

# What Is L'Hopital's Rule?

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## WHAT'S COVERED

In this lesson, you will evaluate limits in special forms by using L'Hopital's rule. Specifically, this lesson will cover:

1. A Brief Review of Frequently Used Limits
2. Limits of the Form  $0/0$
3. Limits of the Form  $\infty/\infty$

## 1. A Brief Review of Frequently Used Limits

In this challenge, we will be seeing limits of exponential and logarithmic functions pretty often, so here is a chart that describes the behavior of some functions. To visualize these limits, use technology to graph the function.

| Behavior of Common Functions                 |  |
|--|--|
| $\lim_{x \rightarrow \infty} e^x = \infty$   | $\lim_{x \rightarrow -\infty} e^x = 0$         |
| $\lim_{x \rightarrow \infty} e^{-x} = 0$     | $\lim_{x \rightarrow -\infty} e^{-x} = \infty$ |
| $\lim_{x \rightarrow \infty} \ln x = \infty$ | $\lim_{x \rightarrow 0^+} \ln x = -\infty$     |

## 2. Limits of the Form $0/0$

Earlier in the course, we encountered limits such as  $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$  and  $\lim_{x \rightarrow 3} \frac{5x-15}{7x-21}$ .

Note that in each limit, the numerator and denominator are both zero when direct substitution is used. We say that these limits are in the form " $\frac{0}{0}$ ". This is an example of an **indeterminate form**, since its value is not known until further analysis is done. There is no way to determine its value just by being " $\frac{0}{0}$ ".

Case in point, if we evaluate these limits, we have:

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$$

$$\lim_{x \rightarrow 3} \frac{5x-15}{7x-21} = \lim_{x \rightarrow 3} \frac{5(x-3)}{7(x-3)} = \lim_{x \rightarrow 3} \frac{5}{7} = \frac{5}{7}$$

These limits both had the form  $\frac{0}{0}$ , but ended up having different values. This is why  $\frac{0}{0}$  is called an indeterminate form.

While these limits were able to be manipulated easily using algebra, how would we go about evaluating

$\lim_{x \rightarrow 0} \frac{e^{2x}-1}{x}$ ? This has the indeterminate form  $\frac{0}{0}$ , but there is no algebraic way to manipulate this expression.

For limits like this, we have L'Hopital's rule.



#### BIG IDEA

Suppose  $f(x)$  and  $g(x)$  are differentiable on an open interval which contains  $x = a$  and  $g'(x) \neq 0$  except possibly at  $x = a$ . If  $f(x)$  and  $g(x)$  both approach 0 as  $x \rightarrow a$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  provided that the limit on the right exists.

Note: " $a$ " can be replaced with either " $-\infty$ " or " $\infty$ " to allow for limits as  $x \rightarrow \pm \infty$ .

⇒ EXAMPLE Evaluate the following limit:  $\lim_{x \rightarrow 0} \frac{e^{2x}-1}{x}$

First check the requirements.

1. The numerator and denominator both approach 0 as  $x \rightarrow 0$ .
2. The numerator and denominator are both differentiable on any interval containing  $x = 0$ .

This means L'Hopital's rule can be used to evaluate the limit.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{e^{2x}-1}{x} && \text{Start with the limit that needs to be evaluated.} \\ &= \lim_{x \rightarrow 0} \frac{2e^{2x}}{1} && D[e^{2x}-1] = 2e^{2x} \text{ and } D[x] = 1 \\ &= \lim_{x \rightarrow 0} \frac{2e^{2x}}{1} = \frac{2e^0}{1} = 2 && \text{Use direct substitution.} \end{aligned}$$

We can conclude that  $\lim_{x \rightarrow 0} \frac{e^{2x}-1}{x} = 2$ .

⇒ EXAMPLE Evaluate the following limit:  $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x^3 - 8}$

First, check the requirements.

1. The numerator and denominator both get closer to 0 as  $x \rightarrow 2$ .
2. The numerator and denominator are both differentiable on any interval containing  $x = 2$ .

