0 Useful Distriubtion and Properties

Random Sum. When S is sum of N r.v. N is r.v., we have $E[S] = E[X_i]E[N]$ and $Var(S) = E_N(Var_{X_i}(S|N)) + Var_{X_i}(E_N(S|N))$ $= E(N)Var(X_i) + E(X_i)^2 Var(N)$. Simplified, $Var(S)$ becomes $[E(N)E(X_i^2)] + [E(X_i)^2(Var(N) - E(N))]$.

Taylor Series Expansion

- $\log(1+x) = \sum_{i\geq 1}(-1)^{i+1}\frac{x^i}{i}$
- $\exp(x) = \sum_{i\geq 0}\frac{x^i}{i!}$
- $\sum_{k\geq 0} kx^k = x(\sum_{k\geq 0} x^k)^2 = x\cdot\left(\frac{1}{1-x}\right)^2$

MGF. We have $E[X^k] = M_X^{(k)}(0) = \frac{d^k}{ds^k}M_X(s)|_{s=0}$, where M_X is the moment generating function $M_X(s) = E[e^{sX}]$.

Poisson Distribution. pmf $\frac{e^{-\lambda}\lambda^k}{k!}$, MGF $\exp(\lambda(e^t-1))$. $\mu = \lambda$, $\sigma^2 = \lambda$. $\text{Pois}(\lambda_1) + \text{Pois}(\lambda_2) \sim \text{Pois}(\lambda_1+\lambda_2)$. $\text{Bin}(n,p)$ can be approximated by $\text{Pois}(np)$ (besides normal/CLT). Let $N \sim \text{Pois}(\mu)$, and conditional on N, let $M \sim \text{Bin}(N,p)$. Then the unconditional distribution of M is $\text{Pois}(\mu p)$

Exponential Distribution. $X \sim$ Exponential param $\theta > 0$, pdf $f(x) = \frac{1}{\theta}\cdot e^{-x/\theta}, x > 0$, MGF $M(t) = \mathbb{E}[e^{tX}] = \frac{1}{1-\theta t}, t < \frac{1}{\theta}$, $\mu = \theta$, $\sigma^2 = \theta^2$.

Gamma Distribution. $X \sim \text{Gamma}(\alpha,\beta)$, params $\alpha,\beta > 0$, pdf $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-x\beta}, x > 0$, MGF $M(t) = (1-\frac{t}{\beta})^{-\alpha}, t < \beta$ where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$ is the "complex continuation" of the factorial function: $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ and $T(\alpha) = \alpha T(\alpha-1)$ for all α. $\mu = \frac{\alpha}{\beta}$, $\sigma^2 = \frac{\alpha}{\beta^2}$. $\text{Exp}(\lambda)$ is identical to $\text{Gamma}(1,\lambda)$. $\text{Gamma}(\alpha_1,\beta) + \text{Gamma}(\alpha_2,\beta) \sim \text{Gamma}(\alpha_1+\alpha_2,\beta)$

Normal Distribution. $X \sim N(\mu,\sigma^2)$, mean μ and variance σ^2 $(\sigma > 0)$. pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], x \in \mathbb{R}$, MGF $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, t \in \mathbb{R}$. $\text{Normal}(\mu_1,\sigma_1^2) + \text{Normal}(\mu_2,\sigma_2^2) \sim \text{Normal}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

Chi-squared Distribution. $X \sim \chi^2(r)$ for $r \in \mathbb{N}$ is given by $X \sim \text{Gamma}(r/2, 2)$. Plug in to Gamma distribution, r commonly referred as "degree of freedom". $\chi^2(1) \sim N(0,1)^2$, $\chi^2(x) + \chi^2(y) = \chi^2(x+y)$.

1 Poisson Process

Poisson Process of intensity/rate is λ is an integer-valued stochastic process $\{X(t); t \geq 0\}$ such that:

- $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent
- $\Pr(X(s+t) - X(s) = k) = \frac{(t\lambda)^k e^{-t\lambda}}{k!}$. It's a poisson dist. with parameter $t\lambda$
- $X(0) = 0$

Four postulates of Poisson Process

- $N((t_0,t_1]), N((t_1,t_2]), \ldots$ are independent.
- For any time t and $h > 0$, the dist. of $N((t,t+h])$ depends on h and not t.
- There is positive constant λ such that $\Pr(N((t,t+h] \geq 1) = \lambda h + o(h)$ as $h \to 0$.
- $\Pr(N((t,t+h]) \geq 2) = o(h)$ as $h \to 0$.

