Antiderivative Applications

by Sophia

WHAT'S COVERED

In this lesson, you will revisit the ideas of area and distance traveled now that we have a more general way to evaluate definite integrals (the fundamental theorem of calculus). Specifically, this lesson will cover:

1. Calculating Areas of Regions

Recall the following about areas and definite integrals:

1. When $f(x)$ is nonnegative on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the area of the region between the graph of $y=f(x)$ and the x-axis on $[a, b]$.

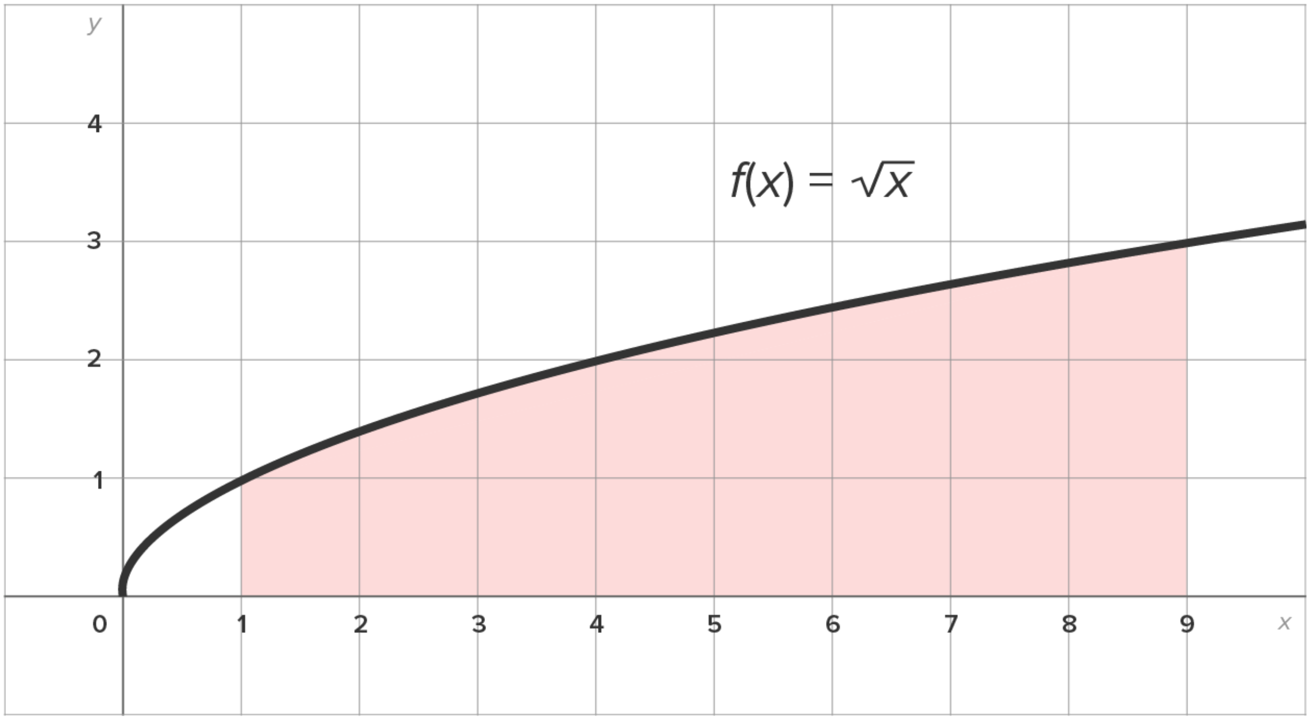
   That is, if the area of the region is *A* (a positive number), then $\int_a^b f(x)dx = A$.

2. When $f(x)$ is negative on the interval $[a, b]$, then $\int_a^b f(x)dx$ is the negative of the area of the region between the graph of $y=f(x)$ and the x-axis on $[a, b]$.

   That is, if the area of the region is *A* (a positive number), then $\int_a^b f(x)dx = -A$.

We use these ideas to find areas of regions that are above the x-axis, below the x-axis, or a combination of the two.

