The Area Between Two Curves That Do Not Intertwine

by Sophia

WHAT'S COVERED

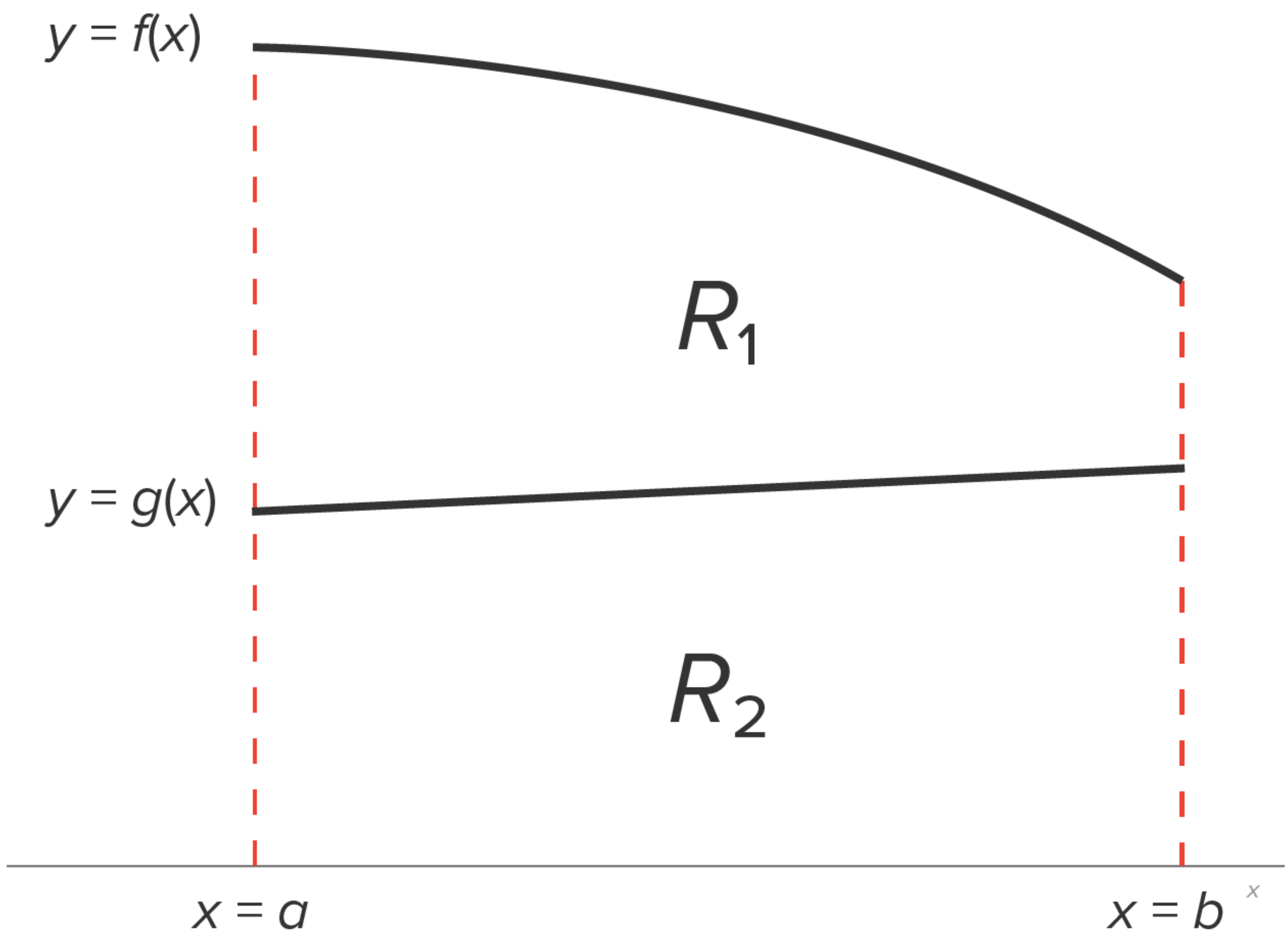
In this lesson, you will use the fundamental theorem of calculus to find the areas of regions bounded by two curves (for now, the regions will not intertwine). Specifically, this lesson will cover:

1. The Idea Behind the Area Between Two Curves That Do Not Intertwine

In addition to the regions we have worked with up to now, we also need to consider regions that do not have an axis as an edge.

