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The Fundamental Theorem of Calculus

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WHAT'S COVERED

In this lesson, you will apply the fundamental theorem of calculus to definite integrals. Specifically, this lesson will cover:

1. The Fundamental Theorem of Calculus

Recall the fundamental theorem of calculus:

FORMULA TO KNOW

Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of a continuous function $f(x)$ on the interval $[a, b]$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$.

To show that we are evaluating $F(x)$ at $x = a$ and $x = b$ and then subtracting, we use the notation:

$$F(x)|_a^b$$

Note that $F(x)$ is any antiderivative of $f(x)$. Recall that the general antiderivative is $F(x) + C$, where C is an arbitrary constant.

Then, evaluating the definite integral, we obtain:

$$(F(x)+C)\Big|_a^b = (F(b)+C) - (F(a)+C)$$

$$= F(b) + C - F(a) - C$$
$$= F(b) - F(a)$$

What this means is that when evaluating a definite integral, the result is the same regardless of the value of C used. Therefore, to keep it simple, use $C = 0$.

BIG IDEA

When evaluating a definite integral, use $C = 0$ (which means not to write the integration constant).

2. Using the Fundamental Theorem of Calculus

We can now use this powerful theorem to evaluate definite integrals, which also includes finding exact areas of regions we could only approximate before.

EXAMPLE Evaluate the definite integral: $\int_1^2 4x^2 dx$

$\int_1^2 4x^2 dx$	Start with the original expression.
$= \frac{4}{3}x^3 \Big\|_1^2$	Find the antiderivative. Remember: no "+C" is required.
$= \frac{4}{3}(2)^3 - \frac{4}{3}(1)^3$	Evaluate at the upper and lower endpoints.
$= \frac{32}{3} - \frac{4}{3}$	Evaluate the exponents.
$= \frac{28}{3}$	Simplify.

Thus, $\int_1^2 4x^2 dx = \frac{28}{3}$.

EXAMPLE Evaluate $\int_0^1 (e^{2x} - 2e^x) dx$.

$\int_0^1 (e^{2x} - 2e^x) dx$	Start with the original expression.
$= \left(\frac{1}{2}e^{2x} - 2e^x\right)\Big\|_0^1$	Find the antiderivative. Remember: no "+C" is required.

$$=\left(\frac{1}{2}e^{2(1)}-2e^{1}\right)-\left(\frac{1}{2}e^{2(0)}-2e^{0}\right)$$ Evaluate at the upper and lower endpoints.

$$=\frac{1}{2}e^{2}-2e-\frac{1}{2}+2$$ Evaluate each parentheses.

$$=\frac{1}{2}e^{2}-2e+\frac{3}{2}$$ Simplify.

Thus, $\int_{0}^{1}(e^{2x}-2e^{x})dx=\frac{1}{2}e^{2}-2e+\frac{3}{2}$.

TRY IT

Consider the definite integral $\int_{4}^{9}(x-\sqrt{x})dx$.

Evaluate the definite integral, giving the exact answer. +

$$\int_{4}^{9}(x-x^{1/2})dx$$ Rewrite radical as a fractional power.

$$=\frac{x^{2}}{2}-\frac{2}{3}x^{3/2}\Big|_{4}^{9}$$ Use the power rule to find the antiderivative of each term.

Note: $\int x^{1/2}dx=\frac{x^{3/2}}{\left(\frac{3}{2}\right)}=\frac{2}{3}x^{3/2}$

$$=\left(\frac{9^{2}}{2}-\frac{2}{3}(9)^{\frac{3}{2}}\right)-\left(\frac{4^{2}}{2}-\frac{2}{3}(4)^{\frac{3}{2}}\right)$$ Evaluate the antiderivative at each endpoint, then subtract. This is the Fundamental Theorem of Calculus.

$$=\left(\frac{81}{2}-18\right)-\left(8-\frac{16}{3}\right)$$ Simplify within each set of parentheses.

$$=\frac{45}{2}-\frac{8}{3}$$ Simplify further.

$$=\frac{119}{6}$$ Combine the fractions.

