Area Under A Curve –– Riemann Sums

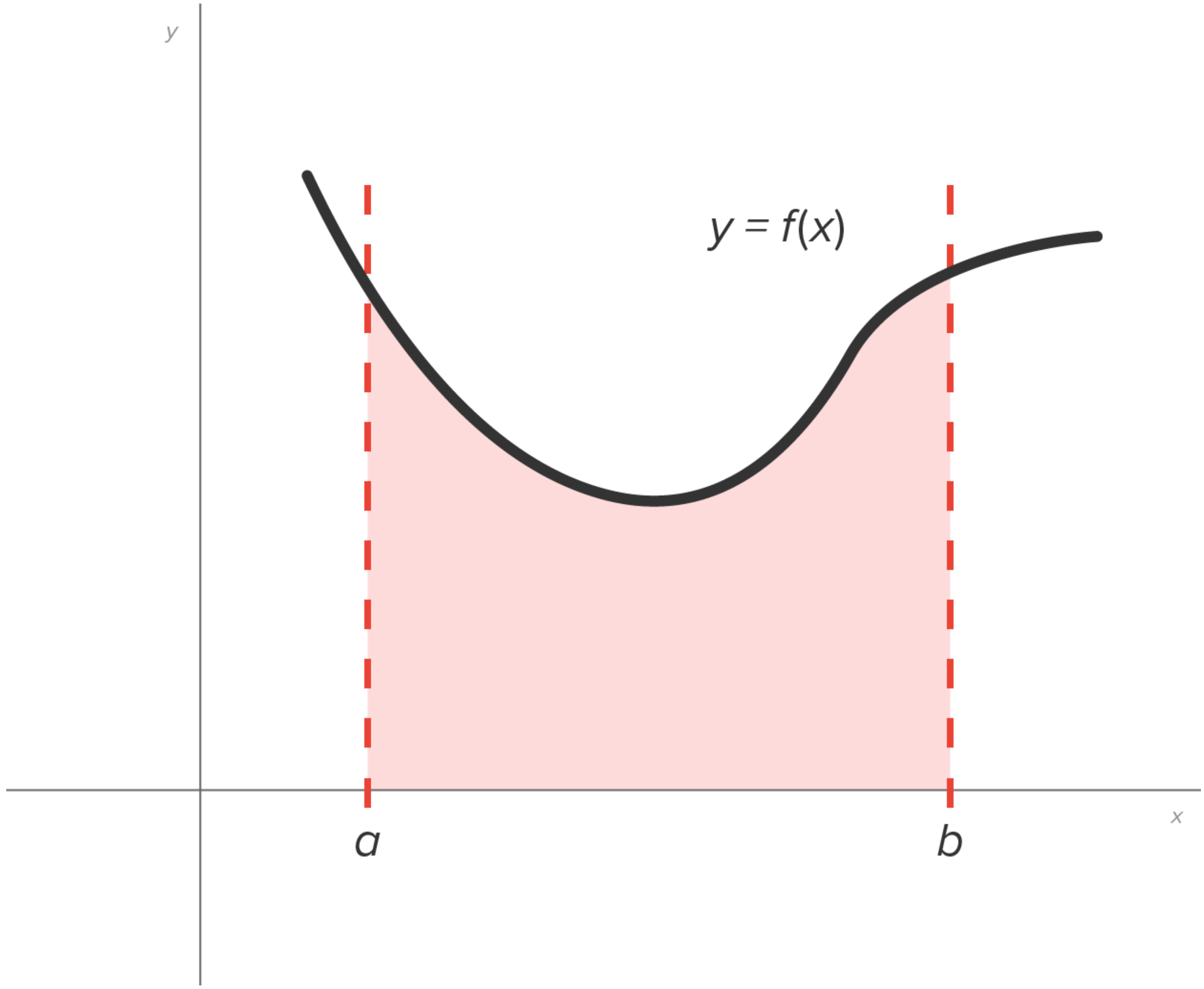
by Sophia

WHAT'S COVERED

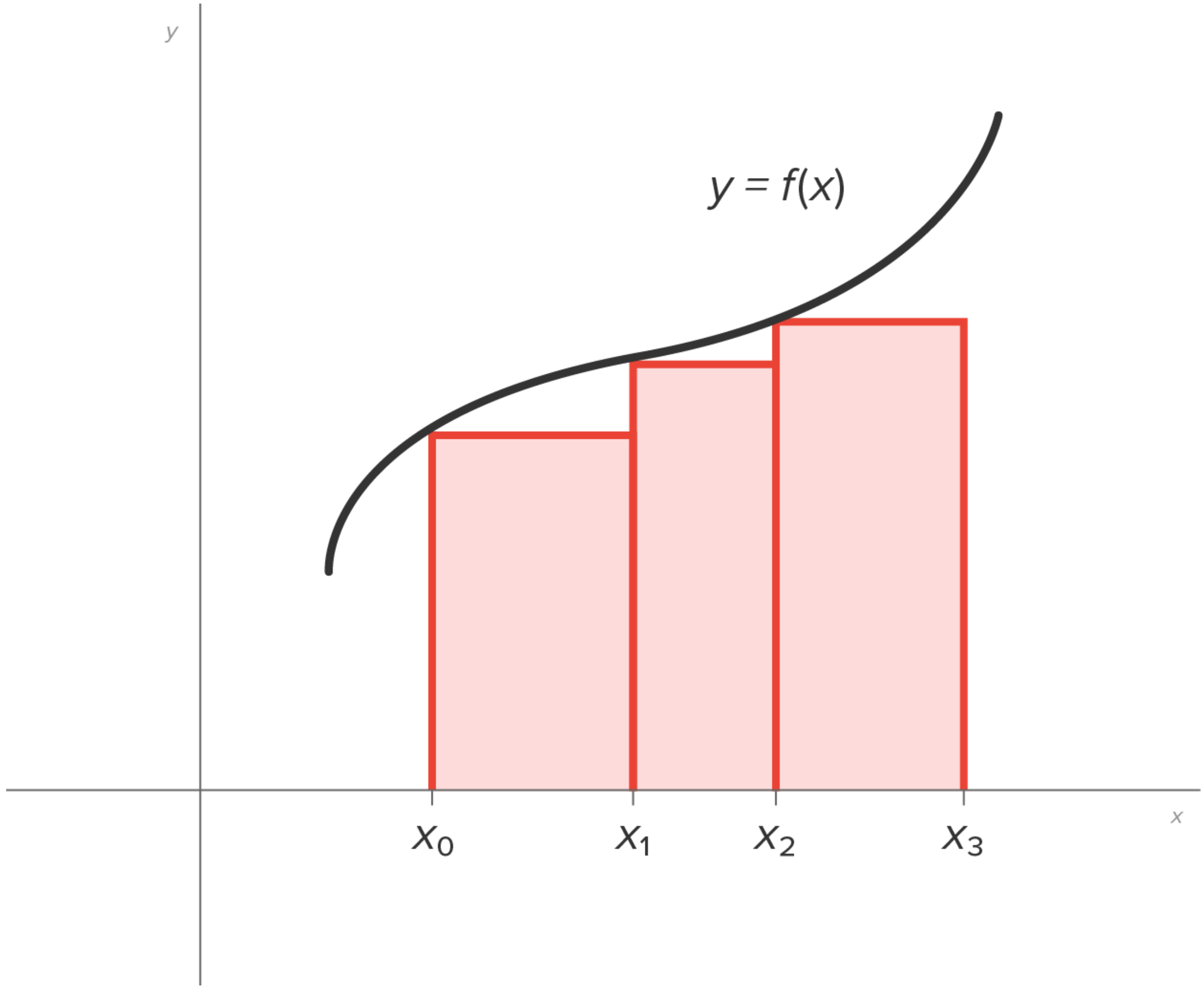
In this lesson, you will form Riemann sums to approximate areas. This idea is very important as it paves the way for some applications in integral calculus. Specifically, this lesson will cover:

1. Definition of Riemann Sum

Suppose we want to calculate the area between the graph of a nonnegative function $f(x)$ and the x-axis interval $[a, b]$, as shown in the figure below.



