Related Rates Problems Using Geometric Formulas

by Sophia

WHAT'S COVERED

In this lesson, you will apply rates of change in a geometric setting. When a solid expands or contracts, there are several rates of change that are related. We can use implicit differentiation to establish those relationships and also find instantaneous rates of change. Specifically, this lesson will cover:

1. Geometric Formulas

In this part of the challenge, we will focus on related rates that come from geometric formulas.

Refer to the geometric formula sheet below for a complete list of the formulas you should know. You can also download this formula sheet as a PDF file at the end of the tutorial.

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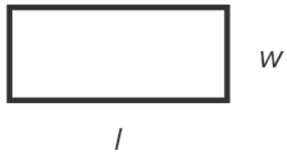
Geometric Formulas

Square

$P = 4s$

$A = s^2$

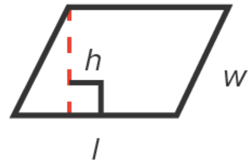
Rectangle



$P = 2l + 2w$

$A = lw$

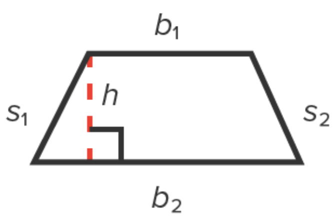
Parallelogram



$P = 2l + 2w$

$A = lh$

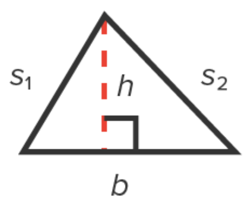
Trapezoid



$P = s_1 + s_2 + b_1 + b_2$

$A = ½h(b_1 + b_2)$

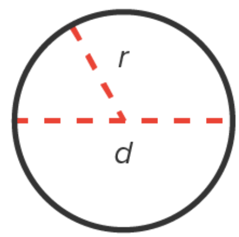
Triangle



$P = s_1 + s_2 + b$

$A = ½bh$

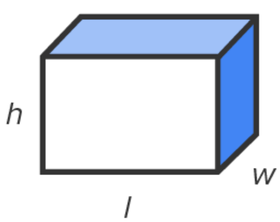
Circle



$C = 2\pi r$ or $C = \pi d$

$A = \pi r^2$

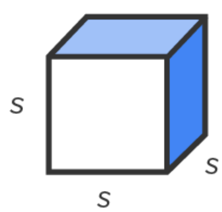
Rectangular Solid



$S = 2lh + 2wh + 2wl$

$V = lwh$

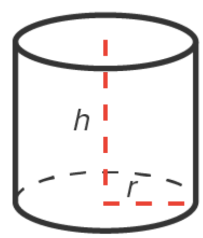
Cube



$S = 6s^2$

$V = s^3$

Right Circular Cylinder



$S = 2\pi rh + 2\pi r^2$

$V = \pi r^2 h$

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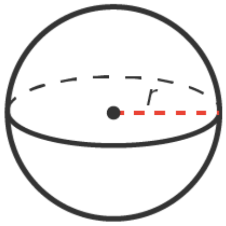
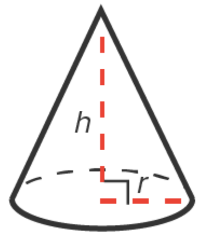
Sphere



$S = 4\pi r^2$

$V = {}^4/_3\pi r^3$

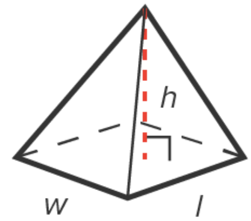
Right Circular Cone



$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$

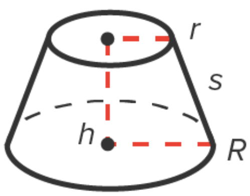
$V = \frac{1}{3}\pi r^2 h$

Square or Rectangular Pyramid



$V = \frac{1}{3}lwh$

Right Circular Cone Frustum



$S = \pi s(R + r) + \pi r^2 + \pi R^2$

$$V = \frac{\pi(r^2 + rR + R^2)h}{3}$$

Geometric Symbols

A = Area

P = Perimeter

V = Volume

S = Surface Area

C = Circumference

π = Pi Constant

2. What Exactly Are Related Rates?

A stone is dropped in a lake. The ripple that is formed is circular, and the radius is increasing at a rate of 2 inches per second. Let’s see how this impacts the area enclosed by the ripple.

