Changing the Variable: u-substitution with Power Rule

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WHAT'S COVERED

In this lesson, you will explore indefinite integrals of composite functions. Specifically, this lesson will cover:

1. Introduction to *u*-Substitution/Review of Chain Rule

One way to view *u*-substitution is "undoing the chain rule." Consider the function $f(x)=(x^2+1)^9$, which is a composite function.

Taking the derivative, we get $f'(x)=9(x^2+1)^8\cdot 2x=18x(x^2+1)^8$.

It follows that $f(x)=(x^2+1)^9+C$ is the antiderivative of $f'(x)=18x(x^2+1)^8$.

In other words, $\int 18x(x^2+1)^8 dx=(x^2+1)^9+C$.

The big question: How do we get the antiderivative without having to guess? The answer to this question lies in the very way we find the derivative.

Let's look again at $f(x)=(x^2+1)^9$.

When we first learned the chain rule, we let $u=x^2+1$. Then, $f(u)=u^9$, and $f'(u)=9u^8\cdot\frac{du}{dx}$.

As you can see, the derivative is less complicated when the "*u*" is used. You can focus on the "inner" function. While you may have gotten used to not using the "*u*" idea in derivatives, it is very useful for antiderivatives. Let's walk through an example.

EXAMPLE Find the indefinite integral: $\int \sqrt{4x^2+3} \cdot 8x dx$

In the spirit of the chain rule, let $u = 4x^2+3$.

Since a chain rule derivative also contains $\frac{du}{dx}$, we'll find that as well: $\frac{du}{dx} = 8x$

Notice that the integrand is written in differential form (a function multiplied by *dx*).

To align with that, we'll write $\frac{du}{dx} = 8x$ in differential form: $du = 8x dx$

The goal is to get an integral that has *u* as its only variable. With these substitutions, $4x^2+3$ gets replaced by *u* and $8x dx$ is replaced by *du*.

This means the integral can now be written as $\int \sqrt{u}\, du$, which is much simpler, and it is clearer what to do:

$\int \sqrt{u}\, du$	Start with the original expression.
$= \int u^{1/2} du$	Rewrite as a power so that the power rule can be used.
$= \frac{u^{3/2}}{\left(\frac{3}{2}\right)} + C$	Apply the power rule with $n = \frac{1}{2}$.
$= \frac{2}{3} u^{3/2} + C$	Simplify.
$= \frac{2}{3}(4x^2+3)^{3/2} + C$	Replace *u* with $4x^2+3$.

The last step is the key step. The original function was in terms of *x*, which means that the final answer should also be in terms of *x*. The *u*-substitution was more or less used to help us to get organized.

Thus, $\int \sqrt{4x^2+3} \cdot 8x dx = \frac{2}{3}(4x^2+3)^{3/2} + C$.

This is the essence with *u*-substitution. When you identify an antiderivative that requires *u*-substitution, here is what you do:

STEP BY STEP

1. Identify the “inner” function, which is the function within another function. For now, it will be the function that is raised to a power (or under a radical). If we call the inside function $g(x)$, then let $u = g(x)$.
2. Find the differential $du = g'(x)dx$. This will only work if $g'(x)$ or some multiple of $g'(x)$ is in the integrand before the substitutions are made.
3. Substitute u and du into the integral so that the integral has u as its only variable.
4. Find the antiderivative with respect to u (and don’t forget $+C$!).
5. Replace $u = g(x)$ in the antiderivative. This is called back-substitution.

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2. Using *u*-Substitution With the Power Rule

Now that we have a process, let’s look at some examples.

EXAMPLE Find the indefinite integral: $\int 2x(x^2+3)^7 dx$

$\int 2x(x^2+3)^7 dx$	Start with the original expression.
$= \int u^7 du$	Make the substitution: $u = x^2+3$ Find the differential: $du = 2xdx$ Replace x^2+3 with u and $2xdx$ with du.
$= \frac{1}{8}u^8 + C$	Apply the power rule with $n = 7$.
$= \frac{1}{8}(x^2+3)^8 + C$	Back-substitute $u = x^2+3$.

Thus, $\int 2x(x^2+3)^7 dx = \frac{1}{8}(x^2+3)^8 + C$.

The next example will illustrate what happens when du is not exactly in the integral, but is a constant multiple.

WATCH

In this video, we will find $\int x^2(4x^3+5)^6 dx$.