It can be proven using Law of Rare Events that the above postulates form a poisson process with rate λ. Recall that Law of Rare Events states:
Let $\epsilon_1, \epsilon_2, \ldots$ be independent Bernoulli r.v. with $\Pr(\epsilon_i = 1) = p_i$ and $S_n = \sum_{i=1}^n \epsilon_i$. With $\lambda = p_1 + \cdots + p_n$, we'll have $|P(S_n = k) - \frac{\lambda^k e^{-\lambda}}{k!}| \leq \sum_{i=1}^n p_i^2$

Nonhomogenous Poisson Process: Unlike the regular poisson process, rate $\lambda = \lambda(t)$ vary with time. Note that in this case, $X(s+t) - X(s)$ will follow a poisson distribution with parameter $\int_s^{s+t}\lambda(u)du$.

Waiting time W_n is the time occurence of the nth event. Note that we have $W_0 = 0$ and the differences $S_n = W_{n+1} - W_n$ are called sojourn times. The following properties are satisfied:

- W_n has a gamma dist. with parameter (n,λ) and pdf $\frac{\lambda^n t^{n-1}}{(n-1)!}e^{-\lambda t}$.
- Sojourn times $S_1, S_2, \ldots$ are independent r.v, each being an exponential dist. with parameter λ and pdf $\lambda e^{-\lambda s}$.

Uniform Distribution in Poisson Process: Conditioned on a fixed total number of events in an interval, the locations of those events are uniformly distributed in a certain way. Mathematically, conditioned on $X(t) = n$, the r.v. $W_1, W_2, \ldots, W_n$ have the joint pdf:

$$f_{W_1,\ldots,W_n|X(t)=n}(w_1,\ldots,w_n) = n!t^{-n}$$

Multi-dimensional Poisson Process: Let S be a subset of $\mathbb{R}, \mathbb{R}^2$, or $\mathbb{R}^3$; Let $\mathcal{A}$ be the family of subsets of S and for any set $A \in \mathcal{A}$ denote $|A|$ as the size (length, area, or volume) of A. Then, $\{N(A): A \in \mathcal{A}\}$ is a homogenous poisson point process of intensity λ if:

- $N(A)$ has a poisson dist. with parameter $\lambda|A|$ for every $A \in \mathcal{A}$.
- For every finite collection $\{A_1,\ldots,A_n\}$ of disjoint subsets of S, $N(A_1),\ldots,N(A_n)$ are independent.
- Consider region A of positive size, then $\Pr(N(B) = 1 \mid N(A) = 1) = \frac{|B|}{|A|}$ for any set $B \subseteq A$.

Four postulates to define a multi-dimensional poisson process mentioned above:

- $N(A) \in \mathbb{N}_0$. Furthermore, $\Pr(N(A) = 0) < 1$ if $0 < |A| < \infty$.
- Distribution of $N(A)$ depends on size $|A|$ only. Furthermore, $\Pr(N(A) \geq 1) = \lambda|A| + o(|A|)$ as $|A| \to 0$.
- If $A_1,\ldots,A_n$ are disjoint, then $N(A_1),\ldots,N(A_n)$ are independent r.v. Furthermore, $N(A_1\cup\cdots\cup A_n) = N(A_1)+\cdots+N(A_n)$.
- $\lim_{|A|\to 0}\frac{\Pr(N(A)\geq 1)}{\Pr(N(A)=1)} = 1$

Compound Poisson Process: A cumulative value process defined by $Z(t) = \sum_{k=1}^{X(t)} Y_k$ where $Y_1, Y_2, \ldots$ are independent r.v with common distribution function $G(y) = P(Y_1 \leq y)$, $\mu = E[Y]$, and $v^2 = Var[Y]$. Using law of total expectation and total variance, we can get:

$$E[Z(t)] = \lambda\mu t, \; Var[Z(t)] = \lambda(v^2+\mu^2)t$$

Marked Poisson Process: Sequence of pairs $(W_1,Y_1),(W_2,Y_2),\ldots$. Theorem 1.6.1 states that $(W_1,Y_1),(W_2,Y_2)$ form a two-dimensional non-homogenous poisson process in the (t,y) plane, where the mean number of points in region A is given by:

$$\mu(A) = \int\int_A \lambda g(y)dydt$$

2 Continuous Time Markov Chain

Pure Birth Process: Markov process satisfying the following postulates, for some birth rate $\{\lambda_k\}$.

