Numerical Chladni Figures

for a vibrating square plate with free boundary condition

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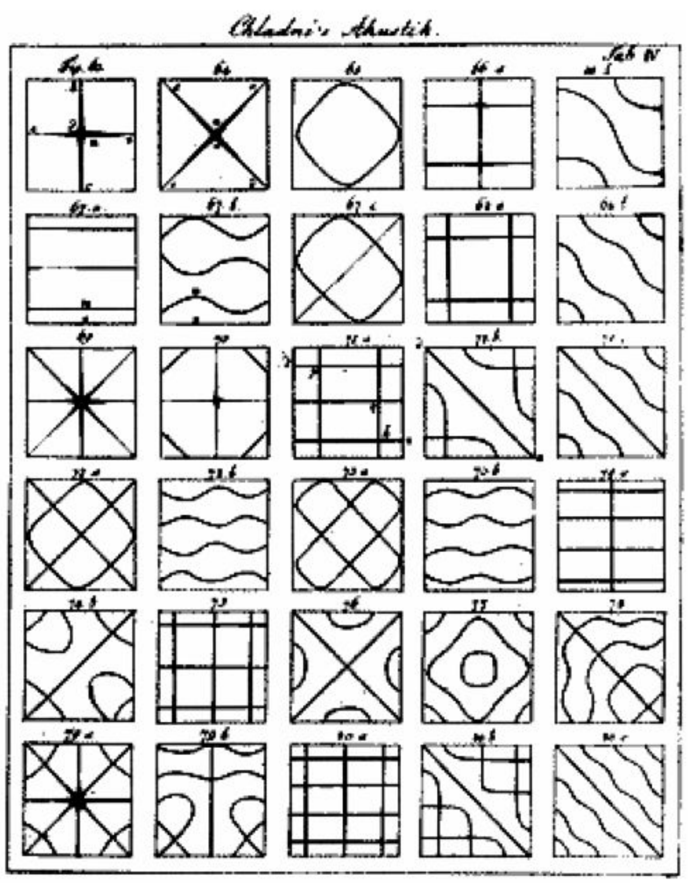
Introduction:

In his famous experiment in 1787, *Ernst Chladni* sprinkled fine sand particles on a square plate before setting it into vibration using a violin bow. As the plate vibrates the sand gathered at the stationary points creating beautiful patterns. It turns out that for this specific case of *free boundary configuration* such nodal patterns are the zero sets of the eigenfunctions of the biharmonic(Δ^2) operator. In here we will briefly discuss how the eigenvalue problem is arrived at, discretized and calculated using the finite volume method(FVM).



Why is this interesting:

1.We get to visualise sound waves

2.Obtaining these approximate patterns needed solving a generalised eigen-value problem of the form $Ax=\lambda Bx$ *where* A & B *are* $10k*10k$ matrices that too for a simple geometry like square. Complexity for irregular shapes such as a guitar will be even more difficult to handle. But in nature these happen instantaneously which is amazing to me.

3.In our day to day life also such patterns are observed. When you slide a glass containing water across a rough table they appear.

4.Singing bowls which are supposed to create healing sounds which have mention in both hindu vedic and buddhist texts also create beautiful patterns when struck.

Mathematical model[#1]:

The force acting on a given point on the plate depends only on the local shape of the plate. So if we let $z = z(x, y, t)$ be the vertical position of the plate at point (x, y) at time t, then the vertical force acting on this point can be written as[#1]:

$$F(x, y, t) = -\mathscr{L}z \quad ..1$$

Where $\mathscr{L}$ is the spatial differential operator acting on z. Applying Newton's Law[#1],

$$\frac{\partial^2 z}{\partial t^2} = -\mathcal{L}z \quad ..2$$

The precise form of $\mathscr{L}$ depends on the model used for the material and can often be derived from energy considerations. After defining $\mathscr{L}$ such motion can be analyzed using standing wave solutions of the form $z(x, y, t) = T(t)u(x, y)$ is a standing wave, where $u(x, y)$ is a solution of the eigenvalue problem[#1]:

$$\mathscr{L}u=\lambda u$$

or when integrated with an arbitrary v,[#1]

$$\iint_\Omega (\mathscr{L}u)v\, dx\, dy = \iint_\Omega \lambda uv\, dx\, dy \quad ...3$$

$\mathscr{L}u$ is simply the restoring force due to the shape of the vibrating plate $u = u(x, y)$.

