Introduction

The discrete time Fourier Transform can be found by taking the continuous time (CT) Fourier Transform of a sampled signal. The basic DTFT is mostly straight forward, but there are a few subtle points considered in this handout.

- What are the steps in finding the DTFT using CTFT operations?
- What is the difference between ω in the CTFT and Ω in the DTFT?
- How does the sample period T effect the height of the DTFT?
- In general the DTFT is just the CTFT of the sampled signal multiplied by T, and frequency normalized. Why does this rule not work for FT which contain $\delta(\omega)$?

Deriving discrete time Fourier Transform

Notation: the continuous time Fourier Transform $X(j\omega)$ has frequency $-\infty < \omega < \infty$. The normalized frequency Ω for the DTFT $X(e^{j\Omega})$ has $0 \leq \Omega < 2\pi$ (or any period of 2π).

Consider a continuous time signal $x(t)$ with Fourier Transform $X(j\omega)$. The signal is sampled at rate T_s, giving

$$x_\delta(t) \; = \; x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \; = \; \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \tag{1}$$

The CT spectrum for the sampled signal (using FT of the product) is:

$$\begin{aligned} \mathcal{F}\{x_\delta(t)\} &= X_\delta(j\omega) && (2) \\ &= X(j\omega) * \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{k2\pi}{T_s}) && (3) \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j\omega - \frac{k2\pi}{T_s}) && (4) \end{aligned}$$

or, taking FT of each term in the sum for $x_\delta(t)$ (eq: 1):

$$X_\delta(j\omega) \; = \; \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j\omega n T_s}$$

To calculate the frequency normalized DTFT, let $\Omega = \omega T_s$. (Check that $\omega = \frac{2\pi}{T_s}$ corresponds to $\Omega = 2\pi$.) Thus

$$X_\delta(\frac{j\Omega}{T_s}) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j\Omega n}$$

This is just the DTFT analysis equation. The frequency normalization can cause some issues when the FT contains $\delta(\omega)$ as shown in Example 1 and 2 below.

$$X(e^{j\Omega}) \; = \; \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = X_\delta(\frac{j\Omega}{T_s})$$

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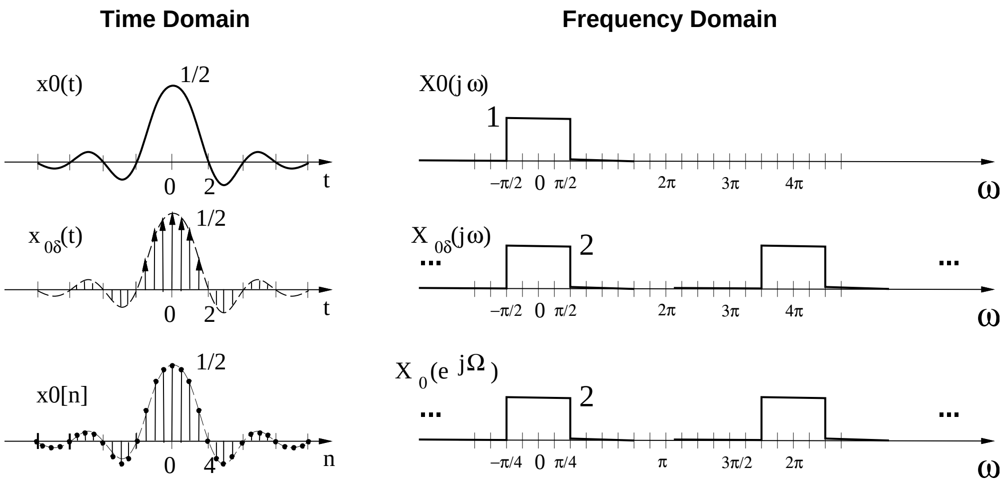


Example 0

Let $x_0(t) = \frac{\sin \pi t/2}{\pi t}$, then $X_0(j\omega) = \Pi(\frac{\omega}{\pi})$. With sampling rate T_0 the sampled signal $x_0(nT_0) = \frac{\sin \pi n T_0/2}{\pi n T_0}$ has FT:

$$
\begin{aligned}
X_{0\delta}(j\omega) &= X_0(j\omega) * \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{k2\pi}{T_0}) && (5)\\
&= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} X_0(j\omega - \frac{k2\pi}{T_0}) && (6)\\
&= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \Pi(\frac{\omega - \frac{k2\pi}{T_0}}{\pi}) && (7)
\end{aligned}
$$

Substituting $\omega = \Omega/T_0$, to normalize frequency range, eq. 7 becomes

$$
\begin{aligned}
X_{0\delta}(j\frac{\Omega}{T_0}) &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \Pi(\frac{\frac{\Omega}{T_0} - \frac{k2\pi}{T_0}}{\pi}) && (8)\\
&= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \Pi(\frac{\Omega - k2\pi}{\pi T_0}) && (9)
\end{aligned}
$$

We can show that the FT of the sampled signal $x_{0\delta}(t)$ is the same as the DTFT of $x_0[n]$, that is that $X_{0\delta}(j\frac{\Omega}{T_0}) = X_0(e^{j\Omega})$. The DTFT is

$$
\begin{aligned}
X_0(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x_0[n] e^{-j\Omega n} && (10)\\
&= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \frac{\sin \frac{\pi T_0}{2} n}{\pi n} e^{-j\Omega n} && (11)\\
&= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \Pi(\frac{\Omega - 2\pi k}{\pi T_0}) && (12)
\end{aligned}
$$

Where $x_0[n]$ was found from the IDTFT of $X_0(e^{j\Omega})$ over one period which is $\frac{1}{T_0}\Pi(\frac{\Omega}{\pi T_0})$. Thus

$$
x_0[n] = \frac{1}{2\pi} \int_{\frac{-\pi T_0}{2}}^{\frac{\pi T_0}{2}} \frac{1}{T_0} e^{j\Omega n} d\Omega \;=\; \frac{1}{2\pi T_0} \frac{2 \sin \frac{\pi T_0}{2} n}{n} \;=\; \frac{\sin \frac{\pi T_0}{2} n}{\pi n T_0} \qquad (13)
$$

We have shown that $X_{0\delta}(j\frac{\Omega}{T_0}) = X_0(e^{j\Omega})$. Also the result agrees with Table 5.2. This example is shown in the figure above, with $T_0 = 1/2$.

