The Algebra Method

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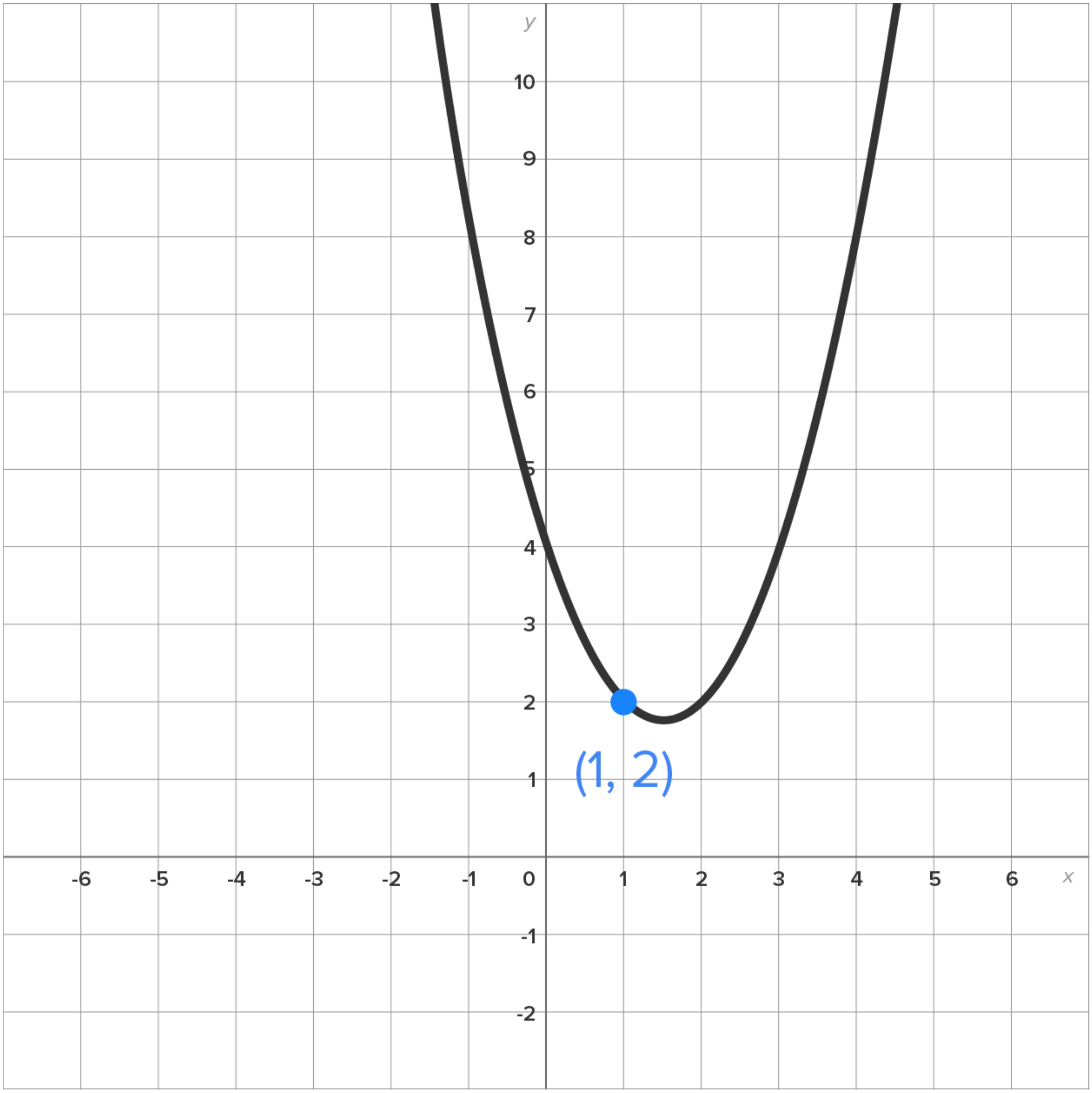
WHAT'S COVERED

In this lesson, you will use algebraic techniques to evaluate limits. Specifically, this lesson will cover:

1. The Direct Substitution Method

Consider the function $f(x) = x^2 - 3x + 4$, with the goal of evaluating $\lim_{x \to 1} (x^2 - 3x + 4)$.

