Problem 1

(a) $F_X(t) = F_{XY}(t,\infty) = 1-e^{-t} + \lim_{u\to\infty} e^{-t-u(1+\theta t)} = 1-e^{-t}$, for $t>0$

$$F_X(t) = \begin{cases} 1-e^{-t}, & \text{if } t>0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_Y(u) = \begin{cases} F_{XY}(\infty,u) = 1-e^{-u}, & \text{if } u>0 \\ 0 & \text{otherwise} \end{cases}$$

if X, Y are independent, $F_{XY}(t,u) = F_X(t)\cdot F_Y(u)$

$$(1-e^{-t})(1-e^{-u}) = 1-e^{-t}-e^{-u}+e^{-(u+t)}$$

$\Rightarrow$ if $\theta=0$, X and Y are independent

(b)

$$\frac{\partial^2}{\partial t\partial u} F_{XY}(t,u) = \frac{\partial}{\partial t}\left(e^{-u} + e^{-(t+u+\theta t u)} \times (-1-\theta t)\right)$$

$$= e^{-(t+u+\theta t u)} \times (-1-\theta u)(-1-\theta t) + e^{-(t+u+\theta t u)} \times (-\theta)$$

$$= e^{-(t+u+\theta t u)} \times \left((\theta u+1)(\theta t+1)-\theta\right)$$

$$f_{XY}(t,u) = \begin{cases} e^{-(t+u+\theta t u)} \times \left((\theta u+1)(\theta t+1)-\theta\right), & \text{if } t>0, u>0 \\ 0, & \text{otherwise} \end{cases}$$

(c)

$$f_X(t) = F_X'(t) = \begin{cases} e^{t}, & \text{if } t>0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(u) = F_Y'(u) = \begin{cases} e^{u}, & \text{if } u>0 \\ 0, & \text{otherwise} \end{cases}$$

(d)

$$F_X(t) = F_{XY}(t,\infty) = 1-e^{-t} + \lim_{u\to\infty} e^{-t-u(1+\theta t)}$$

$\lim_{u\to\infty} e^{-t-u(1+\theta t)}$ must exist $\forall t$, $1+\theta t$ must >0. $\theta t > -1$ for all t

$\Downarrow$

$\boxed{\theta \geq 0}$

$f_{XY}(t,u) \geq 0$ for all (t,u)

$+$

$\lim_{\substack{t\to 0^+ \\ u\to 0^+}} f_{XY}(t,u) \geq 0$, $\lim_{\substack{t\to 0^+ \\ u\to 0^+}} e^{-(t+u+\theta t u)} \times \left[(\theta u+1)(\theta t+1)-\theta\right] = 1 \times (1-\theta) \geq 0 \Rightarrow \boxed{\theta \leq 1}$

$\Rightarrow$ θ is required in $[0,1]$

Problem 2

(a)

$X \sim Unif(-1,3)$, $f_X(t) = \begin{cases} \frac{1}{4}, & \text{if } -1<t<3 \\ 0, & \text{otherwise} \end{cases}$

$$M_X(t) = \int_{-\infty}^{\infty} f_X(t)\, e^{tx}\, dx = \int_{-1}^{3} \frac{1}{4} e^{tx}\, dx = \frac{e^{tx}}{4t}\Big|_{-1}^{3} = \frac{e^{3t}-e^{-t}}{4t}$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = \lim_{t\to 0} \frac{(3e^{3t}+e^{-t})t - (e^{3t}-e^{-t})}{4t^2} \overset{L'H}{=} \lim_{t\to 0} \frac{(9e^{3t}-e^{-t})t + \cancel{(3e^{3t}+e^{-t})} - \cancel{(3e^{3t}+e^{-t})}}{8t}$$

$$= \frac{9-1}{8} = 1$$

$E[X] = 1$ #

$$E[X^2] = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \frac{d}{dx}\left(\frac{(3e^{3t}+e^{-t})t - (e^{3t}-e^{-t})}{4t^2} \right)\Big|_{t=0}$$

$$= \frac{\left[(9e^{3t}-e^{-t})t + \cancel{(3e^{3t}+e^{-t})} - \cancel{(3e^{3t}+e^{-t})}\right]4t^2 - 8t\left[(3e^{3t}+e^{-t})t - (e^{3t}-e^{-t})\right]}{16t^4}$$

$$= \frac{t^2(9e^{3t}-e^{-t}) - 2\left[(3e^{3t}+e^{-t})t - (e^{3t}-e^{-t})\right]}{4t^3}$$

$$\overset{L'H}{=} \frac{\cancel{2t(9e^{3t}-e^{-t})} + \cancel{t^2}(27e^{3t}+e^{-t}) - \cancel{2(9e^{3t}-e^{-t})t}}{12\cancel{t^2}}$$

$$= \frac{28}{12} = \frac{7}{3} = E[X^2]$$

$Var[X] = E[X^2] - E[X]^2 = \frac{7}{3} - 1 = \frac{4}{3}$ #

(b)

$M_X(t) = \sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} \times e^{tk}$, the sum of the series is finite only when $\lim_{k\to\infty} \frac{6}{\pi^2 k^2} e^{kt} = 0$

$\lim_{k\to\infty} \frac{e^{kt}}{k^2} = 0$ only when $t \notin (0,\infty)$

$\Rightarrow M_Y(t)$ is not finite on $t \in (0,\infty)$ $\Rightarrow$ MGF of Y does not exist