Example 1

Let $x_1(t) = \cos(2\pi t)$ with sampling rate $T_1 = \frac{1}{8}$. Then $X_1(j\omega) = \pi(\delta(\omega - 2\pi) + \delta(\omega + 2\pi))$. The sampled signal has FT:

$$
\begin{aligned}
X_{1\delta}(j\omega) &= \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X(j\omega - \frac{k2\pi}{T_1}) && (14) \\
&= 8\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi - \frac{k2\pi}{T_1}) + \delta(\omega + 2\pi - \frac{k2\pi}{T_1}) && (15)
\end{aligned}
$$

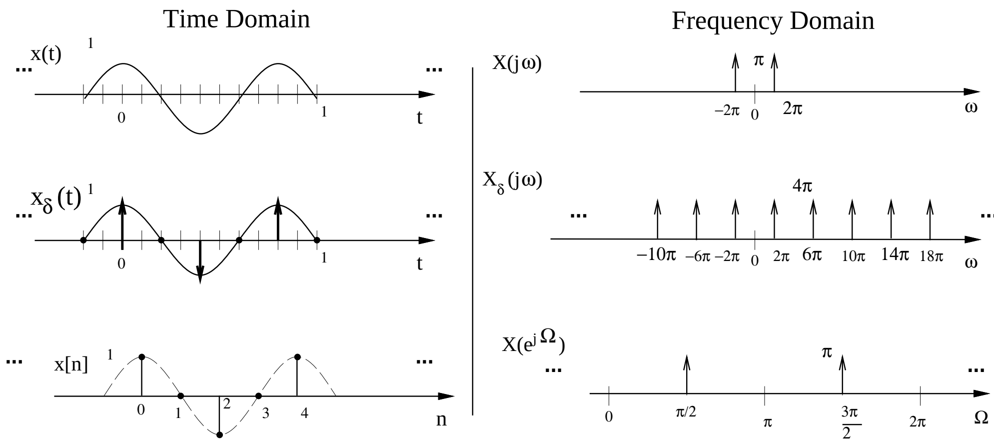
Substituting $\omega = \Omega/T_1$, eq. 15 becomes

$$
\begin{aligned}
X_{1\delta}(j\frac{\Omega}{T_1}) &= 8\pi \sum_{k=-\infty}^{\infty} \delta(\frac{\Omega}{T_1} - 2\pi - \frac{k2\pi}{T_1}) + \delta(\frac{\Omega}{T_1} + 2\pi - \frac{k2\pi}{T_1}) && (16) \\
&= 8\pi \sum_{k=-\infty}^{\infty} \delta(\frac{\Omega - 2\pi T_1 - k2\pi}{T_1}) + \delta(\frac{\Omega + 2\pi T_1 - k2\pi}{T_1}) && (17)
\end{aligned}
$$

Now since $\delta(at) = \frac{1}{|a|}\delta(t)$, $\delta(\frac{\Omega + 2\pi T_1 - k2\pi}{T_1}) = T_1 \delta(\Omega + 2\pi T_1 - k2\pi)$.
Then

$$
\begin{aligned}
X_{1\delta}(j\frac{\Omega}{T_1}) &= 8\pi T_1 \sum_{k=-\infty}^{\infty} \delta(\Omega + 2\pi T_1 - k2\pi) + \delta(\Omega - 2\pi T_1 - k2\pi) && (18) \\
&= \pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \frac{\pi}{4} - k2\pi) + \delta(\Omega - \frac{\pi}{4} - k2\pi) && (19) \\
&= X_{1\delta}(e^{j\Omega}). && (20)
\end{aligned}
$$

This agrees with Table 5.2 in OW.



Example 2
Let $x_2(t) = \cos(2\pi t)$ with sampling rate $T_2 = \frac{1}{4}$. Then $X_2(j\omega) = \pi(\delta(\omega - 2\pi) + \delta(\omega + 2\pi))$. The sampled signal has FT:

$$X_{2\delta}(j\omega) = 4\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi - \frac{k2\pi}{T_2}) + \delta(\omega + 2\pi - \frac{k2\pi}{T_2}) \tag{21}$$

Substituting $\omega = \Omega/T$, eq. 21 becomes

$$\begin{aligned} X_{2\delta}(j\frac{\Omega}{T}) &= 4\pi T_s \sum_{k=-\infty}^{\infty} \delta(\Omega + 2\pi T_2 - k2\pi) + \delta(\Omega - 2\pi T_2 - k2\pi) && (22) \\ &= \pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \frac{\pi}{2} - k2\pi) + \delta(\Omega - \frac{\pi}{2} - k2\pi) && (23) \\ &= X_{2\delta}(e^{j\Omega}). && (24) \end{aligned}$$

This agrees with Table 5.2 in OW.

Comparing example 1 and example 2, it is seen that the DTFT amplitude does not depend on sample period T_i for this example, due the scaling property of the $\delta()$ function.