EXAMPLE Find the indefinite integral: $\int \sqrt{8x+1}\, dx$

$\int \sqrt{8x+1}\, dx$	Start with the original expression.

$= \int \frac{1}{8}\sqrt{u}\,du$	First, make the substitution: $u = 8x + 1$ Write the differential: $du = 8dx$ Solve for *dx*: $dx = \frac{1}{8}du$ Replace $8x + 1$ with *u* and *dx* with $\frac{1}{8}du$.
$= \frac{1}{8}\int \sqrt{u}\,du$	Move the constant $\frac{1}{8}$ outside the integral sign.
$= \frac{1}{8}\int u^{1/2}du$	Rewrite as a power.
$= \frac{1}{8}\frac{u^{3/2}}{\left(\frac{3}{2}\right)} + C$	Use the power rule with $n = \frac{1}{2}$.
$= \frac{1}{12}u^{3/2} + C$	Simplify.
$= \frac{1}{12}(8x+1)^{3/2} + C$	Back-substitute $u = 8x + 1$.

Thus, $\int \sqrt{8x+1}\,dx = \frac{1}{12}(8x+1)^{3/2} + C$.

In the next example, we'll look at a power in the denominator.

EXAMPLE Find the indefinite integral: $\int \frac{2x+1}{(x^2+x+4)^5}dx$

$\int \frac{2x+1}{(x^2+x+4)^5}dx$	Start with the original expression.
$= \int \frac{1}{u^5}du$	First, make the substitution: $u = x^2 + x + 4$ Write the differential: $du = (2x+1)dx$ Replace $x^2 + x + 4$ with *u* and $(2x+1)dx$ with *du*. Note: *dx* is multiplied by the expression, which means it is multiplied by the numerator.
$= \int u^{-5}du$	Rewrite as a negative power so that the power rule can be used.
$= \frac{1}{-4}u^{-4} + C$	Apply the power rule with $n = -5$.
$= \frac{-1}{4}u^{-4} + C$	Simplify.
$= \frac{-1}{4}(x^2+x+4)^{-4} + C$	Back-substitute $u = x^2 + x + 4$.
$= \frac{-1}{4(x^2+x+4)^4} + C$	Write in terms of positive exponents if desired or directed.

Thus, $\int \frac{2x+1}{(x^2+x+4)^5}dx = \frac{-1}{4(x^2+x+4)^4}+C.$

TRY IT

Consider $\int \frac{4x}{(x^2+8)^{3/4}}dx.$

Find the indefinite integral. +

Notice that x^2+8 is the "inner function."

Let $u=x^2+8$, then $du=2xdx.$

Since $2x$ is not directly in the integral, rewrite $du=2xdx$ as $\frac{1}{2}du=xdx.$

Now replace x^2+8 with u and xdx with $\frac{1}{2}du.$

The new integral is $\int \frac{4\left(\frac{1}{2}du\right)}{u^{3/4}}=\int 2u^{-3/4}du.$

Now, evaluate using the power rule, then simplify:

$$\int 2u^{-3/4}du=2\left(\frac{1}{\left(\frac{1}{4}\right)}\right)u^{1/4}+C=8u^{1/4}+C$$

Lastly, replace u with x^2+8. The antiderivative is $8(x^2+8)^{1/4}+C.$

Finally, here is an example where there doesn't appear to be an inner function.

EXAMPLE Find the indefinite integral: $\int \frac{4x}{8x^2+1}dx$

At first glance, it looks like we could manipulate the integrand, but since the denominator has more than one term, this is not possible. Taking a closer look, notice that the substitution $u=8x^2+1$ has differential

$du = 16xdx$, which is a constant multiple of 4*x* (which is in the integral). This is the direction we'll go.

$\int \frac{4x}{8x^2+1}dx$	Start with the original expression.
$= \int \frac{1}{u} \cdot \frac{1}{4}du$	First, make the substitution: $u = 8x^2+1$ Write the differential: $du = 16xdx$ Solve for $x \cdot dx$: $xdx = \frac{1}{16}du$. Replace $8x^2+1$ with *u* and *dx* with $\frac{1}{16}du$. Note: *dx* is multiplied by the expression, which means it is multiplied by the numerator. Note: $4xdx = 4\left(\frac{1}{16}du\right) = \frac{1}{4}du$ We isolate *xdx* since the goal is to rewrite the integrand in terms of *u* and *du*.
$= \frac{1}{4}\int \frac{1}{u}du$	Move the constant $\frac{1}{4}$ outside the integral sign.
$= \frac{1}{4}\ln\|u\| + C$	Apply the natural logarithm rule.
$= \frac{1}{4}\ln\left\|8x^2+1\right\| + C$	Back-substitute $u = 8x^2+1$.
$= \frac{1}{4}\ln(8x^2+1)+C$	It is worth mentioning that since $8x^2+1$ is positive for all real numbers, there is no need to use absolute value. That said, it is not incorrect to use absolute value, but it is not necessary in this case.

