Units for the Definite Integral

by Sophia

WHAT'S COVERED

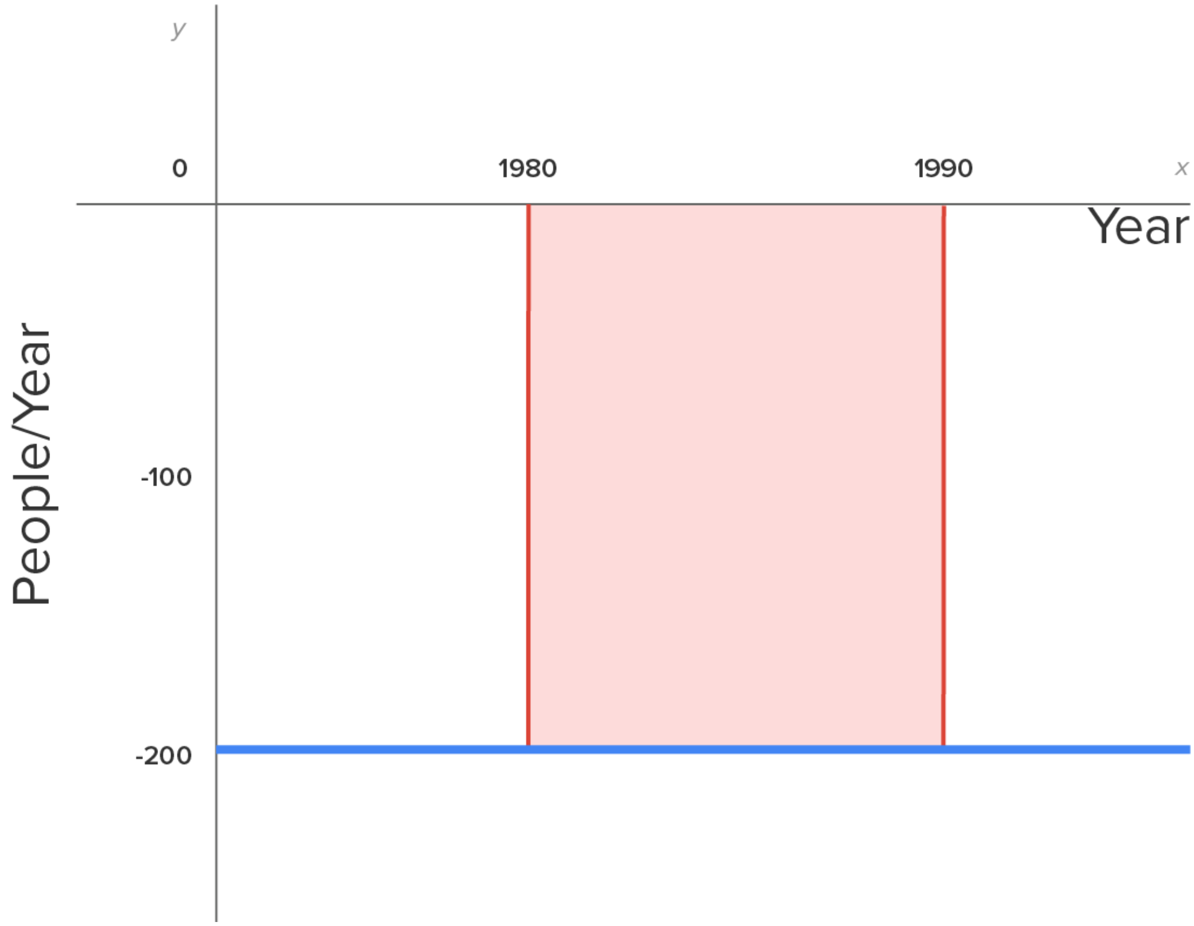
In this lesson, you will investigate the various interpretations of the definite integral. Specifically, this lesson will cover:

1. The Definite Integral As Net Change

Let's say that $f(x) =$ the rate at which the population of a small town is changing in year *t*.

