Problem 1

(a)

$$P_X(t) = P(X \le t,) = P(X_1 = t, X_2 \sim X_n \le t) + P(X_2 = t, X_1 \le t, X_3 \sim X_n \le t) \ldots$$

$$= \sum_{k=1}^{n} P(X_k = t) \cdot P(X_{1\sim k-1} < t) \cdot P(X_{k+1\sim n} \le t) = \sum_{k=1}^{n} p(1-p)^{t-1} \cdot \underbrace{(1-(1-p)^{t-1})}_{a}{}^{k-1} \cdot \underbrace{(1-(1-p)^{t})}_{b}{}^{n-k}$$

let $(1-(1-p)^{t-1}) = a$, $[1-(1-p)^{t}] = b$

$$P_X(t) = p(1-p)^{t-1} \times \frac{b^{n-1} \cdot (1-(\frac{a}{b})^n)}{1-\frac{a}{b}} = p(1-p)^{t-1} \times \frac{b^n - a^n}{a} = b^n - a^n = (1-(1-p)^t)^n - (1-(1-p)^{t-1})^n$$

$$P_Y(t) = P(X \ge t) = \sum_{k=1}^{n} P(X_k = t) \cdot P(X_{1\sim k-1} > t) \cdot P(X_{k+1\sim n} \ge t) = \sum_{k=1}^{n} p(1-p)^{t-1} \cdot (1-p)^{t(k-1)} \cdot (1-p)^{(t-1)(n-k)}$$

$$= \sum_{k=1}^{n} p(1-p)^{\cancel{t}-1+\cancel{tk}-\cancel{t}+tn-\cancel{tk}+k-n} = p(1-p)^{tn-n-1} \cdot \sum_{k=1}^{n} (1-p)^k = \cancel{p}(1-p)^{tn-n-1} \times \frac{(1-p)(1-(1-p)^n)}{\cancel{p}}$$

$$= (1-p)^{n(t-1)} \times (1-(1-p)^n)$$

$P_Y(t) = \text{Geometric}(1-(1-p)^n)$

(b)

when $n=1$, $P(X=1) = P(X=2) = \frac{1}{2}$

let $n=k$ 成立 $\rightarrow P_k(X=1) = P_k(X=2) \ldots = P_k(X=k+1) = \frac{1}{k+1}$

when $n=k+1$, $P_{k+1}(X=1) = \frac{1}{k+1} \times \frac{k+1}{k+2} = \frac{1}{k+2}$, $P_{k+1}(X=a \mid a \in \{2,3,\ldots,k+1\}) = P_k(X=a-1) \times P_k(\text{get a ball}) + P_k(X=a) \cdot P_k(\times \text{ball})$

$$= \frac{1}{k+1} \times \frac{a-1}{k+2} + \frac{1}{k+1} \times \frac{k+2-a}{k+2} = \frac{\cancel{a}-1+k+2-\cancel{a}}{(k+1)(k+2)} = \frac{1}{k+2}$$

$$P_{k+1}(X=k+2) = \frac{1}{k+1} \times \frac{k+1}{k+2} = \frac{1}{k+2}$$

by math induction, $P_n(X) = \frac{1}{n+1} \Rightarrow X$ is a discrete uniform variable with parameter $(1, n+1)$

(c)

$$Y = \begin{cases} aX+b, & \text{if } X > 0 \\ -aX+b, & \text{if } X \le 0 \end{cases}$$

① $t \le b$

if $a > 0$, $P(Y \le t) = P(aX+b \le t, X>0) + P(-aX+b \le t, X \le 0) = P(X \le \underset{\le 0}{\frac{t-b}{a}}, X>0) + P(X \ge \frac{-(t-b)}{a}, X \le 0)$

$$= P(0 < X < \tfrac{t-b}{a}) + P(0 \le X \ge \tfrac{b-t}{a}) = 0 + 0 = 0$$

if $a < 0$, $P(Y \le t) = P(X \ge \frac{t-b}{a}, X>0) + P(X \le \frac{b-t}{a}, X \le 0) = 1 - F_X(\frac{t-b}{a}) + F_X(\frac{b-t}{a})$

