)?

What about if $g(x)$ is negative and very small (far from 0)?

Example 3.39. The graph of the identity function $g(x) = x$ is shown in Figure 3.18, below.

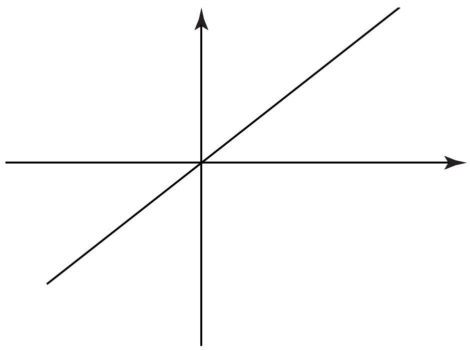


Figure 3.18. Function $g(x) = x$

From the statements above, we know the following facts about the identity function's reciprocal function, $f(x) = 1 \ / \ x$:

- $g(0) = 0$; hence, $f(x) = \dfrac{1}{x}$ is not defined at 0.
- $g(x) > 0$ on $(0,\infty)$; hence, $f(x) = \dfrac{1}{x} > 0$ for $x > 0$.
- $g(x) < 0$ on $(-\infty,0)$; hence, $f(x) = \dfrac{1}{x} < 0$ for $x < 0$.
- $g(x) > 0$ and very small for $x \to 0^+$; hence, $f(x) = \dfrac{1}{x} > 0$ and very large (far from 0) for $x \to 0^+$.
- $g(x) < 0$ and very large for $x \to 0^-$; hence, $f(x) = \dfrac{1}{x} < 0$ and very small (far from 0) for $x \to 0^-$.
- $f(x) = \dfrac{1}{x} > 0$ and near to 0, if x is very large and positive (far from 0). [Why?]
- $f(x) = \dfrac{1}{x} < 0$ and near to 0, if x is very small and negative (far from 0). [Why?]

Using this information, we see that the graph of $f(x) = \dfrac{1}{x}$ is as shown in Figure 3.19, below.

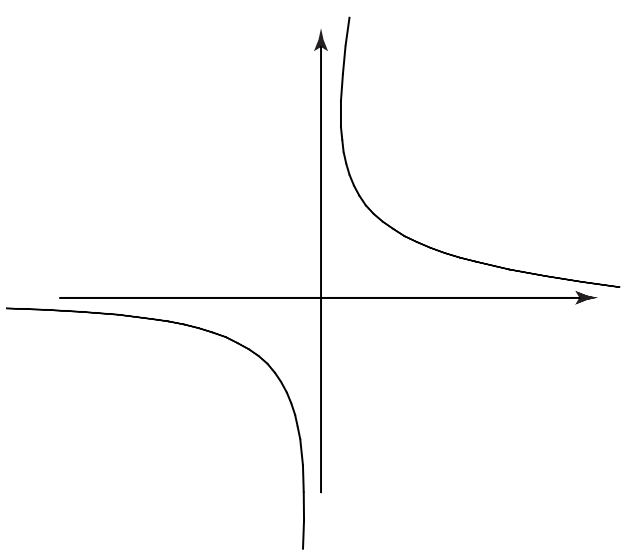


Figure 3.19. Function $f(x) = \dfrac{1}{x}$

Example 3.40. If $g(x) = x^2 - 1$, then the graph of g is as shown in Figure 3.20, below, and its reciprocal is the function $f(x) = \dfrac{1}{x^2 - 1}$, shown in Figure 3.21. Observe why.

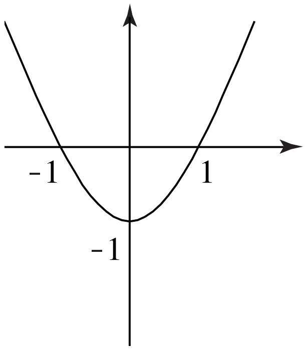


Figure 3.20. Function $g(x) = x^2 - 1$

Applying the statements above to the graph in Figure 3.20, we find the following facts about its reciprocal:

- $f(x) = \dfrac{1}{x^2 - 1}$ is not defined at 1 or -1.
- $f(x) = \dfrac{1}{x^2 - 1} > 0$ on $(-\infty, -1)$ and $(1,\infty)$.
- $f(x) = \dfrac{1}{x^2 - 1} < 0$ on $(-1,1)$.
- $f(x) = \dfrac{1}{x^2 - 1} > 0$ and far from 0 for $x \to 1^+$ and for $x \to -1^-$.
- $f(x) = \dfrac{1}{x^2 - 1} < 0$ and far from 0 for $x \to 1^-$ and for $x \to -1^+$.
- $f(x) = \dfrac{1}{x^2 - 1} > 0$ and very small (close to 0) for very large positive x, and very small negative x.

Using this information, we determine that the graph $f(x) = \dfrac{1}{x^2 - 1}$ is as shown in Figure 3.21, below.

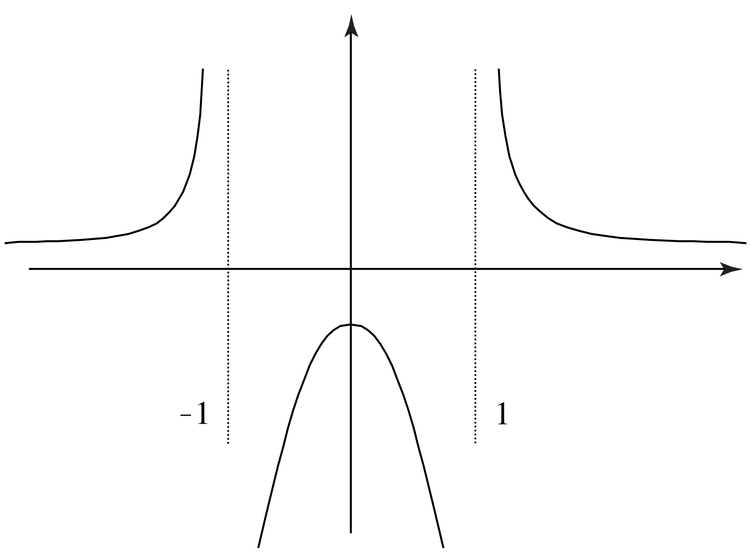


Figure 3.21. Function $f(x) = \dfrac{1}{x^2 - 1}$

Example 3.41. In Figure 3.22, below, we show the graph of a function $g(x)$, and in Figure 3.23 we show the graph of its reciprocal $f(x) = \dfrac{1}{g(x)}$. Examine these graphs. Do you agree with them?

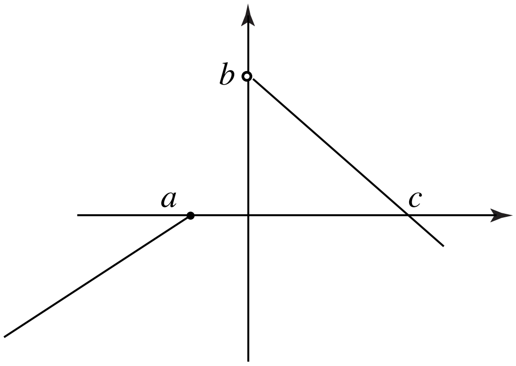


Figure 3.22. Function $g(x)$

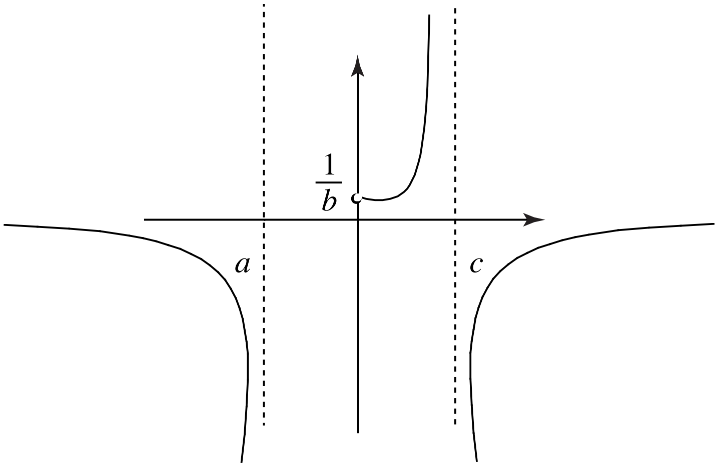


Figure 3.23. Function $f(x) = \dfrac{1}{g(x)}$

Example 3.42. In Figure 3.24, below, we show the graph of the function $g(x) = |x|$, and in Figure 3.25, the graph of its reciprocal. Examine these graphs. Do you agree with them?

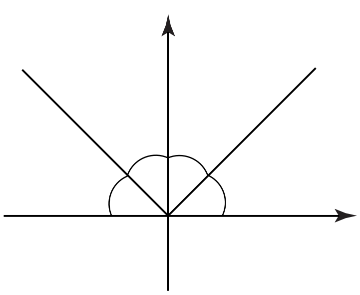


Figure 3.24. Function $g(x) = |x|$

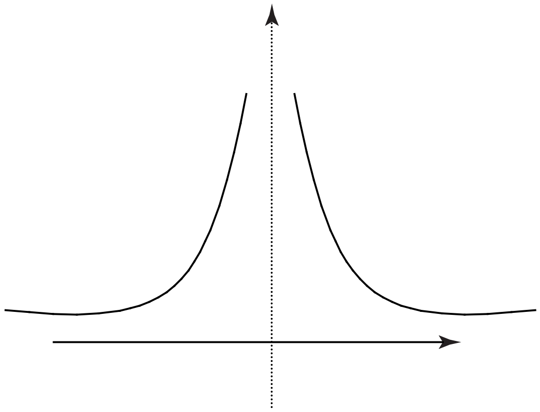


Figure 3.25. Function $f(x) = \dfrac{1}{g(x)}$; that is, $f(x) = \dfrac{1}{|x|}$

From the graphs of Examples 3.39 to 3.42, we derive the following theorem.

Theorem 3.17.

a. If $\lim\limits_{x \to a^+} g(x) = 0$ and $g(x) > 0$ for $x \to a^+$, then $\lim\limits_{x \to a^+} \dfrac{1}{g(x)} = \infty$.

b. If $\lim\limits_{x \to a^-} g(x) = 0$ and $g(x) > 0$ for $x \to a^-$, then $\lim\limits_{x \to a^-} \dfrac{1}{g(x)} = \infty$.

c. If $\lim\limits_{x \to a^+} g(x) = 0$ and $g(x) < 0$ for $x \to a^+$, then $\lim\limits_{x \to a^+} \dfrac{1}{g(x)} = -\infty$.

d. If $\lim\limits_{x \to a^-} g(x) = 0$ and $g(x) < 0$ for $x \to a^-$, then $\lim\limits_{x \to a^-} \dfrac{1}{g(x)} = -\infty$.

Therefore,

e. $\lim\limits_{x \to a} \dfrac{1}{g(x)} = \infty$, if $\lim\limits_{x \to a} g(x) = 0$ and $g(x) > 0$ for $x \to a$.

f. $\lim\limits_{x \to a} \dfrac{1}{g(x)} = -\infty$, if $\lim\limits_{x \to a} g(x) = 0$ and $g(x) < 0$ for $x \to a$.

Example 3.43. The function $f(x) = \dfrac{1}{\sqrt{9+x}-3}$ in the limit

$$\lim_{x \to 0^+} \frac{1}{\sqrt{9+x}-3}$$

is the reciprocal of the function $g(x) = \sqrt{9+x} - 3$.

However, $\lim\limits_{x \to 0^+} \sqrt{9+x} - 3 = 0$ and $\sqrt{9+x} - 3 > 0$, for $x \to 0^+$; hence, by Theorem 3.18(a), we conclude that

$$\lim_{x \to 0^+} \frac{1}{\sqrt{9+x}-3} = \infty.$$

Note: To see that $\sqrt{9+x} - 3 > 0$ for $x \to 0^+$, take an x positive and close to 0, say $x = 0.1$, and check that $\sqrt{9+0.1} - 3 > 0$.

Example 3.44. By Theorem 3.17,

$$\lim_{x \to 0^-} \frac{1}{\sqrt{x+9}-3} = -\infty,$$

because $\lim\limits_{x \to 0^-} \sqrt{x+9}-3 = 0$, and $\sqrt{x+9}-3 < 0$ for $x \to 0^-$.

From these two examples, we conclude that the limit

$$\lim_{x \to 0} \frac{1}{\sqrt{x+9}-3}$$

does not exist.

Example 3.45.

$$\lim_{x \to 1^-} \frac{1}{x^2+x-2} = -\infty$$

because $\lim\limits_{x \to 1^-} x^2+x-2 = 0$ and $x^2+x-2 < 0$ for $x \to 1^-$.

Example 3.46.

$$\lim_{x \to 0^+} \frac{1}{\sin x + \tan x} = \infty$$

because $\lim\limits_{x \to 0^+} \sin x + \tan x = 0$, and $\sin x + \tan x > 0$ for $x \to 0^+$.

Exercises

13. Without looking at the graphs of the trigonometric functions, find the following limits:
    a. $\lim\limits_{x \to \pi/2^+} \sec x$.
    b. $\lim\limits_{x \to 0^-} \csc x$.
14. Without looking at the graph of the secant function, explain why the limit $\lim\limits_{x \to \pi/2} \sec x$ does not exist.
15. Evaluate each of the limits below.
    a. $\lim\limits_{x \to 0} \csc^2 x$
    b. $\lim\limits_{x \to 0^-} \dfrac{1}{\sin x + \tan x}$
    c. $\lim\limits_{x \to 0} \dfrac{1}{x^2 \cos x}$

Answers to Exercises (appendix-a.htm)

Now, let us analyse a limit of the form $\lim\limits_{x \to a} h(x)g(x)$, where

$$\lim_{x \to a} g(x) = L \quad (L \neq 0), \quad \text{and} \quad \lim_{x \to a} h(x) = \pm\infty.$$

We have several cases to consider.

I. $\lim\limits_{x \to a} g(x) = L > 0$ and $\lim\limits_{x \to a} h(x) = \infty$

For $x \to a$, we have $g(x) \to L$ and $h(x) \to \infty$. So, for x close to a, $g(x)$ is close to $L > 0$, and $h(x)$ is very large and positive; hence, the product $g(x)h(x)$ is positive and very large, and we have

$$\lim_{x \to a} h(x)g(x) = \infty.$$

II. $\lim_{x \to a} g(x) = L < 0$ and $\lim_{x \to a} h(x) = -\infty$

For $x \to a$, we have $g(x) \to L$ and $h(x) \to -\infty$. So, for x close to a, $g(x)$ is close to $L < 0$, and $h(x)$ is very small and negative; hence, the product $g(x)h(x)$ is positive and very large, and we have

$$\lim_{x \to a} h(x)g(x) = \infty.$$

III. $\lim_{x \to a} g(x) = L > 0$ and $\lim_{x \to a} h(x) = -\infty$

For $x \to a$, we have $g(x) \to L$ and $h(x) \to -\infty$. So, for x close to a, $g(x)$ is close to $L > 0$, and $h(x)$ is very small and negative; hence, the product $g(x)h(x)$ is negative and very small, and we have

$$\lim_{x \to a} h(x)g(x) = -\infty$$

IV. $\lim_{x \to a} g(x) = L < 0$, and $\lim_{x \to a} h(x) = \infty$

For $x \to a$ we have $g(x) \to L$ and $h(x) \to \infty$. So, for x close to a, $g(x)$ is close to $L < 0$ and $h(x)$ is very large and positive; hence, the product $g(x)h(x)$ is negative and very small, and we have

$$\lim_{x \to a} h(x)g(x) = -\infty$$

Therefore, we have Theorem 3.18.

Theorem 3.18.

a. The limit $\lim_{x \to a} h(x)g(x) = \infty$ if either

I. $\lim_{x \to a} g(x) = L > 0 \quad \text{and} \quad \lim_{x \to a} h(x) = \infty \quad \text{or}$

II. $\lim_{x \to a} g(x) = L < 0 \quad \text{and} \quad \lim_{x \to a} h(x) = -\infty$

b. The limit $\lim_{x \to a} h(x)g(x) = -\infty$ if either

I. $\lim_{x \to a} g(x) = L < 0 \quad \text{and} \quad \lim_{x \to a} h(x) = \infty \quad \text{or}$

II. $\lim_{x \to a} g(x) = L > 0 \quad \text{and} \quad \lim_{x \to a} h(x) = -\infty$

This theorem holds if $x \to a$ is replaced by $x \to a^+$ or $x \to a^-$.

Tutorial 2: The steps we must follow to conclude that a limit does not exist. (../multimedia/tutorials/265Limit02/index.html)

Example 3.47. The limit

$$\lim_{x \to 0^-} \frac{x - 5}{\sqrt{x + 9} - 3}$$

is of the form $\lim_{x \to 0^-} (x - 5)\dfrac{1}{\sqrt{x + 9} - 3}$.

a. $\lim_{x \to 0^-} \dfrac{1}{\sqrt{x + 9} - 3} = -\infty$ by Example 3.43

b. $\lim_{x \to 0^-} x - 5 = -5 < 0$ by Theorem 3.14 (unit03-01.htm#theorem3-14)

Hence, by Theorem 3.18(b), $\lim\limits_{x \to 0^-} \dfrac{x-5}{\sqrt{x+9}-3} = \infty$.

Note: An algebraic manipulation of the function $f(x) = \dfrac{x-5}{\sqrt{x+9}-3}$ is *not* a better approach than the one we use here, try it.

Example 3.48. For the limit

$$\lim_{x \to 0^-} \frac{x^2+1}{x^2 \sin x} = \lim_{x \to 0^-} (x^2+1)\frac{1}{x^2 \sin x},$$

we have $x^2 \sin x < 0$, for $x \to 0^-$.

Furthermore, by Theorem 3.14 (unit03-01.htm#theorem3-14),

$$\lim_{x \to 0^-} x^2+1 = 1 > 0 \quad \text{and} \quad \lim_{x \to 0^-} x^2 \sin x = 0.$$

Therefore, by Theorem 3.17,

$$\lim_{x \to 0^-} \frac{1}{x^2 \sin x} = -\infty.$$

We conclude, by Theorem 3.18, that

$$\lim_{x \to 0^-} \frac{x^2+1}{x^2 \sin x} = -\infty.$$

Example 3.49. Let us evaluate the limit $\lim\limits_{x \to 2} \dfrac{\sqrt{x+3}}{(x-2)^2}$.

a. $\lim\limits_{x \to 2} \sqrt{x+3} = \sqrt{5}$ by Theorem 3.14 (unit03-01.htm#theorem3-14).

b. $\lim\limits_{x \to 2} \dfrac{1}{(x-2)^2} = \infty$ since $\lim\limits_{x \to 2} (x-2)^2 = 0$ and $(x-2)^2 > 0$ for $x \to 2$ (Theorem 3.17).

Hence, by Theorem 3.18,

$$\lim_{x \to 2} \frac{\sqrt{x+3}}{(x-2)^2} = \lim_{x \to 2} \sqrt{x+3}\frac{1}{(x-2)^2} = \infty.$$

Example 3.50. $\lim\limits_{x \to \pi/2^+} \left(x^2-3\right)\sec x = \infty$

because

a. $\lim\limits_{x \to \pi/2^+} x^2-3 = \frac{\pi^2}{4} - 3 < 0$ by Theorem 3.14 (unit03-01.htm#theorem3-14).

b. $\lim\limits_{x \to \pi/2^+} \sec x = -\infty$ by Theorem 3.17.

From the graphs of Examples 3.39 to 3.42, we can see that a vertical line $x = a$ is a vertical asymptote if a limit at a is infinity.

Definition 3.19. The vertical line $x = a$ is a *vertical asymptote* of the function $f(x)$ if any of the following limits is true.

$$\lim_{x \to a^+} f(x) = \infty \quad \lim_{x \to a^+} f(x) = -\infty$$
$$\lim_{x \to a^-} f(x) = \infty \quad \lim_{x \to a^-} f(x) = -\infty$$
$$\lim_{x \to a} f(x) = \infty \quad \lim_{x \to a} f(x) = -\infty$$

To find vertical asymptotes, we take any of the first four limits listed in Definition 3.19 (since these limits always exist), at a point or points a where the function is not defined. But note that, even when a function is not defined at a point a, the line $x = a$ may not be a vertical asymptote.

For help with the concept of vertical asymptotes, view the PowerPoint tutorials below. To access a tutorial:

1. Click on the file link to open it.
2. Save the file (.ppsx) to your computer's hard drive.
3. Click on the PowerPoint file to view.

Note: If you don't have PowerPoint installed on your computer, you can download the PowerPoint Viewer from here: https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6 (https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6).

PowerPoint 3 (../multimedia/powerpoint/265U3VAsymtpte1.ppsx): For x close to a from the left, the values of the function $f(x)$ grow without bound. Hence $x = a$ is a vertical asymptote.

PowerPoint 4 (../multimedia/powerpoint/265U3VAsymtpte2.ppsx): For x close to a from the right, the values of the function $f(x)$ grow without bound. Hence $x = a$ is a vertical asymptote.

PowerPoint 5 (../multimedia/powerpoint/265U3VAsymtpte3.ppsx): For x close to a from the left, the values of the function $f(x)$ decrease without bound. Hence $x = a$ is a vertical asymptote.

PowerPoint 6 (../multimedia/powerpoint/265U3VAsymtpte4.ppsx): For x close to a from the right, the values of the function $f(x)$ decrease without bound. Hence $x = a$ is a vertical asymptote.

Example 3.51. Although the function

$$f(h) = \frac{(4+h)^2 - 16}{h^2 + h}$$

is not defined at 0, the vertical line $x = 0$ is not a vertical asymptote, because, as we showed in Example 3.35 (unit03-02.htm#example3-35), the limit

$$\lim_{h\to 0} \frac{(4+h)^2 - 16}{h^2 + h} = 8$$

is finite.

However, the function is not defined at -1 either. [Why?]

We have

$$\lim_{h\to -1^+} \frac{(4+h)^2 - 16}{h^2 + h} = \lim_{h\to -1^+} \frac{8+h}{h+1} = \infty$$

by Theorem 3.18, since

$$\lim_{h\to -1^+} \frac{1}{h+1} = \infty. \quad \text{[By which theorem?]}$$

Finally,

$$\lim_{h\to -1^+} 8 + h = 7 > 0.$$

Hence, for this function $x = -1$ is a vertical asymptote.

For the composition of functions, we have the following theorem.

Theorem 3.20. If

$$\lim_{x\to a} g(x) = b \quad \text{and} \quad \lim_{x\to b} f(x) = \pm\infty,$$

then

$$\lim_{x\to a} f(g(x)) = \pm\infty.$$

The theorem holds if $x \to a$ is replaced by either $x \to a^+$ or $x \to a^-$; and $x \to b$ is replaced by either $x \to b^+$ or $x \to b^-$.

Theorem 3.20 says that if $u = g(x)$, then $u = g(x) \to b$ as $x \to a$, and

$$\lim_{x\to a} f(g(x)) = \lim_{u\to b} f(u) = \pm\infty$$

Example 3.52. The function in the limit

$$\lim_{x\to\pi^+} \tan\left(x - \frac{\pi}{2}\right)$$

is a composition of functions.

For $x \to \pi^+$, we have

$$x - \frac{\pi}{2} > \frac{\pi^+}{2};$$

hence,

$$u = x - \frac{\pi}{2} \to \frac{\pi^+}{2} \quad \text{as} \quad x \to \pi^+.$$

Furthermore, by Theorem 3.20,

$$\lim_{x\to\pi^+} \tan\left(x - \frac{\pi}{2}\right) = \lim_{x\to\pi/2^+} \tan u = -\infty$$

Observe that if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then for $x \to a$, both $f(x)$ and $g(x)$ increase without bound, and their sum and product also increase without bound.

Thus, $\lim_{x\to a} f(x) + g(x) = \infty$ and $\lim_{x\to a} f(x)g(x) = \infty$.

Moreover, for any nonzero number c, we know, by Theorem 3.18, that

$$\lim_{x\to a} cf(x) = \pm\infty \quad \text{if} \quad c > 0 \quad \text{or} \quad c < 0.$$

This fact explains, in part, the following theorem.

Theorem 3.21.

a. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ and c is a nonzero constant, then

$$\lim_{x\to a} f(x) + g(x) = \infty, \qquad \lim_{x\to a} f(x)g(x) = \infty \quad \text{and}$$
$$\lim_{x\to a} cf(x) = \pm\infty, \quad \text{if} \quad c > 0 \text{ or } c < 0.$$

b. If $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = -\infty$ and c is a nonzero constant, then

$$\lim_{x\to a} f(x) + g(x) = -\infty, \qquad \lim_{x\to a} f(x)g(x) = \infty \quad \text{and}$$
$$\lim_{x\to a} cf(x) = \pm\infty, \quad \text{if} \quad c > 0 \text{ or } c < 0.$$

c. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = -\infty$, then $\lim_{x\to a} f(x)g(x) = -\infty$

Example 3.53. Let us evaluate the limit

$$\lim_{x\to 0^+} \csc x + \cot x.$$

We know that

$$\lim_{x \to 0^+} \csc x = \infty \quad \text{and} \quad \lim_{x \to 0^+} \cot\ x = \infty;$$

so, by Theorem 3.21, we conclude that

$$\lim_{x \to 0^+} \csc x + \cot x = \infty.$$

WARNING: Theorem 3.21 does not say that

$$\lim_{x \to a} f(x) + g(x) = \infty + \infty = \infty.$$

This expression is incorrect because ∞ is a symbol, and it does not make sense to add symbols. Instead, Theorem 3.21 states that if two functions increase without bound, then their sum also increases without bound.

Example 3.54. By Theorem 3.21,

$$\lim_{x \to \pi/2^-} -\frac{\sec x}{3} = -\infty,$$

because $-1/3 < 0$ and $\lim\limits_{x \to \pi/2^-} \sec x = \infty$.

WARNING: Theorem 3.21 does not say that

$$\lim_{x \to a} cf(x) = c\infty.$$

This expression is incorrect because ∞ is a symbol, and it does not make sense to multiply a number by a symbol. Instead, Theorem 3.21 states that if a function increases without bound then the product of the multiplication of the function by a positive constant also increases without bound.

When working with infinite limits, we must be careful to apply theorems correctly, and not to infer more than they indicate. For instance, if $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = -\infty$, it is not true that $\lim\limits_{x \to a} f(x) + g(x) = 0$.

Exercises

16. Do Exercises 23-30 on page 59 of the textbook.
17. Explain why the line $x = 4$ is a vertical asymptote of each of the functions listed below.
    a. $f(x) = \dfrac{x^2 + 3x - 1}{x - 4}$
    b. $g(t) = \dfrac{\sin(t-4) + 3t}{t^2 - 16}$
18. Find the vertical asymptote or asymptotes of the function

    $$f(x) = \frac{x^2 + x - 12}{2x^2 - 5x - 3}.$$

19. If $\lim\limits_{x \to a} f(x) = \infty$ and c is any number, what would be the limit

    $$\lim_{x \to a} f(x) + c?$$

    Explain.
20. If $\lim\limits_{x \to a} f(x) = -\infty$ and $\lim\limits_{x \to a} g(x) = c$ for any number c, what would be the limit

    $$\lim_{x \to a} f(x) + g(x)?$$

    Explain.

21. Explain why

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty, \quad \lim_{x\to 0^+} \; -\frac{1}{x} = -\infty, \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x^2} - \frac{1}{x} = \infty.$$

Answers to Exercises (appendix-a.htm)

The Squeeze Theorem

Prerequisites

To complete this section you must be able to

1. apply the definitions of the trigonometric functions. See Unit 1 in this *Study Guide*.
2. apply the "Rules of Inequalities." See page 338 of the textbook.

In this section, we consider the evaluation of limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where

- the quotient of $\frac{f(x)}{g(x)}$ is not continuous at a, and so Theorem 3.14 (unit03-01.htm#theorem3-14) does not apply.
- the quotient cannot be simplified into a function to which Theorems 3.14, 3.17, 3.18, 3.20 or 3.21 can be applied.
- $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

To analyse such limits, we may need to apply the "Squeeze Theorem" (see Theorem 3 on page 78 of the textbook).

Theorem 3.22. Squeeze Theorem.

If

$$g(x) \leq f(x) \leq h(x) \quad \text{for} \quad x \to a,$$

and

$$\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x),$$

then

$$\lim_{x \to a} f(x) = L.$$

This theorem holds if $x \to a$ is replaced by $x \to a^+$ or $x \to a^-$.

To apply this theorem, we must find two functions g and h to "squeeze" the function of the limit f.

For functions that involve sine and cosine, we use the fact that $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all real numbers x.

Example 3.55. The function $\sqrt{x} \cos\left(\frac{1}{x}\right)$ is continuous on the interval $(0, \infty)$. [Why?]

To evaluate the limit $\lim_{x \to 0^+} \sqrt{x} \cos\left(\frac{1}{x}\right)$, we apply the Squeeze Theorem.

Recall that $-1 \leq \cos\theta \leq 1$. We can multiply this inequality by $\sqrt{x}$ to get the function

$$\sqrt{x} \cos\left(\frac{1}{x}\right)$$

in the middle:

$$-\sqrt{x} \leq \sqrt{x} \cos\left(\frac{1}{x}\right) \leq \sqrt{x}.$$

Note that the inequality does not change, because $\sqrt{x}$ is positive for all $x > 0$.

Since

$$\lim_{x \to 0^+} \sqrt{x} = 0 = \lim_{x \to 0^+} -\sqrt{x}$$

(by Theorem 3.14 (unit03-01.htm#theorem3-14)), we conclude that

$$\lim_{x \to 0^+} \sqrt{x} \cos\left(\frac{1}{x}\right) = 0$$

(by the Squeeze Theorem).

Note: The following example is very important; we will use it to find the derivative of trigonometric functions.

Example 3.56. An argument similar to the one used in Example 3.55 does not work when we attempt to evaluate the limit

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

[Try and see why.]

To find the two functions, we need to apply the Squeeze Theorem, we use the geometric argument given on page 146 of the textbook. You must read and understand this argument.

From the geometric argument, we see that

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}.$$

If $-\dfrac{\pi}{2} < \theta < 0$, then $0 < -\theta < \dfrac{\pi}{2}$, and by the conclusion above, we have

$$\cos(-\theta) < \tfrac{\sin(-\theta)}{-\theta} < 1.$$

Since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, we conclude that

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \quad \text{for} \quad -\frac{\pi}{2} < \theta < 0.$$

Therefore,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \quad \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \text{except} \quad \theta = 0.$$

Theorem 3.14 (unit03-01.htm#theorem3-14) tells us that $\lim\limits_{x \to 0} \cos x = 1 = \lim\limits_{x \to 0} 1$, and by the Squeeze Theorem, we conclude that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Exercises

22. Read Example 11 on page 79 of the textbook.
23. Use the Squeeze Theorem to evaluate the limits of Exercises 34, 35, 37 and 58 on pages 80 and 81 of the textbook.

Answers to Exercises (appendix-a.htm)

Laws of Limits

Prerequisites

To complete this section, you must be able to

1. apply the trigonometric identities. See the addition and subtraction formulas, the double-angle formulas and the half-angle formulas on the "Reference Pages" at the beginning of your textbook or on pages 362-363.
2. simplify algebraic expressions. See Unit 1 in this *Study Guide* and the PDF document titled "Review of Algebra," available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

   Keep in mind as well the factorization of special polynomials as listed in the "Reference Pages" at the front of your textbook.

In this section, we combine all of our previous results and definitions to evaluate limits of functions that are the sum, product, quotient or composition of two or more functions. The next theorem gives the conditions for the arithmetic operations of limits that are known as "Laws of Limits."

Theorem 3.23. Laws of Limits.

If c is a constant and $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

a. $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$
b. $\lim_{x\to a}\ cf(x) = c\lim_{x\to a} f(x) = cL$
c. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x) = LM$
d. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$ if $M \neq 0$

This theorem holds if $x \to a$ is replaced by $x \to a^+$ or $x \to a^-$.

For the composition of functions, we have the following results.

Theorem 3.24. If $\lim_{x\to a} g(x) = b$ and $\lim_{x\to b} f(x) = L$, then

$$\lim_{x\to a} f(g(x)) = L.$$

This theorem also holds if $x \to a$ is replaced by $x \to a^+$ or $x \to a^-$.

We can see that if $u = g(x)$, then $u = g(x) \to b$ as $x \to a$, and Theorem 3.24 states that

$$\lim_{x\to a} f(g(x)) = \lim_{u\to b} f(u) = L.$$

Two particular cases of Theorem 3.24 are important. The first occurs when the function f is continuous at b. In this case, $\lim_{x\to a} f(x) = f(a)$ and we have the following corollary.

Corollary 3.25. If $\lim_{x\to a} g(x) = b$ and f is continuous at b, then

$$\lim_{x\to a} f(g(x)) = f(b).$$

The second case occurs when the outside function f is the power function $f(x) = x^r$. Then the function f is continuous at b if $\lim_{x\to b}\ x^r = b^r$ (the number b^r is well defined). Hence, a direct application of Theorem 3.24 gives the following corollary.

Corollary 3.26. If $\lim_{x\to a} g(x) = b$ and $f(x) = x^r$ is continuous at b, then

$$\lim_{x\to a} f(g(x)) = \lim_{x\to a} (g(x))^r = \left(\lim_{x\to a} g(x)\right)^r = b^r$$

This corollary holds if $x \to a$ is replaced by $x \to a^+$ or $x \to a^-$.

Observe that in order to apply Theorem 3.23, we must know that the limits of f and g are finite. To apply the Laws of Limits, we may need to manipulate the function first, and use Proposition 3.16 (unit03-02.htm#proposition3-16).

Example 3.57. The function

$$f(x) = \frac{x}{\sin x}$$

resembles the function in the limit of Example 3.56 (unit03-04.htm#example3-56), above. We change it so that we can use this limit.

We have

$$f(x) = \frac{x}{\sin x} = \frac{1}{\frac{\sin x}{x}}.$$

Applying Theorem 3.23(d), we get

$$\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} = \frac{\lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0} \frac{\sin x}{x}} = \frac{1}{1} = 1.$$

Example 3.58. Let us evaluate the limit

$$\lim_{\theta\to 0} \frac{\sin(c\theta)}{\theta}$$

where c is a nonzero constant.

We can see that if $t = c\theta$, then $\theta = \dfrac{t}{c}$ and $t \to 0$ as $\theta \to 0$.

Hence, by Theorem 3.23(b),

$$\lim_{\theta\to 0} \frac{\sin(c\theta)}{\theta} = \lim_{t\to 0} \frac{c\sin(t)}{t} = c\lim_{t\to 0} \frac{\sin(t)}{t} = c(1) = c.$$

Example 3.59. Similarly, to evaluate the limit

$$\lim_{\theta\to 0} \frac{\theta}{\sin(c\theta)},$$

where c is a nonzero constant, we use Example 3.57 and Theorem 3.23(c), setting $t = c\theta$.

We evaluate as follows:

$$\lim_{\theta\to 0} \frac{\theta}{\sin(c\theta)} = \lim_{t\to 0} \frac{t}{c\sin(t)} = \frac{1}{c} \lim_{t\to 0} \frac{t}{(\sin t)} = \frac{1}{c}(1) = \frac{1}{c}$$

Example 3.60. If c and b are two nonzero constants, we see from Examples 3.58 and 3.59, above, and Theorem 3.23(c) that

$$\lim_{x\to 0} \frac{\sin(cx)}{\sin(bx)} = \lim_{x\to 0} \frac{\sin(cx)}{x} \frac{x}{\sin(bx)} = \lim_{x\to 0} \frac{\sin(cx)}{x} \lim_{x\to 0} \frac{x}{\sin(bx)} = c\frac{1}{b} = \frac{c}{b}.$$

Example 3.61. We will use Theorem 3.14 (unit03-01.htm#theorem3-14), Proposition 3.16 (unit03-02.htm#proposition3-16), and Theorem 3.23 to evaluate the limit

$$\lim_{x\to 0} \frac{\cos x - 1}{x}.$$

First, we multiply the numerator by its conjugate to obtain the product of two functions whose limits at 0 are finite.

$$\begin{aligned}\left(\frac{\cos x-1}{x}\right)\left(\frac{\cos x+1}{\cos x+1}\right) &= \frac{\cos^2-1}{x(\cos x+1)}\\ &= \frac{-(\sin x)^2}{x(\cos x+1)}\\ &= \frac{\sin x}{x}\,\frac{-\sin x}{\cos x+1}.\end{aligned}$$

By Theorem 3.14 (unit03-01.htm#theorem3-14),

$$\lim_{x\to 0}\frac{\sin x}{\cos x+1}=0,$$

and by Example 3.56 (unit03-04.htm#example3-56),

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Hence, by Proposition 3.16 (unit03-02.htm#proposition3-16), and by Theorem 3.23(c),

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-1}{x} &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{\cos x+1}\right)\\ &= \lim_{x\to 0}\frac{\sin x}{x}\,\lim_{x\to 0}\frac{-\sin x}{\cos x+1}\\ &= (1)(0)\\ &= 0.\end{aligned}$$

Example 3.62.

$$\begin{aligned}\lim_{x\to -1}\frac{\sin\,(x+1)}{x^2-3x-4} &= \lim_{x\to -1}\frac{\sin(x+1)}{(x+1)(x-4)}\\ &= \lim_{x\to -1}\left(\frac{\sin(x+1)}{x+1}\right)\left(\frac{1}{x-4}\right).\end{aligned}$$

a. $\lim_{x\to -1}\frac{1}{x-4}=\frac{1}{-5}$, by Theorem 3.14 (unit03-01.htm#theorem3-14).

b. If $t=x+1$, then $t\to 0$ as $x\to -1$, and by Corollary 3.25 and Example 3.56 (unit03-04.htm#example3-56),

$$\lim_{x\to -1}\frac{\sin(x+1)}{x+1}=\lim_{t\to 0}\frac{\sin t}{t}=1.$$

By Theorem 3.23,

$$\begin{aligned}\lim_{x\to -1}\frac{\sin\,(x+1)}{x^2-3x-4} &= \lim_{x\to -1}\frac{\sin(x+1)}{(x+1)}\,\frac{1}{x-4}\\ &= \lim_{x\to -1}\frac{\sin(x+1)}{x+1}\,\lim_{x\to -1}\frac{1}{x-4}\\ &= (1)\frac{1}{-5} = -\frac{1}{5}\end{aligned}$$

Tutorial 3: Evaluation of a trigonometric limit with algebraic manipulation. (../multimedia/tutorials/265Limit03/index.html)

Exercises

24. Read the section titled “Laws of Limits” on page 72 of the textbook.
25. Do Exercises 1 and 2 on page 79 of the textbook.

26. Do Exercises 35-44 on page 151 of the textbook.
27. Indicate which examples and results are applied in each step in the evaluation of the following limits.

a.
$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x^2-3x} &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{1}{x-3}\right)\\ &= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{x-3}\\ &= -\frac{1}{3}.\end{aligned}$$

b. For $x\to 0^+$, we have $\dfrac{\cos x-1}{x}<0$ and $\lim\limits_{x\to 0}\dfrac{\cos x-1}{x}=0$; hence, $\lim\limits_{x\to 0}\dfrac{x}{\cos x-1}=-\infty$.

c. $\lim\limits_{x\to 3}\dfrac{x^2+9}{x^2-9}=\lim\limits_{x\to 3}\dfrac{1}{x^2-9}\left(x^2+9\right)$ (this limit does not exist).

d.
$$\begin{aligned}\lim_{x\to 0}\frac{\sin(2x)}{(x-1)(\sin(5x))} &= \lim_{x\to 0}\left(\frac{\sin(2x)}{\sin(5x)}\right)\left(\frac{1}{x-1}\right)\\ &= \frac{2}{5}(-1)\\ &= -\frac{2}{5}.\end{aligned}$$

Answers to Exercises (appendix-a.htm)

Limits at Infinity: Horizontal Asymptotes

Prerequisites

To complete this section, you must be able to

1. factor polynomials. See Unit 1 in this *Study Guide* and the PDF document titled “Review of Algebra,” available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

   Keep in mind as well the factorization of special polynomials as listed in the “Reference Pages” at the front of your textbook.
2. apply the definition and properties of the absolute value of a number. See Boxes 3, 4 and 5 on pages 340-341 of the textbook.

We know how to apply theorems to evaluate limits at a finite number, but now we must learn how to evaluate limits at infinity. We use these limits to find the horizontal asymptotes of functions. We begin by learning the definition of a horizontal asymptote from the graph of the function.

The graph of a function with a horizontal asymptote $y = L$ in the positive direction may look like that shown in Figure 3.26, below (see also Figure 2 on page 85 of the textbook).

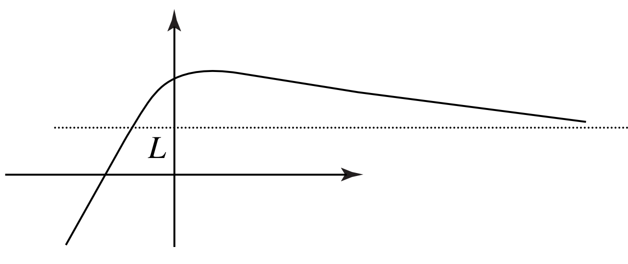


Figure 3.26. Function with horizontal asymptote $y = L$ in the positive direction

The graph of a function with a horizontal asymptote $y = L$ in the negative direction may look like in Figure 3.27, below (see also Figure 3 on page 85 of the textbook).

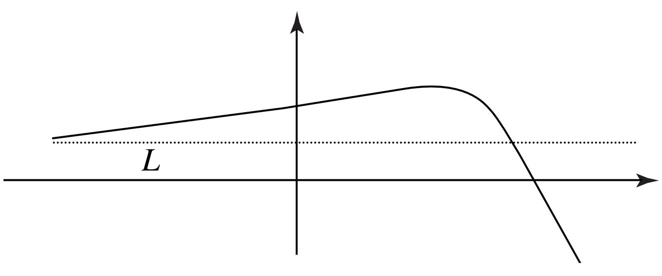


Figure 3.27. Function with horizontal asymptote $y = L$ in the negative direction

What makes the line $y = L$ an asymptote is the fact that the values of the function $f(x)$ are close to L, and the larger and positive or smaller and negative x is, the closer $f(x)$ is to L. To indicate that x is very large and positive—that is, that x increases without bound—we write $x \to \infty$. To indicate that x is very small and negative—that is, that x decreases without bound—we write $x \to -\infty$.

It is a misconception to say that a function has a horizontal asymptote when the function gets close to the horizontal line, but never touches it. As you see in Figure 2 on page 85 of the textbook, the function oscillates around the horizontal line, and the values of the function get close to or are equal to L.

Definition* 3.27.

a. Let f be a function defined on an interval (a, ∞) for some a.

The limit of f, as x increases without bound, is L, if the values of $f(x)$ can be made arbitrarily close to L (as close as we want) by taking x sufficiently large and positive. We write

$$\lim_{x\to\infty} f(x) = L.$$

b. Let f be a function defined on an interval $(-\infty, a)$ for some a.

The limit of f, as x decreases without bound, is L, if the values of $f(x)$ can be made arbitrarily close to L (as close as we want) by taking x sufficiently small and negative. We write

$$\lim_{x\to-\infty} f(x) = L.$$

We also write $f(x) \to L$ as $x \to \infty$. This expression is usually read, "the limit of f, as x approaches infinity, is L," or "the limit of f, as x increases without bound, is L."

Similarly, we write $f(x) \to L$ as $x \to -\infty$, and this expression is usually read, "the limit of f, as x approaches negative infinity, is L," or "the limit of f, as x decreases without bound, is L."

Remember: The symbols ∞ and $-\infty$ do *not* represent numbers, and we must interpret the notation $x \to \infty$ and $x \to -\infty$ correctly. Even when we say that "x approaches infinity," we mean that x increases without bound, no more and no less.

For help with the concept of horizontal asymptotes, view the PowerPoint tutorials below. To access a tutorial:

1. Click on the file link to open it.
2. Save the file (.ppsx) to your computer's hard drive.
3. Click on the PowerPoint file to view.

Note: If you don't have PowerPoint installed on your computer, you can download the PowerPoint Viewer from here: https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6 (https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6).

PowerPoint 1 (../multimedia/powerpoint/265U3HAsymtpte1.ppsx): For a large x, the values of the function $f(x)$ tend to M. Hence $y = M$ is a horizontal asymptote.

PowerPoint 2 (../multimedia/powerpoint/265U3HAsymtpte2.ppsx): For a negative small x, the values of the function $f(x)$ tend to M. Hence $y = M$ is a horizontal asymptote.

Definition 3.28. The line $y = L$ is a *horizontal asymptote* of the curve $y = f(x)$ if

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Example 3.63. From Figure 3.19 (unit03-03.htm#figure3-19), we see that $y = 0$ is a horizontal asymptote in both directions of the function $f(x) = \dfrac{1}{x}$. That is

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Example 3.64. From the graph of the function f shown in Figure 3.28, below, we see that the vertical lines $x = M$ and $x = 0$ are vertical asymptotes, and the horizontal line $y = L$ is a horizontal asymptote.

In terms of limits,

$$\lim_{x\to M^-} f(x) = \infty, \quad \lim_{x\to 0^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = L, \quad \lim_{x\to-\infty} f(x) = \infty.$$

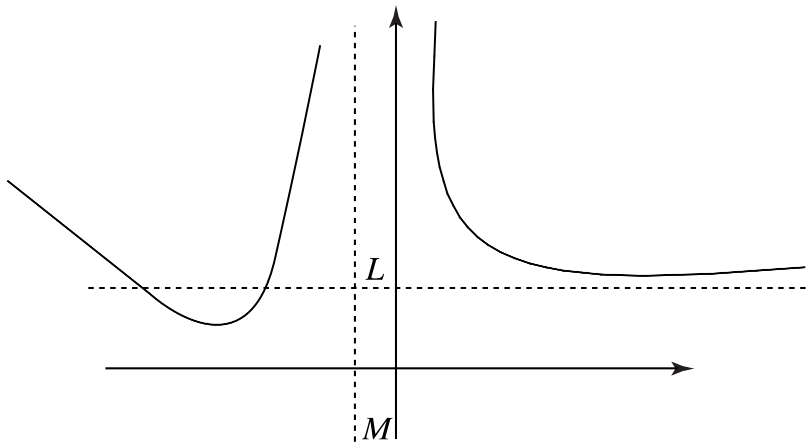


Figure 3.28. Function with two vertical asymptotes and one horizontal asymptote

Exercises

28. Do Exercises 3 and 4 on page 94 of the textbook.

Answers to Exercises (appendix-a.htm)

Observe that, by Definition 3.27, for us to take a limit at infinity, the function must be defined on an interval (c, ∞), for some c; and for us to evaluate limits at negative infinity, the function must be defined on an interval $(-\infty, d)$ for some number d. For instance, we cannot take the limit at infinity or negative infinity of the tangent, cotangent, secant or cosecant functions. We also see that the sine and cosine functions oscillate between -1 and 1 as x increases (positively) or decreases (negatively). Therefore, the limits

$$\lim_{x \to \infty} \sin x, \quad \lim_{x \to -\infty} \sin x, \quad \lim_{x \to \infty} \cos x, \quad \text{and} \quad \lim_{x \to -\infty} \cos x$$

do not exist.

Some of the definitions and theorems we apply to evaluate finite limits also hold for infinite limits. More precisely, the following definition and results are valid if we replace $x \to a$ (and $x \to b$ in Theorem 3.20 (unit03-03.htm#theorem3-20)) by either $x \to \infty$ or $x \to -\infty$.

1. Definition 3.9 (unit03.htm#definition3-9)
2. Proposition 3.16 (unit03-02.htm#proposition3-16)
3. Theorem 3.17 (unit03-03.htm#theorem3-17)—parts (*e*) and (*f*) (limit of the reciprocal of a function)
4. Theorem 3.18 (unit03-03.htm#theorem3-18)
5. Theorem 3.20 (unit03-03.htm#theorem3-20) (composition of functions—infinity limits)
6. Theorem 3.21 (unit03-03.htm#theorem3-21) (sum and product of infinity limits)
7. Theorem 3.22 (unit03-04.htm#theorem3-22) (Squeeze Theorem)
8. Theorem 3.23 (unit03-05.htm#theorem3-23) (Laws of Limits)
9. Theorem 3.24 (unit03-05.htm#theorem3-24)
10. Corollary 3.25 (unit03-05.htm#corollary3-25)
11. Corollary 3.26 (unit03-05.htm#corollary3-26)

Example 3.65. (Theorem 4 on page 87 of the textbook).

The function

$$g(x) = \frac{1}{x}$$

is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$.

If r is a positive rational number, then the function $f(x) = x^r$ is continuous at 0 from the right, and if $u = 1 / x$, then $u \to 0^+$ as $x \to \infty$. [Why?]

By Corollary 3.26 (unit03-05.htm#corollary3-26),

$$\lim_{x \to \infty} \frac{1}{x^r} = \lim_{u \to 0^+} u^r = 0.$$

Similarly, if r is a positive rational number such that $f(x) = x^r$ is defined on an interval $(-\infty, c)$ for some c, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = \lim_{u \to 0^-} u^r = 0.$$

A particular case of Example 3.65 occurs when n is any positive integer. Then

$$f(x) = \frac{1}{x^n}$$

is defined on the intervals $(-\infty, 0)$ and $(0, \infty)$; and therefore,

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0.$$

Example 3.66. Example 3.65, with $r = 1/3$, says that

$$\lim_{x \to -\infty} \frac{1}{\sqrt[3]{x}} = 0.$$

By Corollary 3.25 (unit03-05.htm#corollary3-25),

$$\lim_{x \to -\infty} \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0.$$

To evaluate the limits at infinity of rational functions, we must remember that a polynomial function of degree n with leading coefficient a_n is of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots. + a_1 x + a_0.$$

Example 3.67. Let us identify the degree and the leading coefficient of a few polynomial functions.

a. $Q(x) = 3 - x + 6x^4$, degree 4, leading coefficient 6.
b. $P(x) = -x^3 + 6x^2 + x$, degree 3, leading coefficient -1.
c. $P(x) = 6$, degree 0, leading coefficient 6.

In the next three examples, we consider the strategy used to evaluate infinite limits of rational functions.

Example 3.68. For the limit

$$\lim_{x \to \infty} \frac{3 - x - x^2}{2x^2 + x},$$

we identify the largest power of the polynomials in the numerator and denominator—in this case, 2. We divide the numerator and the denominator by x^2, and we find that

$$\frac{3 - x - x^2}{2x^2 + x} = \frac{\dfrac{3}{x^2} - \dfrac{1}{x} - 1}{2 + \dfrac{1}{x}}.$$

We apply Proposition 3.16 (unit03-02.htm#proposition3-16), Theorem 3.23 (unit03-05.htm#theorem3-23) and Example 3.65, above, to conclude that

$$\lim_{x\to\infty}\frac{3-x-x^2}{2x^2+x} = \lim_{x\to\infty}\frac{\dfrac{3}{x^2}-\dfrac{1}{x}-1}{2+\dfrac{1}{x}} = \frac{\lim\limits_{x\to\infty}\dfrac{3}{x^2}-\dfrac{1}{x}-1}{\lim\limits_{x\to\infty} 2+\dfrac{1}{x}} = -\frac{1}{2}.$$

The line $y = -\dfrac{1}{2}$ is a horizontal asymptote.

Observe that when the degree of the numerator and denominator are equal, the limit is -1 (the leading coefficient of the numerator polynomial) over 2 (the leading coefficient of the denominator polynomial).

Example 3.69. In this example, we proceed as in Example 3.68, above.

In the rational function

$$\frac{3-x}{2x^2+x},$$

the biggest power is 2. Dividing both the numerator and the denominator by x^2, we obtain

$$\frac{3-x}{2x^2+x} = \frac{\dfrac{3}{x^2}-\dfrac{1}{x}}{2+\dfrac{1}{x}}.$$

Applying Proposition 3.16 (unit03-02.htm#proposition3-16), Theorem 3.23 (unit03-05.htm#theorem3-23) and Example 3.65, we find that

$$\lim_{x\to-\infty}\frac{3-x}{2x^2+x} = \lim_{x\to-\infty}\frac{\dfrac{3}{x^2}-\dfrac{1}{x}}{2+\dfrac{1}{x}} = \frac{\lim\limits_{x\to-\infty}\dfrac{3}{x^2}-\dfrac{1}{x}}{\lim\limits_{x\to-\infty} 2+\dfrac{1}{x}} = \frac{0}{2} = 0.$$

The line $y = 0$ is a horizontal asymptote.

In this case, the degree of the numerator is less than the degree of the denominator.

Example 3.70. In the limit

$$\lim_{x\to-\infty}\frac{3-x^3}{2x^2+x},$$

the largest degree is 3.

Dividing both the numerator and the denominator by x^3, we obtain

$$\frac{3-x^3}{2x^2+x} = \frac{\dfrac{3}{x^3}-1}{\dfrac{2}{x}+\dfrac{1}{x^2}} = \left(\frac{3}{x^3}-1\right)\left(\frac{1}{\dfrac{2}{x}+\dfrac{1}{x^2}}\right).$$

We know that

a. by Example 3.65 and Theorem 3.23 (unit03-05.htm#theorem3-23).

$$\lim_{x \to -\infty} \frac{3}{x^3} - 1 = -1 < 0$$

b. $\displaystyle\lim_{x \to -\infty} \frac{2}{x} + \frac{1}{x^2} = 0$ by Example 3.65 and Theorem 3.23 (unit03-05.htm#theorem3-23).

c. $\displaystyle\frac{2}{x} + \frac{1}{x^2} < 0$ for $x \to -\infty\,(x < 0)$.

d. $\displaystyle\lim_{x \to -\infty} \frac{1}{\frac{2}{x} + \frac{1}{x^2}} = -\infty$ by (a) and (b), above, and Theorem 3.17 (unit03-03.htm#theorem3-17)(f).

Applying Theorem 3.18 (unit03-03.htm#theorem3-18), we obtain

$$\lim_{x \to -\infty} \frac{3 - x^3}{2x^2 + x} = \lim_{x \to -\infty} \left(\frac{3}{x^3} - 1\right)\left(\frac{1}{\frac{2}{x} + \frac{1}{x^2}}\right) = \infty.$$

Observe that the rational function

$$R(x) = \frac{3 - x^3}{2x^2 + x} > 0,$$

for $x \to -\infty$, $(x < 0)$. To see this, take a very small negative value of x, say $x = -1000$, and check that $R(-1000) > 0$.

In this case, the degree of the numerator is greater than the degree of the denominator.

The strategy we used in the last three examples should convince us that the degree of the numerator and denominator determine the limits at infinity of rational functions. So, we have the following result.

Theorem 3.29. If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function, then

a. $\displaystyle\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$ if degree of $p(x) =$ degree of $q(x)$.

b. $\displaystyle\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = 0$ if degree of $p(x) <$ degree of $q(x)$.

c. $\displaystyle\lim_{x \to \infty} \frac{p(x)}{q(x)} = \infty(-\infty)$ if degree of $p(x) >$ degree of $q(x)$ and $R(x) = \dfrac{p(x)}{q(x)} > 0(<0)$ for $x \to \infty$.

d. $\displaystyle\lim_{x \to -\infty} \frac{p(x)}{q(x)} = \infty(-\infty)$ if degree of $p(x) >$ degree of $q(x)$ and $R(x) = \dfrac{p(x)}{q(x)} > 0(>0)$ for $x \to -\infty$.

Therefore, we see that a rational function has a horizontal asymptote only when the degree of the numerator is less than or equal to the degree of the denominator.

Example 3.71. From Theorem 3.29, we have

$$\lim_{x \to \infty} \frac{\pi x}{2x + 1} = \frac{\pi}{2}$$

if $t = \dfrac{\pi x}{2x + 1}$, then $t \to \dfrac{\pi}{2}$ as $x \to \infty$.

Hence, by Corollary 3.26 (unit03-05.htm#corollary3-26) and Theorem 3.14 (unit03-01.htm#theorem3-14),

$$\lim_{x\to\infty}\left(\frac{\pi x}{2x+1}\right)^2 = \lim_{t\to\frac{\pi}{2}} t^2 = \frac{\pi^2}{4}.$$

Example 3.72. From the definition of absolute value, we have

$$\sqrt{x^2} = x \quad \text{for} \quad x \ge 0, \quad \text{and} \quad \sqrt{x^2} = -x \quad \text{for} \quad x < 0.$$

Hence,

$$\sqrt{3x^2+4x} = \sqrt{x^2\left(3+\frac{4}{x}\right)} = x\sqrt{3+4x} \quad \text{for} \quad x\to\infty,$$

and

$$\sqrt{3x^2+4x} = \sqrt{x^2\left(3+\frac{4}{x}\right)} = -x\sqrt{3+4x} \quad \text{for} \quad x\to-\infty.$$

By Proposition 3.16 (unit03-02.htm#proposition3-16), Corollary 3.26 (unit03-05.htm#corollary3-26), Theorem 3.23 (unit03-05.htm#theorem3-23) and Example 3.65:

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{3x^2+4x}}{x+7} &= \lim_{x\to\infty}\frac{x\sqrt{3+\frac{4}{x}}}{x\left(1+\frac{7}{x}\right)} = \lim_{x\to\infty}\frac{\sqrt{3+\frac{4}{x}}}{\left(1+\frac{7}{x}\right)}\\ &= \frac{\sqrt{\lim\limits_{x\to\infty} 3+\frac{4}{x}}}{\lim\limits_{x\to\infty}\left(1+\frac{7}{x}\right)} = \sqrt{3}\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{3x^2+4x}}{x+7} &= \lim_{x\to-\infty}\frac{-x\sqrt{3+\frac{4}{x}}}{x\left(1+\frac{7}{x}\right)} = \lim_{x\to-\infty}\frac{-\sqrt{3+\frac{4}{x}}}{\left(1+\frac{7}{x}\right)}\\ &= \frac{-\sqrt{\lim\limits_{x\to-\infty} 3+\frac{4}{x}}}{\lim\limits_{x\to-\infty}\left(1+\frac{7}{x}\right)} = -\sqrt{3}\end{aligned}$$

Example 3.73. For $x\to\infty$

$$\begin{aligned}\frac{3x^3-3x+1}{\sqrt{2x^2+3}} &= \frac{x^3\left(3-\frac{3}{x^2}+\frac{1}{x^3}\right)}{\sqrt{x^2\left(2+\frac{3}{x^2}\right)}}\\ &= \frac{x^3\left(3-\frac{3}{x^2}+\frac{1}{x^3}\right)}{x\sqrt{2+\frac{3}{x^2}}}\\ &= x^2\left(\frac{3-\frac{3}{x^2}+\frac{1}{x^3}}{\sqrt{2+\frac{3}{x^2}}}\right).\end{aligned}$$

a. $\lim_{x\to\infty} x^2 = \infty$ by Theorem 3.21(a) (unit03-03.htm#theorem3-21).

b. $\lim_{x\to\infty} \dfrac{3 - \dfrac{3}{x^2} + \dfrac{1}{x^3}}{\sqrt{2 + \dfrac{3}{x^2}}} = \dfrac{3}{\sqrt{2}} > 0$ by Corollary 3.26 (unit03-05.htm#corollary3-26), Theorem 3.23 (unit03-05.htm#theorem3-23) and Example 3.65.

From points (a) and (b), above, and Theorem 3.21 (unit03-03.htm#theorem3-21), we have

$$\lim_{x\to\infty} x^2\left(\frac{3 - \dfrac{3}{x^2} + \dfrac{1}{x^3}}{\sqrt{2 + \dfrac{3}{x^2}}}\right) = \infty$$

and by Proposition 3.16 (unit03-02.htm#proposition3-16),

$$\lim_{x\to\infty} \frac{3x^3 - 3x + 1}{\sqrt{2x^2+3}} = \lim_{x\to\infty} x^2\left(\frac{3 - \dfrac{3}{x^2} + \dfrac{1}{x^3}}{\sqrt{2 + \dfrac{3}{x^2}}}\right) = \infty.$$

Note: In the last two examples, we factored the x with the biggest power, and simplified until we could apply one or several of the results listed above.

Example 3.74. For the limit

$$\lim_{x\to -\infty} x^2 - \sqrt{2x^4 + x^2},$$

we use a different strategy. Here, we multiply by the conjugate, and then we proceed as in the previous examples.

$$\begin{aligned}\left(x^2 - \sqrt{2x^4+x^2}\right)\left(\frac{x^2 + \sqrt{2x^4+x^2}}{x^2 + \sqrt{2x^4+x^2}}\right) &= \frac{x^4 - \left(2x^4 + x^2\right)}{x^2 + \sqrt{2x^4+x^2}} \\ &= \frac{-x^4 - x^2}{x^2 + \sqrt{2x^4+x^2}}\end{aligned}$$

Factoring,

$$\begin{aligned}\frac{-x^4 - x^2}{x^2 + \sqrt{2x^4+x^2}} &= \frac{-x^4\left(1 + \dfrac{1}{x^2}\right)}{x^2\left(1 + \sqrt{2 + \dfrac{1}{x^2}}\right)} \\ &= -\frac{x^4\left(1 + \dfrac{1}{x^2}\right)}{x^2\left(1 + \sqrt{2 + \dfrac{1}{x^2}}\right)} = -x^2\left(\frac{1 + \dfrac{1}{x^2}}{1 + \sqrt{2 + \dfrac{1}{x^2}}}\right)\end{aligned}$$

a. $\lim_{x\to -\infty} x^2 = \infty$, so, $\lim_{x\to -\infty} -x^2 = -\infty$ by Theorem 3.21(a) (unit03-03.htm#theorem3-21).

b. $\lim_{x\to -\infty} \dfrac{1 + \dfrac{1}{x^2}}{1 + \sqrt{2 + \dfrac{1}{x^2}}} = \dfrac{1}{1+\sqrt{2}} > 0$ by Corollary 3.26 (unit03-05.htm#corollary3-26), Theorem 3.23 (unit03-05.htm#theorem3-23) and Example 3.65.

From points (a) and (b), above, and Theorem 3.21 (unit03-03.htm#theorem3-21), we have

$$\lim_{x\to-\infty} -x^2\left(\frac{1+\dfrac{1}{x^2}}{1+\sqrt{2+\dfrac{1}{x^2}}}\right) = -\infty$$

and by Proposition 3.16 (unit03-02.htm#proposition3-16),

$$\lim_{x\to-\infty} x^2-\sqrt{2x^4+x^2} = \lim_{x\to-\infty} -x^2\left(\frac{1+\dfrac{1}{x^2}}{1+\sqrt{2+\dfrac{1}{x^2}}}\right) = -\infty.$$

Example 3.75. To evaluate the limit

$$\lim_{x\to\infty}\frac{\cos x}{x^2},$$

we use the fact that, for any nonzero x, $-1\le\cos x\le 1$. Since $x^2>0$, we multiply by $\dfrac{1}{x^2}$, and we get

$$-\frac{1}{x^2}\le\frac{\cos x}{x^2}\le\frac{1}{x^2}.$$

By Example 3.65,

$$\lim_{x\to\infty}\frac{1}{x^2}=0=\lim_{x\to\infty}-\frac{1}{x^2},$$

and we conclude, using the Squeeze Theorem (unit03-04.htm#theorem3-22), that

$$\lim_{x\to\infty}\frac{\cos x}{x^2}=0.$$

Example 3.76. By Theorem 3.29,

$$\lim_{x\to\infty}\frac{\pi x}{2x+1}=\frac{\pi}{2},\quad\text{and so}\quad\frac{\pi x}{2x+1}\le\frac{\pi}{2}\quad\text{for}\quad x\to\infty$$

[To see this, take a large positive value of x say 10,000.]

Hence, if

$$t=\frac{\pi x}{2x+1},\quad\text{then}\quad t\to\left(\frac{\pi}{2}\right)^-\quad\text{as}\quad x\to\infty.$$

By Theorem 3.20 (unit03-03.htm#theorem3-20),

$$\lim_{x\to\infty}\tan\left(\frac{\pi x}{2x+1}\right)=\lim_{t\to\frac{\pi}{2}^-}\tan t=\infty.$$

REMARK 3.2

Polynomials have neither vertical nor horizontal asymptotes, because polynomials increase or decrease without bounds for either $x\to\infty$ or $x\to-\infty$. Indeed, if

$$P(x)=c_nx^n+c_{n-1}x^{n-1}+\ldots+c_1x+c_0$$

is a polynomial of degree n, then

$$c_nx^n+c_{n-1}x^{n-1}+\ldots+c_1x+c_0=x^n\left(c_n+\frac{c_{n-1}}{x}+\ldots+\frac{c_1}{x^{n-1}}+\frac{c_0}{x^n}\right)$$

By Proposition 3.16 (unit03-02.htm#proposition3-16), Example 3.65 and Theorem 3.18 (unit03-03.htm#theorem3-18),

$$\lim_{x \to \pm\infty} c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = \lim_{x \to \pm\infty} c_n x^n = \pm\infty.$$

Example 3.77. Let us consider the cases below.

a. $\lim_{x \to \infty} \left(6 - x^2\right)\left(3x^2 + x - 1\right) = \lim_{x \to \infty} -3x^4 = -\infty$

b. $\lim_{x \to -\infty} 3 + 4x^2 + 4x^3 = \lim_{x \to -\infty} 4x^3 = -\infty$

Tutorial 4: The use of rationalization to evaluate a limit. (../multimedia/tutorials/265Limit04/index.html)

Tutorial 5: Application of several results to evaluate an infinite limit. (../multimedia/tutorials/265Limit05/index.html)

Exercises

29. Read Examples 5, 6, 8, 9 and 10 on pages 89-90 of the textbook.
30. Do at least 12 exercises from numbers 9 to 32 on page 95 of the textbook.
31. Explain why each of the limits below is correct.
    a. $\lim_{x \to \infty} \dfrac{4x^3 - 2x + 5}{(x+1)(2x^2+1)} = 2$
    b. $\lim_{x \to -\infty} \dfrac{(1-x)(4x+2)}{2 - 3x^4} = 0$
    c. $\lim_{x \to \infty} \dfrac{(1-x)\left(x^2 - 2\right)}{x^2 + 1} = -\infty$
    d. $\lim_{x \to \infty} \dfrac{2x^4 + 6x}{6 - x^3} = -\infty$
32. Explain why $\lim_{x \to \infty} \cot\left(\dfrac{1}{x}\right) = \infty$.
33. Explain why each of the statements below is true.
    a. $\lim_{x \to \infty} 5x^2 + 4x - 1 = \infty$
    b. $\lim_{x \to -\infty} 3 - 4x + 6x^3 = -\infty$
    c. $\lim_{x \to \infty} (3x - 1)(5 - x) = -\infty$

Answers to Exercises (appendix-a.htm)

The Definition of a Limit

Prerequisites

To complete this section, you must be able to

1. apply the definition and properties of absolute value, including the triangle inequality. Read pages 341 and 342 of the textbook, and do Exercises 65, 66 and 68 on page 343.
2. apply the addition formulas of trigonometric functions. See the Reference pages at the beginning of the textbook.

In earlier sections of this unit, we introduced the basic definitions and results we apply to evaluate limits. We provided some explanations to make them plausible, but it is one thing to accept something as plausible, and another to understand, without any reason for doubt, that something is true. Mathematicians look for logical arguments that explain why a statement or result is true. In this section, we consider how these results came about, so that your understanding of limit becomes deeper, and so that you can appreciate that all of the things we do with mathematics have a solid foundation. It is not our intention to prove all the results we have been using—time constraints prevent that. Instead, we will choose some of them, and we invite you to try to prove the others if your schedule allows it.

Understanding *why* something is true is what makes mathematics exciting. In our efforts to show that something is true, we must start with definitions and axioms.

We know that the statement $f(x) \to L$ as $x \to a$ means that we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficiently close to a, but not equal to a.

This is the same as saying that we can make the values of $f(x)$ arbitrarily close to L (as close as we like) because there are always xs sufficiently close to a such that $f(x)$ is close to L.

Do you agree with this statement? Keep in mind the graph of a function with limit L at a (see Figure 3.29, below).

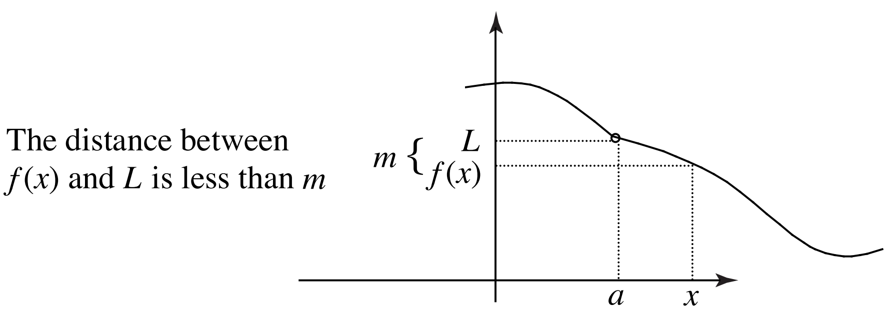


Figure 3.29. Function defined at points a and x close to a

In Figure 3.29, we can see that for a small number m (as small as we like), we can find x close to a, such that the distance between $f(x)$ and L is less than m. The challenge is to write precisely this idea using mathematical concepts; that is, to translate this idea into a mathematical statement.

We need to express the idea of "x close to a" in mathematical terms. Two numbers are close if the distance between them is small, so we need to express the idea of "distance between two numbers." What is the distance between the numbers x and a? How do we measure the distance between x and a?

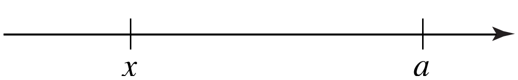


Figure 3.30. Real number line showing $x < a$

If $x < a$, as shown in the real line above, then their distance is $a - x$, regardless of whether both are positive, both are negative, or one is positive and the other negative. Convince yourself of this fact. Figures 7 and 8 on page 341 of the textbook may help you.

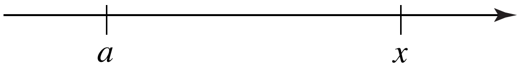


Figure 3.31. Real number line showing $x > a$

If $x > a$, as shown in the real line above, then their distance is $x - a$, regardless of whether both are positive, both are negative, or one is positive and the other negative.

Combining these two results, we can say that the distance between two distinct numbers x and a is $|x - a|$ because

$$|x - a| = \begin{cases} x - a & \text{if } \; x - a \geq 0 \\ -(x - a) & \text{if } \; x - a < 0 \end{cases} = \begin{cases} x - a & \text{if } \; x \geq a \\ a - x & \text{if } \; x < a \end{cases}$$

This is precisely what we say above!

As you can see, $|6 - 6.01| = 0.01$ because the distance between 6 and 6.01 is 0.01, and $|3.5 - 3| = 0.5$, because the distance between 3 and 3.5 is 0.5.

The distance between two distinct numbers x and a can be very small, and this fact can be expressed mathematically as follows:

$|x - a| < \delta$ for a small number $\quad \delta > 0$.

We use the Greek letter δ (delta) to denote a small number. It is a convention, and it has its historical reasons.

If we write $|x - 3| < 0.01$, then the distance between x and 3 is less than 0.01. These numbers are on the interval $(2.99, 3.01)$.

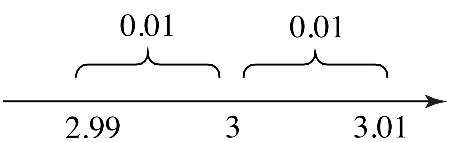


Figure 3.32. Real number line showing the interval $(2.99, 3.01)$

If we write $|x - 3| < \delta$, then we are referring to the numbers x whose distance to 3 is less than the small number δ.

When we write $0 < |x - 3| < \delta$, we are referring to all numbers x very close to 3, but not equal to 3, because $|x - 3| = 0$ only of $x = 3$.

Now we must express the idea of "making the values of $f(x)$ arbitrarily close to L (as close as we like)" mathematically.

We know that "$f(x)$ is close to L" is written as $|f(x) - L| < \varepsilon$ for a small number $\varepsilon > 0$.

Note that ε is the Greek letter epsilon.

But "the distance is as small as we want" is the same as saying that the distance is less than any number we want to give.

So, when we write

$|f(x) - L| < \varepsilon$ for any given small number $\quad \varepsilon > 0$,

we are saying that the distance between $f(x)$ and L is as small as any number we want to give.

Hence, the statement

we can find a $\delta > 0$ such that, for any $0 < |x - a| < \delta$, it is the case that $|f(x) - L| < \varepsilon$. for any given small number $\varepsilon > 0$

is saying that

we can find xs very close to a, but not equal to a, so that $f(x)$ can be as close to L as we like.

This is precisely the definition of a limit.

Definition 3.30. Let f be a function defined around a.

The limit of f, as x approaches a, is L if given any $\varepsilon > 0$, we can find $\delta > 0$ such that

for any $0 < |x - a| < \delta$ it is the case that $|f(x) - L| < \varepsilon$.

Note that this definition expresses what we understood a limit to be.

We will need the following definition in our further discussions.

Definition 3.31. If a and b are two numbers, the number $\min\{a, b\}$ is the smaller of the two; for instance, $\min\{3, -4\} = -4$. Similarly, $\max\{a, b\}$ is the larger of the two; hence, $\max\{3, -4\} = 3$. It is true that $\min\{4, 4\} = 4 = \max\{4, 4\}$.

Definition 3.31 extends naturally to any finite number or numbers; for example,

$$\min\left\{3, -1, \frac{6}{2}\right\} = -1 \quad \text{and} \quad \max\left\{0, -\frac{1}{2}, -\frac{1}{4}\right\} = 0.$$

The results about limits that we presented to you are consequences of Definition 3.30 and the properties of the real numbers. For example, let us see why the first and third laws of limits in Theorem 3.23 (unit03-05.htm#theorem3-23) are true.

