1 Introduction ("Review" of ODE)

Separation of Variables: Set $u(x,t) = T(t)X(x)$. Group like vars on one side, take lin comb.

Solving ODEs: $y'' + ay' + by = 0$, $y = e^{\lambda x} \to \lambda^2 + a\lambda + b = 0$.

- $\gamma^2 = a^2 - 4b > 0$, $e^{(-a+\gamma)t/2}$ and $e^{(-a-\gamma)t/2}$.
- $\gamma^2 = 0. e^{-at/2}, te^{-at/2}$.
- $\gamma^2 = 4b - a^2 > 0$. $e^{-at/2}\cos(\gamma t/2), e^{-at/2}\sin(\gamma t/2)$.

Tip: use det (inverse use gauss jordan) to find if nontrivial sol exists.

2 (Basic) Fourier Series (Computation)

Fourier Series: $f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$.

Pcw cts and smooth fns. f is **uniformly cts** on each interval (x_{i-1}, x_i). Pcw smth on $[a,b]$ if f' and f are pcw cts on $[a,b]$.

L'Hopital rule: Find limit of f' approaches point. Fact: if $\lim_{x\to a^+} f'(x) = l$, then $\lim_{x\to a^+} \frac{f(x)-f(a^+)}{x-a} = l$. Similar fact holds for left limit. (In practice, don't need to specify left or right limit of a, just plug $f(a)$.)

Integration by Parts Shortcut. When P polynomial with degree $< m$ and f cts, $\int Pf dx = PF_1 - P'F_2 + P''F_3 - \cdots + (-1)^m P^{(m)} F_{m+1} + C$, where $f := F_0, P := P^{(0)}$, $P^{(j)} = (P^{(j-1)})'$ and F_{j+1} is antiderivative of F_j.

Note: integral's "sum" is 0 and "sum" of terms at LHS is 1.

Quick Fourier Series Computation.

- Sine on $[0,\pi]$, we have $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$, cosine on $[0,\pi]$ we have a_n similarly defined and $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx$, for Fourier series on $[-\pi,\pi]$, we "switch" the 1 and 2; b_n and a_n has constant $\frac{1}{\pi}$, a_0 has $\frac{2}{\pi}$.
- For fn with period L other than 2π, express as $f := a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi x}{L}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi x}{L}\right)$, change the denominator to L, integral borders to $L/2$ and $-L/2$, and the cosine/sines to be $\frac{2\pi n x}{L}$.

3 Inner Products (and its spaces), Best Approx, Gram Det, Parseval Identity.

Inner Products. We have symmetric, left/right bilinear, $\langle f, f\rangle \geq 0$ and equality iff $f = 0$ (pos-def).

Vector spaces. Examples are $C[-\pi,\pi], PC[-\pi,\pi], PC^1[-\pi,\pi]$, $L^2[-\pi,\pi]$, l^2, $\mathcal{S}_{2\pi}$ infinitely differentiable periodic fns period 2π,

$$S = \{\sum_{k=1}^{\infty} a_k \sin(kx) : \sum_{k=1}^{\infty} a_k^2 < \infty\}.$$

Orthogonal and orthonormal, and their friends. $\mathcal{F}$ orthog if dot prod 0 diff vec, orthon dot itself 1. If inn prod then sqrt of inn prod defines norm. C-S + parallelogram rule, and in general, Pythag too (sum squares of v_j eq square of sum of v_j if $\langle v_j, v_l\rangle = 0$ if $j \neq l$). Conseq: orthog set is *lin indep*, Hilbert space has *maximal/complete orthon set.*

Bessel and Parseval. $\mathcal{F}$ family of orthonormal vectors, so

$$\sum_{v_\alpha \in \mathcal{F}} |\langle v, v_\alpha\rangle|^2 \leq \|v\|^2.$$

If this family is complete, ineq is equality. Proof is considering $v - \sum_{k=1}^{n}\langle v, v_k\rangle v_k$ orthogonal to v_j, and then take inner product with itself.

