The Limit Is Infinite

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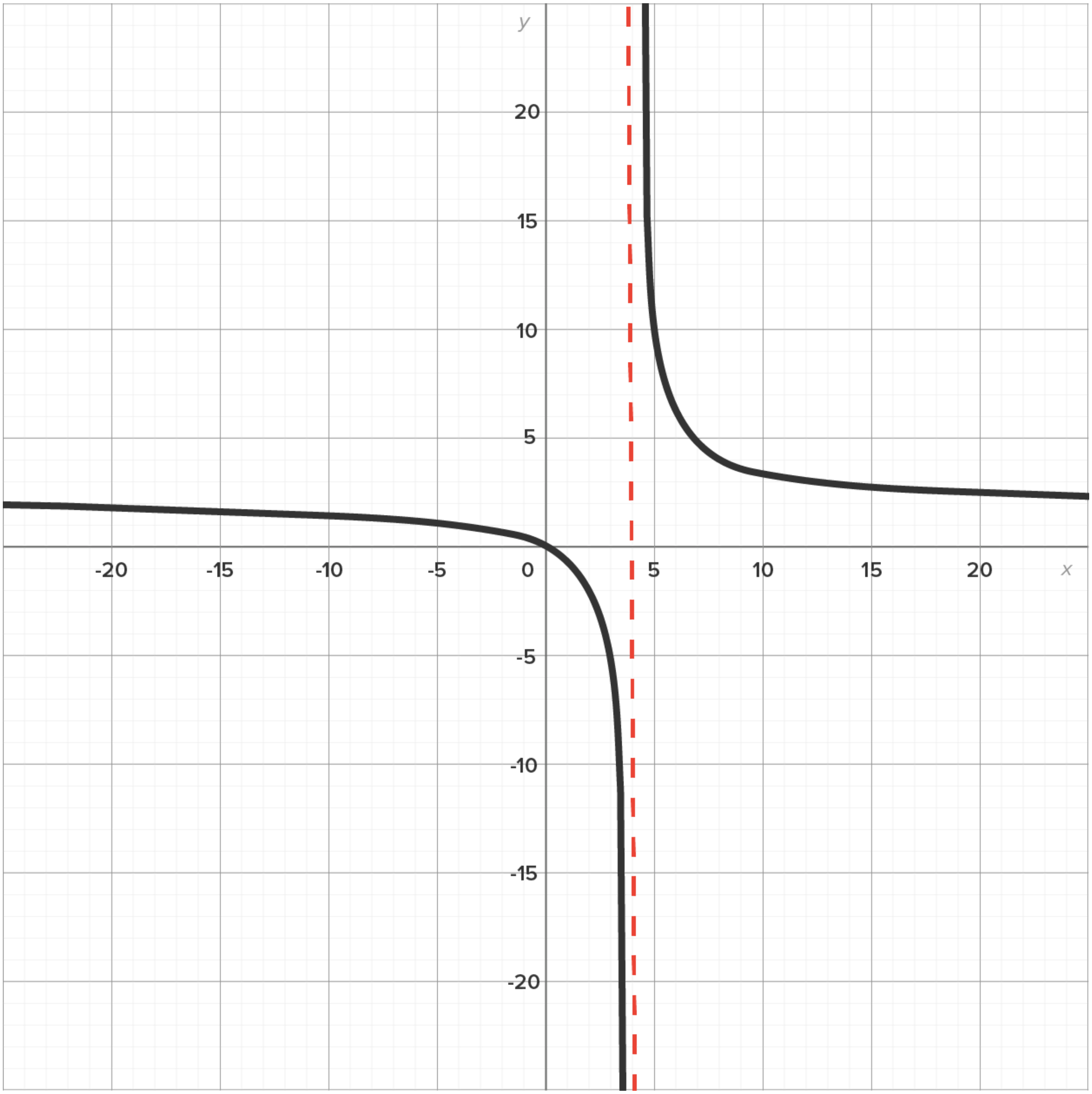
WHAT'S COVERED

In this lesson, you will analyze functions whose values get arbitrarily large as *x* approaches a finite value. Specifically, this lesson will cover:

1. Graphically Finding Infinite Limits

Consider the graph of $f(x)=\frac{2x}{x-4}$, as shown in the figure (the dashed line $x=4$ is drawn for reference).



Notice the behavior of the graph near $x = 4$.

As *x* gets closer to 4 from the left side, the graph decreases in value very quickly.

As a limit, this is written $\lim_{x \to 4^-} \frac{2x}{x-4} = -\infty$.