This means L'Hopital's rule can be used to evaluate the limit.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^5 - 32}{x^3 - 8} & \quad \text{Start with the limit that needs to be evaluated.} \\ &= \lim_{x \rightarrow 2} \frac{5x^4}{3x^2} \quad D[x^5 - 32] = 5x^4 \text{ and } D[x^3 - 8] = 3x^2 \\ &= \lim_{x \rightarrow 2} \frac{5x^4}{3x^2} = \frac{5(2)^4}{3(2)^2} = \frac{20}{3} \quad \text{Use direct substitution.} \end{aligned}$$

Thus,  $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x^3 - 8} = \frac{20}{3}$ .



Consider the following limit:  $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 7x}$

Evaluate the limit.

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Direct substitution yields  $\frac{\sin 0}{\sin 0} = \frac{0}{0}$ , which is indeterminate. This means we can use L'Hopital's Rule to attempt to evaluate the limit.

L'Hopital's Rule states that  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  as long as  $f(c) = g(c) = 0$ .

We have the following:

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 7x} = \lim_{x \rightarrow 0} \frac{3\cos 3x}{7\cos 7x} = \frac{3\cos 0}{7\cos 0} = \frac{3(1)}{7(1)} = \frac{3}{7}$$

L'Hopital's rule can be applied more than once as long as the new limit is also in the form  $\frac{0}{0}$ .

⇒ EXAMPLE Evaluate the following limit:  $\lim_{x \rightarrow 0} \frac{\cos(2x) - 1 + 2x^2}{x^2}$

First, check the requirements. Let  $f(x) = \cos(2x) - 1 + 2x^2$  and  $g(x) = x^2$ .

- $f(x)$  and  $g(x)$  both get closer to 0 as  $x \rightarrow 0$ .
- $f'(x) = -2\sin(2x) + 4x$  and  $g'(x) = 2x$
- $f(x)$  and  $g(x)$  are both differentiable on any interval containing  $x = 0$ .

This means L'Hopital's rule can be used to evaluate the limit.

$$\lim_{x \rightarrow 0} \frac{\cos(2x) - 1 + 2x^2}{x^2}$$

Start with the limit that needs to be evaluated.

$$= \lim_{x \rightarrow 0} \frac{-2\sin(2x) + 4x}{2x}$$

$$f'(x) = -2\sin(2x) + 4x \text{ and } g'(x) = 2x$$

$$= \lim_{x \rightarrow 0} \frac{-4\cos(2x) + 4}{2}$$

The numerator and denominator both approach 0 as  $x \rightarrow 0$ , and the numerator and denominator are both differentiable.

Therefore, L'Hopital's rule can be applied again:

$$D[-2\sin(2x) + 4x] = -4\cos(2x) + 4 \text{ and } D[2x] = 2$$

$$= \lim_{x \rightarrow 0} \frac{-4\cos(2x) + 4}{2} = \frac{-4\cos(0) + 4}{2} = 0$$

Direct substitution gives 0.

Conclusion: After applying L'Hopital's rule twice,  $\lim_{x \rightarrow 0} \frac{\cos(2x) - 1 + 2x^2}{x^2} = 0$ .



#### TERM TO KNOW

##### Indeterminate Form

A form of a limit, such as " $\frac{0}{0}$ ", that doesn't always yield the same value. Further analysis is needed to determine its value, if it exists.

## 3. Limits of the Form $\infty/\infty$

Consider the limit  $\lim_{x \rightarrow \infty} \frac{2x+1}{3x+4}$ , whose value is  $\frac{2}{3}$ , and  $\lim_{x \rightarrow \infty} \frac{2x+1}{x^2+4}$ , whose value is 0.

In each limit, the numerator and denominator go to  $\infty$  as  $x \rightarrow \infty$ . This means that both limits are in the form  $\frac{\infty}{\infty}$ , which is another indeterminate form (since it leads to different values).



#### BIG IDEA

$\frac{\infty}{\infty}$  is another indeterminate form.

As it turns out, L'Hopital's rule can also be applied to this indeterminate form in a similar way:



#### BIG IDEA

Suppose  $f(x)$  and  $g(x)$  are differentiable on an open interval which contains  $x = a$  and  $g'(x) \neq 0$  except possibly at  $x = a$ . If  $f(x)$  and  $g(x)$  both tend toward  $\infty$  or  $-\infty$  as  $x \rightarrow a$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  provided that the limit on the right exists.