Recall that the area enclosed by a circle is $A = \pi r^2$. The table below shows the time elapsed (in seconds), along with the radius and the area inside the ripple.

Time	Radius	Area
1	2	4π
2	4	16π
3	6	36π
4	8	64π

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The table suggests that the radius and the area are functions of time, meaning that as time changes, so do the values of the radius and the area. In addition, it is pretty clear that even though the radius is changing at the same rate each second, the rate at which the area is changing is at different rates (in fact, it is increasing).

This also means that the rate at which the area is changing is affected by the rate of change in the radius and the radius itself.

Formally, let's call $\frac{dA}{dt}$ the rate of change in the area with respect to time and let $\frac{dr}{dt}$ represent the rate of change in the radius with respect to time.

Then, $\frac{dA}{dt}$ and $\frac{dr}{dt}$ are related by some equation. Here is how we get there.

We know that the area of the circle is $A = \pi r^2$. Since *A* and *r* are functions of *t*, we will differentiate both sides with respect to *t*:

$A = \pi r^2$	Start with the area of a circle formula.
$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$	Apply the derivative with respect to *t* to both sides.
$\frac{dA}{dt} = \pi(2r) \cdot \frac{dr}{dt}$	Take the derivative of each side. Multiply $\pi(2r)$ by $\frac{dr}{dt}$ since we are differentiating *r* implicitly.
$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$	Simplify.

Thus, the equation $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ shows us how the rates are related.

Furthermore, we also assumed that the radius was increasing by 2 inches per second, which means $\frac{dr}{dt} = 2$ in/sec.

This means that $\frac{dA}{dt} = 2\pi r(2) = 4\pi r \text{ in}^2/\text{sec}$.

With this relationship, we can answer two questions:

- What is the radius when the area is changing by a certain amount?
- What is the rate of change in the area for a specific radius?

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3. Related Rates Applied to Geometric Situations

In this part of the challenge, we will focus on related rates that come from the geometric formula sheet.

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➦ EXAMPLE A spherical balloon is being inflated at a rate of $20\pi\ \text{cm}^3/\text{sec}$. At what rate is the radius increasing when the radius is 10 cm?

First, identify the geometric formula to use. Since the balloon is spherical, use the formula for the volume of a sphere: $V = \frac{4}{3}\pi r^3$

Now, identify all quantities, and what we are looking for.

- Given : $\frac{dV}{dt} = 20\pi$
- Want to know: $\frac{dr}{dt}$ when $r = 10$

Since the information we have (and need) involves rates, we need to use implicit differentiation to take the derivative:

$V = \frac{4}{3}\pi r^3$	Start with the original formula.
$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.
$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$	Simplify.
$20\pi = 4\pi(10)^2 \cdot \frac{dr}{dt}$	Given $\frac{dV}{dt} = 20\pi$, we want to know $\frac{dr}{dt}$ when $r = 10$.
$20\pi = 400\pi \cdot \frac{dr}{dt}$	Simplify.
$\frac{dr}{dt} = \frac{1}{20}\ \text{cm}/\text{sec}$	Divide both sides by 400π and affix appropriate units.

The radius is increasing at a rate of $\frac{1}{20}$ cm/sec at the exact moment the radius is 10cm.

➦ EXAMPLE A large ice cube is melting in such a way that its volume is decreasing at a rate of $40\ \text{in}^3/\text{hour}$. At what rate is one of its sides changing when it measures 45 inches on each side?

First, identify the geometric formula to use. Since the ice cube is in the shape of a cube, use the formula for the volume of a cube, $V = s^3$.

Note that both quantities (*V* and *s*) are changing with respect to times, so they remain as variables in the equation. Now, identify all quantities, and what we are looking for.

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- Given: $\frac{dV}{dt} = -40$ (Since the ice cube is melting, the volume is decreasing, making $\frac{dV}{dt}$ negative.)
- Want to know: $\frac{ds}{dt}$ when $s = 45$

Since the information we have (and need) involves rates, we need to take the derivative:

$V = s^3$	Start with the original formula.
$\frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt}$	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.
$-40 = 3(45)^2 \cdot \frac{ds}{dt}$	Given $\frac{dV}{dt} = -40$, we want to know $\frac{ds}{dt}$ when $s = 45$.
$-40 = 6075 \cdot \frac{ds}{dt}$	Simplify.
$\frac{ds}{dt} = -\frac{8}{1215}$ in/min	Divide both sides by 6,075 and reduce; affix appropriate units.