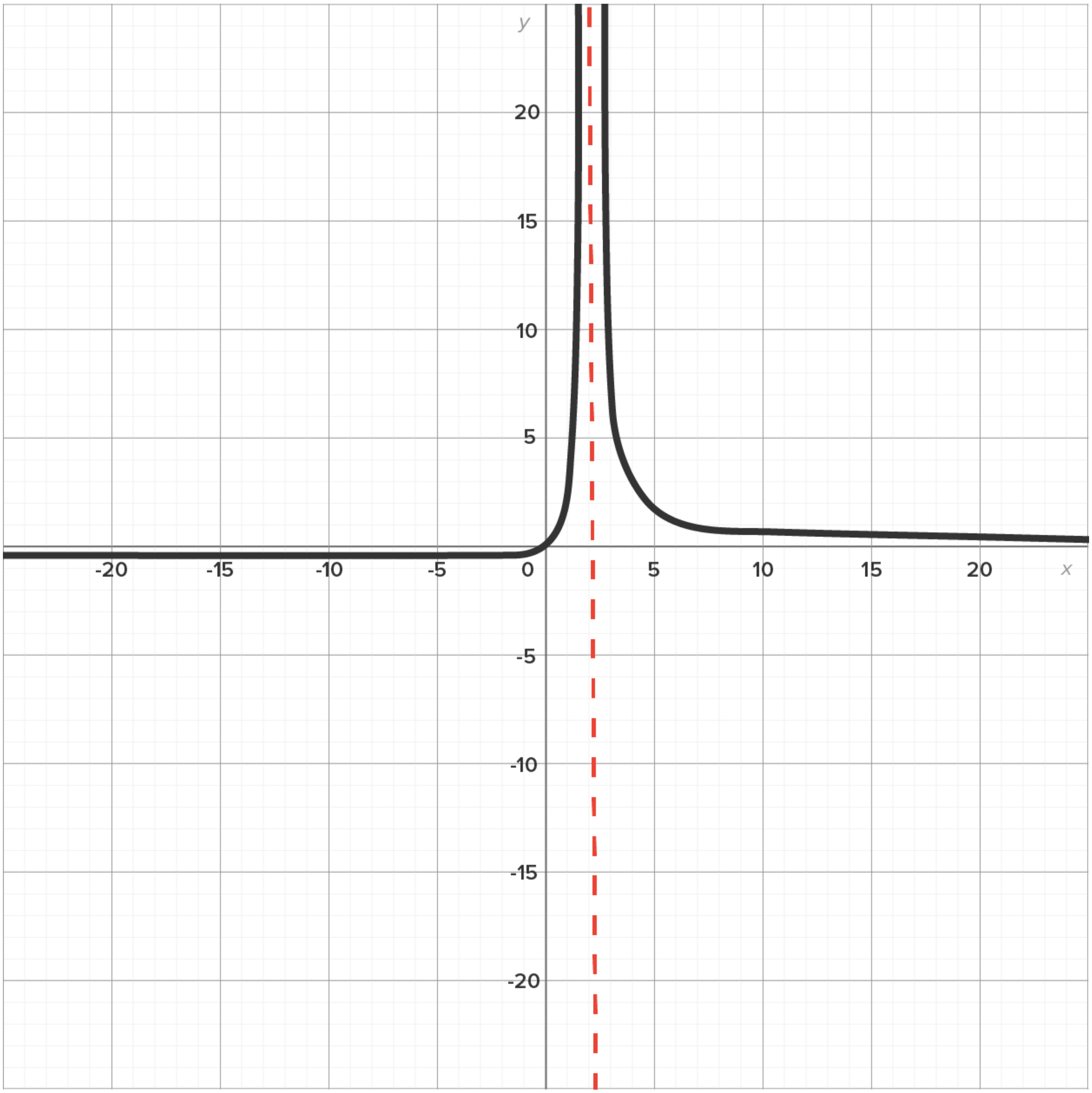
As *x* gets closer to 4 from the right side, the graph increases in value very quickly.

As a limit, this is written $\lim_{x \to 4^+} \frac{2x}{x-4} = \infty$.

Then, since the one-sided limits are not equal, the limit $\lim_{x \to 4} \frac{2x}{x-4}$ does not exist.

TRY IT

Consider the graph of $f(x) = \frac{3x}{(x-2)^2}$ below:



Suppose you want to find each of the following limits:

- $\lim_{x \to 2^-} \frac{3x}{(x-2)^2}$
- $\lim_{x \to 2^+} \frac{3x}{(x-2)^2}$

- $\lim\limits_{x \to 2} \frac{3x}{(x-2)^2}$

Evaluate each limit (if possible). +

All limits are ∞.

