Logarithmic Differentiation

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WHAT'S COVERED

In this lesson, you will find derivatives of combinations of products, quotients, and powers by using a technique called logarithmic differentiation. Specifically, this lesson will cover:

1. Defining Logarithmic Differentiation

While we have a vast set of rules we could use to find the derivative of most any function, there are still functions beyond our reach.

For example, consider the function $y=x^x$. While we can find the derivative of $y=x^n$ (exponent is constant) and $y=a^x$ (base is constant), there is no rule that handles the case when both the base and exponent are variables.

Also, consider the function $y=\frac{x\cdot\cos x}{2x+1}$. While this could be differentiated using the quotient and product rules, it would be very lengthy and cumbersome.

These are examples of functions that could benefit from logarithmic differentiation. Here is how it works:

STEP BY STEP

Given $y=f(x)$:

1. Take the natural logarithm of both sides: $\ln y=\ln f(x)$.
2. Rewrite the right-hand side using properties of logarithms.
3. Take the derivative of both sides with respect to x (implicitly).
4. Solve for $\frac{dy}{dx}$.

HINT

This technique is only useful when $f(x)$ is some combination of powers, products, or quotients.

2. Finding Derivatives Using Logarithmic Differentiation

EXAMPLE Using logarithmic differentiation, find the derivative of $y = x^x$. Since both the base and power are variables, logarithmic differentiation will be useful.

$y = x^x$	Start with the original function.
$\ln y = \ln x^x$	Apply the natural logarithm to both sides.
$\ln y = x \ln x$	Use the property of logarithms: $\ln a^b = b \ln a$
$\frac{1}{y} \cdot \frac{dy}{dx} = (1)\ln x + x\left(\frac{1}{x}\right)$	Take the derivative of both sides: $D[\ln y] = \frac{1}{y} \cdot \frac{dy}{dx}$ Apply the product rule for $D[x \cdot \ln x] = D[x] \cdot \ln x + x \cdot D[\ln x]$.
$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + 1$	Simplify the right-hand side.
$\frac{dy}{dx} = y(\ln x + 1)$	Solve for $\frac{dy}{dx}$ by multiplying both sides by y.
$\frac{dy}{dx} = x^x(1 + \ln x)$	Substitute $y = x^x$ so that the derivative is a function of x alone.

Thus, the derivative is $\frac{dy}{dx} = x^x(1 + \ln x)$.

TRY IT

Consider the function $y = (\sin x)^x$.

Using logarithmic differentiation, find the derivative. +

First, apply the natural logarithm to both sides, then rewrite using properties of logarithms:

$$\ln y = \ln(\sin x)^x$$

$$\ln y = x \cdot \ln(\sin x)$$

Next, take the derivative of both sides with respect to x:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \cdot \ln(\sin x)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\sin x) \cdot D[x] + x \cdot D[\ln(\sin x)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\sin x) \cdot 1 + x \cdot \frac{1}{\sin x} \cdot D[\sin x]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\sin x) + x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\sin x) + x \cdot \cot x$$

To solve for $\frac{dy}{dx}$, multiply both sides by *y*:

$$\frac{dy}{dx} = y[\ln(\sin x) + x \cdot \cot x]$$

To finalize the answer, replace *y* with $(\sin x)^2$:

$$\frac{dy}{dx} = (\sin x)^x[\ln(\sin x) + x \cot x]$$

Now, let's see how logarithmic differentiation can help with a combination of products, quotients, and/or powers.

➯ EXAMPLE Using logarithmic differentiation, find the derivative of $y = \frac{x \cdot \cos x}{2x+1}$. Since this function is a product within a quotient, logarithmic differentiation will be very useful to compute the derivative.

$y = \frac{x \cdot \cos x}{2x+1}$	Start with the original function.
$\ln y = \ln\left(\frac{x \cdot \cos x}{2x+1}\right)$	Apply the natural logarithm to both sides.
$\ln y = \ln x + \ln \cos x - \ln(2x+1)$	Use the properties of logarithms on the right-hand side.
$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cos x} \cdot (-\sin x) - \frac{1}{2x+1} \cdot 2$	Find the derivatives of each term.
$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} - \tan x - \frac{2}{2x+1}$	Simplify the right-hand side.
$\frac{dy}{dx} = y\left(\frac{1}{x} - \tan x - \frac{2}{2x+1}\right)$	Solve for $\frac{dy}{dx}$ by multiplying both sides by *y*.
$\frac{dy}{dx} = \frac{x \cdot \cos x}{2x+1}\left(\frac{1}{x} - \tan x - \frac{2}{2x+1}\right)$	Substitute $y = \frac{x \cdot \cos x}{2x+1}$ so that the derivative is a function of *x* alone.