Sum of limits: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} f(x) + g(x) = L + M.$$

Proof. We start with knowing *what we want to show* (prove).

Given any $\varepsilon > 0$, we can find $\delta > 0$ such that, for any $0 < |x - a| < \delta$, it is the case that

$$|f(x) + g(x) - (L + M)| < \varepsilon.$$

We are given $\varepsilon > 0$, and we must find $\delta > 0$ with the property indicated above.

To proceed from this point, we must determine *what we know*.

In this case, we know that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Definition 3.30 is true for any positive number, so

1. for $\frac{\varepsilon}{2} > 0$, there is a $\delta_1 > 0$ such that, for any $0 < |x - a| < \delta_1$, it is the case that $|f(x) - L| < \frac{\varepsilon}{2}$, and
2. for $\frac{\varepsilon}{2} > 0$, there is a $\delta_2 > 0$ such that, for any $0 < |x - a| < \delta_2$, it is the case that $|g(x) - M| < \frac{\varepsilon}{2}$.

If $\delta = \min(\delta_1, \delta_2)$, then $\delta \le \delta_1$ and $\delta \le \delta_2$. So, for any x such that $0 < |x - a| < \delta$, it is true that $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$.

Hence, statements 1 and 2 above are true for this δ.

If $0 < |x - a| < \delta$, then $|f(x) - L| < \frac{\varepsilon}{2}$, and $|g(x) - M| < \frac{\varepsilon}{2}$.

From the triangle inequality, we have

$$|f(x) + g(x) - (L + M)| \le |f(x) - L| + |g(x) - M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, given $\varepsilon > 0$, we have found $\delta = \min(\delta_1, \delta_2)$, such that if $0 < |x - a| < \delta$ then $|f(x) + g(x) - (L + M)| < \varepsilon$.

Q.E.D

Note: The end of a proof is indicated by the initials "Q.E.D.," an abbreviation of the Latin phrase *quod erat demonstrandum* ("which was to be proved").

Product of limits: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof. We want to show that, for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that for any $0 < |x - a| < \delta$ it is the case that $|f(x)g(x) - LM| < \varepsilon$

We know that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; hence,

1. for any $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that for any $0 < |x - a| < \delta_1$, it is the case that $|f(x) - L| < \varepsilon_1$, and
2. for any $\varepsilon_2 > 0$, there is a $\delta_2 > 0$ such that for any $0 < |x - a| < \delta_2$, it is the case that $|g(x) - M| < \varepsilon_2$.

See how the addition of $0 = g(x)L - g(x)L$ allows us to relate *what we want to show* with *what we know*.

By the triangle inequality,

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - LM + g(x)L - g(x)L| \\ &= |f(x)g(x) - g(x)L + g(x)L - LM| \\ &= |g(x)(f(x) - L) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||(g(x) - M)| \end{aligned}$$

We want the final inequality above to be less than any $\varepsilon > 0$, and we can achieve this result by taking particular values for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ in statements 1 and 2 above.

In statement 1, we take $\varepsilon_1 = \dfrac{\varepsilon}{2(1 + |M|)} > 0$. So, there is a $\delta_1 > 0$ such that

3. $|f(x) - L| < \dfrac{\varepsilon}{2(1 + |M|)}$ for any $0 < |x - a| < \delta_1$.

In statement 2 we take $\varepsilon_2 = 1$. So, for $1 > 0$ there is a $\delta_2 > 0$ such that, for any $0 < |x - a| < \delta_2$, it is the case that $|g(x) - M| < 1$.

Since $|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M|$, we conclude that

4. $|g(x)| \leq 1 + |M|$ for any $0 < |x - a| < \delta_2$.

Again, in statement 2, we take $\varepsilon_3 = \dfrac{\varepsilon}{2|L|} > 0$. So, there is a $\delta_3 > 0$ such that

5. $|g(x) - M| < \dfrac{\varepsilon}{2|L|}$, for any $0 < |x - a| < \delta_3$.

Hence, if $\delta = \min(\delta_1, \delta_2, \delta_3)$, then, for any $0 < |x - a| < \delta$, statements 3, 4 and 5, above, are valid. Therefore,

$$\begin{aligned} |f(x)g(x) - LM| &\leq |g(x)||f(x) - L| + |L||(g(x) - M)| \\ &< (1 + |M|)\tfrac{\varepsilon}{2(1+|M|)} + |L|\tfrac{\varepsilon}{2|L|} = \varepsilon. \end{aligned}$$

Q.E.D

Limit of the reciprocal: If $\lim_{x \to a} g(x) = M$ and $M \neq 0$, then $\lim_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{M}$.

Proof. We want to show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that for any

$0 < |x - a| < \delta$ it is the case that $\left|\dfrac{1}{g(x)} - \dfrac{1}{M}\right| < \varepsilon$.

We know that $\lim_{x \to a} g(x) = M$; hence,

1. for any $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that for any $0 < |x - a| < \delta_1$ it is the case that $|g(x) - M| < \varepsilon_1$.

To relate what we know to what we want to show, we do the operations and obtain

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{g(x)M}\right| = |g(x) - M|\frac{1}{|g(x)M|}$$

From statement 1, if $|g(x) - M| < \varepsilon_1$, then by the properties of the absolute value,

$$\varepsilon_1 > |g(x) - M| = |M - g(x)| \geq |M| - |g(x)|.$$

Therefore,

$$|g(x)| > |M| - \varepsilon_1 \quad \text{and} \quad \frac{1}{|Mg(x)|} < \frac{1}{|M|(|M| - \varepsilon_1)}.$$

Hence,

$$|g(x) - M|\frac{1}{|g(x)M|} < \frac{\varepsilon_1}{|M|(|M| - \varepsilon_1)}.$$

If we want $\varepsilon = \dfrac{\varepsilon_1}{|M|(|M| - \varepsilon_1)}$, we solve for ε_1, and we determine that

$$\varepsilon_1 = \frac{\varepsilon|M|^2}{1 + |M|\varepsilon} > 0.$$

From statement 1, above, we know that for this value of ε_1, there is a δ such that if

$$0 < |x - a| < \delta \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon|M|^2}{1 + |M|\varepsilon}.$$

We then conclude that

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = |g(x) - M|\frac{1}{|g(x)M|} < \frac{\varepsilon_1}{|M|(|M| - \varepsilon_1)} = \varepsilon.$$

Finally, we can use the fact that

$$\frac{f(x)}{g(x)} = f(x)\frac{1}{g(x)},$$

and determine, from the product of limits and the limit of the reciprocal, the limit of the quotient.

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ where $M \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x)\frac{1}{g(x)} = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L\frac{1}{M} = \frac{L}{M}$$

Exercises

34. Use Definition 3.30 and the properties of the absolute value to show that
    a. if c is any number and $\lim\limits_{x \to a} f(x) = L$, then $\lim\limits_{x \to a} cf(x) = cL$.
    b. if $\lim\limits_{u \to b} f(u) = L$ and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} f(g(x)) = L$.
35. Suppose that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} f(x) = K$, use Definition 3.30 to show that $L = K$ (i.e., that the limit of a function is unique).
    Hint: $|L - K| = |f(x) - K + L - f(x)| \leq |f(x) - K| + |f(x) - L|$.

Answers to Exercises (appendix-a.htm)

The formal definition of the side limits should indicate when an x is very close to a from the right (or left) but is not a. We denote this situation by

$$0 < x - a < \delta \quad (\text{or } \; 0 < a - x < \delta).$$

Observe that the statement $0 < x - a$ says that $x \neq a$ and that $a < x$; that is, x is on the right of a.

Definition 3.32.

a. Let f be a function defined on the right of a.

The limit of f, as x approaches a from the right, is L if,

given any $\varepsilon > 0$, we can find $\delta > 0$ such that, for any $0 < x - a < \delta$, we have $|f(x) - L| < \varepsilon$.

b. Let f be a function defined on the left of a.

The limit of f, as x approaches a from the left, is L if,

given any $\varepsilon > 0$, we can find $\delta > 0$ such that, for any $0 < a - x < \delta$, we have $|f(x) - L| < \varepsilon$.

Remember that a function f is continuous at a if, $\lim_{x \to a} f(x) = f(a)$. By Definition 3.30, the formal definition of continuity is as follows.

Definition 3.33. A function f is continuous at a if

given any $\varepsilon > 0$, we can find $\delta > 0$ such that, for any $|x - a| < \delta$, it is the case that $|f(x) - f(a)| < L$.

From the laws of limits we have the following laws for continuous functions.

Proposition 3.34. If f and g are continuous at a, then

a. $\lim_{x \to a} f(x) \pm g(x) = f(a) \pm g(a)$.

b. $\lim_{x \to a} cf(x) = cf(a)$, for any constant c.

c. $\lim_{x \to a} f(x)g(x) = f(a)g(a)$.

d. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$, if $g(a) \neq 0$.

Proposition 3.34 is saying that the sum (difference), constant product, product and quotient of continuous functions at a are continuous at a.

Example 3.78. Using Definition 3.33, above, we can also prove that the identity function $I(x) = x$ is continuous everywhere. We want to show that, for any number a, the function is continuous at a; that is, given $\varepsilon > 0$, we can find $\delta > 0$ such that

for any $0 < |x - a| < \delta$, it is the case that $|I(x) - I(a)| < \varepsilon$.

Since $I(x) = x$, we have $|I(x) - I(a)| = |x - a|$, and for $\varepsilon = \delta$, the statement is true. Since a is any number, the function I is continuous everywhere.

Example 3.79. By Example 3.78 and Proposition 3.34(c), we know that for any positive integer n, the function $I(x)^n = x^n$ is the n product of the everywhere-continuous identity function. Hence, by the second law of limits, the function $f(x) = cx^n$ is continuous everywhere for any constant c. You can conclude, again by the law of limits, that polynomial functions are continuous everywhere.

Example 3.80. Since a rational function is the quotient of polynomial functions, rational functions are continuous on their domains by Example 3.79 and Proposition 3.34(d).

Example 3.81. By the definition of radians, for any numbers (radians) θ and η, it is true that $|\sin\theta - \sin\eta| \leq |\theta - \eta|$.

Let us take any $\varepsilon > 0$. Then, for $\varepsilon = \delta$, we have that, for $|\theta - \eta| < \delta$, it is true that $|\sin\theta - \sin\eta| < \varepsilon$.

This result shows that the sine function is continuous at η; however, since η is arbitrary, we conclude that sine is continuous everywhere.

Theorem 3.35. If the function g is continuous at a, and the function f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a. [Note that this theorem was presented earlier as Theorem 3.13.]

Proof. We have $\lim_{x \to a} g(x) = g(a)$ and $\lim_{x \to b} f(x) = f(b)$ for $b = g(a)$. Then, by Exercise 34(b), which you just completed, $\lim_{x \to a} f(g(x)) = f(g(a))$, and therefore, $\lim_{x \to a} f(g(x))$ is continuous at a.

Q.E.D

Example 3.82. From the addition formula of the sine function, we have

$$\cos x = \sin\left(x + \frac{\pi}{2}\right).$$

Hence, cosine is the composition of the outside function sine and the inside polynomial function $x + \pi / 2$.

By Examples 3.79 and 3.81, these two functions are continuous everywhere; hence, by Theorem 3.35, cosine is continuous everywhere.

We conclude from Examples 3.79 to 3.82 and Proposition 3.34 that Theorem 3.14 (unit03-01.htm#theorem3-14) is true.

At this point, we want to consider the formal definition of

$$\lim_{x \to a^+} f(x) = \pm\infty, \quad \lim_{x \to a^-} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} f(x) = \pm\infty.$$

By Definition 3.6 (unit03.htm#definition3-6), we have $\lim_{x \to a^+} f(x) = \infty$ if we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x to be sufficiently close to a from the right but not equal to a.

This statement is the same as saying that it does not matter how large a number M we have, we can always find an x close to a from the right, but not equal to a, such that $f(x)$ is bigger than M.

Do you agree with this statement? Again, keep in mind the graph of a function with an infinite limit at a from the right.

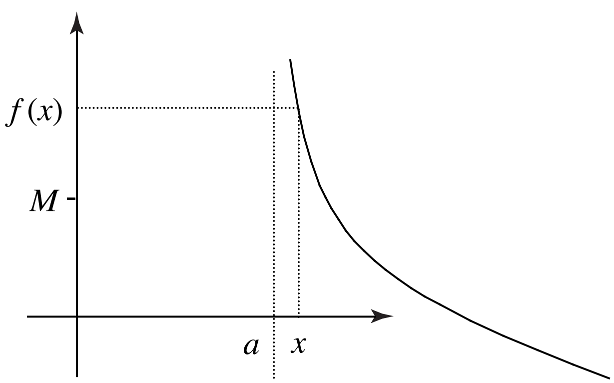


Figure 3.33. Function with an infinite positive limit at a from the right

From Figure 3.33, above, we can see that, given any large $M > 0$, we can find $\delta > 0$ such that, for any $0 < x - a < \delta$, we have $f(x) > M$.

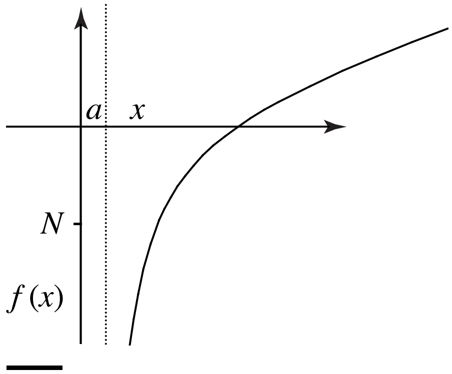


Figure 3.34. Function with an infinite negative limit at a from the right

Figure 3.34, above, is the graph of a function with a negative infinite limit at a from the right. We can see that it does not matter how small and negative a number N we have, we can always find an x close to a from the right, but not equal to a, so that $f(x)$ is less than N.

In Figure 3.34, we can see that, given a small $N < 0$ we can find $\delta > 0$ such that for any $0 < x - a < \delta$, we have $f(x) < N$.

We leave it to you to analyse the limits $\lim\limits_{x \to a^-} f(x) = \pm\infty$ and $\lim\limits_{x \to a} f(x) = \pm\infty$ in the definition below.

Definition 3.36.

a. Let f be a function defined at the right of a. The limit of f, as x approaches a from the right, is infinity if

given any $M > 0$, we can find $\delta > 0$ such that, for any $0 < x - a < \delta$, we have $f(x) > M$.

The limit of f, as x approaches a from the right, is negative infinity if

given any $N < 0$, we can find $\delta > 0$ such that, for any $0 < x - a < \delta$, we have $f(x) < N$.

b. Let f be a function defined at the left of a. The limit of f, as x approaches a from the left, is infinity if

given any $M > 0$, we can find $\delta > 0$ such that, for any $0 < a - x < \delta$, we have $f(x) > M$.

The limit of f, as x approaches a from the left, is negative infinity if

given any $N < 0$, we can find $\delta > 0$ such that, for any $0 < a - x < \delta$, we have $f(x) < N$.

c. Let f be a function defined around a. The limit of f, as x approaches a, is infinity if

given any $M > 0$, we can find $\delta > 0$ such that, for any $0 < |x - a| < \delta$, we have $f(x) > M$.

The limit of f, as x approaches a, is negative infinity if

given any $N < 0$, we can find $\delta > 0$ such that, for any $0 < |x - a| < \delta$, we have $f(x) < N$.

Definition 3.37.

a. If $g(x) > 0$ for $x \to a^+$, then there is a $\delta > 0$ such that $g(x) > 0$ for any $0 < x - a < \delta$.
b. If $g(x) \le f(x)$ for $x \to a^+$, then there is a $\delta > 0$ such that $g(x) \le f(x)$ for any $0 < x - a < \delta$.

In the discussion below, we present some proofs about infinite limits. We strongly recommend that you try other proofs.

Theorem 3.38. If $\lim\limits_{x \to a^+} g(x) = 0$ and $g(x) > 0$ for $x \to a^+$, then

$$\lim_{x \to a^+} \frac{1}{g(x)} = \infty.$$

Proof. We want to show that, given $M > 0$, there is a $\delta > 0$ such that, for any $0 < x - a < \delta$, we have $\dfrac{1}{g(x)} > M$.

We have

1. for any $\varepsilon > 0$, there is $\delta_1 > 0$ such that $|g(x)| < \varepsilon$ for any $0 < x - a < \delta_1$.
2. there is a $\delta_2 > 0$, such that $g(x) > 0$ for any $0 < x - a < \delta_2$.

In particular, for $\varepsilon = \dfrac{1}{M} > 0$ in statement 1, we have

3. there is $\delta_1 > 0$ such that $|g(x)| < \dfrac{1}{M}$ for any $0 < x - a < \delta_1$.

If $\delta = \min(\delta_1, \delta_2)$, then for $0 < x - a < \delta$, statements 2 and 3 hold, and

$$g(x) > 0 \quad \text{and} \quad |g(x)| < \frac{1}{M}.$$

Hence,

$$|g(x)| = \mathrm{g}(x) \quad \text{and} \quad \frac{1}{g(x)} > \mathrm{M}.$$

Q.E.D

Theorem 3.39. If $\lim_{x \to a} f(x) = L > 0$, and $\lim_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} f(x)g(x) = \infty.$$

Proof. We want to show that, given $M > 0$, there is a $\delta > 0$ such that, for any $0 < |x - a| < \delta$, we have $f(x)g(x) > M$.

We have

1. for any $\varepsilon > 0$, there is $\delta_1 > 0$ such that $|f(x) - L| < \varepsilon$ for any $0 < |x - a| < \delta_1$.
2. for any $M' > 0$, there is a $\delta_2 > 0$ such that $g(x) > M'$ for any $0 < |x - a| < \delta_2$.

In particular, in statement 1, for

$$\varepsilon = \frac{L}{2} > 0,$$

there is $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{L}{2} \quad \text{for any} \quad 0 < |x - a| < \delta_1.$$

Hence,

$$-\frac{L}{2} < f(x) - L < \frac{L}{2}.$$

Adding L to each side of the inequality, we conclude that

3. $\dfrac{L}{2} < f(x) < \dfrac{3L}{2}$ for $0 < |x - a| < \delta_1$.

In statement 2, we take $M' = \dfrac{2M}{L} > 0$. There is a $\delta_2 > 0$, such that

4. $g(x) > \dfrac{2M}{L}$ for any $0 < |x - a| < \delta_2$.

If $\delta = \min(\delta_1, \delta_2)$, then statements 3 and 4 hold for all $0 < |x-a| < \delta$, and we have

$$f(x)g(x) > \frac{L}{2}\frac{2M}{L} = M.$$

Q.E.D

Exercises

34. Use Definitions 3.36 and 3.37 to prove the statements below.

a. If $\lim_{x\to a^+} g(x) = 0$ and $g(x) < 0$ for $x \to a^+$, then $\lim_{x\to a^+} \frac{1}{g(x)} = -\infty$.

b. If $\lim_{x\to a} g(x) = 0$ and $g(x) > 0$ for $x \to a$, then $\lim_{x\to a} \frac{1}{g(x)} = \infty$.

c. If $\lim_{x\to a} f(x) = L < 0$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)g(x) = -\infty$.

d. If $\lim_{x\to a} f(x) = L > 0$ and $\lim_{x\to a} g(x) = -\infty$, then $\lim_{x\to a} f(x)g(x) = -\infty$.

Answers to Exercises (appendix-a.htm)

Finally, let us consider the formal definition of

$$\lim_{x\to\pm\infty} f(x) = L.$$

The limit

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large and positive (see Figure 3.35, below).

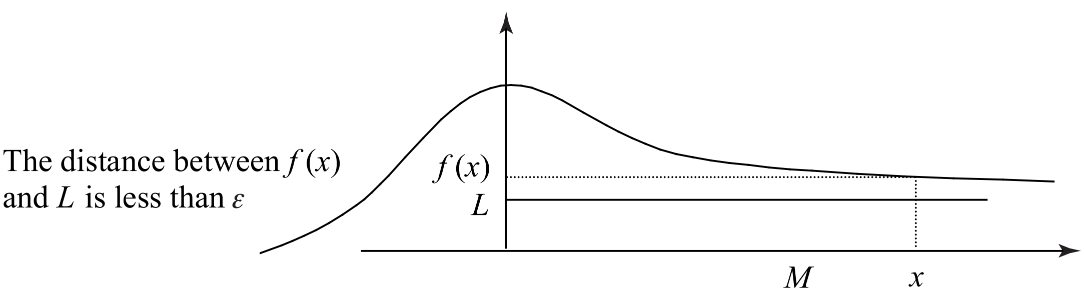


Figure 3.35. Formal definition of $\lim_{x\to+\infty} f(x) = L$

For any small number $\varepsilon > 0$ (as small as we want), we can always find a large number $M > 0$ such that, for any $x > M$, the distance between $f(x)$ and L is less than ε.

Definition 3.40. Let f be a function defined on an interval (c, ∞) for some c. The limit of f, as x approaches infinity, is L if given any $\varepsilon > 0$, we can find an $M > 0$ such that, for any $x > M$, it is the case that $|f(x) - L| < \varepsilon$.

We leave it to you to give the formal definitions of the statements in Definition 3.40 for $x \to a^-$ and $x \to a$.

The limit

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently small and negative (see Figure 3.36, below).

For any small number $\varepsilon > 0$ (as small as we want), we can always find a small number $N < 0$ such that, for any $x < N$, the distance between $f(x)$ and L is less than ε.

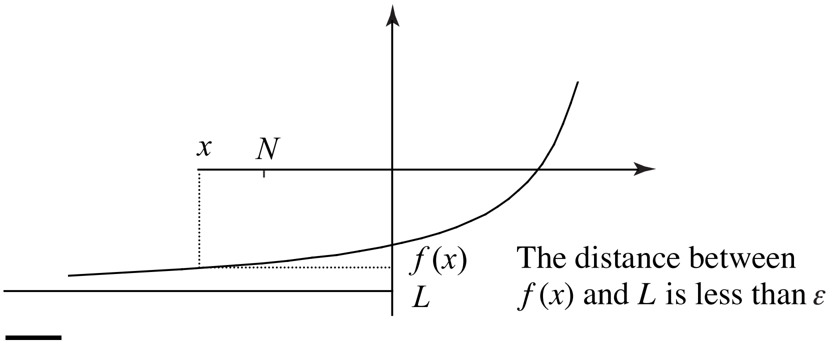


Figure 3.36. Formal definition of $\lim_{x \to -\infty} f(x) = L$

> **Definition 3.41.** Let f be a function defined on an interval $(-\infty, c)$ for some c. The limit of f, as x approaches negative infinity, is L if
>
> given any $\varepsilon > 0$ we can find an $N < 0$ such that, for any $x < N$, it is the case that $|f(x) - L| < \varepsilon$.

We have provided a set of theorems that hold for limits at infinity; one of them is the laws of limits. Compare the following proof with the product of limits we gave before (see Theorem 3.23 (unit03-05.htm#theorem3-23)).

Theorem 3.42. If $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = K$, then

$$\lim_{x\to\infty} f(x)g(x) = LK.$$

Proof. We want to show that, for any given $\varepsilon > 0$, we can find an $M > 0$ such that, for any

$x > M$ it is the case that $|f(x)g(x) - LM| < \varepsilon$.

We know that, for any positive number,

1. $\varepsilon_1 > 0$, there is an $M_1 > 0$ such that, for any $x > M_1$, it is the case that $|f(x) - L| < \varepsilon_1$.
2. $\varepsilon_2 > 0$, there is an $M_2 > 0$ such that, for any $x > M_2$, it is the case that $|g(x) - K| < \varepsilon_2$.

As before, we add $0 = g(x)L - g(x)L$ to relate what we want to show with what we know. By the triangle inequality, we know that

$$\begin{aligned} |f(x)g(x) - LK| &= |f(x)g(x) - g(x)L + g(x)L - LK| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - K| \end{aligned}$$

Since we want this inequality to be less than ε, and since statements 1 and 2 above are true for any positive real number, we take values for ε_1 and ε_2 that fit these criteria.

In statement 2, we take $\varepsilon_2 = 1$, and we find that for $1 > 0$, there is an $M_2 > 0$ such that for any $x > M_2$ it is the case that $|g(x) - K| < \varepsilon_2$.

Since $|g(x)| = |g(x) - K + K| \leq |g(x) - K| + |K|$, we conclude that

3. $|g(x)| < 1 + |K|$ for any $x > M_2$.

Again, in statement 2, we take $\varepsilon_2 = \dfrac{\varepsilon}{2|L|} > 0$, and we find that there is an $M_3 > 0$ such that

4. $|g(x) - K| < \dfrac{\varepsilon}{2|L|}$ for any $x > M_3$.

In statement 1, we take $\varepsilon_1 = \dfrac{\varepsilon}{2(1 + |K|)} > 0$, and we find that there is an $M_1 > 0$ such that

5. $|f(x) - L| < \dfrac{\varepsilon}{2(1+|K|)}$ for any $x > M_1$.

Hence, if $M = \max(M_1, M_2, M_3)$, then for any $x > M$, statements 3, 4 and 5 are valid. Therefore,

$$\begin{aligned} |f(x)g(x) - LK| &\le |g(x)||f(x) - L| + |L||g(x) - K| \\ &< (1+|K|)\tfrac{\varepsilon}{2(1+|K|)} + |L|\tfrac{\varepsilon}{2|L|} = \varepsilon \end{aligned}$$

Q.E.D

From this proof, you can see intuitively why some results for limits at finite numbers are also true for limits at infinity.

Next, we give some other proofs to illustrate the different arguments we can use to prove a statement logically. Observe also how we use the properties of the absolute value.

Theorem 3.43. If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = K$ and $g(x) \le f(x)$ for $x \to a$, then $K \le L$.

This theorem is saying that if $g(x) \le f(x)$ for $x \to a$, then $\lim\limits_{x \to a} g(x) \le \lim\limits_{x \to a} f(x)$.

Proof. We will argue by contradiction. If we assume that $K > L$, this assumption will take us to a contradiction, and from the contradiction, we can conclude that our assumption is false, and that $K \le L$.

By the sum of limits, we know that

$$\lim_{x \to a} g(x) - f(x) = K - L.$$

If $K > L$, then $K - L > 0$. We have two hypotheses:

1. for any $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that $|g(x) - K| < \varepsilon_1$ for any $0 < |x - a| < \delta_1$.
2. for any $\varepsilon_2 > 0$ there is a $\delta_2 > 0$ such that $|g(x) - f(x) - (K - L)| < \varepsilon_2$ for any $0 < |x - a| < \varepsilon_2$.

In particular, we take $\varepsilon_2 = K - L > 0$ in statement 2, and we find that there is $\delta_2 > 0$ such that

3. $|g(x) - f(x) - (K - L)| < K - L$ for any $0 < |x - a| < \delta_1$.

If we take $\delta = \min(\delta_1, \delta_2)$, then statements 1 and 3 are true, and for any $0 < |x - a| < \delta$, we know that

$$-(K - L) < g(x) - f(x) - (K - L) < K - L.$$

Adding $(K - L)$ to both sides, we obtain

$$0 < g(x) - f(x) < 2(K - L).$$

Therefore, $f(x) < g(x)$ for $0 < |x - a| < \delta$. This inequality states that $f(x) < g(x)$ for $x \to a$, contrary to our assumption. Thus, we conclude that $K \le L$.

Q.E.D

Corollary 3.44. Squeeze Theorem.

If $g(x) \le f(x) \le h(x)$ for $x \to a$, and $\lim\limits_{x \to a} g(x) = L = \lim\limits_{x \to a} h(x)$, then $\lim\limits_{x \to a} f(x) = L$.

Proof. From Theorem 3.43, we know that

$$\lim_{x \to a} g(x) \le \lim_{x \to a} f(x) \le \lim_{x \to a} h(x).$$

Hence, $L \le \lim\limits_{x \to a} f(x) \le L$, and $\lim\limits_{x \to a} f(x) = L$.

Q.E.D

Theorem 3.45. If $\lim\limits_{x \to a} g(x) = \infty$ and $g(x) \le f(x)$ for $x \to a$, then $\lim\limits_{x \to a} f(x) = \infty$.

Proof. We want to show that for any $M > 0$, there is a $\delta > 0$ such that $f(x) > M$ for any $0 < |x - a| < \delta$.

We have two assumptions:

1. there is $\delta_1 > 0$ such that $g(x) \leq f(x)$ for any $0 < |x - a| < \delta_1$
2. for $M > 0$, there is a $\delta_2 > 1$ such that $g(x) > M$ for any $0 < |x - a| < \delta_2$.

For $\delta = \min(\delta_1, \delta_2)$, both statements are true; and for any $0 < |x - a| < \delta$, it is the case that $f(x) \geq g(x) > M$.

Q.E.D

Exercises

37. Prove each of the statements below.
    a. If $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} -g(x) = -\infty$.
    b. If $\lim_{x \to a} g(x) = -\infty$ and $g(x) \geq f(x)$ for $x \to a$, then $\lim_{x \to a} f(x) = -\infty$.
    c. If $\lim_{x \to a} g(x) = \infty$ and $\lim_{x \to a} f(x) = \infty$, then $\lim_{x \to a} f(x) + g(x) = \infty$.
    d. If $\lim_{x \to a} g(x) = \infty$ and $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} f(x) + g(x) = \infty$.

Answers to Exercises (appendix-a.htm)

Finishing This Unit

1. Review the objectives of this unit and make sure you are able to meet all of them.
2. If there is a concept, definition, example or exercise that is not yet clear to you, go back and reread it, then contact your tutor for help.
3. Diagrams summarizing strategies for evaluating limits are presented below. Note that these strategies are recommendations—with more practice, you may develop your own strategies. Until then, you may wish to detach these diagrams and pin them above your desk for easy reference.
4. Do the exercises in "Learning from Mistakes" section for this unit.
5. You may want to do Exercises 1-16, 18, 23 parts (a) and (b) and 24-26 from the "Review" (pages 98-99 of the textbook).

Evaluation of the Limit $\lim_{x\to a} f(x)$

Is f continuous at a?	YES $\longrightarrow$	By Theorem 3.14 (unit03-01.htm#theorem3-14), $\lim_{x\to a} f(x) = f(a)$.
$\downarrow$NO		
Can $f(x)$ be simplified so that $f(x) = g(x)$ for x near a, and $\lim_{x\to a} g(x) = M$	YES $\longrightarrow$	By Proposition 3.16 (unit03-02.htm#proposition3-16), $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = M$.
$\downarrow$NO		
Is f equal to, or can it be simplified to a sum, product or quotient of functions each of which has finite limits at a?	YES $\longrightarrow$	Theorem 3.23 (unit03-05.htm#theorem3-23) may apply. If it is a quotient, check that the denominator limit is nonzero.
$\downarrow$NO		
Is $f(x) = h(g(x))$ for x near a, $\lim_{x\to a} g(x) = b$, and either $\lim_{x\to b} h(x) = \pm\infty$ or $\lim_{x\to b} h(x) = L$?	YES $\longrightarrow$	See Theorems 3.20 (unit03-03.htm#theorem3-20) and 3.24 (unit03-05.htm#theorem3-24), and Corollaries 3.25 (unit03-05.htm#corollary3-25) and 3.26 (unit03-05.htm#corollary3-26).
$\downarrow$NO		
Is $f(x) = h(x)g(x)$ with $\lim_{x\to a} g(x) = L$ ($L \neq 0$) and $\lim_{x\to a} h(x) = \pm\infty$?	YES $\longrightarrow$	See Theorem 3.18 (unit03-03.htm#theorem3-18).
$\downarrow$NO		
Are there functions g and h such that $g(x) \le f(x) \le h(x)$ near a and $\lim_{x\to a} h(x) = \lim_{x\to a} g(x)$?	YES $\longrightarrow$	Apply the Squeeze Theorem (unit03-04.htm#theorem3-22).
$\downarrow$NO		

Other techniques may be required; for example, L'Hospital's Rule or power series, which are studied in more advanced courses.

The same flow chart can be used for the left and right limits: $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$.

Evaluation of the limit $\lim_{x\to\infty} f(x)$

Is f a polynomial function?	$\xrightarrow{\text{YES}}$	The limit is ∞ if $f(x) > 0$ for $x \to \pm\infty$; it is $-\infty$ if $f(x) < 0$ for $x \to \pm\infty$.
↓NO		
Is f a rational function?	$\xrightarrow{\text{YES}}$	Apply Theorem 3.29 (unit03-06.htm#theorem3-29).
↓NO		
Is f the quotient of radicals of polynomial functions?	$\xrightarrow{\text{YES}}$	Simplify and apply Proposition 3.16 (unit03-02.htm#proposition3-16) and Theorem 3.23 (unit03-05.htm#theorem3-23).
↓NO		
Is $f(x) = h(g(x))$ for x near a, $\lim_{x\to\infty} g(x) = b$ and $\lim_{x\to b} h(x) = \pm\infty$?	$\xrightarrow{\text{YES}}$	See Theorem 3.20 (unit03-03.htm#theorem3-20).
↓NO		
Are there functions g and h such that $g(x) \le f(x) \le h(x)$ for $x \to \infty$ and $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} g(x)$?	$\xrightarrow{\text{YES}}$	Apply the Squeeze Theorem (unit03-04.htm#theorem3-22).
↓NO		
Other techniques may be required; for example, L'Hospital's Rule or power series, which are studied in more advanced courses.		

The same flow chart can be used for the limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

For help with the concept of limits, view the PowerPoint tutorials below. To access a tutorial:

1. Click on the file to open it.
2. Extract the files and save them to your computer's hard drive.
3. Click on the file folder to open it.
4. Click on the PowerPoint file to view and listen.

Note: If you don't have PowerPoint installed on your computer, you can download the PowerPoint Viewer from here: https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6 (https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6).

Tutorial 5: Limits (../multimedia/powerpoint/limit1.zip)

Tutorial 6: Limits (../multimedia/powerpoint/limit2.zip)

Tutorial 7: Limits (../multimedia/powerpoint/limit3.zip)

Learning from Mistakes

There are mistakes in each of the following solutions. Identify the errors, and give the correct answer.

1. Draw the graph of a single function f, such that
   - $\lim_{x \to 0} f(x) = 3$,
   - $\lim_{x \to -2} f(x) = \infty$, and
   - $\lim_{x \to \infty} f(x) = -1$.

   Erroneous Solution

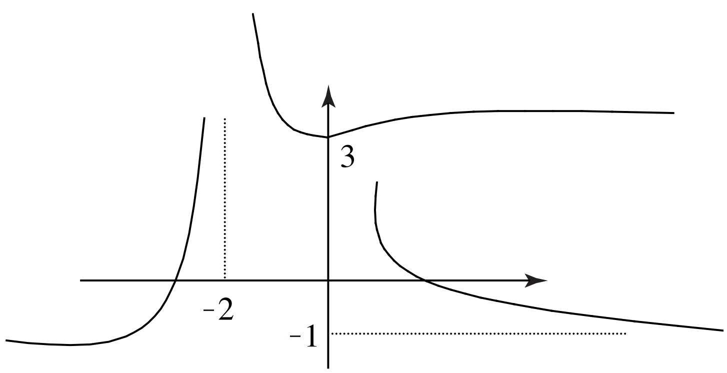


Figure 3.37. Erroneous solution to “Learning from Mistakes” Question 1

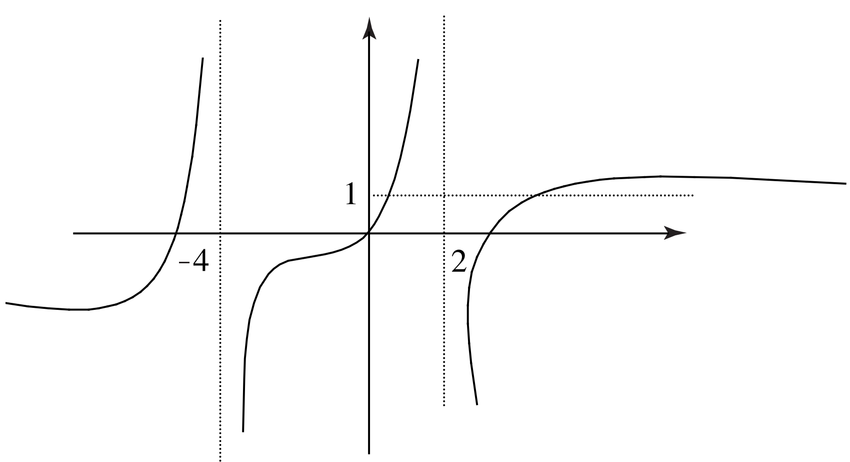


Figure 3.38. Graph for “Learning from Mistakes” Question 2.

2. Find the following limits for the function shown in Figure 3.38, above:
   a. $\lim_{x \to -4^-} f(x)$.
   b. $\lim_{x \to 2} f(x)$.
   c. $\lim_{x \to \infty} f(x)$.
   d. $\lim_{x \to -4^+} f(x)$.

Erroneous Solutions

a. $\lim_{x \to -4^-} f(x) = -\infty$

b. $\lim_{x \to 2} f(x) = \infty$

c. $\lim_{x \to \infty} f(x) = 2$

d. $\lim_{x \to -4^+} f(x) = \infty.$

3. Evaluate each of the following limits. If a limit does not exist, explain why.

a. $\lim_{x \to 0^+} x \sin x + \dfrac{1}{x}.$

b. $\lim_{x \to 1} \dfrac{x^2 + 2x - 15}{x^2 + x - 12}.$

c. $\lim_{x \to 0} \dfrac{x^2 - 1}{x^2 + x^4}.$

d. $\lim_{x \to \pi/2^-} \tan x + \sec x.$

e. $\lim_{x \to 0^+} \dfrac{\tan(3x)}{\sin(4x)}.$

f. $\lim_{x \to 3} \dfrac{\sin(x-3)}{x^2 + x - 12}.$

g. $\lim_{x \to \infty} x \cos\left(\dfrac{1}{x}\right).$

h. $\lim_{x \to \infty} \dfrac{\cos x}{x}.$

i. $\lim_{x \to \infty} \dfrac{x^2 + 3}{\sqrt{x^2 + x - 1}}.$

j. $\lim_{x \to -\infty} \dfrac{\sqrt{(3x+2)(x+1)}}{3 - x}.$

Erroneous Solutions

a. $\lim_{x \to 0^+} x \sin x + \dfrac{1}{x} = \lim_{x \to 0^+} x \lim_{x \to 0^+} \sin x + \lim_{x \to 0^+} \dfrac{1}{x} = 0 - \infty = -\infty.$

b. $\lim_{x \to 1} \dfrac{x^2 + 2x - 15}{x^2 + x - 12} = \dfrac{(x-3)(x+5)}{(x-3)(x+4)} = \dfrac{x+5}{x+4} = \dfrac{6}{5}.$

c. $\lim_{x \to 0} \dfrac{x^2 - 1}{x^2 + x^4} = \dfrac{-1}{0} - \infty.$

d. $\lim_{x \to \pi/2^-} \tan x + \sec x = \infty + \infty = \infty.$

e. $\lim_{x \to 0^+} \dfrac{\tan(3x)}{\sin(4x)} = \dfrac{0}{0}$; the limit does not exist because the numerator and denominator get very close to 0.

f. $\lim_{x \to 3} \dfrac{\sin(x-3)}{x^2 + x - 12} = \infty$ because for x close to 3, $\dfrac{1}{x^2 + x - 12}$ is very large and $\sin(x-3)$ oscillates between -1 and 1.

g. $\lim_{x \to \infty} x \cos\left(\dfrac{1}{x}\right) = \infty$ by the Squeeze Theorem (unit03-04.htm#theorem3-22) because $-x \leq x \cos\left(\dfrac{1}{x}\right) \leq x$ and $\lim_{x \to \infty} \dfrac{1}{x} = \infty$

h. $\lim_{x \to \infty} \dfrac{\cos x}{x}$ does not exist because $\lim_{x \to \infty} \cos x$ does not exist.

i. $\lim_{x \to \infty} \dfrac{x^2 + 3}{\sqrt{x^2 + x - 1}} = 1$, because the degree of the numerator is equal to 2, and the degree of the denominator is equal to 2.

j.
$$
\begin{aligned}
\frac{\sqrt{(3x+2)(x+1)}}{3-x} &= \frac{\sqrt{3x^2+5x+2}}{3-x} \\
&= \frac{\sqrt{x^2\left(3+\frac{5}{x}+\frac{2}{x^2}\right)}}{x\left(3-\frac{1}{x}\right)} \\
&= \frac{\sqrt[x]{\left(3+\frac{5}{x}+\frac{2}{x^2}\right)}}{x\left(3-\frac{1}{x}\right)} \\
&= \frac{\sqrt{\left(3+\frac{5}{x}+\frac{2}{x^2}\right)}}{\left(3-\frac{1}{x}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x \to -\infty} \frac{\sqrt{(3x+2)(x+1)}}{3-x} &= \lim_{x \to -\infty} \frac{\sqrt{\left(3+\frac{5}{x}+\frac{2}{x^2}\right)}}{\left(3-\frac{1}{x}\right)} \\
&= \frac{\sqrt{3}}{-1} \\
&= -\sqrt{3}.
\end{aligned}
$$

4. Find the vertical and horizontal asymptotes of the function

$$g(x) = \frac{\sqrt{x^4+4x^3+4x^2}}{3x^2+5x-2}.$$

Erroneous Solution

The function g is not defined at -2 and $1\,/\,3$, so $x = -2$ and $x = 1\,/\,3$ are vertical asymptotes.

$$\lim_{x \to \infty} \frac{\sqrt{x^4+4x^3+4x^2}}{3x^2+5x-2} = \infty,$$

because the degree of the numerator is 4 and that of the denominator is 2; hence, there are no horizontal asymptotes.

Differentiation

Objectives

When you have completed this unit, you should be able to

1. interpret the derivative as a rate of change.
2. interpret the derivative as a slope of a tangent line.
3. differentiate algebraic and trigonometric functions.
4. apply differentials to estimate numbers and relative errors.
5. apply the derivative to solve problems in the natural and social sciences.
6. apply implicit differentiation to solve related rate problems.

In this unit, we address the question of how, in a function $f(x)$, the dependent variable f changes with respect to the independent variable x. That is, for a specific value of x, we want to know if f is increasing, deceasing or neither, and if $f(x)$ does change, we want to know how quickly or how slowly the change is occurring. The answers to these questions are intrinsically related to the geometric concept of slope.

The Average Rate of Change and the Slope of the Secant Line

Prerequisites

To complete this section, you must be able to

1. find the equation of the slope of a line. Read the section titled "Lines" on pages 346-348 of the textbook.
2. obtain the point-slope form of the equation of a line. Do Exercises 7-10, and odd-numbered Exercises 21-33 and 37 on page 349.

If d is the distance traveled by a moving object (in metres) at time t (in seconds), then d depends on t—in terms of functions, $d(t)$. If we want to know how d is increasing or decreasing with respect to an interval of time, we are referring to the "average velocity" of the object. The *average velocity* is defined as follows:

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}.$$

If the interval is from the time t_0 to the time t_1, then the time elapsed is $t_1 - t_0$, and the distance traveled is $d(t_1) - d(t_0)$; hence,

$$\text{average velocity} = \frac{d(t_1) - d(t_0)}{t_1 - t_0} \quad \text{with units of m/s.}$$

Example 4.1. If $d(2) = 7$ (the distance of the object at 2 seconds is 7 metres), and $d(5) = 12$, then the time elapsed between 5 and 2 is $5 - 2 = 3$ seconds, and the distance traveled during this period of time is $d(5) - d(2) = 12 - 7 = 5$ metres. The average velocity is

$$\text{average velocity} = \frac{d(5) - d(2)}{5 - 3} = \frac{5}{3} \ \text{m/s}$$

On average, from 2 to 5 seconds, the distance is increasing at a rate of $5\ /\ 3$ m/s, that is, the average velocity of the object, from 2 to 5 seconds, is $5\ /\ 3$ m/s.

Example 4.2. If P is the population (in millions) at time t (in minutes), then the average rate of growth in a period of time is given by the population over the time elapsed. Hence, if $P(12) = 120$ and $P(26) = 324$, then

$$\text{average rate of growth} = \frac{P(26) - P(12)}{26 - 12} = \frac{324 - 120}{14} = \frac{102}{7}.$$

From 12 to 26 minutes, the population is growing on average, at a rate of $102\,/\,7$ million/min.

Example 4.3. If C is the cost (in thousands of dollars) of extracting x tonnes of gold, then C depends on x and we evaluate $C(x)$. If $C(20) = 50$ and $C(50) = 96$, then the quotient

$$\frac{C(50) - C(20)}{50 - 20} = \frac{23}{15}$$

is the average cost of gold extraction from 20 to 50 tonnes. Hence, when we are extracting between 20 and 50 tonnes of gold, the total cost is increasing on average, at a rate of 23 / 15 thousand dollars/tonne.

Example 4.4. If S is the number of people who have been ill for a time t (in months), and $S(28) = 568$ and $S(36) = 236$, then the quotient

$$\frac{S(36) - S(28)}{36 - 28} = \frac{236 - 568}{8} = -\frac{83}{2}$$

indicates that from 28 to 36 months, the number of ill people decreases, on average, at a rate of 83 / 2 people/month.

In general,

> **Definition 4.1.** $f(x)$ is a continuous function on the interval $[a, b]$, then the *average rate of change* of f with respect to x, on the interval $[a, b]$, is the quotient
>
> $$\frac{f(b) - f(a)}{b - a} \text{ units of } f/\text{units of } f.$$

Let us consider the geometric interpretation of the average rate of change.

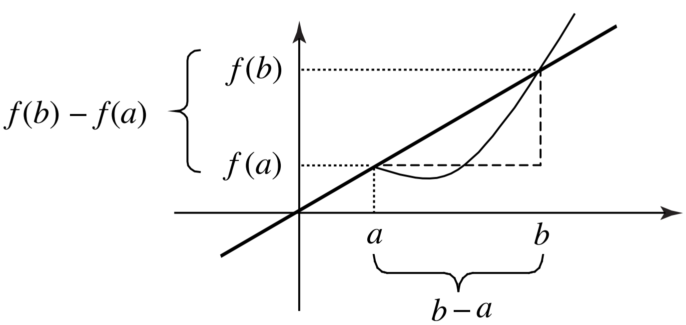


Figure 4.1. Average rate of change

In Figure 4.1, above, we see that the points $(a, f(a))$ and $(b, f(b))$ are on the graph of f, and the line that passes through these points, by definition, has a slope given by the quotient

$$\frac{\text{rise}}{\text{run}} = \frac{f(b) - f(a)}{b - a}.$$

Comparing this equation with Definition 4.1, we conclude that, if $f(x)$ is a continuous curve on an interval $[a, b]$, then the quotient

$$\frac{f(b) - f(a)}{b - a}$$

has two different interpretations:

1. it is the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.
2. it is the average rate of change of f on $[a, b]$.

The rate of change, a physical concept, is equal to the slope of a line, a geometrical concept. It is because of this relationship between the physical and the geometrical that we constantly refer to the graphs of functions in calculus.

Example 4.5. In Example 4.1, the quotient

$$\frac{d(5) - d(2)}{5 - 3} = \frac{5}{2}$$

is the slope of the line passing through the points $(5, d(5))$ and $(2, d(2))$. The slope is positive, so the distance is increasing.

Example 4.6. In Example 4.4, the quotient

$$\frac{S(36)-S(28)}{36-28}=-\frac{83}{2}$$

indicates that the slope of the secant line passing through $(28, S(28))$ and $(36, S(36))$ is $-83\,/\,2$; hence, the average number of people who are ill is decreasing.

Exercises

1. Read Example 3, on pages 100-101 of the textbook.
2. The cost of gas G (in dollars per litre) depends on the demand for litres of gas ℓ (in thousands of litres); hence, $G(\ell)$.
   a. What are the units of the quotient $\dfrac{G(50)-G(30)}{50-30}$?
   b. Give the two different interpretations of the quotient $\dfrac{G(50)-G(30)}{50-30}=0.24$.

Answers to Exercises (appendix-a.htm)

The Instantaneous Rate of Change and the Slope of the Tangent Line

Prerequisites

To complete this section, you must be able to

1. find the equation of the slope of a line. Read the section titled “Lines” on pages 346-348 of the textbook.
2. obtain the point-slope form of the equation of a line. Do Exercises 7-10, and odd-numbered Exercises 21-33 and 37 on page 349.

If average velocity indicates how quickly or slowly an object moves in a period of time, instantaneous velocity indicates how quickly or slowly the object moves at a precise instant.

If $d(t)$ is, again, the distance (in metres) traveled by a moving object at time t (in seconds), then the average velocity on the interval $[t, a]$—or $[a, t]$—corresponds to the quotient

$$\frac{f(a) - f(t)}{a - t} = \frac{f(t) - f(a)}{t - a}.$$

By “the instantaneous velocity at time a,” we mean the velocity at precisely the time a. Intuitively, this value is obtained by taking the average velocity on intervals $[a, t]$—or $[t, a]$—for t very close to a. Hence, the instantaneous velocity at time a is defined as the limit of the average velocity as t approaches a. That is, the velocity of the object at time a is the limit

$$\lim_{t \to a} \frac{f(t) - f(a)}{t - a}.$$

Example 4.7. The displacement (in centimetres) of a particle at time t (in seconds) is given by the function

$$s(t) = t^2 - \frac{t}{2} + 1.$$

The average velocity on the interval $[1, t]$ is

$$\frac{s(t) - s(1)}{t - 1} = \frac{\left(t^2 - \frac{t}{2} + 1\right) - \left(1 - \frac{1}{2} + 1\right)}{t - 1} = \frac{t^2 - \frac{t}{2} - \frac{1}{2}}{t - 1}.$$

The average velocity on the interval $[0, 1]$ is

$$\frac{s(1) - s(0)}{1 - 0} = \frac{\frac{-1}{2}}{-1} = \frac{1}{2}$$

and the average velocity on the interval $[1, 2]$ is

$$\frac{s(2) - s(1)}{2 - 1} = \frac{\frac{5}{2}}{1} = \frac{5}{2}.$$

The limit

$$\begin{aligned} \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} \quad &= \lim_{t \to 1} \frac{t^2 = \frac{t}{2} - \frac{1}{2}}{t - 1} \\ &= \lim_{t \to 1} \frac{(t - 1)\left(t + \frac{1}{2}\right)}{t - 1} \\ &= \lim_{t \to 1} t + \frac{1}{2} = \frac{3}{2} \end{aligned}$$

is the velocity of the particle at 1 second.

That is to say, at precisely 1 second, the velocity of the particle is $3 \ / \ 2$ cm/s, or the distance traveled by the particle at 1 second is increasing at a rate of $3 \ / \ 2$ cm/s.

Example 4.8. The temperature T (in $^\circ$C) at time t (in hours) is given by $T(t) = 5 - 2t - t^3$. The average temperature on the interval $[t, 1]$ is the quotient

$$\begin{aligned}\frac{T(1) - T(t)}{1 - t} &= \frac{(5 - 2 - 1) - \left(5 - 2t - t^3\right)}{1 - t} \\ &= \frac{-3 + 2t + t^3}{1 - t}.\end{aligned}$$

Hence, the average temperature on the interval $[0, 1]$ is -3. On average, during the first hour, the temperature is decreasing at a rate of $3^\circ\mathrm{C/h}$. The limit

$$\lim_{t \to 1} \frac{T(t) - T(1)}{t - 1} = \lim_{t \to 1} \frac{3 - 2t - t^3}{t - 1} = \lim_{t \to 1} \ -t^2 - t - 3 = \ -5$$

indicates that, at precisely 1 hour, the temperature is decreasing at a rate of $5^\circ\mathrm{C/h}$.

In general, then,

Definition 4.2. if f is a continuous function around a, then the *instantaneous rate of change* of f at a is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \text{expressed in units of } f \,/\, \text{units of } x.$$

Figure 4.2, below, shows the geometric interpretation of this limit.

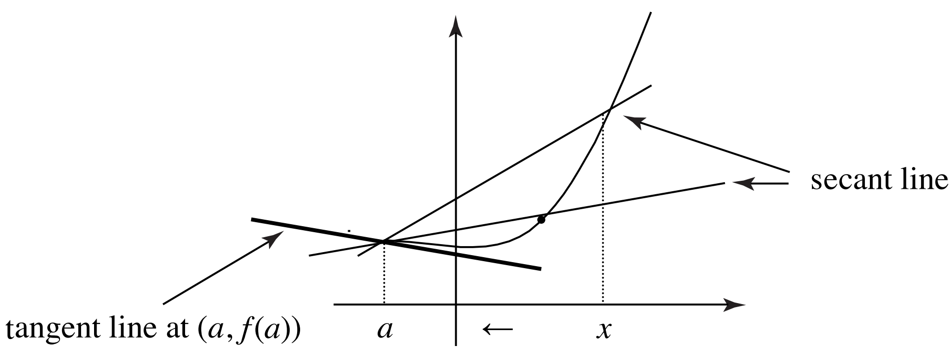


Figure 4.2. Instantaneous rate of change

The quotient

$$\frac{f(x) - f(a)}{x - a}$$

is the slope of the secant line through the points $(a, \ f(a))$ and $(x, \ f(x))$, as shown on Figure 4.2, above.

For each x near a (in Figure 4.2, from the right), we consider the point $(x, \ f(x))$ and the corresponding secant line through $(a, \ f(a))$ and $(x, \ f(x))$. If we move x towards a, we see that the secant line tends toward the line tangent to the curve at the point $(a, \ f(a))$ (see Figure 1 on page 103 of the textbook). The same is true for x near a from the left. Make a similar sketch and convince yourself that this is indeed the case.

Hence, the slope of the secant lines tend to the slope of the tangent line at $(a, \ f(a))$.

We know that the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

has two interpretations:

1. it is the instantaneous rate of change at (precisely) a.
2. it is the slope of the tangent to the curve at the point $(a, f(a))$.

Example 4.9. Consider the graph of the function $f(x) = 4 - x^2$ shown in Figure 4.3, below.

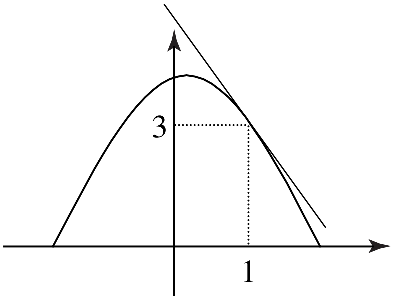


Figure 4.3. Function $f(x) = 4 - x^2$

We can see by the inclination of the tangent line that the slope must be negative, and indeed the slope at the point $(1, f(1)) = (1, 3)$ is given by the limit

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{4 - x^2 - 3}{x - 1} = \lim_{x\to 1} \frac{1 - x^2}{x - 1} = \lim_{x\to 1} -(x + 1) = -2$$

If $f(x) = 4 - x^2$ is the altitude at time x (in seconds) of an object falling freely from a height of 4 m, then at precisely 1 second, the altitude is decreasing at a rate of 2 m/s.

We also see from the graph that the tangent line at $x = 0$ is horizontal; hence, the slope must be 0. Indeed,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{4 - x^2 - 4}{x} = \lim_{x\to 0} -x = 0.$$

At $x = 0$ (the initial position), the height of the object is neither increasing nor decreasing—the object is stationary.

Exercises

3. Do part (a) of Exercises 3, 5, 6 and 7 on pages 101-102 of the textbook
4. Do Exercises 27-29 on pages 118-119 of the textbook

Answers to Exercises (appendix-a.htm)

The limit in Definition 4.2 has a special name and notation: it is called the "derivative of the function f at a," and we write it $f^{'}(a)$.

Definition 4.3. If f is continuous around a, then the *derivative* of f at a is the limit

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f^{'}(a),$$

if this limit exists.

Example 4.10. If C is the cost of producing x tonnes of wheat, then the equation $C^{'}(10) = 12$ has two interpretations. First, the "real world" interpretation: when the production is 10 tonnes, the cost is increasing at a rate of \$12.00 dollars/tonne. The geometric interpretation is that the slope of the tangent line to the curve $C(x)$ at the point $(10, C(10))$ is 12.

Example 4.11. See Exercise 30 on page 119 of the textbook.

The function f represents the quantity (in pounds) of coffee sold at a price p dollars per pound. The derivative function $f'(p)$ is the rate at which the quantity of coffee increases or decreases for a certain price p. When the price of coffee sold is \$8.00, the quantity of coffee increases or decreases at a rate of $f'(8)$ pounds/dollar. We estimate that the higher the value of p, the less coffee is sold. For a price of \$8.00 per pound, we believe that the quantity of coffee sold decreases; hence, $f'(8)$ is negative.

Example 4.12. If the derivative function of a function $s(t)$ that represents the motion of a particle measured in metres at t seconds is $s'(t) = 3t^2 + 5$, then $s'(t)$ is the velocity of the particle at time t and its units are metres per second. Since

$$\lim_{t \to 5} \frac{s'(t) - s'(5)}{t - 5} = \lim_{t \to 5} \frac{3t^2 - 75}{t - 5} = \lim_{t \to 5} 3(t + 5) = 30,$$

we find that at time 5 seconds, the velocity of the particle is increasing at a rate of 30 (m/s)/s. That is, the acceleration of the particle is 30 m/s ^{2}.

On the other hand, the slope of the tangent line at the point $\left(5, s'(5)\right) = (5, 80)$ is 30.

Example 4.13.

a. Refer to Example 4.7. If $s(t) = t^2 - \dfrac{t}{2} + 1$, then $s'(1) = \dfrac{3}{2}$.

b. Refer to Example 4.8. If $T(t) = 5 - 2t - t^3$, then $T'(1) = -5$.

c. Refer to Example 4.9. If $f(x) = 4 - x^2$, then $f'(0) = 0$ and $f'(1) = -2$.

We know that a limit may or may not exist. The limit in Definition 4.3 is no exception; we will use the geometric interpretation of this limit to find out when the limit itself does not exist.

For the limit in Definition 4.3 to exist, we need a tangent line to the curve at the point $(a, f(a))$. If the curve does not have such a tangent line, then the limit cannot exist. The tangent line is the line closest to the curve at the point $(a, f(a))$; that is, the tangent line lies on the point—we do not have tangent lines when the function is not defined at a.

In Figure 4.4, below, it is clear that there cannot be a tangent line at the point $(a, f(a))$ in either graph. But breaks do not constitute the only case where a curve is not defined at a point.

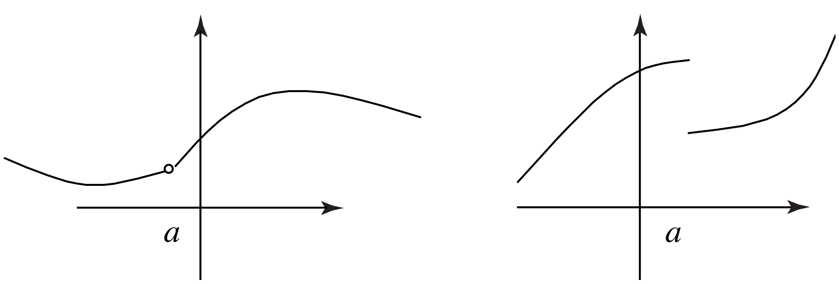


Figure 4.4. Curves that are not defined at the point $(a, f(a))$

For example, it may be that, while the function is continuous, it has a kink at the point $(a, f(a))$ that makes a tangent line impossible, examples of this case are shown in Figure 4.5.

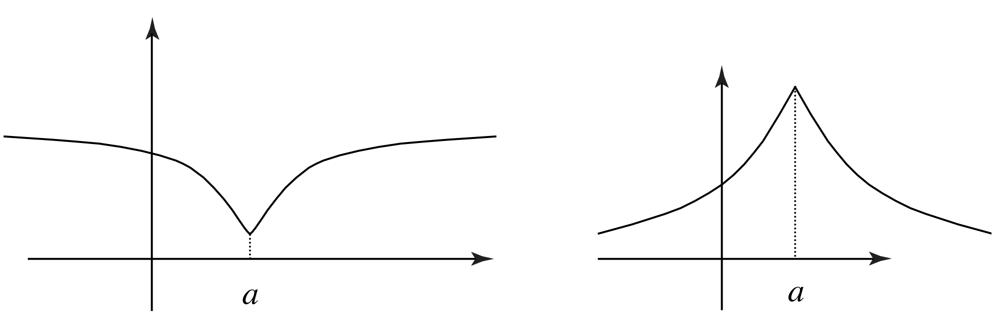


Figure 4.5. Curves showing a kink at the point $(a, f(a))$

The slope of vertical lines is undefined; hence, if the tangent line at the point $(a, f(a))$ is vertical, the limit in Definition 4.3 is also undefined. This case is shown in Figure 4.6.

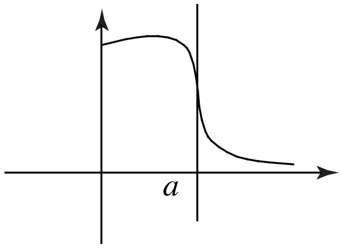


Figure 4.6. Curve with a vertical tangent at the point $(a, f(a))$

Summarizing: If the limit in Definition 4.3 does not exist, it is because

- the function is not continuous at a, or
- the function is continuous, but there is no clearly defined tangent line at the point $(a, f(a))$, or
- the function is continuous and there is a tangent line at the point $(a, f(a))$, but that tangent line is vertical.

Conversely if the limit in Definition 4.3 exists, it is because the curve around the point $(a, f(a))$ is "smooth" enough to have a tangent line.

Definition 4.4. A function f is *differentiable* at a if the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists.}$$

If the function is differentiable at all real numbers, it is then *differentiable everywhere*.

From the graphs of the basic functions in Table 2.1 (unit02-01.htm#table2-1), we see that the absolute value function $g(x) = |x|$ is not differentiable at 0, the root function is not differentiable at 0, the tangent and secant functions are not differentiable at $\pi / 2$, and the sine and cosine functions are differentiable everywhere.

In the next examples, we evaluate the limit using Definition 4.3 for a fixed number a. As you will see, for relatively simple functions, this limit is not difficult to evaluate. In the following examples, you must identify the algebraic manipulations and results we apply to evaluate each limit.

Example 4.14. Let $f(x) = \sqrt{x}$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}.$$

So, for example, $f'(3) = \dfrac{1}{2\sqrt{3}}$ is the slope of the tangent line at $(3, f(3)) = \left(3, \sqrt{3}\right)$.

Since $f'(0)$ is undefined, the function is not differentiable at 0, as we observed earlier.

Example 4.15. Let $f(x) = \dfrac{1}{x + 3}$

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{\frac{1}{x+3} - \frac{1}{a+3}}{x - a} \\
&= \lim_{x \to a} \frac{a - x}{(x - a)(x + 3)(a + 3)} \\
&= \lim_{x \to a} -\frac{1}{(x + 3)(a + 3)} \\
&= -\frac{1}{(a + 3)^2}.
\end{aligned}$$

For example,

$$f'(5) = -\frac{1}{8^2} = -\frac{1}{64}$$

and the slope of the tangent line at the point $(5, f(5)) = \left(2, \frac{1}{8}\right)$ is $-1/64$.

We see that $f'(-3)$ is undefined; hence, the function is not differentiable at -3, because the function f is not continuous at -3.

Example 4.16. Let $f(x) = x^3$.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} x^2 + xa + a^2 = 3a^2.$$

So, for example, the slope of the tangent line at $\left(\frac{1}{2}, \frac{1}{8}\right)$ is

$$f'\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

The function is differentiable everywhere. [Why?]

These examples show that if we have a general expression for the derivative, then we can find the slope of the tangent line or the instantaneous rate of change for any number where $f'(a)$ is defined, without having to evaluate a limit at each particular number.

There is an equivalent form for the limit in Definition 4.3. If we set $x = h + a$, then $h = x - a$, and as $x \to a$, we observe that $h \to 0$, and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

We can use either form to find the derivative at a, depending on which is more convenient for the given function.

Exercises

5. Do Exercises 35 and 36 on page 130 of the textbook.
6. Read the section titled “How Can a Function Fail to be Differentiable?” on page 127.
7. Read Example 4 on page 116 of the textbook.
8. Using Definition 4.4 in this *Study Guide*, do Exercises 13, 15, 16 and 18 on page 118 of the textbook.

Answers to Exercises (appendix-a.htm)

The Graph of the Derivative Function

Prerequisites

To complete this section, you must be able to

1. identify when the slope of a line is positive or negative. Read page 346 of the textbook.
2. obtain the equation of a line. Read the section titled "Lines" on pages 346-348 of the textbook. Do the odd-numbered exercises from 21 to 29 on page 349.

Observe that of the two forms for $f^{'}(a)$, that is,

$$f^{'}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad f^{'}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

we can replace the number a with x only in the former. Doing so gives us

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This general form of $f'(x)$ for any number x gives a function that is called the "derivative function of f."

> **Definition 4.5.** The *derivative function* $f^{'}$ of the function f is given by the limit
>
> $$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of the derivative function is the set of all x for which the limit exists; that is, the set of numbers for which the function f is differentiable.

The value of $f'(x)$ has two interpretations:

1. it is the instantaneous rate of change at x.
2. it is the slope of the tangent line at the point $(x, f(x))$.

Using the geometric interpretation of the derivative function, we can sketch its graph by observing closely how the slope of the tangent line changes as it is "travels" through the curve.

Examine the graph of the function given in Figure 4.7, below, and pay attention to the slopes S_1, S_2, S_3, S_4, and S_5, of the tangent lines. As you can see, all the tangent lines are positive; furthermore,

$$S_1 > S_2 > S_3 \quad \text{and} \quad S_3 < S_4 < S_5.$$

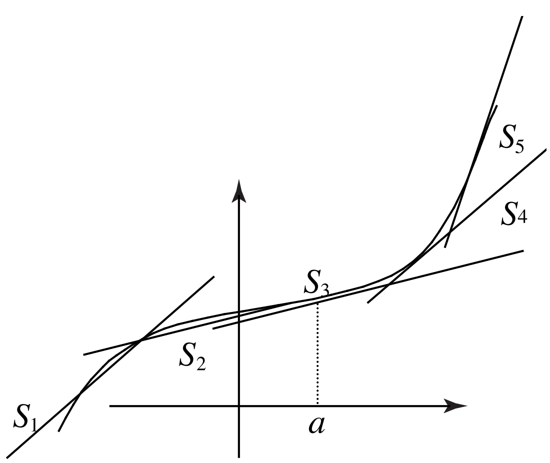


Figure 4.7. Function showing positive slopes of the tangent lines

As the tangent lines "travel" through the curve, the slopes decrease before a and increase after a. Since the derivative function is positive, decreasing before a and increasing after a, its graph might look like that shown in Figure 4.8, below.

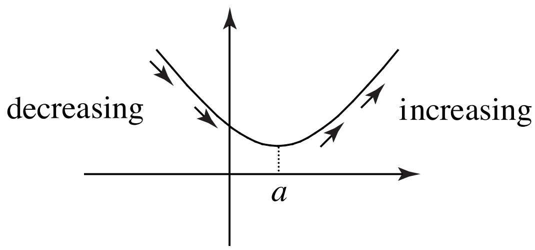


Figure 4.8. Possible derivative function for the function in Figure 4.7

In Figure 4.9, below, as the tangent line travels through the curve, the slopes $S_1,\ S_2,\ S_3,\ S_4$ and S_5 are negative; furthermore,

$$S_1 < S_2 < S_3 \qquad \text{and} \qquad S_3 > S_4 > S_5.$$

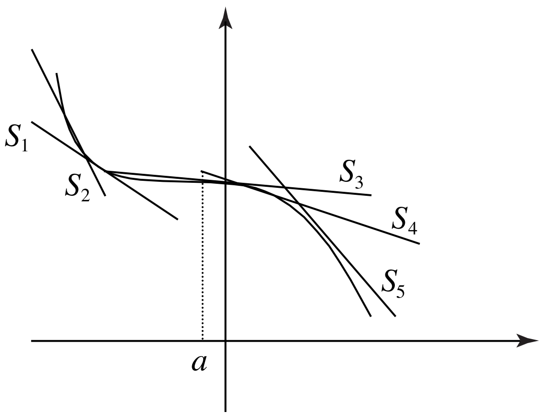


Figure 4.9. Function showing negative slopes of the tangent lines

Thus, the graph of the derivative function is increasing before a and decreasing after a, and its graph might look like that shown in Figure 4.10, below.

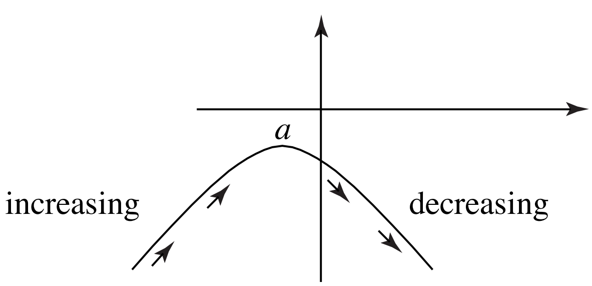


Figure 4.10. Possible derivative function for the function in Figure 4.9

If the tangent line is horizontal, then the slope is zero, and the derivative function is zero at this point. In Figure 4.11, below, the slopes $S_1,\ S_2,\ S_3,\ S_4$ and S_5 are negative, and

$$S_1 < S_2 < S_3 < S_4 < S_5 = 0$$

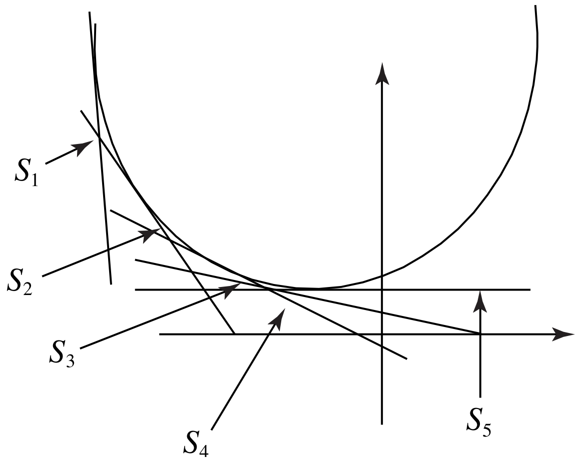


Figure 4.11. Function with negative tangents and one horizontal tangent

In Figure 4.12, below, the slopes S_6, S_7, S_8 and S_9 are positive, and

$$0 = S_5 < S_6 < S_7 < S_8 < S_9.$$

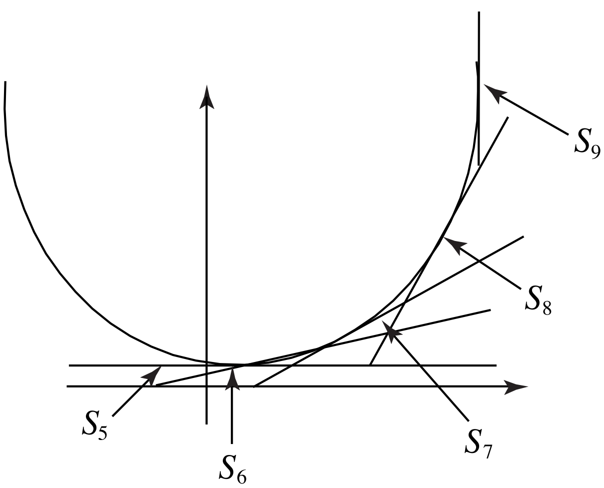


Figure 4.12. Function with positive tangents and one horizontal tangent

The graph of the derivative function might be like that shown in Figure 4.13, below.

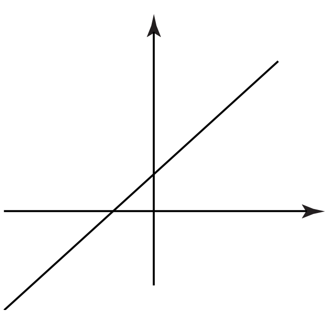


Figure 4.13. Possible derivative function for the function in Figure 4.12

To sketch the graph of the derivative function $f'(x)$ from the graph of the function $f(x)$:

1. identify the points where the derivative is zero; that is, where the tangent line is horizontal.

2. identify the points where the derivative is undefined; that is, where there is no tangent line or where the tangent line is vertical.
3. identify the intervals where the derivative is positive or negative.
4. identify the intervals where the derivative is increasing or decreasing.

Example 4.17. In Figure 4.14, below, we have the graph of a function $f(x)$.

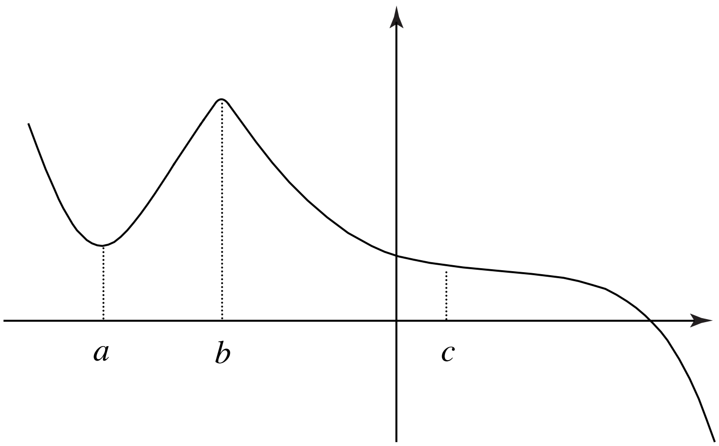


Figure 4.14. Graph of a function $f(x)$, Example 4.17

To sketch the graph of the derivative function, $f'(x)$, we must note that

1. $f^{'}(a) = 0$ and $f^{'}(c) = 0$, because at the points $(a, f(a))$ and $(c, f(c))$, the tangent lines are horizontal.
2. $f^{'}(b)$ is undefined, because there is no tangent line at the point $(b, f(b))$.
3. $f^{'}(x) > 0$ on the interval (a, b), because the tangent line on this interval is positive; and $f^{'}(x) < 0$ on the intervals $(-\infty, a)$, (b, c), and (c, ∞). [Why?]
4. $f'(x)$ is increasing on the intervals $(-\infty, a)$, (a, b) and (b, c). [Why?]
5. $f'(x)$ is decreasing on the interval (c, ∞).