Best approximation by family of orthonormal vectors. $\{v_i\}$ orthonormal, then for $u \in V$, the $v \in M$ (induced spanned subspace) which $\|u - v\|$ minimum (i.e. infimum) is

$$P_M(u) := \sum_{j=1}^{n}\langle u, v_j\rangle v_j - u.$$

Independent of basis. We also have $\|P_M(u) - P_M(v)\| \leq \|u - v\|$,implying P_M cts.

Least Square Approximation in $\mathbb{R}^n$. Want find $\alpha^* \in \mathbb{R}^m$ s.t.

$$\left\|\sum_{k=1}^{m}\alpha_k^* a_k - b\right\| = \min\left\|\sum_{k=1}^{m}\alpha_k a_k - b\right\|$$

with $\{a_i\}_{1\leq i\leq m}$ basis of $M \subseteq \mathbb{R}^n$. In other words, $\|A\alpha^* - b\| = \min_{\alpha\in\mathbb{R}^m}\|A\alpha - b\|$.

The vector $\alpha^* = (\alpha_1^*, \ldots, \alpha_m^*)$ satisfies

$$A^T A\alpha^* = A^T b,$$

where $A^T A$ has entries $x_{ij} = \langle a_i, a_j\rangle$, and $A^T b$ has entries $x_{1j} = \langle b, a_j\rangle$.

Discrete Least Squares Problem. Let $S := \{(x_k, y_k) \in \mathbb{R}^2\}$, x_ks distinct, W fin-dim space of cts fns,

$$\min_{f\in W}\left(\sum_{k=1}^{n}|y_k - f(x_k)|^2\right)^{1/2} = \min_{\alpha^*\in\mathbb{R}^m}\left(\sum_{k=1}^{n}|y_k - \sum_{i=1}^{m}\alpha_i f_i(x_k)|^2\right)^{1/2}.$$

M is subspace of $\mathbb{R}^n$ spanned by the vectors $e_i := (f_i(x_1), \ldots, f_i(x_n)), 1 \leq i \leq m$. Set the a_is to be e_is, and the equation has unique solution iff linearly independent in $\mathbb{R}^n$.

Gram Determinant. $\det(A^T A)$ of $\{a_1, \cdots, a_m\}$, denoted $G(a_1, \cdots, a_m)$. We may express the minimum dist of b to $A\mathbb{R}^m$ as

$$\|b - P_M(b)\|^2 = G(b, a_1, \cdots, a_m)/G(a_1, \cdots, a_m).$$

Remember to take square root.

In Fourier Series, pcw cts fn on $[0,L]$, we have $\int_0^L |f(x) - (a_0' + \sum_{k=1}^{n} a_k' \cos\frac{2k\pi x}{L} + \sum_{k=1}^{n} b_k' \sin\frac{2k\pi x}{L})|^2 dx \geq \int_0^L |f(x) - (\text{Fourier Series of } f)|^2 dx$.

Parseval's identity. Let $\{\varphi_k(x) : k \in \mathbb{N}\}$ orthonormal basis of $L^2[a,b]$. For any $f := \sum_{k=1}^{\infty} c_k\varphi_k$ (henceforth $c_k = \int_a^b f(x)\varphi_k(x)dx$) in $L^2[a,b]$, we have

$$\int_a^b |f(x)|^2 dx = \sum_{k=1}^{\infty} c_k^2.$$

Corollary:

$$\int_0^L |f(x)|^2 dx = L/2\left(2a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right).$$

4 Pointwise Convergence

Theorems. f pcw smth on $[0,L]$, then its fourier series converges to $\frac{f(x^+)+f(x^-)}{2}$ for all $x \in (0,L)$. If $x = 0$ or L, converges to $(f(0^+) + f(L^-))/2$.

Bessel. Basically finite Parseval (in this chpt).

