Definite Integrals of Negative Functions

by Sophia

WHAT'S COVERED

In this lesson, you will connect the ideas from Riemann sums, integrals, and regions that are below the x-axis. Specifically, this lesson will cover:

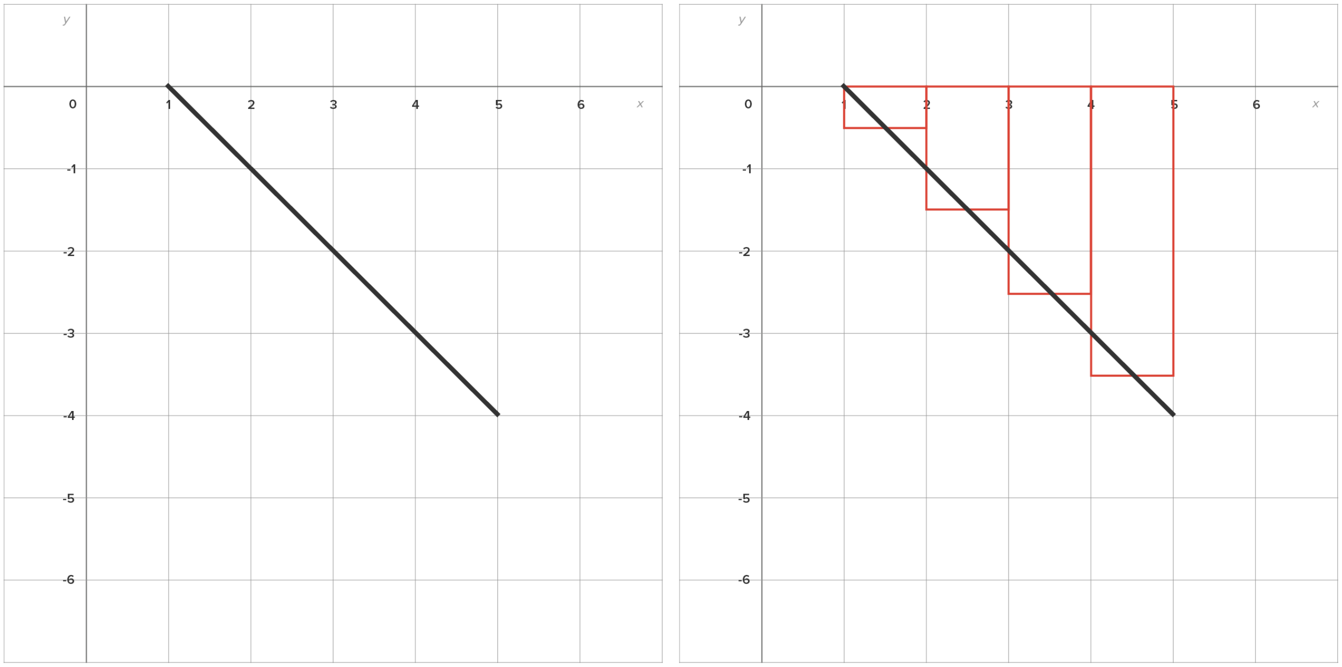
1. Riemann Sums of Functions That Are Below the x-Axis

Up until now, we only integrated functions that were above the x-axis on $[a, b]$. We'll use this example to see what happens when that is not the case.

Consider the function $f(x) = 1 - x$ on the interval $[1, 5]$.

The graph on the left is $f(x)$ on the interval $[1, 5]$, and the graph on the right shows the rectangles that could be used in a Riemann sum. Remember, the rectangles have a base on the x-axis and extend out to the graph of $f(x)$.

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.



Now consider a Riemann sum, $\sum_{k=1}^{n} f(c_k)\Delta x$.

- When $f(c_k)$ is positive, we know the quantity $f(c_k)\cdot\Delta x$ is the area of one rectangle.
- As a result, when $f(c_k)$ is negative, the quantity $f(c_k)\cdot\Delta x$ is the negative of the area of that one rectangle.

Now picture adding these quantities, all of which are negative. The Riemann sum would be an estimate for the negative of the area.

Now, consider the definite integral of this function on the interval $[1, 5]$, written $\int_1^5 (1-x)dx$.

We know the value of this integral is the limit of the Riemann sums as the number of rectangles gets larger and larger $(n \to \infty)$.

Note that the area of the region between $f(x)$ and the x-axis is $\frac{1}{2}(4)(4) = 8 \text{ units}^2$.

Then, $\int_1^5 (1-x)dx = -8$, the negative of the area of the region.