- $\Pr(X(t+h) - X(t) = 1 \mid X(t) = k) = \lambda_k h + o_{1,k}(h)$ as $h \to 0$
- $\Pr(X(t+h) - X(t) = 0 \mid X(t) = k) = 1 - \lambda_k h + o_{2,k}(h)$
- $\Pr(X(t+h) - X(t) = 0 \mid X(t)) = k$
- $X(0) = 0$

Define $P_n(t) = \Pr(X(t) = n)$. By analyzing the probabilities at time t just prior to time $t+h$ $(h \to 0)$, we can get:

- $P_0'(t) = -\lambda_0 P_0(t) \text{ß} P_0(t) = \exp(-\lambda_0 t)$
- $P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1}P_{n-1}(t)$

To solve the above equations, define $Q_n(t) = e^{\lambda_n t}P_n(t)$. We can then get

$$Q_n'(t) = e^{\lambda_n t}\lambda_{n-1}P_{n-1}(t) \text{ß} P_n(t) = \lambda_{n-1}e^{-\lambda_n t}\int_0^t e^{\lambda_n x}P_{n-1}(x)dx$$

It can be proved using induction that when $\{\lambda_k\}$ are distinct, we have $P_n(t) = \lambda_0\ldots\lambda_{n-1}(B_{0,n}\exp(-\lambda_0 t) + \cdots + B_{n,n}\exp(-\lambda_n t))$ where:

$$B_{k,n} = \frac{1}{(\lambda_0-\lambda_k)\ldots(\lambda_{k-1}-\lambda_k)(\lambda_{k+1}-\lambda_k)\ldots(\lambda_n-\lambda_k)}$$

Sojourn Time of Pure Birth Process: Define sojourn time S_k as the time between the kth and $(k+1)$th birth. Using the four postulates above, it can be proven that S_k is an independent exponential dist. with parameter λ_k.

Yule Process A birth process in which the birth rates are proportional to the current population size $(\lambda_k = k\beta)$. Assuming independence, we have:
$\Pr(X(t+h) - X(t) = 1 \mid X(t) = n) = n(\beta h + o(h))(1 - \beta h + o(h))^{n-1} = n\beta h + o_n(h)$
By analyzing the probabilities at time t just priot to time $t+h$ $(h \to 0)$, we can get

$$P_n'(t) = -\beta(nP_n(t) - (n-1)P_{n-1}(t))$$

The solution for the above solution is $P_n(t) = \exp(-\beta t)(1-\exp(-\beta t))^{n-1}$.

Pure Death Process: Markov process $X(t)$ whose state space is $0,1,\ldots,N$ satisfying the four axioms below, for some death rate $\{\mu_k\}$.

- $\Pr(X(t+h) = k-1 \mid X(t) = k) = \mu_k h + o(h)$ for $k = 1,\ldots,N$.
- $\Pr(X(t+h) = k \mid X(t) = k) = 1 - \mu_k h + o(h)$ for $k = 1,\ldots,N$.
- $\Pr(X(t+h) > k \mid X(t) = k) = 0$
- $X(0) = N$

When the death rate $\{\mu_k\}$ are distinct, then we have $\Pr_N(t) = \exp(-\mu_N t)$ and $P_n(t) = \mu_{n+1}\ldots\mu_N(A_{n,n}\exp(-\mu_n t) + \cdots + A_{N,n}\exp(-\mu_N t))$ where:

$$A_{k,n} = \frac{1}{(\mu_N-\mu_k)\ldots(\mu_{k+1}-\mu_k)(\mu_{k-1}-\mu_k)\ldots(\mu_n-\mu_k)}$$

Linear Death Process: Death process in which the death rates are proportional to the current population size $(\mu_k = k\alpha)$. We then have:

$$P_n(t) = \binom{N}{n}\exp(-n\alpha t)(1-\exp(-\alpha t))^{N-n}$$

Note that linear death process can also be viewed as: N individuals, each lifetime follows independent exponential dist. with parameter α. $X(t)$ is number survivors at time t.