If $|f|$'s integral is finite instead of f pcw cts, limit of a_k, b_k is 0. Apply Bessel to family of orthonormal vectors, to get this, for example:

$$\lim_{n\to\infty} f(x)\sin\frac{(2n+1)\pi x}{2T}dx = 0.$$

- If f_n cts on an interval I for each $n \in \mathbb{N}$ and $\sum f_n(x)$ converges uniformly to f (take sup norm), f also cts.
- f_n differentiable on interval J, $\sum_{n=1}^{\infty}$ converges uniformly. If exists $x_0 \in J$ s.t. series converges then series converges uniformly to a *differentiable function* f and $f'(x) = \sum_{i=1}^{\infty} f_n'(x)$ on J.
- (**Cauchy Criterion**) $\sum f_n(x)$ converges uniformly on I iff exists $K(\epsilon)$ for any ϵ s.t. $|f_m(x) + \cdots + f_n(x)| < \epsilon$ for all $x \in I$, $n > m \geq K$.
- (**Weierstrass M-test**) $f_n(x) \leq M_n$ for all $x \in I$, and $\sum M_n < \infty$, then series $\sum f_n(x)$ *converges uniformly* on I.
- f cts, period 2π, derivative f' cts on $[-\pi,\pi]$. Fourier series of f is then differentiable at **each point** $x_0 \in (-\pi,\pi)$ at which f'' exists.
- **Abel's Lemma.** $(a_n), (b_n)$ sequences, $S_n = \sum_{k=1}^{n} b_k$, with $S_0 = 0$. Then

$$\sum_{n+1}^{m} a_k b_k = a_m S_m - a_{n+1}S_n + \sum_{k=n+1}^{m-1}(a_k - a_{k+1})S_k.$$

- **Dirichlet's Test.** $(a_n) \to 0$ decreasing, $|\sum_{k=1}^{N} b_k| \leq M$ for all $N \in \mathbb{N}$. Then series $\sum a_k b_k$ converges.
- **Abel's Test.** (a_n) convergent monotone, $\sum b_k$ converges. Then series $\sum a_k b_k$ converges.

Smoothness of function, rate of convergence. If f period 2π, f' pcw cts on $[-\pi,\pi]$, then $ka_k, kb_k \to 0$ as $k \to \infty$.

Dirichlet Kernel (missed this). $D_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos(kx)$. Equal to $\frac{\sin[(2n+1)x/2]}{2\sin(x/2)}$ when $2\pi \nmid x$. Also, $\int_0^{\pi} D_n(x)dx = \pi/2$, not by integrating the fraction form but by integrating the original (finite) sum.

5 Complex calculus and "Advanced" ODE

Complex F.series. $f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}, c_k \in \mathbb{C}$. If f only real valued, $\hat{f}(k) = \overline{\hat{f}(-k)}$.

Remark 5.1. f pcw cts on $[0,2\pi]$, $\sum_{k=0}^{\infty}\hat{f}(k)z^k$ converges on open unit disk. If we define $\hat{f}(z)$ as above, $\hat{f}(e^{ix}) = f(x)$ if f pcw smth, cts of period 2π.

Convol. f, g period 2π pcw cts. Then, convol is $f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy$. Then, $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$. Convols are assoc, linear and closed.

Convol with Dirichlet Ker for Fejer Ker. We have

$$S_n(f)(x) = \sum_{|k|\leq n} c_k e^{ikx} = f * D_n(x) = \frac{1}{2\pi} f(x-t)[\sum_{|k|\leq n} e^{ikt}]dt.$$

First eq by definition of S_n, second by first convol property (and D_n having $\hat{D}(k) = 1$ for $|k| \leq n$, 0 otherwise), third just expanding definition of convol.

We define $\sigma_n(f)(x) = C(1)(S_n(f))_{n\geq 0}$. Denoting $\sigma_n(t) = 2\left(\frac{\sum_{k=0}^{n} D_k(t)}{n+1}\right)$; $\sigma_n(t) = \frac{1}{n+1}\left[\frac{\sin((n+1)t/2)}{\sin(t/2)}\right]^2$, even if $t = 0$.