BIG IDEA

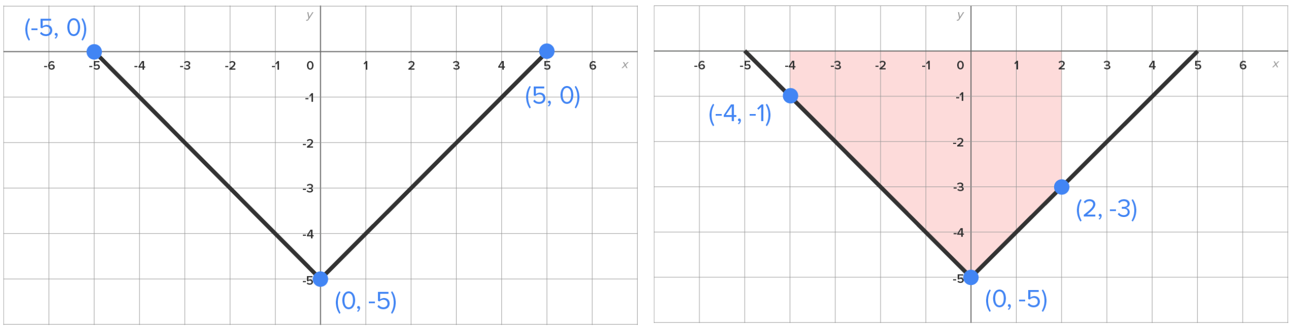
If the graph of $f(x)$ is below the x-axis on $[a, b]$, then $\int_a^b f(x)dx$ is the negative of the area of the region between $f(x)$ and the x-axis on $[a, b]$.

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.

2. Evaluating Definite Integrals When $f(x) \leq 0$ on $[a, b]$

➩ EXAMPLE Evaluate $\int_{-4}^{2} (|x| - 5)dx$.

The graph of $f(x)$ is shown on the left and the region is shown on the right.



Note that the graph of $f(x)$ is below the x-axis on the interval $[-4, 2]$. The region itself is not a standard shape, so let's split the region at $x = 0$.

On $[-4, 0]$, the region is a trapezoid with parallel (vertical) bases 1 and 5, and (horizontal) height 4. The area is $\frac{1}{2}(4)(1+5) = 12 \text{ units}^2$.

On $[0, 2]$, the region is a trapezoid with parallel (vertical) bases 5 and 3, and (horizontal) height 2. The area is $\frac{1}{2}(2)(5+3) = 8 \text{ units}^2$.

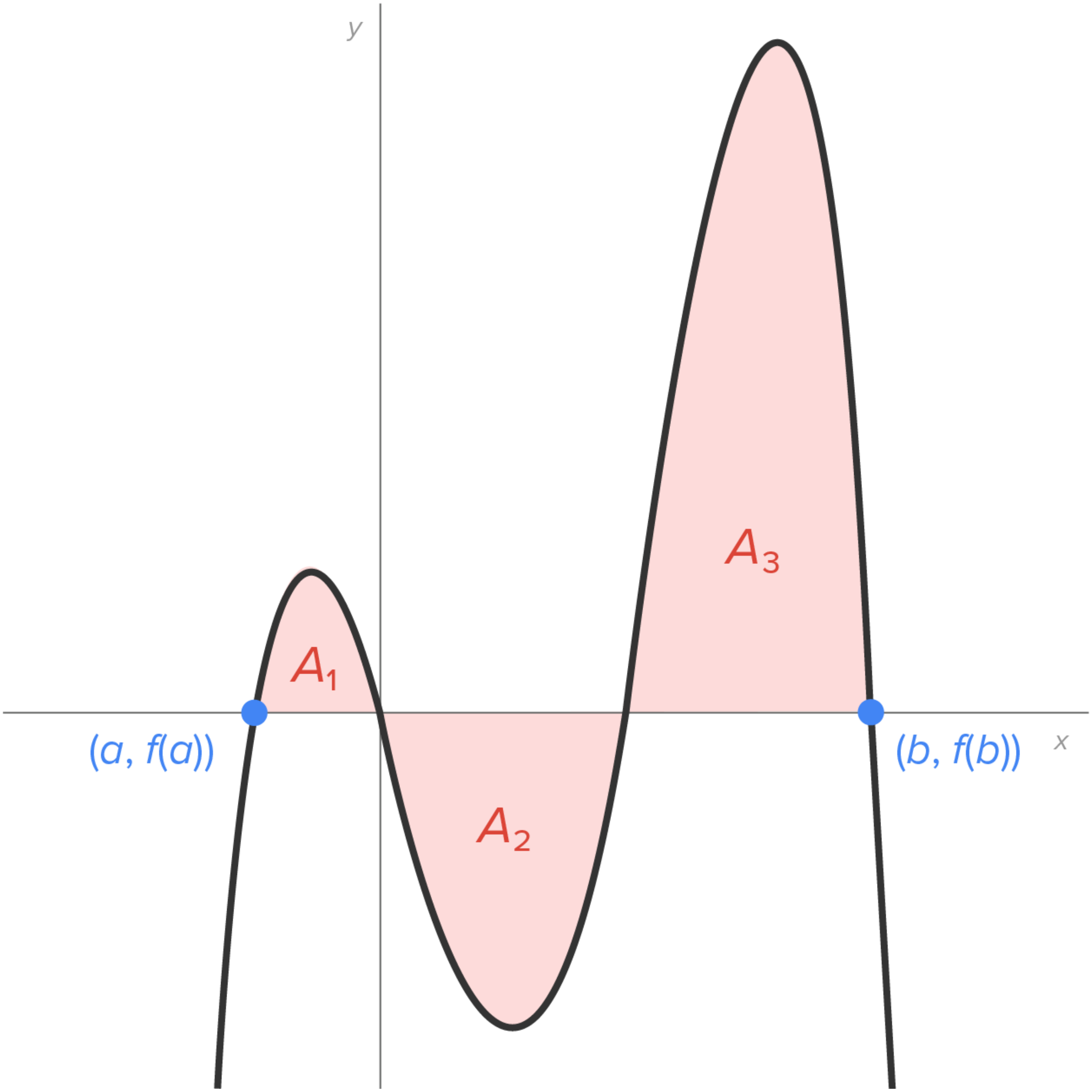
Then, the total area is 20 units^2.