Birth-Death Process: Can be viewed as a continuous-time analogue of discrete random walk. It's transition probabilities $P_{i,j}(t) = \Pr(X(t+s) = j \mid X(s) = i)$ must satify five properties below:

- $P_{i,i+1}(h) = \lambda_i h + o(h)$ as $h \to 0$, $i \geq 0$
- $P_{i,i-1}(h) = \mu_i h + o(h)$ as $h \to 0$, $i \geq 1$
- $P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$ as $h \to 0$, $i \geq 0$
- $P_{i,j} = \delta_{i,j}$
- $\mu_0 = 0, \lambda_0 > 0, \mu_i > 0, \lambda_i > 0$

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

It can be checked that $P_{i,j}(t)$ must satisfies two systems of differential equations. The first is the backward Kolmogorov equations, given by:

- $P_{0,j}'(t) = -\lambda_0 P_{0,j}(t) + \lambda_0 P_{1,j}(t)$
- $P_{i,j}'(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{i,j}(t) + \lambda_i P_{i+1,j}(t)$ for $i \geq 1$

The second is the forward kolmogorov equations, given by:

- $P_{i,0}'(t) = -\lambda_0 P_{i,0}(t) + \mu_1 P_{i,1}(t)$
- $P_{i,j}'(t) = \lambda_{j-1}P_{i,j-1}(t) - (\lambda_j + \mu_j)P_{i,j}(t) + \mu_{j+1}P_{i,j+1}(t)$ for $j \geq 1$

Sojourn Time of Birth-Death Process: As usual, define sojourn time S_i is the amount of time at state i, until $X(t)$ leaves i. Turns out that S_i follows an exponential distribution with parameter $\lambda_i + \mu_i$. Furthermore, when leaving state i, $X(t)$ will enter either $i+1$ or $i-1$ with probability $\frac{\lambda_i}{\lambda_i+\mu_i}$ and $\frac{\mu_i}{\lambda_i+\mu_i}$ respectively.

Birth-Death Process with No Absorbing States: It can be proved that $\lim_{t\to\infty}P_{i,j}(t) = \pi_j$ exists and independent of state i. Furthermore, we'll have

$$\pi_j = \begin{cases} \frac{\theta_j}{\sum_{k=0}^\infty \theta_k} & \text{if } \sum_{k=0}^\infty \theta_k < \infty \\ 0 & \text{if } \sum_{k=0}^\infty \theta_k \end{cases}$$

where $\theta_0 = 1$ and $\theta_j = \frac{\lambda_0\lambda_1\ldots\lambda_{j-1}}{\mu_1\cdots\mu_j}$. To prove the above equations, it suffices to consider the derivative of the forward kolmogorov equations and use induction.

Birth-Death Process with Absorbing States: Theorem 2.5.1 states that in a birth-death process with birth rate λ_n and death rate μ_n with $\lambda_0 = 0$ (0 is an absorbing state), the probability of absorption into state 0 from initial state m is

$$u_m = \begin{cases} \frac{\sum_{i=m}^\infty \rho_i}{1+\sum_{i=1}^\infty \rho_i} & \text{if } \sum_{i=1}^\infty \rho_i < \infty \\ 1 & \text{if } \sum_{i=1}^\infty \rho_i = \infty \end{cases}$$

where $\rho_0 = 1$ and $\rho_i = \frac{\mu_1\ldots\mu_i}{\lambda_1\ldots\lambda_i}$. Furthermore, conditioned on $\sum_{i=1}^\infty \rho_i = \infty$, the mean time to absorption is

$$w_m = \begin{cases} \infty & \text{if } \sum_{i=1}^\infty \frac{1}{\lambda_i\rho_i} = \infty \\ \sum_{i=1}^\infty \frac{1}{\lambda_i\rho_i} + \sum_{k=1}^{m-1}\rho_k\sum_{j=k+1}^\infty \frac{1}{\lambda_j\rho_j} & \text{if } \sum_{i=1}^\infty \frac{1}{\lambda_i\rho_i} < \infty \end{cases}$$