EXAMPLE Find the area of the region bounded by $f(x)=\sqrt{x}$ and the x-axis between $x=1$ and $x=9$. The region is shown in the figure below.



Since the region is above the x-axis, the value of the definite integral is equal to the area of the region.

The definite integral that describes this area is $\int_1^9 \sqrt{x}\,dx$.

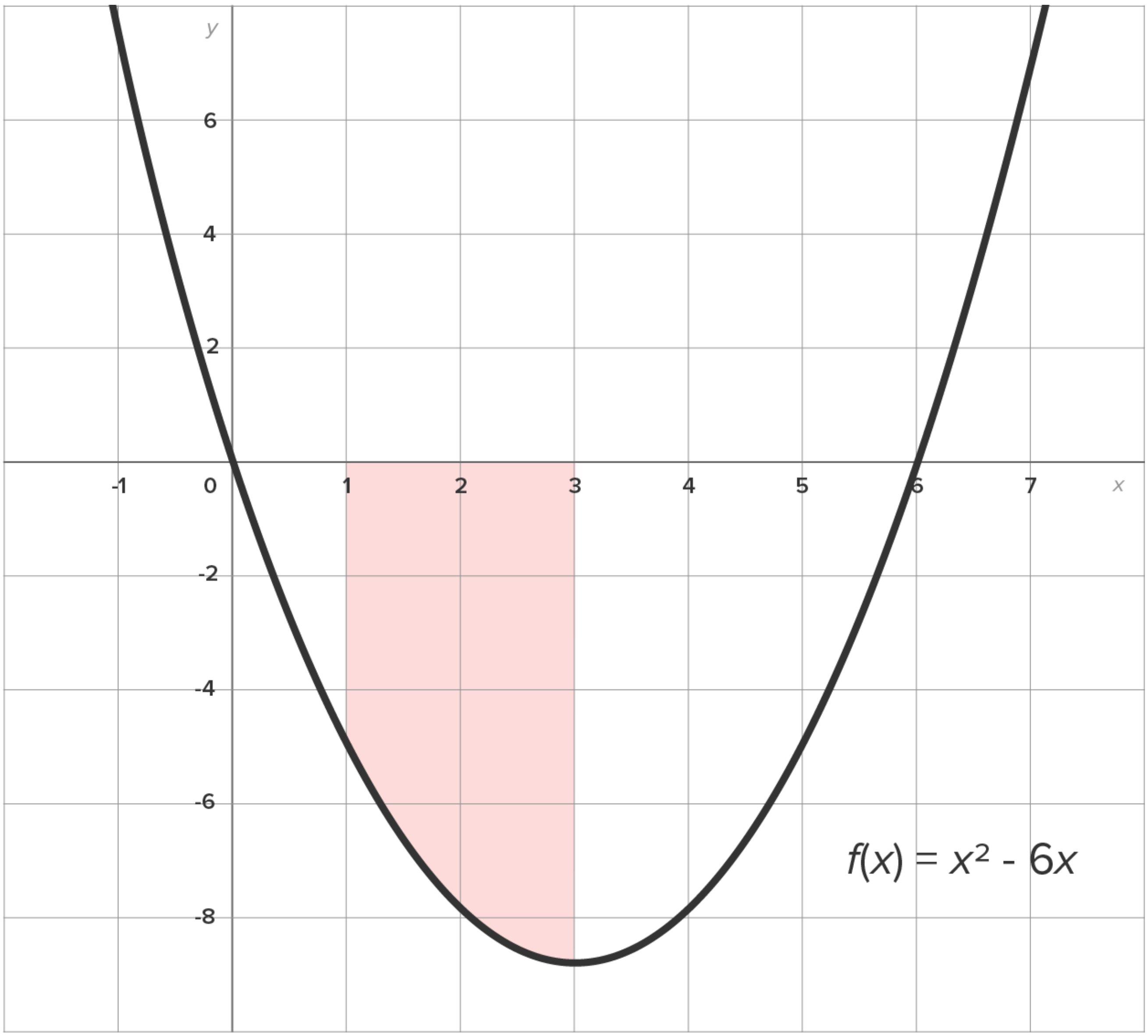
Now we evaluate:

$\int_1^9 \sqrt{x}\,dx$	Start with the original expression.
$= \int_1^9 x^{1/2}dx$	Rewrite as a power.
$= \left.\frac{2}{3}x^{3/2}\right\vert_1^9$	Apply the fundamental theorem of calculus and the power rule for antiderivatives.
$= \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2}$	Substitute the upper and lower endpoints.
$= \frac{2}{3}(27) - \frac{2}{3}$	Evaluate.
$= \frac{52}{3}$	Simplify.

In conclusion, the area of the region bounded by $f(x) = \sqrt{x}$ and the x-axis between $x = 1$ and $x = 9$ is equal to $\frac{52}{3}\text{units}^2$.

Now, let's look at a region that is below the x-axis.

➦ EXAMPLE Find the area of the region between the x-axis and the curve $f(x)=x^2-6x$ on the interval between $x=1$ and $x=3$. The region is shown in the figure below.



Since the region is entirely below the x-axis, we know that the definite integral will be negative. Thus, we'll evaluate $\int_a^b f(x)dx$ as usual, but remember that its value is the negative of the area.

$\int_1^3 (x^2-6x)dx$	Start with the definite integral that is tied to the area of the region.
$=\left(\frac{1}{3}x^3-3x^2\right)\Big\|_1^3$	Apply the fundamental theorem of calculus.

$= \left[\frac{1}{3}(3)^3 - 3(3)^2\right] - \left[\frac{1}{3}(1)^3 - 3(1)^2\right]$ Substitute the limits of integration and subtract. Grouping symbols are used to make the subtraction more clear.

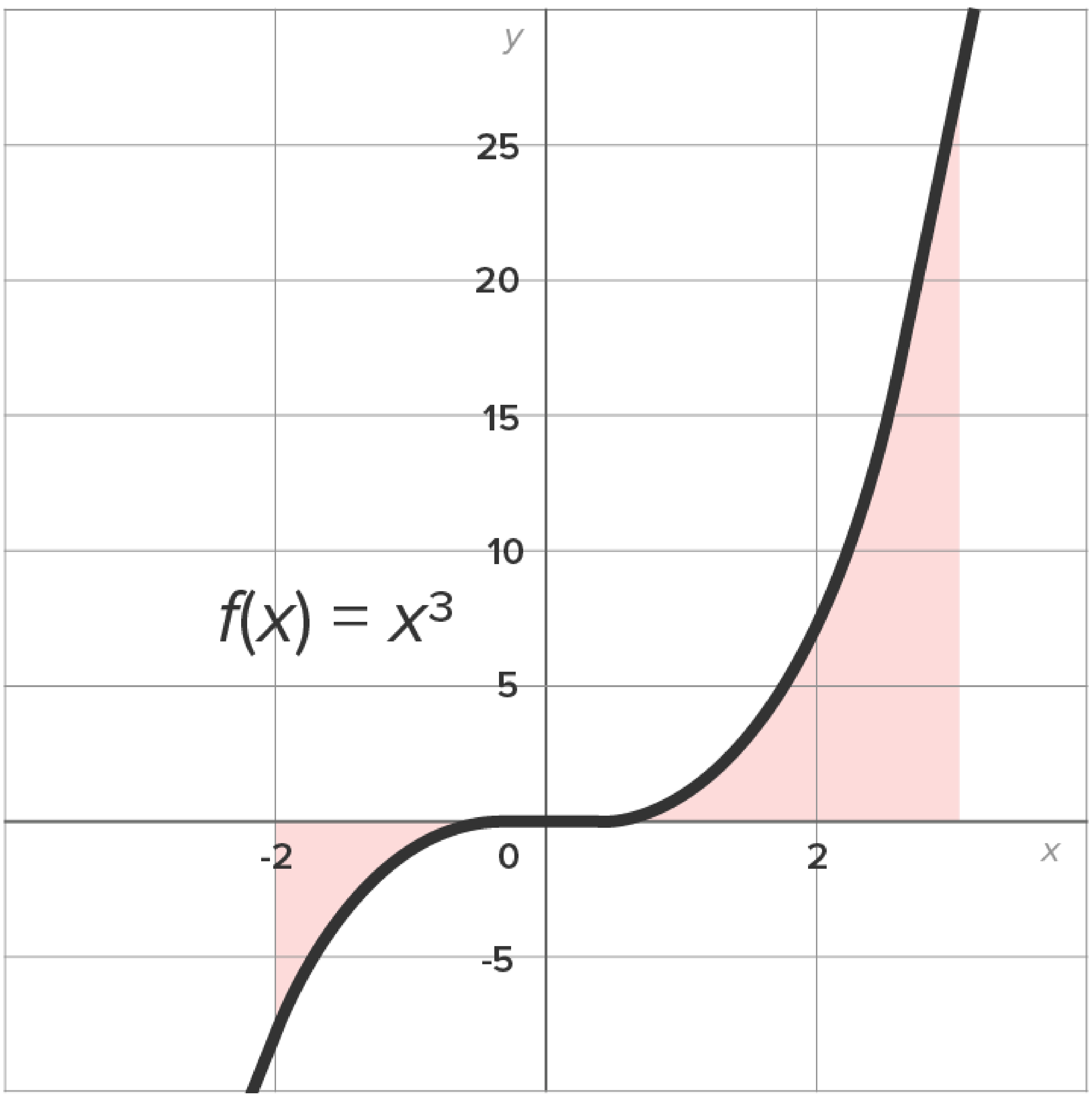
$= -18 - \left(\frac{-8}{3}\right)$ Evaluate each bracket.