A possible graph of the derivative function is shown in Figure 4.15 below. Later, we will learn to sketch this graph more precisely.

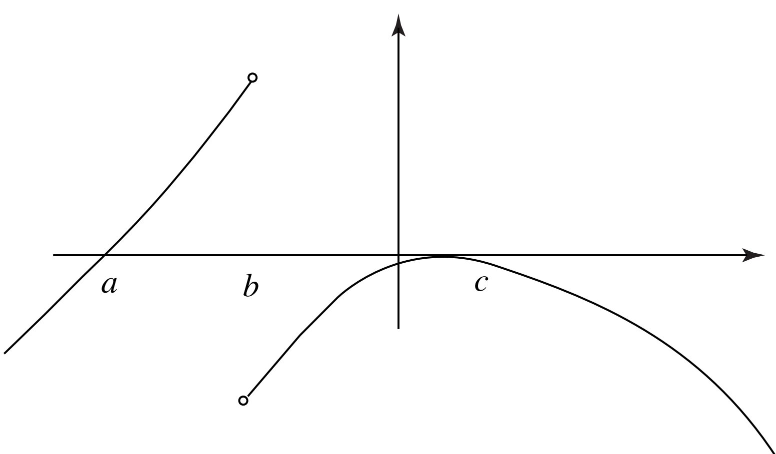


Figure 4.15. Possible derivative function $f'(x)$ for the function in Figure 4.14

Exercises

9. Read Example 1 on page 120 of the textbook.
10. Do Exercises 4-13 on pages 128-129 of the textbook.

Answers to Exercises (appendix-a.htm)

The Derivative Function

The concept of the derivative evolved through a long process, and in part independently, in astronomy, physics and other fields, and has important applications in each of these fields. Not surprisingly, different notations for the derivative developed in each field. Some notations have been superseded by more popular notations; others have remained. In this course, we use the most common notations interchangeably.

For the derivative at a number a, we have

$$f'(a) = \frac{d}{dx} f(x) \bigg|_{x=a} = \frac{df}{dx} \bigg|_{x=a} = D_x f(a).$$

For the derivative function, we have

$$f'(x) = \frac{d}{dx} f(x) = \frac{df}{dx} = D_x f(x).$$

If $y = f(x)$, then we write

$$f'(a) = y'(a) = \frac{dy}{dx} \bigg|_{x=a} \quad \text{and} \quad f'(x) = y' = y'(x) = \frac{dy}{dx}.$$

The notation $\frac{dy}{dx} = \frac{df}{dx}$ is the *Leibnitz notation* for the derivative.

Example 4.18.

a. By Example 4.14 (unit04-01.htm#example4-14), the derivative function of $f(x) = \sqrt{x}$ is

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

We can also write

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{d}{dx}\sqrt{x} \bigg|_{x=3} = \frac{1}{2\sqrt{x}} \bigg|_{x=3} = \frac{1}{2\sqrt{3}}.$$

b. By Example 4.15 (unit04-01.htm#example4-15), the derivative function of $f(x) = \frac{1}{x+3}$ is

$$f'(x) = D_x f(x) = -\frac{1}{(x+3)^2}$$

or

$$D_x \frac{1}{x+3} = -\frac{1}{(x+3)^2};$$

therefore,

$$D_x f(2) = -\frac{1}{(2+3)^2} = -\frac{1}{25}.$$

c. By Example 4.16 (unit04-01.htm#example4-16), the derivative function of $f(x) = x^3$ is

$$f'(x) = 3x^2, \quad \text{and} \quad f'(3) = 3(9) = 27.$$

Example 4.19. To apply Definition 4.5 (unit04-02.htm#definition4-5), we must understand how $f(x+h)$ is obtained.

Note that $x + h$ takes the place of the independent variable x. Hence, if $f(x) = x^2 + x$, then

$$f(x+h) = (x+h)^2 + (x+h) = x^2 + 2xh + h^2 + x + h.$$

The derivative function is

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left(x^2 + 2xh + h^2 + x + h\right) - \left(x^2 + x\right)}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} \\ &= \lim_{h\to 0} (2x + h + 1) = 2x + 1. \end{aligned}$$

That is, $f'(x) = 2x + 1$.

As we note above, the line closest to a curve at a point p is the tangent line that lies on p. This line is very useful for estimating values of functions.

Example 4.20. In Figure 4.16, below, we have the graph of the function $f(x) = \sqrt{x}$, and we can see that the tangent line near the point $(1, f(1)) = (1, 1)$ is close to the curve.

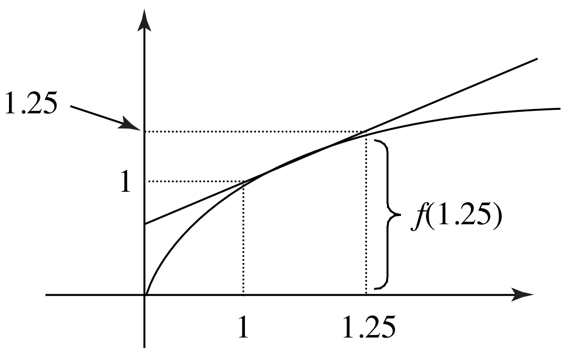


Figure 4.16. Graph of the function $f(x) = \sqrt{x}$

Since $f(1) = 1$, the value of the function near 1 is approximately equal to 1, that is $f(1.25) \approx f(1)$ and $f(0.75) \approx f(1)$; thus, $\sqrt{1.25} \approx 1$ and $\sqrt{0.75} \approx 1$.

To give a first approximation of these values near 1, we use the equation of the tangent line at this point.

Let us assume that the tangent line has the equation

$$y = \frac{x}{2} + \frac{1}{2}$$

(we will see why later on).

Then, from Figure 4.16, we conclude that $f(1.25) \approx y(1.25)$. So,

$$f(1.25) \approx \frac{1.25}{2} + \frac{1}{2} = \frac{9}{8};$$

that is, $f(1.25) = \sqrt{1.25} \approx 1.125$.

Similarly,

$$f(0.75) \approx y(0.75) = \frac{0.75}{2} + \frac{1}{2} = 0.875;$$

that is, $f(0.75) = \sqrt{0.75} \approx 0.875$. Use your calculator to check these square roots. You will see that, at least to the first decimal place, each of these approximations is correct.

Now, the slope of the tangent line at $(1, 1)$ is given by the derivative at 1. From Example 4.14 (unit04-01.htm#example4-14), we have

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of a line with slope $1 / 2$ running through the point $(1, 1)$ is

$$y - 1 = \frac{x-1}{2} \quad \text{or} \quad y = \frac{x}{2} + \frac{1}{2}.$$

Example 4.21. By Example 4.15 (unit04-01.htm#example4-15), we have

$$\frac{d}{dx}\frac{1}{x+3} = -\frac{1}{(x+3)^2}.$$

The slope of the tangent line at the point

$$(1, f(1)) = \left(1, \frac{1}{1+3}\right) = \left(1, \frac{1}{4}\right)$$

is

$$-\frac{1}{(1+3)^2} = -\frac{1}{16}$$

and the equation of the tangent line is

$$y - \frac{1}{4} = -\frac{x-1}{16} \quad \text{or} \quad y = -\frac{x}{16} + \frac{5}{16}.$$

Thus,

$$f(x) \approx -\frac{x}{16} + \frac{5}{16}$$

for any x near 1. For example, for $x = 1.1$, we find that

$$\frac{1}{4.1} = f(1.1) \approx -\frac{1.1}{16} + \frac{5}{16} = \frac{39}{160} \qquad \text{[Why?]}$$

Use your calculator to determine that $39 / 160$ is in fact a good approximation of $1 / 4.1$.

In general, we have the following proposition about the equation of the tangent line.

Proposition 4.6. If the function f is continuous around a, then the equation of the tangent line at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x-a) \qquad \text{or} \qquad y = f'(a)(x-a) + f(a),$$

and for any b close to a,

$$f(b) \approx f'(a)(b-a) + f(a).$$

Exercises

11. Read Example 4 on page 116 of the textbook.
12. Do Exercises 7, 8 and 9 (a) on page 118.
13. Read Examples 3, 4 and 5 on pages 123-124.
14. Do Exercises 19-24 on page 118.

Answers to Exercises (appendix-a.htm)

Rules of Differentiation

We have applied Definition 4.5 (unit04-02.htm#definition4-5) to find the derivative function. However, the similarities among certain types of functions allow for generalizations and for the development of rules or formulas for differentiation, instead of the direct application of Definition 4.3 (unit04-01.htm#definition4-3) or Definition 4.5 (unit04-02.htm#definition4-5).

Since the derivative is the slope of the tangent line, we can see that the graph of the constant function $g(x) = c$ is a horizontal line which coincides with its tangent line. Therefore, the derivative is zero; that is,

$$\frac{d}{dx}\, c = 0.$$

Also, for a linear function $g(x) = mx$, the tangent line coincides with the function, and the slope of this line is m. Hence,

$$\frac{d}{dx}\, mx = m.$$

From Examples 4.14 (unit04-01.htm#example4-14) and 4.16 (unit04-01.htm#example4-16), we have

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} = \frac{1}{2}\left(x^{-1/2}\right) \quad \text{and} \quad \frac{d}{dx}\, x^3 = 3x^2.$$

These two examples suggest a generalization of the form

$$\frac{d}{dx}\, x^r = r\, x^{r-1}.$$

If this generalization is indeed workable, then the derivative of $f(x) = x^{-1}$ should be $f'(x) = -\left(x^{-2}\right)$. [Why?]

Let us see if this surmise is true by applying Definition 4.3 (unit04-01.htm#definition4-3):

$$\lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = \lim_{x \to a} \frac{a - x}{(xa)(x - a)} = \lim_{x \to a} -\frac{1}{xa} = -\frac{1}{a^2} = -a^{-2}.$$

Since this statement is true for any number a, we conclude that $f'(x) = -\left(x^{-2}\right)$, as we suspected.

Pages 132 and 133 of the textbook give a proof (explanation) of why, for any positive integer n,

$$\frac{d}{dx}\, x^n = n\, x^{n-1}.$$

The proof for any real number r is more complex, and is not covered in this course. Instead, we simply present the power rule, shown below.

The Power Rule: For any real number r, $\dfrac{d}{dx}\, x^r = r\, x^{r-1}$.

To apply the power rule, we must identify the power r.

Example 4.22. In the function

$$f(x) = \frac{1}{x^{3/2}},$$

the power is $r = -3/2$, since $f(x) = x^{-3/2}$.

So, applying the power rule, we find that

$$f'(x) = \left(-\frac{3}{2}\right)x^{(-3/2)-1} = \left(-\frac{3}{2}\right)x^{-5/2} = -\frac{3}{2x^{5/2}}.$$

Example 4.23. In the function

$$f(x) = \frac{1}{\sqrt{x}},$$

the power is $r = -1/2$. [Why?]

Then,

$$f'(x) = \left(-\frac{1}{2}\right)x^{(-1/2)-1} = \left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{2x^{3/2}}.$$

Example 4.24. $\frac{d}{dx}\frac{1}{x^{\sqrt{3}}} = \frac{d}{dx}x^{-\sqrt{3}} = -\sqrt{3}\,x^{-\sqrt{3}-1} = -\frac{\sqrt{3}}{x^{\sqrt{3}+1}}.$

By the laws of limits (Theorem 3.23 (unit03-05.htm#theorem3-23)), we derive the sum and constant multiple rules shown below.

The Sum Rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

The Constant Multiple Rule: $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$

Observe how these rules are applied in the differentiation of the functions below.

Example 4.25. We indicated by a geometric argument that the derivative of $f(x) = mx$ is m. Using the rules of differentiation we will confirm this conclusion, using $m = 5$.

$$\begin{aligned} \frac{d}{dx}5x &= 5\frac{d}{dx}x && \text{constant multiple rule} \\ &= 5\,(1)\,x^{1-1} && \text{power rule} \\ &= 5\,x^0 = 5 && \text{as we indicated above} \end{aligned}$$

Example 4.26.

$$\begin{aligned} \frac{d}{dx}\frac{3}{4x} + \sqrt[3]{x} &= \frac{d}{dx}\frac{3}{4}x^{-1} + \frac{d}{dx}x^{1/3} && \text{sum rule} \\ &= \frac{3}{4}\frac{d}{dx}x^{-1} + \frac{d}{dx}x^{1/3} && \text{constant multiple rule} \\ &= \frac{3}{4}(-1)x^{-2} + \left(\frac{1}{3}\right)x^{(1/3)-1} && \text{power rule} \\ &= -\frac{3}{4x^2} + \frac{1}{3x^{2/3}}. \end{aligned}$$

Example 4.27.

$$\begin{aligned} \frac{d}{dx}6x^{12} &= 6\frac{d}{dx}x^{12} && \text{constant multiple rule} \\ &= 6(12)\,x^{11} && \text{power rule} \\ &= 72\,x^{11} \end{aligned}$$

In practical terms, we see from Examples 4.26 and 4.27 that

$$\frac{d}{dx}c\,x^r = (cr)x^{r-1}.$$

Example 4.28.

$$\begin{aligned} &\frac{d}{dx}\frac{2x^4}{3} + 6x^2 - \sqrt{5} \\ &= \frac{2}{3}\frac{d}{dx}x^4 + 6\frac{d}{dx}x^2 - \frac{d}{dx}\sqrt{5} && \text{constant multiple and sum rules} \\ &= \frac{8}{3}x^3 + 12x && \text{power and constant rules} \end{aligned}$$

Exercises

15. Read the proofs of the constant multiple and sum rules on pages 133-134 of the textbook.
16. Do at least 10 of Exercises 1 to 20 on pages 140-141 of the textbook.

Answers to Exercises (appendix-a.htm)

Let us use Definition 4.2 (unit04-01.htm#definition4-2) to find the derivative of the reciprocal function $f(x) = \dfrac{1}{g(x)}$.

$$\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{x\to a}\frac{\dfrac{1}{g(x)}-\dfrac{1}{g(a)}}{x-a} && \text{Definition 4.2 (unit04-01.htm\#definition4-2)}\\
&= \lim_{x\to a}\frac{g(a)-g(x)}{g(x)g(a)(x-a)} && \text{doing the operations}\\
&= \lim_{x\to a}\frac{-(g(x)-g(a))}{x-a}\lim_{x\to a}\frac{1}{g(x)g(a)} && \text{law of limits}\\
&= -\frac{g'(a)}{g(a)^2}
\end{aligned}$$

From this operation, we derive the reciprocal rule.

The Reciprocal Rule: $\dfrac{d}{dx}\dfrac{1}{g(x)} = -\dfrac{g'(x)}{g(x)^2}$

Example 4.29. To apply the reciprocal rule in the function

$$f(x) = \frac{1}{4x^3-\sqrt{x}},$$

we identify $g(x)$ as $g(x) = 4x^3 - \sqrt{x}$.

Since we need $g'(x)$, we differentiate it, using the sum and multiple constant rules:

$$g'(x) = 12x^2 - \frac{1}{2\sqrt{x}}.$$

Therefore,

$$\frac{d}{dx}\frac{1}{4x^3-\sqrt{x}} = -\frac{12x^2-\dfrac{1}{2\sqrt{x}}}{\left(4x^3-\sqrt{x}\right)^2}.$$

The slope of the tangent line at the point $(1, f(1)) = \left(1, \dfrac{1}{3}\right)$ is

$$f'(1) = -\frac{12-\dfrac{1}{2}}{(3)^2} = -\frac{23}{18},$$

and the equation of the tangent line is

$$y - \frac{1}{3} = -\frac{23}{18}(x-1) \quad \text{or} \quad y = -\frac{23}{18}x + \frac{29}{18},$$

Example 4.30.

$$\begin{aligned}\frac{d}{dx}\frac{30}{4x^4-2x+6} &= \frac{30}{2}\frac{d}{dx}\frac{1}{2x^4-x+3} && \text{multiple constant rule}\\ &= (-15)\frac{\frac{d}{dx}\left(2x^4-x+3\right)}{\left(2x^4-x+3\right)^2} && \text{reciprocal rule}\\ &= (-15)\frac{8x^3-1}{\left(2x^4-x+3\right)^2} && \text{power and constant rules}\\ &= -\frac{-120x^3+15}{\left(2x^4-x+3\right)^2}.\end{aligned}$$

The slope of the tangent line at the point $(0, f(0)) = (0, 5)$ is

$$f'(0) = \frac{15}{9} = \frac{5}{3},$$

and the equation of the tangent line at this point is

$$y - 5 = \frac{5x}{3} \quad \text{or} \quad y = \frac{5x}{3} + 5.$$

Unfortunately, the derivative of the product of two functions is *not* the product of the derivatives. You can see why in the proof of the product rule on page 136 of the textbook. The product rule is given below.

The Product Rule: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

Example 4.31. For the derivative of the function $g(x) = (f(x))^2 = f(x)f(x)$, we can use the product rule to determine that

$$g'(x) = f(x)f'(x) + f'(x)f(x) = 2f(x)f'(x).$$

Moreover, we can use this result to find

$$\begin{aligned}\frac{d}{dx}(f(x))^3 &= \frac{d}{dx}(f(x))^2 f(x)\\ &= 2f(x)f'(x)f(x) + (f(x))^2 f'(x) && \text{product rule (unit04-04.htm\#product-rule-desc)}\\ &= 3(f(x))^2 f'(x)\end{aligned}$$

Example 4.32. Since the derivative of the function $f(x) = 3x^2 + 5x$ is $f'(x) = 6x + 5$, we can use Example 4.31 to differentiate the function

$$g(x) = \left(3x^2+5x\right)^2 = f(x)f(x) = \left(3x^2+5x\right)\left(3x^2+5x\right).$$

Hence,

$$g'(x) = 2f(x)f'(x) = 2\left(3x^2+5x\right)(6x+5).$$

We can also go further and find the derivative of

$$h(x) = \left(3x^2+5x\right)^3 = g(x)f(x) = \left(3x^2+5x\right)^2\left(3x^2+5x\right).$$

By Example 4.31,

$$h'(x) = 3(f(x))^2 f'(x) = 3\left(3x^2+5x\right)^2(6x+5).$$

What do you observe in this example? What do you think the derivative of the function $u(x) = \left(3x^2+5x\right)^4$ is?

Observe that the product rule involves the derivatives of both functions. Using the reciprocal and product rules, we can get the rule for the quotient of two functions, as is shown below.

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{d}{dx}f(x)\left(\frac{1}{g(x)}\right) \\
&= \left(\frac{d}{dx}f(x)\right)\frac{1}{g(x)} + f(x)\frac{d}{dx}\frac{1}{g(x)} && \text{product rule (unit04-04.htm\#product-rule-desc)} \\
&= f'(x)\frac{1}{g(x)} + f(x)\left[-\frac{g'(x)}{(g(x))^2}\right] && \text{reciprocal rule} \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}$$

And so we have found the quotient rule.

The Quotient Rule: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{f^{'}(x)g(x) - f(x)g^{'}(x)}{(g(x))^2}.$

WARNING: Observe that in the quotient rule, the numerator *must* be $f'(x)\,g(x) - f(x)\,g'(x)$. It is a common mistake to change the order to $f(x)\,g'(x) - f'(x)\,g(x)$.

You must now practise so that you can apply these rules without making mistakes. When you start to use these rules, you should do as many steps as necessary, so that you eventually memorize them. See how we differentiate the following functions. We will not simplify the resulting function at this point.

Example 4.33. For the function

$$f(x) = \frac{4\sqrt{x} + 6x^4 - 4x}{5x^3 - x},$$

we must apply the quotient rule. Since we need the derivative of the numerator and denominator functions, we obtain these first. By the power rule,

$$\frac{d}{dx}\left(4\sqrt{x} + 6x^4 - 4x\right) = \frac{2}{\sqrt{x}} + 24x^3 - 4$$

and

$$\frac{d}{dx}5x^3 - x = 15x^2 - 1.$$

Now we can apply the quotient rule

$$\frac{d}{dx}\frac{4\sqrt{x} + 6x^4 - 4x}{5x^3 - x} = \frac{\left(\frac{2}{\sqrt{x}} + 24x^3 - 4\right)\left(5x^3 - x\right) - \left(4\sqrt{x} + 6x^4 - 4x\right)\left(15x^2 - 1\right)}{\left(5x^3 - x\right)^2}.$$

Example 4.34. We could use the quotient rule to differentiate the function

$$f(x) = \frac{3x^4 - \sqrt{3x} - 5}{x^2},$$

but it is easier if we simplify it first.

Hence,

$$f(x) = 3x^2 - \sqrt{3}x^{-3/2} - 5x^{-2},$$

and by the power rule,

$$f^{'}(x) = 6x + \frac{3\sqrt{3}}{2x^{5/2}} + \frac{10}{x^3}.$$

Exercises

17. Using Example 4.31, guess the derivative of the functions $g(x) = (f(x))^4$ and $h(x) = (f(x))^5$.
18. Read the proof of the Quotient Rule (unit04-04.htm#quotient-rule-desc) on page 137 of the textbook, and compare it with the one we gave above.
19. Read Examples 5, 6, 7, 8, 9, 10 and 11 on pages 136-140 of the textbook.
20. Do at least 10 of Exercises 23 to 42 on page 141.
21. Do Exercises 51 and 53 on page 141.

Answers to Exercises (appendix-a.htm)

Derivatives of the Trigonometric Functions

Prerequisites

To complete this section, you must be able to

1. use the correct notation for powers of trigonometric functions; for example, $\sin^2 x = (\sin x)^2$.
2. use the correct notation for compositions of trigonometric functions; for example, $\sin 3x + 7 = \sin(3x) + 7$, and $\sin(3x+7) \neq \sin 3x + 7$.
3. use the trigonometric addition formulas. Read the "Reference Pages" at the beginning of your textbook.

Before we find the derivative functions of the trigonometric functions, we should try to find their graphs. See how this is done for the sine function on page 145 of the textbook. We will do the same for the cosine function.

The graphs of cosine and its derivative function are shown in Figure 4.17, below.

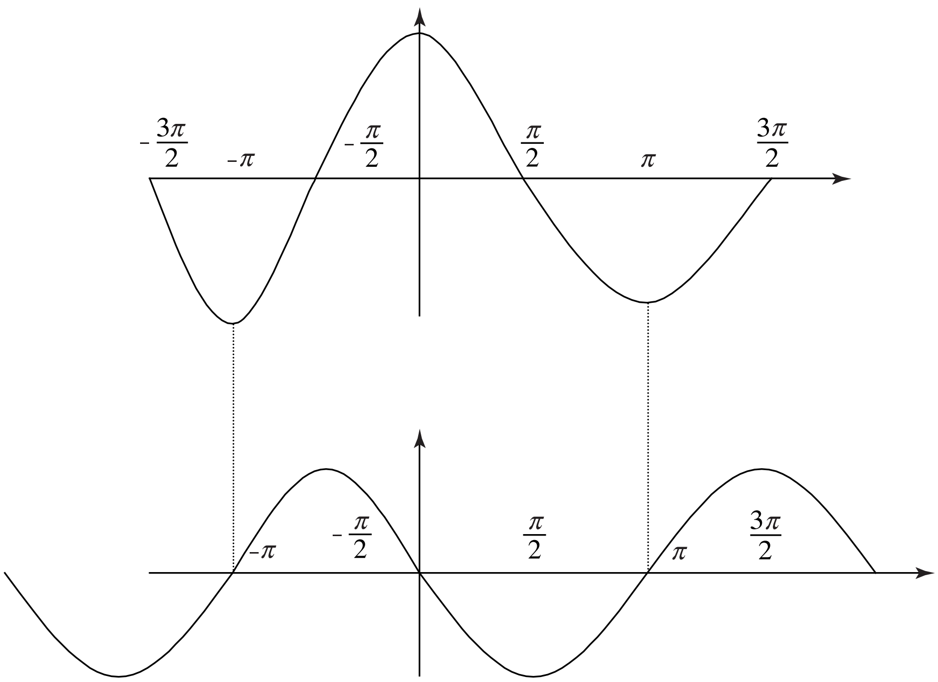


Figure 4.17. Cosine and its derivative function

We suspect from these graphs that the derivative functions of the sine and cosine are "wave" functions.

On page 145 of the textbook, you can see how Definition 4.5 (unit04-02.htm#definition4-5) is applied to find the derivative of the sine function. The derivative of the cosine function is obtained in the same manner.

$$
\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} && \text{Definition 4.5 (unit04-02.htm\#definition4-5)}\\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h} && \text{addition formula (unit03.htm\#example3-82)}\\
&= \lim_{h\to 0}\cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right) && \text{factoring}\\
&= \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} && \text{laws of limits (unit03.htm\#theorem3-23)}\\
&= \cos x(0) - \sin x(1)\\
&= -\sin x.
\end{aligned}
$$

You can see on page 148 of the textbook how the quotient rule is applied to obtain the derivative of

$$\tan x = \frac{\sin x}{\cos x}.$$

We can use the reciprocal rule to obtain the derivative of the cotangent function. If

$$\frac{d}{dx}\tan x = \sec^2 x,$$

then

$$\begin{aligned}\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{1}{\tan x}\\ &= -\frac{\sec^2 x}{(\tan x)^2} && \text{reciprocal rule (unit04-04.htm\#reciprocal-rule-desc)}\\ &= -\frac{1}{\cos^2 x}\left(\frac{\cos^2 x}{\sin^2 x}\right)\\ &= -\frac{1}{\sin^2 x}\\ &= -\csc^2 x\end{aligned}$$

Similarly, the reciprocal rule is used to obtain the derivatives of the secant and cosecant functions. All the derivative functions of the trigonometric functions are listed on page 148 of the textbook. You *must* know all of them. The effective way to memorize them is by doing many exercises.

It should be clear to you that the rules of differentiation can be applied to differentiate the sum, product and quotient of more than two functions.

Example 4.35. The function $f(x) = x^2 \sin x\left(\sqrt{x}+5\right)$ is the product of three functions. [Which ones?] To apply the product rule, we consider it as a product of two functions, $f(x) = \left(x^2 \sin x\right)\left(\sqrt{x}+5\right)$, and apply the product rule to these two functions. We first find the derivative of $g(x) = x^2 \sin x$, again using the product rule.

$$g'(x) = 2x \sin x + x^2 \cos x,$$

and

$$\frac{d}{dx}\sqrt{x}+5 = \frac{1}{2\sqrt{x}}.$$

So,

$$\begin{aligned}f'(x) &= \left[\frac{d}{dx}\left(x^2 \sin x\right)\right]\left(\sqrt{x}+5\right) + \left(x^2 \sin x\right)\frac{d}{dx}\left(\sqrt{x}+5\right)\\ &= \left(2x \sin x + x^2 \cos x\right)\left(\sqrt{x}+5\right) + \left(x^2 \sin x\right)\left(\frac{1}{2\sqrt{x}}\right)\\ &= \left(\frac{5}{2}\sqrt{x}+10\right)x \sin x + \left(\sqrt{x}+5\right)x^2 \cos x.\end{aligned}$$

Example 4.36. When we differentiate, we should simplify the function (if possible) before applying the rules of differentiation. To differentiate the function $f(x) = \cot x \sec^2 x$ we simplify, obtaining

$$f(x) = \frac{\cos x}{\sin x}\,\frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x}.$$

We find the derivative of $g(x) = \sin x \cos x$ using the product rule.

$$\begin{aligned}\frac{d}{dx}\sin x \cos x &= \left(\frac{d}{dx}\sin x\right)\cos x + \sin x\left(\frac{d}{dx}\cos x\right) && \text{product rule (unit04-04.htm\#product-rule-desc)}\\ &= \cos x(\cos x) + (\sin x)(-\sin x)\\ &= \cos^2 x - \sin^2 x.\end{aligned}$$

Applying the reciprocal rule,

$$\begin{aligned}\frac{d}{dx}\frac{1}{\sin x \cos x} &= -\frac{\cos^2 x - \sin^2 x}{(\sin x \cos x)^2} && \text{reciprocal rule (unit04-04.htm\#reciprocal-rule-desc)}\\ &= -\frac{\cos^2 x}{\sin^2 x \cos^2 x} + \frac{\sin^2 x}{\sin^2 x \cos^2 x} && \text{simplifying}\\ &= -\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\\ &= -\csc^2 x + \sec^2 x.\end{aligned}$$

You may want to apply the product rule to the function $f(x) = \cot x \sec^2 x$ and compare the methods to see which procedure is easier.

Example 4.37. To differentiate the function

$$f(x) = \frac{x^4 \tan x + \cos x}{x \sin x + 3}.$$

we start by deciding *which differentiation rule we need to apply first*. In this case, we will need the quotient rule, but to apply it, we must first find the derivatives of the numerator and denominator functions. For the derivative of the functions $g(x) = x^4 \tan x + \cos x$ and $h(x) = x \sin x + 3$, we apply the product and sum rules

$$\begin{aligned} g'(x) &= \left(\frac{d}{dx} x^4\right)\tan x + x^4\left(\frac{d}{dx}\tan x\right) + \frac{d}{dx}\cos x && \text{sum (unit04-04.htm\#sum-rule-desc) and product (unit04-04.htm\#product-rule-desc) rules} \\ &= 4x^3 \tan x + x^4 \sec^2 x - \sin x && \text{power (unit04-04.htm\#power-rule-desc) rule} \\ &= x^3(4\tan x + x\sec^2 x) - \sin x \end{aligned}$$

and

$$h^{'}(x) = \left(\frac{d}{dx}x\right)(\sin x) + x\left(\frac{d}{dx}\sin x\right) = \sin x + x\cos x.$$

Now, we can apply the quotient rule (unit04-04.htm#quotient-rule-desc):

$$\begin{aligned} &\frac{d}{dx}\frac{x^4\tan x + \cos x}{x\sin x + 3} \\ &= \frac{\left(\frac{d}{dx}x^4\tan x + \cos x)(x\sin x + 3\right) - (x^4\tan x + \cos x)\frac{d}{dx}(x\sin x + 3)}{(x\sin x + 3)^2} \\ &= \frac{(x^3(4\tan x + x\sec^2 x) - \sin x)(x\sin x + 3) - (x^4\tan x + \cos x)(\sin x + x\cos x)}{(x\sin x + 3)^2}. \end{aligned}$$

Observe that by applying the quotient rule step by step, we reduce the possibility of mistakes.

Example 4.38. For the function $f(x) = \sin^2 x = (\sin x)(\sin x)$, we use the product rule.

As in Example 4.31 (unit04-04.htm#example4-31), we have

$$f^{'}(x) = 2f(x)f^{'}(x) = 2(\sin x)(\cos x).$$

We use this information to find the derivative of $g(x) = \sin^3 x = f(x)(\sin x)$:

$$\begin{aligned} g'(x) &= f(x)\frac{d}{dx}\sin x + f'(x)(\sin x) \\ &= \left(\sin^2 x\right)(\cos x) + (2\sin x\cos x)(\sin x) \\ &= 3\sin^2 x\cos x \end{aligned}$$

Exercises

22. Do the odd-numbered exercises from 1 to 15 on page 150 of the textbook.
23. Use the reciprocal rule (unit04-04.htm#reciprocal-rule-desc) to find the derivative of the secant and cosecant functions.
24. Use the product rule (unit04-04.htm#product-rule-desc) and Example 4.38 to find the derivative of the function $h(x) = \sin^4 x$. Make a guess at the derivative of the function $u(x) = \sin^5 x$.

Answers to Exercises (appendix-a.htm)

The Chain Rule

Prerequisites

To complete this section, you must be able to recognize the composition of functions. Read the section titled "Composition of Functions" on page 35 of the textbook. Do Exercises 45-50 on page 39.

The chain rule is applied in the differentiation of the composition of two or more functions. We will start by identifying the composition of a power function $f(u) = u^r$ and a function $u = g(x)$. That is,

$$f \circ g(x) = f(g(x)) = (g(x))^r.$$

It is important that you recognize a function of this form as a composition of these two functions.

Example 4.39. The function $h(x) = \sqrt{4x - 6}$ is the composition of

$$f(u) = u^{1/2} \quad \text{and} \quad u = g(x) = 4x - 6.$$

To see this, note that

$$h(x) = f \circ g(x) = f(g(x)) = (4x - 6)^{1/2}.$$

Example 4.40. The function $h(x) = \tan^5(3x)$ is the composition of

$$f(u) = u^5 \quad \text{and} \quad u = g(x) = \tan(3x),$$

since

$$h(x) = f(g(x)) = (\tan(3x))^5.$$

The composition of a trigonometric function (as outside function) and a function $g(x)$ (as an inside function) has the form $\mathrm{TrigFctn}(g(x))$.

Example 4.41. The function $\cos\left(x^2 - 5\right)$ is the composition of $\cos t$ and $g(x) = x^2 - 5$; that is, $\cos(g(x))$.

Example 4.42. The function $\sec(\cot x)$ is the composition of $\sec t$ and $g(x) = \cot x$, that is, $\sec(g(x))$.

Example 4.43. The function

$$R(x) = \frac{1}{\sec^2\left(\sqrt{x}\right)}$$

has the form

$$R(x) = \left(\sec\left(\sqrt{x}\right)\right)^{-2};$$

hence, it is the composition of the power function $f(u) = u^{-2}$ and the function $u = g_1(x) = \sec\left(\sqrt{x}\right)$.

At the same time, the function $g_1(x)$ is the composition of the functions $\sec t$ and $g_2(x) = \sqrt{x}$.

Hence,

$$R(x) = f(\sec(g_2(x))) = [\sec(g(x))]^{-2}.$$

Our problem now is to find the derivative of such functions. The rule that we apply is the chain rule.

The Chain Rule: $\quad \dfrac{d}{dx} f \circ g(x) = f'(g(x))g'(x)$

On page 152 of the textbook, the author makes an attempt to explain why the chain rule works. To apply this rule, we must understand it. The term $f'(g(x))$ is the composition of the derivative function f' (outside function) and the function $g(x)$ (inside function); that is $f'(g(x)) = f' \circ g(x)$. So, we must obtain the derivative of the outside function, and then make a composition of this derivative function (as an outside function) with the function $g(x)$.

Example 4.44. The function $h(x) = \sqrt{4x - 6}$ (see Example 4.39, above) is the composition of $f(u) = u^{1/2}$ and $g(x) = 4x - 6$.

We have

$$f'(u) = \frac{1}{2\sqrt{u}},$$

so

$$f'(g(x)) = \frac{1}{2\sqrt{g(x)}} = \frac{1}{2\sqrt{4x-6}},$$

and since $g'(x) = 4$, the derivative is

$$h'(x) = \left(\frac{1}{2\sqrt{4x-6}}\right)(4) = \frac{2}{\sqrt{4x-6}}.$$

Example 4.45. The function $h(x) = \sin^3 x$ is the composition of $f(u) = u^3$ and $g(x) = \sin x$.

Since $f'(u) = 3u^2$, then $f'(g(x)) = 3(g(x))^2 = 3\sin^2 x$, and $g'(x) = \cos x$. So, $h'(x) = 3\sin^2 x \cos x$. See Example 4.38 (unit04-05.htm#example4-38).

A function of the form $h(x) = (g(x))^r$ is the composition of $f(u) = u^r$ and $u = g(x)$. Since $f'(u) = ru^{r-1}$, we have $f'(g(x)) = r(g(x))^{r-1}$, and then

$$\frac{d}{dx}(g(x))^r = r(g(x))^{r-1}g'(x).$$

This is known as the *general power rule*.

The General Power Rule: $\quad \dfrac{d}{dx}(g(x))^r = r(g(x))^{r-1}g'(x).$

WARNING: When applying the general power rule, do not forget to multiply $r(g(x))^{r-1}$ by the derivative $g'(x)$. Observe that, to apply the general power rule, we must identify the function $g(x)$ and the power r.

Example 4.46. The function $f(x) = \sqrt{\cos^3 x + 5}$ is equal to $f(x) = \left(\cos^3 x + 5\right)^{1/2}$. Hence, the power is $r = 1/2$ and $g(x) = \cos^3 x + 5$.

We will find the derivative of g first. Alas, we must use the general power rule, since $g(x) = (\cos x)^3 + 5$.

Hence, $g'(x) = -3(\cos x)^2 \sin x$. [Agree?]

Then,

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\cos^3 x + 5\right)^{-1/2}\left(-3(\cos x)^2 \sin x\right) \\ &= -\frac{3}{2}\frac{\cos^2 x \sin x}{\sqrt{\cos^3 x + 5}}. \end{aligned}$$

Example 4.47. For the function $R(x) = \left(\sec\left(\sqrt{x}\right)\right)^{-2}$ in Example 4.42, we have

$$\frac{d}{dt}\sec t = \sec t \tan t \quad \text{and} \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}},$$

and by the chain rule, we conclude that

$$\frac{d}{dx}\sec\left(\sqrt{x}\right) = \sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right).$$

Hence, by the general power rule,

$$\begin{aligned} R'(x) &= -2(\sec(\sqrt{x})^{-3}\left[\sec(\sqrt{x})\tan(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)\right] \\ &= \frac{-\tan(\sqrt{x})}{\sec^2(\sqrt{x})\sqrt{x}}. \end{aligned}$$

Some particular cases of the general power rule are worth keeping in mind, because we make frequent use of them in this course.

$$\text{If } f(x) = \frac{1}{g(x)}, \quad \text{then } r = -1 \text{ and}$$

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{d}{dx}(g(x))^{-1} = -\frac{g'(x)}{(g(x))^2}. \tag{4.1}$$

$$\text{If } f(x) = \sqrt{g(x)}, \quad \text{then } r = \frac{1}{2} \text{ and}$$

$$\frac{d}{dx}\sqrt{g(x)} = \frac{d}{dx}(g(x))^{1/2} = \frac{g'(x)}{2\sqrt{g(x)}}. \tag{4.2}$$

$$\text{If } f(x) = \frac{1}{\sqrt{g(x)}}, \quad \text{then } r = -\frac{1}{2} \text{ and}$$

$$\frac{d}{dx}\frac{1}{\sqrt{g(x)}} = \frac{d}{dx}(g(x))^{-1/2} = -\frac{g'(x)}{2(g(x))^{3/2}}. \tag{4.3}$$

Example 4.48. In the function $f(x) = \dfrac{\sqrt{3}}{x^4 \tan x}$, the derivative of $g(x) = x^4 \tan x$ is obtained by applying the product rule

$$g'(x) = x^4(\sec^2 x) + (4x^3)(\tan x) = x^3(x \sec^2 x + 4\tan x)$$

then by Equation (4.1), above,

$$f'(x) = -\frac{\sqrt{3}\,x^3(x\sec^2 x + 4\tan x)}{(x^4 \tan x)^2} = -\frac{\sqrt{3}(x\sec^2 x + 4\tan x)}{x^5 \tan^2 x}.$$

Example 4.49. For the function $h(x) = \dfrac{1}{\sqrt{\cot^3 x - 6x}}$, we have, by the general power rule

$$\frac{d}{dx}\cot^3 x - 6x = -3(\cot^2 x \csc^2 x + 2).$$

Hence, by Equation (4.3), above,

$$h'(x) = \frac{3(\cot^2 x \csc^2 x + 2)}{2(\cot^3 x - 6x)^{3/2}}.$$

Exercises

25. Do Exercises 1, 3 and 5, on page 157 of the textbook.
26. Do at least 21 exercises from 7 to 42 on pages 157-158.

Answers to Exercises (appendix-a.htm)

The Leibniz notation of the derivative plays an important role in the chain rule. If we set $u = g(x)$, then $f \circ g(x) = f(g(x)) = f(u)$. In Leibniz notation

$$\frac{df}{du} = f'(u) = f'(g(x)) \quad \text{and} \quad \frac{du}{dx} = g'(x).$$

Therefore, the chain rule in Leibnitz notation is

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$$

Observe in the next examples how effective this notation is.

Example 4.50. For the function in Example 4.49, the function $h(x)$ is the composition of $f(u) = u^{-1/2}$ and $u = g(x) = \cot^3 x - 6x$; that is $h(x) = f(u) = f \circ g(x) = \left(\cot^3 x - 6x\right)^{-1/2}$.

In Leibnitz notation,

$$\begin{aligned} h'(x) &= \frac{df}{du}\frac{du}{dx} = \frac{-1}{2}u^{-3/2}\left(-3\cot^2 x \csc^2 x - 6\right) \\ &= \frac{3\cot^2 x \csc^2 x + 6}{2u^{3/2}}. \end{aligned}$$

Substituting u, we have $h'(x) = \dfrac{3\cot^2 x \csc^2 x + 6}{2\left(\cot^3 x - 6x\right)^{3/2}}$.

Example 4.51. For the function $R(x) = \left(\sec\left(\sqrt{x}\right)\right)^{-2}$ in Example 4.47, the outside and inside functions ($f_1(u) = u^{-2}$ and $u = g_1(x) = \sec\left(\sqrt{x}\right)$, respectively) give the function $R(x) = f_1(u) = f_1 \circ g_1(x)$.

At the same time, $g_1(x)$ is the composition of $f_2(t) = \sec t$ and $t = g_2(x) = \sqrt{x}$; that is, $g_1(x) = f_2(t) = f_2 \circ g_2(x)$. Hence, $R(x) = f_1 \circ (f_2 \circ g_2(x))$.

The derivative in Leibnitz notation is

$$R'(x) = \frac{df_1}{du}\frac{du}{dt}\frac{dt}{dx} = -2(u)^{-3}(\sec t \tan t)\frac{1}{2\sqrt{x}} = \frac{-\sec t \tan t}{u^3\sqrt{x}}.$$

Substituting u and t, we conclude that

$$R'(x) = \frac{-\sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)}{\sec^3\left(\sqrt{x}\right)\sqrt{x}} = -\frac{\tan\left(\sqrt{x}\right)}{\sec^2\left(\sqrt{x}\right)\sqrt{x}}.$$

For help with the concept of differentiation, view the PowerPoint tutorials below. To access a tutorial:

1. Click on the file to open it.
2. Extract the files and save them to your computer’s hard drive.
3. Click on the file folder to open it.
4. Click on the PowerPoint file to view and listen.

Note: If you don’t have PowerPoint installed on your computer, you can download the PowerPoint Viewer from here: https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6 (https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6).

Tutorial 1: Applying Differentiation Rules (../multimedia/powerpoint/diff1.zip)

Tutorial 2: Differentiation (../multimedia/powerpoint/diff2.zip)

Tutorial 3: Differentiation (../multimedia/powerpoint/diff3.zip)

Exercises

27. Read Examples 1-8 on pages 153-156 of the textbook.
28. Do part (a) of Exercises 47, 49 and 53 on page 158.

Answers to Exercises (appendix-a.htm)

Applications of the Derivative

Prerequisites

To complete this section, you must be able to convert degrees to radians and *vice versa*. See page 358 of the textbook. Do Exercises 1-12 on page 366.

In this unit, we apply the derivative to estimate values of quantities and errors in measurement. Then, we see how the derivative is used in solving problems in the natural and social sciences.

The capital Greek letter delta, Δ, together with a variable, is used to denote a small quantity of the indicated variable. So, Δx indicates a small quantity of the variable x (positive or negative); similarly, Δy is a small quantity of the variable y.

When we write $a + \Delta x$, we indicate the number a plus a small quantity of the variable x, and this quantity is on the x-axis. The quantity $b + \Delta y$ is on the y-axis. The quantity $a + \Delta x$ is close to a, since Δx is small. It is close to a from the right if $\Delta x > 0$, and from the left if Δx is negative.

We have pointed out that the tangent line is the line closest to the curve, and that we use it to approximate values of the function at a point close to $f(a)$. We will go over this concept again, using the delta notation.

First, observe that since $a + \Delta x$ is a quantity, we can evaluate a function at this number; hence, we find $f(a + \Delta x)$ by replacing x by $a + \Delta x$ in the function $f(x)$.

For instance, if $f(x) = 4x^2 + \sqrt{x}$, then

$$\begin{aligned} f(a + \Delta x) &= 4(a + \Delta x)^2 + \sqrt{a + \Delta x} \\ &= 4\left(a^2 + 2a\Delta x + (\Delta x)^2\right) + \sqrt{a + \Delta x}. \end{aligned}$$

Remember that Δx is a number.

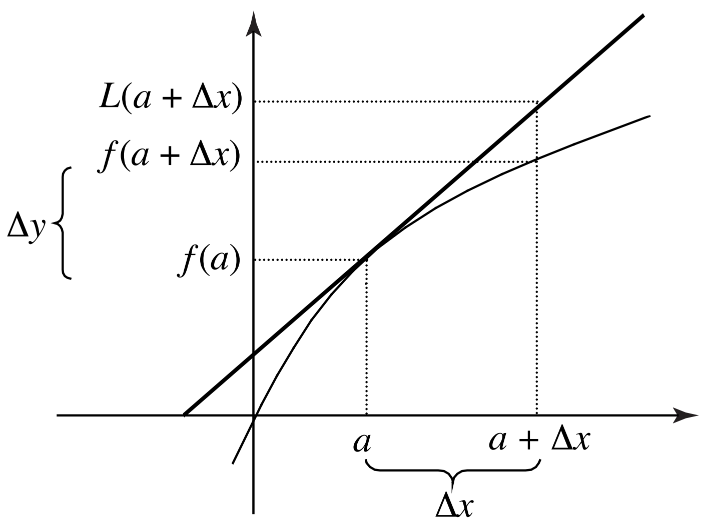


Figure 4.18. $a + \Delta x$ close to a from the right

In Figure 4.18, above, we have $a + \Delta x$ close to a from the right; that is, $\Delta x > 0$.

We see that $f(a + \Delta x) - f(a)$ is a small quantity on the y-axis; hence, we write

$$\Delta y = f(a + \Delta x) - f(a).$$

The equation of the tangent line at the point $(a, f(a))$ is

$$L(x) = f'(a)(x - a) + f(a),$$

as we indicated in Proposition 4.6 (unit04-03.htm#proposition4-6).

Then $L(a+\Delta x) = f^{'}(a)((a+\Delta x)-a)+f(a)$, and

$$L(a+\Delta x) = f^{'}(a)(\Delta x)+f(a)$$

and

$$f(a+\Delta x) \approx L(a+\Delta x) = f^{'}(a)(\Delta x)+f(a). \tag{4.4}$$

Hence,

$$\Delta y = f(a+\Delta x)-f(a) \approx L(a+\Delta x)-f(a) = f^{'}(a)(\Delta x). \tag{4.5}$$

From Figure 4.18, we see that these two approximations are indeed true. There are three important consequences of Equations (4.4) and (4.5).

A. From (4.4), we obtain the *linear approximation* of $f(a+\Delta x)$; that is

$$f(a+\Delta x) \approx f^{'}(a)(\Delta x)+f(a).$$

We used this approximation in Examples 4.20 (unit04-03.htm#example4-20) and 4.21 (unit04-03.htm#example4-21). The equation of the tangent line $y = f^{'}(a)(\Delta x)+f(a)$ is called the *linearization* of f at a.

B. We also find that

$$\Delta y = f(a+\Delta x)-f(a) \approx f^{'}(a)(\Delta x).$$

C. From (4.5), we have $\dfrac{\Delta y}{\Delta x} \approx f^{'}(a)$. Then

$$f^{'}(a) \approx \frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x)-f(a)}{\Delta x}.$$

The next examples show the applications of A, B and C, respectively.

When we know the exact value of $f(a)$, but not of $f(a+\Delta x)$, we estimate the latter value using linear approximation.

Example 4.52. We have

$$\cos\ (30^{\circ}) = \cos\ \left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

but we cannot find an exact value for

$$\cos\ (32^{\circ}) = \cos\ \left(\frac{8\pi}{45}\right).$$

So we use linear approximation to estimate $\cos\ (32^{\circ})$.

Since

$$\frac{8\pi}{45} = \frac{\pi}{6}+\frac{\pi}{90},$$

we can say that

$$a = \frac{\pi}{6} \quad \text{and} \quad \Delta x = \frac{\pi}{90}.$$

From consequence A, above, we have

$$\cos\ \left(\frac{8\pi}{45}\right) = \cos\ \left(\frac{\pi}{6}+\frac{\pi}{90}\right) \approx \left[\frac{d}{dx}\cos\ x\bigg|_{x=\pi/6}\right]\left(\frac{\pi}{90}\right)+\cos\ \left(\frac{\pi}{6}\right).$$

Finding the derivative gives us

$$\frac{d}{dx}\cos x\Big|_{x=\pi/6} = -\sin x\Big|_{x=\pi/6} = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}.$$

Hence,

$$\cos(32^\circ) = \cos\left(\frac{8\pi}{45}\right) \approx -\frac{1}{2}\left(\frac{\pi}{90}\right) + \frac{\sqrt{3}}{2} = -\frac{\pi}{180} + \frac{\sqrt{3}}{2} = \frac{90\sqrt{3}-\pi}{180}.$$

This example is shown graphically in Figure 4.19, below.

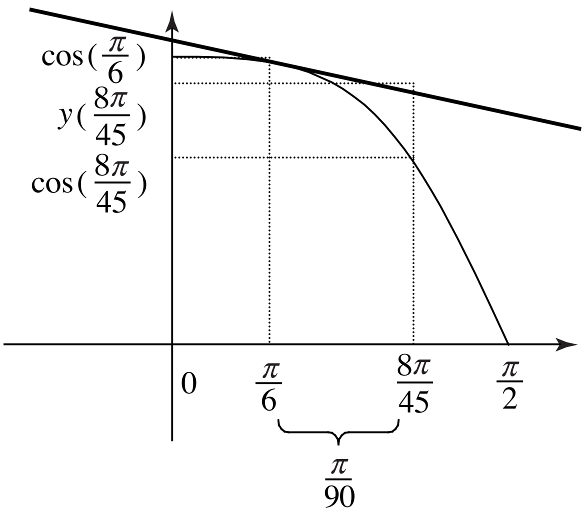


Figure 4.19. Graph showing the tangent line at $8\pi/45$ compared with $\cos 32°$

Since $y\left(\frac{8\pi}{45}\right)$ is above the graph of $\cos x$, we can see that our approximation is an overestimate of the actual value of $\cos(32^\circ)$. That is

$$y\left(\frac{8\pi}{45}\right) = \frac{90\sqrt{3}-\pi}{180} > \cos(32^\circ).$$

Example 4.53. To estimate the value of $(0.89)^{4/3}$, we first recognize that we are looking for $f(0.89)$, which is equal to $(0.89)^{4/3}$. So the function is $f(x) = x^{4/3}$.

Next, we look for a value a close to 0.89 such that $f(a)$ is known. We see that $0.89 = 1 - 0.11$, and we know that $f(1) = 1$. So we let $a = 1$ and $\Delta x = -0.11$. Then from conclusion A, above, we recognize that

$$(0.89)^{4/3} = f(0.89) \approx f'(1)(\Delta x) + f(1).$$

Since $f'(x) = \left(\frac{4}{3}\right)\left(x^{1/3}\right)$, we find that $f'(1) = \frac{4}{3}$.

Hence,

$$(0.89)^{4/3} \approx \left(\frac{4}{3}\right)(-0.11) + 1 = \frac{64}{75}.$$

Exercises

29. Read Examples 2 and 3 on pages 162-163 of the textbook
30. Do Exercises 5, 7 and 8 on page 166.
31. Do Exercises 31, 33, 35 and 36 on page 167.

Answers to Exercises (appendix-a.htm)

The application of conclusion B, above, is used in estimating errors in measurements. When an error in measurement occurs, it affects further calculations, a process called *error propagation*.

Definition 4.7. If a is the exact value of the quantity being measured and f is differentiable at a, then

a. The *measurement error* of x is $\Delta x = x - a$ and $x = a + \Delta x$.
b. The difference $\Delta y = f(x) - f(a) = f(a + \Delta x) - f(a)$ is the *propagated error* of y.
c. The quotient $\dfrac{\Delta y}{f(a)}$ is the *relative error* in measurement.
d. The percentage value of the relative error is the *percentage error* of measurement; that is, $\dfrac{\Delta y}{f(a)}(100)$.

From conclusion B, we know that the propagated error Δy is approximately equal to $f^{'}(a)(\Delta x)$. This expression has a special name.

Definition 4.8. For a function f differentiable at a, the product $f^{'}(a)\Delta x$ is called the *differential* of f at a, and we write

$$df = dy = f^{'}(a)\Delta x.$$

Hence, by conclusion B,

$$\Delta y \approx dy; \quad \text{that is,} \quad f(a + \Delta x) - f(a) \approx f^{'}(a)\Delta x.$$

The relative error can be estimated using differentials:

$$\frac{\Delta y}{f(a)} \approx \frac{dy}{f(a)}.$$

It is always easier to calculate the differential dy than the difference Δy.

Example 4.54. A metal rod 20 cm long and 6 cm in diameter is to be covered (except for the ends) with insulation that is 0.2 cm thick. We want to estimate the amount of insulation needed. The exact amount of insulation is equal to the difference of the volume of the rod plus the insulation (radius 3.2 cm) and the volume of the rod without the insulation (radius 3 cm).

Let V be the volume of the rod, and let r be its radius. So, $V = 20\pi r^2$, and the amount of insulation is

$$V(3.2) - V(3) = \Delta V = 20\pi(3.2)^2 - 20\pi 3^2 = 20\pi\left((3.2)^2 - 3^2\right).$$

We could evaluate this amount, or use differentials to estimate its value, as follows:

$$\Delta V = V(3.2) - V(3) \approx dV = V^{'}(3)(0.2). \quad \text{[Why?]}$$

We determine that $V^{'}(r) = 40\pi r$, then

$$\Delta V \approx 40\pi(3)(0.2) = 24\pi \quad \text{cm}^3.$$

You can check that $\Delta V = \dfrac{124\pi}{5}$, and $\dfrac{124\pi}{5} \approx 24\pi$.

Note the advantage that it was easier to calculate dV than ΔV.

The relative error in this case is approximately equal to

$$\frac{dV}{V(3)} = \frac{24\pi}{180\pi} = \frac{2}{15},$$

and the percentage error is estimated at 13.33%.

Example 4.55. The electrical resistance R of a wire is given by

$$R(r) = \frac{k}{r^2},$$

where k is a constant and r is the radius of the wire. If the percentage error of the radius must be $\pm 4\%$, what is the percentage error of R?

The relative error of $\frac{\Delta R}{R}$ is approximately equal to $\frac{dR}{R}$. Hence,

$$\frac{\Delta R}{R} \approx \frac{dR}{R} = \frac{R'(r)(\Delta R)}{R} = \frac{(-2kr^{-3})(\Delta r)}{kr^{-2}} = (-2)\frac{\Delta r}{r}.$$

Since the relative error $\frac{\Delta r}{r}$ must be between -0.04 and 0.04, we have

$$-0.04 \leq \frac{\Delta r}{r} \leq 0.04.$$

Multiplying by -2 yields

$$-0.04(-2) = 0.08 \geq (-2)\frac{\Delta r}{r} \geq 0.04(-2) = -0.08.$$

We find that the relative error of R is approximately $\pm 8\%$.

Exercises

32. Read Example 5 on page 165 of the textbook.
33. Do Exercises 21-26, 27, 29, 39 and 41 on page 167.

Answers to Exercises (appendix-a.htm)

Going back to the interpretation of the derivative as the instantaneous rate of change, we know, by conclusion C, that

$$f'(a) \approx \frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

Observe that the quotient

$$\frac{f(a+\Delta x) - f(a)}{\Delta x}$$

is the average rate of change. So, the instantaneous rate of change (derivative) is its limit:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

We look into the average rate of change first, and then we find the instantaneous rate of change. Note that in applications in the natural and social sciences—physics, chemistry, biology and economics—the concepts of velocity, linear density, rate of reaction, compressibility, rate of growth, marginal cost and marginal revenue are all instantaneous rates of change.

Example 4.56. The linear density ρ of a nonhomogeneous rod is the limit of the average density of the rod. If the mass measured from its left end point is x, then the mass is $m = f(x)$, and the average density is defined as mass over the length. Hence, the average density is

$$\frac{\Delta m}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2},$$

and the linear density is

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = f'(x).$$

If the mass is $f(x) = 3x^2$ kg, then the linear density is $f'(x) = 6x$ kg/m. The linear density at $x = 1$ m is $f'(1) = 6$ kg/m; at $x = 2$ m, it is $f'(2) = 12$ kg/m; and at $x = 3$ m, it is $f'(3) = 18$ kg/m. Thus, the linear density increases, and it will be at its highest at the end of the rod. See Exercise 17 on page 178 of the textbook.

Example 4.57. See Exercise 28 on page 180 of the textbook.

Let f be the frequency of vibrations of a vibrating violin string

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is the tension, and ρ is the linear density.

a. The rate of change of the frequency with respect to
   i. the length (T and ρ are constant) is
   $$f'(L) = \frac{df}{dL} = \left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)\left(\frac{d}{dL}\frac{1}{L}\right) = -\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}.$$
   ii. the tension (L and ρ are constant) is
   $$f'(T) = \frac{df}{dT} = \left(\frac{1}{2L}\frac{1}{\sqrt{\rho}}\right)\left(\frac{d}{dT}\sqrt{T}\right) = \frac{1}{4L\sqrt{T\rho}}.$$
   iii. the linear density (L and T are constant) is
   $$f'(\rho) = \frac{df}{d\rho} = \left(\frac{1}{2L^2}\sqrt{T}\right)\left(\frac{d}{d\rho}\frac{1}{\sqrt{\rho}}\right) = -\frac{\sqrt{T}}{4L\rho^{3/2}}.$$

b. What happens to the pitch of the note when
   i. the effective length of the string is decreased by placing a finger on the string so that a shorter portion of the string vibrates?
   The frequency is of the form $f(L) = k\ /\ L$ for k a constant (sketch a possible graph of this function). Since the derivative $f'(L)$ is negative, the frequency decreases for any increase in L, giving us a lower note. Conversely, when L decreases, the frequency increases, so we have a higher note.

   ii. the tension is increased by turning a tuning peg?
   The frequency is of the form $f(T) = k\sqrt{T}$ for k a constant (sketch a possible graph of this function). The derivative is positive, so the frequency increases and when T increases, giving us a higher note.

   iii. the linear density is increased by switching to another string?
   The frequency is of the form $f(\rho) = k\frac{1}{\sqrt{\rho}}$ for k a constant (sketch a possible graph of this function). The derivative is negative, so the frequency decreases and when ρ increases, giving a lower note.

Exercises

34. Study Examples 1, 2 and 3 on pages 170-172 of the textbook.
35. Do Exercises 1, 3, 5, 6, 7, 9 and 19 on page 178.
36. Study Example 4 on page 172.
37. Do Exercise 25 on page 179.
38. Study Examples 6 and 8 on pages 174 and 176 of the textbook.

39. Do Exercises 13, 26, 29 and 30 on pages 178-180.

Answers to Exercises (appendix-a.htm)

Implicit Differentiation

Prerequisites

To complete this section, you must be able to

1. apply the vertical line test. See page 13 of the textbook.
2. apply the relations between the slopes of parallel lines and the slopes of perpendicular lines. See the section titled "Parallel and Perpendicular Lines," pages 348-349 of the textbook. Do Exercises 35, 36 and 58 on pages 349-350.

At this point, we want to consider curves that do not correspond to a function. By the vertical line test, such curves can be circles, ellipses or cardiods (see Exercise 27 on page 189 of the textbook). These curves are given by equations; for example the equation $(x-k)^2+(y-l)^2=r^2$ corresponds to a circle with centre at (k,l) and radius r. Such curves have tangent lines at some points, and we want to find the slopes of these tangent lines. But the slopes of tangent lines are given by the derivative, and differentiation is defined only for functions. One way to overcome this problem would be to decompose the curve into parts or pieces such that each piece corresponds to a function, and then differentiate.

Observe, on page 182 of the textbook, how by solving for y in the equation $x^2+y^2=25$, we can decompose the circle into an upper and a lower half-circle, each represented by a function:

$$f(x)=\sqrt{25-x^2} \quad \text{and} \quad g(x)=-\sqrt{25-x^2}.$$

Then,

$$x^2+(f(x))^2=25 \quad \text{and} \quad x^2+(g(x))^2=25.$$

We can differentiate these functions:

$$f'(x)=-\frac{x}{\sqrt{25-x^2}} \quad \text{and} \quad g'(x)=\frac{x}{\sqrt{25-x^2}}.$$

The slope of the tangent line at the point $(3,4)$ in the upper half-circle is

$$f'(3)=-\frac{3}{4};$$

and the tangent at the point $(3,-4)$ in the lower half-circle is

$$g'(3)=\frac{3}{4}.$$

We can see that $f'(5)$ and $f'(-5)$ are undefined; that is, the tangent lines at $(5,0)$ and $(-5,0)$ are vertical.

Figures 2 and 3 on page 183 of the textbook show the graph of the curve $x^3+y^3=6xy$, and its decomposition into different pieces, each of which corresponds to a function.

The problem here is that, *explicitly* to find the functions that correspond to each piece, we would have to solve for y in the equation $x^3+y^3=6xy$, and this is not an easy task. However, we can assume that it can be done, since we have the pieces that show that there are functions corresponding to these curves.

So we say that these functions are *implicit* in the equation; that is, we can interpret the equation to mean that $y=f(x)$ for some function $f(x)$, and $x^3+(f(x))^3=6x(f(x))$.

To differentiate with respect to x, we can apply the differentiation rules, so that

$$3x^2+3(f(x))^2f'(x)=6x\left(f'(x)\right)+6f(x)=6\left(xf'(x)+f(x)\right).$$

Since $y=f(x)$ and $y'=f'(x)$, we conclude that

$$3x^2+3y^2y'=6\left(xy'+y\right),$$

and we can solve for y':

$$3y^2y'-6xy'=6y-3x^2 \qquad \text{and} \qquad y'\left(3y^2-6x\right)=3\left(2y-x^2\right).$$

Therefore,

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

The point $(3, 3)$ is on the curve, because $3^3 + 3^3 = 6(3)(3)$, and for $x = 3$, $y = 3$, the derivative at $(3, 3)$ is

$$y' = \frac{2(3) - 3^2}{3^2 - 2(3)} = -1,$$

so the slope of the tangent line at the point $(3, 3)$ is -1.

In the same manner, you can verify that

- the point $\left(\frac{8}{3}, \frac{4}{3}\right)$ is on the curve, and
- for $x = \frac{8}{3}$ and $y = \frac{4}{3}$,

$$y' = \frac{2\left(\frac{4}{3}\right) - \left(\frac{8}{3}\right)^2}{\left(\frac{4}{3}\right)^2 - 2\left(\frac{8}{3}\right)} = \frac{5}{4}.$$

The slope of the tangent line at the point $\left(\frac{8}{3}, \frac{4}{3}\right)$ is $\frac{5}{4}$.

So, by implicitly differentiating the equation (by assuming that $y = f(x)$), we solved the problem without having to find explicitly the function that corresponds to the curve that passes through the designated point.

Example 4.58. An astroid is shown in Exercise 28 on page 187 of the textbook; its equation is $x^{2/3} + y^{2/3} = 4$.

As before, if we assume that $y = f(x)$, then the equation is

$$x^{2/3} + (f(x))^{2/3} = 4,$$

and by the differentiation rules,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}(f(x))^{-1/3}f'(x) = 0.$$

Replacing $y = f(x)$ and $y' = f'(x)$, we obtain

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0,$$

and solving for y' yields

$$y' = -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}.$$

The points $(0, 8)$ and $(8, 0)$ are on the curve. For $x = 0$ and $y = 8$, we know that y' is undefined, because there is no tangent line at this point. For $x = 8$ and $y = 0$, we know that $y' = 0$, so the tangent line is horizontal. Examine the curve and convince yourself that this is indeed the case.

Example 4.59. If we are careful to remember that $y = f(x)$, we may not need to replace y by $f(x)$ in an equation, and we may be able to differentiate without mistakes.

In the equation $x^4 + xy^4 = x \sin y$, we differentiate, obtaining

$$4x^3 + y^4 + 4xy^3y' = \sin y + xy' \cos y.$$

Solving for y' yields

$$y' = \frac{\sin\ y - 4x^3 - y^4}{4xy^3 - x\cos y}.$$

Families of similar curves are represented by equations that differ only by a constant. For example, circles with centre at the origin $(0, 0)$, and radii 1 and 2 have equations $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, respectively. In general, the family of circles with centre at the origin and different radii r is represented by the equation $x^2 + y^2 = r^2$ for any positive real number r.

Definition 4.9. Two curves are *orthogonal* if at each point of intersection their tangent lines are perpendicular.

Two families of curves have *orthogonal trajectories* if each curve in one family is orthogonal to every curve in the other family.

Example 4.60. To show that the curves represented by the equations $3x^2 + 2y^2 = 5$ and $y^3 = x^2$ are orthogonal, we must first find their points of intersection by solving the two equations simultaneously.

Substituting $x^2 = y^3$ into the other equation, we obtain $3y^3 + 2y^2 = 5$; hence, $y^2(3y + 2) = 5$. Since $5 = 3 + 2$, we find that $y = 1$ is a solution of this equation (there might be others, but we do not need them at this time).

So, if $y = 1$ then $x^2 = y^3 = 1$; and therefore, $x = 1$ and $x = -1$. The points of intersection are $(1, 1)$ and $(-1, 1)$.

Now we have to see whether the slopes of the tangent lines at these points are perpendicular. Implicitly differentiating both equations and solving for y':

$$6x + 4yy' = 0 \quad \text{and} \quad y' = -\frac{6x}{4y} = -\frac{3x}{2y}. \tag{1}$$

$$3y^2y' = 2x \quad \text{and} \quad y' = \frac{2x}{3y^2}. \tag{2}$$

For $x = 1$ and $y = 1$, the slope of the tangent line of the curve in (1) is $y' = -3 / 2$, and the slope of the tangent line of the curve in (2) is $y' = 2 / 3$. Therefore, the tangent lines are perpendicular at $(1, 1)$.

For $x = -1$ and $y = 1$, we have $y' = 3 / 2$ in (1) and $y' = -2 / 3$ in (2).

Therefore, the tangent lines are also perpendicular at $(-1, 1)$, and the curves are orthogonal.

Example 4.61. To see if the families of parabolas $x = ay^2$ and ellipses $2x^2 + y^2 = 2b$ have orthogonal trajectories, we need to see if the slopes of the tangent lines are perpendicular. Implicitly differentiating both equations and solving for y' yields

$$1 = 2ayy' \quad \text{and} \quad y' = \frac{1}{2ay}. \tag{1}$$

$$4x + 2yy' = 0 \quad \text{and} \quad y' = -\frac{2x}{y}. \tag{2}$$

Now observe that from the equation $x = ay^2 = (ay)y$, we have $ay = x / y$; hence, in (1), we have $y' = y / (2x)$.

The slopes of the tangent lines of the curves are $y' = y / (2x)$ and $y' = -2x / y$; hence, the tangent lines are perpendicular, and the curves have orthogonal trajectories.

Exercises

40. Do the odd-numbered exercises from 5 to 19 on pages 186-187 of the textbook.
41. Read Examples 1 and 2 on pages 183-184.
42. Do Exercises 25 and 27 on page 187.
43. Use Definition 4.9, above, to do Exercise 41 on page 187.
44. Read Example 4 on page 186 of the textbook.
45. Do Exercises 45 and 47 on page 188.

Answers to Exercises (appendix-a.htm)

Related Rates

Prerequisites

To complete this section, you must be able to

1. state Pythagoras' Theorem.
2. determine whether two triangles are similar.
3. state the laws of sines and cosines.
4. apply the geometric formulas of areas, volumes and surface areas of regular solids.

Related rate problems deal with quantities that change with respect to time and are related to each other. We will be looking for the rate of change of one quantity in terms of the related rates of the others. The aim is to find the equation that relates these quantities, and to use implicit differentiation with respect to time to find the rate we are looking for.

A possible strategy for approaching such problems is given on page 199 of the textbook. We repeat it here, with additional notes.

1. *Read the problem carefully*. Identify the quantities that change with respect to time and the quantities that are constant. You will be able to do this only if you understand the problem. Pay attention to the units.
2. *Draw a diagram if possible*. It is not always possible to draw a diagram, but try to draw one whenever possible to help you visualize and understand the problem.
3. *Introduce notation*. Assign symbols to all quantities that are functions of time, and indicate the constants in the diagram. Do not include in your diagram the known rates or the rate to be found under the conditions given.
4. *Express the given information and the required rate in terms of the derivatives*. Identify the known rates and the ones you are looking for.
5. *Write an equation that relates the various quantities of the problem*. If necessary, use the geometry of the situation to eliminate one of the variables by substitution. Use the quantities of your diagram to find the corresponding equation.
6. *Use the chain rule to differentiate both sides of the equation with respect to time*. It is important that you identify the quantities that change with respect to time, because you must differentiate the equation implicitly with respect to t.
7. *Substitute the given information into the resulting equation, and solve for the rate you are looking for*.

Example 4.62. A ladder 13 feet long is leaning against the side of a building. If the foot of the ladder is pulled away from the building at a constant rate of 0.2 feet per second, how fast is the angle formed by the ladder and the ground changing at the instant when the top of the ladder is 12 feet above the ground?

We identify the constants and variables that change with respect to time.

x: distance of the ladder to the wall (variable)

y: distance from the top of the ladder to the ground (variable)

θ: angle formed by the ladder and the ground (variable)

13 ft: the length of the ladder (constant)

A picture for this problem is given in Figure 4.20, below.

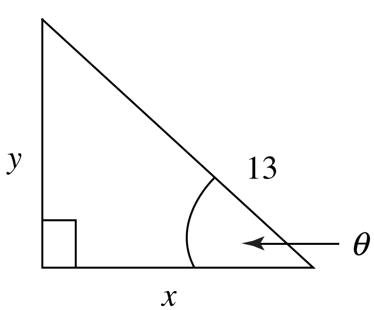


Figure 4.20. Diagram for Example 4.62

The known rate is $\dfrac{dx}{dt} = 0.2$ ft/s.

The rate we want to know is $\dfrac{d\theta}{dt}$, when $y = 12$ ft.

We must relate the variables θ and x and the constant 13. From the diagram we have $\cos\theta = x/13$. This is the equation that we differentiate with respect to time:

$$(-\sin\theta)\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}.$$

Solving for the rate we are looking for,

$$\frac{d\theta}{dt} = -\frac{\csc\theta}{13}\frac{dx}{dt}.$$

We have $dx/dt = 0.2$, and we need to know $\csc\theta$ when $y = 12$.

We refer to the Figure 4.20, above, and we see, by Pythagoras' Theorem, that $x = 5$. Therefore, $\csc\theta = 13/12$.

Hence,

$$\frac{d\theta}{dt} = -\frac{13}{12}\frac{1}{13}(0.2) = -\frac{1}{60}.$$

The angle is decreasing at a rate of $1\ /\ 60$ rad/sec.

Example 4.63. A water trough with vertical cross section in the shape of an equilateral triangle is being filled at a rate of 4 cubic feet per minute. Given that the trough is 12 feet long, how fast is the level of the water rising at the instant the water reaches a depth of 1.5 feet?

The constants and variables are given below.

length of the trough: 12 ft,

volume of water: V, and

depth of water: h.

We have $\dfrac{dV}{dt} = 4\ \text{ft}^3/\text{min}$, and we want to know $\dfrac{dh}{dt}$ when $h = 1.5$ ft.

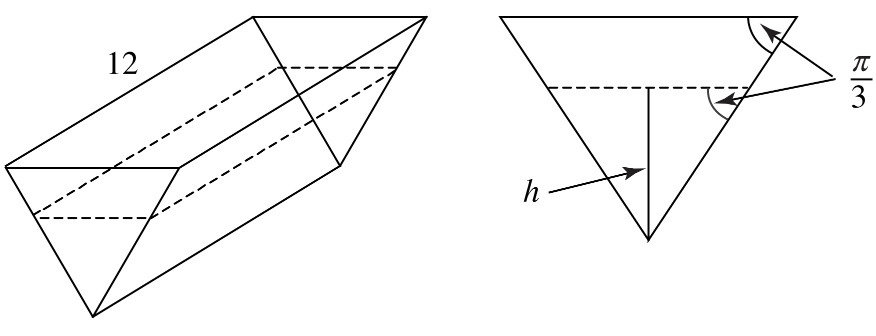


Figure 4.21. Diagram for Example 4.63

Figure 4.21, above, shows the trough as a whole, and in cross-section.

The volume of the water is the product of the area of the lower isosceles triangle shown in the cross-section times the length of the trough.

The area of the isosceles triangle is equal to half the product of the base times the height.

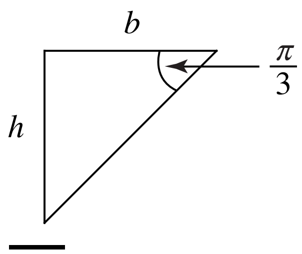


Figure 4.22. Right-angle triangle, one-half of the lower triangle (cross-section), shown in Figure 4.21

Figure 4.22, above, shows half of the lower triangle, with b being the base. Hence,

$$\cot\frac{\pi}{3} = \frac{b}{h}, \quad \text{and so,} \quad b = h\cot\frac{\pi}{3} = \frac{h}{\sqrt{3}}.$$

The area of *half* of the lower triangle is

$$\left(\frac{h}{2\sqrt{3}}\right)h = \frac{h^2}{2\sqrt{3}}.$$

The volume of the water is

$$V = \frac{12h^2}{\sqrt{3}}.$$

This is the equation that relates the variables and the constants.