- If $f(x) > 0$, this means that the population is increasing.
- If $f(x) < 0$, this means that the population is decreasing.

Suppose that $f(x)$ has this graph between years $x = 1980$ and $x = 1990$.



This means that at any point between 1980 and 1990, the population is decreasing at a rate of 200 people per year.

Note that the area of the region is $10(200) = 2000$, and the region is below the x-axis.

This means that the value of the definite integral is $\int_{1980}^{1990} f(x)dx = \int_{1980}^{1990} -200dx = -2000$.

So, what are the units? Since the horizontal scale is measured in years and the vertical scale is measured in people/year, the area is the product of the units, which is the number of people. Thus, the area can be interpreted as 2,000 people. But how does this fit in, especially since this region is below the x-axis?

What does this result mean? The value of the definite integral means that the town lost 2,000 people between 1980 and 1990. So, if the population of the town in 1980 was 14,250, the population in 1990 was $14{,}250 - 2{,}000 = 12{,}250$ people. Thus, the definite integral $\int_{1980}^{1990} f(x)dx$ is interpreted as the change in population over the interval $[1980, 1990]$.

BIG IDEA

The meaning of the definite integral $\int_a^b f(x)dx$ can be interpreted as the net change in $f(x)$ over the interval $[a, b]$.

The units of the definite integral are (units of *x*)(units of $f(x)$).

2. The Definite Integral As It Relates to Everyday Situations

2a. Velocity and Distance

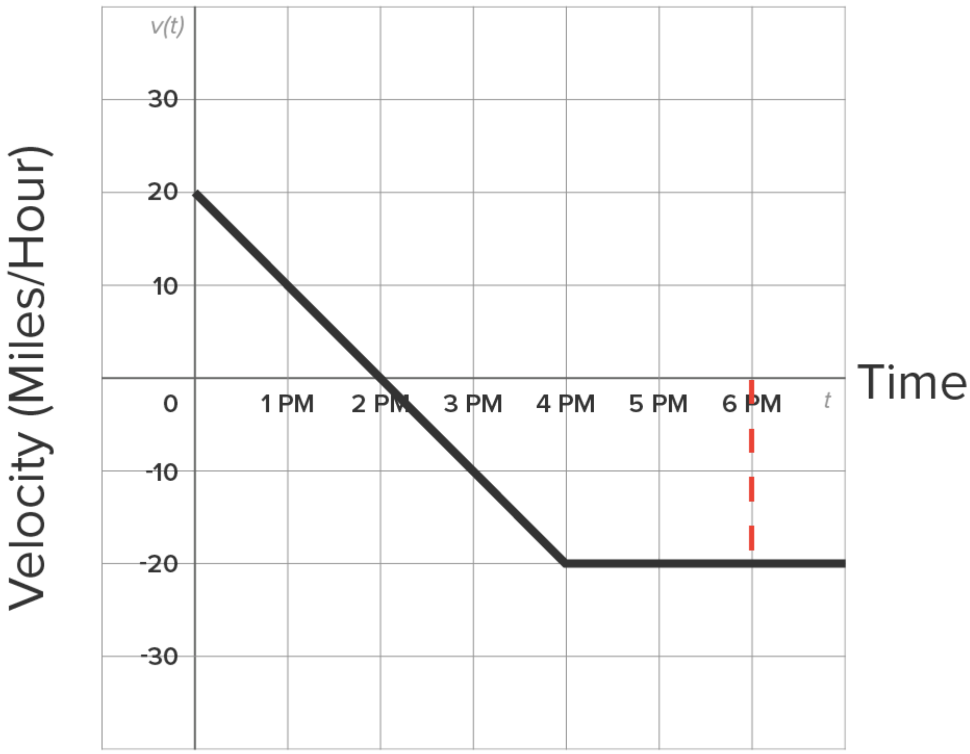
Recall that velocity is the rate of change of distance.

