1 Introduction ("Review" of ODE)

Separation of Variables: Set $u(x,t) = T(t)X(x)$. Note that we will have
$$u_t = T'(t)X(x) \text{ and } u_{xx} = T(t)X''(x)$$
and then we can group the like variables in one side. Solution is linear combination of the $T(t)X(x)$s.

Solving ODEs: $y'' + ay' + by = -$, observe that $y = e^{\lambda x}$ implies $\lambda^2 + a\lambda + b = 0$.

- $\gamma^2 = a^2 - 4b > 0$, we know $e^{(-a+\gamma)t/2}$ and $e^{(-a-\gamma)t/2}$. Usually this will result in no nontrivial sol, since exponential function is weird.
- $\gamma^2 = 0$. Then, $e^{-at/2}, te^{-at/2}$ are solutions.
- $\gamma^2 = 4b - a^2 > 0$. Then, $e^{-at/2}\cos(\gamma t/2), e^{-at/2}\sin(\gamma t/2)$ are solutions.

Tip: use linear algebra (determinant) to determine if linear equation has nontrivial solution.

Special case: $y'' + y = \sin^3 x, 0 < x < \pi, y(0) = y(\pi) = 0$. Set $y'' + y = \lambda y$, then we get $\lambda_k - 1 = -k^2$, so $y_k = sin(kx)$.

Solution is $y = \sum c_k y_k$, and $y'' + y$ is
$$\sum c_k(y_k'' + y_k) = \sum c_k \lambda_k y_k = \sin^3 x = 3\cos^2 x \sin x - \sin^3 x$$

2 (Basic) Fourier Series (Computation)

Fourier Series: $f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$.

Pcw cts and smooth fns. A fn f is pcw cts on $[a,b]$ if it has a partition $a = x_0 < x_1 < \cdots < x_n = b$ s.t. f is **uniformly cts** on each interval (x_{i-1}, x_i). Pcw smth on $[a,b]$ if both f' and f are pcw cts on $[a,b]$.

L'Hopital rule: Useful to find limit of f' as it approaches a certain point.
$$\text{Fact: if } \lim_{x\to a^+} f'(x) = l, \text{then } \lim_{x\to a^+} \frac{f(x) - f(a^+)}{x - a} = l.$$
Similar fact holds for left limit. (In practice, don't need to specify limit of f as $x \to a$ from the right or left, just plug $f(a)$.)

Integration by Parts Shortcut. When P polynomial with degree $< m$ and f cts,
$$\int P f dx = PF_1 - P'F_2 + P''F_3 - \cdots + (-1)^m P^{(m)} F_{m+1} + C$$
where $f := F_0, P := P^{(0)}$, $P^{(j)} = (P^{(j-1)})'$ and F_{j+1} is the antiderivative of F_j.

Note: to remember this, integral's "sum" is 0 and "sum" of terms at LHS is 1.

Quick Fourier Series Computation.

- $\int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = 2\pi\delta(m-n)$ where $\delta(n) = 0$ when $n \neq 0$ and $\delta(0) = 1$.
- $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \pi\delta(m-n)$ as it is equal to the real part of $\int_{-\pi}^{\pi} e^{imx}\cos(nx)dx$. **This is equal to** 2π when $m = n = 0$.
- $\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx = 0$.
- $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \pi\delta(m-n)$
- For Fourier sine series on $[0,\pi]$, we have $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$.
- For Fourier series on $[-\pi,\pi]$, we "switch" the 1 and 2; b_n and a_n has constant $\frac{1}{\pi}$, a_0 has $\frac{2}{\pi}$.
- Similarly, for Fourier cosine on $[0,\pi]$ we have a_n similarly defined and $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx$.
- For fn with period L other than 2π, express as $f := a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi x}{L}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi x}{L}\right)$.
- Correspondingly, change the denominator to L, the integral borders to $L/2$ and $-L/2$, and the cosine/sines to be $\frac{2\pi n x}{L}$.

3 Inner Products (and its spaces), Best Approx, Gram Det, Parseval Identity.

Inner Products. We have $\langle f,g\rangle = \langle g,f\rangle$, $\langle f, g+ch\rangle = \langle f,g\rangle + c\langle f,h\rangle$ (left bilinear) and also right bilinear, $\langle f,f\rangle \geq 0$ and equality iff $f = 0$ (pos-def).

Vector spaces. Examples are $C[-\pi,\pi], PC[-\pi,\pi], PC^1[-\pi,\pi]$ (pcw smth fns), $L^2[-\pi,\pi]$ (square integrable fns), l^2 space of square summable sequences, $\mathbb{S}_{2\pi}$ infinitely differentiable periodic fns period 2π.