Implicitly differentiating with respect to time, we obtain

$$\frac{dV}{dt} = \left(\frac{24h}{\sqrt{3}}\right)\frac{dh}{dt}.$$

We solve for $\dfrac{dh}{dt}$ and get

$$\frac{dh}{dt} = \left(\frac{\sqrt{3}}{24h}\right)\frac{dV}{dt}.$$

When $h = 1.5$ ft the rate of change of the depth h is

$$\frac{dh}{dt} = \frac{\sqrt{3}}{24(1.5)}(4) = \frac{\sqrt{3}}{9}\ \text{ft/min}.$$

When the depth is 1.5 ft, the level of the water is rising at a rate of $\sqrt{3}\,/\,9$ ft/min.

For help with the concept of related rates, view the PowerPoint tutorials below. To access a tutorial:

1. Click on the file to open it.
2. Extract the files and save them to your computer's hard drive.
3. Click on the file folder to open it.
4. Click on the PowerPoint file to view and listen.

Note: If you don't have PowerPoint installed on your computer, you can download the PowerPoint Viewer from here: https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6 (https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6).

Tutorial 4: Related Rates (../multimedia/powerpoint/relrates1.zip)

Tutorial 5: Related Rates (../multimedia/powerpoint/relrates2.zip)

Exercises

46. Read “Principles of Problem Solving,” on pages 44-48 of the textbook.
47. Read Examples 1-5 on pages 196-200 of the textbook.
48. Do Exercises 9, 11, 13, 15, 19, 21, 23 and 24 on pages 201-202.

Answers to Exercises (appendix-a.htm)

Higher-order Derivatives

Prerequisites

To complete this section, you must be able to apply the definition of the factorial for any positive integer n ($n! = 1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n$, and for 0 ($0! = 1$).

As we indicated in Definition 4.5 (unit04-02.htm#definition4-5), the derivative $f'(x)$ of a function $f(x)$ is a function that we may be able to differentiate. Then, the derivative of the function $f'(x)$ is called the *second derivative* function of f, and we write $f''(x)$. [Note that $f'(x)$ is referred to as the first derivative of f.] Notation must be established for this second derivative.

For a real number a,

$$f''(a) = \frac{d^2}{dx^2} f(x)\bigg|_{x=a} = \frac{d^2 f}{dx^2}\bigg|_{x=a} = D_x^2 f(a),$$

and for the second derivative function,

$$f''(x) = \frac{d^2}{dx^2} f(x) = \frac{d^2 f}{dx^2} = D_x^2 f(x).$$

If $y = f(x)$, then

$$f''(a) = \frac{d^2 y}{dx^2}\bigg|_{x=a} \qquad \text{and} \qquad f''(x) = \frac{d^2 y}{dx^2}.$$

Observe that this notation makes sense, because we are saying that

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}.$$

Example 4.64. The first derivative of $\sec x$ is $\sec x \tan x$. So, the second derivative of $\sec x$ is

$$\begin{aligned} \frac{d}{dx} \sec x \tan x &= \sec x \sec^2 x + \sec x \tan x \tan x \\ &= \sec^3 x + \sec x \tan^2 x, \end{aligned}$$

and we write

$$\frac{d^2}{dx^2} \sec x = \sec^3 x + \sec x \tan^2 x.$$

Example 4.65.

$$\frac{d^2}{dx^2}\sqrt{x} = \frac{d}{dx}\left(\frac{d}{dx} x^{1/2}\right) = \frac{d}{dx}\frac{x^{-1/2}}{2} = -\frac{x^{-3/2}}{4} = -\frac{1}{4x^{3/2}}.$$

By analogy, the *third derivative* of f is the derivative of the second derivative f'' and we write f'''. Hence,

$$f'''(x) = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) = \frac{d^3 f}{dx^3} = D_x^3 f(x).$$

In the same manner, we can continue with this process and obtain the fourth, fifth and any other higher-order derivative. In general, the n-th derivative of $y = f(x)$ is

$$y^n = f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d^n y}{dx^n} = D_x^n f(x) \qquad \text{for } n \geq 4.$$

Exercises

49. Read Examples 1-3 on pages 189-191 of the textbook.
50. Do Exercises 5, 9, 11, 13, 17, 23 and 27 on pages 193-194.
51. Read Example 4 on page 191.
52. Do Exercises 35 and 36 on page 194.
53. Read Examples 5 and 6 on page 192.
54. Do Exercises 29, 31 and 39 on page 194.

Answers to Exercises (appendix-a.htm)

Finishing This Unit

1. Review the objectives of this unit and make sure you are able to meet all of them. In particular you should be able to
   - differentiate functions with ease.
   - differentiate trigonometric functions.
   - apply all the rules of differentiation.
   - identify the different notations of the derivative.
   - interpret the derivative in two different ways.
2. If there is a concept, definition, example or exercise that is not yet clear to you, go back and reread it, then contact your tutor for help.
3. Tables of differentiation rules are presented below. You may wish to print these diagrams and pin them above your desk for easy reference.
4. Do the exercises in "Learning from Mistakes" section for this unit.
5. You may want to do Exercises 1-4, 13-42, 45-47, 69, 71, 72, 74, 76, 81(a) and 83 from the "Review" (pages 203-207 of the textbook).

Rules of Differentiation

Name	Rule
Power Rule (unit04-04.htm#power-rule-desc)	$\frac{d}{dx}\,x^r = r\,x^{r-1}$
Reciprocal Rule (unit04-04.htm#reciprocal-rule-desc)	$\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{g(x)^2}$
Product Rule (unit04-04.htm#product-rule-desc)	$\frac{d}{dx}\,f(x)g(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule (unit04-04.htm#quotient-rule-desc)	$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
Chain Rule (unit04-06.htm#chain-rule-desc)	$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
General Power Rule (unit04-06.htm#general-power-rule-desc)	$\frac{d}{dx}\,g(x)^r = r\,g(x)^{r-1}g'(x)$

Particular Cases of the Power Rule

r	$\frac{d}{dx}\,x^r = r\,x^{r-1}$
$r = -1$	$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$
$r = \frac{1}{2}$	$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$
$r = -\frac{1}{2}$	$\frac{d}{dx}\frac{1}{\sqrt{x}} = -\frac{1}{2x^{3/2}} = -\frac{1}{2\sqrt{x^3}}$

Particular Cases of the General Power Rule

r	$\frac{d}{dx}\ g(x)^r = r\ g(x)^{r-1} g^{'}(x)$
$r = -1$	$\frac{d}{dx}\ \frac{1}{g(x)} = -\frac{g^{'}(x)}{g(x)^2}$
$r = \frac{1}{2}$	$\frac{d}{dx}\ \sqrt{g(x)} = \frac{g^{'}(x)}{2\sqrt{g(x)}}$
$r = -\frac{1}{2}$	$\frac{d}{dx}\ \frac{1}{\sqrt{g(x)}} = -\frac{g^{'}(x)}{2g(x)^{3/2}} = -\frac{g^{'}(x)}{2\sqrt{g(x)^3}}$

Learning from Mistakes

There are mistakes in each of the following solutions. Identify the errors, and give the correct answer.

1. The displacement (in km) of a moving car is given by $s(t) = 3t^3 + 4t - 2$, where t is measured in hours.
   a. Give the average velocity in the time period $[1, 5]$.
   b. Give the two different interpretations of the value found in part (a), above.
   c. Give the average velocity in the time period $[1, 1 + h]$, for $h > 0$.

Erroneous Solution

a. $\dfrac{s(5) - s(1)}{1 - 5} = \dfrac{3(75) + 4(5) - 2 - (3 + 4 - 2)}{1 - 5} = \dfrac{238}{-4} = -59.5.$

b. The velocity is decreasing at a rate of 59.5 km/h, and the slope of the secant line passing through $(1, 5)$ and $(5, 245)$ is 59.5.

c. $\dfrac{s(1) - s(1 + h)}{h} = \dfrac{5 - 3(1 + h)^3 + 4(1 + h) - 2}{h} = \dfrac{4 - 9h^2 - 5h - 3h^3}{h}.$

2. Use Definition 4.5 (unit04-02.htm#definition4-5) to find $\dfrac{d}{dx}\dfrac{3}{x}$.

Erroneous Solution

$$\lim_{h\to 0}\frac{3(x + h)^{-1} - 3\left(x^{-1}\right)}{h} = \lim_{h\to 0}\frac{3(x + h) - 3x}{x(x + h)h} = \frac{3}{x(x + h)} = \frac{3}{x^2}.$$

3. Consider the piecewise function

$$f(s) = \begin{cases} |x| & \text{for} \quad x \le 4 \\ (x - 6)^2 & \text{for} \quad x > 4 \end{cases}$$

   a. Sketch the graph of the function f.
   b. Sketch the graph of the derivative function f'.

Erroneous Solution

a.

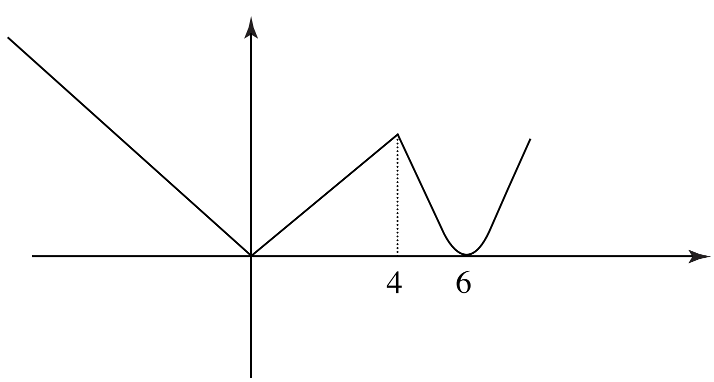


Figure 4.23. Erroneous solution, "Learning from Mistakes," Problem 3(a)

b.

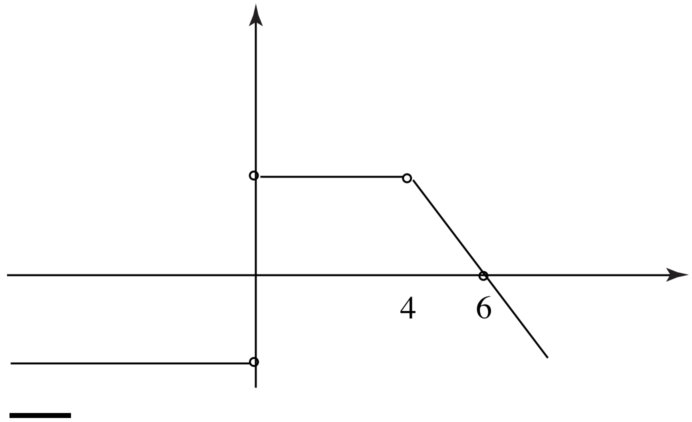


Figure 4.24. Erroneous solution, "Learning from Mistakes," Problem 3(*b*)

4. Find the derivatives indicated below.

   a. $f(x) = \dfrac{3x^2 + 6x - 3}{x - 4x^3}$ find $f^{'}(x)$.

   b. $\dfrac{d}{dx}\sqrt{x^3 \sin\ x}$.

   c. $\dfrac{d}{dx}\ \sec\sqrt{x+1}$.

   d. $\dfrac{d^2}{dx^2}\ \tan(3x)$.

   Erroneous Solution

   a. $f^{'}(x) = \dfrac{\left(3x^2 + 6x - 3\right)\left(1 - 12x^2\right) - (6x+6)\left(x - 4x^3\right)}{\left(x - 4x^3\right)^2}$.

   b. $\dfrac{d}{dx}\sqrt{x^3 \sin\ x} = \dfrac{3x^2\ \cos\ x}{2\sqrt{x^3\ \sin\ x}}$.

   c. $\dfrac{d}{dx}\ \sec\sqrt{x+1} =\ \sec\ x \tan x\left(\dfrac{1}{2\sqrt{x+1}}\right)$.

   d. $\dfrac{d}{dx}\ \tan\ (3x) =\ \sec^2\ (3x)\ \text{ and }\ \dfrac{d}{dx}\ \sec^2\ (3x) = 2\ \sec^2(3x)\ \tan(3x)$.

5. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

   Erroneous Solution

   We have $\dfrac{dx}{dt} = 3$, and we want to know $\dfrac{dy}{dt}$ when $x = 4$, where y is the distance from the origin to the point $\left(x, \sqrt{x}\right)$ on the curve.

   Hence, $y = \sqrt{x^2 + x}$.

   Differentiating, we obtain

   $$\frac{dy}{dt} = \frac{2x+1}{2\sqrt{x^2+1}}.$$

   For $x = 4$,

$$\frac{dy}{dt} = \frac{9}{2\sqrt{17}}.$$

The distance is increasing at a rate of $\frac{9}{2\sqrt{17}}$ cm/s.

6. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 2 mi away from the station.

   Erroneous Solution

   We want $\frac{dy}{dt}$ when $y = 2$ mi.

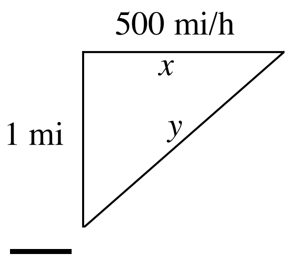
   

   Figure 4.25. Erroneous solution, "Learning from Mistakes," Problem 6

   From the figure, $y^2 = x^2 + 1$. Differentiating yields $2y\frac{dy}{dt} = 2x$.

   Then $\frac{dy}{dt} = \frac{x}{y}$, when $y = 2$

   $$\frac{dy}{dt} = \frac{500}{2} = 250 \quad \text{mi/h}.$$

Applications of the Derivative

Objectives

When you have completed this unit, you should be able to

1. use calculus to sketch the graph of a function in detail.
2. solve optimization problems.
3. apply Newton's method to solve equations of the form $f(x) = 0$.

To sketch the graph of a function, we must be methodical and clear about the information we obtain at each of the steps described in the first seven sections of this unit, and listed on pages 232-234 of the textbook. "Optimization" problems are so named because they involve decisions one can make to optimize the resources available under constraints. We end this unit with a discussion of one more application of the derivative—Newton's method.

Domain, *x-y* Intercepts, and Symmetries

Prerequisites

To complete this section, you must be able to

1. write inequalities in interval notation. See the section titled "Intervals" on pages 337-338 of the textbook.
2. solve inequalities. See the section titled "Inequalities" on pages 338-340 of the textbook. Do Exercises 13-20 on page 343.
3. use the absolute value to solve inequalities. See the section titled "Absolute Value" on pages 340-343 of the textbook. Do Exercises 1-10 on page 343.
4. complete a perfect square. See the section titled "Complete the Square" on page 6 of the PDF file "Review of Algebra" available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

In this section, we discuss how to obtain basic information about the graph of a function *without* using the derivative.

Sketching starts with the domain of a function as defined in Definition 2.6 (unit02.htm#definition2-6). The domain indicates where the function may have a vertical asymptote, and where the graph is located with respect to the x-axis. The functions we work with in this course may have a vertical asymptote in the following two possible cases:

1. points where the function is not defined.
2. the endpoints of the open intervals of the function's domain.

Therefore, a function has no vertical asymptotes if its domain is $\mathbb{R}$, all real numbers.

Example 5.1. The domain of the function

$$f(x) = \sqrt{x^2 - 9}$$

consists of all x such that $x^2 - 9 \geq 0$.

If we solve this inequality, we obtain $x^2 \geq 9$.

Taking the square roots gives us $|x| \geq 3$.

Therefore, we have two inequalities—$x \leq -3$ and $x \geq 3$—and in interval notation, the domain is $(-\infty, -3] \cup [3, \infty)$.

Hence, the graph of this function does not exist between the vertical lines $x = -3$ and $x = 3$. Since intervals of the domain are closed at the endpoints 3 and -3, the function does not have vertical asymptotes.

Example 5.2. The domain of the function

$$h(x) = \frac{4x - 5}{4x^2 - 6x}$$

consists of all x such that the denominator $4x^2 - 6x$ is not zero.

If $4x^2 - 6x = 0$, then $2x(2x - 3) = 0$ and $x = 0$ or $x = \frac{3}{2}$.

The domain is all real numbers except 0 and $3 / 2$. So, this function may have vertical asymptotes at $x = 0$ or $x = 3 / 2$.

Example 5.3. The domain of the function

$$g(x) = \frac{x^2 + 1}{x(2x - 7)^2}$$

consists of x such that $x(2x - 7)^2 \neq 0$, hence $x \neq 0$ and $x \neq 7 / 2$

The domain is

$$D_g = \mathbb{R} - \left\{0, \frac{7}{2}\right\},$$

and again, the vertical lines $x = 0$ and $x = 7 / 2$ may be asymptotes of the function.

Example 5.4. The domain of the function

$$h(x) = \frac{x - 4}{\sqrt{9x^2 - 24x + 10}}$$

consists of all x such that $9x^2 - 24x + 10 > 0$.

Let us solve this inequality:

$$\begin{array}{lllll}
9x^2 - 24x + 10 = (3x - 4)^2 - 6 & & & & \text{completing the square} \\
(3x - 4)^2 > 6 & & & & \text{taking square roots} \\
|3x - 4| > \sqrt{6} & & & & \text{this gives two inequalities} \\
3x - 4 < -\sqrt{6} & \text{and} & 3x - 4 > \sqrt{6} & & \\
4 + \sqrt{6} < 3x & \text{and} & 4 - \sqrt{6} > 3x & & \\
\dfrac{4 + \sqrt{6}}{3} < x & \text{and} & \dfrac{4 - \sqrt{6}}{3} > x & &
\end{array}$$

The domain is the union of the intervals

$$\left(-\infty, \frac{4 - \sqrt{6}}{3}\right) \quad \text{and} \quad \left(\frac{4 + \sqrt{6}}{3}, \infty\right).$$

In this case, the vertical lines

$$x = \frac{4 - \sqrt{6}}{3} \quad \text{and} \quad \frac{4 + \sqrt{6}}{3}$$

may be vertical asymptotes of the function, and there is no graph between these two vertical lines.

Example 5.5. In the function,

$$u(x) = \sqrt{1 - \sin x},$$

we have $1 - \sin x \geq 0$, but since $|\sin x| \leq 1$, it is the case that $1 - \sin x \geq 0$ for any x, and the domain of the function is $\mathbb{R}$. Hence, there are no vertical asymptotes.

Example 5.6. To find the domain of the composition of two functions, we must consider the domain of each function.

In the function $f(x) = \tan(2x + 1)$, the domain of $\tan u$ is the union of the intervals of the form

$$\left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right), \quad \text{for any integer } k.$$

Hence, the inside function $2x + 1$ must be in these intervals, that is

$$\frac{(2k+1)\pi}{2} < 2x + 1 < \frac{(2k+3)\pi}{2}$$

Solving for x:

$$\frac{(2k+1)\pi - 2}{4} < x < \frac{(2k+3)\pi - 2}{4}$$

The domain is all intervals of the form

$$\left(\frac{(2k+1)\pi - 2}{4}, \frac{(2k+3)\pi - 2}{4}\right) \quad \text{for any integer } k.$$

The function may have vertical asymptotes at

$$x = \frac{(2k+1)\pi - 2}{4} \quad \text{or} \quad x = \frac{(2k+3)\pi - 2}{4} \quad \text{for any integer } k.$$

The x and y intercepts are the points of the graph that intersect the x-axis and the y-axis, respectively.

Definition 5.1. For a function $f(x)$

a. the points $(a, 0)$ on the graph of f are the x-intercepts
b. the point $(0, b)$ on the graph of f is the y-intercept.

REMARKS 5.1

a. The x-intercepts of a function are also known as the real "roots" of the function.
b. A function may or may not intersect the axis.
c. A function may have no, finite or infinitely many x intercepts.
d. A function has either no or one y-intercept.

To find the x-intercepts, we must find all a such that $f(a) = 0$.

Example 5.7. For the function

$$f(x) = \sqrt{x^2 - 9},$$

we set $\sqrt{x^2 - 9} = 0$ and solve for x. Thus $x^2 = 9$, if $x = 3$ or $x = -3$, and the x intercepts are $(3, 0)$ and $(-3, 0)$.

Example 5.8. A quotient

$$g(x) = \frac{x^2 + 1}{x(2x - 7)^2}$$

is zero if the numerator is zero, but in this case, $x^2 + 1 \neq 0$ for any real x; hence, the function has no x-intercepts.

Example 5.9. For the function

$$u(x) = \sqrt{1 - \sin x},$$

we set $\sqrt{1 - \sin x} = 0$ and solve for x.

This strategy yields $\sin x = 1$, and we have

$$x = \frac{(4k+1)\pi}{2} \quad \text{for any integer } k.$$

The function has infinitely many x-intercepts:

$$\left(\frac{(4k+1)\pi}{2}, 0\right) \quad \text{for any integer } k.$$

Example 5.10. For the function

$$f(x) = \tan(2x+1),$$

we have $\tan(2x+1) = 0$ if $\sin(2x+1) = 0$.

Then

$$2x + 1 = k\pi,$$

and solving for x, we conclude that

$$\left(\frac{k\pi - 1}{2}, 0\right).$$

There may be instances where we cannot solve $f(x) = 0$ algebraically, and in such cases, we may be able to locate the x-intercepts using the Intermediate Value Theorem (see page 67 of the textbook).

Theorem 5.2. The Intermediate Value Theorem

If f is a continuous function on the interval $[a, b]$ and $f(a) < M < f(b)$, then there is number c in the interval (a, b)—that is, $a < c < b$—such that $f(c) = M$.

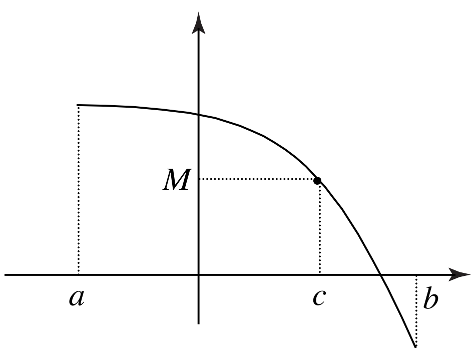


Figure 5.1. Intermediate Value Theorem

In Figure 5.1, above, you can see the interpretation of this theorem.

For $M = 0$, this theorem says that if $f(a) < 0 < f(b)$, then there is a c such that $f(c) = 0$.

Example 5.11. To find the x-intercept of the function

$$g(x) = x^3 + 4x^2 + x - 4,$$

we must solve for x in the equation $x^3 + 4x^2 + x - 4 = 0$. Since it is not easy to do so algebraically, we apply the Intermediate Value Theorem (unit05.htm#theorem5-2). The function is continuous everywhere, because g is a polynomial. So, we must find two numbers, a and b, such that $g(a) < 0 < g(b)$.

We observe that $g(0) = -4 < 0$ and $g(1) = 2 > 0$. Hence, $g(0) < 0 < g(1)$, and there must be a number c between 0 and 1 such that $g(c) = 0$; hence, there is an x-intercept in the interval $(0, 1)$.

You can check that $g(-1) < 0 < g(-2)$; therefore, there is another x-intercept on the interval $(-2, -1)$.

In the last section of this unit, we consider Newton's method, which can also be used to locate x-intercepts.

REMARK 5.2

When sketching the graph of a function, it is convenient but not necessary to find the x-intercepts. If we find that the solution of $f(x) = 0$ is too challenging or beyond our abilities, we do not solve it.

To find the y-intercept, we evaluate the function at zero, if zero is in the domain of the function.

Example 5.12. In Examples 5.1 (unit05.htm#example5-1) to 5.3 (unit05.htm#example5-3), we saw that zero is not in the domain of the functions

$$f(x) = \sqrt{x^2 - 9}, \quad h(x) = \frac{4x - 5}{4x^2 - 6x} \quad \text{or} \quad g(x) = \frac{x^2 + 1}{x(2x - 7)^2}.$$

Hence, these functions do not have y-intercepts—that is, their graphs do not intersect the y-axis.

Example 5.13. In Example 5.4, we have the function

$$h(x) = \frac{x - 4}{\sqrt{(4 - 3x)^2 - 6}},$$

and $h(0) = -\frac{4}{10}$. Hence, the y-intercept of the function is

$$\left(0, -\frac{4}{\sqrt{10}}\right)$$

Example 5.14. In Example 5.6, we have the function

$$f(x) = \tan(2x + 1),$$

and $f(0) = \tan(1)$.

Hence, the y-intercept of the function is $(0, \tan(1)) \approx (0, 1.5574)$.

To sketch the graph of a function, it helps to know when a function has certain symmetries.

Consider the graph in Figure 5.2, below.

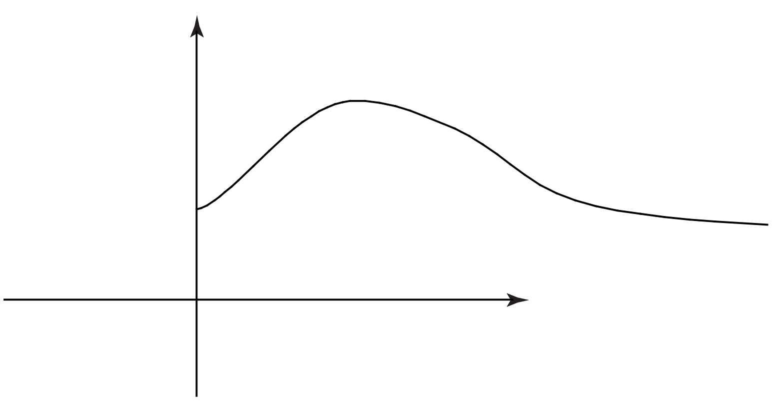


Figure 5.2. Basic graph

If the graph in Figure 5.2 were symmetric with respect to the y-axis, then the complete graph would be as shown in Figure 5.3:

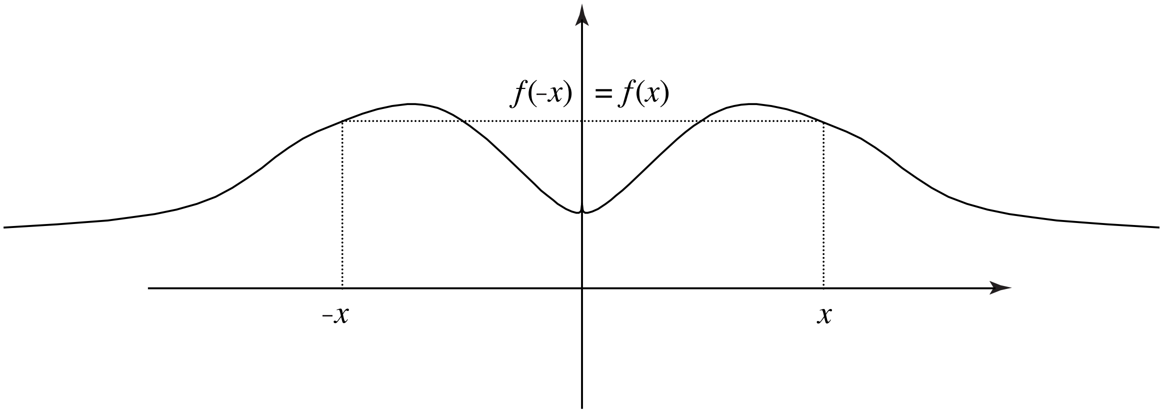


Figure 5.3. Basic graph symmetrical with respect to the y-axis

If the graph in Figure 5.2 were symmetric with respect to the origin (i.e., across the diagonal formed by the identity function), then the complete graph would be as shown in Figure 5.4:

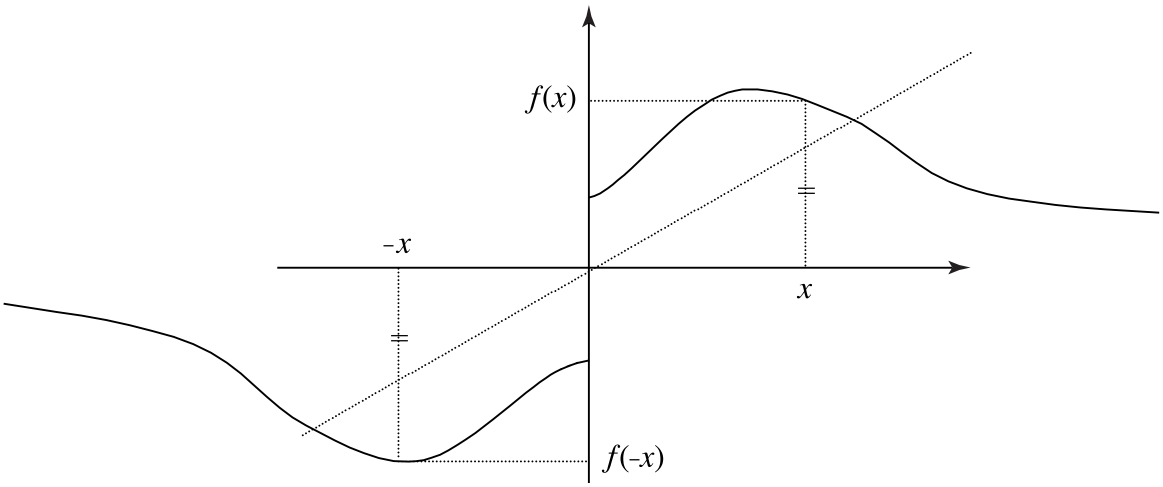


Figure 5.4. Basic graph symmetrical with respect to the origin

If a function is even, then its graph is symmetric with respect to the y-axis (see Figure 5.3, above). If the function is odd, then its graph is symmetric with respect to the origin (see Figure 5.4).

The function in Figure 5.5, below, is even.

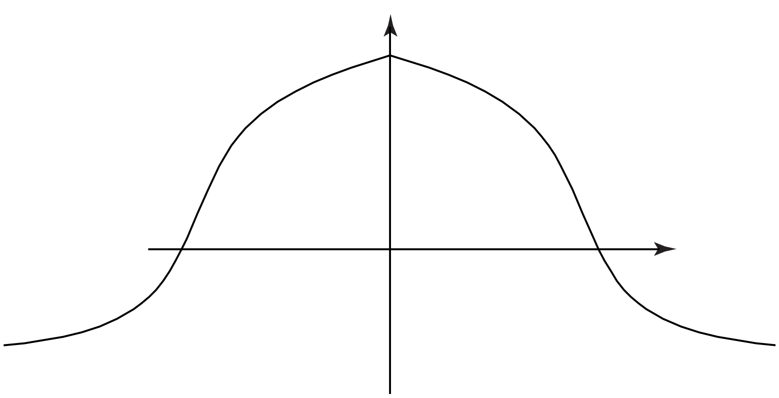


Figure 5.5. Even function

The function in Figure 5.6 is odd:

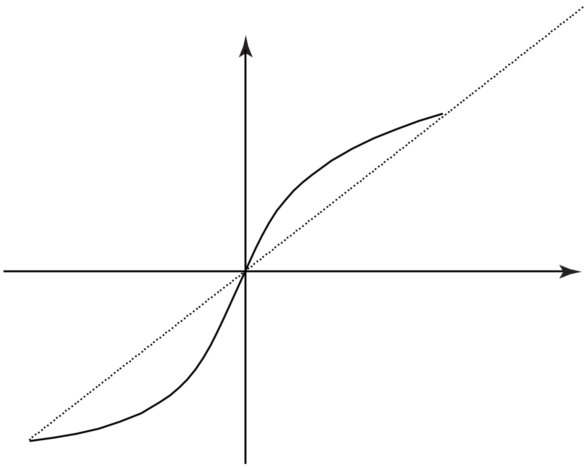


Figure 5.6. Odd function

To identify these symmetries, we use the following definition.

Definition 5.3. A function $f(x)$ is *even* if $f(-x) = f(x)$ for any x in its domain.

A function $f(x)$ is *odd* if $f(-x) = -f(x)$ for any x in its domain.

Example 5.15.

a. The function $f(x) = x^4$ is even because $f(-x) = (-x)^4 = x^4 = f(x)$.
   In general, if n is even, then $E(x) = x^n$ is even.

b. The function $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$.
   In general if m is odd, then $O(x) = x^m$ is odd.

c. From the graphs of the trigonometric functions, we see that sine, tangent, cotangent and cosecant are odd functions; and cosine and secant are even.
   Remember that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$

d. A constant function is even. [Why?]

e. The absolute value function is even.

Example 5.16. To determine whether the function $h(x) = -2x^3 + \sin\left(2x^2\right)$ is even, odd or neither, we evaluate the function at $-x$.

Hence,

$$h(-x) = -2(-x)^3 + \sin\left(2(-x)^2\right) = 2x^3 + \sin\left(2x^2\right).$$

Since

$$h(-x) \neq h(x) \quad \text{and} \quad h(-x) \neq -h(x),$$

the function is neither even nor odd.

Example 5.17. The function $g(x) = \dfrac{6x}{\cos(4x)}$ is odd because

$$g(-x) = \frac{6(-x)}{\cos(4(-x))} = -\frac{6x}{\cos(4x)} = -g(x)$$

Example 5.18. Observe that for $f(x)$ and $g(x)$ even, and $h(x)$ and $p(x)$ odd,

a. $f(x) + g(x)$ is even because $f(-x) + g(-x) = f(x) + g(x)$.
So the function $q(x) = x^4 + \cos(x)$ is even.

b. $h(x) + p(x)$ is odd because $h(-x) + p(-x) = -h(x) - p(x) = -(h(x) + p(x))$.
So the function $q(x) = x + \tan x$ is odd.

c. $f(x)g(x)$ is even because $f(-x)g(-x) = f(x)g(x)$.
So the function $q(x) = x^4\cos(x)$ is even.

d. $h(x)p(x)$ is even because $h(-x)p(-x) = (-h(x))(-p(x)) = h(x)p(x))$.
So the function $q(x) = x\tan x$ is even.

e. $f(x)h(x)$ is odd because $f(-x)h(-x) = f(x)(-h(x)) = -(f(x)h(x))$.
So the function $q(x) = x^4\tan(x)$ is odd.

f. $f(g(x))$ is even because $f(g(-x)) = f(g(x))$.
So the function $q(x) = \cos\left(x^4\right)$ is even.

g. $f(h(x))$ is even because $f(h(-x)) = f(-h(x)) = f(h(x))$.
So the function $q(x) = \cos\left(x^3\right)$ is even.

Example 5.19. Observe how we use Example 5.18 to decide if the functions given are even or odd.

a. $p(x) = 4x^9$ is odd because the constant function 4 is even and the function x^9 is odd—Example 5.18, (e).

b. $q(x) = x \tan x + x^2 \cos x$ is even—Example 5.18, (c), (d), (a).

c. $g(x) = \sec(4x) \cos x \tan x$ is odd—Example 5.18, (d), (e).

Exercises

1. Explain why polynomial functions do not have vertical asymptotes.
2. Determine whether either of the following functions may have a vertical asymptote.
   a. $f(t) = \dfrac{1}{\sqrt{3-t}}$
   b. $g(x) = \dfrac{3x^2 + 5x - 2}{x^2 - 4}$
3. Find the x-intercepts, if any, of each of the following functions.
   a. $f(t) = \dfrac{1}{\sqrt{3-t}}$
   b. $g(x) = \dfrac{3x^2 + 5x - 2}{x^2 - 4}$
   c. $R(x) = x^2 + 2x + 6$
4. Find the y-intercept, if any, of each of the following functions.
   a. $f(t) = \dfrac{1}{\sqrt{3-t}}$
   b. $g(x) = \dfrac{3x^2 + 5x - 2}{x^2 - 4}$
5. Determine whether each of the functions below is even, odd or neither. If f is even or odd, use symmetry to sketch its graph.
   a. $f(x) = x^{-2}$

b. $f(x) = x^2 + x$

c. $f(x) = x^3 - x$

d. $f(x) = 3x^3 + 2x^2 + 1$

6. If $f(x)$ is even and $h(x)$ is odd, determine whether each of the following functions is odd or even.

a. $h(f(x))$

b. $\dfrac{f(x)}{h(x)}$

7. Do Exercises 45-48, 49(a) and 51 on page 70 of the textbook.

Answers to Exercises (appendix-a.htm)

Asymptotes

In Unit 3, we defined and learned how to find the vertical and horizontal asymptotes of a function (see Definitions 3.19 (unit03-03.htm#definition3-19) and 3.28 (unit03-06.htm#definition3-28)). As we mentioned in the previous section, the domain of a function indicates its possible vertical asymptotes. Let us review the procedure for vertical asymptotes once more.

Example 5.20. In Example 5.2 (unit05.htm#example5-2), we decided that the function

$$h(x) = \frac{4x-5}{4x^2-6x}$$

may have a vertical asymptotes at $x = 0$ or $x = 3 / 2$.

To see if it does so, we take any one of the limits listed in Definition 3.19 (unit03-03.htm#definition3-19). We find that

$$\lim_{x\to 0^+} \frac{4x-5}{4x^2-6x} = \lim_{x\to 0^+} \left(\frac{4x-5}{4x-6}\right)\left(\frac{1}{x}\right) = \infty,$$

since

$$\lim_{x\to 0^+} \frac{4x-5}{4x-6} = \frac{5}{6} > 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

Furthermore,

$$\lim_{x\to 3/2^+} \frac{4x-5}{4x^2-6x} = \lim_{x\to 3/2^+} \left(\frac{1}{2x-3}\right)\left(\frac{4x-5}{2x}\right) = \infty,$$

since

$$\lim_{x\to 3/2^+} \frac{4x-5}{2x} = \frac{1}{3} > 0 \quad \text{and} \quad \lim_{x\to 3/2^+} \frac{1}{2x-3} = \infty.$$

Hence, the function has vertical asymptotes at $x = 0$ and $x = 3 / 2$.

Example 5.21. In Example 5.3 (unit05.htm#example5-3), we saw that $x = 0$ and $x = 7 / 2$ may be vertical asymptotes of the function

$$g(x) = \frac{x^2+1}{x(2x-7)^2}.$$

Taking limits, we find that

$$\lim_{x\to 0^+} \frac{x^2+1}{x(2x-7)^2} = \lim_{x\to 0^+} \left(\frac{x^2+1}{(2x-7)^2}\right)\left(\frac{1}{x}\right) = \infty,$$

since

$$\lim_{x\to 0^+} \frac{x^2+1}{(2x-7)^2} = \frac{1}{49} > 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = \infty;$$

and

$$\lim_{x\to 7/2^+} \frac{x^2+1}{x(2x-7)^2} = \lim_{x\to 7/2^+} \left(\frac{x^2+1}{x}\right)\left(\frac{1}{(2x-7)^2}\right) = \infty,$$

since

$$\lim_{x\to 7/2^+} \frac{x^2+1}{x} = \frac{53}{14} > 0 \quad \text{and} \quad \lim_{x\to 7/2^+} \frac{1}{(2x-7)^2} = \infty.$$

Hence, the function has vertical asymptotes at $x = 0$ and $x = 7 / 2$.

Example 5.22. Consider the function

$$f(x) = \tan(2x + 1)$$

(see Example 5.6 (unit05.htm#example5-6).

This function is periodic: its graph repeats itself every interval

$$\left(\frac{(2k+1)\pi - 2}{4}, \frac{(2k+3)\pi - 2}{4}\right).$$

Hence, it is enough to check if the function has a vertical asymptote at the endpoints of the open interval for $k = 0$.

Taking limits, we see that

$$\text{if } t = 2x + 1 \quad \text{then} \quad t \to \frac{\pi}{2}^{+} \quad \text{as} \quad x \to \left(\frac{\pi - 2}{4}\right)^{+}.$$

Hence,

$$\lim_{x \to \left(\frac{\pi-2}{4}\right)^{+}} \tan(2x+1) = \lim_{t \to \left(\frac{\pi}{2}\right)^{+}} \tan t = -\infty.$$

We can check in a similar way that

$$\lim_{x \to \left(\frac{3\pi-2}{4}\right)^{+}} \tan(2x+1) = \lim_{t \to \left(\frac{3\pi}{2}\right)^{+}} \tan t = -\infty. \text{ [Try it.]}$$

Therefore, the function has vertical asymptotes at each endpoint of the open intervals of its domain.

Example 5.23. The function

$$\frac{x^2 + 8x + 15}{x^2 + 5x + 6}$$

may have vertical asymptotes at $x = -3$ and $x = -2$. [Why?]

Taking limits, we find

$$\lim_{x \to -3^-} \frac{x^2 + 8x + 15}{x^2 + 5x + 6} = \lim_{x \to -3^-} \frac{x+5}{x+2} = -2$$

and

$$\lim_{x \to -2^-} \frac{x^2 + 8x + 15}{x^2 + 5x + 6} = \lim_{x \to -2^-} \frac{x+5}{x+2} = -\infty.$$

Hence, $x = -2$ is the only vertical asymptote for this function.

A line $y = mx + b$ with $m \neq 0$ is a slant asymptote of a function $f(x)$ if

$$\lim_{x \to \pm\infty} f(x) - (mx + b) = 0.$$

In this course, we only consider the case of slant asymptotes of rational functions in which the degree of the numerator is one more than the degree of the denominator. See Example 4 on page 237 of the textbook.

Exercises

8. Do Exercises 43 and 45 on page 238 of the textbook.
9. Find the slant asymptote of the functions in Exercises 47, 49 and 51 on page 238.

Answers to Exercises (appendix-a.htm)

Intervals of Increase and Decrease

Prerequisites

To complete this section, you must be able to

1. write inequalities in interval notation. See the section titled “Intervals” on pages 337-338 of the textbook.
2. solve inequalities. See the section titled “Inequalities” on pages 338-340 of the textbook. Do Exercises 13-20 on page 343.
3. use absolute value to solve inequalities. See the section titled “Absolute Value” on pages 340-343 of the textbook. Do Exercises 1-10 on page 343.
4. factor algebraic and trigonometric functions. See Unit 1 in this *Study Guide* and the PDF document titled “Review of Algebra,” available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

In this section, we discuss using the derivative to find the intervals where a function is increasing and decreasing. We must start by defining these terms.

Definition 5.4.

a. A function f is *increasing* (abbreviated inc) on the interval J if for any $a < b$ in the interval J, it is the case that $f(a) < f(b)$.

b. A function f is *decreasing* (abbreviated dec) on the interval J if for any $c < d$ in the interval J, it is the case that $f(c) > f(d)$.

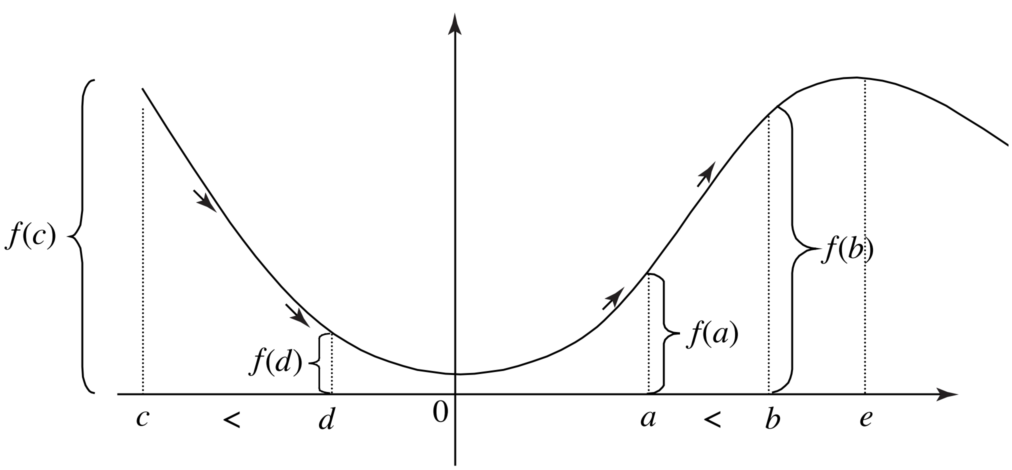


Figure 5.7. Function decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, e)$

In Figure 5.7, above, we see that $f(c) > f(d)$ for *any* $c < d$ in the interval $(-\infty, 0)$ and $f(a) < f(b)$ for any $a < b$ in the interval $(0, e)$. Thus, according to Definition 5.4, the function is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, e)$, as we see on the graph. That is, the definition corresponds to our idea of increasing and decreasing.

When a function is increasing, as in Figure 5.8, below, the tangent lines are positive.

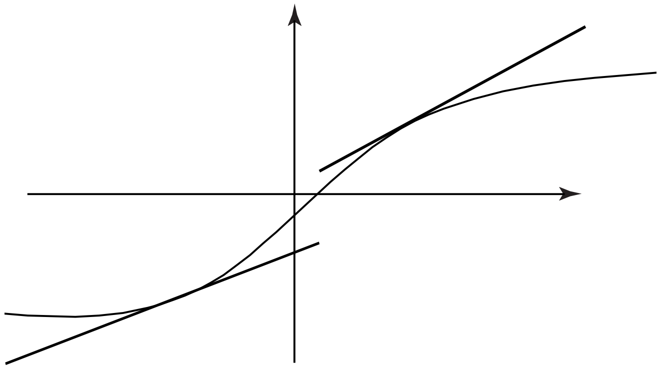


Figure 5.8. Increasing function showing tangent lines

When it is decreasing, as in Figure 5.9, below, then the slopes are negative.

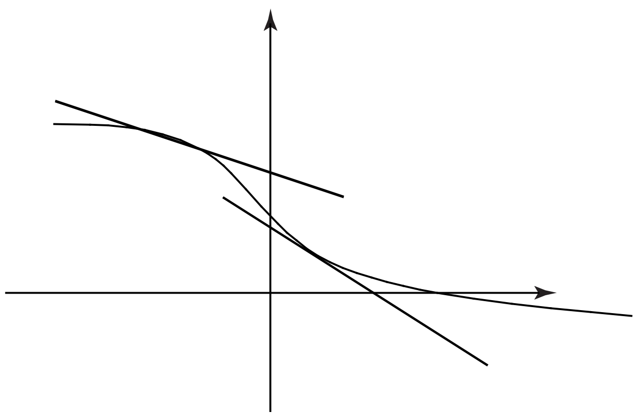


Figure 5.9. Decreasing function showing tangent lines

The following theorem then seems plausible.

Theorem 5.5.

a. If $f^{'}(x) > 0$ for any x on the interval J, then the function f is increasing on the interval J.
b. If $f^{'}(x) < 0$ for any x on the interval J, then the function f is decreasing on the interval J.

Later, we will see how we can prove this theorem, for now we will learn to use it to find the intervals where a function is increasing and decreasing.

Let us think about this carefully. We need to be aware that when a continuous function changes from increasing to decreasing (or from decreasing to increasing) at some point c, it is because, at this point, either the tangent line is horizontal or there is no tangent line. See Figure 5.10, below.

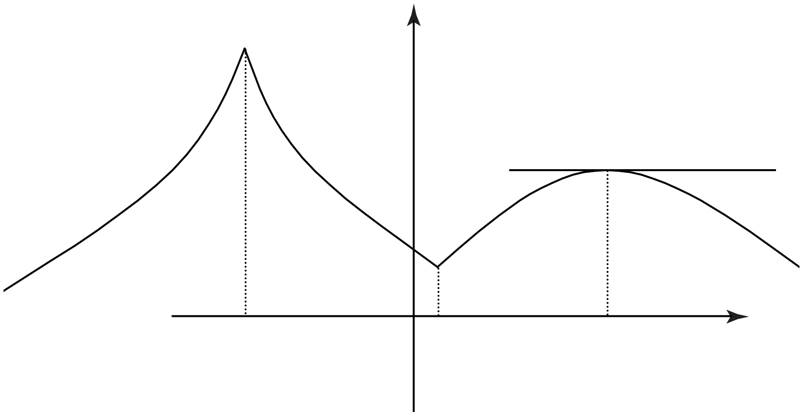


Figure 5.10. Changing continuous function showing no tangent line (left and centre) and horizontal tangent line (right) at points of change

Thus, to find the intervals of increase and decrease, we must identify such numbers c, called "critical numbers."

> **Definition 5.6.** A number c in the domain of a function f is a *critical number* if either $f'(c) = 0$ or $f'(c)$ does not exist.
>
> The point $(c, f(c))$ for a critical number c of f is the *critical point* of f.

To identify the critical points, we must simplify and factor the derivative function as much as possible. Once we identify the critical points, we use Theorem 5.5 to decide if the function is increasing or decreasing near the critical point.

Example 5.24. Consider the function $f(x) = \sqrt{x^2 - 9}$ (see Example 5.1 (unit05.htm#example5-1)). To find the critical points, we use the derivative of this function,

$$f'(x) = \frac{x}{\sqrt{x^2 - 9}}.$$

Then

$$f'(x) = \frac{x}{\sqrt{x^2 - 9}} \neq 0$$

for all x in its domain.

Since $f'(-3)$ and $f'(3)$ are undefined, the critical numbers are -3 and 3. We have two intervals to consider:

$$(-\infty, \ -3) \quad \text{and} \quad (3, \infty). \quad [\text{Why?}]$$

Choose any number s in the interval $(-\infty, \ -3)$, say -5, and see that for this number $f'(-5) < 0$; hence, $f'(x) < 0$ on the interval $(-\infty, \ -3)$.

In the same manner, we can see that $f'(x) > 0$ on the interval $(3, \infty)$.

By Theorem 5.5, the function is decreasing on $(-\infty, \ -3)$ and increasing on $(3, \infty)$.

Example 5.25. The domain of the function

$$h(x) = \frac{4x - 5}{4x^2 - 6x}$$

of Example 5.2 (unit05.htm#example5-2) is $\mathbb{R} - \{0, 3 / 2\}$.

Its derivative is

$$h'(x) = \frac{-2\left(8x^2 - 20x + 15\right)}{\left(4x^2 - 6x\right)^2},$$

and since

$$8x^2 - 20x + 15 \neq 0 \quad \text{for all} \quad x$$

(you may want to use the quadratic formula to see this), we conclude that h does not have critical points in its domain.

Moreover,

$$8x^2 - 20x + 15 = \left(4x^2 - 6x\right)^2 > 0 \quad \text{for all} \quad x;$$

hence, $h'(x) < 0$ for all x in its domain. By Theorem 5.5, the function is decreasing on its domain.

Sometimes it is necessary to make up a table to find the intervals of increase and decrease, as follows:

- in the first column, we indicate the intervals given by the critical numbers.
- in the next columns, we put all of the factors of the derivative that determine the sign (positive or negative) of the derivative.
- in the next column, we place the sign of the derivative according to the product of the signs of its factors.
- in the last column, we indicate the conclusion, whether the function is increasing or decreasing.

Example 5.26. To find the critical numbers of the function

$$y = x^3 - 6x^2 - 15x + 4,$$

we use Definition 5.6 above:

$$y' = 3x^2 - 12x - 15 = 3\left(x^2 - 4x - 5\right) = 3(x - 5)(x + 1).$$

Solving, we find that $y' = 3(x - 5)(x + 1) = 0$, and $x = 5$ and $x = -1$. These critical numbers give three intervals: $(-\infty, -1)$, $(-1, 5)$, and $(5, \infty)$.

Next, we make up a table as follows:

Table summarizing the intervals of increase and decrease for the function $y = x^3 - 6x^2 - 15x + 4$

Interval	$(x-5)$	$(x+1)$	y'	y
$x < -1$	$-$	$-$	$+$	inc
$-1 < x < 5$	$-$	$+$	$-$	dec
$x > 5$	$+$	$+$	$+$	inc

From this table, we conclude that the function is increasing on the intervals $(-\infty, -1)$ and $(5, \infty)$, and decreasing on $(-1, 5)$.

> **Note:** One way to find the sign in the table is to take any number in the indicated interval and test for the sign of each factor. For instance for $x = 0$ in $-1 < x < 5$, we have $(x - 5) = -5 < 0$ and $(x + 1) = 1 > 0$.

Example 5.27. The domain of the function $y = \dfrac{x}{x^2 + 4x + 3}$ is

$$(-\infty, -3) \cup (-3, -1) \cup (-1, \infty).$$

Its derivative is

$$y' = \frac{(x^2 + 4x + 3) - x(2x + 4)}{(x + 3)^2(x + 1)^2} = \frac{-x^2 + 3}{(x + 3)^2(x + 1)^2} = \frac{\left(\sqrt{3} - x\right)\left(\sqrt{3} + x\right)}{(x + 3)^2(x + 1)^2}$$

The critical numbers are $x = \sqrt{3}$ and $x = -\sqrt{3}$.

From the domain and the critical numbers, we have five intervals to consider, as indicated in the table below. The sign of the derivative depends on the factors of the numerator, since the denominator is always positive; therefore, we must consider these factors on the table.

Table summarizing the intervals of increase and decrease for the function $y = \dfrac{x}{x^2+4x+3}$. The table indicates whether the function is increasing or decreasing over each interval.

Interval	$(\sqrt{3}-x)$	$(\sqrt{3}+x)$	y'	y
$x < -3$	$+$	$-$	$-$	dec
$-3 < x < -\sqrt{3}$	$+$	$-$	$-$	dec
$-\sqrt{3} < x < -1$	$+$	$+$	$+$	inc
$-1 < x < \sqrt{3}$	$+$	$+$	$+$	inc
$x > \sqrt{3}$	$-$	$+$	$-$	dec

The function is increasing on the intervals $(-\sqrt{3}, -1)$ and $\left(-1, \sqrt{3}\right)$; it is decreasing on the intervals $(-\infty, -3)$, $(-3, -\sqrt{3})$ and $(\sqrt{3}, \infty)$.

REMARK 5.3

To find the critical points of a function, we may need to solve the equation $f'(x) = 0$, and it may be algebraically difficult or impossible to do so. In such cases, we may be left with the option of locating the critical points by some other method, such as the Intermediate Value Theorem (unit05.htm#theorem5-2) or Newton’s method. It is important that we locate these points if we are to sketch a function. In this course, the exercises and examples are always such that you can find the critical points algebraically, but keep this remark in mind.

Exercises

10. Do part (a) of Exercises 11, 13, 15, 20, 29, 31, 33, 35 and 37 on pages 217-218 of the textbook.
11. Do Exercises 31, 33, 35, 38, 39 and 41 on page 229.

Answers to Exercises (appendix-a.htm)

Local and Absolute Extreme Values

Prerequisites

To complete this section, you must be able to

1. write inequalities in interval notation. See the section titled “Intervals” on pages 337-338 of the textbook.
2. factor algebraic and trigonometric functions. See Unit 1 in this *Study Guide* and the PDF document titled “Review of Algebra,” available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

The intervals of decrease and increase of a function can be used to determine the local extreme points; that is, the points where the function has a local maximum or minimum. The graph in Figure 5.11, below, shows the local extreme points of a function f.

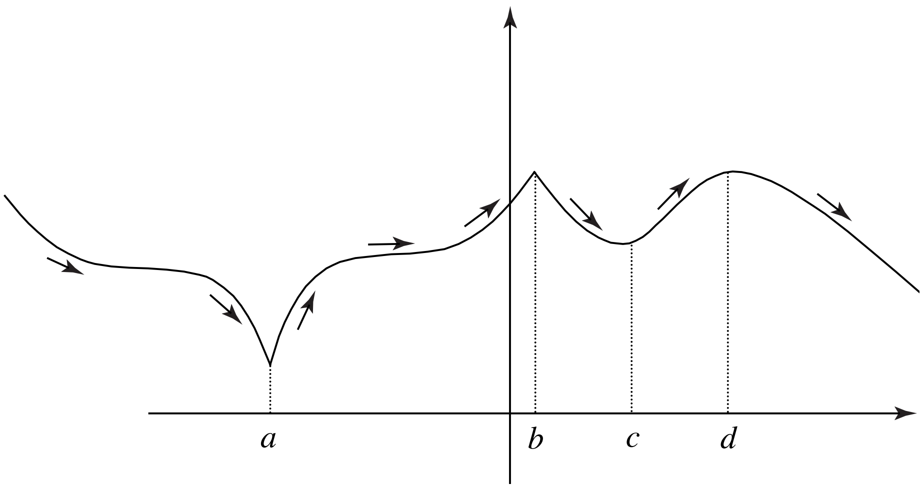


Figure 5.11. Local extreme points of a function f

REMARK 5.4

The critical points where the function changes from increasing to decreasing are the local maximum points; in this case, the function has local maxima at b and d, and the values $f(b)$ and $f(d)$ are the local maximum values of f. The points where the function changes from decreasing to increasing are the local minimum points; in this case, the function has local minima at a and c, and the values $f(a)$ and $f(c)$ are the local minimum values of f.

Definition 5.7. A continuous function on an interval J has

a. a *local maximum* or *relative maximum* (abbreviated loc max) at a in J if

$$f(a) \geq f(x) \text{ for any } x \text{ in } J \text{ such that} |x-a| < \delta \text{ for some } \delta > 0.$$

and $f(a)$ is the *local maximum value* of f at a.[1]

b. a *local minimum* or *relative minimum* (abbreviated loc min) at b in J if

$$f(b) \leq f(x) \text{ for any } x \text{ in } J \text{ such that} |x-b| < \delta \text{ for some } \delta > 0.$$

and $f(b)$ is the *local minimum value* of f at b.

The local maximum and minimum values are called the *local or relative extreme values* of f on J.

REMARKS 5.5

a. Observe that in Definition 5.7, the first condition says that $f(a) \geq f(x)$ for any x that is in an open subinterval in J containing a, and that is near a.

b. Similarly, the second condition says that $f(b) \leq f(x)$ for any x that is in an open subinterval of J containing b, and that is near b.

Example 5.28. From the basic graphs of functions in Table 2 (unit02-01.htm#table2-1), $f(x) = x^2$ and $g(x) = |x|$ have only one local minimum, at 0, and that $f(0) = 0 = g(0)$ is the local minimum value.

The sine function has infinitely many local maxima and minima, at

$$\frac{(4k+1)\pi}{2} \quad \text{and} \quad \frac{(4k-1)\pi}{2},$$

respectively, for any integer k. The local minimum value is -1 and the local maximum value is 1.

The functions $h(x) = x^3$, tangent and cotangent do not have local extreme points.

Example 5.29. In Example 5.26 (unit05-02.htm#example5-26), we found the intervals of decrease and increase of the function

$$y = x^3 - 6x^2 - 15x + 4.$$

The function has a local maximum at -1 with local maximum value of $y(-1) = 12$, and a local minimum at 5 with minimum local value of $y(5) = -96$.

Example 5.30. The function

$$y = \frac{x}{x^2 + 4x + 3}$$

of Example 5.27 (unit05-02.htm#example5-27) has a relative minimum at $-\sqrt{3}$ with relative minimum value of

$$f(-\sqrt{3}) = \frac{-\sqrt{3}}{6 - 4\sqrt{3}}.$$

Although the function changes from increasing to decreasing at -3, the function does not have a relative maximum at this point, because it is not defined at this point.

Exercises

12. Explain why the functions in Examples 5.24 (unit05-02.htm#example5-24) and 5.25 (unit05-02.htm#example5-25) do not have local extreme points.
13. Read Examples 1 and 4 on page 222 of the textbook.

Answers to Exercises (appendix-a.htm)

From Figure 5.11, we can see that the following result is plausible, and we will prove it.

Theorem 5.8. If a function f has a local maximum or local minimum at c, then c is a critical number of f, that is either $f'(c) = 0$ or $f'(c)$ does not exist.

Proof. If we assume that $f'(c)$ exists, then we must explain why $f'(c) = 0$.

By Definition 5.7, above, if f has a minimum at c, then $f(c) \leq f(c+h)$ for any h such that $|h| < \delta$ for some $\delta > 0$. So, for this h,

$$f(c+h) - f(c) \geq 0$$

Observe that $c + h$ is near c from the right if $h > 0$ and from the left if $h < 0$. Hence,

$$\frac{f(c+h) - f(c)}{h} \geq 0 \text{ for } h > 0 \quad \text{and} \quad \frac{f(c+h) - f(c)}{h} \leq 0 \text{ for } h < 0.$$

Then, by Theorem 3.43 (unit03-07.htm#theorem3-43), we conclude that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \geq \lim_{h \to 0^+} 0 = 0$$

and

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^-} 0 = 0$$

Since $f^{'}(c)$ exists,

$$f^{'}(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = 0.$$

The proof for when f has a maximum is given on page 224 of the textbook.

Q.E.D.

The maximum and minimum values of a function correspond to its optimum values. For instance, if $C(x)$ is a cost function of certain commodity x, then the local maximum and minimum values of C on an interval J correspond to the maximum and minimum costs when x is limited to the interval J. The highest and lowest costs of C correspond to the absolute maximum and minimum values of the function C on the interval J.

Definition 5.9.

a. A function f has an *absolute maximum* at c on the interval J, if

$$f(c) \geq f(x) \quad \text{for any } x \text{ in } J.$$

The value $f(c)$ is the *absolute maximum value* (abbreviated abs max val) of f on J.

b. A function f has an *absolute minimum* at c on the interval J, if

$$f(c) \leq f(x) \quad \text{for any } x \text{ in } J.$$

The value $f(c)$ is the *absolute minimum value* (abbreviated abs min val) of f on J.

The absolute maximum and minimum values of f are called the *absolute extreme values* of f.

Compare Definitions 5.7 and 5.9, and note that if a function has absolute extreme values on an interval, they are unique. Conversely, a function may have several local extreme values on an interval. A function may or may not have absolute extreme values on its domain.

Example 5.31. The function $f(x) = x^2$ has an absolute minimum at 0 (with an absolute minimum value of $f(0) = 0$), but does not have an absolute maximum value on its domain.

Example 5.32. The function $f(x) = x^3$ does not have absolute extreme values on its domain. However, it does have them in the closed interval $[-3, 4]$ because $f(-3) \leq f(x)$ and $f(4) \geq f(x)$ for any x in the interval $(-3, 4)$.

Observe that the absolute extreme values $f(-3) = -27$ and $f(4) = 64$ are attained at numbers not in the open interval $(-3, 4)$. [Remember that the end points of an open interval are not included in the interval.]

Example 5.33. Although the function $f(x) = 1 / x$ is positive on the interval $[0, \infty)$—that is $f(x) > 0$ for all x in $(0, \infty)$—it does not have an absolute minimum value, because there is no c such that $f(c) \leq f(x)$ for all x in $(0, \infty)$.

Example 5.34. The absolute maximum value of the cosine function in its domain is 1, and is attained at infinitely many values $2k\pi$, for any integer k.

However, in the interval $\left[\frac{\pi}{3}, \pi\right]$ the absolute maximum value is $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Example 5.35. A function may have local extreme values, but no absolute extreme ones. In the graph shown in Figure 5.12, there is one local maximum and one local minimum, but no absolute extreme values on the domain.

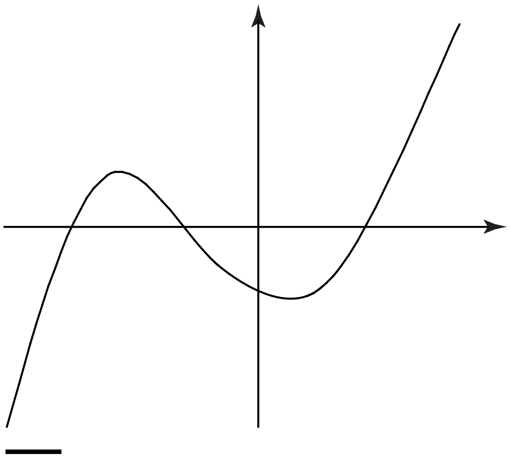


Figure 5.12. Function with local, but not absolute, extreme values on its domain

The following theorem gives conditions for a function to have absolute extreme values.

Theorem 5.10. The Extreme Value Theorem

A continuous function f on a closed interval $[a, b]$ has absolute extreme values in the interval; that is, there are numbers c and d in the interval $[a, b]$ such that

$$f(c) \geq f(x) \quad \text{and} \quad f(d) \leq f(x) \text{ for all } x \text{ in } [a, b].$$

The Extreme Value Theorem assure us that a continuous function on a closed interval has absolute extreme values, and from Figure 5 on page 223 of the textbook, we know that we should look for them at the endpoints, or at the points where the function has critical points. So, to find them, we use the closed interval method, described below.

The Closed Interval Method

Step 1
Find the critical points of the function on the open interval (a, b), if any.

Step 2
Evaluate the function at the critical points found in Step 1, above.

Step 3
Evaluate the function at the endpoints a and b.

Step 4
Identify the biggest and smallest values found in Steps 2 and 3, above, as the absolute maximum and minimum values, respectively.

Example 5.36. The function

$$f(x) = \frac{x}{x^2 - 4}$$

is continuous on $[3, 6]$.

By the closed interval method

1. $f^{'}(x) = -\dfrac{x^2 + 4}{\left(x^2 - 4\right)^2} \neq 0$ for all x in $[3, 6]$.
2. As a result, there are no critical points, and the absolute extreme values are at the endpoints.

3. $f(3) = \frac{3}{5}$ and $f(6) = \frac{3}{16}$
4. The absolute maximum value is $\frac{3}{5}$ and the absolute minimum value is $\frac{3}{16}$.

Example 5.37. The function $g(x) = x^3 - 6x^2 + 9x + 2$ is continuous on the interval $[-1, 2]$.

a. $g'(x) = 3x^2 - 12x + 9 = 3(x - 3)(x - 1) = 0$ if $x = 3$ and $x = 1$.
b. The only critical point in the interval $[-1, 2]$ is $x = 1$, thus $g(1) = 6$.
c. $g(-1) = -14$ and $g(2) = 4$.
d. The absolute maximum value is 6 (at $x = 1$), and the absolute minimum value is -14 (at $x = -1$).

Exercises

14. Read the section titled “Maximum and Minimum Values,” pages 221-227 of the textbook.
15. Do Exercises 3, 4, 5, 7, 11, 13, 15, 17, 19, 21, 23, 45, 47, 49 and 51, on pages 227-229.

Answers to Exercises (appendix-a.htm)

FOOTNOTE

[1] For a discussion of the meaning of δ in this context, read through the section “The Definition of a Limit (unit03-07.htm#limit-definition)” in Unit 3.

Concavity and Points of Inflection

Prerequisites

To complete this section, you must be able to

1. write inequalities in interval notation. See the section titled “Intervals” on pages 337-338 of the textbook.
2. solve inequalities. See the section titled “Inequalities” on pages 338-340 of the textbook. Do Exercises 13-20 on page 343.
3. use absolute values to solve inequalities. See the section titled “Absolute Value” on pages 340-343 of the textbook. Do Exercises 1-10 on page 343.
4. factor algebraic and trigonometric functions. See Unit 1 in this *Study Guide* and the PDF document titled “Review of Algebra,” available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)

A function may be concave up or down. In Figure 5.13, below, the function is concave up (abbreviated CU) on the interval $[a, b]$ and concave down (abbreviated CD) on the interval $[b, c]$.

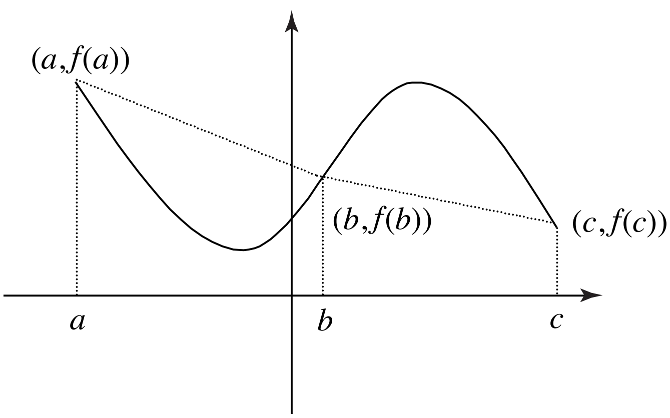


Figure 5.13. Function showing concave up and concave down segments

Definition* 5.11.

1. A continuous function f is *concave up* or *convex* on an interval J if, for all a and b in J, the line segment joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f.
2. A continuous function f is *concave down* or *concave* on an interval J if, for all a and b in J, the line segment joining $(a, f(a))$ and $(b, f(b))$ lies below the graph of f.

The graphs in Figure 5.14, below, show two functions that are concave up and down on the interval $[c, d]$. Observe that these functions satisfy the conditions of Definition 5.11 for *any* a and b in $[c, d]$.

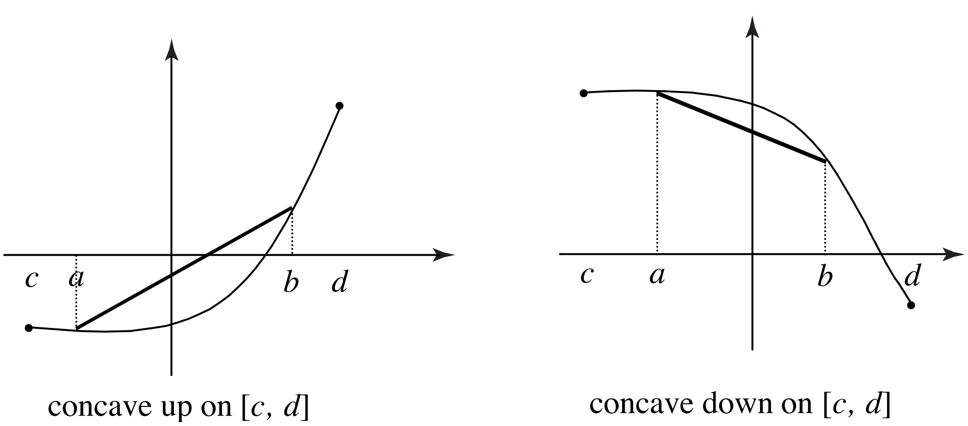


Figure 5.14. Functions that are concave up and concave down on the interval $[c, d]$

The graphs in Figure 5.15, below, show two functions: one is concave up on the interval (c, d), and the other is concave down on the interval $(-\infty, d)$. Observe that these functions satisfy the conditions of Definition 5.11 for *any* a and b in the corresponding intervals.

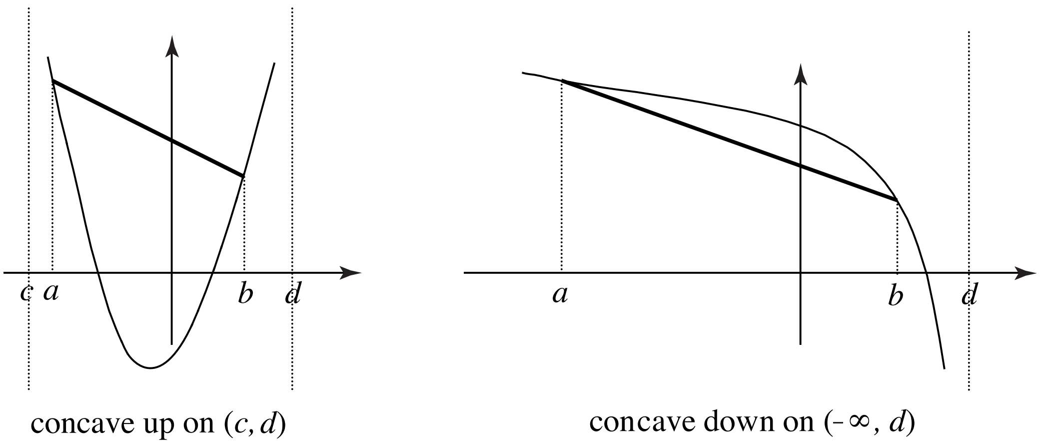


Figure 5.15. Functions that are concave up on the interval (c, d) and concave down on the interval $[-\infty, d)$

The graphs in Figure 5.16, below, show two functions: one is concave up on the interval $[c, d)$, and the other is concave down on the interval $[c, \infty)$. Observe that these functions satisfy the conditions of Definition 5.11 for *any* a and b in the corresponding intervals.

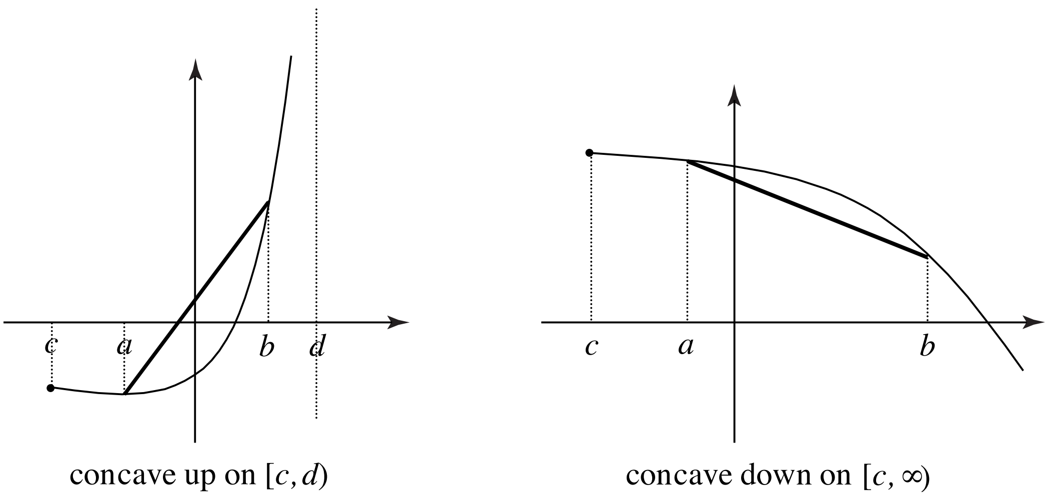


Figure 5.16. Functions that are concave up on the interval $[c, d)$ and concave down on the interval $[c, \infty)$

The graphs in Figure 5.17, below, show two functions: one is concave up on the interval $(c, d]$ and the other is concave down on the interval $(-\infty, d])$. Observe that these functions satisfy the conditions of Definition 5.11 for *any* a and b in the corresponding intervals.

