STARDUST-R ACTIVITY ON DYNAMICAL SYSTEMS AND MACHINE LEARNING

ALESSANDRA CELLETTI, CHRISTOS EFTHYMIOPOULOS,
CATALIN GALES, GIUSEPPE PUCACCO, AND VICTOR RODRIGUEZ

1. The forced pendulum

1.1. The equations of motion. The *forced pendulum* is described by the one-dimensional, time-dependent Hamiltonian system:

$$H(y,x,t) = \frac{y^2}{2} - \varepsilon\, \Big(\cos(x) + \cos(x-t)\Big)\ , \tag{1.1}$$

where $(y,x) \in \mathbb{R} \times \mathbb{T}$ and $\varepsilon \in \mathbb{R}_+$ is a parameter.

Hamilton's equations associated to (1.1) are given by

$$\begin{aligned} \dot y &= -\varepsilon\, \Big(\sin(x) + \sin(x-t)\Big) \\ \dot x &= y\ . \end{aligned} \tag{1.2}$$

The above equations are integrable for $\varepsilon = 0$, since they reduce to

$$\begin{aligned} \dot y &= 0 \\ \dot x &= y\ , \end{aligned} \tag{1.3}$$

which can be integrated as

$$\begin{aligned} y(t) &= y(0) \\ x(t) &= x(0) + y(0)\, t\ . \end{aligned} \tag{1.4}$$

For $\varepsilon \neq 0$, Hamilton's equations (1.2) are non-integrable and display regular and chaotic motion. Such motions are conveniently seen when computing the so-called *Poincaré map*, which consists in plotting the intersection of the solution with the plane $t = 0$ mod. 2π (notice that the system is periodic in time).

As shown by the Poincaré maps, *regular* motions can be either *librational* (typically arising as closed curves around elliptic equilibrium points) and *rotational* (spanning the whole interval $[0, 2\pi)$ in the x variable).

1.2. **Rotational, librational, chaotic motion.** This part of the activity is aimed to familiarize with rotational, librational, chaotic dynamics by integrating (1.2) and plotting the Poincaré map for suitable initial conditions.

Integrate (1.2) with a modified Euler's method (e.g. with MATLAB, C or any other language); this gives a map of the form:

$$
\begin{aligned}
y' &= y + h\,\Big(- \varepsilon\,(\sin(x) + \sin(x-t))\Big) \\
x' &= x + h\,y' \\
t' &= t + h\ ,
\end{aligned}
$$

where x should be computed mod. 2π and h is the time step.

We can take $h = \frac{2\pi}{10^3}$, $\varepsilon = 0.025$ and plot the Poincaré map when the solution crosses the plane $t = 0$ mod. 2π with an accuracy of 10^{-3}.

Iterate for $2 \cdot 10^5$ times and plot the points that cross the Poincaré section in the plane (x, y).

You should find that:

(i) the initial conditions $(x_0, y_0) = (\pi, 1.4)$ correspond to a rotational motion;

(ii) the initial conditions $(x_0, y_0) = (0.1, 1.2)$ correspond to a librational motion;

(iii) the initial conditions $(x_0, y_0) = (\pi, 0.945)$ correspond to a chaotic motion.

1.3. **The Poincaré map.** Once you have familiarized with the rotational, librational and chaotic motions, make a program that computes the Poincaré map for several initial conditions and precisely for $\varepsilon = 0.025$, iterating for 10^5 times with time step $h = \frac{2\pi}{10^3}$, accuracy 10^{-3} of the crossing of the Poincaré map, and the following sets of initial conditions (draw all on the same plot):

(a) $(x_0, y_0) = (0.1, 0.123\,k)$ for $k = 1, ..., 9$;
(b) $(x_0, y_0) = (1.1, 0.123\,k)$ for $k = 1, ..., 9$;
(c) $(x_0, y_0) = (2.1, 0.123\,k)$ for $k = 1, ..., 9$;
(d) $(x_0, y_0) = (3.1, 0.123\,k)$ for $k = 1, ..., 9$.

2. The space debris problem

Being subjected to various perturbations, space debris exhibit complex dynamics that can described, in many cases, by reducing the general theory to simple toy models similar to forced pendulums.

Let us consider the motion of a point-mass body (space debris) under the influence of the Earth's oblateness and rotation. Within the Hamiltonian formalism, the motion is described by the Hamiltonian given by equation (3.2) of [1], which is expressed in terms of the Delaunay variables $L, G, H, M, \omega, \Omega$ and the sidereal time θ, where the Delaunay actions are related to the well known orbital elements (a, e, i) through the relation (3.1) of [1], M is the mean anomaly, ω is the argument of perigee and Ω is the longitude of the ascending node.