Consider the areas in the figure below where:

- R_1 is the area of the region between the graphs of $y=f(x)$ and $y=g(x)$ on the interval $[a, b]$.
- R_2 is the area of the region between the x-axis and $y=g(x)$ on the interval $[a, b]$.
- R_1+R_2 is the area of the region between the x-axis and the graph of $y=f(x)$ on $[a, b]$.



From the picture, we know that $\int_a^b f(x)dx = R_1 + R_2$. We also see that $R_2 = \int_a^b g(x)dx$.

Replacing R_2 with $\int_a^b g(x)dx$ into the first equation, we have $\int_a^b f(x)dx = R_1 + \int_a^b g(x)dx$.

Solving for R_1, we have $R_1 = \int_a^b f(x)dx - \int_a^b g(x)dx$.

By properties of definite integrals, we know this is equal to $R_1 = \int_a^b [f(x) - g(x)]dx$.

Thus, assuming that the graph of $y = f(x)$ is above the graph of $y = g(x)$ on $[a, b]$, we have the following formula to find the area of the region between the graphs.

FORMULA TO KNOW

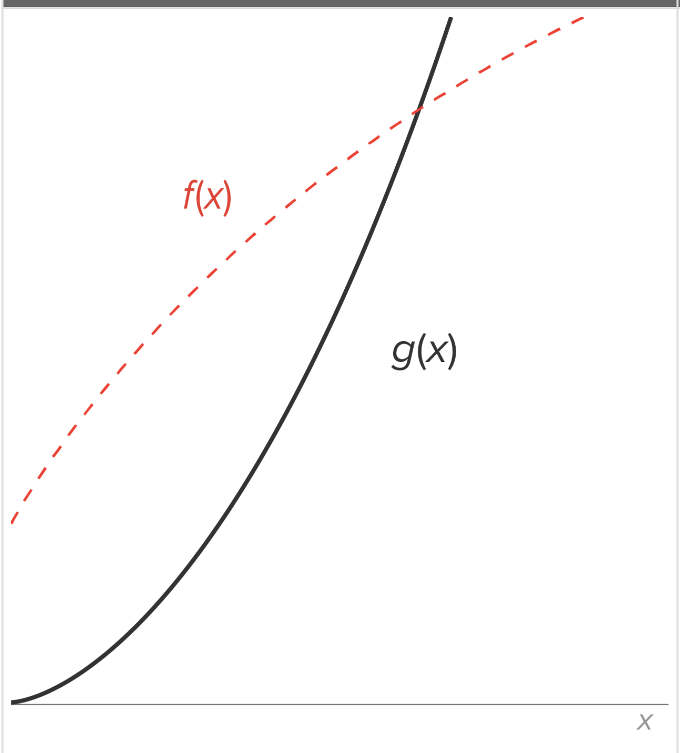
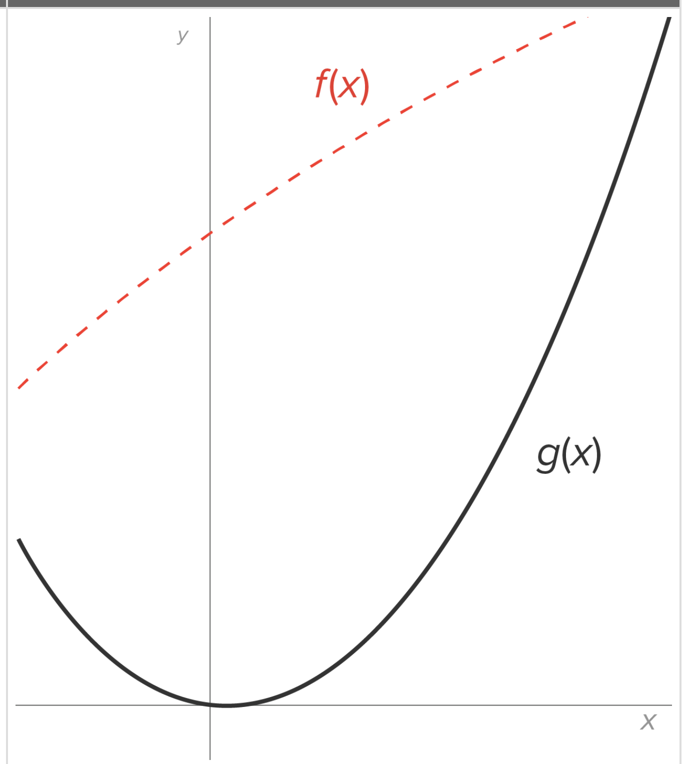
Area Between Two Curves, $y = f(x)$ and $y = g(x)$, Assuming $f(x) \geq g(x)$ on $[a, b]$