6 List of Facts about Haar and Chebyshev

Existence (6.1-1). Y findim subsp of $X = (X, \|\cdot\|)$, for each $x \in X$, best approx exists. Proof by ball and compactness.

Convexity and strict conv. In normed space, set of best approx of x into subsp Y is convex. Use ineq to prove this.

Strict convex norm is $\forall x, y$ with norm 1, $\|x+y\| < 2$. Hilbert spaces are strictly convex.

Haar condition. Extremal point of $x \in C[a,b]$ is a $t_0 \in [a,b]$ such that $|x(t_0)| = \|x\|$.

Findim $Y \subseteq C[a,b]$ satisfy *Haar condition* if $\forall y \in Y, y \neq 0$, has at most $\dim(Y) - 1$ zeros in $[a,b]$.

This is equivalent to condition where for every basis $\{y_i\}$ and $n-$tuple of distinct $\{t_i\} \in [a,b]$, $\det\{y_i(t_j)\} \neq 0$.

Lemma 6.3-3 (Extremal points). Subspace of $C[a,b]$ satisfy Haar. If for given x, y s.t. $x - y$ has $\leq n$ extremal points, y not best approx of x.

Proof use $x - y - (\epsilon \cdot y_0)$ where we pick

$$\sum_{k=1}^{n} \beta_k y_k(t_j) = v(t_j) := (x-y)(t_j), y_0 = \sum_{k=1}^{n} \beta_k y_k.$$

with $\{t_i\}$ superset of $m \leq n$ extremal points.

Lemma 6.4-2 (Best approximation). If given $x \in C[a,b]$, $y \in Y$ s.t. $x - y$ has alternating set of $\dim(Y) + 1$ points, y is best *uniform* approx to x *out of* Y.

Thm 6.4-3 (Chebyshev poly). $T_n(x) + t^n$ is smallest maximum deviation from 0 to the interval $[-1,1]$ of t^n. The polynomials is defined by $T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t)$.

We have $T_n(t) = \cos n\theta$, $\theta = \arccos t$, arccos sends $[-1,1] \to [\pi, -\pi]$, so monotone decreasing fn.

7 Linear operators: Solving DE and ODE

Proof of convergence using Parseval's. Let $u(x,t) = \sum_{k=1}^{\infty} e^{-(k\alpha\pi)^2 t} \sin k\pi x$. This converges $\forall t > 0$, so well-defined $\forall t > 0$.

- Step (1): $\forall t_0 > 0$, series (each differentiated w.r.t t, add $-(k\alpha\pi)^2$ in front of every term) converges uniformly on $[t_0/2, 2t_0]$, and each term on $u(x,t)$ has cts derivative on that interval. So, $u_t(x, t_0)$ behaves as expected.
- Step (2): apply theorem twice for u_{xx}.
- Step (3): Parseval's says that $\lim_{N\to\infty} \int_0^1 |f(x) - \sum_{k=1}^{N} c_k \sin k\pi x|^2$ equals $\lim_{N\to\infty} \sum_{k=N+1}^{\infty} |c_k|^2 \to 0$. Thus, the series converges to f in $L^2[0,1]$ (L^2 means), so solution is "valid". Since $u(\cdot,t) \to u(\cdot,0)$ as $t \to 0$.
- Note: if f' pcw cts and cts s.t. $f(0) = f(1) = 0$, $u(\cdot,t) \to f$ as $t \to 0$ uniformly on $[0,1]$.

We skip ODEs and directly to Sturm-Liouville, its generalisation. Self adj diff eq: $(p(x)y')' + q(x)y + \lambda r(x)y = P_2[y] + \lambda r(x)y = 0$, $a_0 y(a) + a_1 y'(a) = b_0 y(b) + b_1 y'(b) = 0$ with not all a_i, b_i equal 0. If $p, r > 0$ on $[a,b]$, p, q, r cts and p ctsly differentiable, regular.

