SDSS 2019 – Bellevue, Washington

CS32 – Statistical Methods for Analyzing Large Scale or Massive Data

An Application of Linear Programming to Computational Statistics

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Meet Marlene

- Marlene is a data scientist who works at a major consumer packaged goods company
  - She is skilled in several statistical packages but uses R for most of her day-to-day work
  - Marlene is respected by her peers as someone who learns quickly and can be trusted to come through with solutions to difficult problems
- As we join Marlene today, we find her working on tables to communicate statistical differences between a large number of samples on a large number of attributes

Marlene’s Problem

- Marlene is working on a dataset of 31 frozen pizzas evaluated on 25 attributes
  - She seeks to produce tables reflecting multiple statistical comparisons between all pizzas and on all attributes
  - Her solution will be the basis for a tool to be used throughout her company
- Ideally, her solution will be:
  - Flexible enough to allow her to adjust the statistical testing without having to rewrite large portions of her code
  - Fast enough to handle a large number of samples and attributes in reasonable time

Marlene's First Attempt

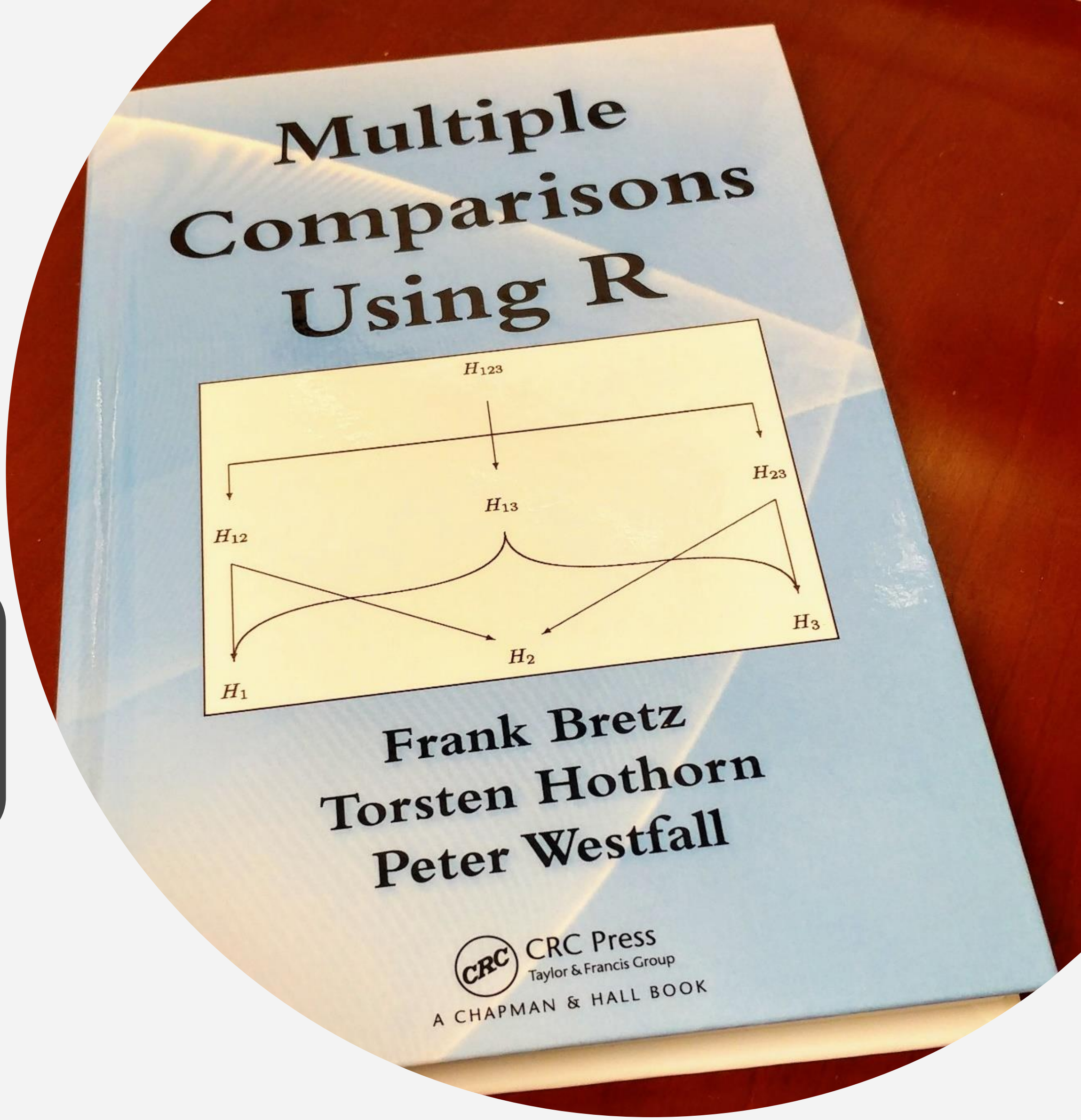
- Marlene begins by following the approach recommended by Bretz et al. (2016)
  - This excellent book recommends the use of compact letter displays to communicate the result of post-hoc Tukey tests:

```
ex_aov <- aov(scale_1 ~ prod_code, data = ex_data)

ex_mc <- ex_aov %>%
   multcomp::glht(linfct = mcp(code = "Tukey"))

cld_results <- multcomp::cld(ex_mc)
```