② $b < t$

if $a > 0$, $P(Y \le t) = P(aX+b \le t, X>0) + P(-aX+b \le t, X \le 0) = P(X \le \frac{t-b}{a}, X>0) + P(X \ge \frac{b-t}{a}, X \le 0)$

$$= F_X(\tfrac{t-b}{a}) - \cancel{F_X(0)} + \cancel{F_X(0)} - F_X(\tfrac{b-t}{a}) = F_X(\tfrac{t-b}{a}) - F_X(\tfrac{b-t}{a})$$

if $a < 0$, $P(Y \le t) = 1 - F_X(0) + F_X(0) = 1$

if $a > 0$, $F_Y(t) = \begin{cases} 0, & \text{if } t \le b \\ F_X(\frac{t-b}{a}) - F_X(\frac{b-t}{a}), & \text{if } t > b \end{cases}$, $f_Y(t) = \begin{cases} 0, & \text{if } t \le b \\ \frac{1}{a}(f_X(\frac{t-b}{a}) + f_X(\frac{b-t}{a})), & \text{if } t > b \end{cases}$

if $a < 0$, $F_Y(t) = \begin{cases} 1 - F_X(\frac{t-b}{a}) + F_X(\frac{b-t}{a}), & \text{if } t \le b \\ 1, & \text{if } t > b \end{cases}$, $f_Y(t) = \begin{cases} -(f_X(\frac{t-b}{a}) + f_X(\frac{b-t}{a})) \cdot \frac{1}{a}, & \text{if } t \le b \\ 0, & \text{if } t > b \end{cases}$

Y can never be a normal random variable.

Problem 2

(a)

$$\sum_{i=1}^{n} P_i \ln P_i = P_1 \ln P_1 + P_2 \ln P_2 \dots P_n \ln P_n \quad , \quad P_1 + P_2 + P_3 \dots P_n = 1$$

$$\frac{P_1 \ln P_1 + P_2 \ln P_2 \dots P_n \ln P_n}{1} \geq (\ln P_1)^{P_1} (\ln P_2)^{P_2} \dots (\ln P_n)^{P_n}$$

equality holds when $P_1 \ln P_1 = P_2 \ln P_2 = \dots = P_n \ln P_n \rightarrow P_1 = P_2 = P_3 = \dots P_n = \frac{1}{n}$

$\sum_{i=1}^{n} P_i \ln P_i$ has minimum $\left(\left(\ln \frac{1}{n}\right)^{\frac{1}{n}}\right)^n = \ln \frac{1}{n}$

$-\sum_{i=1}^{n} P_i \ln P_i$ has maximum $\left(-\ln \frac{1}{n}\right) = \ln n$ #, $P(x) = \frac{1}{n}$ #

(b)

$-\sum_{i=1}^{n} \underset{\substack{\downarrow \\ P_i \geq 0}}{P_i} \underset{\substack{\searrow \\ \ln P_i \leq 0}}{\ln P_i} \geq 0$, if $P_1 = 1$, $-\sum_{i=1}^{n} P_i \ln P_i = 0$, minimum of $-\sum_{i=1}^{n} P_i \ln P_i = 0$ #

minimum happens when $P_i = 1$, and all the other $P_n = 0$. $i = \{1, 2, 3, 4, \dots, n\}$ #

Problem 3

(a)

(i)

$$E[X] = \sum_{i=1}^{\infty} i\, p(1-p)^{i-1} = 1 \times p + 2 \times p(1-p) + 3 \times p(1-p)^2 \dots$$

$$(1-p)E[X] = p(1-p) + 2 \times p(1-p)^2 + 3 \times p \times (1-p)^3 \dots$$

$$p\,E[X] = p + p(1-p) + p(1-p)^2 + \dots = \frac{p}{1-(1-p)} = 1$$

$$E[X] = \frac{1}{p} \#$$

$r = e^t(1-p)$

(ii)

$$E[e^{tx}] = e^t \times p + e^{2t} p(1-p) + e^{3t} p(1-p)^2 \dots$$

$r = e^t(1-p) < 0$ because $t < -\ln(1-p)$

$$\sum_{i=1}^{\infty} e^{tx} p(1-p)^{i-1} = \frac{e^t p \times (1-r^{\infty})}{1-r} = \frac{e^t p}{1-(1-p)e^t} \#$$

(b)

$P(X=t)=P(Y=t) \quad \forall t \in \{a_1 \dots a_n\} \rightarrow E[X^m]=E[Y^m] \quad \forall m \in \{1,2, \dots n-1\}$