$$Area = \int_a^b [f(x) - g(x)]dx$$

We'll use this idea to find the areas of some regions bounded between two curves.

2. Finding the Area Between Two Curves That Do Not Intertwine

In order to find the area of this type of region, first verify that the graphs do not intertwine.

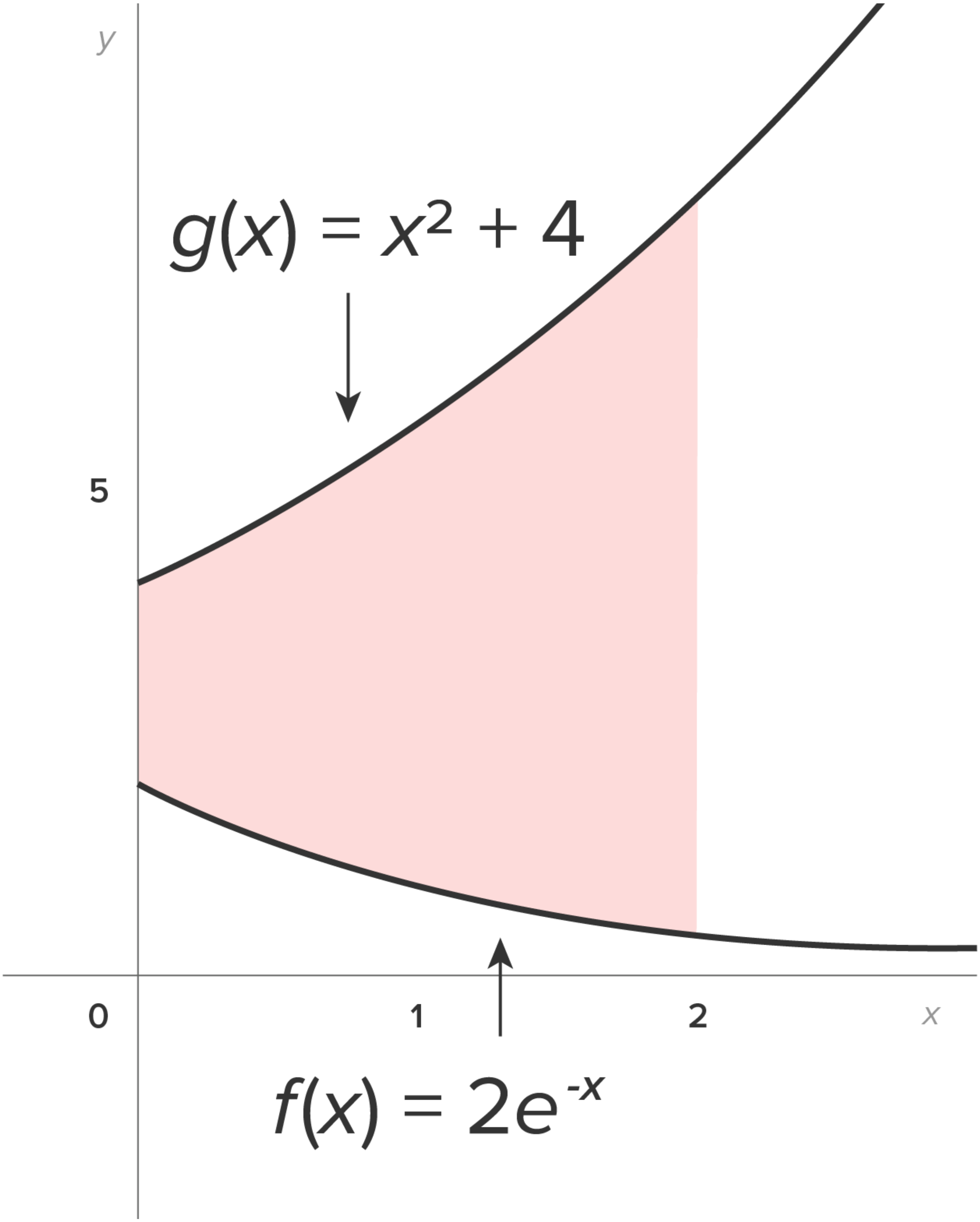
When $f(x)$ and $g(x)$ Intertwine	When $f(x)$ and $g(x)$ Do NOT Intertwine
	

