Derivative of y = e^x

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WHAT'S COVERED

In this lesson, you will learn how to take derivatives of exponential functions. These functions are important since they model population growth, the decay of materials, the temperature change of an object, and much more. Specifically, this lesson will cover:

1. Derivatives of $y = e^x$ and Combinations of Functions With $y = e^x$
2. Derivatives of $y = e^u$ and Combinations of Functions With $y = e^u$, Where u Is a Function of x

1. Derivatives of $y = e^x$ and Combinations of Functions With $y = e^x$

Consider the function $f(x) = e^x$. Using the limit definition of the derivative, we can find $f'(x)$.

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$	Use the limit definition of the derivative.
$f'(x) = \lim\limits_{h \to 0} \dfrac{e^{x+h} - e^x}{h}$	$f(x) = e^x$, $f(x+h) = e^{x+h}$
$f'(x) = \lim\limits_{h \to 0} \dfrac{e^x e^h - e^x}{h}$	Apply the property of exponents: $e^a e^b = e^{a+b}$
$f'(x) = \lim\limits_{h \to 0} \dfrac{e^x(e^h - 1)}{h}$	Remove the common factor of e^x.
$f'(x) = e^x \cdot \lim\limits_{h \to 0} \dfrac{e^h - 1}{h}$	Since e^x is a constant relative to h, it can be factored outside the limit.

Now, let's focus on the limit, which cannot be manipulated algebraically (there is no way to simplify $e^h - 1$). Thus, we will use a table and hopefully be able to get a nice approximation for the limit. The following table shows the behavior of $\dfrac{e^h - 1}{h}$ as $h \to 0$.

h	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$\frac{e^h - 1}{h}$	0.95163	0.99502	0.99950	0.99995	—	1.00005	1.00050	1.00502	1.05171

The table suggests that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. Note that this is not a formal proof, but it is convincing.

It follows that $f'(x) = e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^x(1) = e^x$, which means that e^x is its own derivative!

FORMULA TO KNOW

The Derivative of e^x

$D[e^x] = e^x$

Now we'll incorporate this new derivative rule into the others we already know.

EXAMPLE Consider the function $f(x) = -2x^2 + 3e^x + 6$. Find its derivative.

$f(x) = -2x^2 + 3e^x + 6$	Start with the original function.
$f'(x) = D[-2x^2] + D[3e^x] + D[6]$	Use the derivative of a sum/difference rule.
$f'(x) = -2D[x^2] + 3D[e^x] + D[6]$	Use the constant multiple rule.
$f'(x) = -2(2x) + 3(e^x) + 0$	$D[x^2] = 2x$, $D[e^x] = e^x$, $D[6] = 0$
$f'(x) = -4x + 3e^x$	Simplify.

Thus, $f'(x) = -4x + 3e^x$.

EXAMPLE Write the equation of the line tangent to the graph of $f(x) = \frac{e^x - 1}{e^x + 1}$ at $x = 0$.

Recall that the equation of a tangent line at $x = 0$ is $y = f(0) + f'(0)(x - 0)$. $f'(0)$ will be computed once we find the derivative.

$$f(0) = \frac{e^0 - 1}{e^0 + 1} = \frac{1 - 1}{1 + 1} = 0$$

Let's find $f'(x)$:

$f(x) = \frac{e^x - 1}{e^x + 1}$	Start with the original function.

$$f'(x) = \frac{(e^x+1)\cdot D[e^x-1]-(e^x-1)\cdot D[e^x+1]}{(e^x+1)^2}$$ Apply the quotient rule.

$$f'(x) = \frac{(e^x+1)\cdot e^x-(e^x-1)\cdot e^x}{(e^x+1)^2}$$ $D[e^x-1]=D[e^x]-D[1]=e^x$

$D[e^x+1]=D[e^x]+D[1]=e^x$

$$f'(x) = \frac{e^{2x}+e^x-e^{2x}+e^x}{(e^x+1)^2}$$ Distribute $e^x e^x = e^{x+x} = e^{2x}$.

$$f'(x) = \frac{2e^x}{(e^x+1)^2}$$ Combine like terms.

Then, the slope is $f'(0) = \frac{2e^0}{(e^0+1)^2} = \frac{2(1)}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$.

So, the equation of the tangent line is $y = 0 + \frac{1}{2}(x-0)$, which simplifies to $y = \frac{1}{2}x$.

WATCH

Consider the function $f(x) = \sqrt{e^x+4x}$. Find its derivative.

TRY IT

Find the slope of the tangent line to the function $g(x) = xe^x$ when $x = 2$.

What is the slope? +

Recall that the slope of the tangent line to the graph of a function is the derivative at that point. Therefore, we need to take the derivative of $g(x)$, then find $g'(2)$.