- Marlene runs into difficulty almost immediately
  - Her analysis of the first attribute requires more than 15 minutes to complete!
- Marlene decides she needs to review the literature on compact letter displays



Letter Displays

Letter Displays

- Letter displays enable representation of multiple statistical comparisons in a single table
  - For each attribute, samples with at least one letter in common are not significantly different

Sample	**Attribute 1**	**Attribute 2**	**Attribute 3**	…
1	3.73 a	2.96 ***e***f	3.01 b***c***	…
2	3.57 b	3.57 b	2.73 efghi	…
3	3.46 ***c***	3.16 d	2.79 efg	…
4	3.33 ***c***d	3.77 a	2.96 ***c***d	…
5	3.30 d	2.92 ***e***fg	2.72 fghi	…
6	3.29 d	3.28 cd	2.49 mn	…
7	3.26 d	3.47 b	3.10 b	…
8	3.01 e	3.26 cd	3.64 a	…
9	2.99 e	2.99 ***e***	3.09 b***c***	…
10	2.98 e	3.31 c	3.08 b***c***	…
⋮	⋮	⋮	⋮	

Example of Letter Display Reduction

Sample	Attribute	Sample	Attribute
1	6.25 a	12	5.53 bcdefgh
2	5.95 ab	13	5.52 bcdefghi
3	5.92 abc	14	5.50 bcdefghi
4	5.79 abc	15	5.42 bcdefghi
5	5.75 abcd	16	5.41 cdefghi
6	5.71 bcde	17	5.24 defghi
7	5.68 bcde	18	5.20 efghi
8	5.65 bcdef	19	5.13 fghi
9	5.62 bcdef	20	5.06 ghi
10	5.59 bcdefg	21	5.00 ij
11	5.54 bcdefgh	22	5.00 hij

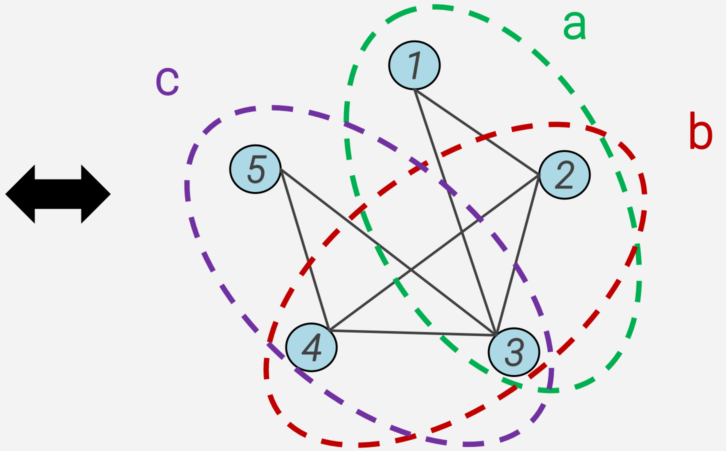
Sample	Attribute	Sample	Attribute
1	6.25 a	12	5.53 bfh
2	5.95 ab	13	5.52 bi
3	5.92 abc	14	5.50 bi
4	5.79 abc	15	5.42 bi
5	5.75 abcd	16	5.41 cefi
6	5.71 be	17	5.24 defi
7	5.68 be	18	5.20 efi
8	5.65 bf	19	5.13 fi
9	5.62 bf	20	5.06 ghi
10	5.59 bfg	21	5.00 ij
11	5.54 bfh	22	5.00 hij