Deriving operator $\mathscr{L}$:

We proceed to derive $\mathscr{L}$ using energy considerations. According to Kirchoff[#1], the potential energy(PE) stored in a deformed thin square plate with shape $u : \Omega \to R$,is given by[#3]

$$J[u(x,y)] = \frac{1}{2}\iint_\Omega (u_{xx}+u_{yy})^2 - 2(1-\mu)(u_{xx}u_{yy} - u_{xy}^2)\, dx\, dy \quad ...4$$

where $0 < \mu < 1$ is a material constant.Here we use $\mu=0.225$ to get result similar to Chladni.

Now, suppose we change the shape of the plate from an initial configuration of $u(x, y)$ to a slightly different shape $u(x, y) + \varepsilon v(x, y)$, where ε is small. Then the change in PE equals the work done on the system, which is the distance travelled by each particle on the plate times the the restoring force $\mathcal{L}u(x, y)$[#1]

$$J[u+\varepsilon v] - J[u] = \iint_\Omega (\mathscr{L}u)(\varepsilon v)\, dx\, dy + O(\varepsilon^2) \quad ...5$$

Dividing both sides by ε and taking limit $\varepsilon \to 0$,we get[#1]:

$$\frac{d}{d\varepsilon}J[u+\varepsilon v]\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac{J[u+\varepsilon v]-J[u]}{\varepsilon} = \iint_\Omega (\mathscr{L}u)v\, dx\, dy \quad ...6$$

Substituting the above into 4 yields[#1]:

$$\iint_\Omega (\mathcal{L}u)v\, dx\, dy = \iint_\Omega \left[u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{yy}v_{xx} + u_{xx}v_{yy}) + 2(1-\mu)u_{xy}v_{xy}\right] dx\, dy \quad ..7$$

Divergence theorem states:

$$\iint_\Omega \nabla \cdot \mathbf{F} \, dx \, dy = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, ds$$

Where F is a vector field and $\boldsymbol{n}(n_x, n_y)$ is the unit outward normal vector. If we substitute $\mathbf{F}$ with $(fg,0)$ *and* $(0,fg)$ where f,g are scalar functions we obtain:

$$\iint_\Omega f g_x \, dx \, dy = \int_{\partial\Omega} f g \, n_x \, ds - \iint_\Omega f_x g \, dx \, dy$$

$$\iint_\Omega f g_y \, dx \, dy = \int_{\partial\Omega} f g \, n_y \, ds - \iint_\Omega f_y g \, dx \, dy \quad ...8$$

For the case of a rectangular plate where $\Omega = (-L, L) \times (-H, H)$, Using 8, the terms on the RHS of 7 can be integrated by parts to remove the derivatives of v. The details are ignored here for the sake of compactness but can be found in the paper #1 in references. After integrating RHS of 7 and using 3 we get the following:

$$\begin{aligned}
0 &= \iint_\Omega (\mathcal{L}u - \lambda u) v \, dx \, dy \\
&= \iint_\Omega (u_{xxxx} + 2u_{xxyy} + u_{yyyy} - \lambda u) v \, dx \, dy \\
&\quad + \int_{\partial\Omega} \left[u_{xx} + \mu u_{yy}\right] v_x n_x \, ds - \int_{\partial\Omega} \left[u_{xxx} + (2-\mu) u_{xyy}\right] v \, n_x \, ds \\
&\quad + \int_{\partial\Omega} \left[\mu u_{xx} + u_{yy}\right] v_y n_y \, ds - \int_{\partial\Omega} \left[u_{yyy} + (2-\mu) u_{xxy}\right] v \, n_y \, ds \\
&\quad + 2(1-\mu)\big[u_{xy}(L,H) v(L,H) - u_{xy}(-L,H) v(-L,H) \\
&\qquad - u_{xy}(L,-H) v(L,-H) + u_{xy}(-L,-H) v(-L,-H)\big]
\end{aligned}$$ #1

By choosing arbitrary functions v that vanish on the boundary of Ω, we obtain the PDE and boundary conditions for the eigenvalue problem which is:

The final problem formulation:

We need to find a nontrivial function u and a corresponding value of λ that satisfy the following conditions:

- $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \lambda u, \qquad (x, y) \in \Omega$ ...c1
- $u_{xx} + \mu u_{yy} = 0, \qquad u_{xxx} + (2-\mu) u_{xyy} = 0, \qquad x = \pm L, \ y \in (-H, H)$
- $u_{yy} + \mu u_{xx} = 0, \qquad u_{yyy} + (2-\mu) u_{xxy} = 0, \qquad y = \pm H, \ x \in (-L, L)$ ...c2
- $u_{xy} = 0, \qquad (x, y) = (\pm L, \pm H)$ ...c3

c1 is just $\mathcal{L}u = \lambda u$ where L is the bi-harmonic(Δ^2) operator and c2 and c3 are the boundary conditions.[#3]

Finite difference discretisation to solve the problem[#1]:

As the PDE contains the bi-harmonic(Δ^2) operator we can be tempted to discretise using the 5 - point Laplacian operator composed on itself. But the free boundary conditions (c2&c3) cannot be implemented in doing so. So we resort to using the Finite Volume Method(FVM) to deal with these systematically.

Let $\Omega = (-1, 1) \times (-1, 1)$ be the square plate, which we discretize on a uniform $(N+1) \times (N+1)$ grid, including nodes on the boundary. Then the grid points (x_i, y_j), $0 \le i, j \le N$, satisfy

$$x_i = -1 + ih, \qquad y_j = -1 + jh, \qquad h = 2/N$$

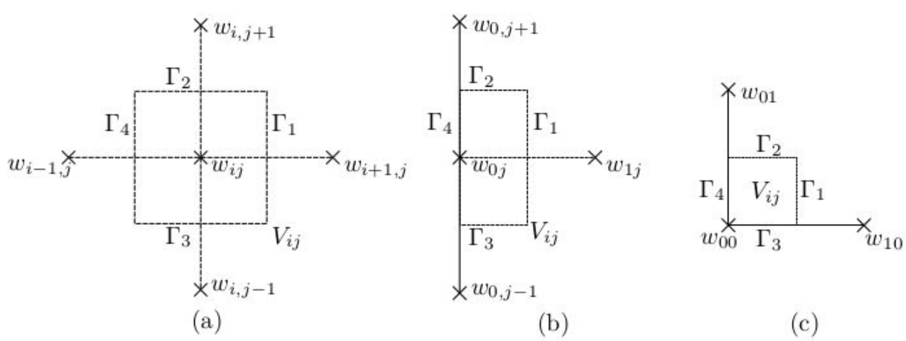
Let $u(x, y)$ be the exact solution of the eigenvalue problem and $u_{ij} \approx u(x_i, y_j)$ be its finite difference approximation. We first define $w(x, y) = -\Delta u(x, y)$ and its discrete analogue:

$$w_{ij} = \frac{4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}{h^2}$$

Note that in order to define w_{ij} along an edge, we need values of u_{ij} that fall outside Ω, i.e., for $i, j \in \{-1, N+1\}$. These are called ghost points and are not part of the original problem; they will need to be eliminated using boundary conditions before we solve the discrete eigenvalue problem. Once we have defined $w(x, y)$,we have

$$\Delta^2 u = \lambda u \iff -\Delta w = \lambda u \quad \text{...9}$$

For the Finite Volume Method we need to integrate the above over control volume V_{ij} around each grid point. The figure below shows the control volume around *(a)interior, (b)edge and (c)corner nodes.*



...fig 1[#1]

Integrating 9, we get

$$-\iint_{V_{ij}} \Delta w \, dx\, dy = \lambda \iint_{V_{ij}} u(x, y)\, dx\, dy$$

Which on applying divergence theorem becomes:

$$-\int_{\partial V_{ij}} \frac{\partial w}{\partial n}\, ds = \lambda \iint_{V_{ij}} u(x, y)\, dx\, dy \quad \text{...10}$$

The RHS can be approximated as

$$\iint_{V_{ij}} u(x, y)\, dx\, dy \approx |V_{ij}| u_{ij} = \begin{cases} h^2 u_{ij} & \text{for interior node} \\ \frac{1}{2} h^2 u_{ij} & \text{for edge nodes,} \\ \frac{1}{4} h^2 u_{ij} & \text{for corner nodes} \end{cases}$$

The flux $\int_{\partial V_{ij}} \frac{\partial w}{\partial n}\, ds$ on the LHS have to be approximated appropriately for different nodes.