Note: " $a$ " can be replaced with either " $-\infty$ " or " $\infty$ " to allow for limits as  $x \rightarrow \pm \infty$ .

⇒ EXAMPLE Evaluate the following limit:  $\lim_{x \rightarrow \infty} \frac{2x+1}{x^2+4}$

Note that both the numerator and denominator are tending toward  $\infty$  as  $x \rightarrow \infty$ . And they are both differentiable.

This means L'Hopital's rule can be used to evaluate the limit.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x+1}{x^2+4} & \quad \text{Start with the limit that needs to be evaluated.} \\ &= \lim_{x \rightarrow \infty} \frac{2}{2x} \quad D[2x+1] = 2, D[x^2+4] = 2x \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \quad \text{Simplify the expression; then, the limit is 0 since } \lim_{x \rightarrow \infty} \frac{c}{x^n} = 0. \end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow \infty} \frac{2x+1}{x^2+4} = 0.$$

Let's look at an example that involves exponential functions.

⇒ EXAMPLE Evaluate the following limit:  $\lim_{x \rightarrow \infty} \frac{e^x + x^2}{x^3 + 8x}$

Note that both the numerator and denominator are tending toward  $\infty$  as  $x \rightarrow \infty$ . And they are both differentiable.

This means L'Hopital's rule can be used to evaluate the limit.

$$\lim_{x \rightarrow \infty} \frac{e^x + x^2}{x^3 + 8x} \quad \text{Start with the limit that needs to be evaluated.}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x + 2x}{3x^2 + 8} \quad D[e^x + x^2] = e^x + 2x, D[x^3 + 8x] = 3x^2 + 8$$

$$= \lim_{x \rightarrow \infty} \frac{e^x + 2}{6x}$$

Numerator and denominator both tend toward  $\infty$  as  $x \rightarrow \infty$ , and both are differentiable.

Apply L'Hopital's rule again.

$$D[e^x + 2x] = e^x + 2, D[3x^2 + 8] = 6x$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{6}$$

Numerator and denominator both tend toward  $\infty$  as  $x \rightarrow \infty$ , and both are differentiable.

Apply L'Hopital's rule again.

$$D[e^x + 2] = e^x, D[6x] = 6$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty$$

L'Hopital's rule no longer applies since the denominator is constant while the numerator tends toward  $\infty$ .

Since the numerator increases without bound, dividing by 6 doesn't affect this, and the limit is  $\infty$ .

Thus,  $\lim_{x \rightarrow \infty} \frac{e^x + x^2}{x^3 + 8x} = \infty$ .



TRY IT

Consider the following limit:  $\lim_{x \rightarrow \infty} \frac{2x^3 + 8x + 5}{5x^3 + x}$

Evaluate the limit.

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Both the numerator and denominator tend toward  $\infty$  as  $x \rightarrow \infty$ , so we can apply L'Hopital's rule, which

states that  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$ .

$$\lim_{x \rightarrow \infty} \frac{2x^3 + 8x + 5}{5x^3 + x} = \lim_{x \rightarrow \infty} \frac{6x^2 + 8}{15x^2}$$

Both the numerator and denominator tend toward  $\infty$  as  $x \rightarrow \infty$ , so we can apply L'Hopital's rule again,

which states that  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$ .

$$\lim_{x \rightarrow \infty} \frac{6x^2 + 8}{15x^2} = \lim_{x \rightarrow \infty} \frac{12x}{30x}$$

At this point, you could use L-Hopital's Rule again, but simplifying is more efficient. Removing the

common factor of  $6x$ , we have:

$$= \lim_{x \rightarrow \infty} \frac{2}{5}$$

This limit is equal to  $\frac{2}{5}$ .



WATCH

In this final example, we'll evaluate  $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ .



## SUMMARY

In this lesson, you began with a **brief review of frequently used limits**. You also learned that if  $f(x)$  and  $g(x)$  are differentiable and  $g'(x) \neq 0$  (except possibly when  $x = a$ ), L'Hopital's rule is a very convenient way to evaluate  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ , that have the indeterminate **form  $0/0$**  or the **form  $\infty/\infty$** .

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## TERMS TO KNOW

### Indeterminate Form

A form of a limit, such as " $\frac{0}{0}$ ", that doesn't always yield the same value. Further analysis is needed to determine its value, if it exists.