Thus, $\frac{dy}{dx} = \frac{x \cdot \cos x}{2x+1}\left(\frac{1}{x} - \tan x - \frac{2}{2x+1}\right)$.

TRY IT

Consider the function $y = x^2 e^{-4x} \sin x$.

Using logarithmic differentiation, find the derivative. +

First, apply the “ln” to both sides, then use properties to expand the right-hand side.

$$\ln y = \ln(x^2 e^{-4x} \sin x)$$

$$\ln y = \ln(x^2) + \ln(e^{-4x}) + \ln(\sin x)$$

$$\ln y = 2\ln x - 4x + \ln(\sin x)$$

Next, take the derivative of both sides with respect to *x*:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[2\ln x - 4x + \ln(\sin x)]$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[2\ln x] - \frac{d}{dx}[4x] + \frac{d}{dx}[\ln(\sin x)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\left(\frac{1}{x}\right) - 4 + \frac{1}{\sin x} \cdot D[\sin x]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{x} - 4 + \frac{\cos x}{\sin x}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{x} - 4 + \cot x$$

To solve for $\frac{dy}{dx}$, multiply both sides by *y*:

$$\frac{dy}{dx} = y\left[\frac{2}{x} - 4 + \cot x\right]$$

To finalize the answer, replace *y* with $x^2 e^{-4x} \sin x$:

$$\frac{dy}{dx} = x^2 e^{-4x} \sin x\left[\frac{2}{x} - 4 + \cot x\right]$$

EXAMPLE Consider the function $f(x) = \frac{\sqrt[4]{x^2+9}}{(2x+5)^7}$.

The goal is to find $f'(x)$, which on the surface looks very complicated. Since the numerator and denominator are both powers of other expressions, we'll use logarithmic differentiation.

$y = \frac{(x^2+9)^{1/4}}{(2x+5)^7}$	Rename the function as y (this makes notation easier to follow as we move through the derivative). Also write the radical in exponential form.
$\ln y = \ln\left(\frac{(x^2+9)^{1/4}}{(2x+5)^7}\right)$	Apply the natural logarithm to both sides.
$\ln y = \ln(x^2+9)^{1/4} - \ln(2x+5)^7$	Apply the property $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
$\ln y = \frac{1}{4}\ln(x^2+9) - 7\ln(2x+5)$	Apply the property $\ln(x^n) = n\ln x$.
$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x^2+9}\right)(2x) - 7\left(\frac{1}{2x+5}\right)(2)$	Apply the derivative of both sides with respect to *x*. Using the derivative formula $D[\ln u] = \frac{1}{u}\cdot u'$.
$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{x}{2(x^2+9)} - \frac{14}{2x+5}$	Simplify each fraction.
$\frac{dy}{dx} = y\left(\frac{x}{2(x^2+9)} - \frac{14}{2x+5}\right)$	Multiply both sides by y in order to solve for $\frac{dy}{dx}$.
$\frac{dy}{dx} = \frac{\sqrt[4]{x^2+9}}{(2x+5)^7}\left(\frac{x}{2(x^2+9)} - \frac{14}{2x+5}\right)$	Replace y with $\frac{\sqrt[4]{x^2+9}}{(2x+5)^7}$.

In terms of the original notation, we would say that $f'(x) = \frac{\sqrt[4]{x^2+9}}{(2x+5)^7}\left(\frac{x}{2(x^2+9)} - \frac{14}{2x+5}\right)$.

WATCH

Use logarithmic differentiation to find the derivative of $y = \sqrt[7]{\frac{x^2(2x-7)^3}{(4x+5)^5}}$.

SUMMARY

In this lesson, you learned that through **logarithmic differentiation**, it is now possible to find the derivative of functions with variable bases and powers (such as $y = x^x$). It is also much simpler to **find derivatives** of combinations of products, quotients, and powers **using logarithmic differentiation**.