Example: for $y'' + \lambda y = 0, y(0) = y(\pi) = 0$, $p, q, r = 1, 0, 1$. Example 2: same ODE, with $y(0) = y'(1) = 0$, can be done by chaning of variable $t = x - 1$, with eigenfunctions $\phi_\lambda(x) = \sin \frac{2k-1}{2}\pi t$.

Legendre Series. $\{P_n(x)\}$ family orthogonal fns on $[-1,1]$. If we write $f(x) = \sum_{k=0}^{\infty} c_n P_n(x)$, then $c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$. Note that $P_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, first 3 terms are $\{1, x, (3x^2-1)/2, (5x^3-3x)/2\}$.

8 Green Fn and Fourier Transform

Delta fn. Fourier series of δ_0 is $\sum_{k\in\mathbb{Z}} e^{ikx}$. We can interpret the two integrals $\int_0^{2\pi}(\sum_{k\in\mathbb{Z}} e^{ikx})\phi dx$ and $\sum_{k\in\mathbb{Z}} \int_0^{2\pi} e^{ikx}\phi(x)dx$.

We can see that $y = \sum_{k\in\mathbb{Z}} \frac{e^{ikx}}{\lambda_k}$ as a solution of $y'' + 2y = \delta_0$ if y periodic.

In particular, could look at $f * \delta_0(x) = f(x)/2\pi$, and so such $f * y$ is a solution (y can be understood as a "y_0"; an "initial solution").

Fourier Transform. $\mathcal{M}(\mathbb{R})$ is f pcw cts on any bounded interval in $\mathbb{R}$, there exists constant M s.t. $|f(x)| \leq M/(1+x^2)$. Call this locally piecewise cts fns of moderate decrease.

Call $\hat{f}(\xi) = \lim_{N\to\infty} \int_{-N}^{N} f(x)e^{-2\pi i\xi x}dx$.

Examples, $\mathcal{F}(e^{-a|x|}) = 2a/(a^2 + 4\pi^2\xi^2)$, $\mathcal{F}(\chi_{[-1,1]}) = \frac{\sin 2\pi\xi}{\pi\xi}$, take value at 0 to be 2 for cty.

Restrict to Schwartz space, inf diff fns $\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty$. Typical examples: $C_0^\infty(\mathbb{R})$ inf diff fn, exists $R > 0$, $f(x) = 0$ when $|x| > R$, Gaussian (e^{-ax^2}), $a > 0$.

Properties. Let $f \in \mathcal{S}$, five properties are true: $\widehat{\tau_h f}(\xi) = \hat{f}(\xi)e^{2\pi ih\xi}$ with $\tau_h f(x) = f(x+h)$; $\mathcal{F}(f(x)e^{-2\pi ixh})(\xi) = \hat{f}(\xi+h)$ if $g(x) = f(x)e^{-2\pi ixh}$; $\widehat{M_\delta f}(\xi)(\xi) = \hat{f}(\xi/\delta)/\delta$ where $M_\delta f(x) = f(\delta x)$; $\mathcal{F} f'(\xi) = 2\pi i\xi\hat{f}(\xi)$; $\mathcal{F}(-2\pi ixf(x))(\xi) = \frac{d}{d\xi}\hat{f}(\xi)$.

Note that $\mathcal{F}(e^{-ax^2})(\xi) = \sqrt{\frac{\pi}{a}}e^{-\pi^2\xi^2/a}$, provided $a > 0$.

Plancherel theorem. $\int_{-\infty}^{\infty} |\hat{f}|^2 = \int_{-\infty}^{\infty} |f|^2$, for all $f \in L^2$.

9 Tutorials and Examples

T5-T6.