The function $g(x)$ is the product of two functions. By the product rule:

$$\begin{aligned} g'(x) &= e^x \cdot D[x] + x \cdot D[e^x] \\ &= e^x(1) + x \cdot e^x \\ &= e^x + xe^x \end{aligned}$$

Then, the slope of the tangent line when $x = 2$ is $g'(2) = e^2 + 2e^2 = 3e^2$.

2. Derivatives of $y = e^u$ and Combinations of Functions With $y = e^u$, Where u Is a Function of x

As a result of the chain rule, we have the following derivative rule for e^u:

FORMULA TO KNOW

The Derivative of e^u, Where u Is a Function of x

$D[e^u] = e^u \cdot u'$

EXAMPLE Consider the function $f(x) = e^{-x^2}$. Find its derivative.

$f(x) = e^{-x^2}$	Start with the original function.
$f'(x) = e^{-x^2}(-2x)$	$u = -x^2,\ u' = -2x$ $f'(x) = e^u \cdot u'$
$f'(x) = -2xe^{-x^2}$	Write "-2x" in front. It is conventional to write the simpler factors before the exponential function.

Thus, $f'(x) = -2xe^{-x^2}$.

TRY IT

Consider the function $f(x) = e^{2\sin x}$.

Find its derivative.

+

Since $f(x)$ is a function within a function, $f'(x) = e^{2\sin x} \cdot D[2\sin x] = e^{2\sin x} \cdot 2\cos x$.

This can also be written $f'(x) = 2\cos x \cdot e^{2\sin x}$.

EXAMPLE Consider the function $f(x) = x^2 e^{-3x}$. Find its derivative.

$f(x) = x^2 e^{-3x}$	Start with the original function.
$f'(x) = D[x^2] \cdot e^{-3x} + x^2 \cdot D[e^{-3x}]$	Use the product rule.

$$f'(x) = 2x \cdot e^{-3x} + x^2 \cdot e^{-3x}(-3) \qquad D[x^2] = 2x$$
$$D[e^u] = e^u \cdot u' \text{ (with } u = -3x)$$
$$D[e^{-3x}] = e^{-3x}(-3)$$

$$f'(x) = 2xe^{-3x} - 3x^2e^{-3x} \qquad \text{Simplify.}$$

Now let's see how these rules are applied within combinations of functions.

EXAMPLE After being attached to a spring, the height of an object (in feet) is modeled by the function $f(t) = 4e^{-2t}\cos(3t)$, where t is the number of seconds since the object was set into motion. Find the initial velocity, which is the velocity when $t = 0$.

Recall that the velocity is $f'(t)$.

$$f(t) = 4e^{-2t}\cos(3t) \qquad \text{Start with the original function.}$$

$$f'(t) = D[4e^{-2t}] \cdot \cos(3t) + 4e^{-2t} \cdot D[\cos(3t)] \qquad \text{Use the product rule.}$$

$$f'(t) = 4(e^{-2t})(-2) \cdot \cos(3t) + 4e^{-2t}(-\sin(3t))(3) \qquad D[e^u] = e^u \cdot u' \text{ (with } u = -2t)$$
$$D[\cos u] = -\sin u \cdot u' \text{ (with } u = 3t)$$

$$f'(t) = -8e^{-2t}\cos(3t) - 12e^{-2t}\sin(3t) \qquad \text{Simplify.}$$

Then, the velocity when $t = 0$ is $f'(0) = -8e^{-2(0)}\cos(3 \cdot 0) - 12e^{-2(0)}\sin(3 \cdot 0) = -8(1)(1) - 12(1)(0) = -8$ ft/s.

TRY IT

Consider the function $f(t) = \sin(2e^{-4t})$.

Find its derivative. +

By the chain rule:

$$\begin{aligned} f'(t) &= \cos(2e^{-4t}) \cdot D[2e^{-4t}] \\ &= \cos(2e^{-4t}) \cdot (2e^{-4t} \cdot D[-4t]) \\ &= \cos(2e^{-4t}) \cdot (2e^{-4t}(-4)) \\ &= \cos(2e^{-4t}) \cdot (-8e^{-4t}) \\ &= -8e^{-4t}\cos(2e^{-4t}) \end{aligned}$$

SUMMARY

In this lesson, you learned how to take **derivatives of exponential functions** ($y = e^x$) **and combinations of functions with** $y = e^x$. You learned that the function $f(x) = e^x$ is quite unique since it is its own

derivative. You also learned the **derivative rule for** $y = e^u$, as a result of the chain rule, as well as **combinations of functions with** $y = e^u$**, where** ***u*** **is a function of** ***x*****.** Knowing both of these derivatives enables you to expand on the types of functions whose derivatives can be found. In a future challenge, we will be exploring more applications in which exponential functions are involved.

FORMULAS TO KNOW

The Derivative of e^x

$D[e^x] = e^x$

The Derivative of e^u, Where u Is a Function of x

$D[e^u] = e^u \cdot u'$