Since the region is completely below the x-axis on $[-4, 2]$, $\int_{-4}^{2} (|x| - 5)dx = -20$.

3. Evaluating Definite Integrals When $f(x)$ Is Both Negative and Positive on $[a, b]$

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.

Suppose we wish to evaluate $\int_a^b f(x)dx$ for the function whose graph is shown in the figure.



Notice how this region is broken into 3 smaller regions with areas A_1, A_2, and A_3.

Now, consider the definite integral on $[a, b]$.

- For the region with area A_1, the definite integral is equal to A_1 since the region is above the x-axis.
- For the region with area A_2, the definite integral is equal to $-A_2$ since the region is below the x-axis.
- For the region with area A_3, the definite integral is equal to A_3 since the region is above the x-axis.

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.

Thus, the definite integral over $[a, b]$ is equal to the sum of the three definite integrals, or $A_1 - A_2 + A_3$. In general, we would add any area above the x-axis and subtract any area below the x-axis.

BIG IDEA

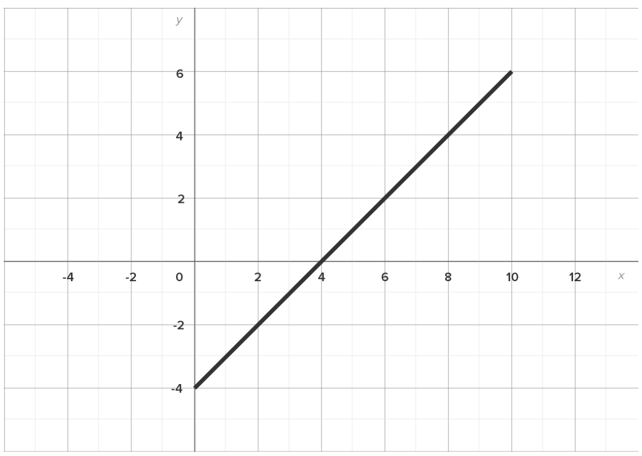
If $f(x)$ takes on both positive and negative values on $[a, b]$, then:

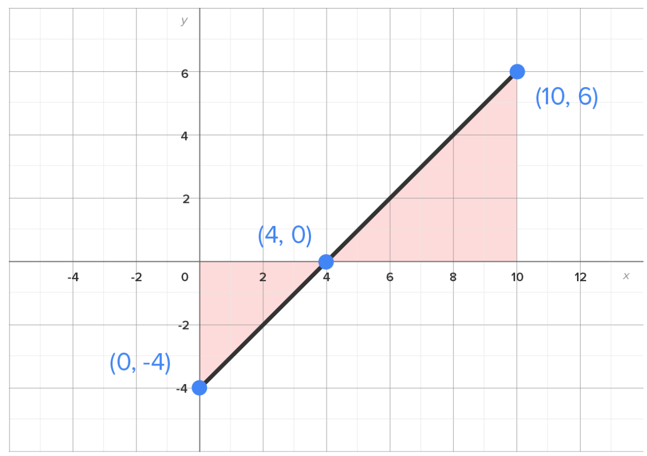
$\int_a^b f(x)dx =$ (sum of all areas above the x-axis) - (sum of all areas below the x-axis)

Let's look at an example.