- A positive velocity means that the distance from a specific point is increasing (moving further away from the point).
- A negative velocity means that the distance from a specific point is decreasing (getting closer to the point).
- A zero velocity (over an interval) means the object is at rest.

Let's say an object in motion has velocity $v(t)$ on the interval $[a, b]$, where velocity is measured in miles per hour and *t* is measured in hours.
Then, the units of the definite integral are (miles/hour)(hours) = miles, and the value of the definite integral means the net distance traveled on $[a, b]$.

EXAMPLE Consider the graph below, which represents the velocity $v(t)$ of a car moving east of its starting point at 12 PM $(t = 0)$. A positive velocity indicates eastward motion, while a negative velocity indicates westward motion.

1. What is the distance traveled from 12 PM to 6 PM?
2. What does $\int_0^6 v(t)dt$ mean in this situation?



Note that the vertical scale is measured in miles per hour and the horizontal scale is measured in hours. Therefore, the area of any region is measured in $\frac{\text{miles}}{\text{hour}} \cdot \text{hours} = \text{miles},$ which is distance. To find the total area, add the area of each region.

- On $[0, 2],$ the region is a triangle, which means its area is $\frac{1}{2}(2)(20) = 20$ miles to the east.
- On $[2, 4],$ the region is a triangle, but under the x-axis. Its area is $\frac{1}{2}(2)(20) = 20$ miles to the west.
- On $[4, 6],$ the region is a rectangle, but under the x-axis. Its area is $(2)(20) = 40$ miles to the west.

1. The total area is the distance traveled (in any direction), $20 + 20 + 40 = 80$ miles.

   The change in position is the value of the definite integral, which considers the location of the region (above or below the t-axis). If the region is above, use the area. If the region is below the t-axis, use the negative of the area.

2. $\int_0^6 v(t)dt = 20 - 20 - 40 = -40$

   This result means that the car is 40 miles west of its starting point after 6 hours (since east is positive, west is negative).