Thus, $\int \frac{4x}{8x^2+1}dx = \frac{1}{4}\ln(8x^2+1)+C$.

The substitution method isn't exclusively used for reversing the chain rule. It can also be used to rewrite expressions that could not otherwise be manipulated for antidifferentiation.

EXAMPLE Find the indefinite integral: $\int x(x+3)^{15}dx$

Since we certainly don't want to multiply $(x+3)^{15}$ out, we'll try a substitution.

$\int x(x+3)^{15}dx$	Start with the original expression.
$= \int (u-3) \cdot u^{15}du$	Make the substitution: $u = x+3$ Use the differential form: $du = dx$ At this point, there is no replacement for the "*x*" in front. Since $x+3$ will be replaced with *u*, we need a replacement for *x*.

	From the substitution $u = x+3$, we can solve for *x* to obtain $x = u-3$. $x \to u-3,\ x+3 \to u,\ dx \to du$
$= \int (u^{16} - 3u^{15})du$	Distribute u^{15}.
$= \frac{1}{17}u^{17} - \frac{3}{16}u^{16} + C$	Apple the power rule and combine constants.
$= \frac{1}{17}(x+3)^{17} - \frac{3}{16}(x+3)^{16} + C$	Back-substitute $u = x+3$.

Thus, $\int x(x+3)^{15}dx = \frac{1}{17}(x+3)^{17} - \frac{3}{16}(x+3)^{16} + C$.

TRY IT

Consider $\int x(x-9)^{10}dx$.

Find the indefinite integral.

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This expression cannot be reasonably simplified, but notice that $x-9$ appears to be an "inner function".

Let $u = x-9$, then $du = dx$.

At this point, we can replace $x-9$ with *u*, and *dx* with *du*.

However, we also need to replace *x* with an expression. From the substitution we made, we can see that $x = u+9$.

The new integral is $\int (u+9)u^{10}du$.

The integrand can be simplified nicely. Perform the multiplication, then find the anti-derivative:

$$\int (u+9)u^{10}du = \int (u^{11} + 9u^{10})du = \frac{1}{12}u^{12} + 9\left(\frac{1}{11}\right)u^{11} + C = \frac{1}{12}u^{12} + \frac{9}{11}u^{11} + C$$

Lastly, remember to replace *u* with its corresponding expression in terms of *x*, which gives $\frac{1}{12}(x-9)^{12} + \frac{9}{11}(x-9)^{11} + C$.

Thus, $\int x(x-9)^{10}dx = \frac{1}{12}(x-9)^{12} + \frac{9}{11}(x-9)^{11} + C$.

WATCH

In this video, we'll find $\int \frac{(2+\ln x)^3}{x}dx$.

TRY IT

Consider $\int \frac{\ln x}{x}dx$.

Find the indefinite integral.

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First, write the integral as $\int \ln x \cdot \frac{1}{x}dx$.

Notice that $\frac{1}{x}$ is the derivative of $\ln x$. This helps us to see that the substitution $u = \ln x$ will work.

Let $u = \ln x$, then $du = \frac{1}{x}dx$. This means that we can replace $\ln x$ with *u* and $\frac{1}{x}dx$ with *du*.

The new integral is $\int u du$, which can be evaluated using the power rule:

$$\int u du = \frac{1}{2}u^2 + C$$

Remember to replace *u* with $\ln x$ for the final answer:

$$\frac{1}{2}(\ln x)^2 + C$$

Thus, $\int \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2 + C$.

SUMMARY

In this lesson, you learned about the ***u*-substitution** method, which is primarily used to reverse **the chain rule**. When the *u* is used, the derivative is less complicated, allowing you to focus on the “inner” function. You also practiced **using *u*-substitution with the power rule**. As you learned, *u*-substitution isn't exclusively used for reversing the chain rule; it can also be used to “rearrange” the expression so that it can be manipulated for antidifferentiation.