$= \frac{-46}{3}$ Simplify.

The value of the definite integral is $\frac{-46}{3}$. Then, the area of the region is $\frac{46}{3}\text{units}^2$.

Let's look at a region that contains parts above and below the x-axis.

EXAMPLE Find the total area between the x-axis and the curve $f(x) = x^3$ between $x = -2$ and $x = 3$. The region is shown in the figure below.



Notice that part of the region is below the x-axis and part of it is above the x-axis.

- On the interval $[-2, 0]$, the region is below the x-axis.
- On the interval $[0, 3]$, the region is above the x-axis.

This means $\int_{-2}^{0} x^3 dx$ will give the negative of the area and $\int_{0}^{3} x^3 dx$ will give the area.

- The region on $[-2, 0]$:
  $\int_{-2}^{0} x^3 dx = \left.\frac{1}{4}x^4\right|_{-2}^{0} = \frac{1}{4}(0)^4 - \frac{1}{4}(-2)^4 = -4.$
- The region on $[0, 3]$:
  $\int_{0}^{3} x^3 dx = \left.\frac{1}{4}x^4\right|_{0}^{3} = \frac{1}{4}(3)^4 - \frac{1}{4}(0)^4 = \frac{81}{4}$

Since the first definite integral has value -4, the actual area of the region is 4.

Then, the total area of the region is $4 + \frac{81}{4} = \frac{97}{4} \text{units}^2$.

WATCH

Check out this video where substitution is required, that shows finding the area bounded by $f(x) = x\sqrt{25 - x^2}$, the x-axis, $x = -4$, and $x = 3$.

Now that you’ve seen a few examples, here are some examples for you to try.

TRY IT

Consider the region bounded by $f(x) = e^{2x} - x$, the x-axis, $x = 0$, and $x = 2$.

Find the exact area of the region. +

The region is completely above the x-axis, so the area is calculated by using the integral $\int_{0}^{2} (e^{2x} - x)dx$.

Now, evaluate the integral:

$$= \left.\frac{1}{2}e^{2x} - \frac{1}{2}x^2\right|_{0}^{2}$$

By the formula $\int e^{kx} dx = \frac{1}{k}e^{kx}$, $\int e^{2x} dx = \frac{1}{2}e^{2x}$.

