Problem 1

(a)

$$\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n = \underbrace{\{x \mid x \in S_n \text{ for infinitely many } n\}}_{A}$$

① $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n \subseteq A$

pick an element $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n$

for all $k>0$, $x \in \bigcup_{n=k}^{\infty} S_n$

$\rightarrow$ x exist in infinitely many S_n

$\rightarrow x \in A \Rightarrow \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n \subseteq A$

② $A \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n$

pick an element $x \in A$

x exist in $\bigcup_{n=k}^{\infty} S_n$ for all k

$\rightarrow x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n \Rightarrow A \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n$

$\Rightarrow \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n \subseteq A$ and $A \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \Rightarrow \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n = \{x \mid x \in S_n \text{ for infinitely many } n\}$

(b)

① $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} (A_n \cup B_n) \subseteq \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \cup \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} B_n\right)$

pick an element $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} (A_n \cup B_n)$

x exist in infinitely many $A_n \cup B_n$

$\rightarrow$ exist in $A_n \cup B_n$ = exist in A_n or B_n

$\rightarrow$ x exist in finitely many A_n or B_n

② $\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \cup \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} B_n\right) \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} (A_n \cup B_n)$

if $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n$, $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} (A_n \cup B_n)$

if $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} B_n$, $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} (A_n \cup B_n)$

$\Rightarrow \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \cup \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\right) \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} (A_n \cup B_n)$

$\Rightarrow \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} (A_n \cup B_n) = \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \cup \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} B_n\right)$

Problem 2

(a)

- $P(A^c) = 1 - P(A)$

  $A + A^c = \Omega$, A and A^c are mutually exclusive

  $1 = P(\Omega) = P(A + A^c) = P(A) + P(A^c)$

  $\Rightarrow P(A^c) = 1 - P(A)$

- $P(A) = P(A-B) + P(A \cap B)$

  $A-B$ and $A \cap B$ are mutually exclusive

  $P(A-B) + P(A \cap B) = P((A-B) \cup (A \cap B)) = P(A)$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  代入 $P(A) = P(A-B) + P(A \cap B)$

  $\rightarrow P(A \cup B) = P(A-B) + P(B)$

  $A-B$ and B are mutually exclusive

  $P(A-B) + P(B) = P((A-B) \cup B) = P(A \cup B)$

(b)

$$\begin{array}{cccccc} ① & ② & ③ & ④ & ⑤ & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0.2 \\ 0 & 1 & 1 & 0 & 0 & 0.35 \\ 0 & 0 & 1 & 1 & 0 & 0.45 \\ 0 & 0 & 0 & 1 & 1 & 0.6 \\ 1 & 0 & 0 & 0 & 1 & 0.4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0.65 \\ 1 & 1 & 1 & 0 & 1 & 0.75 \\ 1 & 0 & 1 & 1 & 1 & 0.85 \\ 0 & 1 & 1 & 1 & 1 & 0.95 \\ 1 & 1 & 0 & 1 & 1 & 0.8 \end{bmatrix} \rightarrow$$

$P(3) = 0.2$, $P(1) = 0.05$, $P(4) = 0.25$, $P(2) = 0.15$

$P(5) = 0.35$, only this set of solution.

Problem 3

(a)

if $\sum_{n=1}^{\infty} P(A_n) < \infty$, $\lim_{n\to\infty} P(A_n) = 0$

$\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_n \subseteq \bigcup_{n=m}^{\infty} A_n$ for all $m \in N$

$P\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_n\right) \leq P\left(\bigcup_{n=m}^{\infty} A_n\right) \leq \sum_{n=m}^{\infty} P[A_n]$

when $m \to \infty$, $\sum_{n=m}^{\infty} P(A_n) = 0$,

$0 \leq P\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_n\right) \leq 0$, $P\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_n\right) = 0$

(b)

A^c is also mutually independent, for $m \in N$

$P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = \prod_{n=m}^{\infty} P(A_n^c) = \prod_{n=m}^{\infty} (1 - P[A_n])$

$\Downarrow \log$ $\qquad \log(1+x) \leq x$

$\log\left(P\left[\bigcap_{n=m}^{\infty} A_n^c\right]\right) = \sum_{n=m}^{\infty} \log(1 - P[A_n]) \leq -\sum_{n=m}^{\infty} P[A_n] = -\infty$

$P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = 0$ → exist at most finite often : ϕ

$P\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_n\right) = 1$

(c)