To prove the above two equations, notice that by first-step analysis, we'll have $u_i = \frac{\lambda_i}{\mu_i+\lambda_i}u_{i+1} + \frac{\mu_i}{\mu_i+\lambda_i}u_{i-1}$ and $w_i = \frac{1}{\lambda_i+\mu_i} + \frac{\lambda_i}{\mu_i+\lambda_i}w_{i+1} + \frac{\mu_i}{\mu_i+\lambda_i}w_{i-1}$. After that, just use induction.

Finite-State Continuous-Time Markov Chain: Markov Process on continuous time and states $\{0, 1, \cdots, N\}$. Four axioms related to $P_{i,j}(t)$ are:

- $P_{i,j}(t) \geq 0$
- $\sum_{j=0}^{N} P_{i,j}(t) = 1$
- $P_{i,k}(s+t) = \sum_{j=0}^{N} P_{i,j}(s)P_{j,j}(t)$. In matrix notation, $P(t+s) = P(t)P(s)$. (Chapman-Kolmogorov equation)
- $\lim_{t\to 0+} P_{i,j}(t) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. In other term, $P(t)$ is continuous at 0.

It can be proven that $P(t)$ is continuous everywhere. Furthermore, it's differentiable in that the limit $\lim_{h\to 0+} \frac{1-P_{i,i}(h)}{h} = q_i$ and $\lim_{h\to 0+} \frac{P_{i,j}(h)}{h} = q_{i,j}$ exists. The parameters $\{q_{i,j}\}$ and $\{q_i\}$ define the infinitesimal description of the process as below:

- $\Pr(X(t+h) = j \mid X(t) = i) = q_{i,j}h + o(h)$ for $i \neq j$
- $\Pr(X(t+h) = i \mid X(t) = i) = 1 - q_i h + o(h)$

It can also be proved that in state i, the process wil stay there for a duration that is exponentially distributed with parameter q_i. Then, it jumps to state $j \neq i$ with probability $p_{i,j} = \frac{q_{i,j}}{q_i}$. The sequence of visited states is called Embeded MC.

Notice that given the parameters $\{q_{i,j}\}$ and $\{q_i\}$ which can be expressed as $A = \lim_{h\to 0+} \frac{P(h)-I}{h}$, we can get $P(t) = e^{At} = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$.

3 Renewal Phenomena

Renewal (Counting) Process: A non-negative integer valued stochastic process $\{N(t), t \geq 0\}$ that registers successive occurences of an event during interval $(0, t]$, where times between consecutive events are independent and identically dist. rv denoted as $\{X_k\}_{k=1}^{\infty}$. We also employ the definitions below:

- $F(x) = \Pr(X_k \leq x)$, the cdf of X
- $W_n = X_1 + X_2 + \cdots + X_n$, the waiting time until nth event. Define also $F_n(x) = \Pr(W_n \leq x)$ as the cdf of W_n.
- $N(t)$ as the number of indices n for which $0 < W_n \leq t$. Define also $M(t) = E[N(t)]$.
- $\gamma_t = W_{N(t)+1} - t$ as excess/residual lifetime
- $\delta_t = t - W_{N(t)}$ as current lifetime
- $\beta_t = \gamma_t + \delta_t$ as total life

Nice Properties of Renewal Process:

- $N(t) \geq k$ iff $W_k \leq t$. Consequently, $\Pr(N(t) \geq k) = F_k(t)$ and $\Pr(N(t) = k) = \Pr(N(t) \geq k+1) - \Pr(N(t) \geq k) = F_k(t) - F_{k+1}(t)$.
- $F_n(x) = \int_0^x F_{n-1}(x-y)f(y)dy = \int_0^x F_{n-1}(x-y)dF(y)$
- $E[W_{N(t)+1}] = E[X](1 + M(t))$. To prove this, can use either LOTE or indicator random variable $I\{N(t) \geq j-1\} = I\{X_1 + \cdots + X_{j-1} \leq t\}$. (Proof: use $M(t) = E[N(t)] = \sum_{k\in\mathbb{N}} P(N(t) \geq k) = \sum_{k\geq 1} F_k(t)$ AND $E\left[X_j \mathbf{I}'\{\sum_{i\leq j-1} X_i \leq t\}\right] = \mu F_{j-1}(t)$).
- $\Pr(\delta_t \geq x, \gamma_t > y) = \Pr(N(t-x) = N(t+y)) = \sum_{k=0}^{\infty} \Pr(W_k < t-x \wedge W_{k+1} > t+y) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-z))dF_k(z)$