If $f(x)$ is nonnegative, the Riemann sum method is to build several rectangles with bases on the interval $[a, b]$ and sides that reach up to the graph of $f(x)$. Then, the areas of the rectangles can be calculated and added together to get a number called a **Riemann sum** of $f(x)$ on $[a, b]$.



The area of the region formed by the rectangles is an approximation of the area between the graph and the x-axis.

TERM TO KNOW

Riemann Sum

The sum obtained from the areas of rectangles that are used to approximate the area between a curve and the x-axis.

2. Finding the Riemann Sum

In order to find the Riemann sum, there are several quantities that need to be established first.

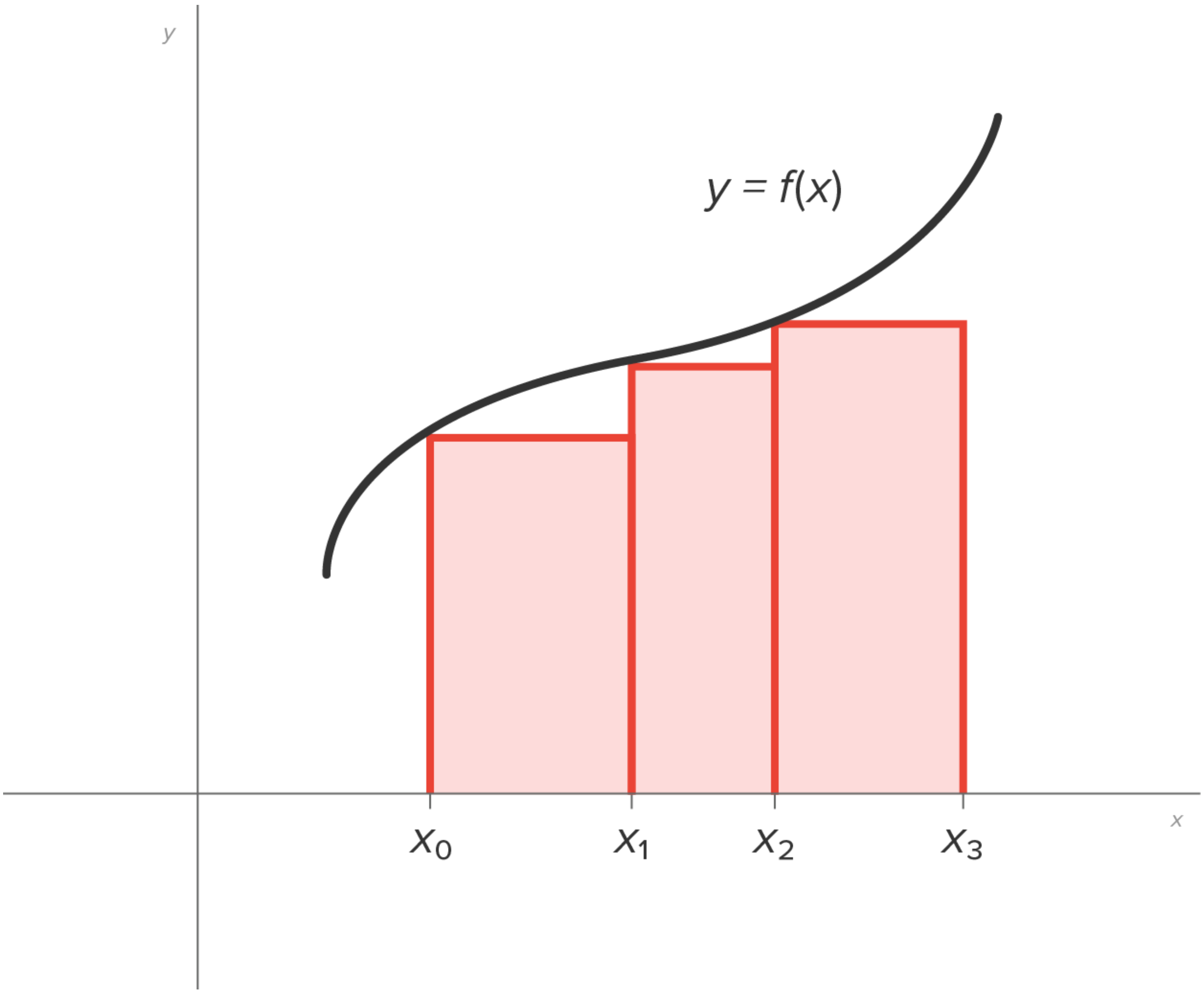
1. Find the partition and subintervals.
2. Find the width of each subinterval.
3. Select x-values within each partition.