Thus, $\int_{4}^{9}(x-\sqrt{x})dx=\frac{119}{6}$.

WATCH

Check out this video to see the example $\int_{1}^{4}\frac{3}{x^{4}}dx$.

3. Fundamental Theorem of Calculus with *u*-Substitutions

When the antiderivative requires *u*-substitution, there are two ways to evaluate the definite integral:

- Finding the indefinite integral first
- Changing the limits of integration to match the substitution

3a. Finding the Indefinite Integral First

EXAMPLE Evaluate the definite integral: $\int_0^3 x^2\sqrt{x^3+9}\,dx$

First, find the indefinite integral:

$\int x^2\sqrt{x^3+9}\,dx$	Start with the original expression.
$=\int x^2(x^3+9)^{1/2}dx$	Rewrite $\sqrt{x^3+9}$ as $(x^3+9)^{1/2}$.
$=\frac{1}{3}\int u^{1/2}du$	Let $u=x^3+9$. Then, $du=3x^2dx$, or $\frac{1}{3}du=x^2dx$.
$=\frac{1}{3}\cdot\frac{2}{3}u^{3/2}+C$	Use the power rule. Dividing by $\frac{3}{2}$ is equivalent to multiplying by $\frac{2}{3}$.
$=\frac{2}{9}u^{3/2}+C$	Simplify.
$=\frac{2}{9}(x^3+9)^{3/2}+C$	Back-substitute $u=x^3+9$.

Now, evaluate the definite integral:

$=\left.\frac{2}{9}(x^3+9)^{3/2}\right\|_0^3$	Apply the fundamental theorem of calculus. Note, "+*C*" is omitted since we are evaluating a definite integral.
$=\frac{2}{9}(3^3+9)^{3/2}-\frac{2}{9}(0^3+9)^{3/2}$	Substitute the upper and lower endpoints.
$=\frac{2}{9}(36)^{3/2}-\frac{2}{9}(9)^{3/2}$	Evaluate parentheses.
$=48-6$	Evaluate.
$=42$	Simplify.

In conclusion, $\int_0^3 x^2\sqrt{x^3+9}\,dx = 42$.

TRY IT

Consider the definite integral $\int_1^2 \frac{1}{(2x+1)^3}dx$.

Evaluate the definite integral. Give an exact answer. +

First, find the antiderivative:

$=\int \frac{1}{u^3}\cdot\frac{1}{2}du$	Let $u=2x+1$. Then, $du=2dx$, and $dx=\frac{1}{2}du$.
$=\frac{1}{2}\int u^{-2}du$	Move the $\frac{1}{2}$ outside and rewrite $\frac{1}{u^3}$ as u^{-3}.
$=\frac{1}{2}\left(\frac{-1}{2}u^{-2}\right)$	Use the power rule to find the antiderivative. Note: No "+C" is used here since we are ultimately evaluating a definite integral.
$=\frac{-1}{4u^2}$	Simplify and rewrite in terms of positive exponents.
$=\frac{-1}{4(2x+1)^2}$	Replace u with $2x+1$.

Now, evaluate at the endpoints:

$\left.\frac{-1}{4(2x+1)^2}\right\|_1^2$	Notation to show the evaluation of the antiderivative at $x=1$ and $x=2$.
$=\left(\frac{-1}{4(2(2)+1)^2}\right)-\left(\frac{-1}{4(2(1)+1)^2}\right)$	Substitute $x=2$ and $x=1$ into the antiderivative, then subtract. This is the Fundamental Theorem of Calculus.
$=\left(\frac{-1}{4(5)^2}\right)-\left(\frac{-1}{4(3)^2}\right)$	Simplify the expression.
$=\frac{-1}{100}+\frac{1}{36}$	Simplify the expression.
$=\frac{4}{225}$	Combine the fractions.

Thus, $\int_1^2 \frac{1}{(2x+1)^3}dx = \frac{4}{225}$.