$E[X^m]=\sum_{i=1}^{n} a_i \cdot [P(X=a_i)]^m = \sum_{i=1}^{n} a_i \cdot [P(Y=a_i)]^m = E[Y^m]$ ✓

$E[X^m]=E[Y^m] \quad \forall m \in \{1,2,3, \dots n-1\} \rightarrow P(X=t)=P(Y=t) \quad \forall t \in \{a_1, a_2 \dots a_n\}$

$a_1 \cdot P(X=a_1)+a_2 \cdot P(X=a_2) \dots = a_1 \cdot P(Y=a_1)+a_2 \cdot P(Y=a_2) \dots$

$$\begin{cases} a_1(P(X=a_1)-P(Y=a_1)) + a_2(P(X=a_2)-P(Y=a_2)) \dots + a_n(P(X=a_n)-P(Y=a_n)) = 0 \\ a_1^2(P(X=a_1)-P(Y=a_1)) + a_2^2(P(X=a_2)-P(Y=a_2)) \dots = 0 \\ \vdots \\ a_1^{n-1}(P(X=a_1)-P(Y=a_1)) + \dots = 0 \\ P(X=a_1)-P(Y=a_1)+P(X=a_2)-P(Y=a_2) \dots = 0 \end{cases}$$

let $b_k = P(X=a_k)-P(Y=a_k)$, $k \in \{1,2 \dots n\}$

$$\begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & \dots & b_n & \end{matrix} \\ \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n & 0 \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 & 0 \\ & & & & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & \dots & a_n^{n-1} & \vdots \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} n \text{ variables, } n \text{ rows} \\ \begin{bmatrix} 1 & \dots & & & 0 \\ 0 & 1 & & & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & 1 \dots & 0 \\ 0 & 0 & 0 & 0 \; 1 & 0 \end{bmatrix} \end{array} \rightarrow \text{exist only 1 solution}$$

$b_k = 0 = P(X=a_k)-P(Y=a_k) \quad \forall k \in \{1,2, \dots n\}$

$\rightarrow P(X=a_k)=P(Y=a_k) \quad \forall k \in \{1,2,3, \dots n\}$

(c)

$Var[Z] = E[Z^2]-E[Z]^2$

$E[Z^2] = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{n} = \infty$, $E[Z] = \frac{6}{\pi}\sum_{k=1}^{\infty}\frac{1}{n^2} < \infty$. $Var[Z] = \infty$. does not exist

if $E[Z^3]$ exist $\rightarrow E[|Z^3|] < \infty, \rightarrow E[|Z|^2] < \infty \rightarrow$ contradict $(E[|Z|^2] = \infty)$

$\Rightarrow E[Z^3]$ does not exist

Problem 4

(a)

$$F_X(t) = \int_0^t \lambda e^{-\lambda x} dx = [-e^{-\lambda x}]_0^t = 1 - e^{-\lambda t} \quad , \text{for } t > 0$$

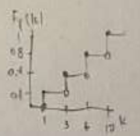
$$y = 1 - e^{-\lambda t} \quad , \quad t = -\frac{\ln(1-y)}{\lambda}$$

$U \sim Unif(0,1)$

$$X = F_X^{-1}(U) = \begin{cases} -\infty, & \text{if } U \le 0 \\ -\frac{\ln(1-U)}{\lambda}, & \text{if } 0 < U < 1 \\ \infty, & \text{if } U \ge 1 \end{cases}$$

(b)

$$F_Y(k) = \begin{cases} 0, & \text{if } k < 1 \\ 0.1, & \text{if } 1 \le k < 3 \\ 0.4, & \text{if } 3 \le k < 6 \\ 0.8, & \text{if } 6 \le k < 10 \\ 1, & \text{if } k \ge 10 \end{cases}$$



$U \sim Unif(0,1)$

$$X = F^{-1}(U) = \begin{cases} -\infty, & \text{if } U \le 0 \\ 1, & \text{if } 0 \le U \le 0.1 \\ 3, & \text{if } 0.1 < U \le 0.4 \\ 6, & \text{if } 0.4 < U \le 0.8 \\ 10, & \text{if } 0.8 < U < 1 \\ \infty, & \text{if } U \ge 1 \end{cases}$$