Not so standard example is
$$S = \{\sum_{k=1}^{\infty} a_k \sin(kx) : \sum_{k=1}^{\infty} a_k^2 < \infty\}.$$

Orthogonal and orthonormal, and their friends. $\mathcal{F}$ orthogonal if $\langle f,g\rangle = 0$ for all $f,g \in \mathcal{F}, f \neq g$. Orthonormal if $\langle f,f\rangle = 1$ and orthogonal.

If inner product then square root of inner product defines norm. C-S and parallelogram rule ($(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$ holds. And, in general, Pythagorean Law holds (sum of squares of v_j equal square of sum of v_j if $\langle v_j, v_l\rangle = 0$ whenever $j \neq l$.

Complete (every cauchy seq converges) inner product space is **Hilbert Space**.

Consequence of this: an orthogonal family of vectors is *linearly independent* (easy) and every Hilbert space has a *maximal or complete orthonormal set* (proof need Gram Schmidt and Zorn's Lemma).

Bessel and Parseval. $\mathcal{F}$ family of orthonormal vectors, so
$$\sum_{v_\alpha \in \mathcal{F}} |\langle v, v_\alpha\rangle|^2 \leq \|v\|^2.$$
If this family is complete, ineq is equality. Proof is considering $v - \sum_{k=1}^{n}\langle v, v_k\rangle v_k$ orthogonal to v_j, and then take inner product with itself.

Best approximation by family of orthonormal vectors. $\{v_i\}_{1\leq i\leq n}$ be orthonormal, then for all $u \in V$, the $v \in M :=$ **Span**$\{v_1, \ldots, v_m\}$ that is the infimum (therefore minimizing) $\|u - v\|$ is
$$P_M(u) := \sum_{j=1}^{n}\langle u, v_j\rangle v_j - u.$$
This is independent of choice of orthonormal basis M, and we have $u - P_M(u)$ perp to M and
$$\|P_M(u) - P_M(v)\| \leq \|u - v\|.$$
The last ineq implies P_M cts.

Least Square Approximation in $\mathbb{R}^n$. We want to find $\alpha^* = (\alpha_1^*, \ldots, \alpha_m^*) \in \mathbb{R}^m$ s.t.
$$\left\|\sum_{k=1}^{m}\alpha_k^* a_k - b\right\| = \min\left\|\sum_{k=1}^{m}\alpha_k a_k - b\right\|$$
where $\{a_i\}_{1\leq i\leq m}$ is a basis of M, a subspace of $\mathbb{R}^n$.

This is can be rewritten to $\|A\alpha^* - b\| = \min_{\alpha\in\mathbb{R}^m}\|A\alpha - b\|$.

This best approx in $M := A\mathbb{R}^m$ is orthogonal projection of b to such subspace $P_M(b) = \sum_{i=1}^{m}\alpha_i^* a_i$, and the vector $\alpha^* = (\alpha_1^*, \ldots, \alpha_m^*)$ satisfies
$$A^T A\alpha^* = A^T b,$$
where $A^T A$ has dimension $m \times m$, with entries $x_i j = \langle a_i, a_j\rangle$, and $A^T b$ is $m \times 1$ matrix, entries $x_1 j = \langle b, a_j\rangle$.

Discrete Least Squares Problem. Let $S := \{(x_k, y_k) \in \mathbb{R}^2\}$, x_ks distinct, W fin-dim space of cts fns,
$$\min_{f\in W}\left(\sum_{k=1}^{n}|y_k - f(x_k)|^2\right)^{1/2} = \min_{(\alpha_1,\cdots,\alpha_m)\in\mathbb{R}^m}\left(\sum_{k=1}^{n}|y_k - \sum_{i=1}^{m}\alpha_i f_i(x_k)|^2\right.$$
It suffices to find w^* s.t. $\|b - w^*\| = \min_{w\in M}\|b - w\|$, where $b = (y_1, \cdots, y_n) \in \mathbb{R}^n$, M is subspace of $\mathbb{R}^n$ spanned by the vectors
$$e_i := (f_i(x_1), \ldots, f_i(x_n)), 1 \leq i \leq m.$$
Set the a_is to be e_is, and the equation has unique solution iff linearly independent in $\mathbb{R}^n$.

Gram Determinant. $A^T A$ plays an important role. Its determinant is the Gram determinant of $\{a_1, \cdots, a_m\}$ and we denote it as $G(a_1, \cdots, a_m)$. We may express the minimum dist of b to $A\mathbb{R}^m$ as
$$\|b - P_M(b)\|^2 = G(b, a_1, \cdots, a_m)/G(a_1, \cdots, a_m).$$
Note that denom is 1 when $\{a_i\}$ is orthonormal.