48 letter assignments of 103 total can be removed, leaving just 55

Letter Displays and Graph Theory

- Each matrix of non-significant differences corresponds to a bidirectional graph
  - Samples correspond to vertices, which are connected when the samples are not significantly different

Sample	Attribute
1	3.73 a
2	3.57 ab
3	3.46 abc
4	3.33 bc
5	3.30 c



- Each letter of the display corresponds to a clique in the corresponding graph
  - An accurate letter display corresponds to a clique covering of the graph's edges
  - Removing a letter assignment from the display corresponds to removing a vertex from a clique
- As long as the graph's edges remain covered, the corresponding letter display is still accurate

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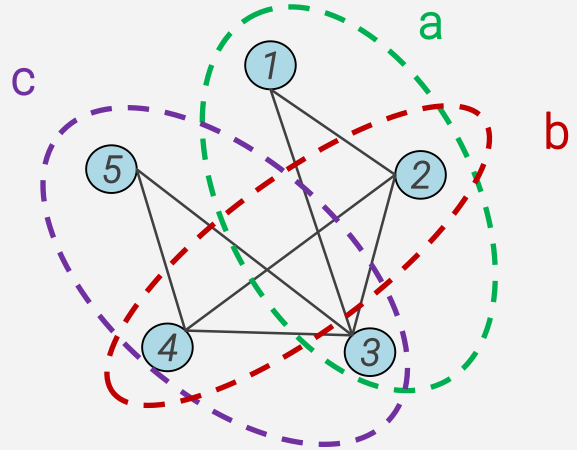
Sample	Attribute
1	3.73 **a**
2	3.57 **ab**
3	3.46 **ac**
4	3.33 **bc**
5	3.30 **c**



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Searching for Minimal Displays

- To find minimal displays in reasonable time for moderate size problems (Ennis et al., 2012; fewer than 50 samples):
  - Start with maximal display
  - Identify all individually removable letter assignments
  - Search for all simultaneously removable assignments
  - Select largest combination of simultaneously removable assignments
- To determine whether assignments are simultaneously removable, Ennis et al. (2012) proposed a characterization of accurate displays:
  - Let a_{ik} indicate with 0 or 1 whether item *i* is assigned letter *k*
  - A matrix of non-significant differences $M = [m_{ij}]$ is accurately described if $m_{ij} = 1$ exactly when $\sum_k a_{ik} a_{jk} \geq 1$
- This condition can be checked quickly as the search proceeds