Problem 3

(a)

$$M_{X_1}(t) = e^{\lambda(e^t-1)} \sim \text{Poisson}(\lambda)$$

$$p_{X_1}(n) = \frac{e^{-\lambda T}(\lambda T)^n}{n!}$$

$$M_{X_2}(t) = \frac{e^{2t}-e^t}{t} \sim \text{Unif}(1,2)$$

$$f_X(t) = \begin{cases} 1, & \text{if } 1<t<2 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$M_{X+2Y}(t) = E[e^{t(X+2Y)}] = E[e^{tX} \cdot e^{t2Y}] = E[e^{tX}] \cdot E[e^{2tY}]$$

$$= e^{e^t-1} \cdot M_Y(2t) = e^{e^t-1} \cdot e^{e^{2t}-1} = e^{e^{2t}+e^t-2}$$

$\Rightarrow$ can't be reduce to the form $e^{\lambda(e^t-1)}$ $\Rightarrow X+2Y$ is not a Poisson Random variable

Problem 4

(a) $X_1 = \sigma_1 Z + \mu_1$

$X_2 = \sigma_2(\rho Z + \sqrt{1-\rho^2}\, W) + \mu_2$

$A = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}$, $|\det(A)| = \sigma_1 \sigma_2 \sqrt{1-\rho^2}$

$A^{-1} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \begin{bmatrix} \sigma_2\sqrt{1-\rho^2} & 0 \\ -\rho\sigma_2 & \sigma_1 \end{bmatrix}$

$f_{ZW}(z, w) = \frac{1}{2\pi} \times e^{-\frac{z^2+w^2}{2}}$

$f_{X_1X_2}(X_1, X_2) = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} f_{ZW}\left(\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\sigma_2\sqrt{1-\rho^2} \times (X-\mu_1), \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\left(-\rho\sigma_2(X_1-\mu_1) + \sigma_1(X_2-\mu_2)\right)\right)$

$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \times \left(\frac{\sigma_2^2(1-\cancel{\rho^2})(X-\mu_1)^2 + \cancel{\rho^2\sigma_2^2(X_1-\mu_1)^2} + \sigma_1^2(X_2-\mu_2)^2 - 2\rho\sigma_1\sigma_2(X_1-\mu_1)(X_2-\mu_2)}{\sigma_1^2\sigma_2^2(1-\rho^2)}\right.\right.$

$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(- \frac{\frac{(X_1-\mu_1)^2}{\sigma_1^2} + \frac{(X_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(X_1-\mu_1)(X_2-\mu_2)}{\sigma_1\sigma_2}}{2(1-\rho^2)}\right.$

(b)

$Cov(Z_1, Z_1) = 2+2\rho_{XY} = Var(Z_1)$

$Cov(Z_2, Z_2) = 2-2\rho_{XY} = Var(Z_2)$

$E[Z_1] = 0$, $E[Z_2] = 0$

$Z_2 = X-Y \sim N(0, 2-2\rho)$

$Z_1 = X+Y \sim N(0, 2+2\rho)$

$Cov(Z_1, Z_2) = E[Z_1Z_2] = E[X^2-Y^2] = 0$

$E[X^2]$ of $N(0,1)$, $E[X]=0$, $Var(X)=1$, $E[X^2]=1$, $E[Y^2]=1$

$\because Cov(Z_1, Z_2) = E[XY] - E[X] \cdot E[Y] = 0$ $\therefore Z_1, Z_2$ are independent

$f_{Z_1Z_2}(Z_1, Z_2) = f_{Z_1}(Z_1) \cdot f_{Z_2}(Z_2) = \frac{1}{\sqrt{2\pi}\,\sigma_{Z_1}} \exp\left(-\frac{Z_1^2}{2\sigma_{Z_1}^2}\right) \times \frac{1}{\sqrt{2\pi}\,\sigma_{Z_2}} \exp\left(-\frac{Z_2^2}{2\sigma_{Z_2}^2}\right)$

$= \frac{1}{2\pi\sigma_{Z_1}\sigma_{Z_2}} \exp\left(-\frac{1}{2}\left(\frac{Z_1^2}{\sigma_{Z_1}^2} + \frac{Z_2^2}{\sigma_{Z_2}^2}\right)\right)$

we know $\rho=0$ by independence $\Rightarrow f_{Z_1Z_2}(Z_1, Z_2)$ is in the form of PDF of bivariate r.v.

Problem 5

(a)

disagree, the idea is correct but $E[T_1|N_1] \neq 3.5N_1$, $E[T_6|N_6] \neq 3.5N_6$

→ so you can't say $E[t_1] = E[t_6]$ with the prove

(b)

disagree

Novak use $E[t_1] = E[t_6]$ as the premisis to prove $E[t_6] > E[t_1]$, which is obviously a contradiction

although the last roll of $E[t_6]$ is bigger than $E[t_1]$, the value in the previous rolls are also different, and he didn't take this into consideration

(c)

$$E[E[T_1|N_1]] = E[\underbrace{(N_1-1) \times \tfrac{(2+6)}{2}}_{\text{for first } (N_1-1) \text{ rolls}} + \underbrace{1}_{\text{last roll}}] = E[4N_1-3] = 21 = E[T_1]$$

$$E[E[T_6|N_6]] = E[(N_6-1) \times 3 + 6] = 21 = E[T_2]$$

$$E[T_1] = E[T_2]$$