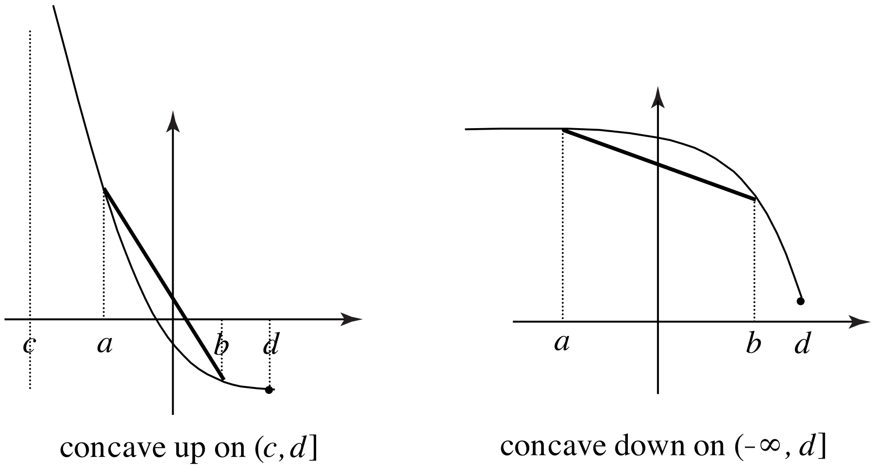


Figure 5.17. Functions that are concave up on the interval $(c, d]$ and concave down on the interval $(-\infty, d]$

Compare Definition 5.11 of this unit, with the equivalent informal definition given on page 213 of the textbook. Note that a function *f* is concave up (down) if and only if the graph of *f* lies above (below) each tangent line (wherever they exist), except at the point of contact.

Example 5.38. The quadratic function $f(x) = x^2$ is concave up on its domain.

The cubic function $g(x) = x^3$ is concave down on $(-\infty, 0]$, and concave up on the interval $[0, \infty)$.

Example 5.39. The tangent function is concave down on the interval $\left(-\frac{\pi}{2}, 0\right]$, and concave up on the interval $\left[0, \frac{\pi}{2}\right)$

To find the intervals of concavity—that is, where the function is concave up or down—we must observe that if the function is concave up, then the tangent lines are increasing.

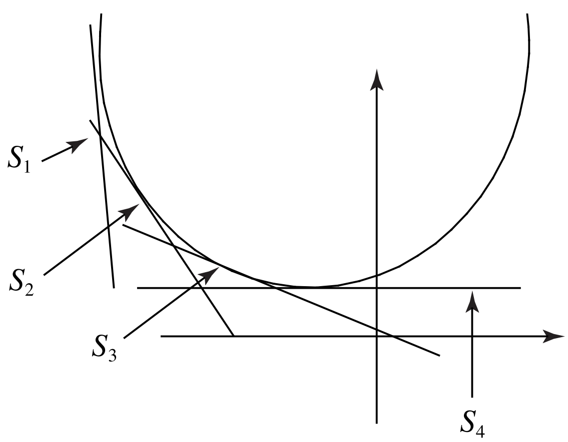


Figure 5.18. Function that is concave up; tangent lines have negative slope

In Figure 5.18, above, the slopes of the tangent lines S_1, S_2 and S_3 are negative, $S_4 = 0$, and $S_1 < S_2 < S_3 < S_4 = 0$.

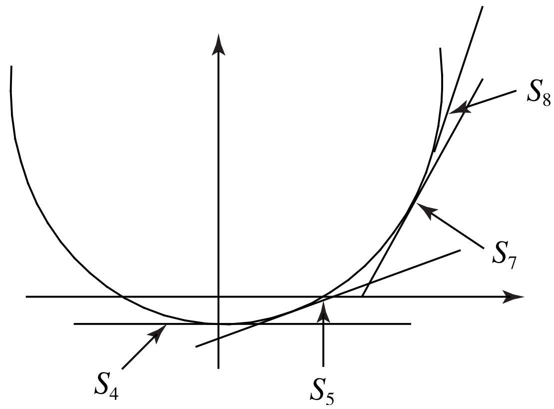


Figure 5.19. Function that is concave up; tangent lines have positive slope

In Figure 5.19, above, the slopes of the tangent lines are positive, $S_4 = 0$, and $0 = S_4 < S_5 < S_6 < S_7 < S_8$.

In both figures, the slopes are increasing.

Therefore, when the function f is concave up, its derivative function f' is increasing, and the derivative of f'—that is the second derivative function, f''—is positive.

If the function is concave down, then the tangent lines are decreasing.

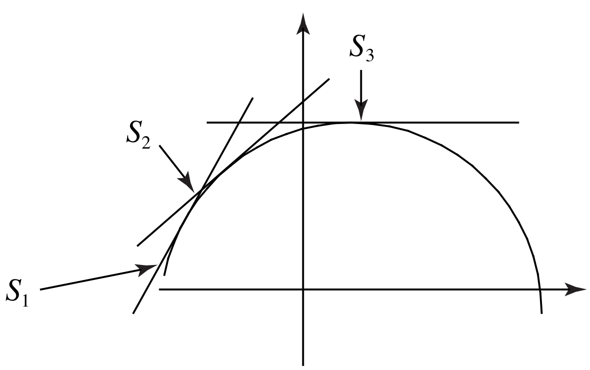


Figure 5.20. Function that is concave down; tangent lines have positive slope

In Figure 5.20, above, the slopes of the tangent lines S_1 and S_2 are positive, $S_3 = 0$ and $S_1 > S_2 > S_3 = 0$.

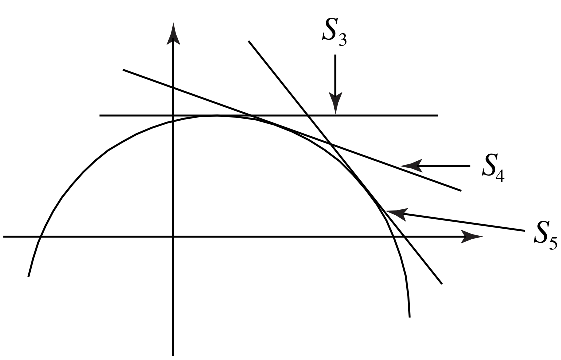


Figure 5.21. Function that is concave down; tangent lines have negative slope

In Figure 5.21, above, the slopes of the tangent lines S_4 and S_5 are negative, $S_3 = 0$ and $0 = S_3 > S_4 > S_5$.

In both figures the slopes are decreasing.

When the function f is concave down, its derivative function $f^{'}$ is decreasing, and the derivative of $f^{'}$—that is, the second derivative function, f''—is negative.

These observations result in the following concavity test.

Definition* 5.12. The Concavity Test

1. If $f''(x) > 0$ for all x on an interval J, then the function is concave up on the interval J.
2. If $f''(x) < 0$ for all x on an interval J, then the function is concave down on the interval J.

Clearly, therefore, to find the concavity of a function f, we must find the intervals where the second derivative is positive or negative. To do so, we

1. find the critical numbers of the derivative function (i.e., all c in the domain of $f^{'}$, such that either $f''(c) = 0$ or $f''(c)$ does not exist); then,
2. simplify and factor the second derivative function as much as possible so that we can
3. make up a table with the intervals given by the critical numbers and the factors that determine the sign of the second derivative function.

Example 5.40. In Example 5.24 (unit05-02.htm#example5-24), we found that the derivative of

$$f(x) = \sqrt{x^2 - 9}$$

is

$$f^{'}(x) = \frac{x}{\sqrt{x^2 - 9}}.$$

Hence,

$$f''(x) = \frac{\sqrt{x^2 - 9} - x\left(\frac{x}{\sqrt{x^2 - 9}}\right)}{x^2 - 9} = -\frac{9}{\sqrt{(x^2 - 9)^3}}$$

We see that $f''(x) < 0$ for all x in the domain of f, so the function is concave down in its domain.

Example 5.41. In Example 5.26 (unit05-02.htm#example5-26), we found the derivative

$$y^{'} = 3x^2 - 12x - 15 = 3\left(x^2 - 4x - 5\right).$$

Hence,

$$y''(x) = 6x - 12 = 6(x - 2) = 0 \text{ for } x = 2,$$

$$y''(x) > 0 \text{ for } x > 2$$

and

$$y''(x) < 0 \text{ for } x < 2.$$

So the function is concave up on the interval $(2, \infty)$, and concave down on the interval $(-\infty, 2)$

Example 5.42. Check that the second derivative function of

$$g(x) = x\sqrt{x + 2}$$

is

$$g''(x) = \frac{3x + 8}{4(x + 2)^{3/2}}.$$

So, $g''(x) = 0$ if $x = -\dfrac{8}{3}$.

The domain of the function is the interval $[-2, \infty)$, and $x = -\dfrac{8}{3}$ is not on this interval. We have $g''(x) > 0$ if $x \geq -2$.

Hence, the function is concave up on the interval $[-2, \infty)$.

Example 5.43. Check that the second derivative of the function

$$h(x) = x^6 - 3x^4 + 3x^2$$

is

$$h''(x) = 6\left(5x^4 - 6x^2 + 1\right).$$

We factor this function, using the quadratic formula, setting $u = x^2$, so that

$$5x^3 - 6x^2 + 1 = 5u^2 - 6u + 1 = (u-1)(5u-1)$$

and

$$h''(x) = 6(x^2-1)(5x^2-1) = 6(x+1)(x-1)(\sqrt{5}x+1)(\sqrt{5}x-1).$$

The critical points of h' are

$$x = 1, \quad x = -1, \quad x = \frac{1}{\sqrt{5}} \quad \text{and} \quad x = -\frac{1}{\sqrt{5}}.$$

The table we must create to find the sign of h'' is shown below.

Summary of the steps used to find the sign of the second derivative $h''(x)$ of the function $h(x) = x^6 - 3x^4 + 3x^2$

Interval	$x-1$	$x+1$	$x-\frac{1}{\sqrt{5}}$	$x+\frac{1}{\sqrt{5}}$	h''	h
$x < -1$	$-$	$-$	$-$	$-$	$+$	CU
$-1 < x < -\frac{1}{\sqrt{5}}$	$-$	$+$	$-$	$-$	$-$	CD
$-\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$	$-$	$+$	$-$	$+$	$+$	CU
$\frac{1}{\sqrt{5}} < x < 1$	$-$	$+$	$+$	$+$	$-$	CD
$x > 1$	$+$	$+$	$+$	$+$	$+$	CU

Hence, the function is concave up on the intervals

$$(-\infty, -1), \left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \quad \text{and} \quad (1, \infty);$$

it is concave down on the intervals

$$\left(-1, -\frac{1}{\sqrt{5}}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{5}}, 1\right).$$

REMARK 5.6

Remark 5.3 (unit05-02.htm#remark5-3) also applies here, since we may have to solve $f''(x) = 0$ to find the inflection points of the function f.

The inflection points of a function are the points in the domain of the function where the concavity changes from up to down or down to up. For example, in Figure 5.13, we can see that $(b, f(b))$ is an inflection point.

Definition 5.13. A continuous function f on an interval J has an *inflection point* at $(a, f(a))$ if the function is concave up (down) on some subinterval (c, a) of J and it is concave down (up) on some subinterval (a, b) of J.

Example 5.44. By Example 5.40 (unit05-04.htm#example5-40), the function in Example 5.24 (unit05-02.htm#example5-24) has no inflection points.

Example 5.45. By Example 5.41 (unit05-04.htm#example5-41), the function in Example 5.26 (unit05-02.htm#example5-26) has an inflection point at $(2, y(2)) = (2, -42)$

Example 5.46. By Example 5.42, the function of $g(x) = x\sqrt{x+2}$ has no inflection points.

Observe that at an inflection point is precisely the instant at which the derivative of the function changes from increasing (decreasing) to decreasing (increasing). This fact has two important consequences:

1. at an inflection point, the *rate of change* changes from increasing (decreasing) to decreasing (increasing).
2. at an inflection point, the second derivative function has a local extreme point.

Example 5.47. If the function $y = x^3 - 6x^2 - 15x + 4$ in Example 5.26 (unit05-02.htm#example5-26) is the position function of an object, then, by Example 5.41 (unit05-04.htm#example5-41), at time $x = 2$, the velocity of the object changed from increasing to decreasing. The acceleration is at its maximum at this instant.

Geometrically the second derivative of this function has a local maximum at $x = 2$ (see Example 5.41 (unit05-04.htm#example5-41)).

Exercises

16. Use Example 5.43 to determine the inflection points of the function $h(x) = x^6 - 3x^4 + 3x^2$.
17. If $h(x) = x^6 - 3x^4 + 3x^2$ is the displacement of a moving particle, use Example 5.43 to find the instant or instants at which the acceleration is at the maximum and minimum.
18. Read Examples 1, 2, 3 and 4, on pages 210-212 and 214 of the textbook.
19. Do Exercises 1, 2, 11, 13, 29, 31, 35, 37 and 47, on pages 217-219.

Answers to Exercises (appendix-a.htm)

Graph Sketching

Sketching involves two steps: the first involves obtaining information from the function, as outlined in earlier sections of this unit; the second involves interpreting the information in terms of the graph of the function. Once we have all the information we need, we proceed as follows:

Step 1
Plot the x and y intercepts, the local maxima and minima, and the inflection points.

Step 2
Draw the asymptotes using a dotted line.

Step 3
If there is a symmetry, sketch the graph on the right side of the y-axis first (see Step 6, below).

Step 4
Identify the intervals of increase and decrease, and the concavity.

Step 5
Join the plotted points according to the shape of the graph given by the increase and decrease and concavity.

Step 6
If there is a symmetry, reflect the graph plotted. Note that there may be more than one symmetry.

Example 5.48. We have been obtaining information about the function

$$f(x) = \sqrt{x^2 - 9}$$

(see Example 5.1 (unit05.htm#example5-1), and we present it in the table below.

Summary of the information about function $f(x) = \sqrt{x^2 - 9}$ collected from previous examples in this unit

Information from f	**Interpretation for the graph of f**	**Example**
Domain $(\infty, 3] \cup [3, \infty)$	no vertical asymptote no graph between the lines $x = -3$ and $x = 3$ no y-intercept	5.1 (unit05.htm#example5-1)
$f(3) = 0 = f(-3)$	x intercepts: $(3, 0)$ and $(-3, 0)$	5.7 (unit05.htm#example5-7)
$f(-x) = f(x)$	even	
$f'(x) < 0$ on $(-\infty, -3)$ $f'(x) > 0$ on $(3, \infty)$	dec on $(-\infty, -3)$ inc on $(3, \infty)$ no local extreme points	5.24 (unit05-02.htm#example5-24)
$f''(x) < 0$ on domain	concave down no inflection points	5.40 (unit05-04.htm#example5-40)

The information from the table takes us to the graph in Figure 5.22, below.

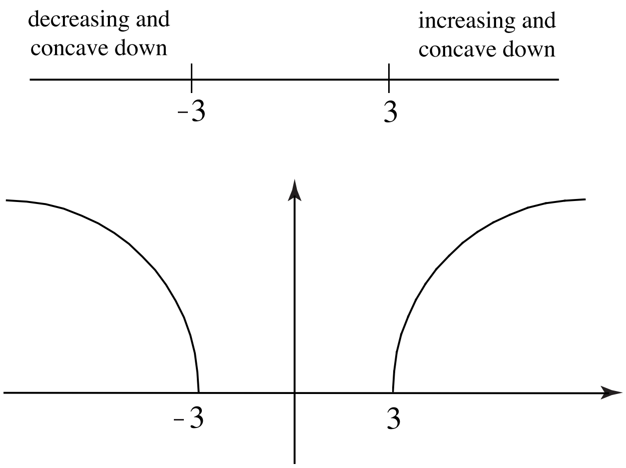


Figure 5.22. Graph of the function $f(x) = \sqrt{x^2 - 9}$

Example 5.49. For the function

$$y = x^3 - 6x^2 - 15x + 4$$

(see Example 5.26 (unit05-02.htm#example5-26)), we use the Intermediate Value Theorem (unit05.htm#theorem5-2) to locate the x-intercepts.

Since

$$y(-3) = -32 < 0 < 2 = y(-2)$$

and

$$y(1) = -16 < 0 < 4 = y(0),$$

the function has two x-intercepts in the intervals $(-3, -2)$ and $(0, 1)$. The y-intercept is $(0, 4)$. You can check that $y(-x)$ is not equal to either $y(x)$ or $-y(x)$. Hence, the function is not symmetric.

The rest of the information is presented in the table below.

Summary of the information about function $y = x^3 - 6x^2 - 15x + 4$ collected from previous examples in this unit

Information from y	**Interpretation for the graph of y**	**Example**
Polynomial	continuous everywhere no asymptotes	
$y(-3) < 0 < y(-2)$ $y(1) < 0 < y(0)$	x intercepts in $(-3, -2)$ and $(0, 1)$	
$y(-x) \neq y(x)$ $y(-x) \neq -y(x)$	no symmetries	
$f'(x) > 0$ on $(-\infty, -1)$ and $(5, \infty)$ $f'(x) < 0$ on $(-1, 5)$	inc on $(-\infty, -1)$ and $(5, \infty)$ dec on $(-1, 5)$ rel max $(-1, 12)$ rel min $(5, -96)$	5.26 (unit05-02.htm#example5-26)
$f''(x) > 0$ on $(2, \infty)$ $f''(x) < 0$ on $(-\infty, 2)$	CU on $(2, \infty)$ CD on $(-\infty, 2)$ inflection point at $(2, -42)$	5.41 (unit05-04.htm#example5-41) 5.45 (unit05-04.htm#example5-45)

The information from the table takes us to the graph in Figure 5.23, below.

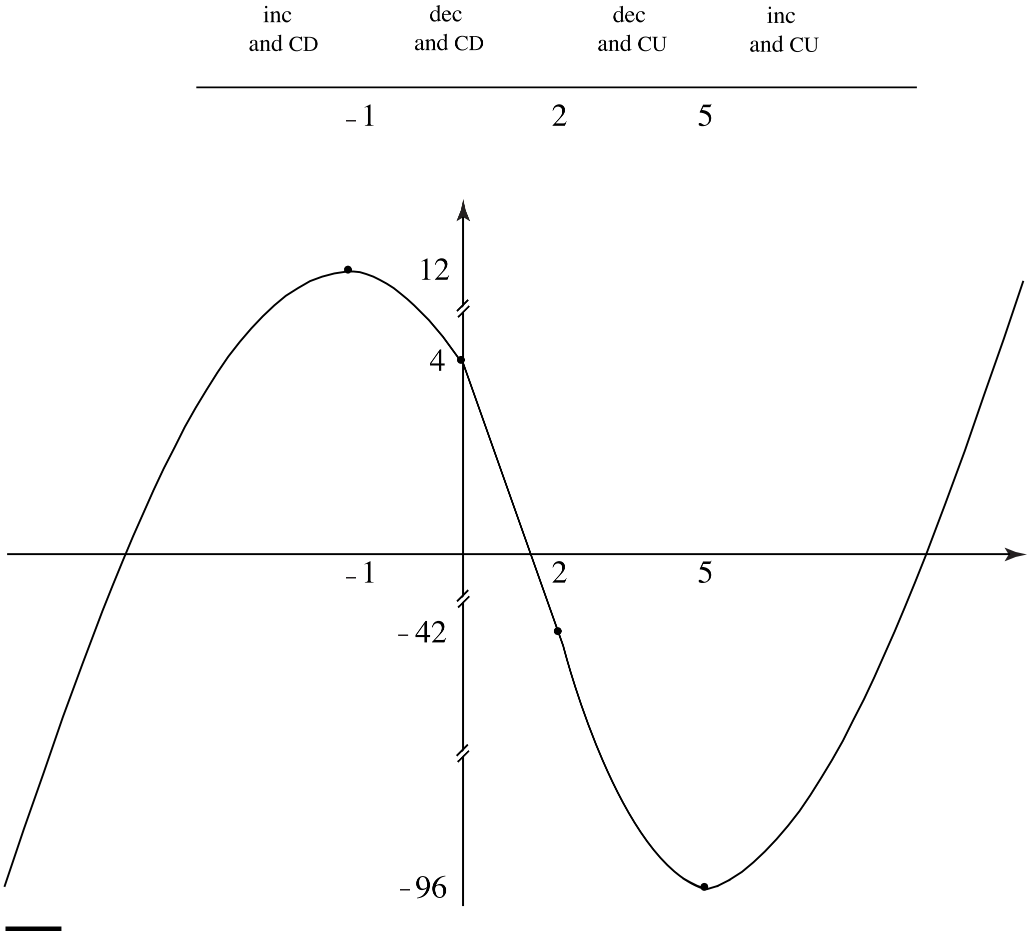


Figure 5.23. Graph of function $y = x^3 - 6x^2 - 15x + 4$—not to scale

Example 5.50. [See Exercise 24 on page 218 of the textbook.]

The interpretation of the information given about the graph is summarized in the table below.

Summary of the interpretation of the information given by the graph of the function $y = x^3 - 6x^2 - 15x + 4$

Information from f	**Interpretation for the graph of f**
$f^{'}(1) = f^{'}(-1) = 0$	horizontal tangent line at $f(1)$ and $f(-1)$
$f^{'}(x) < 0$ on $\|x\| < 1$ $f^{'}(x) > 0$ on $1 < \|x\| < 2$ $f^{'}(x) = -1$ on $\|x\| > 2$	dec on $(-1, 1)$; inc on $(1, 2)$ and $(-2, -1)$ dec on $(-\infty, 2)$ and $(2, \infty)$ $f(x) = -x$ on $(-\infty, 2)$ and $(2, \infty)$ loc min at $x = -2$, $x = 1$ loc max at $x = -1$, $x = 2$ dec and linear on $(-\infty, -2)$ and $(2, \infty)$
$f''(x) < 0$ on $-2 < x < 0$	CD on $(-2, 0)$ CU on $(0, 2) \cup (2, \infty)$ inflection point at $(0, 1)$

The information from the table takes us to the graph in Figure 5.24, below.

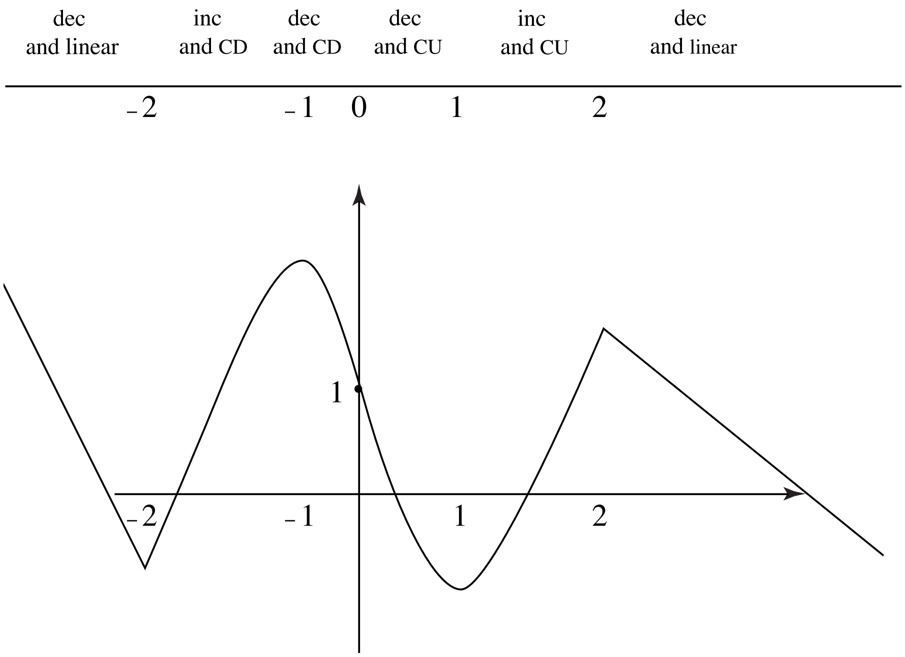


Figure 5.24. Graph of the function satisfying the conditions in Exercise 24, page 218 of the textbook—not to scale

Exercises

20. Do Exercises 51-54 on page 96 of the textbook.
21. Read Examples 6 and 7 on pages 215-216.
22. Do Exercises 23, 25, 26, 29, 31 and 35 on page 218.
23. Read Examples 1, 2 and 4 on pages 234-235 and 237.
24. Do Exercises 11, 13, 17, 19, 21 and 23 on page 238.

Answers to Exercises (appendix-a.htm)

Optimization Problems

Prerequisites

To complete this section, you must be able to

1. state and apply Pythagoras' Theorem.
2. determine when two triangles are similar.
3. state and apply the laws of sines and cosines.
4. apply geometric formulas of areas, volumes and surface areas of regular solids.

As we have seen (see Definition 5.9 (unit05-03.htm#definition5-9)), the extreme values of a function correspond to the optimum values represented by the function. The type of problem we solve in this section requires us to find the maximum or the minimum values—which may turn to be the best or the worse, the largest or the smallest, etc.—of a function. That is, we must find the optimum value for a given situation.

The key to solving optimization problems is establishing the quantity f that we want to maximize or minimize, and then constructing a function $f(x)$ that depends on a single quantity, x. Thus, if we want to maximize or minimize f, we look for the maximum or minimum value of f for certain range of x. There are basically three ways to find these values. We have already considered two of them: finding the intervals of increase and decrease (see Remark 5.4 (unit05-03.htm#remark5-4)), and the closed interval method (unit05-03.htm#closed-interval-method). The third method is known as the second derivative test.

Second Derivative Test

Let f'' be continuous at c. Then,

- if $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

The methods we discussed earlier are always conclusive; however, the second derivative test may not be conclusive if $f''(c) = 0$. For instance, for $f(x) = x^3$ and $g(x) = x^4$, it is true that $f'(0) = 0$ and $g'(0) = 0$, but it is also true that $f''(0) = 0$ and $g''(0) = 0$. In the case of the function f, there is no local extreme value at 0, and in the case of g, there is a local minimum at 0. Hence, when $f''(c) = 0$, we must use one of the other methods to find the extreme values.

Remember that the closed interval method only applies to continuous functions on closed intervals.

The graph of the quadratic function $f(x) = ax^2 + bx + c$ with $a \neq 0$ is a parabola; hence, it has either an absolute maximum or an absolute minimum. We can use the second derivative test to see that this statement is true:

$$f'(x) = 2ax + b;$$

hence,

$$f'(x) = 0 \text{ if } x = -\frac{b}{2a}$$

and

$$f''(x) = 2a.$$

So, at

$$x = -\frac{b}{2a},$$

the function f has an absolute minimum, if $a > 0$, or an absolute maximum, if $a < 0$.

Example 5.51.

a. The function $f(x) = 3x^2 - 7x$ has an absolute minimum, because $a = 3 > 0$, and that point is at

$$x = -\frac{-7}{2(3)} = \frac{7}{6}.$$

So, the absolute minimum value is

$$f\left(\frac{7}{6}\right) = -\frac{49}{12}.$$

b. The function $g(x) = (x - 2)(4 - x)$ has an absolute maximum. [Why?] This value occurs at

$$x = \frac{-6}{2(-1)} = 3,$$

and the absolute maximum value is $g(3) = 1$.

Example 5.52. If we consider the function $f(x) = 3x^2 - 7x$ restricted to the interval $[2, 8]$, we see from Example 5.51 that the absolute minimum is outside this interval. Hence, by the closed interval method, the absolute extreme values are at the endpoints: $f(2) = -2$ is the absolute minimum value, and $f(8) = 136$ is the absolute maximum value.

Optimization problems always have limitations—that is, constraints. These constraints lead to a *constraint equation* that we must often establish to obtain a function $f(x)$ of only one variable x.

On page 240 of the textbook, the author presents a suggested strategy for solving optimization problems. Read it carefully, before continuing with the next examples.

Example 5.53. Read Example 1 on pages 240-241 of the textbook. In this Example the question is, "What are the dimensions of the field that has the largest area?" This tells us that we want to maximize the area A of the field.

The area of the field depends on the dimensions x (width) and y (length); that is,

$$A(x, y) = xy.$$

Since we need a function of A with only one variable, we use the constraint conditions; in this case, the fenced perimeter of the field must be (is limited to) 2400 ft. Hence, the constraint equation is

$$2x + y = 2400.$$

To obtain a function A of one variable, we solve for either x or y from the constraint equation, and we substitute it into the function $A(x, y)$. In this case, we chose y, and $A(x) = 2400x - 2x^2$. Observe that $0 < x < 1200$. [Why?]

The function $A(x)$ is a quadratic function; it has a maximum because the coefficient $a = -2$ is negative.

Thus, the absolute maximum for the function $A(x) = 2400x - 2x^2$ is at

$$x = -\frac{2400}{2(-2)} = 600$$

which is within the limits of 0 and 1200.

Note that $x = 600$ is one of the dimensions of the field that gives the maximum area; the other is $y = 2400 - 2(600) = 1200$.

This result answers the question: the dimensions are 600×1200 ft. The maximum area for these dimensions is $A(600) = 720\,000$ ft.

Compare this solution with the one given in your textbook.

We did not need to use the second derivative test, since we know where the maximum or minimum occurs in the case of quadratic functions.

Example 5.54. [See Exercise 3 on page 245 of the textbook.]

Find two positive numbers whose product is 100 and whose sum is a minimum.

We want to minimize the sum S of two positive numbers, say x and y; hence, $S(x, y) = x + y$ is to be minimized with the constraint that their product must be 100. Therefore, $xy = 100$ is the constraint equation.

Solving for y from this equation we find that

$$y = \frac{100}{x}.$$

Therefore,

$$S(x) = x + \frac{100}{x}$$

is a function of one variable, and $0 < x < 100$. [Why?]

We need to find the minimum of this function. By the second derivative test, we have

$$S^{'}(x) = 1 - \frac{100}{x^2} = 0 \quad \text{if} \quad x = 10$$

and

$$S''(10) = 200 \,/\, (10)^3 > 0,$$

so S has a minimum if $x = 10$ (which is within its range) and $y = 10$.

We cannot use the closed interval method, because x is on the open interval $(0, 100)$.

Example 5.55. [See Exercise 31(a) on page 246 of the textbook.]

A piece of wire 10 m long is cut into two pieces. One piece is bent into a square, and the other into a equilateral triangle. How should the wire be cut such that the total area is a maximum?

We want to maximize the total area A of the square and the triangle. [Recall, from Example 2.27 (unit02.htm#example2-27), that the height of an equilateral triangle of side y is $\frac{\sqrt{3}}{2}\,y$.]

If x is the side of the square and y is the side of the triangle, then

$$A(x, y) = x^2 + \frac{\sqrt{3}}{4}\,y^2.$$

The perimeters of the square and the triangle must be equal to the length of the wire; hence, $4x + 3y = 10$. Solving for y, we find that

$$y = \frac{10 - 4x}{3},$$

and the area is

$$A(x) = x^2 + \frac{\sqrt{3}(10 - 4x)^2}{36}.$$

Doing the operations and simplifying, we find

$$A(x) = \frac{9 + 4\sqrt{3}}{9}x^2 - \frac{20\sqrt{3}}{9}x + \frac{25\sqrt{3}}{9} \text{ for } 0 \leq x \leq 2.5.$$

The area function A is a quadratic with an absolute minimum, and we are looking for the absolute maximum of A. Thus, we use the closed interval method on $[0, 10]$. The maximum is at the endpoints, since the absolute minimum is at the critical point. However,

$$A(0) = \frac{25\sqrt{3}}{9} \approx 4.81 \quad \text{and} \quad A(2.5) = (2.5)^2 + \frac{\sqrt{3}(10 - 4(2.5))^2}{36} = \frac{25}{4}.$$

Therefore, the maximum area occurs when $x = 2.5$. This result means that the wire should be used to do only a square.

Note: Optimization problems may be challenging: you must identify your variables and use whatever you know that relates to the problem. Nothing is gained if you do not learn to overcome challenges. Remember that time and dimensions are positive variables.

For help with the concept of optimization, view the PowerPoint tutorials below. To access a tutorial:

1. Click on the file to open it.
2. Extract the files and save them to your computer's hard drive.
3. Click on the file folder to open it.
4. Click on the PowerPoint file to view and listen.

Note: If you don't have PowerPoint installed on your computer, you can download the PowerPoint Viewer from here: https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6 (https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6).

Tutorial 1: Optimization (../multimedia/powerpoint/optimz1.zip)

Tutorial 2: Optimization (../multimedia/powerpoint/optimz2.zip)

Exercises

25. Consider the polynomial $p(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$. Use the second derivative test to show that $p(x)$ has a local extreme value if $b^2 - 3ac \geq 0$.
26. Read Examples 2, 3, 4 and 5 on pages241-244 of the textbook.
27. Do at least the odd-numbered exercises from 3 to 33 on pages 245-246.

Answers to Exercises (appendix-a.htm)

Newton's Method

As we mentioned earlier, it is not always possible to solve algebraically the equation $f(x) = 0$ for some function f. But we have methods to locate or approximate the value of the exact solution. Newton's method is one of them. The idea is based on linear approximation, discussed in Unit 4. We use the lines that are tangent to the curve to construct a sequence that converges to a solution to the equation $f(x) = 0$.

A *sequence* is a set of numbers listed in a specific order. For instance, the sequence $\{1, 2, 3, 4, 5 \ldots\}$ is not equal to the sequence $\{2, 1, 3, 4, 5, 6 \ldots\}$—the set of numbers may be the same, but the order is different. The n-th term of a sequence is denoted with a subindex n; for example, x_n or s_n.

For the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\},$$

the first term is $x_1 = 1$, the second is $x_2 = 1 / 2$, the third is $x_3 = 1 / 3$, and the n-th term is $x_n = 1 / n$.

Intuitively, we say that a sequence $\{x_n\}$ converges to a number s if, for a large number n, the distance between s and x_n is very small. That is $s \approx x_n$ for a large n. The larger the n, the better the approximation, and we write

$$\lim_{n \to \infty} x_n = s.$$

The difficult part in Newton's method is to choose the first term of the sequence (i.e., x_1). In making our choice, we may want to use the graph of the function, the Intermediate Value Theorem (unit05.htm#theorem5-2), or trial and error. Once the first term is chosen, the others are deduced recursively by applying the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 5.56. Finding the value of $\sqrt[3]{30}$ is the same as finding an x such that $x^3 = 30$; that is, x is the solution of $x^3 - 30 = 0$.

Let us consider the function $f(x) = x^3 - 30$. To apply Newton's method, we must choose x_1.

Since $f(3) = 27 - 30 = -3$ we choose $x_1 = 3$.

We have $f'(x) = 3x^2$; hence,

$$x_{n+1} = x_n - \frac{(x_n)^3 - 30}{3(x_n)^2}$$

and

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{(3^3) - 30}{3(3)^2} = \frac{28}{9} \approx 3.111111.$$

$$x_3 = \frac{28}{9} - \frac{\left(\frac{28}{9}\right)^3 - 30}{3\left(\frac{28}{9}\right)^2} = \frac{32887}{10584} \approx 3.107237.$$

You can check that $x_4 \approx 3.10723251$.

Example 5.57. Let us find a solution of $x^3 - 3x + 6 = 0$.

At first, we consider $f(x) = x^3 - 3x + 6$, and we choose $x_1 = 1$. We have $f'(x) = 3x^2 - 3$. But $f'(1) = 0$, and we cannot divide by 0; thus, Newton's method is not applicable for this value of x_1.

However, we can see that $f(-3) = -12 < 0 < 10 = f(-2)$, and by the Intermediate Value Theorem (unit05.htm#theorem5-2), we know that one solution is in the interval $(-3, -2)$. Hence, we choose $x_1 = -3$, and

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 6}{3x_n^2 - 3}.$$

$$x_2 = -3 - \frac{(-3)^3 - 3(-3) + 6}{3(-3)^2 - 3} = -\frac{5}{2}.$$

So,

$$x_3 = -\frac{5}{2} - \frac{(-2.5)^3 - 3(-2.5) + 6}{3(-2.5)^2 - 3} \approx -2.365079.$$

So one solution is approximately equal to -2.3650.

You may want to find x_4.

In the examples above, we illustrated the application of Newton's method. Now you should read the section titled "Newton's Method," on pages 252-256 of the textbook to see how this method was developed.

Exercises

28. Read pages 252-256 of the textbook.
29. Do Exercises 15, 17 and 19 on page 256.

Answers to Exercises (appendix-a.htm)

Finishing This Unit

1. Review the objectives of this unit and make sure you are able to meet all of them.
2. If there is a concept, definition, example or exercise that is not yet clear to you, go back and reread it, then contact your tutor for help.
3. Do the exercises in "Learning from Mistakes" section for this unit.
4. You may want to do Exercises 1, 5, 13, 15, 17, 21, 23, 26, 38, 41, 49 and 51 from the "Review" (pages 260-262 of the textbook).

Learning from Mistakes

There are mistakes in each of the following solutions. Identify the errors, and give the correct answer.

1. Sketch the graph of the function $f(x) = (x - 1)^{3/2}$.

   Erroneous Solution

   $f^{'}(x) = 3/2(x - 1)^{1/2}$ the critical number is $x = 1$, because $f^{'}(1) = 0$.

   Since $f^{'}(x) > 0$ for all x, the function is increasing for all x.

   Since $f''(x) = \dfrac{3}{4(x-1)^{-1/2}}$, the critical number is $x = 1$, because $f''(1)$ is undefined.

   Since $f''(x) > 0$ for all x, the function is concave up for all x.

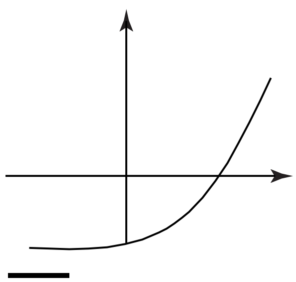
   

   Figure 5.25. Erroneous solution to "Learning from Mistakes" Question 1

2. Find the local extreme values of the function $g(x) = x^6 + 6x^4$.

   Erroneous Solution

   $g^{'}(x) = 6x^5 + 4x^3 = 2x^3\left(x^2 + 4\right) = 0$ if $x = 0$; therefore, $x = 0$ is the only critical number.

   $$g''(x) = 30x^4 + 12x^2 \quad \text{and} \quad g''(0) = 0.$$

   By the second derivative test, the function has neither a minimum nor a maximum; hence, it has an inflection point at $x = 0$
   .

3. Find the extreme values of the function $h(x) = x^3 - 6x$ on the interval $[0, 2]$.

   Erroneous Solution

   To find the critical points $h^{'}(x) = 3x^2 - 6 = 0$, we solve for x. The critical numbers are $x = \sqrt{2}$ and $x = -\sqrt{2}$. Therefore,

   $$h\left(\sqrt{2}\right) = \left(\sqrt{2}\right)3 - 6\sqrt{2} \approx -5.656$$

   and

   $$h\left(-\sqrt{2}\right) = \left(-\sqrt{2}\right)3 + 6\sqrt{2} \approx 5.656.$$

   At the endpoints $h(0) = 0$ and $h(2) = -4$.

   So the absolute minimum value is

   $$\sqrt{2}^3 - \frac{6}{\sqrt{2}} \approx -5.656 \text{ at } x = \sqrt{2},$$

   and the absolute maximum value is

$$-\sqrt{2}^3 + \frac{6}{\sqrt{2}} \approx 5.656 \text{ at } x = -\sqrt{2}.$$

4. Find a number greater than or equal to 2 such that the sum of the number and its reciprocal is as small as possible.

Erroneous Solution

Let x be such a number. Hence, we want to minimize the function $f(x) = x + \dfrac{1}{x}$.

$$f'(x) = 1 - \frac{1}{x^2} = 0 \text{ if } x = 1 \text{ or } x = -1.$$

Since

$$f''(x) = \frac{2}{x^3}$$

we have $f''(1) > 0$ and $f''(-1) < 0$. Then, by the second derivative test, f has a minimum at $x = 1$.

Integration

Objectives

When you have completed this unit, you should be able to

1. find antiderivative functions.
2. evaluate indefinite integrals.
3. evaluate definite integrals.
4. state the Mean Value Theorem and apply it to prove the Fundamental Theorem of Calculus.
5. explain and use the inverse relationship between differentiation and integration when applying the Fundamental Theorem of Calculus.

Integration is the inverse of the operation of differentiation, and as does the derivative, the integral has two different interpretations. In this unit, we consider the process of integration, and in Unit 7, we study the applications of integration that arise from these two different interpretations.

Antiderivatives

Prerequisites

To complete this section, you must be able to

1. do algebraic operations with functions, as explained in the section titled "Combination of Functions" on pages 34 and 35 of the textbook. See Unit 1 in this *Study Guide* and the PDF document titled "Review of Algebra," available through the website that accompanies your textbook:

   http://www.stewartcalculus.com/media/1_home.php (http://www.stewartcalculus.com/media/1_home.php)
2. state and apply trigonometric identities. See the addition and subtraction formulas, the double angle formulas and the half-angle formulas on the "Reference Pages" at the beginning of your textbook.

In this section, we do the opposite of differentiation; that is, given a function $f(x)$, we want to find a function $F(x)$ such that $F^{'}(x) = f(x)$.

Definition 6.1. A function $F(x)$ is the *antiderivative* of a function $f(x)$ defined on an interval J if $F^{'}(x) = f(x)$ for all x in the interval J.

Note: Observe that the antiderivative of the function $f'(x)$ is the function $f(x)$, and the antiderivative of $f''(x)$ is the function $f'(x)$.

Since we know how to differentiate, we can use trial and error to find antiderivatives of several functions.

Example 6.1.

a. The antiderivative of a zero constant function $f(x) = 0$ is any constant function $F(x) = C$, because $F^{'}(x) = 0 = f(x)$. Later, we prove that $F(x)$ is the only antiderivative of the zero constant function. That is, if the derivative of a function F is zero, then $F(x) = C$ for some constant C.

   Since C is any constant, this example shows that the antiderivative is not unique, there are many antiderivatives for a given function.

b. The antiderivative of any constant function $f(x) = m$ is the function $F(x) = mx$, because $F^{'}(x) = m = f(x)$, but it also the function $F_1(x) = mx + 6$, because its derivative is equal to m, as is the derivative $F_2(x) = mx - 123$. As before, we have many antiderivatives, but the only difference among them is a constant. [We prove this statement later.] Hence, the most general antiderivative of $f(x) = m$ is the function $F(x) = mx + C$, for any constant C.

We use rules of differentiation to find derivative functions, but there are no rules for finding antiderivatives. Instead, we must rely on what we know about differentiation, paying attention to particular cases, to generalize whenever possible.

Example 6.2.

a. The antiderivative of the function $f(x) = 2x$ is a function $F(x)$ such that $F^{'}(x) = 2x$.

   If we make a search among all the functions whose derivatives we know, we realize that the derivative of $F(x) = x^2$ is $f(x) = 2x$.

   Hence, the most general antiderivative of $f(x) = 2x$ is $F(x) = x^2 + C$.

b. The antiderivative of $f(x) = x$ is a function $F(x)$ such that $F^{'}(x) = x$.

   The derivative of $F(x) = x^2$ is $2x$, which is not x, but is close.

   To obtain x, we can consider the function $F_1(x) = \frac{1}{2}\, x^2$; its derivative is $f(x) = x$. Hence, the most general antiderivative is $F_1(x) = \frac{1}{2}\, x^2 + C$.

c. What would be the antiderivative of $f(x) = x^2$? We know that the derivative of $F(x) = x^3$ is $3x^2$, again close, and we must consider the function $F_2(x) = \frac{1}{3}\, x^3 + C$ as the most general antiderivative.

We can make a general statement based on these examples. We see that the (general) antiderivative of the function $f(x) = x^r$ is the function

$$F(x) = \frac{x^{r+1}}{r+1} + C.$$

Let us check this conclusion with differentiation:

$$\frac{d}{dx}\, x^{r+1} = (r+1)x^r \quad \text{so} \quad \frac{d}{dx}\, \frac{x^{r+1}}{r+1} = x^r.$$

However, this statement is not true if $r = -1$. [Why?]

So our final conclusion is that the (general) antiderivative of $f(x) = x^r$ for any $r \neq -1$ is the function

$$F(x) = \frac{x^{r+1}}{r+1} + C.$$

Note: The antiderivative of the function $f(x) = \frac{1}{x}$ is not covered in this course.

Example 6.3. We know that the derivative of the sine function is the cosine function; hence, the antiderivative of the cosine function is the sine function. Similarly the derivative of the cosine function is the negative sine function; hence, the antiderivative of the sine function is the negative cosine function. Moreover, we know that

$$\frac{d}{dx}\, \sin(ax) = a \cos(ax).$$

Hence,

$$\frac{d}{dx}\, \frac{\sin(ax)}{a} = \cos(ax).$$

We conclude that the antiderivative of the function $f(x) = \cos(ax)$ is the function

$$F(x) = \frac{\sin(ax)}{a} + C.$$

You can use differentiation to see that the antiderivative of the function $g(x) = \sin(ax)$ is the function

$$G(x) = -\frac{\cos(ax)}{a} + C.$$

Note: In what follows, by "antiderivative," we mean the most general antiderivative. The letter C will be any constant.

We know that

$$\frac{d}{dx}\, kf(x) = k\,\frac{d}{dx}\, f(x);$$

therefore, the antiderivative of the function $g(x) = kf(x)$ is $G(x) = kF(x)$, where $F(x)$ is the antiderivative of the function $f(x)$.

We also know that

$$\frac{d}{dx} f(x) + g(x) = f^{'}(x) + g^{'}(x);$$

therefore, the antiderivative of the function $h(x) = f(x) + g(x)$ is the function $H(x) = F(x) + G(x)$, where $F^{'}(x) = f(x)$ and $G^{'}(x) = g(x)$.

Example 6.4. To find the antiderivative of the function

$$f(x) = \frac{6x^3 - 2x + 6}{\sqrt{x}},$$

we simplify first:

$$f(x) = 6x^{5/2} - 2x^{1/2} + 6x^{-1/2}.$$

We then apply our general conclusion from Example 6.2, and we find that the antiderivative is

$$F(x) = 6\left(\frac{x^{7/2}}{7/2}\right) - 2\left(\frac{x^{3/2}}{3/2}\right) + 6\left(\frac{x^{1/2}}{1/2}\right).$$

So,

$$F(x) = \frac{12x^{7/2}}{7} - \frac{4x^{3/2}}{3} + 12\sqrt{x} + C.$$

Exercises

1. Read Examples 1 and 2 on pages 264-265 of the textbook.
2. Do the odd-numbered exercises from 1 to 15 on page 269 of the textbook.

Answers to Exercises (appendix-a.htm)

To make a unique determination of the constant in a general antiderivative, we need to know at least one exact value of the antiderivative function. These values are referred to as "extra conditions."

Example 6.5. If $f^{'}(x) = \sin(3x) + x^4$ and we know that $f(0) = -1$, then we can find that the antiderivative is

$$F(x) = -\frac{\cos(3x)}{3} + \frac{x^5}{5} + C.$$

Since $-1 = f(0) = -\dfrac{1}{3} + C,$ we conclude that $C = -\dfrac{2}{3}$.

So, the constant is determined, and the antiderivative is

$$f(x) = F(x) = -\frac{\cos(3x)}{3} + \frac{x^5}{5} - \frac{2}{3}.$$

Example 6.6. If $f''(x) = \cos(4x) - 6\sqrt{x} - 1$, $f'(0) = 1$ and $f(0) = -1$, what is $f(x)$?

The antiderivative of the function f'' is f'. So,

$$f'(x) = \frac{\sin(4x)}{4} - \frac{12x^{3/2}}{3} - x + C.$$

Since $1 = f'(0) = C$,

$$f'(x) = \frac{\sin(4x)}{4} - 4x^{3/2} - x + 1.$$

The antiderivative of f' is f. So,

$$f(x) = -\frac{\cos(4x)}{16} - \frac{8x^{5/2}}{5} - \frac{x^2}{2} + x + C.$$

Since $-1 = f(0) = -\dfrac{1}{16} + C$, we conclude that $C = -\dfrac{15}{16}$, and

$$f(x) = -\frac{\cos(4x)}{16} - \frac{8x^{5/2}}{5} - \frac{x^2}{2} + x - \frac{15}{16}.$$

Exercises

3. Read Examples 3 and 4 on page 266 of the textbook.
4. Do the odd-numbered exercises from 17 to 35 on page 269.
5. Read Example 7 on page 268 of the textbook.
6. Do Exercises 53, 55 and 57 on page 270.

Answers to Exercises (appendix-a.htm)

If $F(x)$ is the antiderivative of $f(x)$, then we write

$$\int f(x)dx = F(x) + C$$

and we say that the *indefinite integral* of the *integrand function* $f(x)$ is the function $F(x)$. We justify this notation later. The constant C is called the *constant of integration*. See Figure 6.1, below.

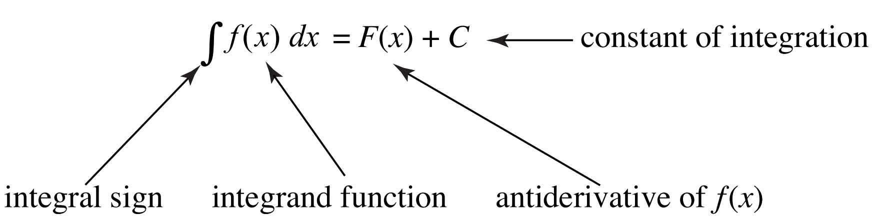


Figure 6.1: The antiderivative function

With this notation, we can state that

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C \quad \text{for} \quad r \neq -1. \tag{6.1}$$

$$\int \sin(ax)dx = -\frac{\cos(ax)}{a} + C \quad \text{for} \quad a \neq 0. \tag{6.2}$$

$$\int \cos(ax)dx = \frac{\sin(ax)}{a} + C \quad \text{for} \quad a \neq 0. \tag{6.3}$$

$$\int kf(x)\, dx = k\int f(x)\, dx. \tag{6.4}$$

$$\int f(x) + g(x) = \int f(x)\, dx + \int g(x)\, dx. \tag{6.5}$$

A table of basic indefinite integrals is given on page 273 of the textbook. You must know all of them. Use what you know about differentiation to help you remember them.

Example 6.7. From Example 6.4 we know that

$$\int \frac{6x^3 - 2x + 6}{\sqrt{x}}\, dx = \frac{12x^{7/2}}{7} - \frac{4x^{3/2}}{3} + 12\sqrt{x} + C.$$

The process of finding the antiderivative of a function is called integration. Hence, when you are asked to integrate, what you need to do is to find the antiderivative of the integrand function.

Example 6.8. To see if the indefinite integral

$$\int \frac{1}{(x^2+4)^{3/2}}\, dx = \frac{x}{\sqrt{x^2+4}} + C$$

is correct, we differentiate, and we see that

$$\frac{d}{dx}\frac{x}{\sqrt{x^2+4}} = \frac{4}{(x^2+4)^{3/2}}.$$

So, the antiderivative is not correct; instead, it should be

$$F(x) = \frac{x}{4\sqrt{x^2+4}}.$$

To use the basic indefinite integrals, we must identify the given integrand function with the integrand function of one of the basic integrals. Algebraic operations and trigonometric identities are used to achieve this identification.

Example 6.9. As it stands, the integrand function in the indefinite integral

$$\int \left(x + \frac{4}{x}\right)^2 dx$$

does not fit any of the types of the integrand functions of the basic integrals. But, if we multiply out, we get

$$\int x^2 + 8 + 16x^{-2}\, dx.$$

This integrand function fits the basic integral (see Equation 6.1), and we have

$$\int x^2 + 8 + 16x^{-2}\, dx = \frac{x^3}{3} + 8x - \frac{16}{x} + C.$$

Example 6.10. The integrand function in the indefinite integral

$$\int \frac{\sin(2x)}{\cos x}\, dx$$

does not fit any of the integrand functions of the basic integrals. But, if we use a trigonometric identity, we find that

$$\frac{\sin(2x)}{\cos x} = \frac{2\sin x\cos x}{\cos x} = 2\sin x.$$

Hence, we can apply Equation 6.2 with $a = 1$:

$$\int \frac{\sin(2x)}{\cos x}\,dx = 2\int \sin x\,dx = -2\cos x + C.$$

Example 6.11. To find the antiderivative of $\sin^2 x$ we use the double angle formula and we apply Equation 6.3, with $a = 2$:

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos(2x)}{2}\,dx\\ &= \frac{1}{2}\int 1-\cos(2x)\,dx\\ &= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C\\ &= \frac{2x-\sin(2x)}{4} + C\end{aligned}$$

Exercises

7. Do Exercises 1-4, 5, 7, 9, 11 and 13 on pages 276-277 of the textbook.
8. Use the half-angle formula to find the antiderivative $\int \cos^2 x\,dx$.

Answers to Exercises (appendix-a.htm)

The Antiderivative and the General Power Rule

Observe that for every derivative we have an antiderivative.

A. Since $\frac{d}{dx}(x+1)^2 = 2(x+1)$, we know that

$$\int (x+1)\,dx = \frac{(x+1)^2}{2} + C.$$

B. Similarly, since $\frac{d}{dx}(2x+1)^3 = 6(2x+1)^2$, we know that

$$\int (2x+1)^2\,dx = \frac{(2x+1)^3}{6} + C.$$

C. Also, $\frac{d}{dx}\left(x^2+1\right)^4 = 8x\left(x^2+1\right)^3$, so we know that

$$\int x\left(x^2+1\right)^3\,dx = \frac{\left(x^2+1\right)^4}{8} + C.$$

In all of these examples, we use the general power rule to differentiate; hence, the integrand function in all these cases is (or would be, except for a constant) a function of the form $f(x)^r f'(x)$.

In (A), $f(x) = (x+1)$, $r = 1$ and $f'(x) = 1$, so

$$\int (x+1)\,dx = \int f(x)\,f'(x)\,dx.$$

In (B), $f(x) = 2x+1$, $r = 2$, and $f'(x) = 2$, so

$$\int (2x+1)^2\,dx = \frac{1}{2}\int (2x+1)^2(2)\,dx = \frac{1}{2}\int f(x)^r\,f'(x)\,dx.$$

In (C), $f(x) = \left(x^2+1\right)$, $r = 3$ and $f'(x) = 2x$, so

$$\int x\left(x^2+1\right)^3\,dx = \frac{1}{2}\int \left(x^2+1\right)^3(2x)\,dx = \frac{1}{2}\int f(x)^r\,f'(x)\,dx.$$

From the general power rule, we know that

$$\frac{d}{dx}\frac{f(x)^{r+1}}{r+1} = f(x)^r f'(x).$$

We conclude that

$$\int f(x)^r f'(x)\,dx = \frac{f(x)^{r+1}}{r+1} + C \quad \text{for} \quad r \neq -1. \tag{6.6}$$

Note: An integral of the form $\int \frac{f'(x)}{f(x)}\,dx$ is not covered in this course.

The antiderivatives presented above are found using Equation (6.6).

(A)

$$\begin{aligned}\int (x+1)\,dx &= \int f(x)f'(x)\,dx\\ &= \frac{f(x)^2}{2}\\ &= \frac{(x+1)^2}{2} + C.\end{aligned}$$

$$\begin{aligned}\int (2x+1)^2\,dx &= \frac{1}{2}\int (2x+1)^2(2)\,dx\\ &= \frac{1}{2}\int f(x)^r f'(x)\,dx\\ &= \frac{1}{2}\frac{f(x)^3}{3}\\ &= \frac{(2x+1)^3}{6} + C.\end{aligned} \tag{B}$$

$$\begin{aligned}\int x(x^2+1)^3\,dx &= \frac{1}{2}\int (x^2+1)^3(2x)\,dx\\ &= \frac{1}{2}\int f(x)^r f'(x)\,dx\\ &= \frac{1}{2}\frac{f(x)^4}{4}\\ &= \frac{(x^2+1)^4}{8} + C.\end{aligned} \tag{C}$$

Observe that once we identify the function f and the power r, we take the derivative $f^{'}$, and modify the integrand function with a constant to fit the form $f(x)^r f^{'}(x)$. If we cannot modify it with a constant, then we must use another method of integration.

Example 6.12. In the integral $\int \sin x \cos x\,dx$, we identify $f(x) = \sin x$ and $r = 1$; thus, $f'(x) = \cos x$ and

$$\int \sin x \cos x\,dx = \int f(x)\,f'(x)\,dx = \frac{f(x)^2}{2} = \frac{\sin^2 x}{2} + C.$$

We can also identify $f(x) = \cos x$ and $r = 1$; thus, $f'(x) = -\sin x$ and

$$\int \sin x \cos x\,dx = -\int f(x)\,f'(x)\,dx = -\frac{f(x)^2}{2} = -\frac{\cos^2 x}{2} + C.$$

So, we have found two antiderivatives of the function $\sin x \cos x$. However, if we use a trigonometric identity we see that

$$\frac{\sin^2 x}{2} = \frac{1-\cos^2 x}{2} = -\frac{\cos^2 x}{2} + \frac{1}{2}.$$

Therefore, the antiderivatives are equal except for a constant, as we said earlier. The promised proof is given next.

Theorem 6.2. If two functions f_1 and f_2 are the antiderivative of a function f, then $f_1 = f_2 + K$ for some constant K.

Proof. Since

$$\frac{d}{dx} f_1(x) - f_2(x) = f_1^{'}(x) - f_2^{'}(x) = f(x) - f(x) = 0,$$

by Example 6.1(a) (unit06.htm#example6-1), we know that $f_1(x) - f_2(x) = K$, for some constant K; and therefore,

$$f_1(x) = f_2(x) + K.$$ **Q.E.D**

Example 6.13. In the integral $\int \tan x \sec^2 x\,dx$, we identify $f(x) = \tan x$ and $r = 1$, because $f^{'}(x) = \sec^2 x$. Thus,

$$\int \tan x \sec^2 x\,dx = \frac{\tan^2 x}{2} + C.$$

Example 6.14. To identify the function f and the power r in the integral

$$\int \frac{x}{\sqrt{3x^2-5}}\,dx,$$

we observe that

$$\frac{x}{\sqrt{3x^2-5}} = x\left(3x^2-5\right)^{-1/2}.$$

Hence, $f(x) = 3x^2 - 5$ and $r = -\dfrac{1}{2}$.

Since $f'(x) = 6x$, we proceed as follows:

$$\begin{aligned}\int \frac{x}{\sqrt{3x^2-5}}\,dx &= \int x(3x^2-5)^{-1/2}\,dx\\ &= \frac{1}{6}\int (3x^2-5)^{-1/2}(6x)\,dx\\ &= \frac{1}{6}\frac{(3x^2-5)^{1/2}}{1/2}\\ &= \frac{\sqrt{3x^2-5}}{3}+C.\end{aligned}$$

Example 6.15. The function raised to a power in the integral

$$\int \cos^3(4x)\sin(4x)\,dx$$

is

$$f(x) = \cos(4x).$$

Thus $r = 3$, and $f'(x) = -4\sin(4x)$.

Then,

$$\begin{aligned}\int \cos^3(4x)\sin(4x)\,dx &= -\frac{1}{4}\int \cos^3(4x)(-4\sin(4x))\,dx\\ &= -\frac{1}{4}\frac{\cos^4(4x)}{4}+C\\ &= -\frac{\cos^4(4x)}{16}+C.\end{aligned}$$

Example 6.16. In this example, we ask you to identify the function f and the power r.

$$\begin{aligned}\int \frac{2x^2}{\sqrt{4x^3+4}}\,dx &= \int (4x^3+4)^{-1/2}(2x^2)\,dx\\ &= \frac{1}{6}\int (4x^3+4)^{-1/2}(12x^2)\,dx\\ &= \frac{(4x^3+4)^{1/2}}{3}+C.\end{aligned}$$

Exercises

9. Identify the function f and power r in the following indefinite integrals

   a. $\int 2x\left(x^2+3\right)^4\,dx$

b. $\int (3x-2)^{20}\ dx$

c. $\int \frac{1+4x}{\sqrt{1+x+2x^2}}\ dx$

d. $\int \frac{x}{(x^2+1)^2}\ dx$

e. $\int \frac{3}{(2y+1)^5}\ dy$

f. $\int \frac{1}{(5t+4)^{27}}\ dt$

g. $\int \sqrt{4-t}\ dt$

h. $\int \cos\theta \sin^6\theta\ d\theta$

i. $\int \frac{z^2}{\sqrt[3]{1+z^3}}\ dz$

10. Determine the indefinite integrals in question 9, above, using Equation 6.6.

Answers to Exercises (appendix-a.htm)

Change of Variable: *u*-identification

As you may suspect, integration is more challenging than differentiation. Fortunately, many people have been integrating before us, and their findings have produced tables of integrals that we can use to facilitate our work. But we must learn to change the variable of the integral given by means of a u-identification before we can use them. A table of integrals is provided on page 6 of the textbook. In the discussion that follows, we refer to these integrals by their identification numbers.

WARNING: Integration tables facilitate our work, but

- their use is not always the most efficient way of determining an integral.
- they do not include all of the possible integrals we may encounter.

So, you must learn as many techniques of integration as possible.

If $u = f(x)$, then its differential is $f'(x)\ dx$. Thus, by identifying the integrand function $f(x)$ with u, we find that the integral

$$\int f(x)^r f'(x) dx = \frac{(f(x))^{r+1}}{r+1}$$

is the integral

$$\int u^r du = \frac{u^{r+1}}{r+1}.$$

This is Integral 2 from the table of integrals on page 6 of the textbook.

Example 6.17. In the integral

$$\int (x+1)^2\ dx$$

the u identification is $u = x + 1$, $r = 2$, and $du = 1\ dx$. Then,

$$\int (x+1)\ dx = \int u^2\ du = \frac{u^3}{3} = \frac{(x+1)^3}{3} + C.$$

Comparing this with the method we use in part A (unit06-01.htm#equation6-6A), we see that the u-identification is not more efficient in this case, because the integrand function is relatively simple.

Example 6.18. In the integral $\int x(x^2+1)^3\ dx$ of part C (unit06-01.htm#equation6-6C), we identify $u = x^2 + 1$, $r = 3$.

Hence, $du = 2x\ dx$, and $\frac{1}{2}\ du = x\ dx$.

Thus,

$$\int x(x^2+1)^3\ dx = \int u^3\ \frac{1}{2}\ du = \frac{1}{2}\int u^3\ du = \frac{1}{2}\ \frac{u^4}{4} = \frac{(x^2+1)^4}{8} + C.$$

Example 6.19. In the integral $\int \cos^3(4x)\ \sin(4x)\ dx$, we identify $u = \cos(4x)$, $r = 3$.

Then $du = -4\ \sin(4x)\ dx$, and $-\frac{1}{4}\ du = \sin(4x)\ dx$.

Thus,

$$\begin{aligned}\int \cos^3(4x)\,\sin(4x)\,dx &= \int u^3 - \frac{1}{4}\,du \\ &= \left(-\frac{1}{4}\right)\int u^3\,du \\ &= -\frac{u^4}{4(4)} \\ &= -\frac{\cos^4(4x)}{16}.\end{aligned}$$

Compare this result with Example 6.15 (unit06-01.htm#example6-15).

Observe that, once we identify u, we obtain the differential du, and in order to apply the corresponding integral from the table, we must rewrite the given integral only in terms of u. However, the answer must be given in terms of the original variable of the integrand function, not in terms of u.

Example 6.20. If we compare the integral $\int x\,\sin\left(1 - 2x^2\right)\,dx$ with the integrals in the table of integrals, we see that Integral 6 is the closest, if we identify $u = 1 - 2x^2$, because $du = -4x\,dx$.

So, we have $\dfrac{1}{-4}\,du = x\,dx$; therefore,

$$\begin{aligned}\int x\,\sin\left(1 - 2x^2\right)\,dx &= \int \sin u\left(\frac{1}{-4}\right)\,du = -\frac{1}{4}\int \sin u\,du \\ &= -\left(\frac{1}{4}\right)(-\cos u) = \frac{\cos\left(1 - 2x^2\right)}{4} + C.\end{aligned}$$

To be able to use integration tables efficiently, you must be aware that the differential

$$du = f'(x)\,dx$$

is the derivative of a function. So, you must identify a function and its derivative (or an expression that is close to its derivative, up to a constant), in the integrand function. Your ability to do so is the key in applying integration tables properly. Observe the general forms of the integrals in the table, because you will always start with a general form in terms of a variable other than u. For instance,

Integral 7. $\int \cos u\,du$ the general form is $\int \cos(f(x))f'(x)\,dx.$

Integral 8. $\int \sec^2 u\,du$ the general form is $\int \sec^2(f(x))f'(x)\,dx.$

The integral

$$\int \cos\left(x^3\right)x\,dx$$

does not fit the form of Integral 7, because we have x^3 and x in the integrand function, and neither one of them is, or almost is, the derivative of the other. Hence, we do not have $f(x)$ and $f'(x)$ in the integrand function.

However, the integral

$$\int \cos\left(x^3\right)x^2\,dx$$

does fit the form of Integral 7, because we have x^3 and x^2, the latter being almost the derivative of the former. Hence, the u-identification that we use is $u = x^3$ and $du = 3x^2\,dx$:

$$\int \cos\left(x^3\right)x^2\,dx = \frac{1}{3}\int \cos u\,du = \frac{\sin\left(x^3\right)}{3} + C.$$

Note: Whenever you use an integral from the table of integrals provided in the textbook, you must identify it by number, and give the explicit identification of u and the differential du.

Exercises

11. Use u-identification to solve the following indefinite integrals:

a. $\int \cos(3x)\ dx$

b. $\int x\left(4+x^2\right)^{10}\ dx$

c. $\int x^2\sqrt{x^3+1}\ dx$

d. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx$

e. $\int \frac{4}{(1+2x)^3}\ dx$

f. $\int \cos^4\theta \sin\theta\ \theta$

g. $\int 2x\left(x^2+3\right)^4\ dx$

h. $\int x^2\left(x^3+5\right)^9\ dx$

i. $\int (3x-2)^{20}\ dx$

j. $\int (2-x)^6\ dx$

k. $\int \frac{1+4x}{\sqrt{1+x+2x^2}}\ dx$

l. $\int \frac{3}{(2y+1)^5}\ dy$

m. $\int \sqrt{4-t}\ dt$

n. $\int \sin \pi t\ dt$

o. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\ dt$

p. $\int \cos\theta \sin^6\theta\ d\theta$

q. $\int \frac{z^2}{\sqrt[3]{1+x^3}}\ dz$

r. $\int \sqrt{\cot x}\ \csc^2 x\ dx$

s. $\int \sec^2 x \tan x\ dx$

Answers to Exercises (appendix-a.htm)

The *u*-substitution

Basically, there are two techniques of integration: substitution and parts. We will learn the basic principles of substitution in this course; other calculus courses provide an in-depth discussion of this technique and that of parts.

You can check that u-identification is not the correct approach to integrating

$$\int x\sqrt{x+1}\,dx.$$

Instead, we use substitution, that is we make a substitution $u = f(x)$ such that each term of the integrand function can be substituted to obtain an integral in terms of u only. In this case, if $u = x + 1$, then $du = dx$. To replace x, we must see that $u - 1 = x$. Hence, the integral is

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du.$$

The next step is to integrate the integral obtained in terms of u, assuming that this can be done. In this case, we find that

$$\begin{aligned}\int u^{1/2}(u-1)\,du \quad &= \int u^{3/2} - u^{1/2}\,du \\ &= \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} + C.\end{aligned}$$

Since the answer must be in terms of x, we substitute once more, and we obtain

$$\int x\sqrt{x+1}\,dx = \frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{2} + C.$$

The challenge with this method is to choose the function u correctly. There is no "recipe" for doing this; it can only be done by practice—that is, experience. We may have to try different substitutions to get the correct one. In the discussion that follows, if more than one substitution is possible, we tend to present the most efficient one, but there is no one substitution that is better than another: as long as they are correct, any one is acceptable.

We can summarize the steps in the substitution method as follows.

Step 1

Make a substitution of $u = f(x)$ for $f(x)$ a function in the integrand function, or any other function that relates to the integrand function.

Step 2

Replace each term of the integrand function in terms of u, including the differential.

Step 3

Integrate the resulting integral in terms of u.

Step 4

Substitute $f(x)$ for u, and give the antiderivative in terms of the original variable x.

Example 6.21. In the integral

$$\int x^5\ \sqrt[3]{x^3+2}\,dx,$$

we choose $u = x^3 + 2$. Then $du = 3x^2\,dx$.

We can replace $x^2\sqrt[3]{x^3+2}\,dx$ with $\sqrt[3]{u}\ \dfrac{1}{3}\ du$, and x^3 by $u - 2 = x^3$.

With these substitutions, we get

$$\begin{aligned}\frac{1}{3}\int (u-2)u^{1/3}\,du &= \frac{1}{3}\int u^{4/3} - 2u^{1/3}\,du\\ &= \frac{u^{7/3}}{7} - \frac{2u^{4/3}}{4} + C.\end{aligned}$$

The antiderivative is

$$\frac{\left(x^3+2\right)^{7/3}}{7} - \frac{\left(x^3+2\right)^{4/3}}{2} + C.$$

Example 6.22. We use substitution with the integral

$$\int x^2\sqrt{x+1}\,dx.$$

Let $u = x+1$. Then $du = dx$, and $(u-1)^2 = x^2$

$$\begin{aligned}\int (u-1)^2\,u^{1/2}\,du &= \int u^{5/2} - 2u^{3/2} + u^{1/2}\\ &= \frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3} + C.\end{aligned}$$

Hence,

$$\int x^2\sqrt{x+1}\,dx = \frac{2(x+1)^{7/2}}{7} - \frac{4(x+1)^{5/2}}{5} + \frac{2(x+1)^{3/2}}{3} + C.$$

Exercises

12. Use u-substitution to evaluate the following indefinite integrals:
    a. $\displaystyle\int \frac{x}{\sqrt[4]{x+2}}\,dx.$
    b. $\displaystyle\int \frac{x^2}{\sqrt{x-1}}\,dx.$
13. Integrate $\displaystyle\int \frac{x^3}{\sqrt{x^2-3}}\,dx.$

Answers to Exercises (appendix-a.htm)

For help with the concept of integration, view the PowerPoint tutorials below. To access a tutorial:

1. Click on the file to open it.
2. Extract the files and save them to your computer's hard drive.
3. Click on the file folder to open it.
4. Click on the PowerPoint file to view and listen.

Note: If you don't have PowerPoint installed on your computer, you can download the PowerPoint Viewer from here: https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6 (https://support.office.com/en-za/article/View-a-presentation-without-PowerPoint-2010-2f1077ab-9a4e-41ba-9f75-d55bd9b231a6).

Tutorial 1: Integration (../multimedia/powerpoint/integ1.zip)

Tutorial 2: Integration (../multimedia/powerpoint/integ2.zip)

The Mean Value Theorem

The Mean Value Theorem relates a function and its derivative. Several important results in calculus are proved using this theorem, among them a very important theorem we will study in the next section: the Fundamental Theorem of Calculus.

The geometric interpretation of the Mean Value Theorem is intuitively simple. This theorem says that if f is a continuous function on a closed interval $[a, b]$ and its graph is smooth, then the line segment from $(a, f(a))$ to $(b, f(b))$ is parallel to at least one tangent line to the curve. See Figure 6.2, below.

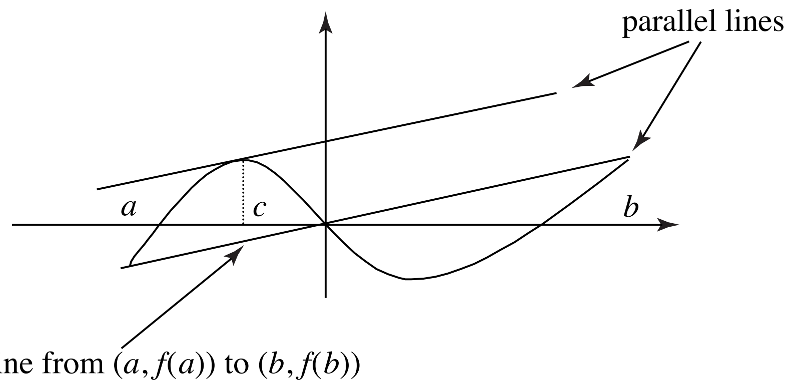


Figure 6.2: The Mean Value Theorem

So we know that the slopes of the line segment from $(a, f(a))$ to $(b, f(b))$ and the indicated tangent line are equal.

The slope of the line segment from $(a, f(a))$ to $(b, f(b))$ is $\dfrac{f(b) - f(a)}{b - a}$.

The slope of the tangent line is $f'(c)$ for some $a < c < b$.

Therefore,

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Theorem 6.3. The Mean Value Theorem.

Let f be a function satisfying the two conditions identified below.

a. It is continuous on a closed interval $[a, b]$.
b. It is differentiable on the open interval (a, b).

Then

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{for some} \quad a < c < b$$

or, equivalently

$$f(b) - f(a) = f'(c)(b - a).$$

The second condition in this theorem guarantees that the graph of the function is smooth.

In terms of rate of change, this theorem is saying that for at least one $a < c < b$, the *instantaneous* rate of change, $f'(c)$, is equal to the *average* rate of change,

$$\frac{f(b) - f(a)}{b - a},$$

on the interval $[a, b]$.