By the power rule, $\int x dx = \frac{1}{2}x^2$.

$= \left(\frac{1}{2}e^{2(2)} - \frac{1}{2}(2)^2\right) - \left(\frac{1}{2}e^{2(0)} - \frac{1}{2}(0)^2\right)$ Substitute $x = 2$ and $x = 0$, then subtract.

$= \left(\frac{1}{2}e^4 - 2\right) - \left(\frac{1}{2}e^0 - 0\right)$ Simplify within the parentheses.

$= \frac{1}{2}e^4 - 2 - \frac{1}{2}$ Continue to simplify.

$= \frac{1}{2}e^4 - \frac{5}{2}$ Combine like terms.

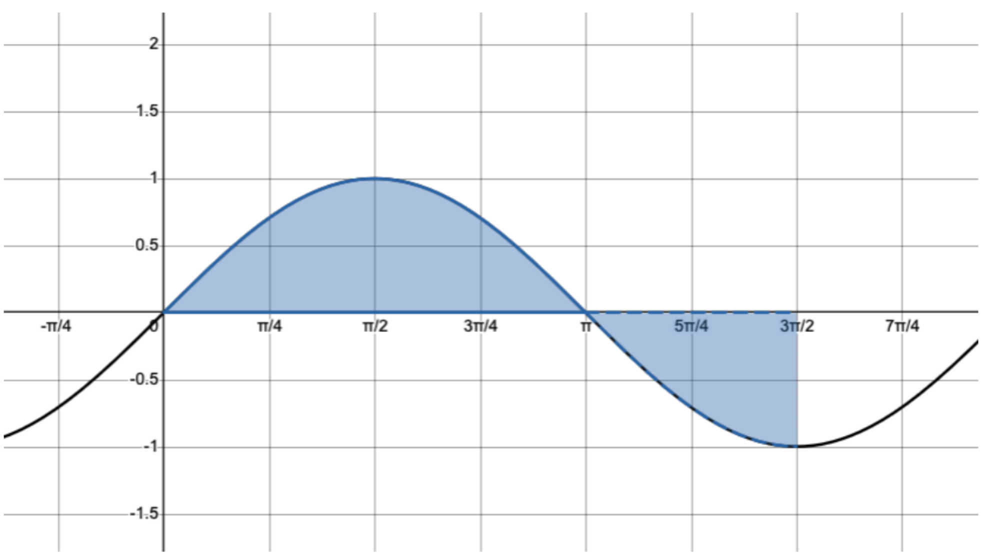
In conclusion, the area between $y = e^{2x} - x$ and the x-axis between $x = 0$ and $x = 2$ is equal to $\frac{1}{2}e^4 - \frac{5}{2}$ square units.

TRY IT

Consider the region bounded by $f(x) = \sin x$, the x-axis, $x = 0$, and $x = \frac{3\pi}{2}$.

Find the exact area of the region. +

The region is shown in the figure below.



The region is above the x-axis between $x = 0$ and $x = \pi$. The area of that region is found by evaluating the integral $\int_0^\pi \sin x\,dx$.

Evaluating gives:

$-\cos x\big\vert_0^{\pi}$	The antiderivative of $\sin x$ is $-\cos x$.
$= -\cos\pi - (-\cos 0)$	Evaluate $-\cos x$ at π and 0, then subtract.
$= -(-1) - (-1)$	$\cos\pi = -1$ and $\cos 0 = 1$.
$= 1 + 1$	Simplify.
$= 2$	Simplify.

Thus, the area of the region that's above the x-axis is 2 square units.

The region is below the x-axis between $x = \pi$ and $x = \frac{3\pi}{2}$. The area of that region is found by evaluating the integral $\int_0^{\pi} \sin x dx$, then finding the opposite.