EXAMPLE Evaluate $\int_0^{10} (x-4)dx$.

Consider the graph of $f(x) = x-4$ on the interval $[0, 10]$, as shown in the figure on the left. The figure on the right shows the graph with the relevant regions.





The triangle between $x=0$ and $x=4$ has area $\frac{1}{2}(4)(4) = 8$, and is below the x-axis.

The triangle between $x=4$ and $x=10$ has area $\frac{1}{2}(6)(6) = 18$, and is above the x-axis.

Then, $\int_0^{10} (x-4)dx = -8+18 = 10$.

WATCH

Given the graph of $y = f(x)$, we'll find $\int_0^6 f(x)dx$.

BIG IDEA

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.

In this context, the definite integral can be thought of as a “net area.” If $\int_a^b f(x)dx > 0,$ there is more area above the x-axis than below the x-axis on $[a, b]$.

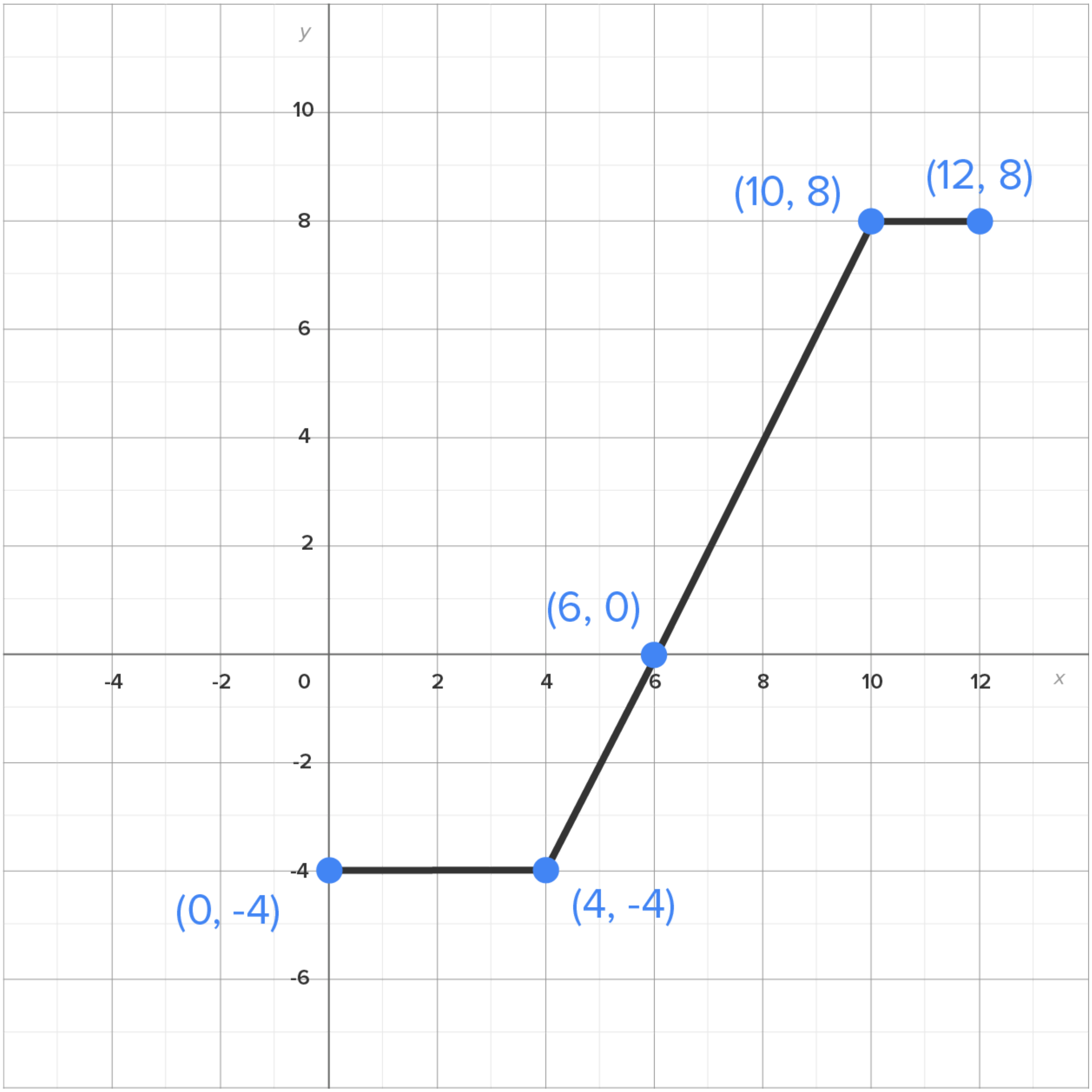
If $\int_a^b f(x)dx < 0,$ there is more area below the x-axis than above the x-axis on $[a, b]$.