Example 6.23. If a car travels 250 km in 2 hours and 15 minutes, then its average velocity is

$$\frac{250}{2.25} = 111.11 \text{ km/hr},$$

and by the Mean Value Theorem (unit06-04.htm#mean-value-theorem), at some time the car's velocity is exactly equal to its average velocity. Hence, the car exceeded the legal speed limit of 100 km/hr at least once during the trip.

Example 6.24. In 2005, Asafa Powell's world record in the 100 m race was 9.77 s, and in 1996, Michael Johnson's world record for the 200 m race was 19.32 s. The Mean Value Theorem (unit06-04.htm#mean-value-theorem) says that at some time during his race, Powell's velocity was equal to his record's average velocity of

$$\frac{100}{9.77} = 10.235 \text{ m/s};$$

and also at some time during his race, Johnson's velocity was equal to his record's average velocity of

$$\frac{200}{19.32} = 10.352 \text{ m/s}.$$

Therefore, over the 200 m distance, Johnson was running faster than Powell did over the 100 m distance.

You may want to check previous records and see, based on the Mean Value Theorem (unit06-04.htm#mean-value-theorem), how often it is the case that a 200 m runner is faster than a 100 m runner.[1]

For a 100 m runner to be as fast as Johnson was in 1996 when he established his record, his 100 m record would have to be

$$\frac{100}{10.352} = 9.66 \text{ s}.$$

The second-best speed in the 200 m race is 19.89 s, established by Tyson Gay in 2007; his average velocity was

$$\frac{200}{19.62} = 10.19 \text{ m/s}.$$

Example 6.25. In 1988, Florence Griffith-Joyner established the world's records for the 100 m and 200 m races at 10.49 s, and 21.34 s, respectively. By the Mean Value Theorem (unit06-04.htm#mean-value-theorem), at some time during each of these races, Griffith-Joyner was running at a velocity equal to her average velocity for that race. That is, in the 100 m race, she ran at

$$\frac{100}{10.49} = 9.53 \text{ m/s};$$

and in the 200 m race, she ran at

$$\frac{200}{21.34} = 9.37 \text{ m/s}.$$

At some point, she was faster in the 100 m race than in the 200 m race.

For a 200 m runner to run as fast as Griffith-Joyner ran in the 100 m race, her record would have to be

$$\frac{200}{9.53} = 20.98 \text{ s}.$$

The second-best speed for the 100 m race is 10.84 s, established by Chandra Sturrup in 2005, with an average velocity of

$$\frac{100}{10.84} = 9.22 \text{ m/s},$$

and the second best world record of the 200 m race is 22.13 s, established by Allyson Felix in 2006 with an average velocity of

$$\frac{200}{22.11} = 9.04 \text{ m/s}.$$

By the Mean Value Theorem (unit06-04.htm#mean-value-theorem), if we are told that $f(a) = f(b)$, then the line segment from $(a, f(a))$ to $(b, f(b))$ is horizontal, with slope 0.

Then, $f'(c) = 0$ for some $a < c < b$. This particular case is known as Rolle's Theorem. See Figure 6.3.

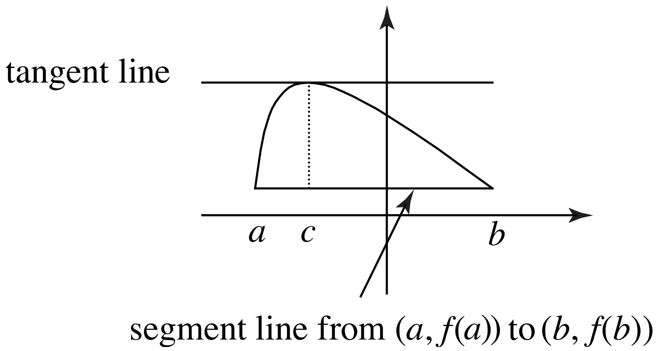


Figure 6.3: Rolle's Theorem

Theorem 6.4. Rolle's Theorem

Let f be a function satisfying the conditions identified below.

a. It is continuous on a closed interval $[a, b]$.
b. It is differentiable on the open interval (a, b).
c. $f(a) = f(b)$.

Then

$$f'(c) = 0 \quad \text{for some} \quad a < c < b.$$

Exercises

14. Read the proofs of Rolle's Theorem and Mean Value Theorem (unit06-04.htm#mean-value-theorem) on pages 282-284 of the textbook.
15. Read Examples 1 and 2 on pages 282-283 of the textbook.
16. Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.
17. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.
18. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.
19. Read Examples 3 and 4 on pages 284-285 of the textbook.
20. For each of the functions below, verify that the function satisfies the conditions of the Mean Value Theorem (unit06-04.htm#mean-value-theorem) on the given interval. Then, find all numbers c that satisfy the conclusion of the Mean Value Theorem.
    a. $f(x) = 3x^2 + 2x + 5$, $[-1, 1]$
    b. $f(x) = \sqrt[3]{x}$, $[0, 1]$
21. Let
    $$f(x) = \frac{x+1}{x-1}.$$
    Show that there is no value of c such that $f(2) - f(0) = f'(c)(2 - 0)$.
    Why does this conclusion not contradict the Mean Value Theorem (unit06-04.htm#mean-value-theorem)?
22. Read Example 5 on page 285 of the textbook.
23. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ be?
24. Does there exist an everywhere continuous function f such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x?

Answers to Exercises (appendix-a.htm)

Let us apply the Mean Value Theorem (unit06-04.htm#mean-value-theorem) to prove several results we have already learned.

Theorem 6.5. If $f'(x) = 0$ for all x on an interval (a, b), then $f(x) = C$ for some constant C.

Proof. We need to show that $f(x_1) = f(x_2)$ for any two numbers x_1 and x_2 in the interval (a, b).

We take $a < x_1 < x_2 < b$, and we consider the closed interval $[x_1, x_2]$. We must check the conditions of the Mean Value Theorem (unit06-04.htm#mean-value-theorem) for this interval. Since f is differentiable on (a, b), it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). So, we conclude that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{for some} \quad x_1 < c < x_2.$$

For this c, we know that $f'(c) = 0$. [Why?]

So, $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$, and $f(x_2) - f(x_1) = 0$.

This result implies that $f(x)$ is constant on the interval (a, b).

Q.E.D

This theorem says that the antiderivative of the zero function is a constant function, as we indicated before.

Theorem 6.6. [See Theorem 5.5 (unit05-02.htm#theorem5-5).]

a. If $f'(x) > 0$ for any x on the interval J, then the function f is increasing on the interval J.
b. If $f'(x) < 0$ for any x on the interval J, then the function f is decreasing on the interval J.

Proof. To prove statement (a), by Definition 5.4 (unit05.htm#definition5-4), we must show that $f(a) < f(b)$ for any $a < b$ in the interval J.

We consider the subinterval $[a, b]$ of J. Since we are assuming that $f'(x) > 0$ for all x in J, the function is differentiable in the interval J, and so it is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem (unit06-04.htm#mean-value-theorem)

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{for some} \quad a < c < b.$$

For this c, we know that $f'(c) > 0$; hence,

$$\frac{f(b) - f(a)}{b - a} > 0.$$

We also know that $b - a > 0$. [Why?].

Therefore, the quotient is positive only if

$$f(b) - f(a) > 0,$$

which implies that $f(a) < f(b)$.

The proof of statement (b) is similar.

Q.E.D

The second derivative test is also a consequence of the Mean Value Theorem (unit06-04.htm#mean-value-theorem). To see why, we must formalize the definition of concavity and prove a lemma.

In Definition 5.11 (unit05.htm#definition5-11), we indicated that a function is concave up on an interval J if the graph of the function is below the line segment from $(a, f(a))$ to $(b, f(b))$ for any $a < b$ in J. This is the same as saying that $f(x)$ is below the line segment from $(a, f(a))$ to $(b, f(b))$ for any $a < x < b$.

The line segment from $(a, f(a))$ to $(b, f(b))$ is given by the following equation for $a \le x \le b$. [Why?]

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Thus, a function is concave up if

$$f(x) < \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

for any $a < x < b$.

This statement is equivalent to saying that

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}$$

The following definition is equivalent to Definition 5.11 (unit05.htm#definition5-11).

Definition 6.7.

a. A continuous function f on an interval J is concave up if for any $a < x < b$ in J

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}.$$

b. A continuous function f on an interval J is concave down if for any $a < x < b$ in J

$$\frac{f(x) - f(a)}{x - a} > \frac{f(b) - f(a)}{b - a}.$$

Lemma 6.8. Let f be a differentiable function with $f^{'}$ increasing. If $a < b$ and $f(a) = f(b)$, then $f(x) < f(a) = f(b)$ for any $a < x < b$.

Proof. Let us argue by contradiction. We suppose that $f(x) \ge f(a)$ for some x in (a, b). This assumption will yield a contradiction; therefore, $f(x) < f(a)$ for any $a < x < b$.

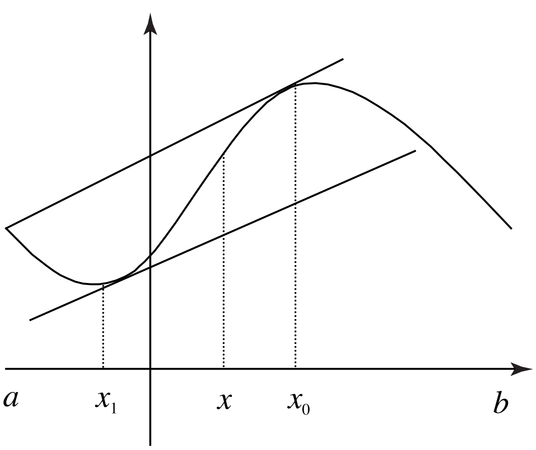


Figure 6.4: Lemma 6.8, proof diagram a, $f(x) > f(a) = f(b)$

If $f(x) > f(a) = f(b)$ for some $a < x < b$, then the maximum of f occurs at some point x_0 in (a, b). See Figure 6.4, above.

Hence, $f(x_0) > f(a)$ and $f^{'}(x_0) = 0$ (local maxima occur at critical numbers). Applying the Mean Value Theorem (unit06-04.htm#mean-value-theorem) in the interval $[a, x_0]$, we find that there is an $a < x_1 < x_0$ such that

$$f^{'}(x_1) = \frac{f(x_0) - f(a)}{x_0 - a} > 0 = f^{'}(x_0).$$

This equation shows that $f^{'}(x_1) > f^{'}(x_0)$, but we know that $x_1 < x_0$ and $f^{'}$ is increasing. By Definition 5.4 (unit05.htm#definition5-4), this cannot be. Therefore, $f(x) \leq f(a)$ for any $a < x < b$.

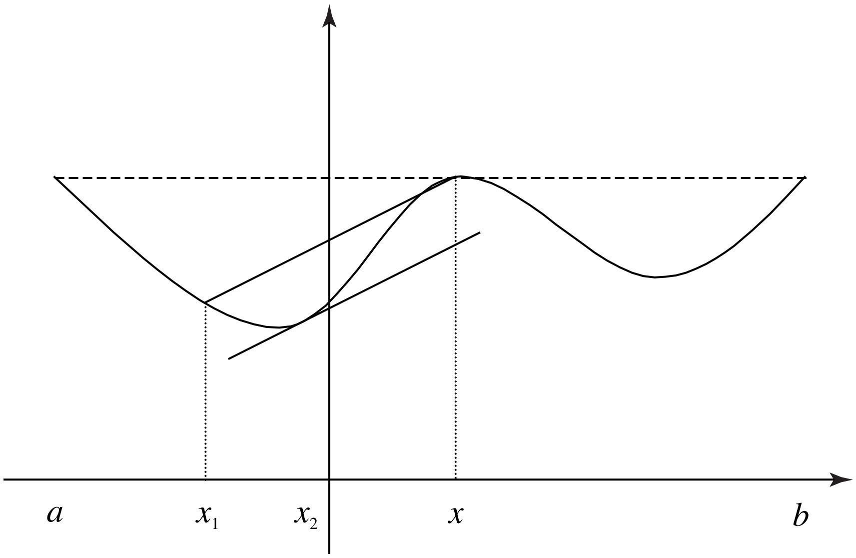


Figure 6.5: Lemma 6.8, proof diagram b, $f(x) = f(a)$

Suppose that $f(x) = f(a)$ for some $a < x < b$. The function must then be as in Figure 6.5, above, and f has a local maximum at x, hence $f^{'}(x) = 0$.

Since $f^{'}$ is increasing, it is not a constant on the interval $[a, x]$. Therefore, there is $a < x_1 < x$ such that $f(x_1) < f(a) = f(x)$. Applying the Mean Value Theorem (unit06-04.htm#mean-value-theorem) on the interval $[x_1, x]$, we find that there is an $x_1 < x_2 < x$ such that

$$f^{'}(x_2) = \frac{f(x) - f(x_1)}{x - x_1} > 0 = f^{'}(x).$$

This equation says that $f^{'}(x_2) > f^{'}(x)$ for $x_2 < x$, again contradicting the fact that $f^{'}$ is increasing.

Therefore it must be the case that $f(x) < f(a)$ for any $a < x < b$.

Q.E.D

Theorem 6.9. The Concavity Test

a. If $f''(x) > 0$ for all x on an interval J, then the function is concave up on the interval J.
b. If $f''(x) < 0$ for all x on an interval J, then the function is concave down on the interval J.

Proof. To prove the first statement, we must recognize that, if $f''(x) > 0$ for all x in J, then $f^{'}(x)$ is increasing on the interval J. Let $a < b$ in J and let $g(x)$ be the function defined for $a \leq x \leq b$ as follows:

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Thus, $g''(x) = f''(x) > 0$ (check it), and $g^{'}$ is increasing. Moreover, $g(a) = g(b) = f(a)$.

Applying Lemma 6.8 to g, we find that for any $a < x < b$,

$$g(x) < g(a) = f(a).$$

This equation states that, if $a < x < b$, then

$$f(x) - \frac{f(b) - f(a)}{b - a}(x - a) < f(a)$$

or

$$\frac{f(a) - f(x)}{x - a} < \frac{f(b) - f(a)}{b - a}.$$

The proof for the second statement is left as an exercise (see below).

Q.E.D

Exercises

25. Prove the second statement of Theorem 6.6.
26. Explain why the second statement of Definition 6.7 is equivalent to the corresponding part of Definition 5.11 (unit05.htm#definition5-11).
27. Prove the second statement of Theorem 6.9.
    Hint: Show that if f is CD, then $-f$ is CU.

Answers to Exercises (appendix-a.htm)

FOOTNOTES

[1] Note that the same athlete can usually run a 200 m race faster than double their time in a 100 m race, largely because they enter the second hundred metres already running at full speed.

The Fundamental Theorem of Calculus

Prerequisites

To complete this section, you must be able to use sigma notation. Read pages 368-371 in the textbook, and do Exercises 1, 3, 5, 13, 15 and 19 on page 372.

For a long time, scientists struggled with the problem of finding areas of regions bounded by irregular curves. The answer required the work of the best scientific minds of the eighteenth and nineteenth centuries, and it is known as the "Fundamental Theorem of Calculus."

In this section, we study the problem of finding the area of regions bounded by a continuous functions on a closed interval and the x -axis. Such areas are referred as "areas under curves."

Example 6.26. If $f(x) = 2x + 5$, then the area under the curve on the interval $[-4, 1]$ consists of the areas of the two triangles shown in Figure 6.6, below.

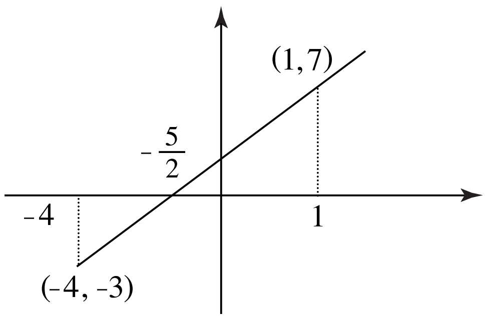


Figure 6.6: Area under the curve defined by $f(x) = 2x + 5$ on the interval $[-4, 1]$—not to scale

Hence, the area under the curve is

$$\frac{|-4+5/2|(3)}{2} + \frac{|1+5/2|(7)}{2} = \frac{29}{2}.$$

Example 6.27. If $f(x) = 2x + 5$, then area under the curve on the interval $[2, 6]$ is the area of a rectangle plus the area of a triangle, as shown in Figure 6.7, below.

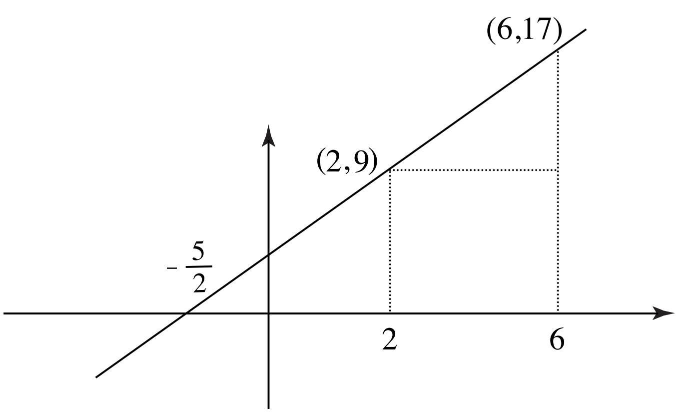


Figure 6.7: Area under the curve defined by $f(x) = 2x + 5$ on the interval $[2, 6]$—not to scale

Hence, the area under the curve is

$$(4)(9) + \frac{(4)(8)}{2} = 52.$$

Finding areas under curves when the function is not a line is more challenging. We consider this problem for functions that are continuous on a closed interval and that have smooth curves; that is, for differentiable functions.

We will approach this problem by approximation, and as we look for better approximations, we will arrive at the solution; that is, we will come to the Fundamental Theorem of Calculus.

Let f be a continuous function on the interval $[a, b]$, and let f be differentiable on the interval (a, b).

A possible approximation to the area under this curve is the area of a rectangle with base $(b - a)$ and height $f(x*)$, where $x*$ is any number $a < x* < b$, as shown in Figure 6.8, below.

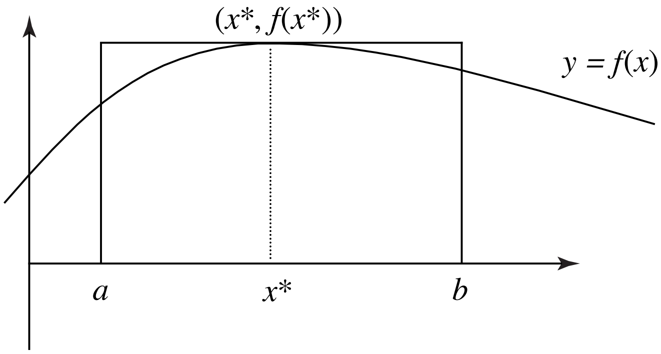


Figure 6.8: Estimating the area under the curve of a continuous function f—first approximation

If A is the area under the curve, then we know that

$$A \approx f(x*)(b - a).$$

If we want a better approximation, we should consider two rectangles instead of one. So we must divide the interval (a, b) into two parts, say

$$a = x_1 < x_2 < x_3 = b.$$

The bases of the rectangles are $(x_2 - x_1)$ and $(x_3 - x_2)$; their heights are $f(x_1*)$ and $f(x_2*)$ for any $x_1 < x_1* < x_2 < x_2* < x_3$, as shown in Figure 6.9, below.

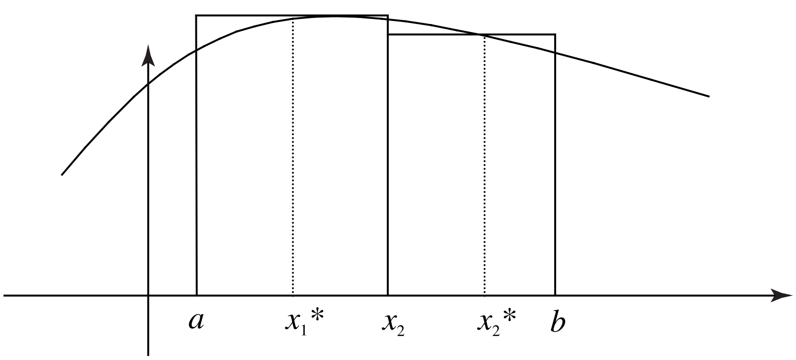


Figure 6.9: Estimating the area under the curve of a continuous function f—second approximation

For these two rectangles.

$$A \approx f(x_1*)(x_2 - x_1) + f(x_2*)(x_3 - x_2).$$

Three rectangles give an even better approximation. We divide the interval as follows:

$$a = x_1 < x_2 < x_3 < x_4 = b.$$

The bases of the rectangles are $(x_2 - x_1)$, $(x_3 - x_2)$ and $(x_4 - x_3)$; their heights are $f(x_1*)$, $f(x_2*)$ and $f(x_3*)$ for any $x_1 < x_1* < x_2 < x_2* < x_3 < x_3* < x_4$, as shown in Figure 6.10, below.

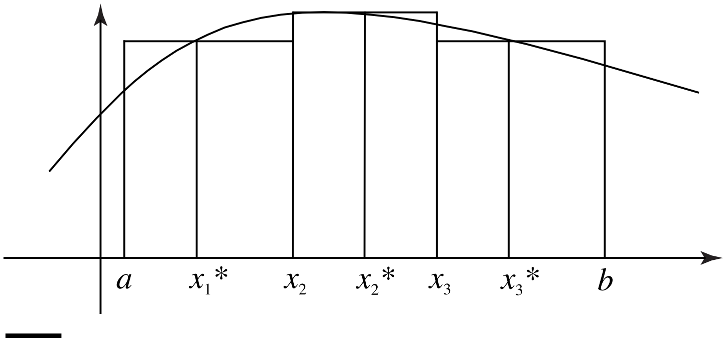


Figure 6.10: Estimating the area under the curve of a continuous function f—third approximation

For these three rectangles

$$A \approx f(x_1 *)(x_2 - x_1) + f(x_2 *)(x_3 - x_2) + f(x_3 *)(x_4 - x_3).$$

We can continue with four, five and more rectangles, obtaining better and better approximations—the greater the number of rectangles, the better the approximation. In general, if we take n rectangles, then we divide the interval (a, b) into n parts

$$a = x_1 < x_2 < x_3 < \ldots\ldots < x_n < x_{n+1} = b.$$

The bases of the n rectangles are

$$(x_2 - x_1), (x_3 - x_2), (x_4 - x_3), \ldots\ldots, (x_{n+1} - x_n).$$

Their heights are

$$f(x_1 *), f(x_2 *), f(x_3 *), \ldots\ldots, f(x_n *),$$

for any

$$x_1 < x_1 * < x_2 < x_2 * < x_3 < x_3 * < x_4 < \ldots\ldots < x_n * < x_{n+1}.$$

See Figure 6.11, below.

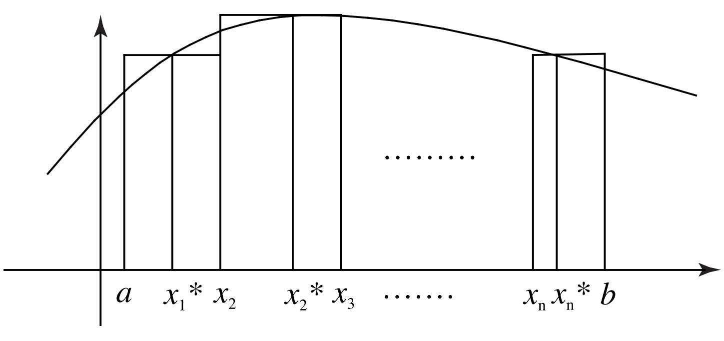


Figure 6.11: Estimating the area under the curve of a continuous function f—nth approximation

For these n rectangles

$$\begin{aligned} A \approx f(x_1 *)(x_2 - x_1) \quad &+ f(x_2 *)(x_3 - x_2) \\ &+ f(x_3 *)(x_4 - x_3) + \ \ldots \ + f(x_n *)(x_{n+1} - x_n). \end{aligned}$$

In sigma notation,

$$A \approx \sum_{i=1}^{n} f(x_i *)(x_{i+1} - x_i). \tag{6.7}$$

Let us now use the Mean Value Theorem (unit06-04.htm#mean-value-theorem) to find the sum in Equation 6.7.

Let F be the antiderivative of the function f; that is, $F^{'} = f$. Hence, F is continuous in the interval $[a, b]$ and differentiable in (a, b), and for any $1 \le i \le n$, the function F is continuous in the interval $[x_i, x_{i+1}]$, and differentiable in (x_i, x_{i+1}) for $i = 1, 2, 3 \ldots n$.

By the Mean Value Theorem (unit06-04.htm#mean-value-theorem)

$$F(x_{i+1}) - F(x_i) = F^{'}(x_i{*})(x_{i+1} - x_i) = f(x_i{*})(x_{i+1} - x_i)$$

for some

$$x_i < x_i{*} < x_{i+1}.$$

Hence,

$$\sum_{i=1}^{n} f(x_i{*})(x_{i+1} - x_i) = \sum_{i=1}^{n} F(x_{i+1}) - F(x_i).$$

The sum $\sum_{i=1}^{n} F(x_{i+1}) - F(x_i)$ is telescopic, because

$$\begin{aligned}\sum_{i=1}^{n} F(x_{i+1}) - F(x_i) &= [F(x_2) - F(x_1)] + [F(x_3) - F(x_2)] + \ldots \\ &\ldots + [F(x_n) - F(x_{n-1})] + [F(x_{n+1}) - F(x_n)] \\ &= F(x_{n+1}) - F(x_1) \\ &= F(b) - F(a)\end{aligned}$$

Therefore,

$$A \approx \sum_{i=1}^{n} f(x_i{*})(x_{i+1} - x_i) = F(b) - F(a).$$

As we said, the more rectangles (the larger the n), the better the approximation, and the more rectangles, the smaller $(x_{i+1} - x_i)$. Since $(x_{i+1} - x_i)$ is a small quantity of the variable x, it is a differential $\Delta x_i = (x_{i+1} - x_i)$.

Hence, as $\sum_{i=1}^{n} f(x_i{*})\Delta x_i$ approaches the area A. This is precisely a limit, and we have found that

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i{*})\Delta x_i = \lim_{n \to \infty} F(b) - F(a) = F(b) - F(a).$$

This is the Fundamental Theorem of Calculus. It is fundamental, because a simple geometrical concept—area—is given by the antiderivative of a function that is not a geometrical concept. That these two concepts are so intimately related is what makes this theorem so outstanding. Moreover, to arrive to this theorem was not easy, we needed to learn about limits—a concept that took humankind a long time to understand fully—together with the Mean Value Theorem (unit06-04.htm#mean-value-theorem).

We write $F(x)\Big|_a^b$ to indicate the difference $F(b) - F(a)$.

That is, $F(x)\Big|_a^b = F(b) - F(a)$.

What we have shown so far is that, if f is a continuous positive function on the interval $[a, b]$ and has antiderivative F, then the area under the curve on the interval is the limit.

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i{*})\Delta x_i = F(x)\Big|_a^b = F(b) - F(a).$$

The notation for this limit must indicate the function f, the interval $[a, b]$ and the fact that the limit is the limit of a sum; therefore, instead of Σ, we use the integral symbol $\int$, and we indicate the interval $[a, b]$ by writing b as an *upper limit of integration* and a as a *lower limit of integration*, as is shown below.

$$\int_a^b$$

We next indicate the integrand function as f and the differential as dx. Once we have done so, we have the definite integral of the function f on the interval $[a, b]$, as shown in Definition 6.10, below.

Definition 6.10. If f is a continuous function on the interval $[a, b]$, then the *definite integral* of f on the interval $[a, b]$ is the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i*)\Delta x_i = \int_a^b f(x)\ dx$$

where

$\Delta x_i = x_{i+1} - x_i$ and $a \le x_i < x_i* < x_{i+1} \le b$ for $i = 1\ .\ .\ .\ n.$

The sum

$$\sum_{i=1}^{n} f(x_i*)\Delta x_i$$

is called the *Riemann sum* of the function f.

Area Under Curve Graph (../multimedia/area-under-curve/math265-area-under-curve-graph.swf)

Example 6.28. We recognize the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i^* \sin(x_i^*)\Delta x_i$$

on the interval $[0, \pi]$ as the definite integral

$$\int_0^\pi x \sin x\ dx.$$

Observe that the function $x \sin x$ is positive on the interval $[0, \pi]$; thus, the definite integral is the area under this function on the interval $[0, \pi]$.

Example 6.29. The function $f(x) = 2x + 5$ is positive on the interval $[2, 6]$. So, by the Fundamental Theorem of Calculus, the definite integral

$$\int_2^6 2x + 5\ dx$$

is the area under the curve on the interval $[2, 6]$. Hence, the area is

$$\int_2^6 2x + 5\ dx = x^2 + 5x \Big|_2^6 = (6)^2 + 5(6) - \Big((2)^2 + 5(2)\Big) = 66 - 14 = 52,$$

as we found in Example 6.27.

Example 6.30. If $f(x) = m$ is any constant function, then it is continuous on any interval $[a, b]$, and the area under the curve is equal to the area of a rectangle with base $(b - a)$ and height $|m|$. Hence, $|m|(b - a)$. On the other hand, $|m|$ is a positive constant, and its antiderivative is $F(x) = |m|x$. So, by the Fundamental Theorem of Calculus:

$$\int_a^b |m|\ dx = |m|x \Big|_a^b = |m|b - |m|a = |m|(b - a).$$

For any constant m, the definite integral

$$\int_a^b m\ dx = m(b - a)$$

is the area above the x-axis if $m > 0$; it is minus the area below the x-axis if $m < 0$.

We know that for a negative function $g(x)$ on an interval $[a, b]$, with antiderivative $G(x)$, the function $-g(x)$ is positive on $[a, b]$, its antiderivative is $-G(x)$, and the areas under the curves of g and $-g$ are equal. [See Definition 2.12 (unit02-02.htm#definition2-12).] Thus, the definite integral

$$\int_a^b -g(x)\,dx = -G(b) + G(a) = -(G(b) - G(a))$$

is positive.

Since the areas under the curves of g and $-g$ are equal, we know that the area under the curve of g on the interval $[a, b]$ is the positive number

$$\int_a^b -g(x)\,dx = -(G(b) - G(a)).$$

Therefore, if the function g is negative on $[a, b]$, the definite integral of g is the negative number

$$\int_a^b g(x)\,dx = G(b) - G(a),$$

and the area under the curve of g in the interval $[a, b]$ is

$$-\int_a^b g(x)dx.$$

Therefore,

$$\int_a^b -g(x)\,dx = -\int_a^b g(x)\,dx.$$

Theorem 6.11. The Fundamental Theorem of Calculus, Part 2.

[Note that we discuss Part 1 later in this unit.]

If $f(x)$ is continuous on the interval $[a, b]$, with antiderivative $F(x)$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i*)\Delta x_i = \int_a^b f(x)\,dx = F(b) - F(a)$$

where

$\Delta x_i = x_{i+1} - x_i$ and $a \le x_i < x_i* < x_{i+1} \le b$ for $i = 1 \ . \ . \ . \ n.$

Proof. Observe that the Mean Value Theorem (unit06-04.htm#mean-value-theorem) holds for the antiderivative function F, for any continuous function f not necessarily positive. Hence, the argument we presented earlier is the proof of this theorem. We used a positive function, because we wanted to give a geometric meaning to the limit of the definite integral.

Q.E.D

Example 6.31. The function $f(x) = 2x + 5$ on the interval $[-4, \ -5\,/\,2]$ is negative, and its graph is below the x-axis (see Figure 6.6).

Hence, the definite integral

$$\begin{aligned}\int_{-4}^{-5/2} 2x + 5\,dx &= x^2 + 5x\Big|_{-4}^{-5/2} \\ &= \frac{25}{4} - \frac{25}{2} - \left((-4)^2 + 5(-4)\right) = -\frac{9}{4}\end{aligned}$$

is a negative number, and the area under the curve is

$$\left|-\frac{9}{4}\right| = \frac{9}{4}.$$

The area under the curve on the interval $[-5\ /\ 2,\ 1]$ is the definite integral

$$\int_{-5/2}^{1} 2x+5\ dx = x^2+5x\Big|_{-5/2}^{1} = 1+5-\left(\frac{25}{4}-\frac{25}{2}\right) = \frac{49}{4}.$$

The area under the curve on the interval $[-4,\ 1]$ is the sum of these two areas:

$$\frac{9}{4}+\frac{49}{4}=\frac{29}{2},$$

as we found in Example 6.26.

However, if we consider the definite integral on the interval $[-4,\ 1]$, we see that

$$\int_{-4}^{1} 2x+5\ dx = x^2+5x\Big|_{-4}^{1} = 1+5-(16-20) = 10.$$

This result is not the same as the area under the curve on the interval $[-4,\ 1]$. Rather, it is the area under the curve on $[-5\ /\ 2,\ 1]$, *minus* the area under the curve on $[-4,\ -5\ /\ 2]$. And as we can see,

$$\frac{49}{4}-\frac{9}{4}=10.$$

REMARK 6.1

Example 6.31 shows that the Fundamental Theorem of Calculus gives the area under the curve *if* the integrand function is positive on the interval of integration. Hence, to find the area under the curve of a function on a closed interval, we must find the intervals where the function is positive and negative, and set up the corresponding definite integrals. We do this in the next unit.

Example 6.32. If we look at the graph of the cosine function on the interval $[0,\ \pi]$, we understand why the definite integral on this interval is zero: on this interval, the area above the x-axis is equal to the area below the x-axis.

$$\int_0^{\pi} \cos x\ dx = \sin x\Bigg|_0^{\pi} = \sin\ (\pi) - \sin\ (0) = 0.$$

Example 6.33. Let us evaluate $\int_{-1}^{4} \sqrt{x+4}+x\ dx$:

$$\begin{aligned}\int_{-1}^{4} \sqrt{x+4}+x\ dx &= \frac{2(x+4)^{3/2}}{3}+\frac{x^2}{2}\Bigg|_{-1}^{4}\\ &= \frac{2(8)^{3/2}}{3}+\frac{4^2}{2}-\left(\frac{2(3)^{3/2}}{3}+\frac{1}{2}\right)\\ &= \frac{32\sqrt{2}}{3}-2\sqrt{3}+\frac{15}{2}.\end{aligned}$$

Now let us evaluate $\int_0^{\pi/4} \sec^2 x - \sin\ (2x)\ dx$

$$\begin{aligned}\int_0^{\pi/4} \sec^2 x-\sin(2x)\ dx &= \tan x+\frac{\cos(2x)}{2}\Bigg|_0^{\pi/4}\\ &= \tan\left(\frac{\pi}{4}\right)+\frac{\cos\left(\frac{\pi}{2}\right)}{2}-\left(\tan(0)+\frac{\cos(0)}{2}\right)\\ &= 1-\frac{1}{2}=\frac{1}{2}.\end{aligned}$$

When we use the u-identification or the u-substitution to evaluate definite integrals, we have two choices to deal with the limits of integration.

1. We can change the limits of integration according to the function u.
2. We can find the antiderivative in terms of the original variable, and then evaluate at the original limits of integration.

Example 6.34. Let us use the u-substitution to find the antiderivative of the integrand function

$$\int_{-1}^{4} x\sqrt[3]{x+1}\,dx.$$

Set $u = x+1$, then $du = dx$ and $x = u-1$.

1. We can change the limits of integration according to the function u. So, if $x=-1$, $u=0$ and if $x=4$, $u=5$. These are the limits of integration according to the variable u, and the integral is

$$\begin{aligned}\int_0^5 (u-1)(u)^{1/3}\,du &= \int_0^5 u^{4/3} - u^{1/3}\,du\\ &= \frac{3u^{7/3}}{7} - \frac{3u^{4/3}}{4}\Big|_0^5\\ &= \frac{3(5^{7/3})}{7} - \frac{3(5^{4/3})}{4}\\ &= 5^{4/3}\left(\frac{15}{7} - \frac{3}{4}\right)\\ &= \frac{39\sqrt[3]{5^4}}{28}\end{aligned}$$

2. We can find the antiderivative in terms of the original variable (in this case, x), and then we can evaluate it at the original limits of integration.

$$\begin{aligned}\int x\sqrt[3]{x+1}\,dx &= \int u^{4/3} - u^{1/3}du = \frac{3u^{7/3}}{7} - \frac{3u^{4/3}}{4} + C\\ &= \frac{3(x+1)^{7/3}}{7} - \frac{3(x+1)^{4/3}}{4} + C.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{-1}^{4} x\sqrt[3]{x+1}\,dx &= \frac{3(x+1)^{7/3}}{7} - \frac{3(x+1)^{4/3}}{4}\Big|_{-1}^{4}\\ &= \frac{3(5)^{7/3}}{7} - \frac{3(5)^{4/3}}{4} = \frac{39\sqrt[3]{5^4}}{28}.\end{aligned}$$

WARNING

The most common mistakes in the evaluation of definite integrals are

1. evaluating the antiderivative in the wrong order.

   Pay attention: The value of the antiderivative should be at the upper limit of integration minus at the lower limit of integration.
2. mixing up the positive and negative signs. Always use parentheses to evaluate the antiderivative to avoid this type of error.
3. incorrect use of the limits of integrations when using the u identification or substitution methods.

Note: A definite integral $\int_a^b f(x)\,dx$ is equal to $F(b) - F(a)$ where $F'(x) = f(x)$, only if the integrand function $f(x)$ is continuous on the interval $[a,b]$

.

The Fundamental Theorem of Calculus cannot be applied to the evaluation of the integral

$$\int_0^{\pi/2} \tan x \; dx$$

because $\tan x$ is continuous on $[0, \pi / 2)$, not on $[0, \pi / 2]$. This type of integral is called an *improper integral*, and its evaluation is not covered in this course.

When the author of your textbook says that a definite integral exists or does not exist, he is referring to the existence of the limit of the Riemann sum in the Fundamental Theorem of Calculus, Part 2. But this limit always exists if the integrand function is continuous on the closed interval given by the limits of integration, and this is what he wants you to check.

Exercises

28. Read Examples 5, 6, 7 and 8 on pages 293-294 of the textbook.
29. Do Exercises 17-20 on page 305.
30. Do Exercises 33, 35, 38 and 39 on page 306.
31. Do the odd numbered exercises from 19 to 33 on page 298.
32. Do Exercises 9-26 on page 309.

Answers to Exercises (appendix-a.htm)

When we have difficulties finding the antiderivative of a function, we have the option of using the properties of the definite integral to find at least an estimate of its value.

Properties of the Definite Integral

1. $\int_a^b c\; f(x)\; dx = c \int_a^b f(x)\; dx$ for any constant c.
2. $\int_a^b f(x) \pm g(x)\; dx = \int_a^b f(x) \pm \int_a^b g(x)\; dx$
3. $\int_a^b m\; dx = m(b-a)$
4. $\int_a^b f(x)\; dx = \int_a^c f(x)\; dx + \int_c^b f(x)\; dx$ for any constant c
5. $\int_a^b f(x)\; dx \geq 0$ if f is positive on $[a, b]$
6. If $f(x) \leq g(x)$ for all x in $[a, b]$, then
$$\int_a^b f(x)\; dx \leq \int_a^b g(x)\; dx$$
7. If $m \leq f(x) \leq M$ for all x in $[a, b]$, then
$$m(b-a) \leq \int_a^b f(x)\; dx \leq M(b-a).$$
8. $-\int_a^b f(x)\; dx = \int_b^a f(x)\; dx.$

Proof. Property 3 is Example 6.30. Property 5 is true because the definite integral is the area under the function f on the interval $[a, b]$.

To prove Properties 1, 2, and 4, let F and G be the antiderivatives of f and g respectively.

Property 1

$$\begin{aligned}\int_a^b cf(x)\,dx &= cF(x)\Big|_a^b = cF(b) - cF(a)\\ &= c(F(b) - F(a)) = c\int_a^b f(x)\,dx.\end{aligned}$$

Property 2

$$\begin{aligned}\int_a^b f(x) \pm \mathrm{g}(x)\,dx &= F(x) \pm G(x)\Big|_a^b\\ &= F(b) \pm G(b) - (F(a) \pm G(a))\\ &= (F(b) - F(a)) \pm (G(b) - G(a))\\ &= \int_a^b f(x) \pm \int_a^b g(x)\,dx.\end{aligned}$$

Property 4

$$\begin{aligned}\int_a^c f(x)\,dx + \int_c^b f(x)\,dx &= F(c) - F(a) + F(b) - F(c)\\ &= F(b) - F(a) = \int_a^b f(x)\,dx.\end{aligned}$$

Property 6

We know that $g(x) - f(x) \geq 0$ on the interval $[a, b]$. So, by Properties 5 and 2,

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b g(x) - f(x)\,dx \geq 0;$$

thus,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Property 7

By Properties 3 and 6, we have

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a).$$

Property 8

$$-\int_a^b f(x)\,dx = -(F(b) - F(a)) = F(a) - F(b) = \int_b^a f(x)\,dx.$$

Q.E.D

Example 6.35. We do not know how to find the antiderivative of the function $x \sin x$, but we know that $|\sin x| \leq 1$.

If x is in the interval $[0, \pi / 3]$, then

$$0 \leq x \leq \frac{\pi}{3},$$

and since $-1 \leq \sin x \leq 1$, we multiply this inequality by $x \geq 0$ and we get $-x \leq x \sin x \leq x$.

Since $x \sin x \geq 0$ on this interval, we have

$$0 \leq x \sin x \leq x \quad \text{for any} \quad x \quad \text{in} \quad \left[0, \frac{\pi}{3}\right].$$

By Properties 5 and 6, above,

$$0 \leq \int_0^{\pi/3} x \sin x \, dx \leq \int_0^{\pi/3} x \, dx = \frac{\pi^2}{18}.$$

That is, the value of the integral $\int_0^{\pi/3} x \sin x \, dx$ is in the interval $\left[0, \frac{\pi^2}{18}\right]$.

If f is continuous on the interval $[a, b]$, then by the Extreme Value Theorem (unit05.htm#theorem5-10), there are numbers m and M, such that $m \leq f(x) \leq M$ for all x in $[a, b]$. Hence, by Property 7, we can always find lower and upper bounds for a definite integral.

Example 6.36. On the interval $[1, 6]$, the function

$$f(x) = \frac{1}{x}$$

is decreasing and continuous.

Hence, it is bounded by 1 and $\frac{1}{6}$; that is, $\frac{1}{6} \leq \frac{1}{x} \leq 1$, and by Property 7,

$$\frac{1}{6}(5) \leq \int_1^6 \frac{1}{x} \, dx \leq 5.$$

The value of the integral is in the interval $\left[\frac{5}{6}, 5\right]$.

Exercises

33. Read Examples 7 and 8 on pages 303-304 of the textbook.
34. Do Exercises 47, 49, 51, 53, 55, 57 and 61 on page 306.

Answers to Exercises (appendix-a.htm)

Part 1 of the Fundamental Theorem of Calculus refers to the inverse nature of differentiation and integration. We know that the integral of the derivative function is equal to the function; that is, $\int f'(x) \, dx = f(x)$. Now we will see that the converse also holds; that is, the derivative of an integral is the integrand function

Theorem 6.12. The Fundamental Theorem of Calculus, Part 1.

Let $f(x)$ be a continuous function on the interval $[a, b]$. For $a \leq x \leq b$, we define the function

$$g(x) = \int_a^x f(t) \, dt.$$

Then

$$g'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

Proof. If F is the antiderivative of f, then

$$g(x) = \int_a^x f(t)\ dt = F(t)\bigg|_a^x = F(x) - F(a).$$

Therefore $g'(x) = F'(x) = f(x)$.

Q.E.D

If we compose the function $g(x) = \int_a^x f(t)\ dt$ with a differentiable function $h(x)$, then we have

$$g(h(x)) = \int_a^{h(x)} f(t)\ dt.$$

By the Chain Rule (unit04-06.htm#chain-rule-desc),

$$\frac{d}{dx}\int_a^{h(x)} f(t)\ dt = \frac{d}{dx}\ g(h(x)) = g'(h(x))h'(x) = f(h(x))h'(x).$$

Example 6.37.

a. $\frac{d}{dx}\int_0^{3x^4} \frac{3}{x^2+6}\ dx = \frac{3}{(3x^4)^2+6}\left(12x^3\right) = \frac{12x^3}{3x^8+2}.$

b. Observe that Part 1 of the Fundamental Theorem of Calculus holds only if the lower limit of integration is a constant. If it is not, we use Property 8 to change the limits of integration to apply this theorem.

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^4 t^2 \cos t\, dt &= -\int_4^{\sqrt{x}} t^2 \cos t\, dt \\ &= -((\sqrt{x})^2 \cos(\sqrt{x}))\frac{1}{2\sqrt{x}} \\ &= -\frac{\sqrt{x}\ \cos(\sqrt{x})}{2}.\end{aligned}$$

c. If the upper and lower limits of integration are functions, then we use Property 4 to split the integral into two separate integrals, each of which has an arbitrarily chosen constant as a lower limit of integration.

$$\begin{aligned}\frac{d}{dx}\int_{3x^2}^{\sin x} t^4+6t^3\, dt &= \frac{d}{dx}\int_{3x^2}^{0} t^4+6t^3\, dt + \frac{d}{dx}\int_0^{\sin x} t^4+6t^3\, dt \\ &= -\frac{d}{dx}\int_0^{3x^2} t^4+6t^3\, dt + \frac{d}{dx}\int_0^{\sin x} t^4+6t^3\, dt \\ &= -[(3x^2)^4+6(3x^2)^3](6x) + (\sin^4 x + 6\sin^3 x)(\cos x) \\ &= -486x^7(x^2+2) + (\sin x + 6)\sin^3 x \cos x\end{aligned}$$

Exercises

35. Read Examples 1-4 on pages 288 and 291-292 of the textbook.
36. Do Exercises 7, 9, 11 and 15, on page 298.

Answers to Exercises (appendix-a.htm)

Finishing This Unit

1. Review the objectives of this unit and make sure you are able to meet all of them.
2. If there is a concept, definition, example or exercise that is not yet clear to you, go back and reread it, then contact your tutor for help.
3. Do the exercises in the "Learning from Mistakes" section for this unit.
4. You may want to do Exercises 53-58 from the "Review" on pages 260-262 of the textbook.
5. You may want to do Exercises 4, 5 and 8, the odd-numbered exercises from 9 to 23, and Exercises 33, 35, 39 and 41 from the "Review" (pages 308-311 of the textbook).

Learning from Mistakes

There are mistakes in each of the following solutions. Identify the errors, and give the correct answer.

1. Integrate $\int \tan^2(3x)\sec^2(3x)\,dx$.

   Erroneous Solution

   $f(x) = \tan(3x)$, $r = 3$ and $f^{'}(x) = \sec^2(3x)$, then

   $$\int \tan^2(3x)\ \sec^2(3x)\ dx = \frac{\tan^3(3x)}{3}.$$

2. Integrate $\int_1^5 x^3\sqrt{x^2+1}\,dx$.

   Erroneous Solution

   Let $u = x^2+1$, $du = 2u\ dx$ and $x^2 = u-1$. Hence,

   $$\begin{aligned}\frac{1}{2}\int_1^5 (u-1)u^{1/2}\,du &= \frac{1}{2}\int_1^5 u^{3/2} - u^{1/2}\,du \\ &= \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\Big|_1^5 \\ &= \frac{2(5)^{5/2}}{3} - \frac{2(5)^{3/2}}{3} - \left(\frac{2}{5} - \frac{2}{3}\right) \\ &= \frac{4\sqrt{5^3}}{3} + \frac{4}{15}\end{aligned}$$

3. Integrate $\int_0^1 \left(x^2 - \sqrt{x}\right)^2 dx$.

   Erroneous Solution

   $$\begin{aligned}\int_0^1 (x^2-\sqrt{x})^2\,dx &= \int_0^1 x^4 - 2x^{5/2} + x\,dx \\ &= \frac{x^5}{5} - \frac{4x^{7/2}}{7} + \frac{x^2}{2}\Big|_0^1 \\ &= 0 - \left(\frac{1}{5} - \frac{4}{7} + \frac{1}{2}\right) \\ &= -\frac{9}{70}\end{aligned}$$

4. Find a single function $f(x)$ that satisfies the following two conditions:
   a. $f^{'}(x) = \cos x + \sqrt{3x}$, and
   b. $f(1) = 1$.

Erroneous Solution

To find f, we integrate its derivative

$$\int \cos x + \sqrt{3}\sqrt{x}\ dx = \sin x + \frac{2\sqrt{3}x^{3/2}}{3}.$$

Hence,

$$f(x) = \sin x + \frac{2\sqrt{3}x^{3/2}}{3}.$$

5. Let $f(x) = 4x^3 - 6x^2 + 3$ be a continuous function. Find the value c in the interval $[-1, 0]$ which satisfies the Mean Value Theorem (unit06-04.htm#mean-value-theorem).

Erroneous Solution

$f^{'}(x) = 12x^2 - 12x$ and $\dfrac{f(0) - f(-1)}{0 - (-1)} = 10$. Hence, $12x^2 - 12x = 10$, and solving for x (using the quadratic formula)

$$x = \frac{3 \pm \sqrt{39}}{6}.$$

There are two values satisfying the Mean Value Theorem (unit06-04.htm#mean-value-theorem).

6. Find the area below the curve of the function $f(x) = x^3 + 4x$ on the interval $[-1, 3]$.

Erroneous Solution

$$\int_{-1}^{3} x^3 + 4x\ dx = \frac{x^4}{4} + 2x\,\Bigg|_{-1}^{4} = \frac{4^4}{4} + 2(4) - \left(\frac{1}{4} - 2\right) = \frac{291}{4}.$$

The area is $\dfrac{291}{4}$.

7. Find the derivative of the function $g(x) = \displaystyle\int_{3x}^{4} t\tan t\ dt$.

Erroneous Solution

$g^{'}(x) = (3x)\ \tan\ (3x)$.

Applications of the Definite Integral

Objectives

When you have completed this unit, you should be able to

1. use integration to solve problems of
    a. rectilinear motion.
    b. net change.
    c. areas between curves.
    d. work.
    e. average of a function.

In this unit, we apply the definite integral to solve problems with applications in physics (rectilinear motion, net change, work and average of a function), and problems with a geometric application (area between curves).

Rectilinear Motion

If $a(t)$ is an acceleration function, then its antiderivative is the velocity function $v(t)$, and the antiderivative of the velocity function $v(t)$ is the displacement function $s(t)$. That is,

$$v(t) = \int a(t)dt \quad \text{and} \quad s(t) = \int v(t)dt.$$

So, the change of the velocity over the time interval $[t_0, t_1]$ is given by the definite integral

$$\int_{t_0}^{t_1} a(t)dt;$$

and the displacement over the time interval $[t_0, t_1]$ is given by the definite integral

$$\int_{t_0}^{t_1} v(t)dt = \int_{t_0}^{t_1} s^{'}(t)dt = s(t_2) - s(t_1).$$

Since $|v(t)|$ is the speed, the distance traveled over the time interval $[t_0, t_1]$ is the definite integral

$$\int_{t_0}^{t_1} |v(t)|dt.$$

Definition* 7.1. *Uniform accelerated motion* occurs when a particle moves with a constant acceleration. If we write $s_0 = s(0)$ for the initial position, and $v_0 = v(0)$ for the initial velocity, both at time $t = 0$, then for a constant of acceleration $a(t) = a$, we have

$$v(t) = \int a \ dt = at + C.$$

Since $v_0 = v(0) = C$, the velocity function is

$$v(t) = v_0 + at. \tag{7.1}$$

Thus

$$s(t) = \int v_0 + at \ dt = v_0 t + a\,\frac{t^2}{2} + C,$$

and

$$s_0 = s(0) = C.$$

Therefore, the displacement function is

$$s(t) = s_0 + v_0\, t + \frac{a}{2}\, t^2. \tag{7.2}$$

Example 7.1. Spotting a police car, a driver hits the brakes to reduce his speed from 90 mi/h to 60 mi/h at a constant rate over a distance of 200 ft. What is the acceleration in $\mathrm{ft/s^2}$? How long would it take for the driver to bring his car to a complete stop from 90 mi/h?

Let t_0 be the time the breaks are applied, and t_1 be the time the car's velocity reaches 60 mi/h. Hence, $s(0) = 0$, $v(t_0) = 90$ mi/h $= 132$ ft/s, and $v(t_1) = 60$ mi/h $= 88$ ft/s.

$$v(t_1) - v(t_0) = 88 - 132 = \int_0^{t_1} a\, dt = at_1.$$

Hence, $at_1 = -44$, and

$$s(t_1) - s(t_0) = 200 = \int_0^{t_1} v(t)\, dt = v_0\, t_1 + \frac{a}{2}\, t_1^2.$$

Solving for t_1, we get

$$200 = 132\, t_1 + \frac{-44}{2}\, t_1 \quad \text{and} \quad t_1 = \frac{100}{55};$$

and therefore, $a = -\dfrac{44}{t_1} = -\dfrac{121}{5}$ ft/s ^{2}.

The car comes to a complete stop when $v(t) = 0$, by Equation 7.1, above,

$$0 = v(t) = 132 - \frac{121}{5} t.$$

Solving for t, we get $t = 5.45$.

The constant of deceleration is

$$a = -\frac{121}{5}\ \mathrm{ft/s^2},$$

and the time to a complete stop is 5.45 seconds.

Example 7.2. [See Exercise 68 on page 271 of the textbook.]

A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 22 $\mathrm{ft/s^2}$. What is the distance covered before the car comes to a stop?

The deceleration is negative (the acceleration decreases); hence, $a(t) = -22$. The distance traveled is $s(t_2) - s(t_1)$, where t_2 is the instant when the car stops, and $t_1 = 0$ is the initial time when the brakes are applied.

We are looking for

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\, dt.$$

Then, $v(t_1) = 50$ mi/h and $v(t_1) = 50(5280)/3600 = 220/3$ ft/s. The car stops at the time t_2; thus, $v(t_2) = 0$ and we have

$$0 - \frac{220}{3} = v(t_2) - v(t_1) = \int_{t_1}^{t_2} a(t) dt = \int_0^{t_2} -22\, dt = -22(t_2 - 0) = -22t_2.$$

Solving for t_2, we find that $t_2 = 10/3$. We need the velocity function, so

$$v(t) = v_0 - 22t = \frac{220}{3} - 22t,$$

and

$$\int_0^{10/3} -22t + \frac{220}{3}dt = -11t^2 + \frac{220t}{3}\Big|_0^{10/3} = -11\left(\frac{10}{3}\right)^2 + \frac{2200}{9} = \frac{1100}{9}.$$

The distance is approximatively 122.22 ft.

An object that is falling freely is also an example of uniformly accelerated motion; in this case, the acceleration g due to gravity is approximately $9.8\,\mathrm{m/s^2}$ or $32\,\mathrm{ft/s^2}$. Furthermore, since we have chosen the up direction to be positive and the down direction to be negative, $a(t) = -g$. It follows from Equations 7.1 and 7.2 that

$$v(t) = v_0 - gt \tag{7.3}$$

and

$$s(t) = s_0 + v_0 t - \frac{g}{2}t^2. \tag{7.4}$$

Example 7.3. A penny is released from the CN tower at a point 300 m above the ground. Assuming that the free-fall model applies, how long does it take for the penny to hit the ground, and what is its speed at the time of impact?

We take $g = 9.8\,\mathrm{m/s^2}$. Initially, we have $s_0 = 300$ and $v_0 = 0$, so from Equation 7.4, above, $s(t) = 300 - 9.8\,\frac{t^2}{2} = 300 - 4.9t^2$.

The impact occurs when $s(t) = 0$. Solving the equation $300 - 4.9\,t^2 = 0$ for t, we find that $t \approx 7.82$ seconds; it takes the penny about 7.82 seconds to hit the ground.

By Equation 7.3, the velocity at this time is $v(7.82) = -9.8(7.82) = -76.64$ and the speed at the time of impact is $|v(7.82)| = 76.64$ m/s.

Exercises

1. Read the section titled “Rectilinear Motion” on page 268 of the textbook.
2. Do Exercises 69, 70 and 71 on page 271.

Answers to Exercises (appendix-a.htm)

Net Change

In this type of problem, we must identify a rate of change. The key to doing so is to pay attention to the units.

Example 7.4. [See Exercise 58 on page 318 of the textbook.]

Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ litres per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.

The given rate is $r(t) = 200 - 4t$ litres/min, and its antiderivative is the amount of water $w(t)$ at time t. The amount of water during the first 10 min is $w(10) - w(0)$; hence,

$$w(10) - w(0) = \int_0^{10} 200 - 4t\ dt = 200t - 2t^2 \Big|_0^{10} = 1800.$$

The amount of water is 1800 litres.

Exercises

3. Read the section titled "Applications" on pages 313-316 of the textbook.
4. Do Exercises 45, 47, 48 and 53-56 on pages 317-318.

Answers to Exercises (appendix-a.htm)

Areas Between Curves

In the first remark in Unit 6, we indicated that a definite integral gives the area under a curve only if the integrand function is positive on the interval given by the limits of integration. If the integrand function is not positive, then the definite integral is the areas under the curve on the interval where the function is positive minus the areas under the curve where the function is negative. In this section, we find areas between curves. We start with a single curve only.

Example 7.5. Consider the function $f(x) = x^2 - 4x$ on the interval $[-2, 5]$. We want to find the area under this curve. Since the graph of this function is a parabola that is concave upwards, we find the x-intercepts to see where the function is positive and where it is negative. The x-intercepts are $(0, 0)$ and $(4, 0)$, it is positive on the intervals $[-2, 0)$ and $(4, 5]$, and it is negative on $(0, 4)$. So, the area is given by the following definite integrals

$$\int_{-2}^{0} x^2 - 4x\,dx = \frac{x^3}{3} - 2x^2\Big|_{-2}^{0} = -\left(\frac{-8}{3} - 8\right) = \frac{32}{3}.$$

$$\int_{4}^{5} x^2 - 4x\,dx = \frac{x^3}{3} - 2x^2\Big|_{4}^{5} = \left(\frac{125}{3} - 2(25)\right) - \left(\frac{64}{3} - 2(16)\right) = \frac{7}{3}.$$

$$\int_{0}^{4} x^2 - 4x\,dx = -\left(\frac{x^3}{3} - 2x^2\right)\Big|_{0}^{4} = -\left(\frac{64}{3} - 2(16)\right) = \frac{32}{3}.$$

The area is

$$\frac{32}{3} + \frac{7}{3} + \frac{32}{3} = \frac{71}{3}.$$

It is always useful to use symmetry when solving a problem. If a function $f(x)$ is even, then on an interval $[-a, a]$, the areas under the curve on the left and right of the y-axis are equal. Therefore,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

The integral on the right may be easier than the integral on the left.

If a function $f(x)$ is odd, then on an interval $[-a, a]$, the areas under the curve on the left and right of the y-axis are equal, but one is under the x-axis and the other is above it. Therefore,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

and the area under the curve on the interval $[-a, a]$ is again equal to

$$2\int_{0}^{a} f(x)\,dx.$$

Example 7.6. The function

$$g(x) = x\left(x^2 + 1\right)^2$$

is odd, and the area under this curve on the interval $[-1, 1]$ is

$$2\int_{0}^{1} x\left(x^2 + 1\right)^2\,dx = \int_{0}^{1} 2x\left(x^2 + 1\right)^2\,dx = \frac{\left(x^2 + 1\right)^3}{3}\Bigg|_{0}^{1} = \frac{7}{3}.$$

Example 7.7. Analyse the following definite integral

$$\int_{-3}^{2} x\cos\left(x^2\right)\,dx.$$

The function $f(x) = x\cos\left(x^2\right)$ is odd, and the definite integral on the interval $[-2, 2]$ is zero, so the integral is equal to

$$\begin{aligned}\int_{-3}^{-2} x\cos(x^2)\,dx &= \frac{1}{2}\int_{-3}^{-2} 2x\cos(x^2)\,dx\\ &= \left.\frac{\sin(x^2)}{2}\right|_{-3}^{-2}\\ &= \frac{\sin(4)-\sin(9)}{2}.\end{aligned}$$

We can also consider functions in terms of y, and apply a definite integral to find the area between the y-axis and the curve, as we did with functions in terms of x.

Example 7.8. The function $g(y) = y^2 - 3y$ is a parabola open to the right, and its y-intercepts are $(0, 0)$ and $(0, 3)$. Therefore, the parabola is to the left of the y-axis on the interval $[0, 3]$, that is $g(y) < 0$ for y in the interval $(0, 3)$, and it is on the right of the y-axis on $[3, 5]$, that is $g(y) > 0$ for y in the interval $(3, 5)$, thus the area under this curve on the interval $[0, 5]$ is

$$-\int_0^3 y^2 - 3y\,dy = -\left.\left(\frac{y^3}{3} - \frac{3y^2}{2}\right)\right|_0^3 = -9 + \frac{27}{2}.$$

$$\int_3^5 y^2 - 3y\,dy = \left.\left(\frac{y^3}{3} - \frac{3y^2}{2}\right)\right|_3^5 = \frac{125}{3} - \frac{75}{2} - 9 + \frac{27}{2}.$$

The area is

$$-9 + \frac{27}{2} + \frac{125}{3} - \frac{75}{2} - 9 + \frac{27}{2} = -18 - \frac{21}{2} + \frac{125}{3} = \frac{79}{6}.$$

If two functions f and g are positive and $g(x) \le f(x)$ for all x in the interval $[a, b]$, then the graph between them is equal to the area under f minus the area under g, as is shown in Figure 7.1, below.

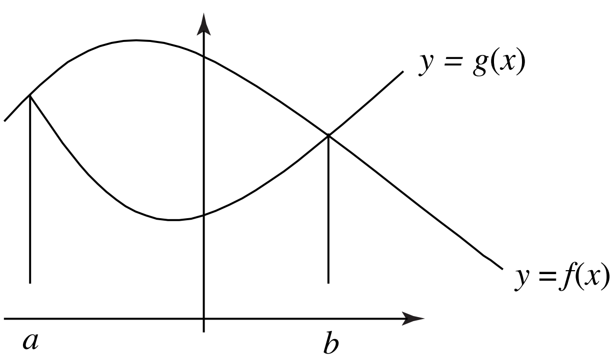


Figure 7.1: Area between functions f and g, where f and g are positive and $g(x) \le f(x)$ for all x in the interval $[a, b]$

Then by Property 2 (unit06-05.htm#properties1-8), the area is

$$\int_a^b f(x) - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx.$$

Example 7.9. The area between $\sin x$ and $\cos x$ on the interval $[0, \pi / 4]$, is shown in Figure 7.2, below.

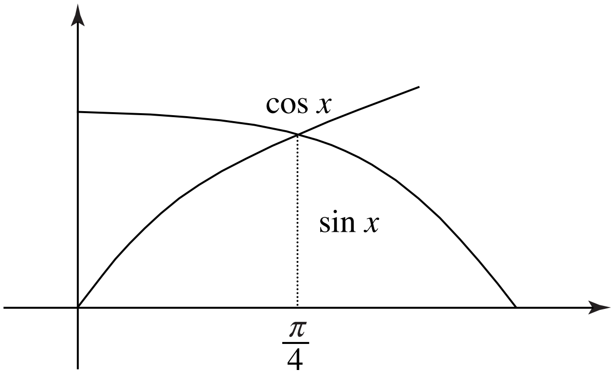


Figure 7.2: Area between $\sin x$ and $\cos x$ on the interval $\left[0, \frac{\pi}{4}\right]$

Since $\cos x \geq \sin x$ on the interval $[0, \pi / 4]$, the area is given by the definite integral

$$\int_0^{\pi/4} \cos x - \sin x \, dx = \sin x + \cos x \Big|_0^{\pi/4} = \sqrt{2} - 1.$$

If two functions f and g are not positive and $g(x) \leq f(x)$ for all x in the interval $[a, b]$, as in Figure 7.3, below, then to obtain the area between their graphs, we consider the area between the graphs of the functions $f(x) + C$ and $g(x) + C$, where C is a positive constant such that the functions $f(x) + C$ and $g(x) + C$ are positive (see Figure 7.3).

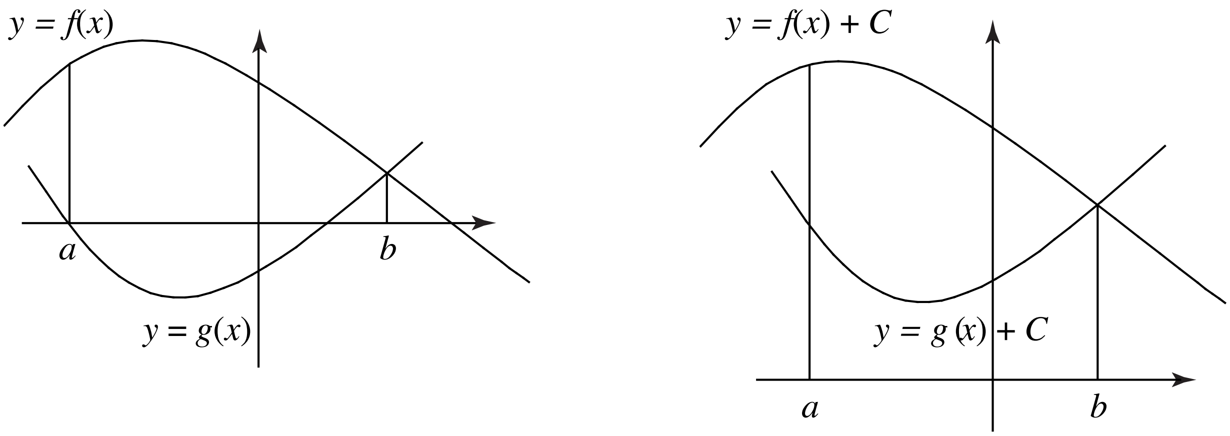


Figure 7.3: Technique for determining the area between functions f and g, where f and g are not positive and $g(x) \leq f(x)$ for all x in the interval $[a, b]$

Since the functions $f(x) + C$ and $g(x) + C$ are a shift of the functions $f(x)$ and $g(x)$, the areas between them are equal. But now $f(x) + C$ and $g(x) + C$ are positive, and the area between them is

$$\int_a^b f(x) + C - (g(x) + C) \, dx = \int_a^b f(x) - g(x) \, dx.$$

This example shows that the area between any two function $f(x)$ and $g(x)$, such that $g(x) \leq f(x)$ on the interval $[a, b]$, is equal to

$$\int_a^b f(x) - g(x) \, dx.$$

Example 7.10. The area between the functions $f(x) = x - x^2$ and $g(x) = 3x - 1$ is given by a definite integral. To find it, we must find their points of intersection; that is, $x - x^2 = 3x - 1$.

Solving for x, $x = -1 - \sqrt{2}$, and $x = -1 + \sqrt{2}$.

From their graphs, we see that $g(x) \leq f(x)$ on the interval $\left[-1 - \sqrt{2}, -1 + \sqrt{2}\right]$ (see Figure 7.4, below).

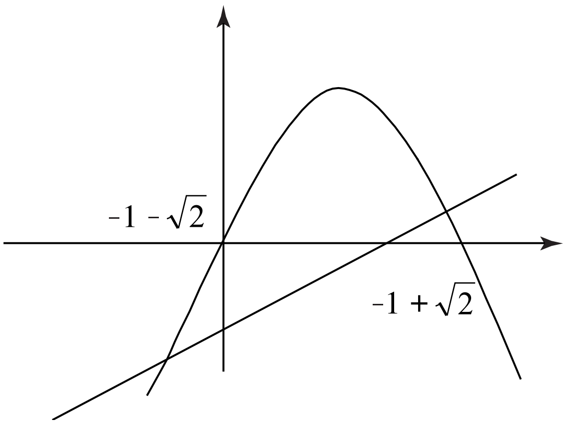


Figure 7.4: Area between the functions $f(x) = x - x^2$ and $g(x) = 3x - 1$ (not to scale)

Hence the area is given by

$$\begin{aligned}
\int_{-1-\sqrt{2}}^{-1+\sqrt{2}} x - x^2 - (3x-1)\,dx &= \int_{-1-\sqrt{2}}^{-1+\sqrt{2}} x - x^2 - 3x + 1\,dx \\
&= \int_{-1-\sqrt{2}}^{-1+\sqrt{2}} -2x - x^2 + 1\,dx \\
&= -x^2 - \frac{x^3}{3} + x\Big|_{-1-\sqrt{2}}^{-1+\sqrt{2}} \\
&= x\left(-x - \frac{x^2}{3} + 1\right)\Big|_{-1-\sqrt{2}}^{-1+\sqrt{2}} \\
&= \left(-1+\sqrt{2}\right)\left[2 - \sqrt{2} - \frac{\left(-1+\sqrt{2}\right)^2}{3}\right] \\
&\quad + \left(1+\sqrt{2}\right)\left[2 + \sqrt{2} - \frac{\left(1+\sqrt{2}\right)^2}{3}\right] \\
&= \frac{8\sqrt{2}}{3}.
\end{aligned}$$

Example 7.11. To find the area between the parabola $f(x) = x^2 - 6x + 9$ and the line $g(x) = 4x$ on the interval $[0, 3]$, we must make a sketch of these functions (see Figure 7.5, below).

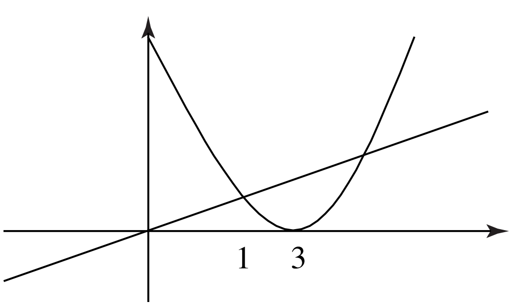


Figure 7.5: Area between the the parabola $f(x) = x^2 - 6x + 9$ and the line $g(x) = 4x$ on the interval $[0, 3]$ (not to scale)

We find the point of intersection of these two functions on the interval $[0, 3]$; thus, $x^2 - 6x + 9 = 4x$.

Solving for x, we find that $\quad x = 1 \quad$ and $\quad x = 9$.

We see that we have two different areas to consider, one on the interval $[0, 1]$ where $f(x) \geq g(x)$, and another on $[1, 3]$ where $f(x) \leq g(x)$. The areas are given by the following definite integrals:

$$\int_0^1 x^2 - 6x + 9 - 4x\, dx + \int_1^3 4x - \left(x^2 - 6x + 9\right) dx$$
$$= \int_0^1 x^2 - 10x + 9\, dx + \int_1^3 -x^2 + 10x - 9\, dx.$$

Evaluating each of these integrals, we obtain

$$\int_0^1 x^2 - 10x + 9\ dx = \frac{x^3}{3} - 5x^2 + 9x\,\Big|_0^1 = \frac{13}{3},$$

and

$$\int_1^3 -x^2 + 10x - 9\ dx = -\frac{x^3}{3} + 5x^2 - 9x\,\Big|_1^3 = \frac{40}{3}.$$

The area is $\dfrac{13}{3} + \dfrac{40}{3} = \dfrac{53}{3}$.

The area between these functions is given by the integral

$$\begin{aligned} \int_1^9 4x - (x^2 - 6x + 9)\ dx &= \int_1^9 10x - x^2 - 9\ dx \\ &= -\frac{x^3}{3} + 5x^2 - 9x\Big|_1^9 \\ &= \frac{256}{3} \end{aligned}$$

For functions in terms of the variable y, we also know that if $g(y) \leq f(y)$ on the interval $[c, d]$, then the graph of the function g is on the left of the graph of f, and the area between these functions is given by the definite integral

$$\int_c^d f(y) - g(y)\ dy.$$

Example 7.12. See Exercise 18 on page 325 of the textbook.

The sketch of the curves $4x + y^2 = 12$ and $y = x$ is shown in Figure 7.6, below.

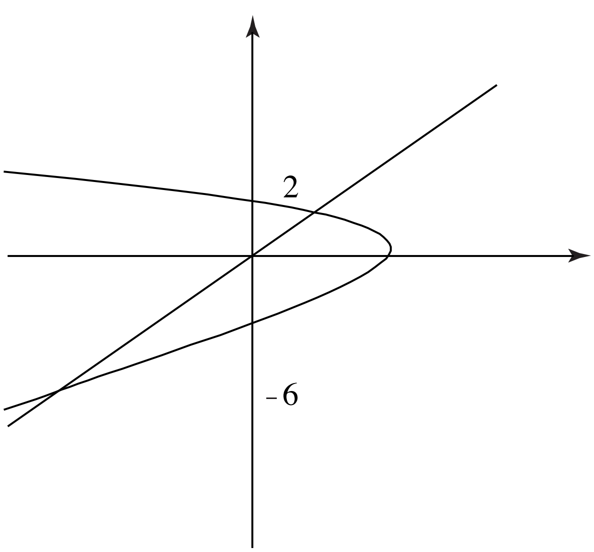


Figure 7.6: Curves $4x + y^2 = 12$ and $y = x$

The points of intersection of these two curves are the solutions of the equation $4y + y^2 = 12$, since $y = x$. Solving for y, we find that $y = -6$ and $y = 2$. [Check this.]

Since the parabola is on the right of the line, the area is given by the integral

$$\begin{aligned}\int_{-6}^{2} 3 - \frac{y^2}{4} - y\,dy &= 3y - \frac{y^3}{12} - \frac{y^2}{2}\bigg|_{-6}^{2} \\ &= 3(2) - \frac{8}{12} - 2 - (-18 + 18 - 18) \\ &= \frac{64}{3}.\end{aligned}$$

Observe that solving for x in the equation $4x + y^2 = 12$ gives the function

$$g(y) = 3 - \frac{y^2}{4}.$$

Note that you will find the theorem below helpful as you complete the assigned exercises.

Theorem 7.2. Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$, then

a. if f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

b. if f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$.