Focusing on the GEO 1:1 and MEO 2:1 resonances, which occur when $\dot{M} - \dot{\theta} \simeq 0$, and respectively $\dot{M} - 2\dot{\theta} \simeq 0$, namely at around 42164 km and respectively at 26560 km from the center of the Earth, the dynamics can be studied by a reduced (averaged) Hamiltonian whose secular part is given by the relation (3.9) of [1] and resonant part is provided in Section 3.2.2, pages 1240, 1241 of [1] for the 1:1 resonance, and respectively in Section 3.2.3, pages 1243, 1244 of [1] for the 2:1 resonance.

Integrating the canonical equations and the associated variational equations, we derive the Fast Lyapunov Indicator up to time $T = 19$ years (see Section 5.1 of [1] for more details on this chaos indicator). We compute a grid of 100x100 points of the $\sigma - a$ plane, where for the 1:1 resonance the critical angle $\sigma = \lambda = M - \theta + \omega + \Omega$ ranges in the interval $[-220^o, 180^o]$ and the semimajor axis a spans the domain $[42120, 42220]$ km, while for the 2:1 resonance the critical angle $\sigma = M - 2\theta + \omega + 2\Omega$ varies in the interval $[-200^o, 300^o]$, while the semimajor axis a covers the domain $[26545, 26590]$ km.

The files $left\_fig3\_index012.plt$, $right\_fig3\_index012.plt$, linked to the 1:1 resonance, and $left\_fig9\_index012.plt$, $right\_fig9\_index012.plt$ obtained for the 2:1 resonance (which can be downloaded from [2]) provide the data computed by integrating the canonical equations and the associated variational equations for the following initial conditions:

a) for the 1:1 resonance: $i = 0^o$, $e = 0.005$, $\omega = 0$ and $\Omega = 0$. The data in the file $left\_fig3\_index012.plt$ are obtained for a model that takes into account the effects of J_2 and J_{22}, while the data in the file $right\_fig3\_index012.plt$ are computed for a model that considers the effects of all harmonics up to degree and order $n = m = 3$;

b) for the 2:1 resonance: $i = 30^o$, $\omega = 0$, $\Omega = 0$ and respectively $e = 0.1$ ($left\_fig9\_index012.plt$) and $e = 0.5$ ($right\_fig9\_index012.plt$). The model takes into account the effects of all harmonics up to degree and order $n = m = 4$.

The (nine) columns of these four files provide the following data: 1) the resonant angle σ (in degrees); 2) the semi-major axis a (in km); 3)-8) the value of FLI after 1 year, 2, 3, 5, 10 and 19 years, respectively; 9) a (dynamical) index with the following meaning: 0–chaotic motion, 1–regular non–resonant (rotational) motion, 2–regular resonant (librational) motion;

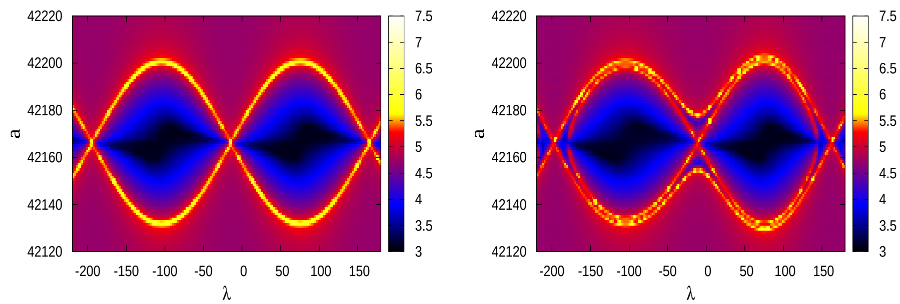

FIGURE 1. FLI (using Hamilton's equations) for the GEO 1:1 resonance for $e = 0.005$, $i = 0^o$, $\omega = 0$, $\Omega = 0$, under the effects of the J_2 and J_{22} terms (left); all harmonics up to degree and order $n = m = 3$ (right). (Reproduced from [1], see the top plots of Figure 3.)