$\sum_{k=1}^{\infty} \frac{1}{100} \times \frac{1}{k^N} = \frac{1}{100}\sum_{k=1}^{\infty} \frac{1}{k^N}$

when $N \leq 1$, $\sum_{k=1}^{\infty} \frac{1}{k^N}$ diverges, $\frac{1}{100}\sum_{k=1}^{\infty} \frac{1}{k^N} = \infty$

$\Rightarrow P(I) = 1$

when $N > 1$, $\sum_{k=1}^{\infty} \frac{1}{k^N}$ converges, $\frac{1}{100}\sum_{k=1}^{\infty} \frac{1}{k^N} < \infty$

$\Rightarrow P(I) = 0$

(d)

(1) As proved in (a)(c), $P(I) = 0$ if $N > 1$

(2) It is possible.

$P_1 = \frac{1}{2}$, $P_k = \begin{cases} P_{k-1} + \frac{(1-P_{k-1})}{10}, & \text{if } k\text{-1th toss is "Yes"} \\ \frac{P_{k-1}}{10}, & \text{if } k\text{-1th toss is not "Yes"} \end{cases}$

Problem 4

(a)

$P(B) = \frac{1.2}{3} = 0.4$, $P(A_1|B) = \frac{\frac{1}{3}\times 0.3}{0.4} = 0.25$, $P(A_2|B) = \frac{\frac{1}{3}\times 0.6}{0.4} = 0.5$, $P(A_3|B) = 0.25$

(b)

$P(A_1|C) = \frac{P(A_1)\times P(C|A_1)}{P(C)} = \frac{\frac{1}{3}\times 0.1^3\times 0.3^6\times 0.6}{\frac{1}{3}\times(0.1^3\times 0.3^6\times 0.6 + 0.3^3\times 0.6^6\times 0.1 + 0.6^3\times 0.3^6\times 0.1)} = \frac{3^6\times 6}{3^6\times 6 + 3^3\times 6^6 + 6^3\times 3^6} = \frac{1}{1+36+288} = \frac{1}{325}$

$P(A_2|C) = \frac{P(A_2)\times P(C|A_2)}{P(C)} = \frac{288}{325}$

$P(A_3|C) = \frac{P(A_3)\times P(C|A_3)}{P(C)} = \frac{36}{325}$ $\Rightarrow$ A_2 is the most probable value

(c)

$P(A_1|C) = \frac{\frac{1}{3}\times 0.1^3\times 0.3^6\times 0.6}{\frac{1}{3}\times(0.1)^3\times 0.3^6\times 0.6 + \alpha\cdot 0.3^3\times 0.6^6\times 0.1 + (\frac{2}{3}-\alpha)\cdot 0.6^3\times 3^6\times 0.1}$

$\alpha\times 0.3^3\times 0.6^6\times 0.1 > \frac{1}{3}\times 0.1^3\times 0.3^6\times 0.6$

$\alpha > \frac{1}{3}\times 3^3\times 6\times\frac{1}{6^6} = \frac{3^3\times 2}{3^6\times 2^6} = \frac{1}{3^3 2^5}$

$\alpha\cdot 0.3^3\times 0.6^6\times 0.1 > (\frac{2}{3}-\alpha)\, 0.6^3\times 0.3^6\times 0.1$

$\alpha\times 6^3 > \frac{2}{3}\times 3^3 - 3^3\alpha = 18 - 27\alpha$

$243\alpha > 18$

$\alpha > \frac{18}{243} = \frac{2}{27}$

$\Rightarrow \frac{2}{3} > \alpha > \frac{2}{27}$

Problem 5

- $P(|X| \le t) = 2 \cdot F_X(t) - 1$ for all $t \ge 0$

$$2 \cdot F_X(t) - 1 = 2 \times \big(P(-t \le X \le t) + \underbrace{P(X < -t)}_{= P(X > t)}\big) - 1$$

$$= 2P(|X| \le t) + P(X < -t) + P(X > t) - 1$$

$$= P(|X| \le t) + P(X, X \in R) - 1$$

$$= P(|X| \le t)$$

- $P(X = t) = F(t) + F(-t) - 1$

$$F(t) + F(-t) = P(|X| \le |t|) + P(X < -|t|) + P(X \le -|t|)$$

$$= P(X \in R) + P(X = -|t|)$$

$$F(t) + F(-t) - 1 = \cancel{P(X \in R) - 1} + P(X = -|t|)$$

$\hookrightarrow P(X = t) = P(X = -t)$

$$= P(X = t)$$