Horizontal and Vertical Asymptotes

by Sophia

WHAT'S COVERED

In this lesson, you will connect limits with horizontal and vertical asymptotes. Specifically, this lesson will cover:

1. Horizontal Asymptotes and Limits

The graph of $f(x)$ has a **horizontal asymptote** $y = c$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

EXAMPLE Find all horizontal asymptotes of the graph of $f(x) = \frac{7x+1}{8x+3}$.

To find horizontal asymptotes, evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

$\lim_{x \to \infty} \frac{7x+1}{8x+3}$ Start with the required limit to evaluate.

$= \lim_{x \to \infty} \frac{\frac{7x}{x} + \frac{1}{x}}{\frac{8x}{x} + \frac{3}{x}}$ Divide each term by the highest power of *x* in the denominator, which is *x*.

$= \lim_{x \to \infty} \frac{7 + \frac{1}{x}}{8 + \frac{3}{x}}$ Simplify.

$= \frac{\lim_{x \to \infty} \left(7 + \frac{1}{x}\right)}{\lim_{x \to \infty} \left(8 + \frac{3}{x}\right)}$ $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)}$

$$= \frac{7}{8} \quad \begin{matrix} \lim\limits_{x \to \infty}\left(7+\frac{1}{x}\right) = \lim\limits_{x \to \infty} 7 + \lim\limits_{x \to \infty} \frac{1}{x} = 7+0=7 \\ \lim\limits_{x \to \infty}\left(8+\frac{3}{x}\right) = \lim\limits_{x \to \infty} 8 + \lim\limits_{x \to \infty} \frac{3}{x} = 8+0=8 \end{matrix}$$

Thus, the graph of $f(x)$ has a horizontal asymptote at $y = \frac{7}{8}$.

Now, check $\lim\limits_{x \to -\infty} \frac{7x+1}{8x+3}$.

$\lim\limits_{x \to -\infty} \frac{7x+1}{8x+3}$	Start with the required limit to evaluate.
$= \lim\limits_{x \to -\infty} \frac{\frac{7x}{x}+\frac{1}{x}}{\frac{8x}{x}+\frac{3}{x}}$	Divide each term by the highest power of *x* in the denominator, which is *x*.
$= \lim\limits_{x \to -\infty} \frac{7+\frac{1}{x}}{8+\frac{3}{x}}$	Simplify.
$= \frac{\lim\limits_{x \to -\infty}\left(7+\frac{1}{x}\right)}{\lim\limits_{x \to -\infty}\left(8+\frac{3}{x}\right)}$	$\lim\limits_{x \to -\infty} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to -\infty} f(x)}{\lim\limits_{x \to -\infty} g(x)}$

$$= \frac{7}{8} \quad \begin{matrix} \lim\limits_{x \to -\infty}\left(7+\frac{1}{x}\right) = \lim\limits_{x \to -\infty} 7 + \lim\limits_{x \to -\infty} \frac{1}{x} = 7+0=7 \\ \lim\limits_{x \to -\infty}\left(8+\frac{3}{x}\right) = \lim\limits_{x \to -\infty} 8 + \lim\limits_{x \to -\infty} \frac{3}{x} = 8+0=8 \end{matrix}$$

This produces the same result as the other limit, so there is no additional horizontal asymptote.

We can conclude that the graph of $f(x)$ has one horizontal asymptote at $y = \frac{7}{8}$. If you graph this function, you would see that the graph approaches the horizontal line $y = \frac{7}{8}$ as $x \to \pm\infty$.

TRY IT

Consider the function $f(x) = \frac{3x}{x^2+3}$.

Find all horizontal asymptotes of the graph of the function. +

The horizontal asymptotes are found by finding $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

$$\lim_{x \to \infty} \frac{3x}{x^2+3} = \lim_{x \to \infty} \frac{\frac{3x}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \lim_{x \to \infty} \frac{\frac{3}{x}}{1 + \frac{3}{x^2}} = \frac{0}{1+0} = 0$$

Similarly:

$$\lim_{x \to -\infty} \frac{3x}{x^2+3} = \lim_{x \to -\infty} \frac{\frac{3x}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \lim_{x \to -\infty} \frac{\frac{3}{x}}{1 + \frac{3}{x^2}} = \frac{0}{1+0} = 0$$

Therefore, the horizontal asymptote is $y = 0$.

If $f(x) = \frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are both polynomials, the following are results of limits:

- If $N(x)$ and $D(x)$ have the same degree, then the horizontal asymptote is $y = \frac{a}{b}$, where *a* is the leading coefficient of the numerator and *b* is the leading coefficient of the denominator.
- If the degree of $N(x)$ is less than the degree of $D(x)$, then the horizontal asymptote is $y = 0$.
- If the degree of $N(x)$ is more than the degree of $D(x)$, then there is no horizontal asymptote, and this case is discussed in the next tutorial.

EXAMPLE The function $f(x) = \frac{2x^2+4x+1}{3x^2+5x}$ has the horizontal asymptote $y = \frac{2}{3}$ since the degrees are the same.