- T5Q1: fact, if f pcw cts on $[0,\pi]$ s.t $\sum_{k=1}^{\infty} |c_k| < \infty$, where the c_ks are Fourier Sine coefficients. If f cts at $t \in (0,\pi)$, $f(t) = \sum_{k=1}^{\infty} c_k \sin kt$. Proof use Weierstrass M-test, so g cts by unif convergence. Then, Fourier Sine coefficients are $c_k \frac{\pi}{2}$, so the Fourier Sine coef of 2 fns are equal. We then get $\int f(x) - g(x)\sin nx dx = 0$, and parseval says $\{b_k(f(x) - g(x))\} \to 0$.
- Q4: use Parseval, remember that constant is twice for a_0.
- Q6: Bessel. If seq not converge cannot even be Fourier series. Then, by $\langle f, f\rangle \geq \sum c_k^2$ when $\{\phi_k(x)\}$ orthonormal, we can (un)bound the sum and deduce cannot be Fourier series.
- T6Q4: Remember that **Parseval for complex numbers** follow norm $\langle f, f\rangle = \int f \cdot \overline{f} = \int \|f\|^2$, the [modulus square of the function] integrated over some interval.
- Prelude of Q8: $\cos^3(t) = \left(\frac{e^{it}+e^{-it}}{2}\right)^3 = \frac{2\cos(3t)+6\cos(t)}{8}$, $(i\sin(t))^3 = \left(\frac{e^{it}-e^{-it}}{2}\right)^3$. So, $\sin^3(t) = (-2\sin(3t) + 6\sin(t))/8$.
- **Q8 (convolution) hell**: We have $f * P_r(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx}r^{|k|}$. For $f := \cos^3(x)$, we have

$$\hat{f}(k) = \frac{3}{8}\delta(|k|-1) + \frac{1}{8}\delta(|k|-3),$$

hence $f * P_r(t) = \frac{3}{4}\cos(t) + \frac{1}{4}r^3\cos(3t) = \frac{3}{4}x + \frac{x^3 - 3y^2x}{4}$ by $x = r\cos(t)$, $y = r\sin(t)$.

T7-T8.

- T7Q1: Consider T-periodic function f. $\hat{(f)}(k) = \int_0^T f(x)e^{\frac{-2\pi ikx}{T}} = \frac{T}{2\pi ik}\hat{f}'(k)$. Parseval's says that

$$\int_0^T |f(t)|^2 dt = T\sum_{k=1}^{\infty} = \frac{T^2}{4\pi^2}T\sum_{k=1}^{\infty}\frac{|\hat{f}'(k)|^2}{k^2} \leq \frac{T^2}{4\pi^2}T\sum_{k=1}^{\infty}|\hat{f}'(k)|^2$$

$= \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2$.

- Q2: to approximate, can use Taylor series (for simple fns, like $\sin(x)$),

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In this case, the sequence of diff of $\sin(x)$ at 0 is $0, 1, 0, -1, \ldots$.

- Still Q2: if $f = (1/\sin(x) - 2/x)$ is pcw cts on $[0,\pi]$ then $\int_0^\pi f(x)\sin(((2n+1)x)/2) \to 0$ as $n \to \infty$.
- Q6: **Fact.** $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{H}$ imply $\|x_n - x\| \to 0$.
- T8Q6,Q7: For best approx, remember that extremal points consist of (possible) endpoints and turning points; i.e. points whose derivative are 0.

Sum-to-product formulas.

- $\sin(a)\cos(b) = 1/2(\sin(a+b) + \sin(a-b))$, $\cos(a)\sin(b) = 1/2(\sin(a+b) - \sin(a-b))$.
- This is equivalent since $\sin(a-b) = -\sin(b-a)$ (sine is odd).
- $\cos(a)\cos(b) = 1/2(\cos(a+b) + \cos(a-b))$, $\sin(a)\sin(b) = -1/2(\cos(a+b) - \cos(a-b))$.
- The second formula can also be "permuted" as $\sin(b)\sin(a)$ gives second term to be $\cos(b-a)$ (cosine is even).