4. Form the Riemann sum.

Let's take a deeper look at each step.

2a. Find the Partition and Subintervals

First, a **partition** of the interval $[a, b]$ is needed to establish the bases of the rectangles. Consider the graph in the figure.



The endpoints of the interval are x_0 and x_3. In order to form three rectangles, two more values (x_1 and x_2) are added to form a partition of the interval $[x_0, x_3]$.

We label the partition by the x-coordinates, namely $\{x_0, x_1, x_2, x_3\}$. The numbers are listed in increasing order.

Note that there are 4 x-values in the partition for three rectangles. In general, if n rectangles are desired, there would be $n+1$ x-values in the partition. This is why the first one is labeled as x_0 (which is the left-hand endpoint of the interval), so the last one can be called x_n (to match the number of rectangles).

The **subintervals** for this partition are $[x_0, x_1]$, $[x_1, x_2]$, and $[x_2, x_3]$.

TERMS TO KNOW

Partition

A set of x-values that are used to split the interval $[a, b]$ into smaller intervals.

Subinterval

A smaller interval that is part of a larger interval.

2b. Find the Width of Each Subinterval

It is most convenient to select a partition where each x-value is the same distance apart from its neighbor, but that is not necessary.

Continuing with this partition, we use the notation Δx_k to represent the width of the k^{th} subinterval. Recall that the width of an interval is the difference between its endpoints.

Subinterval	Width
$[x_0, x_1]$	$\Delta x_1 = x_1 - x_0$
$[x_1, x_2]$	$\Delta x_2 = x_2 - x_1$
$[x_2, x_3]$	$\Delta x_3 = x_3 - x_2$