HINT

If the graphs meet at either endpoint, they are not considered intertwining. We will discuss intertwining graphs in the next tutorial.

EXAMPLE Find the exact area between the graphs of $f(x) = 2e^{-x}$ and $g(x) = x^2 + 4$ between $x = 0$ and $x = 2$.

The graph of the two curves on $[0, 2]$ is shown in the figure below.



The figure shows that $g(x) = x^2 + 4$ is higher than $f(x) = 2e^{-x}$ on the entire interval. Then, the area of the region is $\int_0^2 (x^2 + 4 - 2e^{-x})dx$.

Now we evaluate the integral:

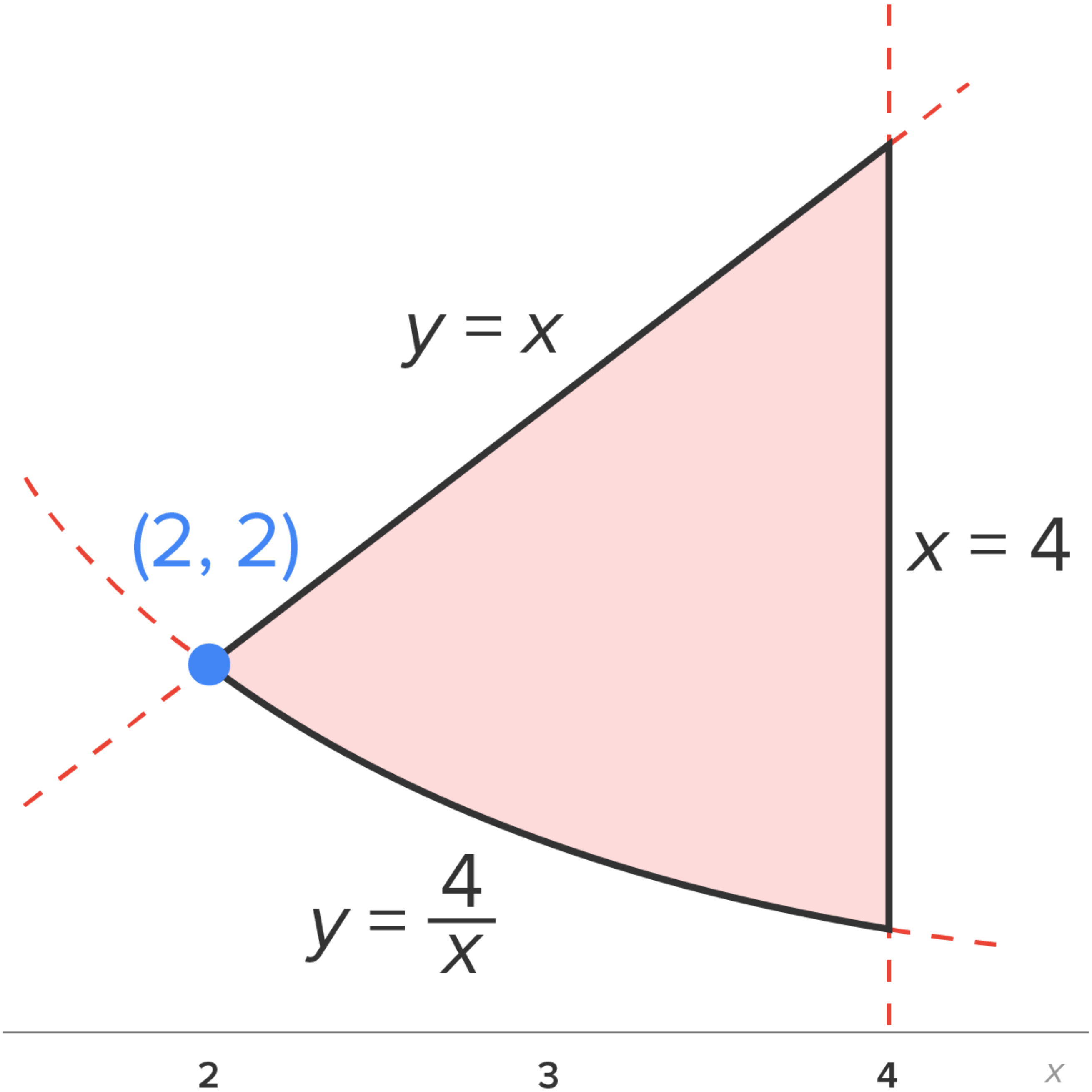
$\int_0^2 (x^2 + 4 - 2e^{-x})dx$	Start with the original expression.
$= \left(\frac{1}{3}x^3 + 4x + 2e^{-x}\right)\Big\vert_0^2$	Apply the fundamental theorem of calculus. Note: $\int 2e^{-x}dx = \frac{2}{-1}e^{-x} + C = -2e^{-x} + C$
$= \left[\frac{1}{3}(2)^3 + 4(2) + 2e^{-2}\right] - \left[\frac{1}{3}(0)^3 + 4(0) + 2e^{-0}\right]$	Substitute the upper and lower endpoints.
$= \frac{8}{3} + 8 + 2e^{-2} - 2e^0$	Evaluate the brackets.
$= \frac{8}{3} + 8 + \frac{2}{e^2} - 2$	Note: $e^0 = 1$
$= \frac{26}{3} + \frac{2}{e^2}$	Simplify.

Then, the area of the region is $\frac{26}{3} + \frac{2}{e^2}$ units^2.

Here is an example where the interval is not given.