EXAMPLE Evaluate the definite integral: $\int_1^{e^2} \frac{(\ln x)^3}{x}dx$

First, find the indefinite integral:

$\int \frac{(\ln x)^3}{x}dx$	Start with the original expression.
$=\int u^3 du$	First, write $\frac{(\ln x)^3}{x} = (\ln x)^3 \cdot \frac{1}{x}$. Let $u = \ln x$. Then, $du = \frac{1}{x}dx$.
$=\frac{1}{4}u^4 + C$	Use the power rule.
$=\frac{1}{4}(\ln x)^4 + C$	Back-substitute $u = \ln x$.

Now, evaluate the definite integral:

$=\frac{1}{4}(\ln x)^4\Big\vert_1^{e^2}$	Apply the fundamental theorem of calculus. The "+*C*" is omitted since we are evaluating a definite integral.
$=\frac{1}{4}(\ln e^2)^4 - \frac{1}{4}(\ln 1)^4$	Substitute the upper and lower endpoints.
$=\frac{1}{4}(2)^4 - \frac{1}{4}(0)^4$	Evaluate parentheses. Recall that $\ln(e^k) = k$.
$=4-0$	Evaluate.
$=4$	Simplify.

In conclusion, $\int_1^{e^2} \frac{(\ln x)^3}{x}dx = 4$.

WATCH

In this video, the example $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x dx$ is presented.

3b. Changing the Limits of Integration to Match the Substitution

Now we will look at a method where we change the limits of integration. This method allows us to avoid having to back-substitute.

We'll look at the same examples as before to help make the connection.

EXAMPLE Evaluate the definite integral: $\int_0^3 x^2\sqrt{x^3+9}\,dx$

First, make the *u*-substitution:

$\int_0^3 x^2\sqrt{x^3+9}\,dx$	Start with the original expression.
$=\frac{1}{3}\int u^{1/2}du$	First, write $\sqrt{x^3+9}=u^{1/2}$. Let $u=x^3+9$. Then, $du=3x^2dx$, or $\frac{1}{3}du=x^2dx$.

At this point, we write the original definite integral as a new definite integral, this time with *u*-values as the limits of integration.

The original definite integral is taken from $x=0$ to $x=3$. What are the corresponding values of *u*? We use the substitution $u=x^3+9$:

- When $x=0$, $u=0^3+9=9$.
- When $x=3$, $u=3^3+9=36$.

Thus, through our substitution, $\int_0^3 x^2\sqrt{x^3+9}\,dx=\frac{1}{3}\int_9^{36} u^{1/2}du$. We can now continue.

$=\frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big\|_9^{36}$	Apply the fundamental theorem of calculus. The "+*C*" is omitted since we are evaluating a definite integral.
$=\frac{2}{9}u^{3/2}\Big\|_9^{36}$	Simplify before evaluating.
$=\frac{2}{9}(36)^{3/2}-\frac{2}{9}(9)^{3/2}$	Substitute the upper and lower endpoints.
$=48-6$	Evaluate.
$=42$	Simplify.

Thus, $\int_0^3 x^2\sqrt{x^3+9}\,dx=42$, which is the same as the result from the last method.

Having done the same problem two different ways now, let's compare the methods.

By using the indefinite integral, we had these steps:

$$\int_0^3 x^2(x^3+9)^{1/2}dx$$

$$= \frac{2}{9}(x^3+9)^{3/2}\Big|_0^3$$

$$= \frac{2}{9}(36)^{3/2} - \frac{2}{9}(9)^{3/2}$$

$$= 42$$

After making the *u*-substitution and replacing the *x*-values with *u*-values, we had these steps:

$$\int_0^3 x^2(x^3+9)^{1/2}dx$$

$$= \frac{1}{3}\int_9^{36} u^{1/2}du$$

$$= \frac{2}{9}u^{3/2}\Big|_9^{36}$$

$$= \frac{2}{9}(36)^{3/2} - \frac{2}{9}(9)^{3/2}$$

$$= 42$$

REFLECT

Now that you have seen two methods to evaluate definite integrals with *u*-substitution, which method do you think is easier, and why? If you can't answer this yet, wait to answer this question until you have gone through more examples.