In Fourier Series, pcw cts fn on $[0,L]$, we have
$$\int_0^L |f(x) - (a_0' + \sum_{k=1}^{n} a_k' \cos\frac{2k\pi x}{L} + \sum_{k=1}^{n} b_k' \sin\frac{2k\pi x}{L})|^2 dx$$

$$\geq \int_0^L |f(x) - (\text{Fourier Series of } f)|^2 dx.$$

Parseval's identity. Let $\{\varphi_k(x) : k \in \mathbb{N}\}$ orthonormal basis of $L^2[a,b]$. For any $f := \sum_{k=1}^{\infty} c_k\varphi_k$ (henceforth $c_k = \int_a^b f(x)\varphi_k(x)dx$) in $L^2[a,b]$, we have

$$\int_a^b |f(x)|^2 dx = \sum_{k=1}^{\infty} c_k^2.$$

Corollary:

$$\int_0^L |f(x)|^2 dx = L/2\left(2a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right).$$

For Fourier sine and cosine on $[0,\pi]$,

$$\int_0^{\pi} |f(x)|^2 dx = \frac{\pi}{2}\sum_{k=1}^{\infty} b_k^2, \text{or } \frac{\pi}{2}\left(2a_0^2 + \sum_{k=1}^{\infty} a_k^2\right).$$

Lagrange's Interpolation. Just note it exists.

4 Pointwise Convergence

Theorems. f pcw smth on $[0,L]$, then its fourier series converges to $\frac{f(x^+)+f(x^-)}{2}$ for all $x \in (0,L)$. If $x = 0$ or L, converges to $(f(0^+)+f(L^-))/2$. In particular, if f cts at $x_0 \in (0,L)$, f converges to $f(x_0)$ since $f(x_0) = f(x_0^+) = f(x_0^-)$.

Bessel. Let $\{\psi_k : k \in \mathbb{N}\}$ orthonormal on $[a,b]$. Analogue of formula in Chapter 3 holds.

In particular, if f pcw cts, limit of its a_k and b_k is 0 as $k \to \infty$ (enough to assume $|f|$'s integral is finite instead of f pcw cts). In particular (2), try to apply on family of orthonormal to prove lim integral(f times n^{th} fn) is 0.

MA3110 stuff.

- If f_n cts on an interval I for each $n \in \mathbb{N}$ and $\sum f_n(x)$ converges uniformly to f (take sup norm), f also cts.
- f_n differentiable on interval J, $\sum_{n=1}^{\infty}$ converges uniformly. If exists $x_0 \in J$ s.t. series convergesm then series converges uniformly to a *differentiable function* f and $f'(x) = \sum_{i=1}^{\infty} f_n'(x)$ on J.
- (**Cauchy Criterion**) $\sum f_n(x)$ converges uniformly on I iff exists $K(\epsilon)$ for any ϵ s.t. $|f_m(x) + \cdots + f_n(x)| < \epsilon$ for all $x \in I$, $n > m \geq K$.
- (**Weierstrass M-test**) $f_n(x) \leq M_n$ for all $x \in I$, and $\sum M_n < \infty$, then series $\sum f_n(x)$ *converges uniformly* on I.
- Consequently, the Fourier Series of any function $f \in \mathcal{S}_{2\pi}$ converges uniformly and absolutely to f.
- f cts, period 2π, derivative f' cts on $[-\pi,\pi]$. Fourier series of f is then differentiable at **each point** $x_0 \in (-\pi,\pi)$ at which f'' exists.

- **Abel's Lemma.** $(a_n),(b_n)$ sequences, $S_n = \sum_{k=1}^n b_k$, with $S_0 = 0$. Then

$$\sum_{n+1}^{m} a_k b_k = a_m S_m - a_{n+1}S_n + \sum_{k=n+1}^{m-1}(a_k - a_{k+1})S_k.$$

- **Dirichlet's Test.** $(a_n) \to 0$ decreasing, $|\sum_{k=1}^N b_k| \leq M$ for all $N \in \mathbb{N}$. Then series $\sum a_k b_k$ converges.
- **Abel's Test.** (a_n) convergent monotone, $\sum b_k$ converges. Then series $\sum a_k b_k$ converges.
- With the help of Dirichlet's Test and Abel's Lemma, we get $a_k \to 0$, Fourier sine $\sum_{k=1}^{infty} a_k \sin kx$ converges uniformly on $[\delta, \pi - \delta]$ for all $0 < \delta < \pi/2$.
- Proof uses summation of sine $1x$ to $nx = \cos(n + (1/2)x) - \cos(x/2)$ divided by $2\sin(x/2)$. Then, sum is equal to

$$|a_m S_m(x) - a_{n+1}S_n(x) + \sum_{k=n+1}^{m-1}(a_k - a_{k+1})S_k(x)|.$$

  Also, we have $S_i(x) \leq \pi/\delta$, so whole sum is less than $\pi/\delta \times 2a_{n+1}$.