HINT

In the previous problem, notice that we say that $\lim\limits_{x \to 2} \frac{3x}{(x-2)^2} = \infty$ rather than saying the limit doesn't exist.

If both one-sided limits approach the same value, then the two-sided limit approaches that value as well. In the context of infinite limits, we only say that a limit doesn't exist if one side goes to ∞ and the other side goes to $-\infty$.

2. Numerically Finding Infinite Limits

Using tables is another way to get some understanding about a limit.

EXAMPLE Consider the function $f(x) = \frac{x+4}{x^2-1}$. Use a table of values to evaluate $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$, and $\lim\limits_{x \to 1} f(x)$.

In Unit 2, you may recall that the expression can be simplified before evaluating the limit. That is not the case here (if you factor the denominator, none of its factors cancel with the numerator).

To evaluate the one-sided limits, we choose x-values that successively get closer to 1 on each side:

From the left:

x	0.9	0.99	0.999	0.9999
$f(x) = \frac{x+4}{x^2-1}$	-25.7895	-250.7538	-2500.7504	-25000.7500

From the right:

x	1.1	1.01	1.001	1.0001
$f(x) = \frac{x+4}{x^2-1}$	24.2857	249.2537	2499.2504	24999.2500

As x gets closer to 1 from the left, the value of $f(x)$ decreases quickly. This means $\lim\limits_{x \to 1^-} \frac{x+4}{x^2-1} = -\infty$.

As x gets closer to 1 from the right, the value of $f(x)$ increases quickly. This means $\lim\limits_{x \to 1^+} \frac{x+4}{x^2-1} = \infty$.

As a result, since the one-sided limits do not agree, $\lim\limits_{x \to 1} \frac{x+4}{x^2-1}$ does not exist.

TRY IT

Consider the function $f(x) = \frac{2x}{x^2-4x+3}$ and we want to find $\lim\limits_{x \to 3^-} f(x)$, $\lim\limits_{x \to 3^+} f(x)$, and $\lim\limits_{x \to 3} f(x)$.

Use appropriate tables to evaluate the limits. +

From the left:

x	2.9	2.99	2.999	2.9999
$f(x) = \frac{2x}{x^2-4x+3}$	-30.5263	-300.5025	-3000.5003	-30000.5000

From the right:

x	3.1	3.01	3.001	3.0001
$f(x) = \frac{2x}{x^2-4x+3}$	29.5238	299.5025	2999.5002	29999.5000

$\lim\limits_{x \to 3^-} f(x) = -\infty$, $\lim\limits_{x \to 3^+} f(x) = \infty$, and $\lim\limits_{x \to 3} f(x)$ does not exist.

3. Analytically Finding Infinite Limits

To evaluate an infinite limit analytically, one-sided limits are used. Instead of making a table, the numerator and denominator are analyzed separately.

EXAMPLE Evaluate $\lim\limits_{x \to 4} \frac{3x}{x-4}$ analytically.

Since $x = 4$ cannot be substituted directly and the function doesn't simplify, we'll examine the one-sided limits near $x = 4$.

Left-sided limit:
If $x \to 4^-$, then $x < 4$ but is getting closer to 4.

- The numerator, $3x$, is close to 12.
- The denominator, $x - 4$, is a small negative number.
- Thus, we have $\frac{\text{Close to 12}}{\text{small negative}}$, which is equal to a large negative number.
- This means that $\lim\limits_{x \to 4^-} \frac{3x}{x-4} = -\infty$.

Right-sided limit:
If $x \to 4^+$, then $x > 4$ but is getting closer to 4.

- The numerator, $3x$, is close to 12.
- The denominator, $x - 4$, is a small positive number.
- Thus, we have $\frac{\text{Close to 12}}{\text{small positive}}$, which is equal to a large positive number.
- This means that $\lim\limits_{x \to 4^+} \frac{3x}{x-4} = \infty$.

This means that $\lim\limits_{x \to 4} \frac{3x}{x-4}$ does not exist. Since the left- and right-sided limits do not have the same unbounded behavior, we cannot write the overall behavior.

WATCH

In this video, we'll evaluate $\lim\limits_{x \to 2} \frac{4x}{(x-2)^2}$ analytically.

SUMMARY

In this lesson, you learned that **infinite limits can be determined graphically, numerically, and analytically**. All methods produce reliable results, with the graphical approach being the simplest. When a graph is not available or is too difficult to read, it is useful to use either the numerical approach, using the appropriate tables to evaluate the limits, or the analytical approach, using one-sided limits and analyzing the numerator and denominator separately.