Finally when we vectorise u the above RHS can be written as λBu where B will be a diagonal matrix containing the control volumes of the nodes *(h^2, $h^2/2$ and $h^2/4$)* and for LHS we have a matrix A which when operates on u, gives the corresponding fluxes $\int \frac{\partial w}{\partial n}\, dx$ so that in the end we have a generalised eigenvalue problem $Au=\lambda Bu$ which is solved by *Lanczos* algorithm.

So we now see pointers on how to get LHS, ie flux approximations for each type separately:

Interior nodes[#1]: The flux along each piece of ∂V_{ij} is simply approximated by a finite difference. For example, along Γ_1 shown in *fig 1* in previous page

$$\int_{\Gamma_1} \frac{\partial w}{\partial n}\, ds \approx h \cdot \frac{w_{i+1,j} - w_{ij}}{h} \qquad \text{...11}$$

Which leads to the finite difference stencil

$$-\int_{\partial V_{ij}} \frac{\partial w}{\partial n}\, ds \approx 4w_{ij} - w_{i-1,j} - w_{i+1,j} - w_{i,j-1} - w_{i,j+1} \qquad \text{...12}$$

So it is evident that the interior stencil is the discrete laplacian squared i.e we operate Δ on u to get w and then the same Δ on w i.e. $\mathcal{L}u=\Delta^2 u$.

Edge nodes[#1]***:***

We consider a control volume along the left edge $x = x_0 = -1$, as shown in *Fig 1*; the other edges are treated similarly. For edges Γ_2, $\Gamma_{2'}$ and Γ_3 the approximation is similar to interior node except for a factor of ½ for top and bottom edges.

$$-\int_{\partial V_{0j}} \frac{\partial w}{\partial n}\, ds \approx 2w_{0j} - w_{1j} - \frac{1}{2}w_{0,j-1} - \frac{1}{2}w_{0,j+1} - \int_{\Gamma_4} \frac{\partial w}{\partial n}\, ds$$

For Γ_4 , we make use of boundary edge boundary condition c2 to get:

$$\begin{aligned}
\int_{\Gamma_4} \frac{\partial w}{\partial n}\, ds &= \int_{y_{j-1/2}}^{y_{j+1/2}} \frac{\partial}{\partial x}(u_{xx} + u_{yy})\, dy \\
&= \int_{y_{j-1/2}}^{y_{j+1/2}} \big[\underbrace{u_{xxx} + (2-\mu)u_{xyy}}_{=0} - (1-\mu)u_{xyy}\big]\, dy \\
&= -(1-\mu)\int_{y_{j-1/2}}^{y_{j+1/2}} u_{xyy}\, dy \\
&= -(1-\mu)\big[u_{xy}(x_0, y_{j+1/2}) - u_{xy}(x_0, y_{j-1/2})\big].
\end{aligned} \qquad \text{...13}$$

Next the mixed derivatives u_{xy} are approximated by finite differences as:

$$u_{xy}(x_0, y_{j-1/2}) \approx \frac{u_{1,j} - u_{-1,j} - u_{1,j-1} + u_{-1,j-1}}{2h^2} \qquad \text{...14}$$

Corner nodes*[#1]*:

For the corner node in fig 1 we proceed as follows,

$$-\int_{\partial V_{ij}} \frac{\partial w}{\partial n}\, ds \approx w_{00} - \frac{1}{2}w_{10} - \frac{1}{2}w_{01} - \int_{\Gamma_3} \frac{\partial w}{\partial n}\, ds - \int_{\Gamma_4} \frac{\partial w}{\partial n}\, ds$$

For Γ_3 and Γ_4 , we perform the same calculations as 13 to get, for instance,

$$\int_{\Gamma_4} \frac{\partial w}{\partial n}\, ds = -(1-\mu)\left[u_{xy}(x_0, y_{1/2}) - u_{xy}(x_0, y_0)\right]$$

But here we have $u_{xy}=0$ from the boundary condition c3.

$$\int_{\partial V_{ij}} \frac{\partial w}{\partial n}\, ds \approx w_{00} - \frac{1}{2}w_{10} - \frac{1}{2}w_{01} + (1-\mu)u_{xy}(x_0, y_{1/2}) + (1-\mu)u_{xy}(x_{1/2}, y_0) \quad \text{...15}$$

The mixed derivative is calculated as 14 and rest of the corners are treated similarly.