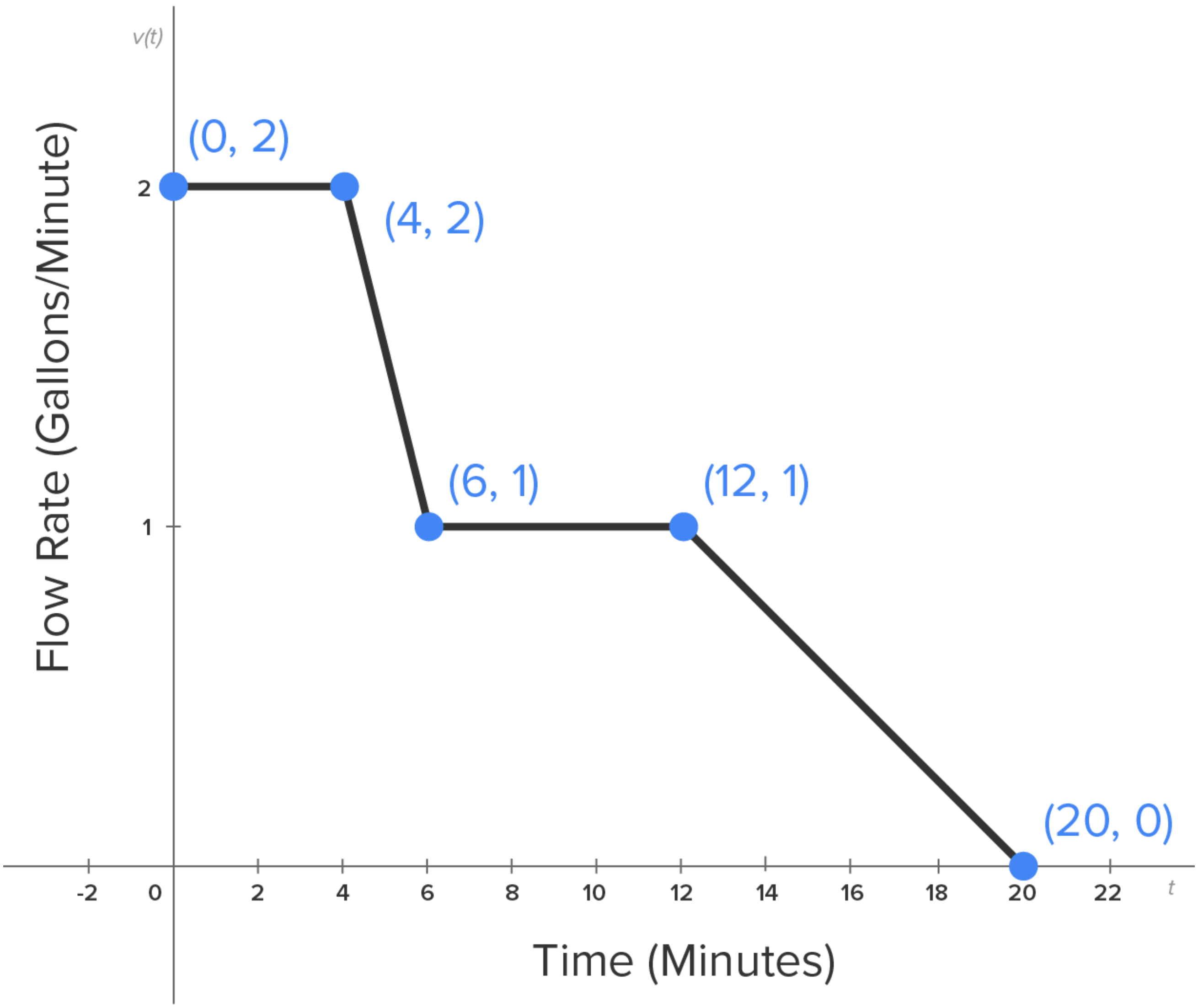
WATCH

In this video, we'll use another graph to find the distance traveled and other interpretations based on the results.

2b. Other Situations

TRY IT

Shown below is the graph of the flow rate $f(t)$ of a pipe, in gallons per minute. Here, $t =$ the number of minutes.



Using this information, we can find the total number of gallons of water that was flowing through this pipe in 20 minutes. To get this, we need to find $\int_0^{20} f(t)dt$.

Calculate the value of the definite integral and give units for your answer. +

The total number of gallons is the value of the definite integral, which is related to the areas of the regions between the graph and the t-axis.

Let's find the areas on a few subintervals, which correspond the transition points of the graph:

On $[0, 4]$, the region is a rectangle with base is 4 and whose height is 2. The area of this region is 8 units^2.

On $[4, 6]$, the region is a trapezoid with bases 2 and 1, and with height 2. The area of this region is $\frac{2}{2}(2+1) = 3 \text{ units}^2$.

On $[6, 12]$, the region is a rectangle with base 6 and height 1. The area of this region is $6(1) = 6 \text{ units}^2$.

On $[12, 20]$, the region is a triangle with base 8 and height 1. The area of this region is $\frac{1}{2}(8)(1) = 4 \text{ units}^2$.

All regions are above the x-axis, which means we simply add the areas to get the value of the definite integral.

As far as units, keep in mind that the area is calculated by multiplying (Gallons/Minute) by minutes, which results in gallons.

Thus, $\int_0^{20} f(t)dt = 8+3+6+4 = 21$ gallons.

SUMMARY

In this lesson, you learned that **the definite integral is interpreted as a net change** over an interval. The units of $\int_a^b f(x)dx$ are (units of *x*)(units of $f(x)$). Next, you explored some examples using **the definite integral as it relates to everyday situations**, involving **velocity and distance** and **other situations**. For example, when given a **velocity** function $v(t)$, the definite integral of $v(t)$ is the net distance traveled on the interval.

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