Exercises

5. Read Theorem 6 “Integrals of Symmetric Functions,” on page 296 of the textbook.
6. Do the odd numbered exercises from 5 to 17, and Exercise 21, on page 324.

Answers to Exercises (appendix-a.htm)

Work

To find work, we must find the corresponding force function.

Study the section titled "Work" on pages 326-329 of the textbook.

The following table gives the units we use when dealing with work.

System	Force	$\times$	Distance	$=$	Work
SI	newton (N)	$\times$	metre (m)	$=$	joule (J)
CGS	dyne (dyn)	$\times$	centimetre (cm)	$=$	erg
BE	pound (lb)	$\times$	foot (ft)	$=$	foot-pound (ft-lb)

The conversion factors are given below.

$1 \text{ N} = 10^5 \text{ dyn} \approx 0.225 \text{ lb} \qquad 1 \text{ lb} \approx 4.45 \text{ N}$

$1 \text{ J} = 10^7 \text{ erg} \approx 0.738 \text{ ft-lb} \qquad 1 \text{ ft-lb} \approx 1.36 \text{ J} = 1.36 \times 10^7 \text{ erg}$

Example 7.13. In 1976 Vasili Alexeev lifted a record-breaking 562 lb from the floor to above his head (about 2 m). In 1957, Paul Anderson braced himself against the floor and used his back to lift 6720 lb of lead and automobile parts a distance of 1 cm. Who did more work?

The force that the Earth exert on an object is the object's weight; thus, Alexeev applied a force of 562 lb over a distance of 2 m and Anderson applied a force of 6720 lb over a distance of 1 cm.

We choose SI units. Converting, we have

$$562 \text{ lb} \approx 562 \text{ lb} \times 4.45 \text{ N/lb} \approx 2500 \text{ N},$$

and

$$6270 \text{ lb} \approx 6270 \times 4.45 \text{ N/lb} \approx 27900 \text{ N}.$$

Since $1 \text{ cm} = 0.01 \text{ m}$, we have

$$\text{Alexeev's work } = (2500\,\text{N/lb}) \times (2\,\text{m}) = 5000\,\text{J},$$

and

$$\text{Anderson's work } = (27900\,\text{N}) \times (0.01\,\text{m}) = 279\,\text{J}.$$

Therefore, Alexeev did more work. For Anderson to have done as much work as Alexeev, he would have needed to move the 6270 lb a distance of

$$d = \frac{5000}{27900} \approx 0.179\,\text{m};$$

that is, about 18cm.

Example 7.14. [See Exercise 16 on page 329 of the textbook.]

A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well that is 80 ft deep. The bucket is filled with 40 lb of water and is pulled up at a rate of 2 ft/s, but water leaks out of a hole in the bucket at a rate of 0.2 lb/s. Find the work done by pulling the bucket to the top of the well.

Observe that the work needed to lift the bucket is $4 \text{ lb}(80 \text{ ft}) = 320$ ft-lb. Thinking through the process we, see that at time t seconds the bucket is $2t$ ft above the original 80 ft depth of the well. Hence, $x_i * = 2t$, and at this instant the bucket holds $(40 - 0.2t)$ lb of water. When the bucket is $x_i *$ ft above its original 80 ft, the bucket holds

$$\left[40 - 0.2\left(\frac{1}{2}\right) x_i{*}\right] \text{ lb of water.}$$

To move this water by a distance of Δx, requires

$$\left(40 - \frac{1}{10} x_i{*}\right) \Delta x \text{ ft-lb of work.}$$

Thus, the work needed to lift the water is

$$\begin{aligned} W &= \lim_{x\to\infty} \sum_{i=1}^{n} \left(40 - \left(\frac{1}{10}\right) x_i{*}\right) \Delta x \\ &= \int_0^{80} 40 - \frac{x}{10}\, dx \\ &= 40x - \frac{x^2}{20}\Bigg|_0^{80} \\ &= 3200 - 320. \end{aligned}$$

Adding the work of lifting the bucket gives a total of 3200 ft-lb.

Exercises

7. Do Exercises 3, 7, 9, 13 and 17 on page 329 of the textbook.

Answers to Exercises (appendix-a.htm)

Average Value of a Function

The "average value of a function" is the last application of the definite integral that we cover in this course.

Definition* 7.3. The *arithmetic mean* or *arithmetic average* of a finite sequence of n terms $x_1, x_2, x_3, \ldots, x_n$ is given by their sum divided by n. Thus,

$$x_{\text{avg}} = \frac{x_1 + x_2 + \ldots. + x_n}{n}.$$

If the values of the sequence correspond to values of a function $x_i = f(a_i)$ for $1 \le i \le n$, then the arithmetic average is

$$f_{\text{avg}} = \frac{f(a_1) + f(a_2) + \ldots.. + f(a_n)}{n}.$$

To extend this concept to infinitely many values of $f(x)$ on a closed interval $[a, b]$, we use the definite integral. In this case, the number of terms corresponds to the length of the interval $[a, b]$, that is $b - a$, and the sum of the values corresponds to the definite integral.

Hence,

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Example 7.15. Find the average value of the function $f(x) = \sqrt{x}$ on the interval $[1, 5]$, and find the values of x for which the value of f is equal to this average.

$$f_{\text{avg}} = \tfrac{1}{5-1}\int_1^5 \sqrt{x}\, dx = \tfrac{2}{12}x^{3/2}\Big|_1^5 = \tfrac{1}{6}(5^{3/2} - 1) \approx 1.7.$$

The values of x for which

$$f(x) = \sqrt{x} = \frac{\left(5^{3/2} - 1\right)}{6}$$

is the solution of

$$x = \frac{\left(5\sqrt{5} - 1\right)^2}{36}$$

and this is

$$x = \frac{7}{3} - \frac{5\sqrt{5}}{18} \approx 1.71.$$

For a continuous functions f on an interval $[a, b]$ it is always possible to find a number c such that $f(c) = f_{\text{avg}}$. This statement is known as the Mean Value Theorem for Integrals.

Theorem 7.4. Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$, such that

$$\int_a^b f(x)\, dx = f(c)(b-a).$$

Proof. Let

$$g(x) = \int_a^x f(t)\, dt \quad \text{for} \quad a \le x \le b.$$

By Part 1 of the Fundamental Theorem of Calculus (unit06.htm#theorem6-12), the function g is continuous on $[a, b]$ and differentiable on (a, b). Hence, by the Mean Value Theorem (unit06.htm#mean-value-theorem), there is a number c, $a \leq c \leq b$, such that

$$g(b) - g(a) = g'(c)(b - a).$$

Since $g'(c) = f(c)$, we conclude that

$$\int_a^b f(t)\,dt - \int_a^a f(t)\,dt = f(c)(b - a).$$

Therefore, for c in $[a, b]$

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Q.E.D

The geometric interpretation of this theorem is given on page 332 of the textbook.

Exercises

8. Read the section titled "Average Value of a Function" on pages 330-332 of the textbook.
9. Do Exercises 1, 3, 5 and 7, parts (a) and (b) of Exercise 9, and Exercises 11, 17 and 19 on page 333.

Answers to Exercises (appendix-a.htm)

Finishing This Unit

1. Review the objectives of this unit and make sure you are able to meet all of them.
2. If there is a concept, definition, example or exercise that is not yet clear to you, go back and reread it, then contact your tutor for help.
3. Do the exercises in "Learning from Mistakes" section for this unit.
4. Find the area of the region bounded by the curves given below.
   a. $y = x^2 - x - 6, y = 0$
   b. $y = 1 - x^2, y = 1 - \sqrt{x}$
   c. $x + y = 0, x = y^2 + 3y$
   d. $y = \sin(\pi x / 2), y = x^2 - 2x$
5. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?
6. Find the average value of the function $f(t) = t \sin\left(t^2\right)$ on the interval $[0, 1]$.
7. If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x + h]$?

Learning from Mistakes

There are mistakes in each of the following solutions, identify them and give the right answer.

1. A spacecraft uses a sail and the "solar wind" to produce a constant acceleration of 0.032 m/s^2. Assuming that the spacecraft has a velocity of 600 km/h when the sail is first raised, how far will the spacecraft travel in 1 hour, and what will its velocity be at the end of the hour?

   Erroneous Solution

   We define $t_0 = 0$ to be the time the sail is raised, and $t_1 = 1$ to be one hour later. The acceleration $a = 0.032$. At time t, the velocity is $v(t)$ and the displacement is $s(t)$. We then have $v(0) = 600$ and $s(0) = 0$.

   $$v(1) - v(0) = \int_0^1 a\ dt = \int_0^1 0.032\ dt = 0.032.$$

   Thus,

   $$v(1) = 0.32 + v(0) = 0.032 + 600 = 600.032 \quad \text{and} \quad v(t) = 60 + 0.032t.$$

   $$s(1) - s(0) = \int_0^1 600 + 0.032t\, dt = 600t + .016t^2 \Big|_0^1 = 600.016.$$

   Answer: The velocity is 600.032 km/h, and the distance is 600.016 km.

2. Find the area under the curve $y = x^2 + x - 2$ on the interval $[0, 3]$.

   Erroneous Solution

   $$\int_0^3 x^2 + x - 2\, dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x \Big|_0^3 = \frac{15}{2}.$$

3. Find the area between the curves $y = \dfrac{1}{x^2}$, $y = x$ and $y = \dfrac{x}{8}$.

Erroneous Solution

The point of intersection of $y = \dfrac{1}{x^2}$ and $y = x$ is the solution of $x = \dfrac{1}{x^2}$; that is $x = 1$ and $(1, 1)$. The point of intersection of $y = frac1x^2$ and $y = \dfrac{x}{8}$ is the solution of $\left(x^2\right)\dfrac{x}{8} = 1$; that is, $x = 2$ and $\left(2, \dfrac{1}{4}\right)$.

$$\begin{aligned}\int_0^1 \frac{1}{x^2} - x\,dx + \int_1^2 -\frac{1}{x^2} + \frac{x}{8}\,dx &= -\frac{1}{x} - \frac{x^2}{2}\bigg|_0^1 + \frac{1}{x} + \frac{x^2}{16}\bigg|_1^2 \\ &= -\frac{3}{2} + \frac{12}{16} - \frac{17}{16} \\ &= -\frac{35}{16}.\end{aligned}$$

4. A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?

Erroneous Solution

$$\int_{.2}^{.8} 100x\,dx = 50x^2\Big|_{.2}^{.8} = 50(.64 - .04) = 30.$$

Answer: 30 J.

Answers to Exercises

Note: Answers to the odd-numbered exercises from the textbook are provided in the Student Solution Manual.

Unit 2

1. At a given time, the number of children in a family may be either changing or constant; in all cases, however, it eventually becomes a constant.
2. The acceleration due to gravity is a changing variable as gravity is different at the poles than at the equator, and different on mountaintops than at sea level. However, we treat it as a constant variable with a value of 9.8 metres per second per second.
3. The number of days in a year is a changing variable, because of leap years.
4. Any symbol is adequate, say V. We would say, "Let V be the volume of an ice cube."
5. Let T be the temperature of a living human being. Given that clinical hypothermia (a fatal condition if not treated) occurs at 33°C and hyperpyrexia (also fatal if untreated) occurs at 41.1°C, we could say: range $31 \leq T < 44$ degrees Celsius. The range you estimate may vary substantially from that given.
6. For the distance d at time t, the relation D is $D = \{(t, d) \mid d \text{ is the distance travelled at time } t\}$.
7. T is the set of all pairs $((b, h), A)$ such that A is the area of a triangle with base b and height h.
8. A is related by T to the pair (b, h) if and only if A is the area of a triangle with base b and height h.
9. For the weight w of a person of age a, we have that the relation $A = \{(w, a) \mid w \text{ is the weight of a person of age } a\}$.
10. T is the set of all pairs (A, y) such that A is your age in the year y, or A is related to y by the relation T if and only if A is your age in the year y.
11. Any pair of the form $\left(s, s^2\right)$ is in M; any pair not of this form is not in M.
12. If you were 34 years old in the year 2000, the answer is yes; if not, the answer is no.
13. Yes, $A(b, h)$.
14. $s(30) = 50$
15. a. $T(c)$
    b. $L(t)$
    c. $r(P)$
16. Let L be the length, W the width and A the area of the rectangle. Thus $A(L) = L(10 - L) = 10L - L^2$.
17. Let V be the volume, L be the length and S be the surface area of the cube. Thus $S(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$.
18. Let x be the length of each side of the box, h its height and S its surface area. Then, $S(x) = x^2 + 4x\left(2 / x^2\right) = x^2 + (8 / x)$.
19. a. Let C be the cost of renting the car, and n the number of kilometres driven. So, $C(n) = 50 + 0.43n$
    b. $D_C = [0, \infty)$, $R_C = [50, \infty)$.
    c. Taking the distance from Edmonton to Calgary as 294 km, we calculate $C(n) = 50 + 0.43(294) = 176.42$. Adding 5% GST gives us $\$185.24$.
20. i. a. $D_f = \mathbb{R}$
       b. $f(1) = 2$, $f(-1) = -1$
    ii. a. $D_f = \mathbb{R}$
       b. $f(1) = 2$, $f(-1) = 4$

iii. a. $D_f = \mathbb{R}$

b. $f(1) = 1, f(-1) = 1$

iv. a. $D_f = \mathbb{R}$

b. $f(1) = 5, f(-1) = -1.$

21. a. When the temperature increases from 70 to 80 the volume also increases; when the temperature decreases from 80 to 72, the volume also decreases.

b. $V(80) = 21 \text{ m}^3$

c. $V(73)$ is between 17 m^3 and 18 m^3

22. a. $\mathbb{R} - \{1/3\}$ in interval notation $(-\infty, 1/3) \cup (1/3, \infty)$.

b. $\mathbb{R} - \{2, 1\} = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$.

c. $[0, \infty)$.

d. $[0, 4]$.

e. $(-\infty, 0) \cup (5, \infty)$

23. a. $D_R = (\infty, -3) \cup (-3, 0) \cup (0, \infty)$.

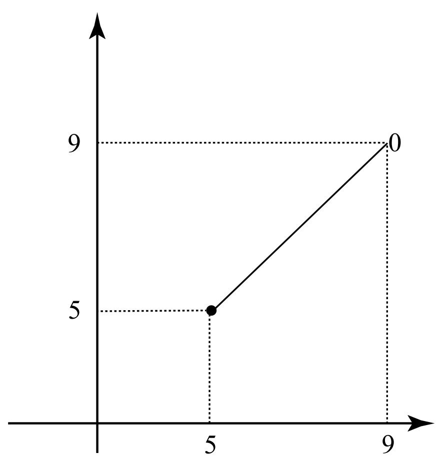
b. $D_F = (-\infty, \infty) = \mathbb{R}$.

24. a. $R(3x+1) = \dfrac{9x^2 + 6x - 8}{9x^2 + 15x + 4}$.

b. $F(3x+1) = \dfrac{4}{\sqrt{27x^2 + 18x + 7}}$.

25. $h(x) = \dfrac{1}{(x-2)(x-5)}$

26.



27. a. $D_f = \mathbb{R}$.

b. $f(3) = 3$.

28. a. No.

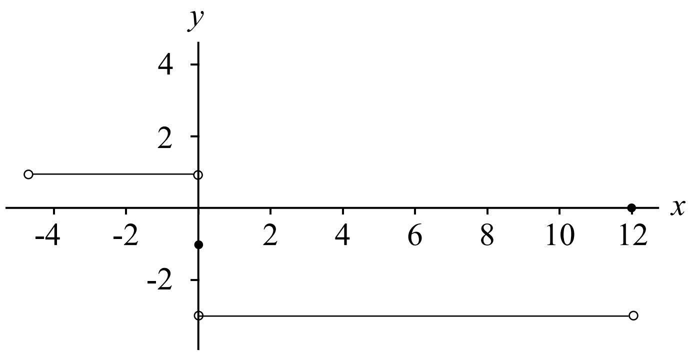
b. $g(-2) = 1$.

c. $g(2) = 2$.

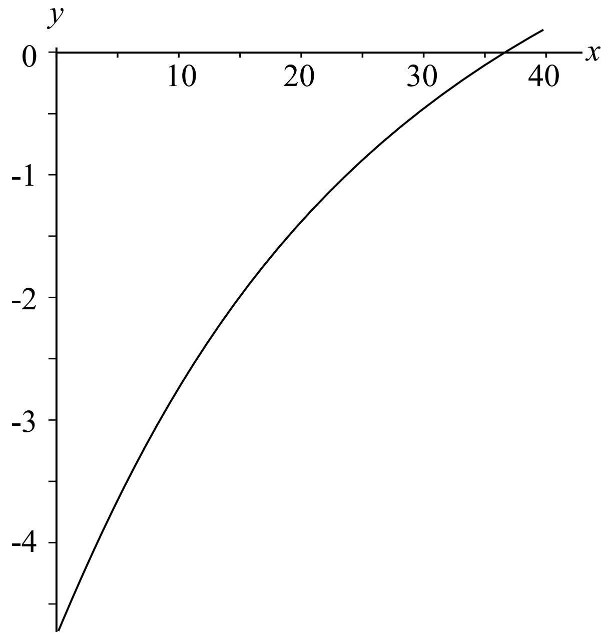
d. $D_g = \mathbb{R} - 0, 4$.

29. The vertical lines $x = -3$, $x = 2$ and $x = 5$ are vertical asymptotes.

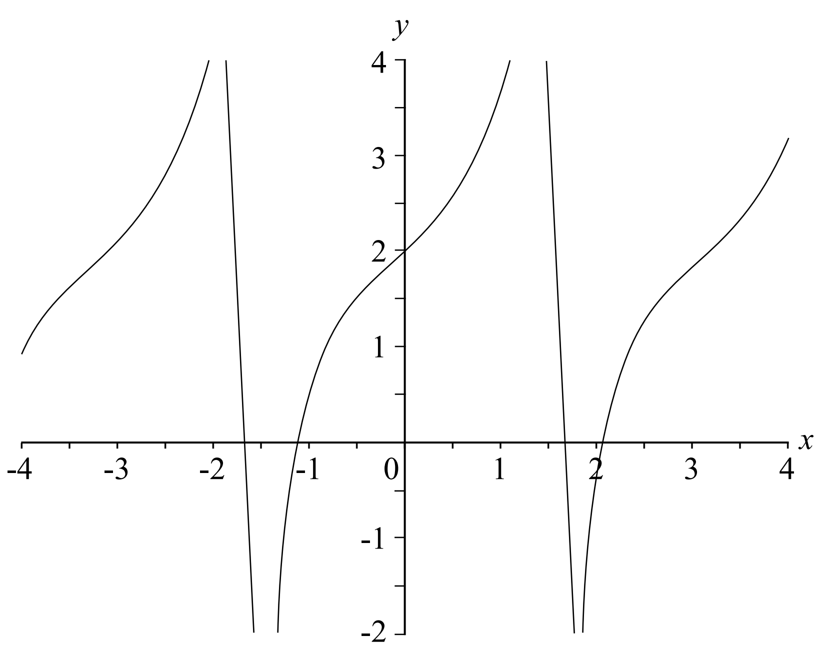
30.



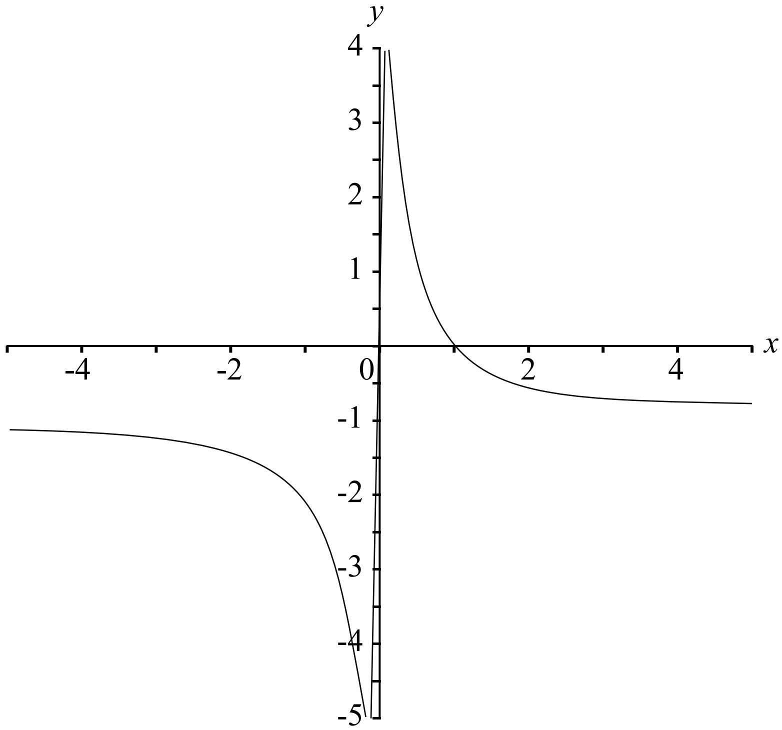
31. a.



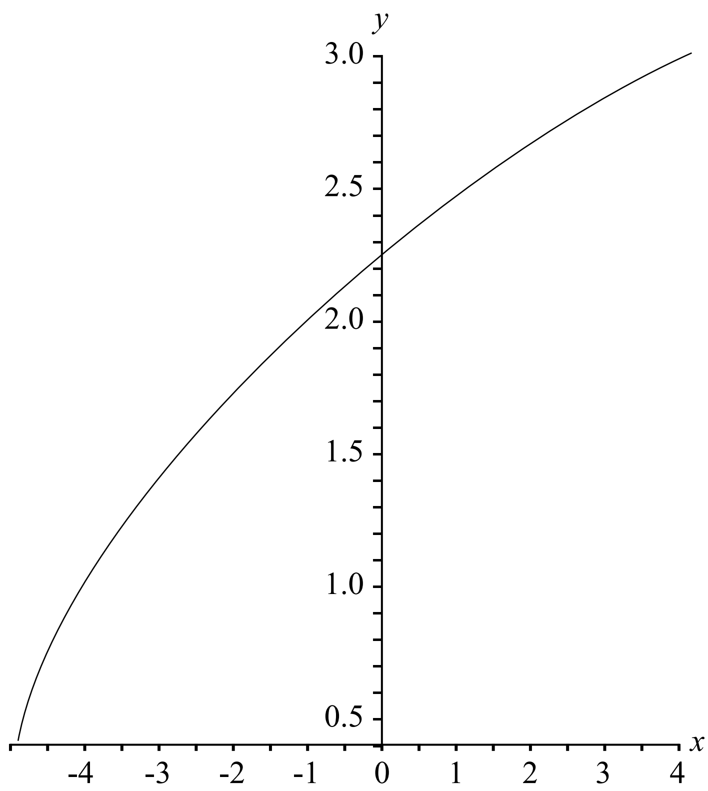
b.



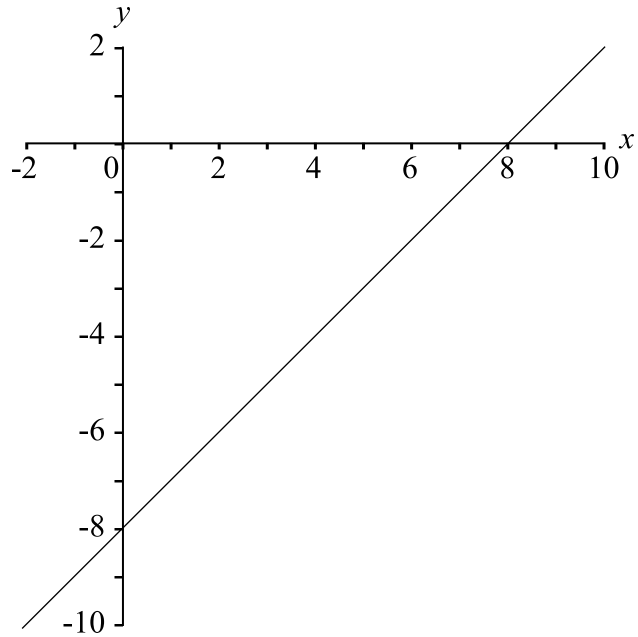
c.



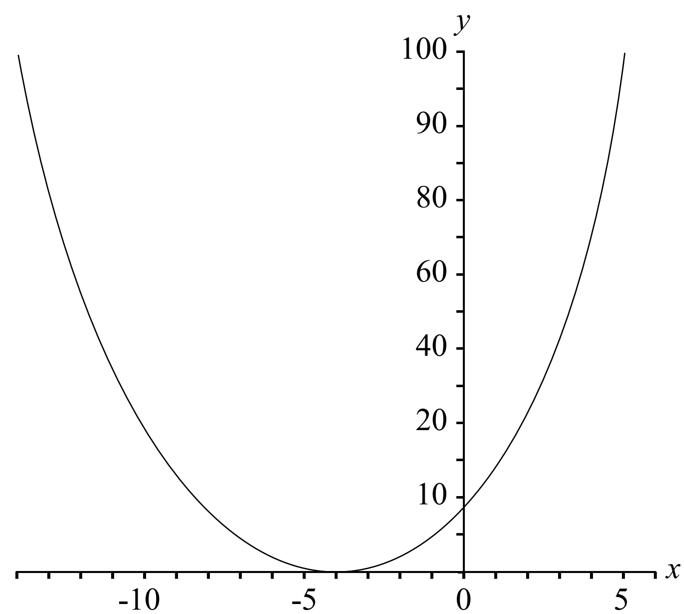
d.



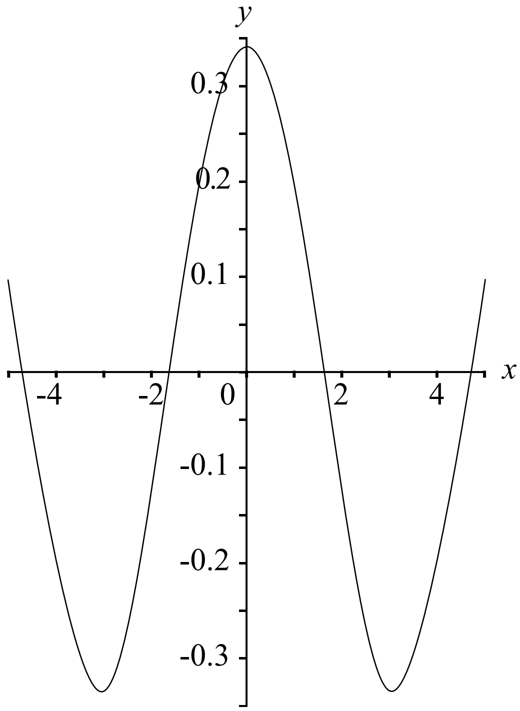
32. a. $F(x)$ is a shift down by 8 units of the function $f(x) = x$.



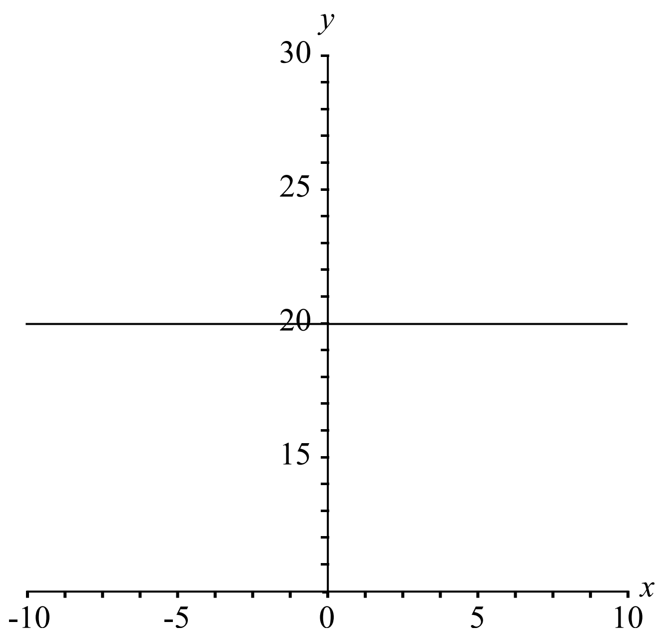
b. $F(x)$ is a shift to the left by 4 units of the quadratic function $f(x) = x^2$.



c. $F(x)$ is a vertical compression of $\cos x$ by a factor of 3.

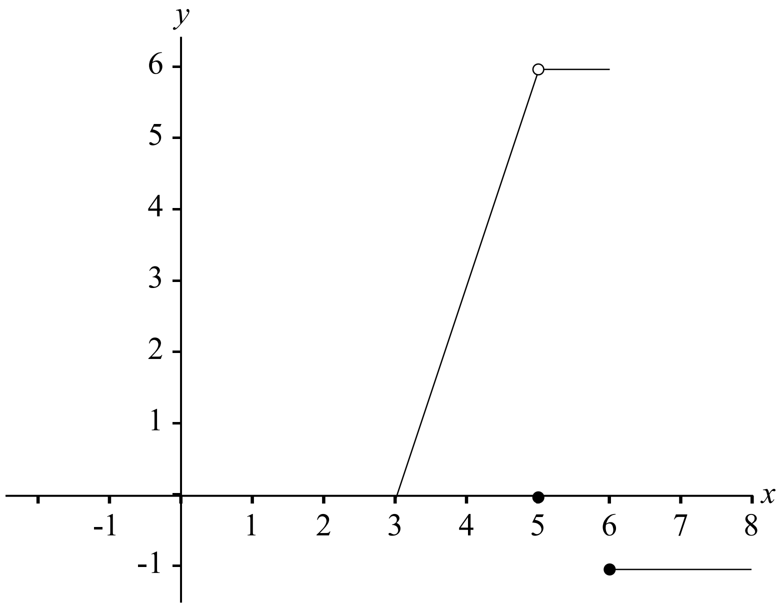


d. $F(x)$ is a vertical stretch of the constant function $f(x) = 4$ by a factor of 5.



Unit 3

2.

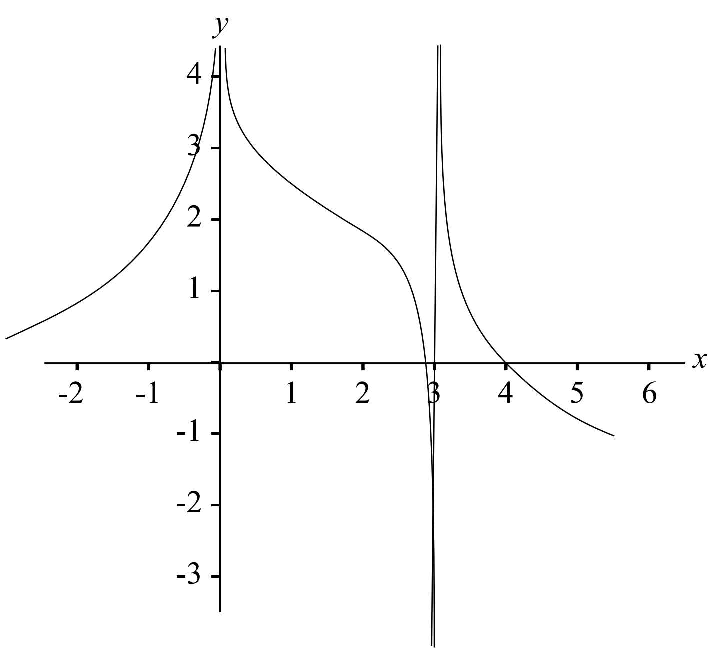


4. a. $\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty \neq \infty = \lim_{x \to \frac{\pi}{2}^-} \tan x$

   b. $\lim_{x \to 0^+} \cot x = \infty \neq -\infty \lim_{x \to 0^-} \cot x$

   c. $\lim_{x \to 0^+} \csc x = \infty \neq -\infty = \lim_{x \to 0^-} \csc x$

5.



7. The functions in numbers 3, 5 and 6, are polynomials; hence, they are continuous everywhere, and their limits are 59, 205 and 256, respectively.

The function in 4 is rational and 2 is in its domain, so the limit is $\dfrac{3}{4}$.

The functions in 7, 8 and 9 are the composition of functions: in 7 the function is continuous at 1 and the limit is $\dfrac{1}{8}$; in 8 the function is continuous at -2 and the limit is 16; in 9 the function is continuous at 4 and the limit is 0.

13. a. $\lim\limits_{x \to \pi/2^+} \sec x = \lim\limits_{x \to \pi/2^+} \dfrac{1}{\cos x} = -\infty$ since $\lim\limits_{x \to \pi/2^+} \cos x = 0$ and $\cos x < 0$ for $x \to \pi/2^+$.

b. $\lim\limits_{x \to 0^-} \csc x = \lim\limits_{x \to 0^-} \dfrac{1}{\sin x} = -\infty$ since $\lim\limits_{x \to 0^-} \sin x = 0$ and $\sin x < 0$ for $x \to 0^-$

14. $\lim\limits_{x \to \pi/2^+} \sec x = -\infty$ by part (a) of Exercise 13, above. $\lim\limits_{x \to \pi/2^-} \sec x = \infty$ since $\lim\limits_{x \to \pi/2^-} \cos x = 0$ and $\cos x > 0$ for $x \to \pi/2^-$. Hence, $\lim\limits_{x \to \pi/2^+} \sec x \neq \lim\limits_{x \to \pi/2^-} \sec x$. The limit does not exist.

15. a. $\lim\limits_{x \to 0} \csc^2 x = \infty$

b. $\lim\limits_{x \to 0^-} \dfrac{1}{\sin x + \tan x} = -\infty$

c. $\lim\limits_{x \to 0} \dfrac{1}{x^2 \cos x} = \infty$

17. a. $\lim\limits_{x \to 4^+} \dfrac{x^2 + 3x - 1}{x - 4} = \infty$ since $\lim\limits_{x \to 4^+} x^2 + 3x - 1 = 27 > 0$ and $\lim\limits_{x \to 4^+} \dfrac{1}{x - 4} = \infty$.

b. $\lim\limits_{t \to 4^+} \dfrac{\sin(t-4) + 3t}{t^2 - 16} = \infty$ since $\lim\limits_{t \to 4^+} \sin(t-4) + 3t = 12 > 0$ and $\lim\limits_{t \to 4^+} \dfrac{1}{t^2 - 16} = \infty$.

18. $x = -1/2$

19. The limit would be ∞ since the function $f(x) + c$ increases without bound as $x \to a$, regardless of the value of c.

20. The limit would be $-\infty$ since the function $f(x) + g(x)$ decreases without bound as $g(x)$ get close to c a fixed number, while $f(x)$ decreases without bound.

21. The limit would be $\lim\limits_{x \to 0^+} \dfrac{1}{x^2} = \infty$ since $\lim\limits_{x \to 0^+} x^2 = 0$ and $x^2 \geq 0$ for any x.

The limit

$$\lim_{x \to 0^+} -\frac{1}{x} = -\infty$$

since $\lim\limits_{x \to 0^+} x = 0$ and $-x < 0$ for $x \to 0^+$.

The limit

$$\lim_{x \to 0^+} \frac{1}{x^2} - \frac{1}{x} = \lim_{x \to 0^+} \frac{1 - x}{x^2} = \infty,$$

since $\lim\limits_{x \to 0^+} 1 - x = 1 > 0$ and $\lim\limits_{x \to 0^+} \dfrac{1}{x^2} = \infty$.

27. a. Laws of Limits, part 3 of Theorem 3.23

b. Example 3.61 and part (a) of Theorem 3.17

c. Theorem 3.18

d. part 3 of Law of Limits 3.23 and Example 3.60

31. a. By part (a) of Theorem 3.29, the limit is $\frac{4}{2} = 2$

b. By part (b) of Theorem 3.29, the limit is 0.

c. By part (c) of Theorem 3.29.

d. By part (c) of Theorem 3.29.

32. $\frac{1}{x} \to 0^+$ as $x \to \infty$; hence, if $u = \frac{1}{x}$, then $\lim_{x\to 0^+} \cot(u) = \infty$.

33. By Remark 2.

a. $5x^2 + 4x - 1 > 0$ for $x \to \infty$

b. $3 - 4x + 6x^3 < 0$ as $x \to -\infty$

c. $(3x-1)(5-x) = -3x^2 + 16x - 5 < 0$ for $x \to \infty$.

34. a. If $c = 0$, then $cf(x) = 0$ and $\lim_{x\to a} cf(x) = 0 = cL$.

Let $c \neq 0$ and $\varepsilon > 0$. Since

$$\lim_{x\to a} f(x) = L \quad \text{for} \quad \frac{\varepsilon}{|c|} > 0,$$

there exists a $\delta > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{|c|} \quad \text{for} \quad 0 < |x-a| < \delta.$$

Hence,

$$|cf(x) - cL| = |c||f(x) - L| < \frac{\varepsilon}{|c|}(|c|) = \varepsilon \quad \text{for} \quad 0 < |x-a| < \delta.$$

b. Let $\varepsilon > 0$. Since $\lim_{u\to b} f(u) = L$, there exists a $\delta_1 > 0$ such that

$$|f(u) - L| < \varepsilon \quad \text{for any} \quad 0 < |u-b| < \delta_1. \tag{5}$$

Since

$$\lim_{x\to a} g(x) = b \quad \text{for} \quad \delta_1 > 0,$$

there exists a $\delta_2 > 0$ such that

$$|g(x) - b| < \delta_1 \quad \text{for} \quad 0 < |x-b| < \delta_2.$$

Therefore if $0 < |x-a| < \delta_2$, then $|g(x) - b| < \delta_1$. Substituting $g(x)$ for u in Equation 1 gives us $|f(g(x)) - L| < \varepsilon$.

35. Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = K$ for $\frac{\varepsilon}{2} > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

I. $|f(x) - L| < \frac{\varepsilon}{2}$ for $0 < |x-a| < \delta_1$, and

II. $|f(x) - K| < \frac{\varepsilon}{2}$ for $0 < |x-a| < \delta_2$.

Hence, for $\delta = \min(\delta_1, \delta_2)$ and $0 < |x-a| < \delta$, statements I and II, hold and

$$\begin{aligned} &|K - L| \\ &= |f(x) - K + L - f(x)| \leq |f(x) - K| + |f(x) - L| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $|K - L| < \varepsilon$ for any $\varepsilon > 0$, and we conclude that $|K - L| = 0$. Otherwise, if $|K - L| > 0$ for $\varepsilon = \frac{|K-L|}{2}$, we find that

$$|K-L| < \frac{|K-L|}{2},$$

and this implies that $1 < \frac{1}{2}$, a contradiction.

So $|K-L| = 0$, and we conclude that $K = L$.

36. a. Since $g(x) < 0$ for $x \to a^+$, there exists a $\delta_1 > 0$ such that

$$g(x) < 0 \quad \text{for} \quad x-a < \delta_1. \tag{I(a)}$$

Since $\lim\limits_{x \to a^+} g(x) = 0$ for $N < 0$, we know that $-N > 0$, and for this positive number, there exists a $\delta_2 > 0$ such that

$$|g(x)| < -N \quad \text{for} \quad 0 < x-a < \delta_2. \tag{II(a)}$$

Hence, for $\delta = \min(\delta_1, \delta_2)$, Statements I($a$) and II($a$) hold, and

$$|g(x)| = -g(x) < -N \quad \text{for} \quad 0 < x-a < \delta.$$

We conclude that

$$\frac{1}{g(x)} < N \quad \text{for} \quad 0 < x-a < \delta.$$

b. Since $g(x) > 0$ for $x \to a$, there exists a $\delta_1 > 0$ such that

$$g(x) > 0 \quad \text{for} \quad |x-a| < \delta_1. \tag{I(b)}$$

Since $\lim\limits_{x \to a} g(x) = 0$ for $M > 0$, there exists a $\delta_2 > 0$ such that

$$|g(x)| < M \quad \text{for} \quad 0 < |x-a| < \delta_2. \tag{II(b)}$$

Hence, for $\delta = \min(\delta_1, \delta_2)$, Statements I($b$) and II($b$) hold, and

$$|g(x)| = g(x) < M \quad \text{for} \quad 0 < |x-a| < \delta.$$

We conclude that

$$\frac{1}{g(x)} > M \quad \text{for} \quad 0 < |x-a| < \delta.$$

c. Since $\lim\limits_{x \to a} f(x) = L < 0$ for $-L\,/\,2 > 0$, there exists a $\delta_1 > 0$ such that

$$|f(x) - L| < -\frac{L}{2} \quad \text{for} \quad 0 < |x-a| < \delta_1. \tag{I(c)}$$

Since $\lim\limits_{x \to a} g(x) = \infty$ for any $N < 0$, the number $\frac{2N}{L} > 0$, and there exists a $\delta_2 > 0$ such that

$$g(x) > \frac{2N}{L} \quad \text{for} \quad 0 < |x-a| < \delta_2. \tag{II(c)}$$

Hence, for $\delta = \min(\delta_1, \delta_2)$, Statements I($c$) and II($c$) hold, and for $0 < |x-a| < \delta$ we find that

$$\frac{L}{2} < f(x) - L < -\frac{L}{2} \quad \text{and} \quad \frac{3L}{2} < f(x) < \frac{L}{2} < 0.$$

Since $g(x) > \frac{2N}{L}$ and $f(x) < \frac{L}{2} < 0$, we conclude that

$$f(x)g(x) < \frac{L}{2}\frac{2N}{L} = N.$$

d. Since $\lim_{x \to a} f(x) = L > 0$ for $\frac{L}{2} > 0$, there exists a $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{L}{2} \quad \text{for} \quad 0 < |x - a| < \delta_1. \tag{I(d)}$$

Since $\lim_{x \to a} g(x) = -\infty$ for any $N < 0$, the number $\frac{2N}{L} < 0$, and there exists a $\delta_2 > 0$ such that

$$g(x) < \frac{2N}{L} \quad \text{for} \quad 0 < |x - a| < \delta_2. \tag{II(d)}$$

Hence, for $\delta = min(\delta_1, \delta_2)$, Statements I($d$) and II($d$) hold, and for $0 < |x - a| < \delta$, we find that

$$-\frac{L}{2} < f(x) - L < \frac{L}{2} \quad \text{and} \quad 0 < \frac{L}{2} < f(x) < \frac{3L}{2}.$$

Since $g(x) < \frac{2N}{L}$ and $f(x) > \frac{L}{2} > 0$, we conclude that

$$f(x)g(x) < \frac{L}{2}\frac{2N}{L} = N.$$

37. a. Since $\lim_{x \to a} g(x) = \infty$ for any $N < 0$, the number $-N > 0$, and there exists a $\delta > 0$ such that $g(x) > -N$ for $0 < |x - a| < \delta$.

Hence, $-g(x) < N$ for $0 < |x - a| < \delta$.

b. Since $g(x) \geq f(x)$ for $x \to a$, there exists a $\delta_1 > 0$ such that

$$g(x) \geq f(x) \text{ for } |x - a| < \delta_1. \tag{I(b)}$$

Since $\lim_{x \to a} g(x) = -\infty$ for any $N < 0$, there exists a $\delta_2 > 0$ such that

$$g(x) < N \quad \text{for} \quad 0 < |x - a| < \delta_2. \tag{II(b)}$$

Hence, for $\delta = \min(\delta_1, \delta_2)$ Statements I(b) and II(b) hold, and

$$f(x) \leq g(x) < N \quad \text{for} \quad 0 < |x - a| < \delta.$$

c. Since $\lim_{x \to a} g(x) = \infty$ and $\lim_{x \to a} f(x) = \infty$ for any $M > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$f(x) > \frac{M}{2} \quad \text{for} \quad 0 < |x - a| < \delta_1. \tag{I(c)}$$

and

$$g(x) > \frac{M}{2} \quad \text{for} \quad 0 < |x - a| < \delta_2. \tag{II(c)}$$

Hence, for $\delta = min(\delta_1, \delta_2)$ both statements are true and

$$f(x) + g(x) > \frac{M}{2} + \frac{M}{2} = M \quad \text{for} \quad 0 < |x - a| < \delta.$$

d. Let M_1 be a positive number such that $M_1 + L > 0$. Since $\lim_{x \to a} f(x) = L$, there exists a $\delta_1 > 0$ such that

$$|f(x) - L| < M_1 + L \quad \text{for} \quad 0 < |x - a| < M_1 + L.$$

Hence,

$$-(M_1 + L) < f(x) - L < M_1 + L,$$

and adding L we find that

$$-M_1 < f(x) < M_1 + 2L \quad \text{for} \quad 0 < |x - a| < M_1 + L. \tag{I(d)}$$

Since $\lim_{x \to a} g(x) = \infty$ for any $M > 0$, the number $M_1 + M > 0$, and there exists a $\delta_2 > 0$ such that

$$g(x) > M_1 + M \quad \text{for} \quad 0 < |x - a| < \delta_2. \qquad \mathrm{II}(d)$$

If $\delta = \min(\delta_1, \delta_2)$, Statements I($d$) and II($d$) hold, and for $0 < |x - a| < \delta$ we conclude that

$$f(x) + g(x) > -M_1 + M_1 + M = M.$$

Unit 4

2. a. dollars/thousand of litres

   b. When the demand for gas is between 30 and 50 thousand litres the cost of gas increases at a rate of 24 cents per thousand litres. The slope of the line passing through (30, G(30) and (50, G(50)) is 0.24.

17. $g'(x) = 4(f(x))^3 f'(x)$ and $h'(x) = 5(f(x))^4 f'(x)$.

23. $\dfrac{d}{dx}\dfrac{1}{\cos x} = -\dfrac{-\sin x}{\cos^2 x} = \dfrac{\sin x}{\cos x}\dfrac{1}{\cos x} = \tan x \sec x.$

    $\dfrac{d}{dx}\dfrac{1}{\sin x} = -\dfrac{\cos x}{\sin^2 x} = -\dfrac{\cos x}{\sin x}\dfrac{1}{\sin x} = -\cot x \csc x.$

24. $\dfrac{d}{dx}\sin^4 x = \dfrac{d}{dx}\sin^3 x \sin x = 3\sin^2 x \sin x \cos x + \sin^3 x \cos x$

    $= \sin^3 x(3\cos x + \cos x) = 4\sin^3 x \cos x.$

    $\dfrac{d}{dx}\sin^5 x = 5\sin^4 x \cos x.$

Unit 5

1. A polynomial function is continuous everywhere, so it cannot have a vertical asymptote.

2. a. $f(t) = \dfrac{1}{\sqrt{3-t}}$ is not defined at $t = 3$ and $D_f = (-\infty, 3)$. The vertical line $x = 3$ is an asymptote because

   $$\lim_{x \to 3^-} f(t) = \frac{1}{\sqrt{3-t}} = \infty,$$

   since $\lim_{x \to 3^-} f(t) = \sqrt{3-t} = 0$ and $\sqrt{3-t} > 0$ for $x \to 3^-$.

   b. $g(x) = \dfrac{3x^2 + 5x - 2}{x^2 - 4}$ is not defined at $x = 2$ and $x = -2$ and $D_g = \mathbb{R} - \{2, -2\}$.

   $$\lim_{x \to -2^+} \frac{3x^2 + 5x - 2}{x^2 - 4} = \lim_{x \to -2^+} \frac{(3x-1)(x+2)}{(x-2)(x+2)} = \lim_{x \to -2^+} \frac{3x-1}{x-2} = \frac{7}{4}.$$

   $$\lim_{x \to 2^+} \frac{3x^2 + 5x - 2}{x^2 - 4} = \lim_{x \to 2^+} \frac{(3x-1)(x+2)}{(x-2)(x+2)} = \lim_{x \to 2^+} \frac{3x-1}{x-2} = \infty,$$

   since $\lim_{x \to 2^+} 3x - 1 = 5 > 0$, $\lim_{x \to 2^+} x - 2 = 0$ and $x - 2 > 0$ for $x \to 2^+$.

   Hence, the vertical line $x = 2$ is the vertical asymptote.

3. a. no x-intercepts
   b. $(-2, 0)$ and $(1 / 3, 0)$
   c. no x intercepts

4. a. $\left(0, \dfrac{1}{\sqrt{3}}\right)$
   b. $\left(0, \dfrac{1}{2}\right)$

5. a. even
   b. neither
   c. odd
   d. neither

6. a. even
   b. odd

12. Domain is $(-\infty, -3]$ and $[3, \infty)$, and the function is decreasing on $(-\infty, -3]$ and increasing on $[3, \infty)$. Therefore, there are no extreme points.

16. $(-1, 1), \left(-\dfrac{1}{\sqrt{5}}, \dfrac{61}{125}\right), \left(\dfrac{1}{\sqrt{5}}, \dfrac{61}{125}\right), (1, 6).$

17. Acceleration is at the maximum when $x = 0$, it is at the minimum when

$$x = \sqrt{\frac{3}{5}}.$$

25. $p^{'}(x) = 3ax^2 + 2bx + c = 0$ only if $4b^2 - 4(3a)(c) = 4\left(b^2 - 3ac\right) > 0$. By the second derivative test, $p(x)$ has a critical number only if $b^2 - 3ac > 0$.

Unit 6

8.
$$\begin{aligned}\int \cos^2 x\,dx &= \int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\int 1+\cos(2x)\,dx \\ &= \frac{1}{2}\left(x + \frac{\sin(2x)}{2}\right) + C \\ &= \frac{2x+\sin(2x)}{4} + C\end{aligned}$$

9. a. $f(x) = x^2 + 3, r = 4.$
   b. $f(x) = 3x - 2, r = 20.$
   c. $f(x) = 1 + x + 2x^2, r = -\dfrac{1}{2}.$
   d. $f(x) = x^2 + 1, r = -2.$
   e. $f(x) = 2y + 1, r = -5.$
   f. $f(x) = 5t + 4, r = -2.7.$
   g. $f(x) = 4 - t, r = \dfrac{1}{2}.$
   h. $f(x) = \sin x, r = 6.$
   i. $f(x) = 1 + z^3, r = -\dfrac{1}{3}.$

10. a. $f(x) = x^2 + 3, f'(x) = 2x, r = 4$, hence

$$\frac{(x^2+3)^5}{5} + C$$

b. $f(x) = 3x - 2, f'(x) = 3, r = 20$, hence

$$\frac{(3x-2)^{21}}{3(21)} = \frac{(3x-2)^{21}}{63} + C$$

c. $f(x) = 1 + x + 2x^2, f'(x) = 1 + 4x, r = -1/2$, hence

$$2(1+x+2x^2)^{1/2} = 2\sqrt{1+x+2x^2} + C$$

d. $f(x) = x^2 + 1, f'(x) = 2x, r = -2$, hence

$$\frac{(x^2+1)^{-1}}{-1(2)} = -\frac{1}{2(x^2+1)} + C$$

e. $f(y) = 2y + 1, f'(y) = 2, r = -5$, hence

$$\frac{3(2y+1)^{-4}}{-4(2)} = -\frac{3}{8(2y+1)^4} + C$$

f. $f(t) = 5t + 4, f'(t) = 5, r = -27$, hence

$$\frac{(5t+4)^{-26}}{-26(5)} = -\frac{1}{130(5t+4)^{26}} + C$$

g. $f(t) = 4 - t, f'(t) = -1, r = 1/2$, hence

$$\frac{2(4-t)^{3/2}}{-(3)} = -\frac{2(4-t)^{3/2}}{3} + C$$

h. $f(\theta) = \sin\theta, f'(\theta) = \cos\theta, r = 6$, hence

$$\frac{\sin^7\theta}{7} + C$$

i. $f(z) = 1 + z^3, f'(z) = 3z^2, r = -1/3$, hence

$$\frac{3(1+z^3)^{2/3}}{2(3)} = \frac{\sqrt[3]{(1+z^3)^2}}{2} + C$$

Note: You may differentiate the answers given, to see whether they are correct or not.

11. Note that the constant of integration has been omitted.

a. $u = 3x;\ du = 3dx \quad \frac{1}{3}\int \cos u\ du = \frac{1}{3}\sin u = \frac{1}{3}\sin(3x).$

b. $u = 4 + x^2;\ du = 2xdx \quad \frac{1}{2}\int u^{10}\ dx = \frac{1}{2}\frac{u^{11}}{11} = \frac{\left(4+x^2\right)^1 1}{22}.$

c. $u = x^3 + 1;\ du = 3x^2dx \quad \frac{1}{3}\int u^{1/2}\ du = \frac{2}{9}u^{3/2} = \frac{2\sqrt{\left(x^2+1\right)^3}}{9}.$

d. $u = \sqrt{x}$; $du = \dfrac{1}{2\sqrt{x}}\, dx \quad 2\displaystyle\int \sin u \, du = -2\cos u = -2\cos\sqrt{x}.$

e. $u = 1 + 2x$; $du = 2dx \quad 2\displaystyle\int u^{-3}\, du = 2\frac{u^{-2}}{-2} = -\frac{1}{(1+2x)^2}.$

f. $u = \cos\theta$; $du = -\sin\theta \, d\theta \quad -\displaystyle\int u^4 \, du = -\frac{u^5}{5} = -\frac{\cos^5\theta}{5}$

g. $u = x^2 + 3$; $du = 2xdx \quad \displaystyle\int u^4 \, du = \frac{u^5}{5} = \frac{(x^2+3)^5}{5}$

h. $u = x^3 + 5$; $du = 3x^2 \, dx \quad \dfrac{1}{3}\displaystyle\int u^9 \, du = \frac{u^{10}}{30} = \frac{(x^3+5)^{10}}{30}.$

i. $u = 3x - 2$; $du = 3\, dx \quad \dfrac{1}{3}\displaystyle\int u^{20}\, du = \frac{u^{21}}{63} = \frac{(3x-2)^{21}}{63}.$

j. $u = 2 - x$; $du = -dx \quad -\displaystyle\int u^6 \, du = -\frac{u^7}{7} = -\frac{(2-x)^7}{7}.$

k. $u = 1 + x + 2x^2$; $du = (1+4x)dx$;

$$\int u^{-1/2}\, du = 2u^{1/2} = 2\sqrt{1+x+2x^2}.$$

l. $u = 2y + 1$; $du = 2dy \quad \dfrac{3}{2}\displaystyle\int u^{-5}\, du = -\frac{3u^{-4}}{8} = -\frac{3}{8(2y+1)^4}.$

m. $u = 4 - t$; $du = -dt \quad -\displaystyle\int u^{1/2}\, du = -\left(\frac{2}{3}\right)u^{3/2} = \frac{2(4-t)^{3/2}}{3}.$

n. $u = \pi t$; $du = \pi \, dt \quad \dfrac{1}{\pi}\displaystyle\int \sin u \, du = -\frac{\cos u}{\pi} = -\frac{\cos(\pi t)}{\pi}.$

o. $u = \sqrt{t}$; $du = \dfrac{1}{2\sqrt{t}}\, dt \quad 2\displaystyle\int \cos u \, du = 2\sin u = 2\sin\sqrt{t}.$

p. $u = \sin\theta$; $du = \cos\theta \, d\theta \quad \displaystyle\int u^6 \, du = \frac{u^7}{7} = \frac{\sin^7\theta}{7}.$

q. $u = 1 + z^3$; $du = 3z^2 \, dz \quad \dfrac{1}{3}\displaystyle\int u^{-1/3}\, du = \frac{3u^{2/3}}{6} = \frac{(1+z^3)^{2/3}}{2}$

r. $u = \cot x$; $du = -\csc^2 x \, dx \quad -\displaystyle\int u^{1/2}\, du = -\frac{2u^{3/2}}{3} = -\frac{2\cot^{3/2} x}{3}.$

s. $u = \sec x$; $du = \sec\tan x \, dx \quad \displaystyle\int u \, du = \frac{u^2}{2} = \frac{\sec^2 x}{2}$ also $u = \tan x$ and $\displaystyle\int u \, du = \frac{u}{2} = \frac{\tan x}{2}.$

12. Note that the constant of integration has been omitted.

a. $u = x + 2$; $du = dx$ and $u - 2 = x$

$$\int \frac{u-2}{\sqrt[4]{u}}\, du = \frac{4u^{7/4}}{7} - \frac{8u^{3/4}}{3} = \frac{4(x+2)^{7/4}}{7} - \frac{8(x+2)^{3/4}}{3}$$

b. $u = x - 1$; $du = dx$ and $x^2 = (u+1)^2$

$$\int \frac{(u+1)^2}{\sqrt{u}} du = \int u^{3/2} + 2u^{-1/2} du = \frac{2u^{5/2}}{5} + \frac{4u^{3/2}}{3} + 2u^{1/2} + C$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + 2\sqrt{x-1} + C$$

13. Note that the constant of integration has been omitted.

$u = x^2 - 3$; $du = 2x\ dx$ and $x^2 = u + 3$

$$\int \frac{x^3}{\sqrt{x^2-3}}\ dx = \tfrac{1}{2}\int u^{-1/2}(u+3)\ du = \frac{1}{2}\int u^{1/2} + 3u^{-1/2}$$

$$= \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2} + 6u^{1/2}\right] = \frac{(x^2-3)^{3/2}}{3} + 3\sqrt{x^2-3} = \frac{\sqrt{x^2-3}\,(x^2+6)}{3}.$$

16. Assume that the equation has two roots r_1, r_2, hence the function $f(x) = 1 + 2x + x^3 + 4x^5$ has these zero roots. Hence $f(r_1) = 0 = f(r_2)$ and it is a polynomial, thus continues and differentiable. By Rolle's Theorem $f'(c) = 0 = 2 + 3x^2 + 20x^4$. Since $2 + 3x^2 + 20x^4 > 0$ for any x. This contradiction proves that no two such roots exist.

17. Let $f(x) = 2x - 1 - \sin x$. We have that $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. Hence by the Intermediate Value Theorem f has a zero in the interval $(0, \pi/2)$. Assume f has two zeros r_1, r_2. Hence $f(r_1) = 0 = f(r_2)$. Since the function is continues and differentiable by Rolle's theorem there $f'(c) = 0$ for some $c \in (r_1, r_2)$. Hence $0 = 2 - \cos c$ and therefore $\cos c = 2$ this contradiction shows that f cannot have two roots.

18. If f has two roots $a < b$, then $f(a) = f(b) = 0$. Since f is polynomial, it is continuous on $[a, b]$ and differentiable on (a, b). Rolle's theorem implies that $f'(r) = 0$ for some $a < r < b$. But $f'(r) = 3r^2 - 15$ and $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3(4) - 15 = -3 < 0$. This contradiction indicates that f cannot have two roots in $[-2, 2]$.

20. a. The function $f(x) = 3x^2 + 2x + 5$ is a polynomial, hence continuous and differentiable on $[-1, 1]$. By the Mean Value Theorem, there is $-1 < c < 1$ so that

$$f(1) - f(-1) = f'(c)1 - (-1) \qquad \text{and} \qquad 4 = (6c+2)(-2)$$

Thus $c = -0$.

b. The function $x) = \sqrt[3]{x}$ is continuous and differentiable for all x. By the Mean Value Theorem, there is $0 < c, 1$ so that

$$f(1) - f(0) = f'(c)(1) \qquad \text{and} \qquad 1 = \frac{1}{3c^{2/3}}$$

Thus $c = \dfrac{1}{3\sqrt{3}}$

21. $f(2) - f(0) = 3 - (-1) = 4$, $f'(x) = -\dfrac{2}{(x-1)^2}$.

Since $f'(x) < 0$ for all x (except $x = 1$), $f'(c)(2 - 0)$ is always < 0, and hence cannot equal 4. This conclusion does not contradicts the Mean Value Theorem since f is not continuous at $x = 1$.

23. $f(4) - f(1) = f'(c)(4 - 1)$ for some $1 < c < 4$.

But for every c in $(1, 4)$, we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3(2) = 16$. So the smallest possible value of $f(4)$ is 16.

24. If the function is continuous and differentiable on $[0, 2]$, then by the Mean Value Theorem

$$\frac{f(2) - f(0)}{2} = f'(c) \qquad \text{for some} \qquad x \in [0, 2]$$

Hence $\dfrac{5}{2} = f'(x) \leq 2$ this contradiction indicates that if the function exists it must be noncontinuous or no differentiable.

25. To prove the second statement we have to show that $f(a) > f(b)$ for any $a < b$ in the interval J.

We consider the interval $[a, b]$ in J. Since $f'(x) < 0$ for any $x \in [a, b]$, the function f is differentiable in J, and therefore it is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c) \qquad \text{for some} \qquad a < c < b$$

for this c, we know that $f'(c) < 0$; hence,

$$\frac{f(b) - f(a)}{b - a} < 0$$

We also know that $b - a > 0$, hence $f(b) - f(a) < 0$ and we conclude that $f(a) > f(b)$.

26. In Definition 5.11, we indicated that a function f is concave down on an interval J if the graph of f is *above* the segment from $(a, f(a))$ to $(b, f(b))$ for any $a < b$ in J. This is the same as saying that for any x in the interval $[a, b]$ we have that $f(x)$ is below the line segment from $(a, f(a))$ to $(b, f(b))$.

    The line segment from $(a, f(a))$ to $(b, f(b))$ is given by the equation

    $$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(b) \quad \text{for} \quad a \leq x \leq b.$$

    Thus, a function is concave down if

    $$f(x) > \frac{f(b) - f(a)}{b - a}(x - a) + f(b)$$

    for any $a < x < b$.

    This is equivalent to the second statement of Definition 6.7.

27. Let $h(x) = -f(x)$. If $f''(x) < 0$ for all x in J, then $h''(x) = -f''(x) > 0$ for any x in J. Hence by part *a* of Theorem 6.9 we conclude that h is concave up. So by Definition 6.7 part *a* for any $a < x < b$ in J

    $$\frac{h(x) - h(a)}{x - a} < \frac{h(b) - h(a)}{b - a}$$

    thus

    $$\frac{-f(x) + f(a)}{x - a} < \frac{-f(b) + f(a)}{b - a}$$

    and

    $$\frac{f(x) - f(a)}{x - a} > \frac{f(b) - f(a)}{b - a}$$

    By Definition 6.7 the function f is concave down.

Unit 7

4. a. $\frac{125}{6}$

   b. $\frac{1}{3}$

   c. $\frac{32}{3}$

   d. $\frac{4}{3} + \frac{4}{\pi}$

5. 3.2 N-m = 3.2 J

6. $\frac{1 - \cos(1)}{20} \approx 0.23$

7. $\lim_{h \to 0} f_{\text{ave}} = \lim_{h \to 0} \frac{1}{(x+h)-x} \int_x^{x+h} f(t)\ dt = \lim_{h \to 0} \frac{F(x+h)-F(x)}{h}$

where $F(x) = \int_0^x f(t)\ dt$.

This limit is $f^{'}(x)$ by the definition of the derivative. Therefore, $\lim_{h \to 0} f_{\text{ave}} = F^{'}(x) = f(x)$ by the Fundamental Theorem of Calculus (Part 1).

Learning from Mistakes—Hints

Unit 1

1. If x is any number in the interval $[4, \infty)$, in which interval is $1 - \frac{x}{2}$?

   Hint: See “Rules for Inequalities” (page 338 of the textbook).

2. Simplify each of the expressions below.

   a. $(a + 3b)^2 ab^{-1}$
   b. $(x^3 y^{-3})^2$
   c. $(ab^2 - bc^3)abc$

   Hint: See the laws of exponents on the “Reference Pages” at the beginning of the textbook.

3. Factor each of the expressions below.

   a. $2x^2 - 7x - 4$
   b. $3x^5 + 24x^2$
   c. $81x^4 y - y^5$

   Hint: See “Factoring Special Polynomials” on the “Reference Pages” at the beginning of your textbook.

4. a. Simplify the expression $\dfrac{x^2 + 9x + 20}{x^2 + 5x}$.

   b. Rationalize the expression $\dfrac{\sqrt{4 + h} - 2}{h}$.

   Hint: Review factorization and rationalization.

5. Give the exact values of each of the trigonometric functions below.

   a. $\sec\left(\dfrac{7\pi}{6}\right)$
   b. $\sin\left(\dfrac{7\pi}{12}\right)$

   Hint: See the table of exact values on page 358 of the textbook, and locate in the unit circle the angles given.

Unit 2

1. Define each of the functions below as a set of pairs.
   a. The velocity v depends on the time t.
   b. The bacteria population B depends on the amount of oxygen o.

   Hint: See Definition 2.3.

2. Let $S(t) = 3.75t^2 + 500$ be a function where S is the salary for the number of units sold t. Find the value of S at $t = 4 - \sqrt{14}$.

   Hint: Review Unit 1.

3. Find the domain of each of the functions below.

   a. $g(t) = \dfrac{4 - t}{6 - \sqrt{t^2 - 9}}$

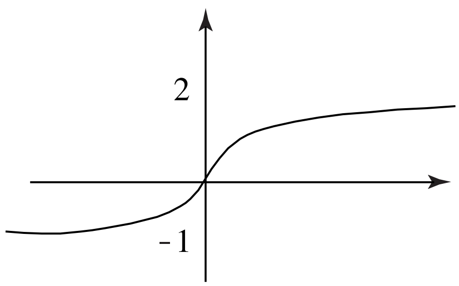
   b. $h(x) = \tan(2x)$

   Hint: See Definition 2.6.

4. Sketch the graph of a single function that satisfies all of the conditions listed below.
   - Domain is [3, 12].
   - The pairs (4, 5) and (5, 4) are not in the function.
   - The function is constant equal to 1 on the interval (8, 12].

   Hint: See the section on the graph of a function in Unit 2.

5. If the graph of the function f is shown below, give a sketch of each of the following functions:



   a. $F(x) = 2f(x) - 1.$

   b. $F(x) = f(x + 2) + 3.$

   Hint: Review the section titled “Transformations of Functions” on pages 30-34 of the textbook.

Unit 3

1. Draw the graph of a single function f such that

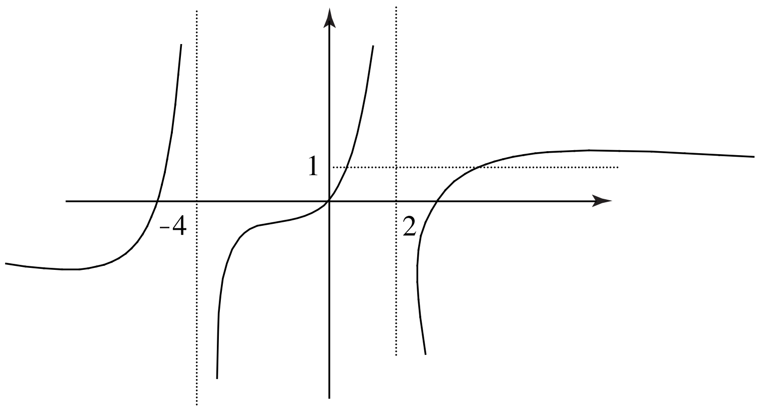
   a. $\lim_{x \to 0} f(x) = 3$

   b. $\lim_{x \to -2} f(x) = \infty$

   c. $\lim_{x \to \infty} f(x) = -1.$

   Hint: See the vertical line test.

2. The graph of a function f is shown below. Give the following limits:



a. $\lim\limits_{x \to -4^-} f(x)$

b. $\lim\limits_{x \to 2} f(x)$

c. $\lim\limits_{x \to \infty} f(x)$

d. $\lim\limits_{x \to -4^+} f(x)$.

Hint: See the section on visual evaluation of limits in Unit 3.

3. Evaluate each of the limits below. If a limit does not exist, explain why.

a. $\lim\limits_{x \to 0^+} x \sin x + \dfrac{1}{x}$

b. $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 15}{x^2 + x - 12}$

c. $\lim\limits_{x \to 0} \dfrac{x^2 - 1}{x^2 + x^4}$

d. $\lim\limits_{x \to \pi/2^-} \tan x + \sec x$

e. $\lim\limits_{x \to 0^+} \dfrac{\tan(3x)}{\sin(4x)}$

f. $\lim\limits_{x \to 3} \dfrac{\sin(x - 3)}{x^2 + x - 12}$

g. $\lim\limits_{x \to \infty} x \cos\left(\dfrac{1}{x}\right)$

h. $\lim\limits_{x \to \infty} \dfrac{\cos x}{x}$

i. $\lim\limits_{x \to \infty} \dfrac{x^2 + 3}{\sqrt{x^2 + x - 1}}$

j. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{(3x + 2)(x + 1)}}{3 - x}$

Hint: Study the examples and warnings in this unit. Each step in the evaluation of a limit must be justified by the application of a definition, theorem, proposition or corollary.

4. Find the vertical and horizontal asymptotes of the function

$$g(x) = \frac{\sqrt{x^4 + 4x^3 + 4x^2}}{3x^2 + 5x - 2}.$$

Hint: See Definition 6 on page 56 of the textbook.

Unit 4

1. The displacement (in km) of a moving car is given by $s(t) = 3t^3 + 4t - 2$, where t is measured in hours.
   a. Give the average velocity in the time period $[1, 5]$.
   b. Give the two different interpretations of value found in part (a).
   c. Give the average velocity in the time period $[1, 1 + h]$ for $h > 0$.

   Hint: See the section on the average rate of change and the slope of the secant line in Unit 4.

2. Use Definition 4.4 to find $\frac{d}{dx}\frac{3}{x}$.
   Hint: See Example 4.15.

3. Consider the piecewise function

$$f(x) = \begin{cases} |x| & \text{for } x \le 4 \\ (x-6)^2 & \text{for } x > 4 \end{cases}$$

   a. Sketch the graph of the function f.
   b. Sketch the graph of the derivative function $f^{'}$.

   Hint: Sketch each piece independently, and then sketch them together at the point where $x = 4$.

4. Give the indicated derivatives.
   a. $f(x) = \dfrac{3x^2 + 6x - 3}{x - 4x^3} f'(x)$.
   b. $\dfrac{d}{dx}\sqrt{x^3}\sin x$.
   c. $\dfrac{d}{dx}\sec\sqrt{x+1}$.
   d. $\dfrac{d^2}{dx^2}\tan(3x)$.

   Hint: Review the rules of differentiation.

5. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

   Hint: Check the section titled "Implicit Differentiation," pages 182-186 of the textbook.

6. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

   Hint: Check the section titled "Implicit Differentiation," pages 182-186 of the textbook.

Unit 5

1. Sketch the graph of the function $f(x) = (x-1)^{3/2}$.

   Hint: See the section titled "Guidelines for Sketching a Curve" on pages 232-236 of the textbook.

2. Find the local extreme points of the function $g(x) = x^6 + 6x^4$.

   Hint: Read "The Second Derivative Test," on page 215 of the textbook.

3. Find the extreme values of the function $h(x) = x^3 - 6x$ on the interval $[0, 2]$.

   Hint: Read "The Closed Interval Method" on page 225 of the textbook.

4. Find a number greater or equal to 2 such that the sum of the number and its reciprocal is as small as possible.

   Hint: Reread the problem carefully.

Unit 6

1. Integrate

   $$\int \tan^2(3x)\sec^2(3x)\ dx.$$

   Hint: See the section on antiderivatives and the general power rule in Unit 6.

2. Solve $\displaystyle\int_1^5 x^3\sqrt{x^2+1}\ dx$.

   Hint: See Example 6.34.

3. Solve $\displaystyle\int_0^1 \left(x^2 - \sqrt{x}\right)^2\ dx$.

   Hint: See the Fundamental Theorem of Calculus, part 2 in Unit 6.

4. Find a function $f(x)$ that satisfies the following two conditions:
   - $f'(x) = \cos x + \sqrt{3x}$
   - $f(1) = 1$.

   Hint: See Example 6.5.

5. Let $f(x) = 4x^3 - 6x^2 + 3$ be a continuous function. Find the value c in the interval $[-1, 0]$ which satisfies the Mean Value Theorem.

   Hint: See the Mean Value Theorem in Unit 6.

6. Find the area below the curve of the function $f(x) = x^3 + 4x$ on the interval $[-1, 3]$.

   Hint: See Example 6.31.

7. Find the derivative of the function $g(x) = \displaystyle\int_{3x}^4 t\tan t\ dt$.

   Hint: See Example 6.37.

Unit 7

1. A spacecraft uses a sail and the "solar wind" to produce a constant acceleration of 0.032 m/s^2. Assuming that the spacecraft has a velocity of 60 km/h when the sail is first raised, how far will the spacecraft travel in 1 hour, and what will its velocity be at the end of this hour?

   Hint: Check the units.

2. Find the area under the curve $y = x^2 + x - 2$ on the interval $[0, 3]$.

   Hint: An area under a curve must be positive.

3. Find the area between the curves $y = \dfrac{1}{x^2}$, $y = x$ and $y = \dfrac{x}{8}$.

   Hint: Sketch a graph of the region bounded by these curves.

4. A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?

   Hint: Review Hooke's Law on page 327 of the textbook.

Correct Solutions: Learning from Mistakes

Unit 1

1. If x is any number in the interval $[4, \infty)$, in which interval is $1 - \frac{x}{2}$?

   MISTAKE

   An inequality changes when it is multiplied (or divided) by a negative number.

   SOLUTION

   If x is in $[4, \infty)$, then $4 \leq x$, and $-4 \geq -x$; thus $-2 \geq \frac{-x}{2}$. Therefore, $-1 \geq 1 - \frac{x}{2}$, and we conclude that $1 - \frac{x}{2}$ is in the interval $(-\infty, -1]$.

2. Simplify the following expressions.

   a. $(a + 3b)^2 ab^{-1}$
   b. $(x^3 y^{-3})^2$
   c. $(ab^2 - bc^3)abc$

   MISTAKE

   Improper use of the laws of exponents.

   It is *not* true that $(a + b)^n = a^n + b^n$, or that $(x^n)^r = x^{n+r}$.