EXAMPLE Find the exact area of the region in the first quadrant that is bounded by the graphs of $y = x$, $y = \frac{4}{x}$, and $x = 4$.

The graph of the region in the first quadrant is shown in the figure.



To find the intersection point, set $x = \frac{4}{x}$ and solve:

$$x = \frac{4}{x}$$

$$x^2 = 4$$

$$x = \pm 2$$

Since the region is in the first quadrant, only $x = 2$ is considered.

Also, since the graph of $y=x$ is above the graph of $y=\frac{4}{x}$ on the interval $[2, 4]$, the definite integral that gives the area is $\int_2^4\left(x-\frac{4}{x}\right)dx$.

Now evaluate the definite integral.

$\int_2^4\left(x-\frac{4}{x}\right)dx$	Start with the original expression.
$=\left(\frac{1}{2}x^2-4\ln\lvert x\rvert\right)\Big\rvert_2^4$	Apply the fundamental theorem of calculus.
$=\left[\frac{1}{2}(4)^2-4\ln\lvert 4\rvert\right]-\left[\frac{1}{2}(2)^2-4\ln\lvert 2\rvert\right]$	Substitute the upper and lower endpoints.
$=[8-4\ln\lvert 4\rvert]-[2-4\ln\lvert 2\rvert]$	Evaluate the parentheses.
$=6-4\ln 4+4\ln 2$	Simplify.
$=6-4(\ln 4-\ln 2)$	Factor.
$=6-4\ln\left(\frac{4}{2}\right)$	Use the property $\ln a-\ln b=\ln\left(\frac{a}{b}\right)$.
$=6-4\ln 2$	Simplify.