Evaluating gives:

$-\cos x\big\vert_{\pi}^{3\pi/2}$	The antiderivative of $\sin x$ is $-\cos x$.
$= -\cos\left(\frac{3\pi}{2}\right) - (-\cos\pi)$	Evaluate $-\cos x$ at $\frac{3\pi}{2}$ and π, then subtract.
$= -(0) - (1)$	$\cos\left(\frac{3\pi}{2}\right) = 0$ and $\cos\pi = -1$.
$= 0 - 1$	Simplify.
$= -1$	Simplify.

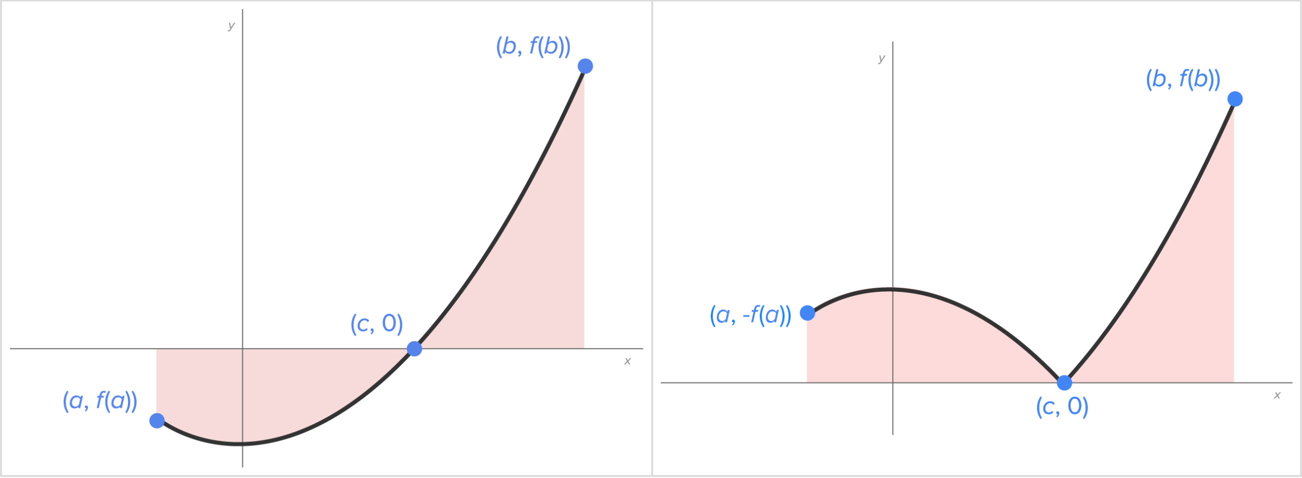
Thus, the area of the region that's below the x-axis is 1 square unit.

In conclusion, the area of the entire region is $2 + 1 = 3$ square units.

BIG IDEA

Consider the graphs of $y = f(x)$ and $y = |f(x)|$ shown below.

The Graph of $y = f(x)$ on $[a, b]$	The Graph of $y = \lvert f(x)\rvert$ on $[a, b]$



The regions on $[c, b]$ are identical. The regions on $[a, c]$ have the same area; one is above the x-axis, and the other is below the x-axis.

Since the graph of $y = |f(x)|$ is nonnegative on $[a, b]$, the definite integral $\int_a^b |f(x)|\,dx$ gives the area of the region between the graph of $y = |f(x)|$ and the x-axis between $x = a$ and $x = b$.

The drawback, however, is that $\int_a^b |f(x)|\,dx$ can be difficult to compute since finding antiderivatives with absolute value can be difficult if $f(x)$ changes sign over the interval $[a, b]$. However, if using technology, using $\int_a^b |f(x)|\,dx$ to calculate area is a nice way to find area, since it doesn't require a graph to calculate the area.

EXAMPLE Consider the region bounded by $f(x) = \sin x$, the x-axis, $x = 0$, and $x = \frac{3\pi}{2}$.

In a previous "TRY IT," you calculated the total area to be 3, but that was by using two integrals since part of the region is below the x-axis.

Using technology, $\int_0^{3\pi/2} |\sin x|\,dx = 3$.

As it turns out, $\int_a^b |f(x)|\,dx$ can be extended to represent distance, as you'll see in the next portion of this tutorial.