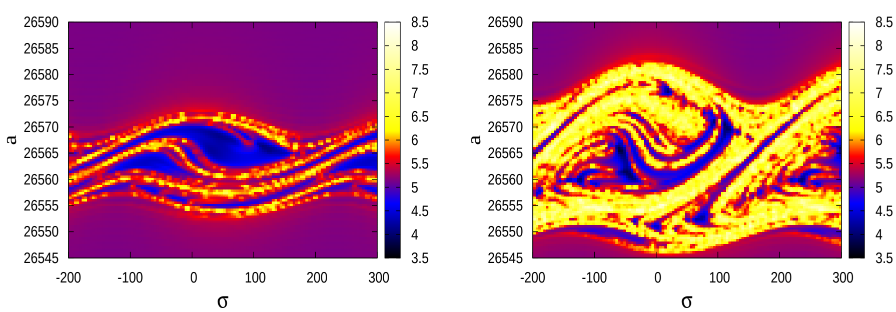

FIGURE 2. FLI (using Hamilton's equations) for the MEO 2:1 resonance, under the effects of all harmonics up to degree and order $n = m = 4$, for $i = 30^o$, $\omega = 0$, $\Omega = 0$: $e = 0.1$ (left); $e = 0.5$ (right). (Reproduced from [1], see the bottom plots of Figure 9.)

The values of FLI at $T = 19$ years (8th column) are used to make a cartography the phase space of the 1:1 and 2:1 resonances as in Figures 1 and 2 reproduced from [1] (top plots of Figure 3 for the GEO 1:1 resonance, and respectively bottom plots of Figure 9 for the MEO 2:1 resonance), where darker colors denote a regular dynamics (either rotational or librational motions), while lighter colors denote chaotic behavior. The values of FLI at $T = 19$ years (8th column) are also used to assign the (dynamical) index of the 9th column.

Machine learning algorithm are used in connection with the values of FLI provided in the columns 3)–6). Since these columns are obtained after integrating the equations for a very short interval of time, this approach could be a very fast and powerful tool to investigate various topics in the space debris dynamics.

3. A machine learning approach to detect chaos

Here we briefly explain the part corresponding to the machine learning application. This part follows the tabular classification model tutorial [1], but applies it to the problem of classifying types of space debris motions automatically, based on its initial conditions and the *FLI* after short intervals of time.

(1) Load the data corresponding to the right plot of Figure 9 in the reference paper [1] (file `right_fig9_index012.plt`) [2]. Merge the indices 1 (non–resonant) and 2 (resonant) into a single index that represents regular motions, to turn the problem into a binary classification (chaotic / regular). Check the class ratio. As you will see, this is the most balanced of the four datasets of this project. Fit a classifier that predicts the dynamical index based on the initial conditions and all the FLIs except FLI-19 (FLI-19 is used for the labelling). Then, produce a plot that shows how the accuracy of the classifier varies if less FLIs are used for the prediction. Do this incrementally, starting by dropping FLI-10, then FLI-10 and FLI-5, until you have dropped all of them. Discuss what would be a good trade-off between the accuracy and the required interval of FLIs.

(2) Load the data corresponding to the right plot of Figure 3 in the reference paper [1] (file `right_fig9_index012.plt`). Merge the

[1]The tutorial can be found in the Github repository of this project `https://github.com/stardust-r/LTW-I/tree/master/ML_space_debris_motions`

[2]In Python, `plt` files can be read with `pandas` as if they were csv files, with the separator `sep='`
`s+'`. Column names are not provided within the file.

indices 1 (non–resonant) and 2 (resonant) into a single index that represents regular motions, to turn the problem into a binary classification (chaotic / regular). Check the class ratio. As you will see, this is the least balanced of the four datasets of this project. Fit a classifier that predicts the dynamical index based on the initial conditions and the FLIs at $T = 1$, $T = 2$ and $T = 3$. Check the final accuracy of the classifier and its confusion matrix. Discuss whether one can state that this is a good classifier based on the accuracy. Re-fit the model using *recall* and *F2* as additional metrics. Try to improve these values by using one of these approaches for class imbalanced problems (if possible, make a comparison of the performance of each of them):

- Adjusting class weights in the loss function.
- Oversampling (any of the techniques available in the `imblearn` package).
- Undersampling (any of the techniques available in the `imblearn` package).

Note that it for this exercise it is important that your validation set has the preserves the class balance ratio of the training set. You can ensure this enabling the stratification while splitting the original data (see an example in the tutorial).

(3) (Optional) Choose any of the datasets provided in the project (the ones for Figure 9 should be easier to study), and fit a multi-class classifier that recognizes resonant, non resonant and chaotic motions, using FLIs at $T = 1$, $T = 2$, and $T = 3$. Interpret the resulting classifier to get the most important features for predicting each of the classes, making a special focus on the effect of the initial conditions on those predictions.

References

[1] A. Celletti, C. Galeş, *On the dynamics of space debris: 1:1 and 2:1 resonances,* J. Nonlinear Science **24**, n. 6, 1231–1262 (2014)

[2] $https://1drv.ms/u/s!AmQS4GfsC6_1h1cu9bIFzuBqGX39?e=qKgOhY$