Thus, the area of the region is equal to $6-4\ln 2$ units^2.

WATCH

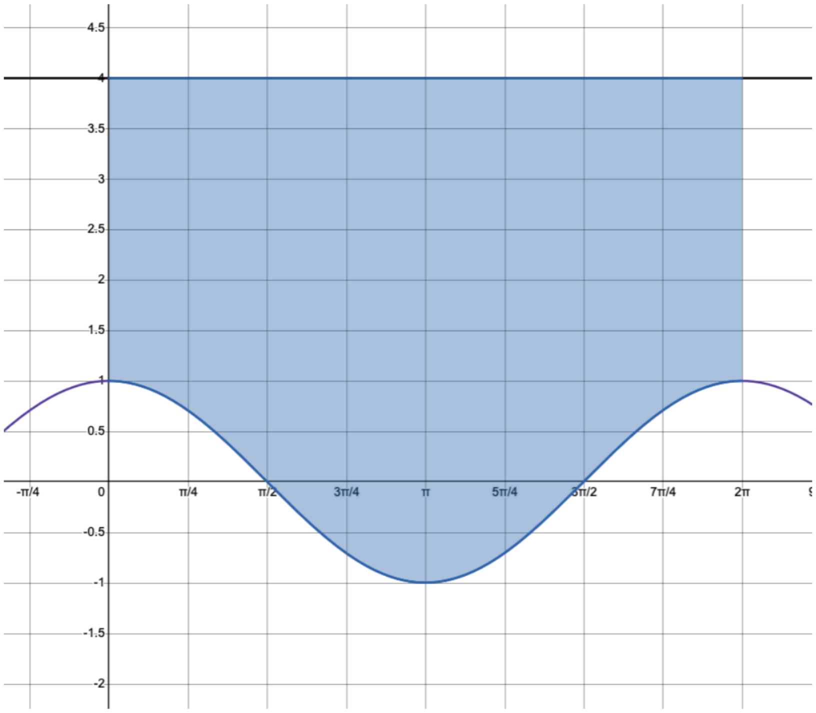
Check out this video to learn how to find the exact area between $f(x)=\sin x$ and $g(x)=\cos x$ over the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

TRY IT

Consider the region bounded by the graphs of $y=4$ and $y=\cos x$ on the interval $[0, 2\pi]$.

Find the exact area of the region. +

The region is shown in the graph below.



Notice that the graphs do not intertwine, and the graph of $y = 4$ is always above the graph of $y = \cos x$.

Then, the area of the region between the graphs is found by calculating the integral $\int_0^{2\pi} (4 - \cos x)dx$.

Now, evaluate the integral:

$= 4x - \sin x\|_0^{2\pi}$	$\int 4dx = 4x;\ \int \cos x dx = \sin x$.
$= (4(2\pi) - \sin 2\pi) - (4(0) - \sin 0)$	Evaluate at $x = 2\pi$ and $x = 0$, then subtract.
$= (8\pi - 0) - (0 - 0)$	$\sin 2\pi = 0,\ \sin 0 = 0$
$= 8\pi$	Simplify.

In conclusion, the area between $y = 4$ and $y = \cos x$ between $x = 0$ and $x = 2\pi$ is 8π square units.

TRY IT

Consider the region bounded by the graphs of $y = 4x$ and $y = x^2 + 3$.

Find the exact area of the region.