2c. Select x-Values Within Each Partition

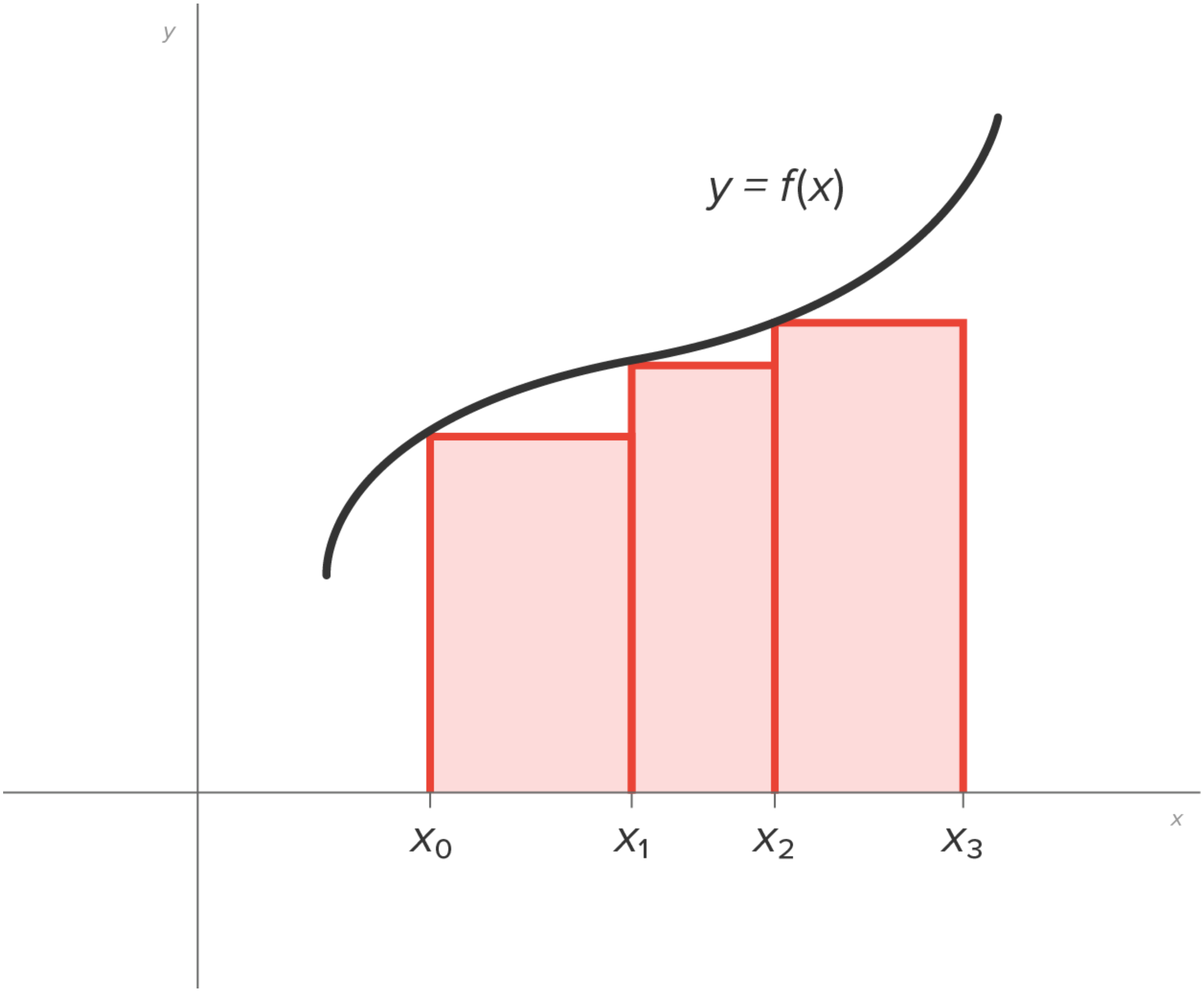
Let $c_k =$ the value of x used in the k^{th} subinterval. There are popular choices for c_k:

- The left endpoint of each subinterval
- The right endpoint of each subinterval
- The midpoint of each subinterval

Of course, we are not forced to use any one of these, but these are the most convenient.

2d. Form the Riemann Sum

Consider the figure shown below:



As the rectangles suggest, the left-hand endpoint was used in each sub-interval to set the height of the rectangle. This means:

Subinterval	Value Chosen	Width
$[x_0, x_1]$	$c_1 = x_0$	$\Delta x_1 = x_1 - x_0$
$[x_1, x_2]$	$c_2 = x_1$	$\Delta x_2 = x_2 - x_1$
$[x_2, x_3]$	$c_3 = x_2$	$\Delta x_3 = x_3 - x_2$

So, we can say:

- Area of the first rectangle: $f(c_1) \cdot \Delta x_1$
- Area of the second rectangle: $f(c_2) \cdot \Delta x_2$
- Area of the third rectangle: $f(c_3) \cdot \Delta x_3$

Then, the approximation for the area between $f(x)$ and the x-axis is the sum of these areas. Written using sigma notation, the Riemann sum is:

$$\sum_{k=1}^{3} f(c_k) \cdot \Delta x_k$$

In general, here is the definition (formula) for a Riemann sum.