2. Calculating Distance Traveled and Net Change in Distance

Let $v(t)$ equal the velocity of an object at time *t*.

- If $v(t) > 0,$ the object is moving in a forward direction.
- If $v(t) < 0,$ the object is moving in a negative direction.

So, if $v(t)$ is the velocity of an object at time *t*, then $\int_a^b v(t)dt$ is the change in position between $t = a$ and $t = b.$

- If $\int_a^b v(t)dt$ is positive, then the object's final position is ahead of its starting point.

  (Example: if $v(t)$ represents upward velocity, then the object finishes above its starting position at $t = a$).
- If $\int_a^b v(t)dt$ is negative, then the object's final position is behind its starting point.

  (Example: if $v(t)$ represents upward velocity, then the object finishes below its starting position at $t = a$).
- If $\int_a^b v(t)dt = 0,$ then the object's final position is the same as its starting point.

It follows that $\int_a^b |v(t)|\, dt$ gives the total distance traveled (in either direction) between $t = a$ and $t = b.$ We will still compute this by examining regions.

➩ EXAMPLE An object has velocity $v(t) = 60 - 12\sqrt{t}$ feet per second, where *t* is the number of seconds.

a. What is the object's change in position after its first 100 seconds of travel?
b. What is the total distance traveled after the first 100 seconds?

Let's first find the object's change in position.

a. What is the object's change in position after its first 100 seconds of travel?

Note: since $v(t)$ is measured in feet per second and *t* is measured in seconds, distance is measured in feet.

If we are looking for a change in position, this is found by evaluating $\int_0^{100} (60 - 12\sqrt{t})dt.$

$\int_0^{100} (60 - 12\sqrt{t})dt$	Start with the original expression.
$= \int_0^{100} (60 - 12t^{1/2})dt$	Rewrite the square root as a power so that the power rule can be used.
$= (60t - 8t^{3/2})\Big\vert_0^{100}$	Apply the fundamental theorem of calculus. Note: $\int 12t^{1/2}dt = 12\left(\frac{2}{3}\right)t^{3/2} = 8t^{3/2}$

$$= \left[60(100) - 8(100)^{3/2}\right] - \left[60(0) - 8(0)^{3/2}\right]$$ Substitute the upper and lower endpoints.

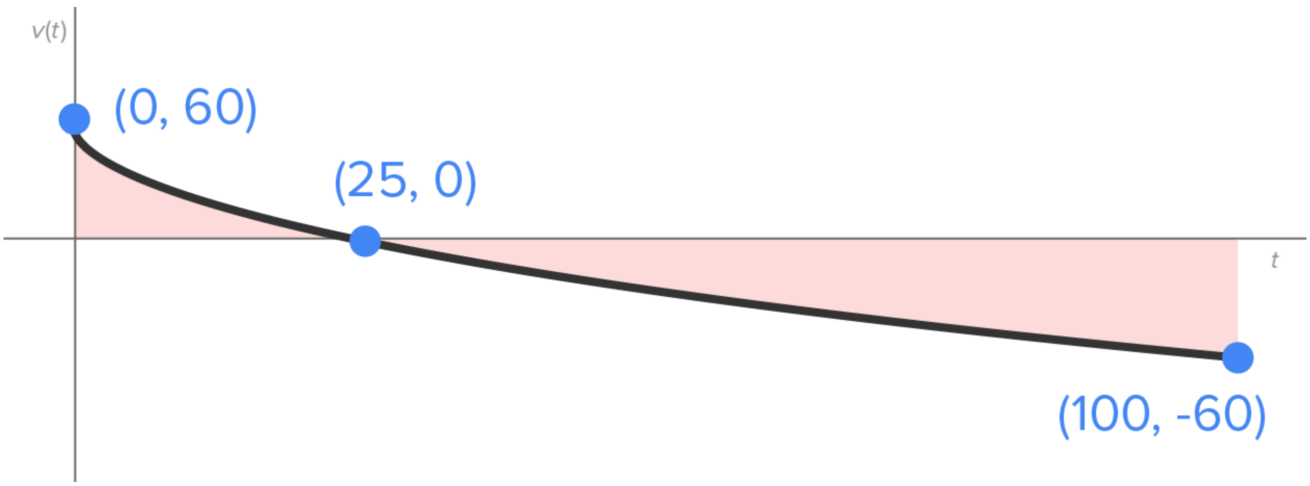
$$= 6000 - 8(1000)$$ Evaluate.

$$= -2000$$ Simplify.