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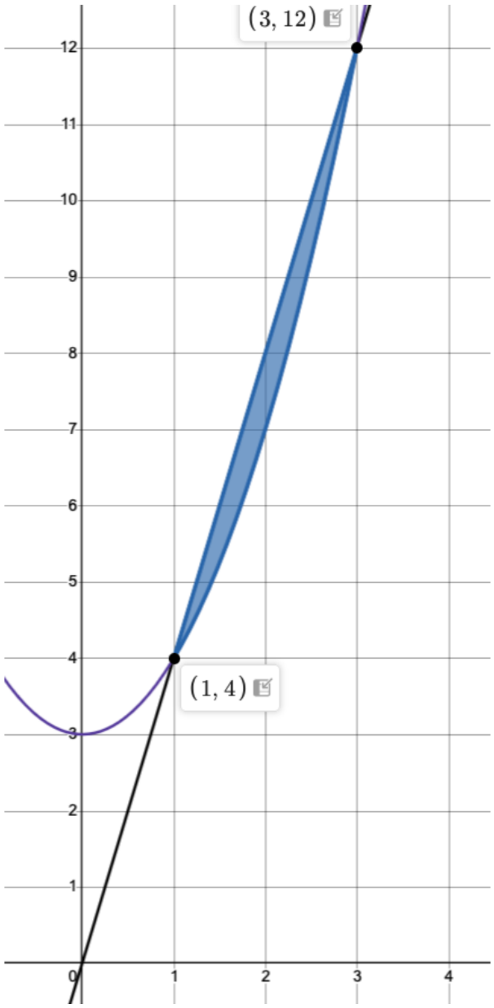
The region is shown in the graph below.



The intersection points can be found either graphically or by using algebra. As you can see in the graph, the intersection points are $(1, 4)$ and $(3, 12)$, so the bounds on the integral are $x = 1$ and $x = 3$.

Since the line $y = 4x$ is above the curve $y = x^2 + 3$, the integral that represents the area is $\int_1^3 (4x - (x^2 + 3))dx$, which can be simplified to $\int_1^3 (4x - x^2 - 3)dx$.

Now, evaluate the integral to find the area.

$$= 2x^2 - \frac{1}{3}x^3 - 3x\Big|_1^3 \qquad \int 4x dx = 4\left(\frac{1}{2}\right)x^2 = 2x^2$$

$$\int x^2 dx = \frac{1}{3}x^3$$

$$\int 3 dx = 3x$$

$$=\left(2(3)^2-\frac{1}{3}(3)^3-3(3)\right)-\left(2(1)^2-\frac{1}{3}(1)^3-3(1)\right)$$ Substitute $x=3$ and $x=1$, then subtract.

$$=(0)-\left(-\frac{4}{3}\right)$$ Simplify within each group of parentheses.

$$=\frac{4}{3}$$ Simplify.

In conclusion, the area of the region that lies between the graphs of $y=4x$ and $y=x^2+3$ is equal to $\frac{4}{3}$ square units.

WATCH

There are situations where it is advantageous to use horizontal subrectangles instead of vertical ones. Check out this video to learn how to find the exact area between $x=y^2-4$ and $x=-2y-1$.

To summarize the previous video, we have the following formula to calculate the area between two curves using horizontal subrectangles:

FORMULA TO KNOW

Area Between Two Curves, $x=h(y)$ and $x=k(y)$, Assuming $h(y)\geq k(y)$ on $[c, d]$ (Horizontal Subrectangles)

$$Area=\int_c^d[h(y)-k(y)]dy$$

SUMMARY

In this lesson, you learned how to apply the fundamental theorem of calculus to compute the area of a region that does not have the x-axis as a boundary, **finding the area between two curves that do not intertwine**. Note that if the graphs meet at either endpoint, they are not considered intertwining. In the next tutorial, we will apply what we've learned in this tutorial to tackle areas where the boundary curves do intertwine.

FORMULAS TO KNOW

Area Between Two Curves, x = h(y) and x = k(y), Assuming h(y) ≥ k(y) on [c, d] (Horizontal Subrectangles)

$$Area = \int_c^d [h(y) - k(y)]dy$$

Area Between Two Curves, y = f(x) and y = g(x), Assuming f(x) ≥ g(x) on [a, b]

$$Area = \int_a^b [f(x) - g(x)]dx$$