Poisson Process: Note that it can be viewed as a renewal process with exponential distribution $F(x) = 1 - e^{-\lambda x}$. It can be proved that the following properties are satisfied:

- Excess life possesses the exponential distribution $\Pr(\gamma_t \leq x) = 1 - e^{-\lambda x}$. This is identical to X.
- Current life follows the truncated exponential distribution $\Pr(\delta_t \leq x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x < t \\ 1 & \text{if } x \geq t \end{cases}$.
- $\Pr(\gamma_t > x, \delta_t > y) = \Pr(\gamma_t > x)\Pr(\delta_t > y) = \begin{cases} e^{-\lambda(x+y)} & \text{if } 0 < y < t \\ 9 & \text{if } y \geq t \end{cases}$. Hence, excess and current life is independent in poisson process.
- Mean total life $E[\beta_t] = \frac{1}{\lambda} + \frac{1}{\lambda}(1 - e^{-\lambda t})$ is significantly larger than the mean life $E[X_k] = \frac{1}{\lambda}$. This paradox is known as length biased sampling, and occurs because a fixed time t is used.

Elementary Renewal Theorem: All renewal functions are asymptotically linear. Furthermore, poisson process is the only continuous-time renewal process where $M(t)$ is exactly linear.

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{E[X]}$$

2nd Term Elementary Renewal Theorem: If X is continuous r.v. with finite mean and variance, we also have

$$\lim_{t\to\infty} (M(t) - \frac{t}{E[X]}) = \frac{Var[X] - E[X]^2}{2E[X]^2} = \frac{E[X^2]}{2E[X]^2} - 1$$

Asymptotic Distribution of $N(t)$: Besides the mean, the variance of $N(t)$ is also asymptotic to some function in X. More specifically, $\lim_{t\to\infty} \frac{Var[N(t)]}{t} = \frac{Var(x)}{E[x]^3}$. Indeed $N(t)$ is asymptotic to normal distribution as described below:

$$\lim_{t\to\infty} \Pr\left(\frac{N(t) - t/E[X]}{\sqrt{tVar[x]/E[x]^3}} \leq x\right) = \phi(x)$$

Limiting Distributions of Age and Excess Life: it can be proved that the excess life has the limiting distribution below.

$$\lim_{t\to\infty} \Pr(\gamma_t \leq x) = \frac{1}{E[X]} \int_0^x (1 - F(y))dy = H(y)$$

Let $h(y) = \frac{1}{E[X]}(1 - F(y))$ be the corresponding pdf. Then,

$$\lim_{t\to\infty} E[\gamma_t] = \frac{1}{E[X]} \int_0^{\infty} y(1 - F(y))dy = \frac{E[X^2]}{2E[X]} = \frac{Var[X] + E[X]^2}{2E[X]}$$

Notice that $\gamma_t \geq x$ and $\delta_t \geq y$ iff $\gamma_{t-y} \geq x + y$. Hence,

$$\lim_{t\to\infty} \Pr(\delta_t \geq y) = \lim_{t\to\infty} \Pr(\gamma_{t-y} \geq y) = 1 - H(y)$$

Therefore, $\lim_{t\to\infty} \Pr(\delta_t \leq y) = H(y)$ and $\lim_{t\to\infty} E[\delta_t] = \frac{Var[X] + E[X]^2}{2E[X]}$

Delayed Renewal Process: Renewal process where $X_2, X_3, \ldots$ are identically distributed with cdf F, while X_1 has a different pdf G. Denote $M_D(t)$ as the expected value of $N(t)$ in this delayed renewal process. Then, two properties below are satisfied:

- $\lim_{t\to\infty} \frac{M_D(t)}{t} = \frac{1}{E[X_2]}$
- $\lim_{t\to\infty} M_D(t) - M_D(t-h) = \frac{h}{E[X_2]}$

Stationary Renewal Process: A delayed renewal process for which the first life has the distribution function $G(x) = \frac{1}{\lambda}\int_0^x (1 - F(y))dy$. Denote γ_t^D as the excess life in this delayed renewal process. Then, two properties below are satisfied:

- $M_D(t) = \frac{t}{E[X_2]}$
- $\Pr(\gamma_t^D \leq x) = G(x)$

4 Brownian Motion and Gaussian Process

Postulates of Brownian Motion:

- $B(t)$ is the y component (as a function of time) of a particle in Brownian motion.
- The probability $\Pr(B(t+t_0) = y \mid B(t_0) = x) = p(y, t \mid x)$ is independent of t_0.
- $p(y, t \mid x) \geq 0$ and $\int_{-\infty}^{\infty} p(y, t \mid x)dy = 1$.
- $B(t_0 + t)$ should be near $B(t_0)$ for small t. Formally, $\lim_{t\to 0} p(y, t \mid x) = 0$ for $y \neq x$

Note that from physical principles, it's proved that $p(y, t|x)$ must satisfy the diffusion equation: $\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}$. From this, we'll have $p(y, t \mid x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp(-\frac{1}{2\sigma^2 t}(y-x)^2)$.

Properties of Brownian Motion:

- $B(s+t) - B(s)$ is a normal distribution with mean 0 and variance $\sigma^2 t$, where σ^2 is a fixed parameter (diffusion coefficient).
- For disjoint interval $(t_1, t_2], (t_3, t_4]$, the increments $B(t_4) - B(t_3)$ and $B(t_2) - B(t_1)$ are independent.
- $B(0) = 0$ and $B(t)$ is a continuous function in t
- $Cov[B(s), B(t)] = \sigma^2 \min(s, t)$

Invariance Principle: Let $S_n = \xi_1 + \cdots + \xi_n$ where $\xi_1, \ldots, \xi_n$ are i.i.d with zero means and unit variance. As a function of continous variable, define

$$B_n(t) = \frac{S_{\lceil nt \rceil}}{\sqrt{n}}$$

By CLT, it's reasonable to believe that $B_n(t)$ should behave much like a standard Brownian motion when n is large. The convergence of such sequence to a standard Brownian motion is termed the invariance principle.

Multivariate Normal Distribution Random vector $X_1, \ldots, X_n$ is a multivariate normal distribution if for every $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, then $\sum \alpha_i X_i$ is univariate normal distribution. Note that multivariate normal distribution is specified by two (sets of) params: $\{\mu_i = E[X_i], \Gamma_{ij} = \text{Cov}(X_i, X_j)\}$.

Gaussian Processs Let T abstract set. A stochastic process $\{X(t)\}$ is a gaussian process if for every finite subset $\{t_i\}_{i\leq n}$ of T, the random vector $(X(t_i))_{i\leq n}$ has a multivariate normal distribution. Then, the following properties are satisfied:

- It can be described uniquely by two functions, $\mu(t) = E[X(t)]$ and $\Gamma(s, t) = E[(X(s) - \mu(s))(X(t) - \mu(t))]$
- The covariance function Γ is positive definite, in a sense that $\forall n \geq 1, \alpha_i \in \mathbb{R}, t_i \in T$, we have $\sum_{i\leq n}\sum_{j\leq n} \alpha_i \alpha_j \Gamma(t_i, t_j) \geq 0$

5 Examples

Shot Noise. Arrive P-Proc rate λ, which intensity after x time is $h(x)$. Intensity of current at t is shot noise $I(t) = \sum_{k=1}^{X(t)} h(t - W_k)$. Define U_i iid uniform; define $\epsilon_k = h(U_k), k \geq 1$. Then $I(t) = S(t) := \sum_{i\leq X(t)} \epsilon_i$. We can then get $E[I(t)] = \lambda \int_0^t h(u)du$ and $Var[I(t)] = \lambda t E(h^2(U_1))$.