Since the result is negative, this means that the object's final position is 2000 ft behind its starting point at $t = 0$.

b. What is the total distance traveled after the first 100 seconds?

This requires us to look at the graph of $v(t)$ and the t-axis over the interval $[0, 100]$. The graph along with the region between $v(t)$ and the t-axis is shown in the figure below.



On the interval $[0, 25]$, the region is above the t-axis, and on the interval $[25, 100]$, the region is below the t-axis.

Remember also that you can find the t-intercept using algebra:

$$60 - 12\sqrt{t} = 0$$
$$60 = 12\sqrt{t}$$
$$5 = \sqrt{t}$$
$$25 = t$$

To find the total distance traveled, we'll need to compute two integrals. Luckily, from part (a), we already know the antiderivative.

Interval	Calculation	Explanation	Distance Traveled

Distance traveled on [0, 25]	$\int_0^{25} (60 - 12t^{1/2})dt$ $= (60t - 8t^{3/2})\Big\vert_0^{25}$ $= [60(25) - 8(25)^{3/2}] - [60(0) - 8(0)^{3/2}]$ $= 1500 - 8(125)$ $= 500$	This result means that the object traveled 500 feet in the positive direction on the interval $[0, 25]$.	500 feet
Distance traveled on [25, 100]	$\int_{25}^{100} (60 - 12t^{1/2})dt$ $= (60t - 8t^{3/2})\Big\vert_{25}^{100}$ $= [60(100) - 8(100)^{3/2}] - [60(25) - 8(25)^{3/2}]$ $= [6000 - 8(1000)] - [1500 - 8(125)]$ $= -2000 - 500$ $= -2500$	This result means that the object traveled 2500 feet in the negative direction on the interval $[25, 100]$.	2500 feet

Thus, the total distance traveled on $[0, 100] = 500 + 2500 = 3000$ feet.

WATCH

This video walks you through an example of finding an object's change in position and total distance traveled using a definite integral.

TRY IT

The velocity of an object in motion after t minutes is given by the function $v(t) = 20 - 10e^{-t}$ feet per minute on the interval $[0, 5]$.

Find the change in position on the interval [0, 5]. Give both the exact answer and rounded to the nearest whole foot. +

The change in position is given by the integral $\int_0^5 (20 - 10e^{-t})dt$.

Evaluating, we have:

$20t + 10e^{-t}\Big\vert_0^5$	The antiderivative of 20 is $20t$. The antiderivative of $10e^{-t}$ is $-10e^{-t}$, by using the formula $\int e^{kt}dt = \frac{1}{k}e^{kt}$.
$= (20(5) + 10e^{-5}) - (20(0) + 10e^{0})$	Substitute $t = 5$ and $t = 0$, then subtract.

$= (100 + 10e^{-5}) - (10)$ Simplify within the parentheses.

$= 90 + 10e^{-5}$ Combine like terms.

Thus, the change in position is $90 + 10e^{-5}$ feet.

After rounding this result to the nearest whole number, the approximate answer is 90 feet.

Is the distance traveled on the interval [0, 5] equal to the change in position after 5 minutes? Why or why not? +

They are equal since the graph of $v(t)$ is above the t-axis on the interval $[0, 5]$, indicating that $v(t)$ is positive on $[0, 5]$.

SUMMARY

In this lesson, you learned that by applying the fundamental theorem of calculus, you are now able to **calculate areas of regions** as well as **calculate distance traveled and net change in distance** exactly rather than using approximation techniques.