Ghost Points:

As it was mentioned earlier in the starting of this section that the values of u_{ij} that fall outside the square Ω are simply defined for convenience to calculate w_{ij} along the edges which use *5-point stencil*. These must be eliminated before the eigenvalue problem is solved. Fortunately, this can be done easily using the edge boundary conditions

$$u_{xx} + \mu u_{yy} = 0 \quad \text{for } x = \pm 1, \qquad u_{yy} + \mu u_{xx} = 0 \quad \text{for } y = \pm 1$$

So for a ghost point along the edge $x = -1$, we have

$$u_{-1,j} = 2(1+\mu)u_{0j} - u_{1j} - \mu u_{0,j-1} - \mu u_{0,j+1} \quad \text{...16}$$

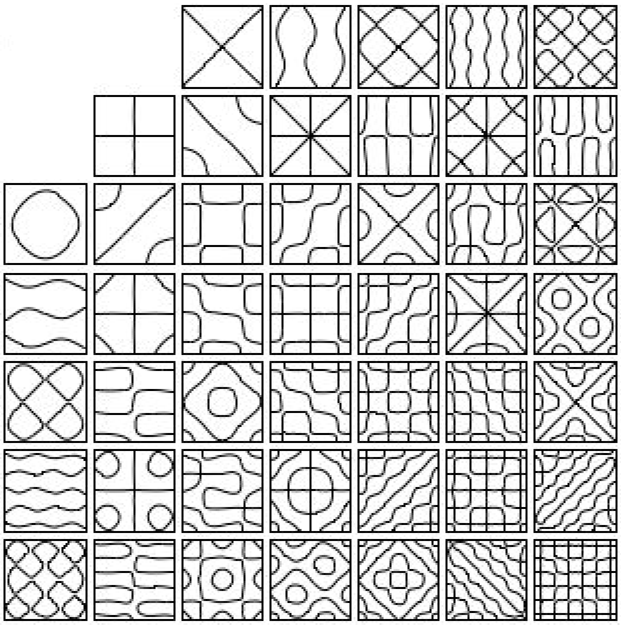
Similar relations are then obtained for ghost points away from the corner. Near the corner, there will be a coupling between the two ghost points attached to the corner[#1]

$$\begin{aligned} u_{-1,0} + \mu u_{0,-1} &= 2(1+\mu)u_{00} - u_{10} - \mu u_{01} \\ \mu u_{-1,0} + u_{0,-1} &= 2(1+\mu)u_{00} - \mu u_{10} - u_{01} \end{aligned} \quad \text{...17}$$

Since $\mu \neq 1$, we can solve a *2 × 2* system to obtain equations for $u_{-1,0}$ *and* $u_{0,-1}$ that depend only on u_{00} , u_{01} , and u_{10}.[#1] In the program the ghost points are eliminated more easily using a trick, more details of which will be sent with the code.

Result:

Computed chladni patterns :



Computed Frequencies:#1

λ values where λ = (freq.)²		
Computed values		Experiment #3
By program	From #3	
12.4	12.5	12.4
25.9	26.0	26.4
35.6	35.6	36.2
80.8	80.9	77.5
234.9	235.4	215.0
269.0	269.3	260.0
320.1	320.7	310.0
374.6	375.2	364.0
728.7	730.0	698.0

Conclusion:

From the results it is evident that there is good agreement between the computed results and those obtained experimentally.[#1]

References:

#1 *Chladni Figures and the Tacoma Bridge: Motivating PDE Eigenvalue Problems via Vibrating Plates,* Gander2012

#2 Image of pattern on water glass taken from
http://cymatica.com/wp-content/uploads/2010/03/dearprudence

#3 *Pg132-142, vol 720, Spectral theory and Applications, CRM Summer School July 4–14, 2016, Université Laval, Québec, Canada*

#4*From Euler, Ritz, and Galerkin to Modern Computing,Vol. 54,SIAM review,2020 gander et al*