1 Problem

”Imagine taking a number and moving its last digit to the front. For example, 1,234 would become 4,123. What is the smallest positive integer such that when you do this, the result is exactly double the original number? (For bonus points, solve this one without a computer.)”

2 Solution

Let’s begin by writing out numbers in the following form

$$10^N d_N + 10^{N-1} d_{N-1} + \cdots + 10^1 d_1 + 10^0 d_0 = \sum_{n=0}^{N} 10^n d_n$$
where $d_n \in \{0, 1, 2, \ldots, 9\}$

We are looking for a number that satisfies the following equality

$$2 \sum_{n=0}^{N} 10^n d_n = 10^N d_0 + \sum_{n=1}^{N} 10^{n-1} d_n$$

Rearranging we get

$$\sum_{n=1}^{N} (2 * 10^n - 10^{n-1}) d_n = (10^N - 2 * 10^0) d_0$$

Note that $2 * 10^n - 10^{n-1} = 19 * 10^{n-1}$. As such we rewrite the equation as follows

$$\sum_{n=1}^{N} 10^{n-1} d_n = \frac{(10^N - 2 * 10^0) d_0}{19}$$

Since we know that the left side of the equation is an integer, we also know that the numerator of the right side had to be divisible by 19! As such, we know that the following must be true

$$10^N - 2 \equiv 0 \ (mod\ 19)$$

[More explination of how to solve this]

... we get

$$N = 17n + 18 \text{ where } n \in \mathbb{Z}_{\geq 0}$$

This shows us that the smallest possible number that satisfies the problem is 18 digits! Once we set $N = 18$ we observe that the right side of equation - can only take on 10 different values (i.e. each of the 10 possible values for d_0). Starting from 0 we find that $d_0 = 2$ is the first value to satisfy equation -. We can now construct our number

$$10 * \frac{(10^{18} - 2) * 2}{19} + 2 = \ldots$$