Smoothness of function, rate of convergence. If f cts period 2π, s.t. f pcw cts on $[\pi,\pi]$, then $ka_k, kb_k \to 0$ as $k \to \infty$.

Applying repeatedly, (idk really if this is true or not, haven't verified) if $f \in \mathcal{S}_{2\pi}$, then $k^n a_k, k^n b_k \to 0$ as $k \to \infty$ for all $n \in \mathbb{N}$.

5 Tutorials and Examples

T1.

- $y'' + \lambda y = 0, y(0) = y(2\pi), y'(0) = y'(2\pi)$. Nothing happens when this has real root ($a^2 - 4b = -4\lambda > 0$, $y = c$ happens when $\lambda = 0$, $\sqrt{\lambda} \in \mathbb{N}$ (or $\sin(2\pi\sqrt{\lambda}) = 0$).
- $u_t = 1.71u_{xx}$ implies $X''(x) - \lambda X(x) = 0$ and $T'(t) - 1.71\lambda T(t) = 0$. We have $u(0,t) = u(2,t) = 0$, implying $X(0) = X(2) = 0$.
- **Since we don't know anything about t yet**, we take care of first $X(x)$ equation $\to \lambda = -\beta^2$, and $X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$. This implies $c_1 = 0, \beta = (k\pi)/2$.
- Then we find $T(t) = e^{\text{sth}\cdot t}$, or solve the equation $\theta - 1.71\lambda = 0$ (value of λ alr known), for $T(t) = e^{-\frac{k^2\pi^2}{4}\cdot 1.71}$ for a fixed β.
- Now we use final equation $u(x,0) = X(x)T(0) = \sin\left(\frac{\pi x}{2}\right) + \sin\left(\frac{3\pi x}{2}\right)$, and *we can just throw away* $T(0)$ since we alr know its fixed value.
- $x\sin\frac{1}{x^2}$ is **(pcw)** cts, not smooth, e^{1/x^2} has limit ∞ near zero, $\sqrt{x}\cos\frac{1}{x^2}$ cts not smth, $x^2\sin^2(1/x)$ cts not smth as $f'(x) = 2x\sin^2(1/x) - 2\sin(1/x)\cos(1/x)$, $\frac{1-e^x}{x}$ cts and smth by L'Hopital twice, $x(\cos(\ln(x)) + \sin(\ln(x)))$ has derivative $2x\cos(\ln(x))$ so $f(0^+)$ not exist (cts not smth),

T4.

- Minimum value of $\int_0^{\pi} |x - a\cos(x) - b\cos(3x)|^2 dx$ is

$$\int_0^{\pi} |x|^2 dx - \int_0^{\pi} |a\cos(x)|^2 dx - \int_0^{\pi} |b\cos(3x)|^2 dx$$

  for the appropriate a and b.
- This is because LHS vector is perpendicular to RHS' second and third vectors. Now, for easier computation, express $a\cos(x)$ as $a' * \sqrt{2/\pi}\cos(x)$ as the last vector has norm 1. Then, it is $\frac{x^3}{3}$ evaluated on $[0,\pi]$ minus $((a')^2 + (b')^2)$.
- Main idea is consider finite dimensional vector space (or countably infinite ones) and then use the fact that it is a **Hilbert space** and apply Parseval's identity.
- Note that $\left\{\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}\cos(kx) : k \in \mathbb{N}\right\}$ is an orthonormal basis of $L^2[0,\pi]$.

Sum-to-product formulas.

- $\sin(a)\cos(b) = 1/2(\sin(a+b) + \sin(a-b))$, $\cos(a)\sin(b) = 1/2(\sin(a+b) - \sin(a-b))$.
- This is equivalent since $\sin(a-b) = -\sin(b-a)$ (sine is odd).
- $\cos(a)\cos(b) = 1/2(\cos(a+b) + \cos(a-b))$, $\sin(a)\sin(b) = -1/2(\cos(a+b) - \cos(a-b))$.
- The second formula can also be "permuted" as $\sin(b)\sin(a)$ gives second term to be $\cos(b-a)$ (cosine is even).