TRY IT

Consider the definite integral $\int_0^3 \frac{1}{\sqrt{5x+1}}dx$.

Write the definite integral with values of *u* as limits of integration. +

$$\int \frac{1}{\sqrt{u}} \cdot \frac{1}{5}du$$

Let $u = 5x + 1$.

Then, $du = 5dx$, and $dx = \frac{1}{5}du$.

Note: this integral doesn't have limits yet. We will work on this in the next steps.

$= \int_1^{16} \frac{1}{\sqrt{u}} \cdot \frac{1}{5} du$ When $x = 0$, $u = 5(0) + 1 = 1$.

When $x = 3$, $u = 5(3) + 1 = 16$.

Replace 0 with 1 and 3 with 16 in the integral.

$= \frac{1}{5} \int_1^{16} u^{-1/2} du$ Move the $\frac{1}{5}$ outside and rewrite $\frac{1}{\sqrt{u}}$ with $u^{-1/2}$.

The simplified integral is $\frac{1}{5} \int_1^{16} u^{-1/2} du$.

Evaluate the integral in the answer from the first question. +

Now, use the Fundamental Theorem of Calculus to evaluate.

$\frac{1}{5} \int_1^{16} u^{-1/2} du = \frac{1}{5} [2u^{1/2}]_1^{16}$ Using the power rule, $\int u^{-1/2} du = \frac{1}{1/2} u^{1/2} = 2u^{1/2}$.

$= \frac{2}{5}(16^{1/2} - 1^{1/2})$ Simplify $\frac{1}{5} \cdot 2 = \frac{2}{5}$, apply the Fundamental Theorem of Calculus.

$= \frac{2}{5}(4 - 1)$ Simplify the ½ powers.

$= \frac{6}{5}$ Simplify.

In conclusion, $\int_0^3 \frac{1}{\sqrt{5x+1}} dx = \frac{6}{5}$.

Now, let’s take another look at a previous example so we can compare.

➩ EXAMPLE Evaluate the definite integral: $\int_1^{e^2} \frac{(\ln x)^3}{x} dx$

$\int_1^{e^2} \frac{(\ln x)^3}{x} dx$ Start with the original expression.

$= \int_0^2 u^3 du$ First, write $\frac{(\ln x)^3}{x} = (\ln x)^3 \cdot \frac{1}{x}$. Let $u = \ln x$, then $du = \frac{1}{x} dx$. When $x = 1$, $u = \ln 1 = 0$. When $x = e^2$, $u = \ln e^2 = 2$.

$$= \frac{1}{4}u^4\Big|_0^2$$ Apply the fundamental theorem of calculus. The "+*C*" is omitted since we are evaluating a definite integral.

$$= \frac{1}{4}(2)^4 - \frac{1}{4}(0)^4$$ Substitute the upper and lower endpoints.

$$= 4 - 0$$ Evaluate.

$$= 4$$ Simplify.

Just as before, $\int_1^{e^2} \frac{(\ln x)^3}{x}dx = 4$.

WATCH

Here is a video showing an example writing the limits of integration with values of *u* for the definite integral $\int_0^{\frac{5\pi}{6}} \sin^4 x \cos x dx$.

REFLECT

How do you feel about this method for evaluating definite integrals with *u*-substitution versus the indefinite integral with back-substitution?

SUMMARY

In this lesson, you learned that **the fundamental theorem of calculus** uses antiderivatives to evaluate definite integrals exactly rather than with a Riemann sum. When *u*-substitution is not required, this is a rather straightforward process. When applying the **fundamental theorem of calculus with *u*-substitution**, you learned that there are two methods that could be used to evaluate the integral: 1) **finding the indefinite integral first**, and 2) **changing the limits of integration to match the substitution**. In later sections, we will revisit applications that utilized definite integrals, but in a different light now that we know how to evaluate definite integrals more generally rather than relying on using geometric formulas.

FORMULAS TO KNOW

Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of a continuous function $f(x)$ on the interval $[a, b]$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$.