Sum Quota Sampling. Estimate mean service life of a part, given a preassigned TIME quota t. Define $N(t)$ is max n such that $\sum_{i\leq n} X_i < t$. The "natural" sample mean $\overline{X}$ is $\sum_{i\leq n} X_i / N(t)$. Note that we must assume $N(t) > 0$. Computation can be carried out when X exponential distribution with common parameter λ.

Key is use theorem to evaluate that $E(W|N(t) = n)$ is $tn/(n+1)$, and $E(W_{N(t)}/N_t|N(t) > 0)$ is $1/\lambda \cdot (1 - \lambda t/(e^{\lambda t} - 1))$. Expressing this as a ratio to the mean bias $EX_1 = 1/\lambda$, it equals $\frac{EX1 - E\overline{X}}{EX1} = \lambda t/e^{\lambda t} - 1$. Name the fraction that solely depends on λt a special term called "fraction bias". The values are $(1, 2, 3, 4, 5, 6, 10)$ to $(0.58, 0.31, 0.16, 0.07, 0.03, 0.015, 0.0005)$.

Example: if $N(t) = 3$ and sample mean is $\overline{X} = 7$, a more accurate estimate is $7/0.84$ (sample/(1-fraction bias)).

Population Processes. Model is $\lambda_n := n\lambda$ when $n < K$ and 0 when $n \geq K$ (K is carrying capacity). Mean time to extinction is $\omega_1 = \sum_{i\geq 1} \frac{1}{\lambda_i \rho_i} = \frac{1}{\mu}\sum_{i\geq 1}^{K} \frac{1}{i}\left(\frac{\lambda}{\mu}\right)^{i-1}$.

Mean generations $M_g = \lambda \cdot \omega_1$, (when $K > 100$) $M_g \sim 1/\theta \ln(1/(1-\theta))$ if $\theta < 1$, $0.57721157 + \ln K$ for $\theta = 1$ and $1/K(\theta^k/(\theta - 1))$ for $\theta > 1$.

Block Replacement. Light bulb, whose life (discrete units), is r.v. X where $P\{X = k\} = p_k$ for $k \geq 1$. Due to scaling, may better replace it at blocks (after a certain specified time), whether or not it is broken [ofc, we also replace it when it breaks before that specified time].

Mean total cost (of 1 bulb) during the (whole) cycle is $c_1 + c_2 \cdot M(K-1)$, where $M(K-1) = E[N(K-1)]$ is mean num of failure-replacements up to $K-1$ (when K is the replacement time, with each cycle consisting of K periods).

$M(n) = \sum_{k\geq n+1} p_k(0) + \sum_{k=1}^{n} p_k[1 + M(n-k)] = F_X(n) + \sum_{k\leq n} p_k M(n-k)$.

Age Replacement. Replaced at min(failure/age T). Mean-renewal-duration is $\mu_T = \int_0^{\infty}[1 - F_T(x)]dx = \int_0^T [1 - F(x)]dx < \mu$.

If $\{Y_i\}$ time between actual failures, e.g. $Y_1 \sim NT + Z$ where $P(N \geq k) = [1 - F(T)]^k$ and $P(Z \leq z) = F(z)/F(T)$, $0 \leq z \leq T$). We have $E[Y_1] = \mu_T / F(T)$.

If each repl (whether planned/not) costs \$K + each failure have penalty of \$c. Long-run-mean-cost-per-unit-time is:

$$C(T) = \frac{K}{\mu_T} + \frac{c}{EY_1} = \frac{K + cF(T)}{\int_0^T [1 - F(x)]dx}$$

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Replacement Model. Assume replacement is not instantaneous. Let Y_i and Z_i be the operating time and lag period respectively. Sequence of times between successive replacements $X_k = Y_k + Z_k$ is a renewal process. Prob. that the system is in operation at a specific time will converge to $\frac{E[Y_1]}{E[X_1]}$

Risk Theory. $W(t) = \sum_{k=1}^{N(t)+1} Y_k$ represents the cumulative amount claimed up to time t, then the long-run-mean-claim-rate is:

$$\lim_{t\to\infty} \frac{E[W(t)]}{t} = \frac{E[Y_1]}{E[X_1]}$$