FORMULA TO KNOW

Riemann Sum

When approximating the area between a nonnegative function $y = f(x)$ and the x-axis by using n rectangles, the summation $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called the Riemann sum, where c_k is a value of x in the k^{th} subinterval, and Δx_k is the width of the k^{th} subinterval.

3. Using Riemann Sums to Calculate Area

Now that we have all the definitions, let's compute a few Riemann sums.

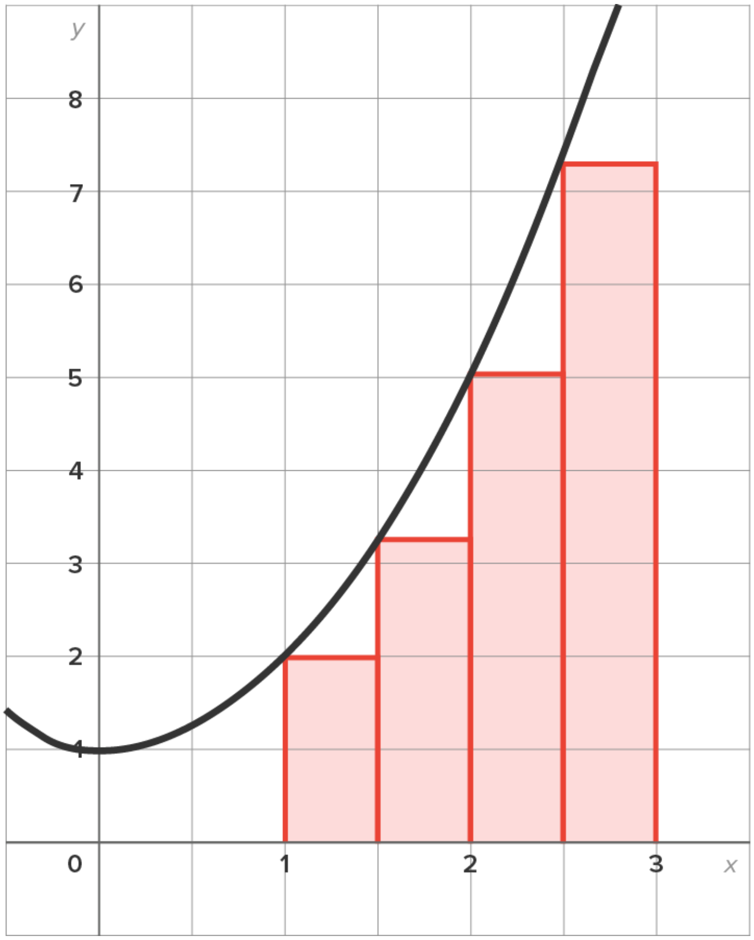
EXAMPLE Use a Riemann sum with 4 rectangles of equal width to approximate the area between $y = x^2 + 1$ and the x-axis on the interval $[1, 3]$. Use the left-hand endpoint of each subinterval.

Since each subinterval will have equal width, that width is $\frac{\text{width of } [1, 3]}{4} = \frac{2}{4} = 0.5$.

Based on the problem, we have the following information:

Subinterval	Width of Subinterval	Value Chosen in Each Subinterval
$[1, 1.5]$	0.5	1
$[1.5, 2]$	0.5	1.5
$[2, 2.5]$	0.5	2
$[2.5, 3]$	0.5	2.5

Here's a picture of the graph with the rectangles that were used:



Then, the Riemann sum is:

$\sum_{k=1}^{4} f(c_k)\Delta x_k = f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5$	Use the Riemann sum formula.
$= 0.5[f(1)+f(1.5)+f(2)+f(2.5)]$	Factor out 0.5.
$= 0.5(2+3.25+5+7.25)$	Substitute values: $f(1)=2$, $f(1.5)=3.25$, $f(2)=5$, $f(2.5)=7.25$
$= 8.75$	Simplify.

Thus, an approximation of the area is 8.75 units^2.