EXAMPLE The function $f(x) = \frac{3x}{x^2+3}$ has the horizontal asymptote $y = 0$, as you saw in the Try It above, since the degree of the numerator is less than the degree of the denominator.

WATCH

In this video, we'll use limits to find the equations of all horizontal asymptotes of the function $f(x) = \frac{|x|}{2x+3}$.

TERM TO KNOW

Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

2. Vertical Asymptotes and Limits

The graph of $f(x)$ has a **vertical asymptote** $x = a$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

The "$\pm\infty$" in the above definition means that the limit could be either $-\infty$ or ∞ for there to be a vertical asymptote when $x = a$.

HINT

If $f(x)$ is a rational function, the only values of *x* where a vertical asymptote could occur are those values where the denominator is equal to 0.

EXAMPLE Determine the vertical asymptotes of $f(x) = \frac{x^2 - 2x}{x^2 - 4x}$.

First, find all values of *x* for which the denominator is 0:

$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$x = 0,\ x = 4$$

Thus, the possible vertical asymptotes are $x = 0$ and $x = 4$. To determine which are vertical asymptotes, we need to evaluate a one-sided limit for each x-value. For this example, we'll choose right-sided limits.

Is $x = 0$ a vertical asymptote?

$\lim_{x \to 0^+} \frac{x^2 - 2x}{x^2 - 4x}$	Start with the required limit to evaluate.
$\lim_{x \to 0^+} \frac{x(x-2)}{x(x-4)} = \lim_{x \to 0^+} \frac{x-2}{x-4}$	Factor, then remove the common factor.
$= \frac{0-2}{0-4} = \frac{1}{2}$	Direct substitution works!

Since the limit is not $\pm\infty$, there is no vertical asymptote at $x = 0$. (Note: The left-sided limit would produce the same result.)

Is $x = 4$ a vertical asymptote?

$\lim_{x \to 4^+} \frac{x^2 - 2x}{x^2 - 4x}$	Start with the required limit to evaluate.

$$\lim_{x \to 4^+} \frac{x(x-2)}{x(x-4)} = \lim_{x \to 4^+} \frac{x-2}{x-4}$$ Factor, then remove the common factor.

$$= \infty$$ As x approaches 4 from the right, x - 2 is around 2, and x - 4 is a small positive number.

$\frac{\text{around 2}}{\text{small positive number}} =$ a large number, so the limit is ∞.

We can conclude that there is a vertical asymptote at $x = 4$, but not at $x = 0$. If you were to graph the function, this would confirm this result.

TRY IT

Consider the function $f(x) = \frac{2x+1}{x^2 - 6x + 5}$.

Find the equations of all vertical asymptotes of the function. +

The possible locations of the vertical asymptotes are the values of x for which the denominator is equal to 0.

$$x^2 - 6x + 5 = 0$$

$$(x-1)(x-5) = 0$$

$$x - 1 = 0, \;\; x - 5 = 0$$

$$x = 1, 5$$

Now, we analyze the behavior of $f(x)$ around each of these values of x using limits. In this example we'll choose right-hand limits, but left-hand limits can be used as well.

$x = 1$:

$$\lim_{x \to 1^+} \frac{2x+1}{(x-1)(x-5)} = \frac{3}{(\text{small positive number})(-4)} = -\infty$$

$x = 5$:

$$\lim_{x \to 5^+} \frac{2x+1}{(x-1)(x-5)} = \frac{11}{(4)(\text{small positive number})} = \infty$$

Since both limits result in either ∞ or $-\infty$, there are vertical asymptotes at $x = 1$ and $x = 5$.

Here is a final example in which we pull everything together and take note of a few things.

➦ EXAMPLE Find equations of all asymptotes of $f(x)=\frac{2x^2-2x-12}{x^2-9}$.

First, notice that the degrees of the numerator and denominator of $f(x)$ are the same (2). This means that the equation of the horizontal asymptote is $y=\frac{2}{1}=2$.

Next, we look for the vertical asymptotes.

$x^2-9=0$	Set the denominator equal to 0.
$x^2=9$	Add 9 to both sides.
$x=\pm 3$	Solve by applying the square root.

This means that vertical asymptotes possibly occur when $x=3$ and $x=-3$. To check this, we use limits. We only need to check one side, so we'll use right-hand limits for each:

Behavior at $x=-3$:

$\lim_{x\to -3^+}\frac{2x^2-2x-12}{x^2-9}$	Set up the limit.
$=\lim_{x\to -3^+}\frac{2(x+2)(x-3)}{(x+3)(x-3)}$	Factor the numerator and denominator.
$=\lim_{x\to -3^+}\frac{2(x+2)}{x+3}$	Remove the common factor of $x-3$.
$=\frac{-2}{\text{small positive number}}$	Direct substitution doesn't work since the denominator is 0 when $x=-3$. As $x\to -3^+$, the numerator gets closer to $2(-1)=-2$, the denominator is a very small positive number.
$=-\infty$	When a negative number is divided by a very small positive number, the result is a negative number that is large in magnitude.

Since this limit is infinite, it follows that there is vertical asymptote when $x=-3$.