   SOLUTION

   a. $$\begin{aligned}(a + 3b)^2 ab^{-1} &= (a^2 + 6ab + 9b^2)ab^{-1} \\ &= a^3 b^{-1} + 6a^2 + 9ab \\ &= \frac{a^3}{b} + 6a^2 + 9ab.\end{aligned}$$

   b. $(x^3 y^{-3})^2 = x^6 y^{-6} = \frac{x^6}{y^6}$.

   c. $(ab^2 - bc^3)abc = a^2 b^3 c - ab^2 c^4$.

3. Factor the following expressions.

   a. $2x^2 - 7x - 4$
   b. $3x^5 + 24x^2$
   c. $81x^4 y - y^5$

   MISTAKE

   Improper use of the factorization rules.

   It is *not* true that $2x^2 - 7x - 4 = (x - 4)(x + \frac{1}{2})$, or that $(x^3 + 8) = (x + 2)(x^2 + 2x + 4)$. In part ($c$), the factorization of $\sqrt{y}$ is incorrect.

SOLUTION

a. By the quadratic formula,

$$x = \frac{7 \pm \sqrt{49 - 4(2)(-4)}}{4} = \frac{7 \pm 9}{4};$$

hence, the solutions are $x = 4$ and $x = -\frac{1}{2}$, and the factors are

$$x^2 - \frac{7}{2}x - 2 = (x-4)(x+\frac{1}{2}).$$

However, $2x^2 - 7x - 4 = (x-4)(2x+1)$.

b. $3x^5 + 24x^2 = 3x^2(x^3+8) = 3x^2(x+2)(x^2-2x+4)$

c.
$$\begin{aligned} 81x^4y - y^5 &= (9x^2\sqrt{y} - y^{5/2})(9x^2\sqrt{y} + y^{5/2}) \\ &= \sqrt{y}\,(9x^2+y^2)\sqrt{y}\,(9x^2-y^2) \\ &= y(3x-y)(3x+y)(9x^2+y^2) \end{aligned}$$

Another way:

$$\begin{aligned} y(81x^4 - y^4) &= y(9x^2-y^2)(9x^2+y^2) \\ &= y(3x-y)(3x+y)(9x^2+y^2). \end{aligned}$$

4. a. Simplify the expression $\dfrac{x^2+9x+20}{x^2+5x}$.

b. Rationalize the expression $\dfrac{\sqrt{4+h}-2}{h}$.

MISTAKE

Improper cancelation in rational expressions.

It is *not* true that $\dfrac{x^2+9x+20}{x^2+5x} = \dfrac{9x+20}{5x}$, or that $\sqrt{4+h} = 2 + \sqrt{h}$.

SOLUTION

a. $\dfrac{x^2+9x+20}{x^2+5x} = \dfrac{(x+5)(x+4)}{x(x+5)} = \dfrac{x+4}{x}$, for $x \neq 0$ and $x \neq -5$.

b.
$$\begin{aligned} \frac{\sqrt{4+h}-2}{h} &= \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} \\ &= \frac{4+h-4}{h(\sqrt{4+h}+2)} \\ &= \frac{1}{\sqrt{4+h}+2}. \end{aligned}$$

5. Give the exact values of each of the following trigonometric functions.

a. $\sec\left(\dfrac{7\pi}{6}\right)$

b. $\sin\left(\dfrac{7\pi}{12}\right)$

MISTAKE

It is *not* true that $\cos\left(\frac{7\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$, or that $\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right)$.

SOLUTION

a. $\sec\left(\frac{7\pi}{6}\right) = \frac{1}{\cos\left(\frac{7\pi}{6}\right)} = \frac{1}{-\cos\left(\frac{\pi}{6}\right)}$

$$= -\frac{1}{\sqrt{3}/2} = -\frac{2\sqrt{3}}{3}.$$

b. $\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Unit 2

1. Define each of the functions below as a set of pairs.
   a. The velocity v depends on the time t.
   b. The bacteria population B depends on the amount of oxygen o.

MISTAKE

The order of the variables is incorrect. In part (b) the variable B depends on the variable o.

SOLUTION

a. $\{(t, v) \mid v \text{ is the velocity at time } t\}$

b. $\{(o, B) \mid B \text{ is the population for the amount of oxygen } o\}$

2. Let $S(t) = 3.75t^2 + 500$ be a function, where S is the salary for the number of units sold t. Find the value of S at $t = 4 - \sqrt{14}$.

MISTAKE

It is *not* true that $(4 - \sqrt{14})^2 = 16 - 14$.

SOLUTION

$$\begin{aligned} S(4 - \sqrt{14}) &= 3.75(4 - \sqrt{14})^2 + 500 \\ &= 3.75(16 - 8\sqrt{14} + 14) + 500 \\ &= 612.50 - 30\sqrt{14}. \end{aligned}$$

3. Find the domain of each of the following functions.

a. $g(t) = \dfrac{4 - t}{6 - \sqrt{t^2 - 9}}$

b. $h(x) = \tan(2x)$

MISTAKE

In part (a) the denominator (i.e., that $6 - \sqrt{t^2 - 9}$) must be nonzero. In part (b) $\tan u$ is not defined for $u = \dfrac{k\pi}{2}$ for k an odd integer.

SOLUTION

a. We need $t^2 - 9 \geq 0$; hence, $t \geq 3$ and $t \leq -3$. We also need $6 - \sqrt{t^2 - 9} \neq 0$; so $t \neq \pm\sqrt{45}$. Therefore, the domain, in interval notation, is the union of the intervals, $(-\infty, -\sqrt{45})$, $(-\sqrt{45}, -3]$, $[3, \sqrt{45})$ and $(\sqrt{45}, \infty)$.

b. Since $\tan(u)$ is undefined for $u = \dfrac{k\pi}{2}$, where k is any odd integer, we need $2x \neq \dfrac{k\pi}{2}$, and $x \neq \dfrac{k\pi}{4}$. The domain is all numbers except $\dfrac{k\pi}{4}$ for any odd integer k.

4. Sketch the graph of a single function that satisfies all of the conditions listed below.

a. The domain is $[3, 12]$.

b. The pairs $(4, 5)$ and $(5, 4)$ are in the function.

c. The function is constant and equal to 1 over the interval $(8, 12]$.

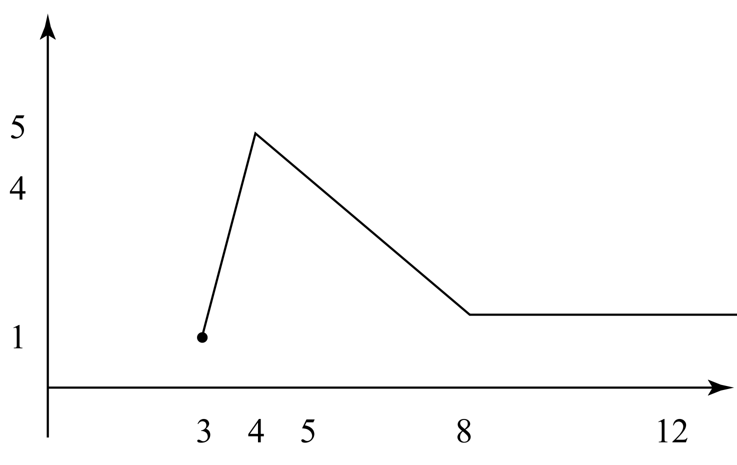


Figure 1. Erroneous solution to “Learning from Mistakes” Question 4.

MISTAKE

The graph does not show the points at the endpoints of the intervals.

SOLUTION

The function must be defined for all numbers in the interval $[3, 12]$; it is constant at 1 on $(8, 12]$.

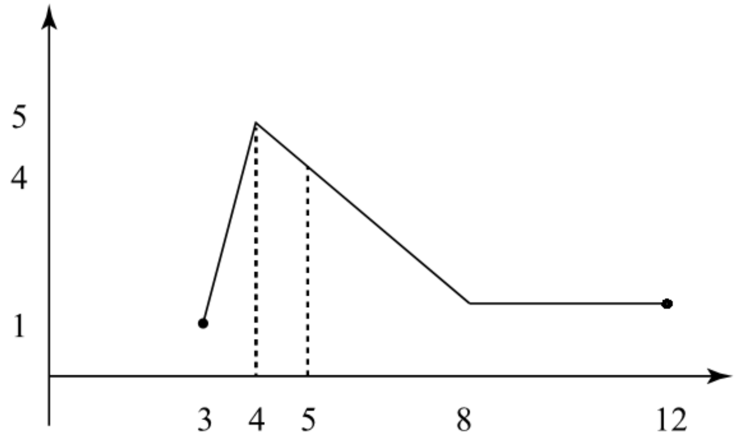


Figure 2. Correct solution to “Learning from Mistakes” Question 4.

5. If the graph of the function f is shown below, give a sketch of each of the following functions.
   a. $F(x) = 2f(x) - 1$
   b. $F(x) = f(x + 2) + 3$.

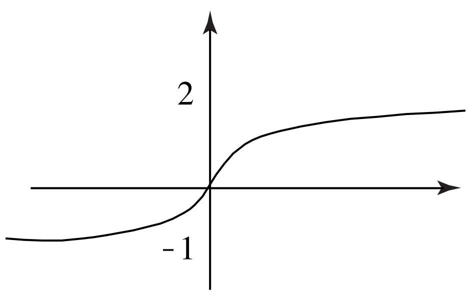


Figure 3. Graph for “Learning from Mistakes” Question 5.

MISTAKE

The transformations are incorrect.

SOLUTION

a. The transformations are stretched by 2 units first, and a shift down by 1 unit second.

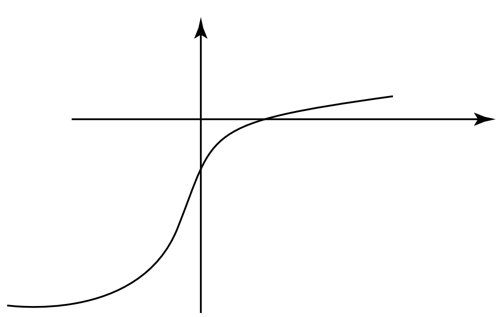


Figure 4. Correct solution to “Learning from Mistakes” Question 5(*a*).

b. The transformations are a shift by 2 units to the left first, and a shift up by 3 units second.

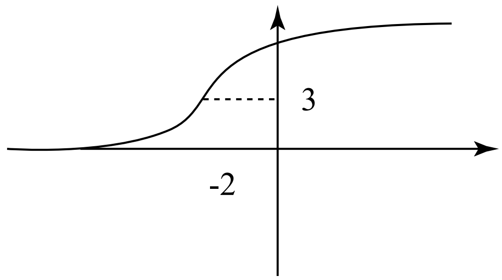
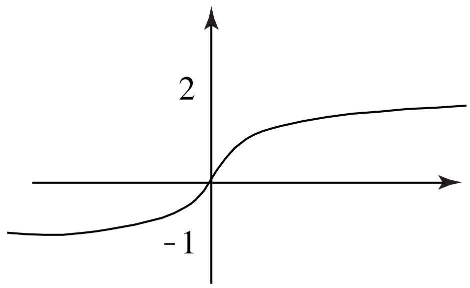


Figure 5. Correct solution to “Learning from Mistakes” Question 5(*b*).



Unit 3

1. Draw the graph of a single function f, such that
   a. $\lim_{x \to 0} f(x) = 3$
   b. $\lim_{x \to -2} f(x) = \infty$
   c. $\lim_{x \to \infty} f(x) = -1.$

MISTAKE

The graph given is not the graph of a function because it fails the vertical line test. A possible answer is given in the figure below.

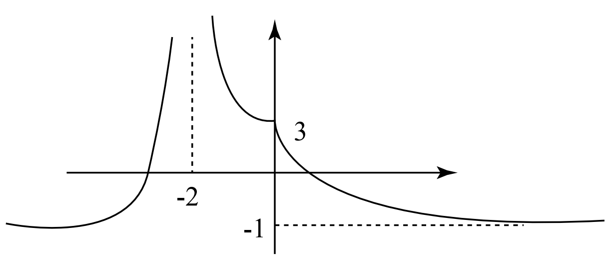


Figure 6. Correct solution to “Learning from Mistakes” Question 1.

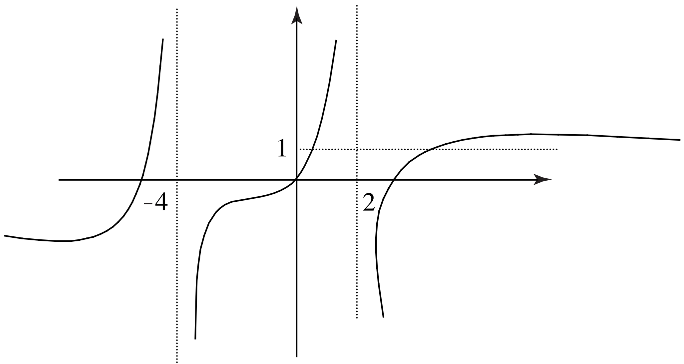


Figure 7. Graph for "Learning from Mistakes" Question 2.

2. Find the following limits for the function shown in Figure 7, above.

   a. $\lim\limits_{x \to -4^-} f(x)$

   b. $\lim\limits_{x \to 2} f(x)$

   c. $\lim\limits_{x \to \infty} f(x)$

   d. $\lim\limits_{x \to -4^+} f(x)$.

Mistakes and Solutions

a. The limit is from the left at -4; hence, it is ∞.

b. Around 2, the function increases (on the left of 2) and decreases (on the right of 2); hence, the limit does not exist.

c. The limit at infinity is a horizontal asymptote, hence the limit is 1.

d. On the right of 4, the function decreases without bounds, hence the limit is negative infinity.

3. Evaluate each of the following limits. If a limit does not exist, explain why.

   a. $\lim\limits_{x \to 0^+} x\sin x + \dfrac{1}{x}$

   b. $\lim\limits_{x \to 1} \dfrac{x^2 + 2x - 15}{x^2 + x - 12}$

   c. $\lim\limits_{x \to 0} \dfrac{x^2 - 1}{x^2 + x^4}$

   d. $\lim\limits_{x \to \pi/2^-} \tan x + \sec x$

   e. $\lim\limits_{x \to 0^+} \dfrac{\tan(3x)}{\sin(4x)}$

   f. $\lim\limits_{x \to 3} \dfrac{\sin(x - 3)}{x^2 + x - 12}$

   g. $\lim\limits_{x \to \infty} x\cos\left(\dfrac{1}{x}\right)$

   h. $\lim\limits_{x \to \infty} \dfrac{\cos x}{x}$

   i. $\lim\limits_{x \to \infty} \dfrac{x^2 + 3}{\sqrt{x^2 + x - 1}}$

j. $\lim_{x \to -\infty} \frac{\sqrt{(3x+2)(x+1)}}{3-x}$

Mistakes and Solutions

a. $\lim_{x \to 0^+} x\sin x + \frac{1}{x}$

MISTAKE

Incorrect use of the laws of limits (Theorem 3.23), because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

SOLUTION

i. Identifying the definitions and results of this unit:

$\lim_{x \to 0^+} x\sin x = 0$ by Theorem 3.14

$\lim_{x \to 0^+} \frac{1}{x} = \infty$ by part (a) of Theorem 3.17

$\lim_{x \to 0^+} x\sin x + \frac{1}{x} = \infty$ by Theorem 3.21.

ii. Without identifying the definitions and results of this unit:

$\lim_{x \to 0^+} x\sin x = 0\sin(0) = 0$ since $x\sin x$ is an everywhere continuous function

$\lim_{x \to 0^+} \frac{1}{x} = \infty$ since $\lim_{x \to 0^+} x = 0$ is an everywhere continuous function, and $x > 0$ for $x \to 0^+$

Hence, $\lim_{x \to 0^+} x\sin x + \frac{1}{x} = \infty$.

b. $\lim_{x \to 1} \frac{x^2+2x-15}{x^2+x-12}$

MISTAKE

$\lim_{x \to 1} \frac{x^2+2x-15}{x^2+x-12} = \frac{(x-3)(x+5)}{(x-3)(x+4)}$ is an incorrect statement.

SOLUTION

i. Identifying the definitions and results of this unit:

$$\lim_{x \to 1} \frac{x^2+2x-15}{x^2+x-12}$$

$$= \lim_{x \to 1} \frac{(x-3)(x+5)}{(x-3)(x+4)} = \lim_{x \to 1} \frac{x+5}{x+4} \quad \text{by Proposition 3.16}$$

$$= \frac{6}{5} \quad \text{by Theorem 3.14.}$$

ii. Without identifying the definitions and results of this unit:

$$\lim_{x\to 1} \frac{x^2+2x-15}{x^2+x-12}$$

$$= \lim_{x\to 1} \frac{(x-3)(x+5)}{(x-3)(x+4)} = \lim_{x\to 1} \frac{x+5}{x+4}$$

$$= \frac{6}{5} \qquad \text{Since } \frac{x+5}{x+4} \text{ is continuous at } 1$$

c. $\lim_{x\to 0} \frac{x^2-1}{x^2+x^4}$

MISTAKE

It is incorrect to divide by zero, and the expression $\frac{-1}{0} = \infty$ is nonsense.

Solution

i. Identifying the definitions and results of this unit:

$\lim_{x\to 0} x^2-1 = -1$ and $\lim_{x\to 0} x^2+x^4 = 0$ by Theorem 3.14.

$\lim_{x\to 0} \frac{1}{x^2+x^4} = \infty$ by Theorem 3.21, since $x^2+x^4 > 0$ for $x \to 0$.

$\lim_{x\to 0} \frac{x^2-1}{x^2+x^4} = -\infty$ by part (b) of Theorem 3.18.

ii. Without identifying the definitions and results of this unit.

$$\lim_{x\to 0} \frac{x^2-1}{x^2+x^4}$$

$$= \lim_{x\to 0} (x^2-1)\frac{1}{x^2+x^4} = \infty$$

because

1. $\lim_{x\to 0} x^2-1 = -1 < 0$, since x^2-1 is everywhere continuous.
2. $\lim_{x\to 0} \frac{1}{x^2+x^4} = \infty$, since $\lim_{x\to 0} x^2+x^4 = 0$ and $x^2+x^4 > 0$ for $x \to 0$.

d. $\lim_{x\to \pi/2^-} \tan x + \sec x$

MISTAKE

It is incorrect to state that $\infty + \infty = \infty$, even when both limits are infinite.

Solution

i. Identifying the definitions and results of this unit:

$$\lim_{x\to \pi/2^-} \tan x = \lim_{x\to \pi/2^-} (\sin x)\frac{1}{\cos x} = \lim_{x\to \pi/2^-} \sin x \sec x$$

$\lim_{x\to \pi/2^-} \sin x = \sin\left(\frac{\pi}{2}\right) = 1$ and

$\lim_{x\to \pi/2^-} \cos x = \cos\left(\frac{\pi}{2}\right) = 0$ by Theorem 3.14.

$$\lim_{x \to \pi/2^-} \frac{1}{\cos x} = \infty$$ by part (a) of Theorem 3.17, since $\cos x > 0$ for $x \to \frac{\pi}{2}^-$

Hence,

$$\lim_{x \to \pi/2^-} \tan x = \infty = \lim_{x \to \pi/2^-} \sec x$$ by Theorem 3.18.

Therefore,

$$\lim_{x \to \pi/2^-} \tan x + \sec x = \infty$$ by part (a) of Theorem 3.21.

ii. Without identifying the definitions and results of this unit.

$$\lim_{x \to \pi/2^-} \tan x = \lim_{x \to \pi/2^-} (\sin x)\frac{1}{\cos x} = \lim_{x \to \pi/2^-} \sin x \sec x$$

$$\lim_{x \to \pi/2^-} \sin x = \sin\left(\frac{\pi}{2}\right) = 1 > 0$$ since $\sin x$ is continuous everywhere.

$$\lim_{x \to \pi/2^-} \frac{1}{\cos x} = \lim_{x \to \pi/2^-} \sec x = \infty$$ since $\lim_{x \to \pi/2^-} \cos x = \cos\left(\frac{\pi}{2}\right) = 0$ and $\cos x > 0$ for $x \to \frac{\pi}{2}^-$.

XX XX

Hence,

$$\lim_{x \to \pi/2^-} \tan x = \infty = \lim_{x \to \pi/2^-} \sec x,$$

Therefore,

$$\lim_{x \to \pi/2^-} \tan x + \sec x = \infty$$

e. $\displaystyle \lim_{x \to 0^+} \frac{\tan(3x)}{\sin(4x)}$

MISTAKE

A side limit always exists so the answer is incorrect. It is also incorrect to write $\frac{0}{0}$; this expression is nonsense.

Solution

i. Identifying the definitions and results of this unit:

$$\lim_{x \to 0^+} \frac{\tan(3x)}{\sin(4x)} = \lim_{x \to 0^+} \frac{\sin(3x)}{\sin(4x)} \frac{1}{\cos(3x)}$$

$$\lim_{x \to 0^+} \frac{\sin(3x)}{\sin(4x)} = \frac{3}{4}$$ by Example 3.60

$$\lim_{x \to 0^+} \frac{1}{\cos(3x)} = \frac{1}{\cos(0)} = 1$$ by Theorem 3.14

$$\lim_{x \to 0^+} \frac{\tan(3x)}{\sin(4x)}$$

$$= \lim_{x \to 0^+} \frac{\sin(3x)}{\sin(4x)} \lim_{x \to 0^+} \frac{1}{\cos(3x)} = \frac{3}{4}$$ by part (c) of Theorem 3.23

ii. Without identifying the definitions and results of this unit.

$$\begin{aligned}\lim_{x\to 0^+}\frac{\tan(3x)}{\sin(4x)} &= \lim_{x\to 0^+}\frac{\sin(3x)}{\sin(4x)}\frac{1}{\cos(3x)}\\ &= \lim_{x\to 0^+}\frac{\sin(3x)}{\sin(4x)}\lim_{x\to 0^+}\frac{1}{\cos(3x)}\\ &= \frac{3}{4}(1)\\ &= \frac{3}{4}\end{aligned}$$

We know that

$$\lim_{x\to 0^+}\frac{\sin(3x)}{\sin(4x)} = \frac{3}{4} \quad\text{and}\quad \lim_{x\to 0^+}\frac{1}{\cos(3x)} = \frac{1}{\cos(0)} = 1,$$

because cosine is continuous everywhere.

f. $\lim_{x\to 3}\frac{\sin(x-3)}{x^2+x-12}$

MISTAKE

This type of “guessing” argument is deceiving, as you can see in this “solution” (see also Examples 4 and 5 on page 74 of the textbook). The only sure way to evaluate limits is by applying definitions and proven results.

Solution

i. Identifying the definitions and results of this unit:

$$\lim_{x\to 3}\frac{\sin(x-3)}{x^2+x-12} = \lim_{x\to 3}\frac{\sin(x-3)}{(x-3)(x+4)} = \lim_{x\to 3}\left(\frac{\sin(x-3)}{x-3}\right)\left(\frac{1}{x+4}\right)$$

If $u = x-3$, then $u \to 0$ as $x \to 3$ and

$$\lim_{x\to 3}\frac{\sin(x-3)}{x-3} = \lim_{x\to 3}\frac{\sin u}{u} = 1 \qquad \text{by Theorem 3.24 and Example 3.56}$$

$$\lim_{x\to 3}\frac{1}{x+4} = \frac{1}{7} \qquad \text{by Theorem 3.14}$$

Hence,

$$\begin{aligned}&\lim_{x\to 3}\frac{\sin(x-3)}{x^2+x-12}\\ &= \lim_{x\to 3}\frac{\sin(x-3)}{x-3}\lim_{x\to 3}\frac{1}{x+4} = \frac{1}{7} \qquad \text{by part } (c) \text{ of Theorem 3.23}\end{aligned}$$

ii. Without identifying the definitions and results of this unit:

$$\lim_{x\to 3}\frac{\sin(x-3)}{x^2+x-12} = \lim_{x\to 3}\frac{\sin(x-3)}{(x-3)(x+4)} = \lim_{x\to 3}\left(\frac{\sin(x-3)}{x-3}\right)\left(\frac{1}{x+4}\right)$$

If $u = x-3$, then $u \to 0$ as $x \to 3$ and

$$\lim_{x\to 3}\frac{\sin(x-3)}{x-3} = \lim_{x\to 3}\frac{\sin u}{u} = 1$$

$$\lim_{x\to 3}\frac{1}{x+4} = \frac{1}{3+4} = \frac{1}{7} \qquad \text{since } \frac{1}{x+4} \text{ is continuous at 3}$$

Hence, by part (c) of Theorem 3.23,

$$\lim_{x\to 3}\frac{\sin(x-3)}{x^2+x-12}=\lim_{x\to 3}\frac{\sin(x-3)}{x-3}\lim_{x\to 3}\frac{1}{x+4}=(1)\frac{1}{7}=\frac{1}{7}.$$

g. $\lim\limits_{x\to\infty} x\cos\left(\frac{1}{x}\right)$

MISTAKE

The inequality $-1\leq\cos\left(\frac{1}{x}\right)\leq 1$ is correct for $x\to\infty$, and the limit $\lim\limits_{x\to\infty} x=\infty$ is also correct, but $\lim\limits_{x\to\infty} -x=-\infty$ by Theorem 3.21. Hence, the Squeeze Theorem does not apply.

Solution

i. Identifying the definitions and results of this unit:

If $u=\frac{1}{x}$, then $u\to 0$ as $x\to\infty$ by Example 3.65.

$\lim\limits_{x\to\infty}\cos\left(\frac{1}{x}\right)=\lim\limits_{x\to 0}\cos u=\cos 0=1$ and

$\lim\limits_{x\to\infty} x=\infty$ by Corollary 3.25

Hence,

$\lim\limits_{x\to\infty} x\cos\left(\frac{1}{x}\right)=\infty$ by part (a) of Theorem 3.18.

ii. Without identifying the definitions and results of this unit:

If $u=\frac{1}{x}$, then $u\to 0$ as $x\to\infty$; hence,

$$\lim_{x\to\infty}\cos\left(\frac{1}{x}\right)=\lim_{x\to 0}\cos u=\cos 0=1>0.$$

Since $\lim\limits_{x\to\infty} x=\infty$, we conclude that

$$\lim_{x\to\infty} x\cos\left(\frac{1}{x}\right)=\infty.$$

h. $\lim\limits_{x\to\infty}\frac{\cos x}{x}$

MISTAKE

It is true that $\lim\limits_{x\to\infty}\cos x$ does not exists, but this not make the given limit also nonexistent.

Solution

i. Identifying the definitions and results of this unit:

For $x\to\infty$, x is positive, and since
$-1\leq\cos x\leq 1$,
$-\frac{1}{x}\leq\frac{\cos x}{x}\leq\frac{1}{x}$, and

$\lim\limits_{x\to\infty} \frac{1}{x} = 0 = \lim\limits_{x\to\infty} -\frac{1}{x}$ by Example 3.65

Hence,

$\lim\limits_{x\to\infty} \frac{\cos x}{x} = 0$ by the Squeeze Theorem.

ii. Without identifying the definitions and results of this unit:

For $x \to \infty$, x is positive, and since $-1 \le \cos x \le 1$, we know that

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}.$$

Since

$$\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to\infty} -\frac{1}{x},$$

we conclude that

$$\lim_{x\to\infty} \frac{\cos x}{x} = 0$$

by the Squeeze Theorem.

i. $\lim\limits_{x\to\infty} \frac{x^2+3}{\sqrt{x^2+xv-1}}$

MISTAKE

The answer is an attempt to use Theorem 3.29, but the function is not rational so the theorem does not apply.

Solution

i. Identifying the definitions and results of this unit:

We know that $\sqrt{x^2} = x$ for $x \to \infty$, and

$$\begin{aligned}\frac{x^2+3}{\sqrt{x^2+x-1}} &= \frac{x^2(1+3/x^2)}{\sqrt{x^2(1+1/x-1/x^2)}} \\ &= \frac{x^2(1+3/x^2)}{x\sqrt{(1+1/x-1/x^2)}} \\ &= x\frac{(1+3/x^2)}{\sqrt{1+1/x-1/x^2}}\end{aligned}$$

$\lim\limits_{x\to\infty} \frac{(1+3/x^2)}{\sqrt{1+1/x-1/x^2}} = 1$ by Example 65 and part (d) of Theorem 3.23

Since $\lim\limits_{x\to\infty} x = \infty$, we know that $\lim\limits_{x\to\infty} \frac{x^2+3}{\sqrt{x^2+x-1}} = \infty$ by part (a) of Theorem 3.18.

ii. Without identifying the definitions and results of this unit:

We know that $\sqrt{x^2} = x$ for $x \to \infty$, and

$$\begin{aligned}\frac{x^2+3}{\sqrt{x^2+x-1}} &= \frac{x^2(1+3/x^2)}{\sqrt{x^2(1+1/x-1/x^2)}}\\ &= \frac{x^2(1+3/x^2)}{x\sqrt{(1+1/x-1/x^2)}}\\ &= x\,\frac{(1+3/x^2)}{\sqrt{1+1/x-1/x^2}}\end{aligned}$$

$$\lim_{x\to\infty}\frac{(1+3/x^2)}{\sqrt{1+1/x-1/x^2}} = \frac{1+0}{\sqrt{1+0-0}} = 1$$

and

$$\lim_{x\to\infty} x = \infty.$$

Hence,

$$\lim_{x\to\infty}\frac{x^2+3}{\sqrt{x^2+x-1}} = \infty.$$

j. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{(3x+2)(x+1)}}{3-x}$

MISTAKE

The mistake is not taking into account that for $x \to -\infty$, $\sqrt{x^2} = -x$.

Solution

i. Identifying the definitions and results of this unit:

$$\begin{aligned}\frac{\sqrt{(3x+2)(x+1)}}{3-x} &= \frac{\sqrt{3x^2+5x+2}}{3-x}\\ &= \frac{\sqrt{x^2\left(3+\frac{5}{x}+\frac{2}{x^2}\right)}}{x\left(\frac{3}{x}-1\right)}\\ &= \frac{-x\sqrt{3+\frac{5}{x}+\frac{2}{x^2}}}{x\left(\frac{3}{x}-1\right)}\\ &= -\frac{\sqrt{3+\frac{5}{x}+\frac{2}{x^2}}}{\frac{3}{x}-1}\end{aligned}$$

$$\lim_{x\to-\infty}\frac{\sqrt{(3x+2)(x+1)}}{3-x}$$

$$= \lim_{x \to -\infty} - \frac{\sqrt{3 + \frac{5}{x} + \frac{2}{x^2}}}{\frac{3}{x} - 1}$$

$$= -\frac{\sqrt{3}}{-1} = \sqrt{3}$$ by Proposition 3.16, Example 3.65, Corollary 3.26 and part (*d*) of Theorem 3.23

ii. Without identifying the definitions and results of this unit:

$$\begin{aligned}
\frac{\sqrt{(3x+2)(x+1)}}{3-x} &= \frac{\sqrt{3x^2+5x+2}}{3-x} \\
&= \frac{\sqrt{x^2\left(3 + \frac{5}{x} + \frac{2}{x^2}\right)}}{x\left(\frac{3}{x} - 1\right)} \\
&= \frac{-x\sqrt{3 + \frac{5}{x} + \frac{2}{x^2}}}{x\left(\frac{3}{x} - 1\right)} \\
&= -\frac{\sqrt{3 + \frac{5}{x} + \frac{2}{x^2}}}{\frac{3}{x} - 1}
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to -\infty} \frac{\sqrt{(3x+2)(x+1)}}{3-x} &= \lim_{x \to -\infty} - \frac{\sqrt{3 + \frac{5}{x} + \frac{2}{x^2}}}{\frac{3}{x} - 1} \\
&= -\frac{\sqrt{3+0+0}}{0-1} = \sqrt{3}
\end{aligned}$$

4. Find the vertical and horizontal asymptotes of the function

$$g(x) = \frac{\sqrt{x^4 + 4x^3 + 4x^2}}{3x^2 + 5x - 2}.$$

Mistake and Solution

It is true that the function is not defined at $\frac{1}{3}$ and -2; however, by Definition 3.19, for the function to have vertical asymptotes, the limit at these numbers must be infinite.

$$\frac{\sqrt{x^4+4x^3+4x^2}}{3x^2+5x-2} = \frac{\sqrt{x^4+4x^3+4x^2}}{(3x-1)(x+2)}$$
$$= \left(\frac{\sqrt{x^4+4x^3+4x^2}}{x+2}\right)\left(\frac{1}{3x-1}\right)$$

$$\lim_{x\to 1/3} \frac{\sqrt{x^4+4x^3+4x^2}}{x+2} = \frac{\sqrt{\frac{1}{81}+\frac{4}{27}+\frac{4}{9}}}{\frac{1}{3}+2} > 0$$

$\lim_{x\to 1/3^+} \frac{1}{3x-1} = \infty$ since $3x-1>0$ for $x\to 1/3^+$ and $\lim_{x\to 1/3^+} 3x-1 = 0.$

Hence,

$\lim_{x\to 1/3^+} \frac{\sqrt{x^4+4x^3+4x^2}}{3x^2+5x-2} = \infty$ by Theorem 3.18

and $x = \frac{1}{3}$ is a vertical asymptote.

For

$$\frac{\sqrt{x^4+4x^3+4x^2}}{3x^2+5x-2} = \left(\frac{\sqrt{x^2(x^2+4x+4)}}{3x-1}\right)\left(\frac{1}{x+2}\right)$$
$$= \left(\frac{\sqrt{x^2}\sqrt{(x+2)^2}}{3x-1}\right)\left(\frac{1}{x+2}\right)$$

we have two cases to consider.

Case 1

For $x\to -2^+$, we know that $x<0$ and $x+2>0$; hence,

$$\sqrt{x^2} = -x \text{ and } \sqrt{(x+2)^2} = x+2.$$

Therefore,

$$\frac{\sqrt{x^4+4x^3+4x^2}}{3x^2+5x-2} = \left(\frac{-x(x+2)}{3x-1}\right)\left(\frac{1}{x+2}\right)$$
$$= -\frac{x}{3x-1} \text{ for } x \neq -2$$

$$\lim_{x\to -2^+} \frac{\sqrt{x^4+4x^3+4x^2}}{3x^2+5x-2} = \lim_{x\to -2^+} -\frac{x}{3x-1} = -\frac{2}{7}$$

by Proposition 3.16 and Theorem 3.14.

Case 2

For $x\to -2^-$, we know that $x<0$ and $x+2<0$; hence,

$$\frac{\sqrt{x^4 + 4x^3 + 4x^2}}{3x^2 + 5x - 2} = \left(\frac{x(x+2)}{3x-1}\right)\left(\frac{1}{x+2}\right) = \frac{x}{3x-1} \text{ (Why?).}$$

Therefore,

$$\lim_{x \to -2^-} \frac{\sqrt{x^4 + 4x^3 + 4x^2}}{3x^2 + 5x - 2} = \lim_{x \to -2^-} \frac{x}{3x-1} = \frac{2}{7}$$

by Proposition 3.16 and Theorem 3.14. The line $x = -2$ is not a vertical asymptote.

For horizontal asymptotes we have

$$\begin{aligned}\frac{\sqrt{x^4 + 4x^3 + 4x^2}}{3x^2 + 5x - 2} &= \frac{\sqrt{x^4(1 + \frac{4}{x} + \frac{4}{x^2})}}{x^2(3 + \frac{5}{x} - \frac{2}{x^2})} \\ &= \frac{x^2\sqrt{1 + \frac{4}{x} + \frac{4}{x^2}}}{x^2(3 + \frac{5}{x} - \frac{2}{x^2})} \\ &= \frac{\sqrt{1 + \frac{4}{x} + \frac{4}{x^2}}}{3 + \frac{5}{x} - \frac{2}{x^2}}\end{aligned}$$

$$\lim_{x \to \infty} \frac{\sqrt{x^4 + 4x^3 + 4x^2}}{3x^2 + 5x - 2} = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{4}{x} + \frac{4}{x^2}}}{3 + \frac{5}{x} - \frac{2}{x^2}} = \frac{1}{3}$$

by Example 3.65, Corollary 3.26 and part (d) of Theorem 3.23.

Hence $y = \frac{1}{3}$ is a horizontal asymptote.

Observe that the limit for $x \to -\infty$ is also $\frac{1}{3}$.

Unit 4

1. The displacement (in km) of a moving car is given by $s(t) = 3t^3 + 4t - 2$, where t is measured in hours.
   a. Give the average velocity in the time period $[1, 5]$.
   b. Give the two different interpretations of value found in part (a), above.
   c. Give the average velocity in the time period $[1, 1 + h]$ for $h > 0$.

MISTAKES

Parts a) and (c)—incorrect order of the quotient. Part (b)—it is not the velocity that is increasing or decreasing, it is the distance. The slope of the secant line is incorrect.

SOLUTIONS

a. $\dfrac{s(1)-s(5)}{1-5} = \dfrac{(3+4-2)-3(125)-4(5)+2}{1-5} = 97.$

The average velocity is 97 km/h.

b. The average velocity from 1 to 5 hours is 97 km/h. On average, the distance is increasing at a rate of 97 km/h. The slope of the secant line through (1, 5) and (5, 393) is 97.

c. $\dfrac{s(1+h)-s(1)}{h} = \dfrac{3(1+h)^3+4(1+h)-2-5}{h} = 13+9h+3h^2.$

2. Use Definition 4.5 of the *Study Guide* to find $\dfrac{d}{dx}\dfrac{3}{x}$.

MISTAKE

Errors in the algebraic manipulations and the missing term $\lim\limits_{h \to 0}$.

SOLUTION

$$\lim_{h\to 0}\frac{3(x+h)^{-1}-3(x^{-1})}{h} = \lim_{h\to 0}\frac{3x-3(x+h)}{x(x+h)h} = \lim_{h\to 0}\frac{-3}{x(x+h)} = -\frac{3}{x^2}.$$

3. Consider the piecewise function

$$f(x) = \begin{cases} |x| & \text{for } x \le 4 \\ (x-6)^2 & \text{for } x > 4 \end{cases}$$

a. Sketch the graph of the function f.

b. Sketch the graph of the derivative function f'.

MISTAKE

There is no mistake in part (*a*). In part (*b*), the derivative function for $x > 4$ is incorrect. For $4 < x < 6$ the derivative is negative, and for $x > 6$ the derivative is positive.

SOLUTION

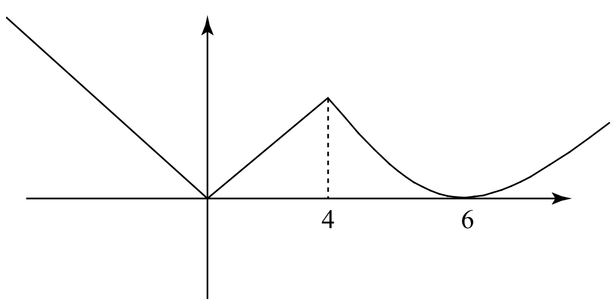


Figure 8. Correct solution to “Learning from Mistakes” Question 3(*a*).

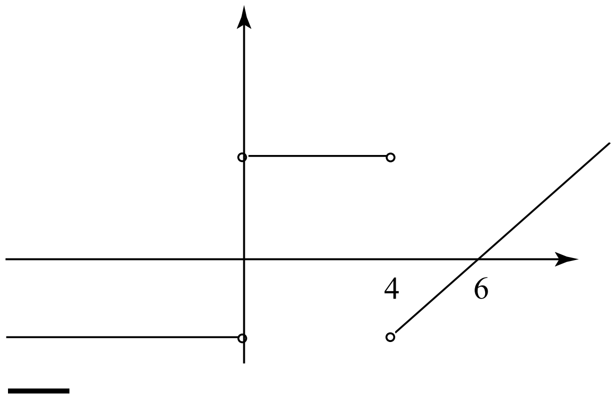


Figure 9. Correct solution to "Learning from Mistakes" Question 3(*b*).

4. Find the derivatives indicated below.

a. $f(x) = \dfrac{3x^2 + 6x - 3}{x - 4x^3}$, find $f'(x)$.

b. $\dfrac{d}{dx}\sqrt{x^3 \sin x}$.

c. $\dfrac{d}{dx}\sec\sqrt{x+1}$.

d. $\dfrac{d^2}{dx^2}\tan(3x)$.

Mistakes and Solutions

a. **MISTAKE**

The numerator of the quotient is in the wrong order.

SOLUTION

$$f'(x) = \frac{(6x+6)(x-4x^3) - (3x^2+6x-3)(1-12x^2)}{(x-4x^3)^2}.$$

b. **MISTAKE**

The derivative of $x^3 \sin x$ is incorrect.

SOLUTION

$$\frac{d}{dx}\sqrt{x^3 \sin x} = \frac{3x^2 \sin x + x^3 \cos x}{2\sqrt{x^3 \sin x}}.$$

c. **MISTAKE**

The chain rule is not applied properly.

SOLUTION

$$\frac{d}{dx}\sec\sqrt{x+1} = \sec\sqrt{x+1}\tan\sqrt{x+1}\left(\frac{1}{2\sqrt{x+1}}\right).$$

d. **MISTAKE**

The chain rule is not applied properly.

SOLUTION

$$\frac{d}{dx}\tan(3x) = 3\sec^2(3x) \text{ and } \frac{d}{dx}3\sec^2(3x) = 18\sec^2(3x)\tan(3x).$$

5. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

MISTAKE

Here x is changing with respect to time, so the implicit differentiation is incorrect.

SOLUTION

We know that $\frac{dx}{dt} = 3$, and we want to know $\frac{dD}{dt}$ when $x = 4$, where D is the distance from the origin to the point (x, y) on the curve.

Hence, $D = \sqrt{x^2 + y^2}$. Differentiating yields

$$\frac{dD}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2 + y^2}}.$$

For $x = 4$ and $\frac{dx}{dt} = 3$, we have $y' = \frac{x'}{2\sqrt{x}}$, hence $y'' = \frac{3}{2\sqrt{4}} = \frac{3}{4}$; and

$$\frac{dD}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}} = \frac{4(3) + 2\left(\frac{3}{4}\right)}{\sqrt{20}} = \frac{18}{2\sqrt{5}}$$

The distance is increasing at a rate of $\frac{18}{2\sqrt{5}}$ cm/s.

6. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 2 mi away from the station.

MISTAKE

Here y and x change with respect to time and the differentiation is incorrect; also, by putting the rate of x in the diagram the substitution is incorrect.

SOLUTION

We want $\frac{dy}{dt}$ when $y = 2$ mi.

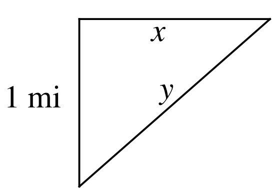


Figure 10. Correct solution to "Learning from Mistakes" Question 3(*a*).

From the figure, $y^2 = x^2 + 1$. Differentiating yields

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

and solving gives us

$$\frac{dy}{dt} = \left(\frac{x}{y}\right)\left(\frac{dx}{dt}\right).$$

When $y = 2$ we have that $x = \sqrt{3}$ and $\frac{dx}{dt} = 500$~mi/h, then

$$\frac{dy}{dt} = \frac{\sqrt{3}}{2}(500) = 250\sqrt{3}.$$

The distance is increasing at a rate of $250\sqrt{3}$ mi/h.

Unit 5

1. Sketch the graph of the function $f(x) = (x-1)^{3/2}$.

MISTAKE

The mistake is in ignoring the domain of the function, and the x and y intercepts.

SOLUTION

A. The domain is the interval $[1, \infty)$

B. No y-intercept (why?), and $f(1) = 0$, hence $(1, 0)$ is the x-intercept.

C. The function cannot have symmetries (why?)

D. No asymptotes (why?)

E. $f'(x) = (3/2)(x-1)^{1/2}$ the critical number is $x = 1$ because $f'(1) = 0$. Since $f'(x) > 0$ for all $x \geq 1$, the function is increasing on $(0, \infty)$. Hence no local extreme values.

F. $f''(x) = 3/4(x-1)^{-1/2}$ and $f''(x) > 0$ for all $x > 1$, the function is concave up on its domain.

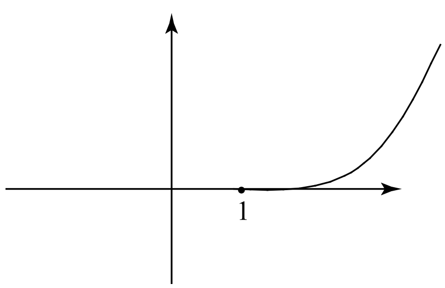


Figure 11. Correct solution to "Learning from Mistakes" Question 1.

2. Find the local extreme points of the function $g(x) = x^6 + 6x^4$.

MISTAKE

The Second derivative test is inconclusive in this case because $g''(0) = 0$, hence we need to use the intervals of increase and decrease of the function.

SOLUTION

It is true that the only critical number is $x = 0$ because $g'(x) = 6x^5 + 24x^3 = 6x^3(x^2 + 4)$ and $g(0) = 0$ and $x^2 + 4 > 0$ for all x.

But we have that $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$, hence the function has a local minimum at $x = 0$.

3. Find the extreme values of the function $h(x) = x^3 - 6x$ on the interval $[0, 2]$.

MISTAKE

The value $-\sqrt{2}$ which is not in the interval $[0, 2]$.

SOLUTION

Since $h'(x) = 3x^2 - 6$ it is true that $x = \sqrt{2}$ and $x = -\sqrt{2}$ are the critical numbers, but only $\sqrt{2}$ is in the interval $[0, 2]$.

Thus $h(\sqrt{2}) = (\sqrt{2})^3 - 6\sqrt{2} \approx -5.656$.

The values at the endpoints are $h(0) = 0$ and $h(2) = -4$.

By the closed interval method the absolute minimum value is $(\sqrt{2})^3 - 6\sqrt{2} \approx -5.656$ at $x = \sqrt{2}$ and the absolute maximum value is 0 at $x = 0$.

4. Find a number greater or equal to 2 such that the sum of the number and its reciprocal is as small as possible.

MISTAKE

The answer has to be a number greater or equal to 2.

SOLUTION

Let x be such a number. Hence we want to minimize the function

$f(x) = x + \frac{1}{x}$ with x in the interval $[2, \infty)$.

Since $f'(x) = 1 - \frac{1}{x^2} = 0$ if $x = 1$ or $x = -1$

and both are less than 2 the number we are looking for is not a critical number.

Since $f'(x) > 0$ for all $x \geq 2$ the function is increasing on $[2, \infty)$, hence the minimum is at $x = 2$, and this is the answer.

Unit 6

1. Integrate $\int \tan^2(3x)\sec^2(3x)\,dx$.

MISTAKE

The derivative of f is $f'(x) = 3\sec^2(3x)$.

SOLUTION

$f(x) = \tan(3x)$, $r = 2$ and $f'(x) = 3\sec^2(3x)$,

then

$$\frac{1}{3}\int 3\tan^2(3x)\sec^2(3x)\,dx = \frac{\tan^3(3x)}{9} + C.$$

2. Integrate $\int_1^5 x^3\sqrt{x^2+1}\,dx$.

MISTAKE

The limits of integration have to change according to the function u.

SOLUTION

Let $u = x^2 + 1$, $du = 2u\,dx$, $x^2 = u - 1$. If $x = 1$, $u = 2$ and if $x = 5$, $u = 26$, then

$$\begin{aligned}
\frac{1}{2}\int_2^{26}(u-1)u^{1/2}\,du &= \frac{1}{2}\int_2^{26} u^{3/2} - u^{1/2}\,du \\
&= \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}\Bigg|_2^{26} \\
&= \frac{(26)^{5/2}}{5} - \frac{(26)^{3/2}}{3} - \left(\frac{2^{5/2}}{5} - \frac{2^{3/2}}{3}\right) \\
&= \sqrt{8}\left(\frac{73\sqrt{13^3} - 1}{15}\right)
\end{aligned}$$

3. Integrate $\int_0^1 (x^2 - \sqrt{x})^2\,dx$.

MISTAKE

The evaluation of the limits of integration is incorrect.

SOLUTION

$$\begin{aligned}
\int_0^1 (x^2 - \sqrt{x})^2\,dx &= \int_0^1 x^4 - 2x^{5/2} + x\,dx \\
&= \frac{x^5}{5} - \frac{4x^{7/2}}{7} + \frac{x^2}{2}\Bigg|_0^1 \\
&= \left(\frac{1}{5} - \frac{4}{7} + \frac{1}{2}\right) \\
&= \frac{9}{70}
\end{aligned}$$

4. Find a single function $f(x)$ that satisfies the following two conditions:

a. $f'(x) = \cos x + \sqrt{3x}$, and

b. $f(1) = 1$.

MISTAKE

The constant of integration is missing.

SOLUTION

To find f we integrate its derivative

$$\int \cos x + \sqrt{3}\sqrt{x}\,dx = \sin x + \frac{2\sqrt{3}x^{3/2}}{3} + C$$

$$f(x) = \sin x + \frac{2\sqrt{3}x^{3/2}}{3} + C$$

$$f(1) = \sin(1) + \frac{2\sqrt{3}}{3} + C = 1$$

solving for C, the answer is

$$f(x) = \sin x + \frac{2\sqrt{3}x^{3/2}}{3} + \frac{3 - 2\sqrt{3}}{3} - \sin(1).$$

5. Let $f(x) = 4x^3 - 6x^2 + 3$ be a continuous function. Find the value c in the interval $[-1, 0]$ which satisfies the Mean Value Theorem.

MISTAKE

Only one of the values found is in interval $[-1, 0]$.

SOLUTION

$f'(x) = 12x^2 - 12x$ and $\dfrac{f(0) - f(-1)}{0 - (-1)} = 10$; hence, $12x^2 - 12x = 10$ and solving for x (using the quadratic formula) $x = \dfrac{3 \pm \sqrt{39}}{6}$.
The value $x = \dfrac{3 - \sqrt{39}}{6}$ in the interval $[-1, 0]$ satisfies the Mean Value Theorem.

6. Find the area below the curve of the function $f(x) = x^3 + 4x$ on the interval $[-1, 3]$.

MISTAKE

The area under the curve is given by two integrals, one for the interval where the function is negative and another where the function is positive.

SOLUTION

The function $f(x) = x(x^2 + 4)$ is negative for $x < 0$ and positive for $x > 0$. Hence the area under this curve in the interval $[-1, 3]$ is given by the integrals

$$\begin{aligned} -\int_{-1}^{0} x^3 + 4x\,dx + \int_{0}^{3} x^3 + 4x\,dx &= -\left(\frac{x^4}{4} + 2x^2\right)\Bigg|_{-1}^{0} + \left(\frac{x^4}{4} + 2x^2\right)\Bigg|_{0}^{3} \\ &= \left(\frac{1}{4} - 2\right) + \left(\frac{3^4}{4} - 2(3)\right) \\ &= 40.5 \end{aligned}$$

The area is 40.5.

7. Find the derivative of the function $g(x) = \int_{3x}^{4} t\tan t\,dt$.

MISTAKE

Part 1 of the Fundamental Theorem of Calculus is applied incorrectly, and the derivative of the lower limit of integration is missing.

SOLUTION

$$g'(x) = -\frac{d}{dx}\int_{4}^{3x} t\tan t\,dt = -(9x)\tan(3x).$$

Unit 7

1. A spacecraft uses a sail and the "solar wind" to produce a constant acceleration of 0.032 m/s^2. Assuming that the spacecraft has a velocity of 600 km/h when the sail is first raised, how far will the spacecraft travel in 1 hour, and what will its velocity be at the end of this hour?

   MISTAKE

   Units are not properly used.

   SOLUTION

   We define $t_0 = 0$ to be the time the sail is raised, $t_1 = 1$ one our later, the acceleration $a = 0.032$, and at time t the velocity is $v(t)$ and displacement is $s(t)$. We then have $s(0) = 0$ and $v(0) = 600$ km/h. We choose metres and seconds, hence $v(0) = 166.66$ m/s and $t_1 = 3600$.

   Hence $v(t) = 166.66 + .032t$, and $v(3600) = 166.66 + .032(3600) = 281.86$.

$$\begin{aligned} s(3600) - s(0) = s(3600) &= \int_0^{3600} 166.66 + .032t\, dt \\ &= 166.66t + .016t^2 \Big|_0^{3600} \\ &= 166.66(3600) + .016(3600)^2 \\ &= 807336. \end{aligned}$$

   Answer: Velocity is 281.86 m/s or 807336 km/h and the distance is 807336 m or 807.336 km.

2. Find the area under the curve $y = x^2 + x - 2$ on the interval $[0, 3]$.

   MISTAKE

   The function given is negative on the interval $[0, 1]$ and positive on $[1, 3]$. Hence the area under the curve is given by two integrals.

   SOLUTION

$$-\int_0^1 x^2 + x - 2\, dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_0^1 = \frac{7}{6}.$$

$$\int_1^3 x^2 + x - 2\, dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x \Big|_1^3 = \frac{26}{3}.$$

   The area is $\frac{7}{6} + \frac{26}{3} = \frac{59}{6}$.

3. Find the area between the curves $y = \frac{1}{x^2}$, $y = x$ and $y = \frac{x}{8}$.

   MISTAKE

   The area is given by two integrals and in each one of them we must take the difference between the function that is above minus the curve that it is below.

   SOLUTION

   Point of intersection of $y = \frac{1}{x^2}$ and $y = x$ is the solution of $x = \frac{1}{x^2}$; that is, $x = 1$. Hence, $(1, 1)$. Point of intersection of $y = \frac{1}{x^2}$ and $y = \frac{x}{8}$ is the solution of $(x^2)\frac{x}{8} = 1$ that is $x = 2$ and $(2, 1/4)$.

On the interval [0, 1] the line $y = x$ is above the curve $y = \dfrac{x}{8}$; hence, the area is

$$\int_0^1 x - \frac{x}{8} = \frac{7}{8}\int_0^1 x\,dx = \left(\frac{7}{8}\right)\frac{x}{2}\,\Big|_0^1 = \frac{7}{16}.$$

On the interval [1, 2] the curve $y = \dfrac{1}{x^2}$ is above the line $y = \dfrac{x}{8}$, hence the area is

$$\int_1^2 \frac{1}{x^2} - \frac{x}{8} = -\frac{1}{x} - \frac{x^2}{16}\,\Big|_1^2 = \frac{5}{16}.$$

The answer is $\dfrac{7}{16} + \dfrac{5}{16} = \dfrac{12}{16} = \dfrac{3}{4}$.

4. A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?

MISTAKE

The spring constant has to be found using Hooke's Law before finding work. Also the spring is stretched from 0 to 0.8 m.

SOLUTION

The force is $F = kx$ by Hooke's Law. Hence $100 = k(.2)$ and we have that the spring constant is $k = 500$ N/m.

$$W = \int_0^{.8} 500x\,dx = 250x^2\,\Big|_0^{.8} = 250(.64) = 160.$$

The answer is 160 J.

MATH 265 Index/Glossary

$\max\{a, b\}$ - Compression

Concept	*Ref / Unit / Section*	*Definition*
$\max\{a, b\}$	Definition 3.31: Unit 3 / The Definition of a Limit	the number larger of the two
$\min\{a, b\}$	Definition 3.31: Unit 3 / The Definition of a Limit	the number smaller of the two
u-identification	Unit 6 / Change of Variable: u-identification	application of the tables of integration
u-substitution	Unit 6 / The u-substitution	method of integration
angle		
– positive	Unit 1 / Trigonometry	angle measured in the counterclockwise direction
– negative	Unit 1 / Trigonometry	angle measured in the clockwise direction
antiderivative	Definition 6.1: Unit 6 / Integration	F is antiderivative of f if the derivative of F is equal to f i.e. $F' = f$
asymptote		
– horizontal	Definition 3.28: Unit 3 / Limits at Infinity: Horizontal Asymptotes	If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote
– vertical	Definition 3.19: Unit 3 / Infinite Limits: Vertical Asymptotes	if the limit at a, or at a from the right or from the left is either infinity or negative infinity, then $x = a$ is a vertical asympotte
average arithmetic	Definition 7.3: Unit 7 / Average Value of a Function	the sum of finitely many numbers, n, divided by n
chain rule	Unit 4 / The Chain Rule	rule to differentiate the composition of functions
Closed Interval Method	Unit 5 / Local and Absolute Extreme Values	method to find the extreme values of a continuous function on a closed interval
compression		
– horizontal	Definition 2.12: Unit 2 / Transformations	F is the horizontal compression of f if $F(x) = f(cx)$ for $c > 1$
– vertical	Definition 2.11: Unit 2 / Transformations	F is the vertical compression of f by a factor of c if $F(x) = \dfrac{1}{c}f(x)$

Concavity Test - Domain

Concept	*Ref / Unit / Section*	*Definition*
Concavity Test	Definition* 5.12: Unit 5 / Concavity and Points of Inflection	Test to determine the concavity of a function
– proof of	Theorem 6.9: Unit 6 / The Mean Value Theorem	
conjugate terms	Unit 1 / Algebra	The binomial terms $(a+b)$ and $(a-b)$ are conjugates of each other
constant multiple rule	Unit 4 / Rules of Differentiation	rule to differentiate the product of a function and a constant
constrain equation	Unit 5 / Optimization Problems	limitation of a optimization problem given by an equation
critical		
– number	Definition 5.6: Unit 5 / Intervals of Increase and Decrease	it is number c, such that either $f'(c) = 0$ of f' is undefined
– point	Definition 5.6: Unit 5 / Intervals of Increase and Decrease	$(c, f(c))$ such that c is a critical number
curves		
– areas between	Unit 7 / Areas Between Curves	integral used to find the area between two curves
– orthogonal	Definition 4.9: Unit 4 / Implicit Differentiation	Two curves are orthogonal if at each point of intersection their tangent lines are perpendicular
delta definition	Unit 3 / The Definition of a Limit	small number denoted by δ
derivative	Definition 4.3: Unit 4 / The Instantaneous Rate of Change and the Slope of the Tangent Line	limit of the average quotient
– high order	Unit 4 / High-order Derivatives	derivative of a derivative function
differential	Definition 4.8: Unit 4 / Applications of the Derivative	$f'(a)\Delta x$ for a function f differentiable at a
differentiation implicit	Unit 4 / Implicit Differentiation	differentiation of a implicit function in an equation
domain	Unit 2 / Relations and Functions	domain of a function is the largest set of acceptable values of the independent variable
– mathematical	Unit 2 / Relations and Functions	domain of a function is the largest set of acceptable values of the independent variable
– physical	Unit 2 / Relations and Functions	the set of acceptable values of the independent variable, according to what the independent and dependent variables involved represent

Error - Function

Concept	*Ref / Unit / Section*	*Definition*
error		
– in measurement	Unit 4 / Applications of the Derivative	
– measurement	Definition 4.7: Unit 4 / Applications of the Derivative	$\Delta x = x - a$
– percentage	Definition 4.7: Unit 4 / Applications of the Derivative	$\frac{\Delta y}{f(a)}(100)$
– propagated	Unit 4 / Applications of the Derivative	error in measurement which affects further calculations
– propagation	Definition 4.7: Unit 4 / Applications of the Derivative	$\Delta y = f(x) - f(a)$
Extreme Value Theorem	Theorem 5.1: Unit 5 / Local and Absolute Extreme Values	existence of extreme values
extreme values		
– absolute	Definition 5.9: Unit 5 / Local and Absolute Extreme Values	the max / min value of a function on an interval
– local	Definition 5.7: Unit 5 / Local and Absolute Extreme Values	a max / min value of a function on an interval
factorization		
– conjugates	Unit 1 / Algebra	$a^2 - b^2 = (a+b)(a-b)$
– quadratic	Unit 1 / Algebra	$ax^2 + bx + c = (x+u)(x+v)$ for some constants u and v if and only if $b^2 - 4ac \geq 0$
– rational expressions	Unit 1 / Algebra	$\frac{p(x)}{q(x)} = u(x)$ if and only if $p(x) = u(x)q(x)$
function		
– function	Unit 2 / Relations and Functions	$f(v)$ is a function if and only if whenever $f(v) = a$ and $f(v) = b$, then $a = b$
– absolute value graph	Unit 2 / The Graph of a Function	graph of the absolute value function
– concave	Definition* 5.11: Unit 5 / Concavity and Points of Inflection	concave shape of the graph of a function
– down	Definition* 5.11: Unit 5 / Concavity and Points of Inflection	concave down shape of the graph of a function
– up	Definition* 5.11: Unit 5 / Concavity and Points of Inflection	concave up shape of the graph of a function
– constant	Unit 2 / The Graph of a Function	$f(v) = d$ for any v in the domain of f

Function (cont'd.)

Concept	*Ref / Unit / Section*	*Definition*
– continuous	Definition* 3.10: Unit 3 / Continuous Functions	conditions for the continuity of a function
– formal definition of	Definition 3.33: Unit 3 / The Definition of a Limit	
– at the left	Definition* 3.15: Unit 3 / Continuous Functions	conditions for the continuity at the left of a function
– at the right	Definition* 3.15: Unit 3 / Continuous Functions	conditions for the continuity at the right of a function
– continuous everywhere	Unit 3 / Continuous functions	continuous at every real number
– convex	Definition* 5.11: Unit 5 / Concavity and Points if Inflection	convex shape of the graph of a function
– cubic graph	Unit 2 / The Graph of a Function	graph of the cubic function
– decreasing	Definition 5.4: Unit 5 / Intervals of Increase and Decrease	if $a < b$, then $f(a) > f(b)$
– defined at the left	Definition 3.1: Unit 3 / Evaluating Limits Visually	defined at the left of a number
– defined at the right	Definition 3.1: Unit 3 / Evaluating Limits Visually	defined at the right of a number
– derivative	Definition 4.5: Unit 4 / The Graph of the Derivative Function	limit of the average rate of change
– differentiable	Definition 4.4: Unit 4 / The Instantaneous Rate of Change and the Slope of the Tangent Line	existence of the limit of the average rate of change
– even	Definition 5.3: Unit 5 / Domain, $x - y$ Intercepts, and Symmetries	$f(x) = f(-x)$ on its domain
– graph of	Unit 2 / The Graph of a Function	the graph of a function F, is the set of all points $(a, F(a))$ on the Cartesian plane
– identity	Unit 2 / The Graph of a Function	$f(x) = x$ for any x in the domain of f
– graph of	Unit 2 / The Graph of a Function	graph of the identity function
– increasing	Definition 5.4: Unit 5 / Intervals of Increase and Decrease	if $a < b$, then $f(a) < f(b)$
– integrand	Unit 6 / Integrand	function of the indefinite integral
– odd	Definition 5.3: Unit 5 / Domain, $x - y$ Intercepts, and Symmetries	$f(x) = -f(x)$ on its domain

Function (cont'd.) - Integral

Concept	*Ref / Unit / Section*	*Definition*
– piecewise	Unit 2 / Relations and Functions	function defined with different mathematical models for different ranges of the independent variable
– **quadratic**	Unit 5 / Optimization Problems	$f(x) = ax^2 + bx + c$
– graph of	Unit 2 / The Graph of a Function	graph of the quadratic function
– reciprocal	Unit 3 / Infinity Limits: Vertical Asymptotes	$\dfrac{1}{f(x)}$ is the reciprocal of f
– roots of	Remark 5.1: Unit 5 / Domain, $x - y$ Intercepts, and Symmetries	see x-intercepts
– transformation of	Unit 2 / Transformations	shifts, stretches, compressions, reflections
– trigonometric	Unit 1 / Trigonometry	functions defined by a unit circle, named, sine, cosine, tangent, secant, cosecant, and cotangent
– winding	Unit 1 / Trigonometry	function which gives the correspondence between radians and degrees
Fundamental Theorem of Calculus		
– part I	Theorem 6.12: Unit 6 / The Fundamental Theorem of Calculus	derivative of $\int_a^x f(u)\, du$
– part II	Theorem 6.11: Unit 6 / The Fundamental Theorem of Calculus	$\int_a^b f(x)\, dx = F(b) - F(a)$
general power rule	Unit 4 / The Chain Rule	rule to differentiate a function risen to a power
inflection point	Unit 5 / Concavity and Points of Inflection	points where the concavity changes
integral		
– definite	Definition 6.10: Unit 6 / The Fundamental Theorem of Calculus	limit of the definite integral
– properties of	Unit 6 / The Fundamental Theorem of Calculus	rules governing the definite integral
– indefinite	Unit 6 / Antiderivatives	antiderivative as an integral
– symmetric functions	Theorem 7.2: Unit 7 / Areas Between Curves	definite integrals of even and odd functions

Integration - Limit

Concept	*Ref / Unit / Section*	*Definition*
integration	Unit 6 / Antiderivatives	process to determine antiderivatives
– constant of	Unit 6 / Antiderivatives	constant of indefinite integral
– lower limit	Unit 6 / The Fundamental Theorem of Calculus	left endpoint of the interval of integration
– upper limit	Unit 6 / The Fundamental Theorem of Calculus	right endpoint of the interval of integration
intercepts	Definition 5.1: Unit 5 / Domain, $x-y$ Intercepts, and Symmetries	points on the x and y axis
Intermediate Value Theorem	Theorem 5.2 / Domain, $x-y$ Intercepts, and Symmetries	if $f(a) < c < f(b)$, then $f(u) = c$ for some $a < u < b$
Laws of Limits	Theorem 3.23: Unit 3 / Laws of Limits	arithmetic operations of finite limits
Leibniz notation	Unit 4 / The Derivative Function	notation $\dfrac{dy}{dx}$ of the derivative
limit		
– at a number	Definition* 3.4: Unit 3 / Evaluating Limits Visually	$f(x)$ approaches L as x approaches a
– formal definition of	Definition 3.30: Unit 3 / The Definition of a Limit	definition of limit at a with ϵ and δ
– at a number from the left	Definition* 3.3: Unit 3 / Evaluating Limits Visually	limit from the left
– formal definition of	Definition 3.32: Unit 3 / The Definition of a Limit	definition of limit at a^- with ϵ and δ
– at a number from the right	Definition* 3.2: Unit 3 / Evaluating Limits Visually	limit from the right
– formal definition of	Definition 3.32: Unit 3 / The Definition of a Limit	definition of limit at a^+ with ϵ and δ
– at negative infinity	Definition* 3.27: Unit 3 / Limits at Infinity: Horizontal Asymptotes	$f(x)$ approaches L as x decreases without bounds
– formal definition of	Definition 3.41: Unit 3 / The Definition of Limits	definition of a limit at negative infinity with ϵ and δ
– at positive infinity	Definition* 3.27: Unit 3 / Limits at Infinity: Horizontal Asymptotes	$f(x)$ approaches L as x increases without bounds
– formal definition of	Definition 3.41: Unit 3 / The Definition of Limits	definition of a limit at positive infinity with ϵ and δ

Limit (cont'd.) - Minimum

Concept	*Ref / Unit / Section*	*Definition*
– does not exist	Definition 3.9: Unit 3 / Evaluating Limits Visually	the limit from the right is not equal to the limit from the left
– negative infinity	Definition* 3.8: Unit 3 / Evaluating Limits Visually	$f(x)$ decreases without bounds as x approaches a number
– formal definition of	Definition 3.36: Unit 3 / The Definition of Limits	definition of a limit which is negative infinity with ϵ and δ
– positive infinity	Definition* 3.8: Unit 3 / Evaluating Limits Visually	$f(x)$ decreases without bounds as x approaches a number
– formal definition of	Definition 3.36: Unit 3 / The Definition of Limits	definition of a limit which is positive infinity with ϵ and δ
– side limits	Definition* 3.2: Unit 3 / Evaluating Limits Visually	limit from the right or left
– formal definition of	Definition 3.32: Unit 3 / The Definition of a limit	definition of limit at left and right with ϵ and δ
linear approximation	Unit 4 / Applications of the Derivative	approximation of $f(a)$ using the tangent line at $(a, f(a))$
linearization	Unit 4 / Applications of the Derivative	equation of the tangent line used to approximate a value of a function
long division	Unit 1 / Algebra	process to divide algebraic expressions
maximum		
– absolute value	Definition 5.9: Unit 5 / Local and Absolute Extreme Values	the maximum value of a function on an interval
– local or relative value	Definition 5.7: Unit 5 / Local and Absolute Extreme Values	a maximum maximum value of a function on an interval
mean arithmetic	Unit 7 / Average Value of a Function	sum of finitely many n numbers, divided by n
Mean Value Theorem		
– for functions	Theorem 6.3: Unit 6 / The Mean Value Theorem	theorem for functions
– for integrals	Theorem 7.4: Unit 7 / Average Value of a Function	theorem for integrals
minimum		
– absolute value	Definition 5.9: Unit 5 / Local and Absolute Extreme Values	the minimum value of a function on an interval
– local or relative value	Definition 5.7: Unit 5 / Local and Absolute Extreme Values	a minimum value of a function on an interval

Motion - Quadratic

Concept	*Ref / Unit / Section*	*Definition*
motion		
– rectilinear	Unit 7 / Rectilinear Motion	use of integration for particles in motion
– uniform accelerated	Definition* 7.1: Unit 7 / Rectilinear Motion	a particle moving with a constant acceleration
Newton's Method	Unit 5 / Newton's Method	approximation to solve an equation $f(x) = K$
orthogonal		
– curves	Definition 4.9: Unit 4 / Implicit Differentiation	the tangent lines of two curves at each point of intersection are perpendicular
– trajectories	Definition 4.9: Unit 4 / Implicit Differentiation	two families of curves have orthogonal trajectories if each curve in the family is orthogonal to every curve in the family
polynomial		
– degree of	Unit 3 / Limits at Infinity: Horizontal Asymptotes	highest power of the variable x with nonzero coefficient
– leading coefficient	Unit 3 / Limits at Infinity: Horizontal Asymptotes	nonzero coefficient of the highest power of x
power rule	Unit 4 / Rules of Differentiation	rule to differentiate $f(x) = x^r$
product rule	Unit 4 / Rules of Differentiation	rule to differentiate the product of functions
Q.E.D.	Unit 3 / The Definition of Limit	quod erat demonstrandum which was to be proved
quadratic		
– expression	Unit 1 / Algebra	polynomial of the form $ax^2 + bx + c$
– formula	Unit 1 / Algebra	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
– perfect square	Unit 1 / Algebra	$ax^2 \pm 2\sqrt{ac}\,x + c = (\sqrt{a}x \pm \sqrt{c})^2$
quotient rule	Unit 4 / Rules of Differentiation	rule to differentiate the quotient of functions
radians	Unit 1 / Trigonometry	real number that corresponds to an angle measured in degrees
range	Unit 2 / Relations and Functions	range of a function is the largest set of acceptable values of the dependent variable

Rate of Change - Shift

Concept	*Ref / Unit / Section*	*Definition*
rate of change		
– average	Definition 4.1: Unit 4 / The Average Rate of Change and the Slope of the Secant Line	rate of change on an interval
– instantaneous	Definition 4.2: Unit 4 / The Average Rate of Change and the Slope of the Secant Line	rate of change at a number
rationalization	Unit 1 / Algebra	process to change quotient expressions with square roots
reciprocal rule	Unit 4 / Rules of Differentiation	rule to differentiate the reciprocal of a function
reflection		
– x-axis	Definition 2.13: Unit 2 / Reflections	$F(x) = -f(x)$ is the reflection of f with respect to the x-axis
– y-axis	Definition 2.13: Unit 2 / Reflections	$F(x) = f(-x)$ is the reflection of f with respect to the y-axis
relation	Unit 2 / Relations and Functions	it is an association of dependency between two or more variables
Riemann Sum	Definition 6.10: Unit 3 / The Fundamental Theorem of Calculus	sums to approximate an area under a curve
Rolle’s Theorem	Theorem 6.4: Unit 6 / The Mean Value Theorem	particular case of the Mean Value Theorem
Second Derivative Test	Unit 5 / Optimization Problems	test to determine local extreme
sequence	Unit 5 / Newton’s Method	list of numbers with an specific order
shift		
– downwards	Definition 2.9: Unit 2 / Transformations	$F(x) = f(x) - c$ is the shift downwards of f by c units
– left	Definition 2.18: Unit 2 / Transformations	$F(x) = f(x + c)$ is the shift to the left of f by c units
– right	Definition 2.18: Unit 2 / Transformations	$F(x) = f(x - c)$ is the shift to the left of f by c units
– upwards	Definition 2.9: Unit 2 / Transformations	$F(x) = f(x) + c$ is the shift upwards of f by c units

Squeeze Theorem - Work

Concept	*Ref / Unit / Section*	*Definition*
Squeeze Theorem	Theorem 3.22: Unit 3 / The Squeeze Theorem	theorem used to evaluate the limit of a function between two others
– proof of	Corollary 3.44: Unit 3 / The Definition of a Limit	proof of the Squeeze Theorem
stretch		
– horizontal	Definition 2.12: Unit 2 / Transformations	$F(x) = f\left(\frac{x}{c}\right)$ is the horizontal stretch of f by c units
– vertical	Definition 2.12: Unit 2 / Transformations	$F(x) = f(cx)$ is the vertical stretch of f by c units
sum rule	Unit 4 / Rules of Differentiation	rule to differentiate the sum of functions
tangent line	Unit 4 / The Instantaneous Rate of Change and the Slope of the Tangent Line	tangent line to a curve
– equation of	Proposition 4.6: Unit 4 / The Derivative Function	equation of the line tangent to a curve
variable	Unit 2 / Relations and Functions	it is a quantity that may or may not change (vary) according to what it represents
– dependency of	Unit 2 / Relations and Functions	in $f(v)$ the variable f depends on the variable v
– dependent	Unit 2 / Relations and Functions	f is the dependent variable in $f(v)$
– independent	Unit 2 / Relations and Functions	v is the independent variable in $f(v)$
vertical line test	Definition 2.8: Unit 2 / The graph of a Function	test to determine whether a curve is the graph of a function
work	Unit 7 / Work	definition of work as an integral
– unit of	Unit 7 / Work	units: joule, erg, ft-lb