The length of one side is decreasing at a rate of $-\frac{8}{1215}$ in/min, or approximately -0.00658 in/min at the exact moment the radius is 45 in.

TRY IT

A graphic designer takes an image of a square and enlarges it in such a way that each side increases by 0.25 inch every second.

When the length of each side is 6 inches, at what rate is the area changing? +

First, write the formula for the area of a square in terms of its side length, *s*: $A = s^2$

Next, take the derivative of both sides with respect to *t*: $\frac{dA}{dt} = 2s \cdot \frac{ds}{st}$

We are given that the side is changing at a rate of 0.25 inches per second, which means $\frac{ds}{dt} = 0.25$.

We are also concerned with the rate of change in the area at the instant that the side length is 6, which means $s = 6$.

Then, $\frac{dA}{dt} = 2(6)(0.25) = 3\text{in}^2/\text{sec}$.

We can also use related rates to examine how the distance between two objects is changing.

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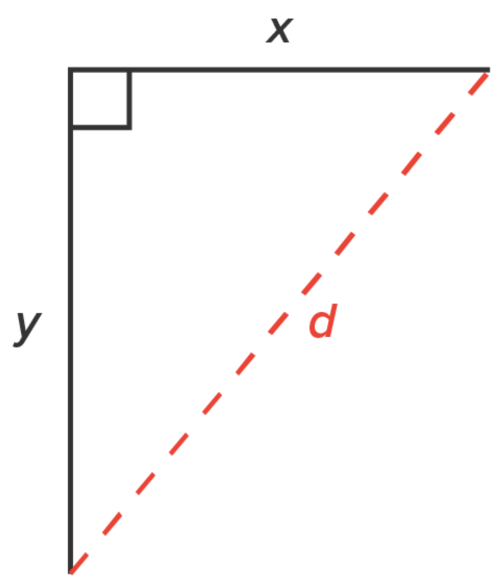
EXAMPLE Two joggers start from the same point in the middle of a large field, one heading east and one heading south. If the eastbound jogger runs at a rate of 5 mph and the southbound jogger runs at a rate of 6 mph, at what rate is the distance between them changing after 2 hours?

- Let x = the distance traveled by the eastbound jogger.
- Let y = the distance traveled by the southbound jogger.
- Let d = the distance between the joggers.



Then, the variables are related by the Pythagorean theorem, $x^2+y^2=d^2$.

Since all three quantities are changing with respect to time, t, in this problem, they will remain as variables when we find the related rates.

- Given: The rates of change in the distances (the speeds), $\frac{dx}{dt}=5$ mi/hr, $\frac{dy}{dt}=6$ mi/hr
- Want to know: $\frac{dd}{dt}$ after 2 hours

Since $\text{distance}=\text{rate}\cdot\text{time}$, this means $x=10$ and $y=12$.

Because the information we have (and need) involves rates, we need to take the derivative:

$x^2+y^2=d^2$	Start with the original formula.
$2d\cdot\frac{dd}{dt}=2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}$	Take the derivative of both sides, remembering that each variable is being

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	differentiated implicitly.
$d^2 = x^2 + y^2$ $d^2 = 10^2 + 12^2$ $d^2 = 244$ $d = \sqrt{244}$	The value of *d* wasn't given, but we can use the original equation to find it.
$2\sqrt{244} \cdot \frac{dd}{dt} = 2(10)(5) + 2(12)(6)$	Given $\frac{dx}{dt} = 5$ and $\frac{dy}{dt} = 6$, we want to know $\frac{dd}{dt}$ when $x = 10$, $y = 12$, and $d = \sqrt{244}$.
$\frac{dd}{dt} = \frac{244}{2\sqrt{244}} \approx 7.81$ mi/hr	Solve for $\frac{dd}{dt}$.

The distance between the joggers is increasing at a rate of 7.81 mph after 2 hours.

SUMMARY

In this lesson, you learned how to **apply related rates to geometric situations**, using a comprehensive list of **geometric formulas**. You learned that derivatives can be applied to geometric formulas to produce **related rates**, which are equations that establish a relationship between two (or more) rates of change.

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