Behavior at $x=3$:

$\lim_{x\to 3^+}\frac{2x^2-2x-12}{x^2-9}$	Set up the limit.
$=\lim_{x\to 3^+}\frac{2(x+2)(x-3)}{(x+3)(x-3)}$	Factor the numerator and denominator.

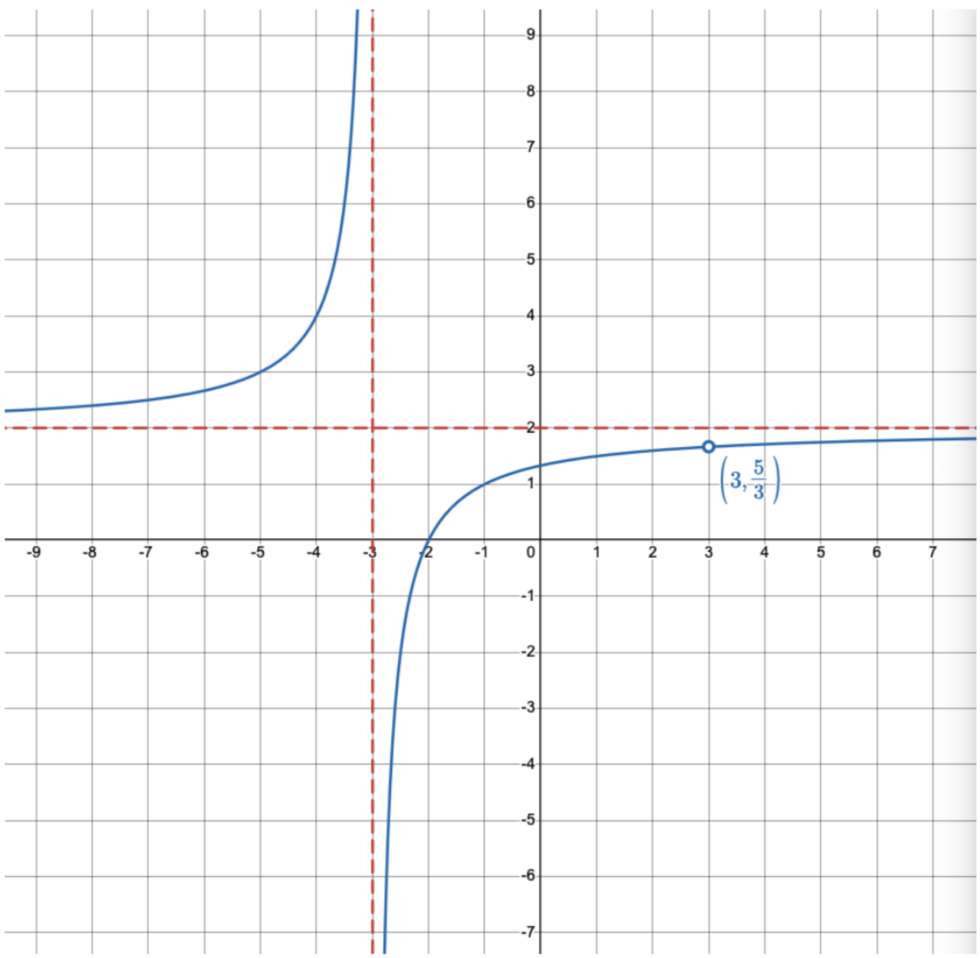
$$= \lim_{x \to 3^+} \frac{2(x+2)}{x+3} \quad \text{Remove the common factor of } x-3.$$

$$= \frac{2(5)}{6} = \frac{5}{3} \quad \text{Substitute } x=3 \text{ and simplify.}$$

Since $\lim_{x \to 3^+} \frac{2x^2-2x-12}{x^2-9} = \frac{5}{3}$, which is not infinite, we conclude that there is no vertical asymptote when $x=3$.

In conclusion, $f(x)$ has a horizontal asymptote at $y=2$ and a vertical asymptote at $x=-3$.

The graph of $f(x)$ is shown below. Note the hole in the graph at $\left(3, \frac{5}{3}\right)$.



The process for identifying vertical asymptotes can be a bit redundant. Here is a hint that you might find helpful.

HINT

When determining which values of x corresponded to vertical asymptotes in the last example, notice that the simplest form of $f(x)$ was useful when evaluating the limit. This leads to the following "shortcut" to determining which values correspond to vertical asymptotes (and which do not, which means they correspond to holes in the graph).

1. Find the values of x where the original rational function is undefined (denominator equal to 0).
2. Write $f(x)$ in simplest form (remove common factors). Call the simplest form $g(x)$.
3. For each value you found in part 1:
   a. If $g(x)$ is undefined, then there is a vertical asymptote.
   b. If $g(x)$ is defined, then there is not a vertical asymptote (there is a hole in the graph instead).

TERM TO KNOW

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

SUMMARY

In this lesson, you learned that **the horizontal and vertical asymptotes of a function are related to limits** of a function where infinity is involved. Specifically, a function $f(x)$ has a horizontal asymptote at $y = c$ if $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$, and a function $f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

TERMS TO KNOW

Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.