Changing the Variable: u-Substitution with Trigonometric Functions

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WHAT'S COVERED

In this lesson, you will continue using substitutions to find antiderivatives, but now we will focus on the trigonometric functions. Specifically, this lesson will cover:

1. The Inner Function Is Trigonometric
2. The Outer Function Is Trigonometric

1. The Inner Function Is Trigonometric

Since we have already been through *u*-substitution, there really is nothing too different about these indefinite integrals. The idea is the same, it's just that we have to use rules for trigonometric functions now.

Let's take a look at an example.

EXAMPLE Find the indefinite integral: $\int \sin^2 x \cos x dx$

First, note that the integral can be written $\int (\sin x)^2 \cos x dx$. At this point, notice that sin*x* is the "inner" function since it is raised to a power. Notice also that its derivative, cos*x*, is also in the integral. This means that *u*-substitution should work!

$\int \sin^2 x \cos x dx$	Start with the original expression.
$= \int u^2 du$	First, make the substitution: $u = \sin x$ Write the differential: $du = \cos x dx$ Replace $\sin x$ with *u* and $\cos x dx$ with *du*.
$= \frac{1}{3} u^3 + C$	Find the antiderivative with respect to *u*.

$$= \frac{1}{3}\sin^3 x + C \quad \text{Back-substitute } u = \sin x.$$

Thus, $\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x + C$.

TRY IT

Consider $\int \frac{\sin x}{\sqrt{\cos x}} dx$.

Find the indefinite integral.

+

Notice that $\cos x$ is an inner function. Let $u = \cos x$, then $du = -\sin x dx$. Since $\sin x dx$ is part of the integral, rewrite the "du" equation as $-du = \sin x dx$.

The new integral is $\int -\frac{du}{\sqrt{u}}$.

To continue, rewrite the integrand in exponential form, find the antiderivative, then replace $u = \cos x$ once finding the antiderivative:

$$\int -\frac{du}{\sqrt{u}} = \int -u^{-1/2} du$$
$$= -\left(\frac{1}{\left(\frac{1}{2}\right)}\right) u^{1/2} + C$$
$$= -2u^{1/2} + C$$
$$= -2(\cos x)^{1/2} + C$$

While this answer is perfectly simplified, sometimes it is preferred to write fractional exponents in radical form. The final answer could also be written $-2\sqrt{\cos x} + C$.

Thus, $\int \frac{\sin x}{\sqrt{\cos x}} dx = -2\sqrt{\cos x} + C$.

We can also use substitution to find antiderivatives of certain trigonometric functions.

WATCH

In this video, we'll find $\int \tan x dx$.

TRY IT

Consider $\int \cot x dx$.

Find the indefinite integral. +

First, write $\int \cot x dx = \int \frac{\cos x}{\sin x} dx$. It may also be helpful to write this as $\int \frac{1}{\sin x} \cdot \cos x dx$.

Since $\sin x$ is part of a larger expression, let $u = \sin x$. Then, $du = \cos x dx$. Next, replace $\sin x$ with *u* and $\cos x dx$ with *du*.

The new integral is $\int \frac{1}{u} du$.

Next, find the antiderivative, simplify, then replace *u* with $\sin x$.

$$\int \frac{1}{u} du = \ln|u| + C = \ln|\sin x| + C$$

Thus, $\int \cot x dx = \ln|\sin x| + C$.

TRY IT

Consider $\int (2+\tan x)^4 \sec^2 x dx$.

Find the indefinite integral. +

Since $2 + \tan x$ is the "inner function", let $u = 2 + \tan x$, then $du = \sec^2 x dx$. Next, replace $2 + \tan x$ with *u* and $\sec^2 x dx$ with *du*.

The integral becomes $\int u^4 du$.

Next, find the antiderivative, simplify, then replace *u* with $2 + \tan x$.

$$\int u^4 du = \frac{1}{5} u^5 + C = \frac{1}{5}(2 + \tan x)^5 + C$$

Thus, $\int (2+\tan x)^4 \sec^2 x\,dx = \frac{1}{5}(2+\tan x)^5 + C$.