BIG IDEA

When the width of each subinterval is the same, we call the width of the interval Δx since they are all the same, then $\Delta x = \frac{b-a}{n}$.

TRY IT

Use a Riemann sum with 4 rectangles of equal width to approximate the area between $y = x^2 + 1$ and the x-axis on the interval $[1, 3]$. Use the right-hand endpoint of each subinterval. Note, this is the same information as in the last example, except that right-hand endpoints are used.

Approximate the area. +

Refer to the figure in the last example. The right-hand x-values are 1.5, 2, 2.5, and 3.

Recall that the width of each rectangle is 0.5 (side along the x-axis).

From the function $f(x) = x^2 + 1$, we have $f(1.5) = 3.25$, $f(2) = 5$, $f(2.5) = 7.25$, and $f(3) = 10$.

Then, the right-hand area estimate is:

$$\begin{aligned} 0.5[f(1.5) + f(2) + f(2.5) + f(3)] &= 0.5[3.25 + 5 + 7.25 + 10] \\ &= 0.5[25.5] \\ &= 12.75 \text{ units}^2 \end{aligned}$$

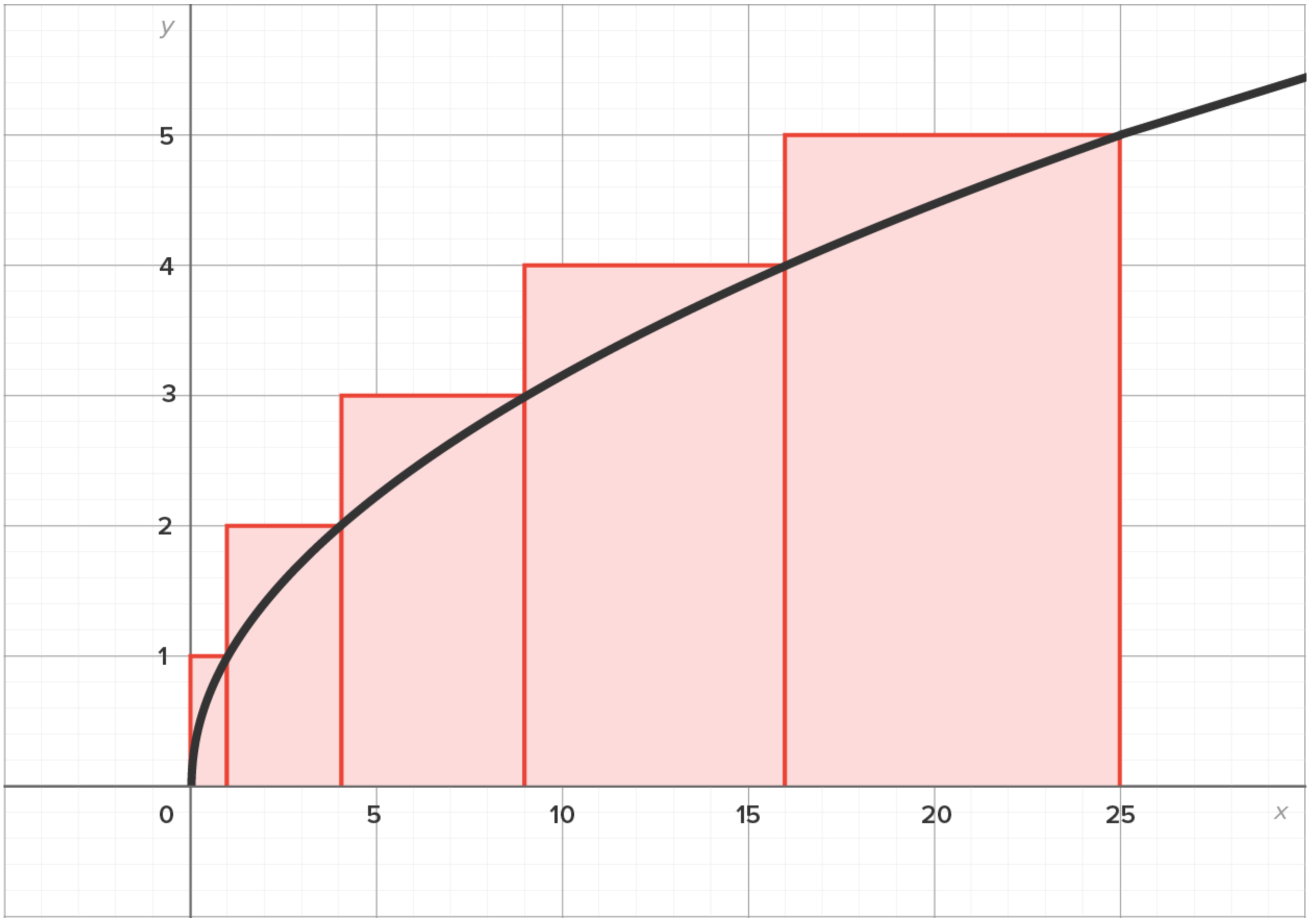
Let’s look at an example where the widths of the intervals are not the same.

