MATHEMATICS

Prove that $E(\overline{X} - \mu)^2 = \frac{1}{n}\sigma^2$

Asked 4 years, 6 months ago Active 4 years, 6 months ago Viewed 3k times

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How to prove $E(\overline{X} - \mu)^2 = \frac{1}{n}\sigma^2$ (from wiki), where $\overline{X}$ - is the sample mean ?
What I have so far:

$$\begin{aligned}E(\overline{X} - \mu)^2 &= \frac{1}{n}\sigma^2 = \\ &= E(\overline{X}^2 + \mu^2 - 2\mu\overline{X}) = E(\overline{X}^2) - \mu^2 = \\ &= \frac{\sum_{i=1}^n \sum_{j=1}^n E(X_i X_j)}{n^2} - \mu^2\end{aligned}$$

Could you give me some tips how to proceed?

statistics means

edited Jul 17 '15 at 12:24

asked Jul 16 '15 at 16:40
Temak
113 1 6

2 Answers

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Let $\{X_i\}_{1\le i\le n}$ be n independent random variables with mean μ and variance σ^2. We need 2 results

1) $var(aX_i) = a^2 \times var(X_i)$

2) $var(\sum_{i=1}^n X_i) = \sum_{i=1}^n var(X_i)$.

Also, Expectation is a linear operator. That is

$E[aX + bY] = aE[X] + bE[Y]$

Hence, $E[\bar{X}] = E[\frac{\sum_{i=1}^n X_i}{n}] = \frac{1}{n}\sum_{i=1}^n E[X_i] = \frac{1}{n}\sum_{i=1}^n \mu = \mu$.

Hence,

$$\begin{aligned}E(\bar{X} - \mu)^2 &= E(\bar{X} - E[\bar{X}])^2 = var(\bar{X}) \\ &= var\left(\frac{\sum_{i=1}^n X_i}{n}\right) \\ &= \frac{1}{n^2} var\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^n var\left(X_i\right) \\ &= \frac{n\sigma^2}{n^2} \\ &= \frac{\sigma^2}{n}\end{aligned}$$

answered Jul 16 '15 at 17:09
user2808118
648 3 10

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We know that $E(X^2) = [E(X)]^2 + \mathrm{Var}(X)$, so

$$\begin{aligned}E((\overline{X} - \mu)^2) &= [E(\overline{X} - \mu)]^2 + \mathrm{Var}(\overline{X} - \mu) \\ &= \mathrm{Var}(\overline{X}) \\ &= \mathrm{Var}\left(\frac{1}{n}(X_1 + \cdots + X_n)\right) \\ &= \frac{n\sigma^2}{n^2},\end{aligned}$$

where we use the following:

$$E(\overline{X}) = \mu, \qquad Var(a + X) = Var(X), \qquad \mathrm{Var}(aX) = a^2\mathrm{Var}(X),$$

and the independence of the X_i so that the variance adds linearly.

answered Jul 16 '15 at 16:54
user217285
5,177 1 10 28