2. The Outer Function Is Trigonometric

EXAMPLE Find the indefinite integral: $\int \cos(4x)\,dx$

$\int \cos(4x)\,dx$	Start with the original expression.
$= \int \cos u \cdot \frac{1}{4}\,du$	Make the substitution: $u = 4x$ Find the differential: $du = 4dx$ Solve for *dx*: $dx = \frac{1}{4}du$ Replace $4x$ with *u* and *dx* with $\frac{1}{4}du$.
$= \frac{1}{4}\int \cos u\,du$	Move the constant $\frac{1}{4}$ outside the integral sign.
$= \frac{1}{4}\sin u + C$	Use the antiderivative rule for $\cos x$.
$= \frac{1}{4}\sin(4x) + C$	Back-substitute $u = 4x$.

Thus, $\int \cos(4x)\,dx = \frac{1}{4}\sin(4x) + C$.

EXAMPLE Find the indefinite integral: $\int x\sec(3x^2)\tan(3x^2)\,dx$

$\int x\sec(3x^2)\tan(3x^2)\,dx$	Start with the original expression.
$= \int \sec u \tan u \cdot \frac{1}{6}\,du$	Make the substitution: $u = 3x^2$ Find the differential: $du = 6x\,dx$ Solve for *xdx*: $x\,dx = \frac{1}{6}du$ Replace $3x^2$ with *u* and *xdx* with $\frac{1}{6}du$.
$= \frac{1}{6}\int \sec u \tan u\,du$	Move the constant $\frac{1}{6}$ outside the integral sign.
$= \frac{1}{6}\sec u + C$	Use the antiderivative rule for $\sec x \tan x$.

$= \frac{1}{6}\sec(3x^2) + C$ Back-substitute $u = 3x^2$.

Thus, $\int x\sec(3x^2)\tan(3x^2)dx = \frac{1}{6}\sec(3x^2) + C$.

TRY IT

Consider $\int \frac{\sin(4\ln x)}{x}dx$.

Find the indefinite integral. +

First, rewrite the integral as $\int \sin(4\ln x) \cdot \frac{1}{x}dx$.

Since $4\ln x$ appears to be an “inner function”, let $u = 4\ln x$. Then, $du = \frac{4}{x}dx$. Since the integrand doesn’t contain the factor of 4, rewrite the *du* equation as $\frac{1}{4}du = \frac{1}{x}dx$.

Next, replace $4\ln x$ with *u* and $\frac{1}{x}dx$ with $\frac{1}{4}du$.

The new integral is $\int \sin u \cdot \frac{1}{4}du$, which could be written $\frac{1}{4}\int \sin u du$.

Next, find the antiderivative, simplify, then replace *u* with $4\ln x$.

$$\frac{1}{4}\int \sin u du = \frac{1}{4}(-\cos u) + C = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos(4\ln x) + C$$

Thus, $\int \frac{\sin(4\ln x)}{x}dx = -\frac{1}{4}\cos(4\ln x) + C$.

THINK ABOUT IT

Consider the antiderivative $\int \cos(x^2)dx$.

A first instinct is to let $u = x^2$, then $du = 2xdx$, which means $xdx = \frac{1}{2}du$. Here is the problem: the original integral only has a "*dx*" term, not an "*x dx*" term. From the substitution, we could also write $x=\sqrt{u}$. Let's see where that takes us.

$x dx=\frac{1}{2}du$ becomes $\sqrt{u}\,dx=\frac{1}{2}du$, so $dx=\frac{1}{2\sqrt{u}}du$.

Making all the substitutions, our integral becomes $\int \frac{\cos u}{\sqrt{u}}du$, which is much more complicated. As it turns out, there is no substitution that would solve $\int \cos(x^2)dx$ because there is no antiderivative for $f(x)=\cos(x^2)$. There are several other functions that do not have antiderivatives.

So, when making substitutions, be careful that all the variables are covered in your substitution. If they aren't, you either should check your work, try another substitution, or it is possible that the antiderivative doesn't exist.

SUMMARY

In this lesson, you learned how to apply your knowledge of *u*-substitution to find indefinite integrals when **the inner function is trigonometric** and **the outer function is trigonometric**. With the addition of trigonometric functions, your abilities to find antiderivatives expands even further. As you saw in one example, however, there are several functions that do not have antiderivatives.