EXAMPLE Consider the function $f(x) = \sqrt{x}$ on the interval $[0, 25]$. Estimate the area between $f(x)$ and the x-axis by using the partition $\{0, 1, 4, 9, 16, 25\}$. Use the right-hand endpoint of each subinterval.

Since there are 6 numbers in the partition, there are 5 rectangles. The table shows the information we need to set this up:

Subinterval	Width of Subinterval	Value Chosen in Each Subinterval
$[0, 1]$	1	1
$[1, 4]$	3	4
$[4, 9]$	5	9
$[9, 16]$	7	16
$[16, 25]$	9	25

The graph of $f(x)$ along with the rectangles is shown below.



Then, the Riemann sum is $\sum_{k=1}^{5} f(c_k)\Delta x_k$.

$\sum_{k=1}^{5} f(c_k)\Delta x_k = f(1)\cdot 1 + f(4)\cdot 3 + f(9)\cdot 5 + f(16)\cdot 7 + f(25)\cdot 9$	Use the Riemann sum formula.
$= 1(1) + 2(3) + 3(5) + 4(7) + 5(9)$	Substitute values: $f(1) = 1$, $f(4) = 2$, $f(9) = 3$, $f(16) = 4$, $f(25) = 5$
$= 95$	Simplify.

Thus, the approximation for the area is 95 units^2.

WATCH

In this video, we will use a Riemann sum to approximate the area below the graph of $f(x) = x^2 + 2$ on the interval $[0, 4]$ using 4 rectangles of equal width, using the left-hand endpoints of each subinterval.

THINK ABOUT IT

What effect would increasing the number of rectangles (partitions) have on the estimate in terms of the actual area?

SUMMARY

In this lesson, you learned that a **Riemann sum** provides a systematic way to approximate the area between a curve $y=f(x)$ and the x-axis on the interval $[a, b]$, by obtaining the sum from the areas of rectangles. You learned that when **finding the Riemann sum**, there are several quantities that need to be established first: **find the partition and subintervals**; **find the width of each subinterval**; **select x-values within each partition**; and finally, **form the Riemann sum**. Using this knowledge, you were then able to explore several examples of **using Riemann sums to calculate area**. Many applications we will investigate later in this course are based on Riemann sums, which makes this a very important topic to understand.

TERMS TO KNOW

Partition

A set of x-values that are used to split the interval $[a, b]$ into smaller intervals.

Riemann Sum

The sum obtained from the areas of rectangles that are used to approximate the area between a curve and the x-axis.

Subinterval

A smaller interval that is part of a larger interval.

FORMULAS TO KNOW

Riemann Sum

When approximating the area between a nonnegative function $y=f(x)$ and the x-axis by using *n* rectangles, the summation $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called the Riemann Sum, where c_k is a value of *x* in the k^{th} subinterval, and Δx_k is the width of the k^{th} subinterval.