We next improve speed using linear programming

Linear Programming

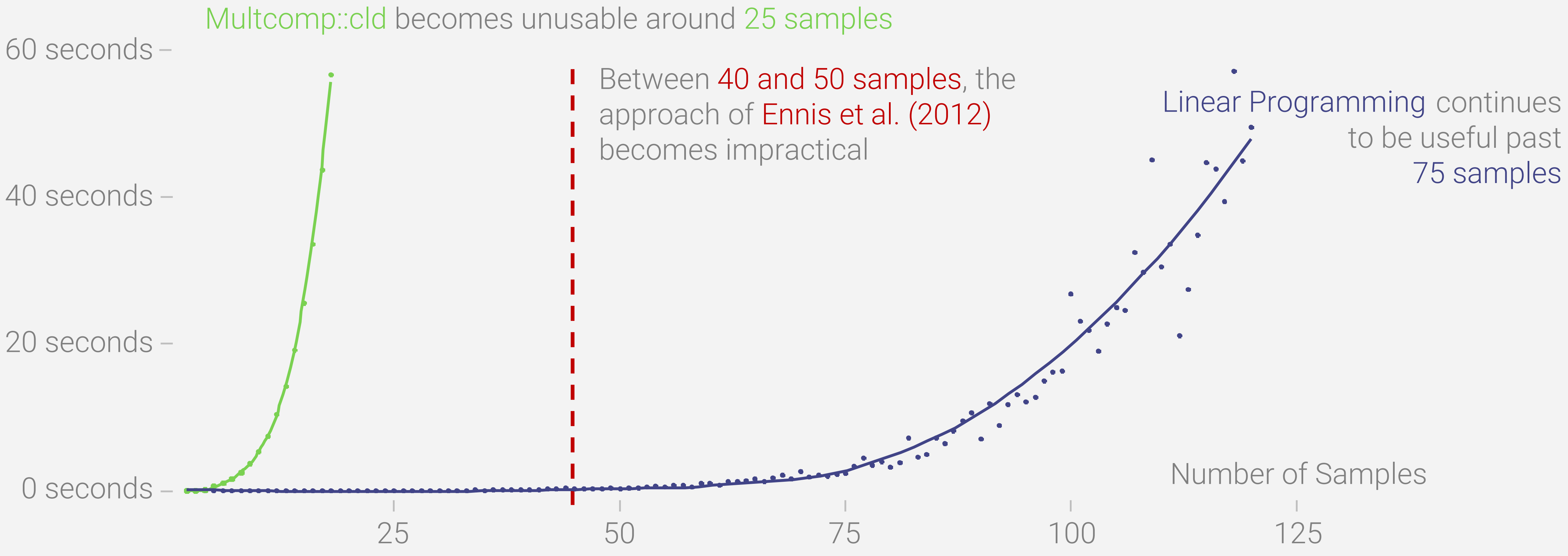
Minimizing Assignments with Linear Programming

1. Start with matrix $M = [m_{ij}]$ of non-significant differences
2. Create an binary indicator variable for every possible letter assignment a_{ik}
   - Goal: minimize the total number of letter assignments $T = \sum_{i,k} a_{ik}$
3. Find the maximal display by computing the complete set of maximal cliques
   - For example, using igraph::maximal.cliques (Csardi & Nepusz, 2006)
4. Define constraints:
   a) Set a_{ik} = 0 if assignment a_{ik} doesn't appear in the maximal display
   b) Set a_{ik}= 1 if assignments a_{ik} can't be removed individually
   c) For every *i* and *j* for which $m_{ij} = 1$, require $\sum_k a_{ik} a_{jk} \geq 1$
5. To implement step 4b, a trick is required because $\sum_k a_{ik} a_{jk}$ is quadratic
   a. For every i and j for which $m_{ij} = 1$, create an intermediate variable $e_{ijk} = a_{ik} a_{jk}$
   b. Require $e_{ijk} \leq a_{ik}, e_{ijk} \leq a_{jk}$, and $e_{ijk} \geq a_{ik} + a_{jk} - 1$ for every *i, j, k*

This problem can be solved using lpsolve::lp (Berkelaar, 2007)

Timing Comparison

- Data simulated according to an approach specified in Gramm et al. (2007)
  - 100 replications for each fixed number of samples
  - 90% quantiles for time to complete shown for each case



Back to Marlene

Back to Our Story

- Marlene writes scripts to implement a linear programming solution to her multiple statistical comparisons problem
- Her analysis of all 25 attributes runs in under 15 seconds
  - Recall: Her original analysis of just one attribute required more than 15 minutes
  - The median time for each attribute is less than half a second
  - The median number of letter assignments removed per attribute is 17
- Marlene's fame within her company grows
  - Her scientific colleagues look forward to using her scripts
  - Her colleagues in packaging and marketing insights invite her to discuss other problems that can be solved using linear programming

But, most important, **Marlene had fun**!

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Thank you for attending!

References

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