If $\int_a^b f(x)dx = 0,$ there is as much area above the x-axis as there is below the x-axis on $[a, b]$.

TRY IT

Consider the graph of $f(x)$ as shown in the figure below that can be used to evaluate $\int_0^{12} f(x)dx$.

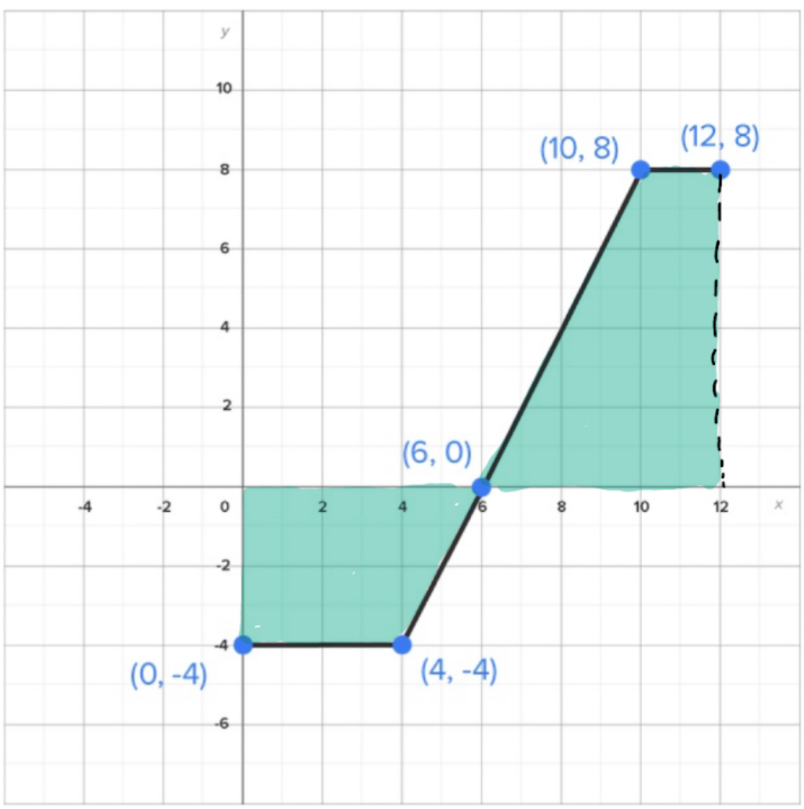
© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.



Evaluate this integral. +

To evaluate this integral, we must pay attention to the graph's location relative to the x-axis. Here is the graph, but with the regions shaded.

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.



Consider the region between $x = 0$ and $x = 6$. This region is in the shape of a trapezoid with bases 4 and 6 and height 4. Its area is $A_1 = \frac{4}{2}(4 + 6) = 20 \text{ units}^2$.

Now consider the region between $x = 6$ and $x = 12$. This is also a trapezoid, but with bases 2 and 6, and with height 8. Its area is $A_2 = \frac{8}{2}(2 + 6) = 32 \text{ units}^2$.

Since the first region is below the x-axis and the second region is above the x-axis, the value of the definite integral is $-A_1 + A_2 = -20 + 32 = 12$.

SUMMARY

In this lesson, you learned about the **Riemann sums of functions that are below the x-axis**, understanding how to evaluate $\int_a^b f(x)dx$ when the graph of f(x) is below the x-axis. You applied this knowledge in **evaluating definite integrals when $f(x) \le 0$ on** [*a*, *b*]. Lastly, you learned that when **evaluating definite integrals when $f(x)$ is both negative and positive on** [*a*, *b*], you can interpret the value of the definite integral as “net area,” considering regions that are above and below the x-axis. This will be very useful when investigating applications in the next tutorial.

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.

Source: THIS TUTORIAL HAS BEEN ADAPTED FROM CHAPTER 4 OF "CONTEMPORARY CALCULUS" BY DALE HOFFMAN. ACCESS FOR FREE AT **WWW.CONTEMPORARYCALCULUS.COM**. LICENSE: **CREATIVE COMMONS ATTRIBUTION 3.0 UNITED STATES**.

© 2025 SOPHIA Learning, LLC. SOPHIA is a registered trademark of SOPHIA Learning, LLC.