The graph of the function is shown below.



Things to notice:

- The graph contains the point $(1, 2)$.
- As x gets closer to 1 from either side, the value of $f(x)$ gets closer to 2.

Thus, we can say that $\lim_{x \to 1} (x^2 - 3x + 4) = 2$.
We could have found this same answer by substituting 1 in for x in the function. This would have given us the point $(1, 2)$ along with the value of the limit, 2.

BIG IDEA

When given a continuous function $f(x)$, one way to evaluate $\lim_{x \to a} f(x)$ is to substitute a in for *x* and simplify. This works when $f(x)$ is defined on both sides of $x = a$.

EXAMPLE Evaluate each limit below by using the direct substitution method.

Limit	Solution
$\lim_{x \to 3} \frac{x^2-4}{x+7}$	$\lim_{x \to 3} \frac{x^2-4}{x+7} = \frac{3^2-4}{3+7} = \frac{5}{10} = \frac{1}{2}$
$\lim_{x \to 2} (3x^4-x^2)$	$\lim_{x \to 2} (3x^4-x^2) = 3(2)^4-2^2 = 44$
$\lim_{x \to 1} \sin \pi x$	$\lim_{x \to 1} \sin \pi x = \sin \pi(1) = \sin \pi = 0$

TRY IT

Consider the function $f(x) = \frac{x^3-8}{x+1}$.

Evaluate the limit as *x* approaches 1. +

$$\lim_{x \to 1} \frac{x^3-8}{x+1} = \frac{1^3-8}{1+1} = -\frac{7}{2}$$

2. Evaluating Limits by Simplifying the Expression

What happens when direct substitution doesn't work? That is, what happens when $f(a)$ is undefined? There are other methods that could be helpful in evaluating the limit.

In previous parts of this challenge, we evaluated $\lim_{x \to 1} \frac{x^2-1}{x-1}$ by using graphs and tables. Both times we concluded that the limit is 2. While these methods were fairly straightforward to use, algebraic techniques are more convincing.

Remember that the limit $\lim_{x \to 1} \frac{x^2-1}{x-1}$ means that *x* is getting closer to 1, but not equal to 1.

Also notice that direct substitution will not work for this limit since $\frac{x^2-1}{x-1}$ is undefined when $x = 1$.

However, $\frac{x^2-1}{x-1}$ can be simplified, keeping in mind that $x \neq 1$.

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1$$

This means $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} (x+1) = 1+1 = 2$.

Once we simplified $\lim_{x \to 1} \frac{x^2-1}{x-1}$ to $\lim_{x \to 1} (x+1)$, we were able to use direct substitution and evaluate the limit.

The big question is: How do we know to simplify?

Notice that when using direct substitution with $\lim_{x \to 1} \frac{x^2-1}{x-1}$, the numerator and denominator are both 0. This is a signal that the expression can be simplified.

EXAMPLE Evaluate the limit: $\lim_{x \to 5} \frac{x^2-6x+5}{x^2-25}$

Attempting direct substitution, notice that the numerator and denominator are both 0. This means we should try to simplify:

$\lim_{x \to 5} \frac{x^2-6x+5}{x^2-25}$	Start with the original limit.
$= \lim_{x \to 5} \frac{(x-5)(x-1)}{(x-5)(x+5)}$	Factor the numerator and denominator.
$= \lim_{x \to 5} \frac{(x-1)}{(x+5)}$	Remove the common factor.
$= \frac{4}{10}$	Substitute $x=5$.
$= \frac{2}{5}$	Simplify the expression.

Conclusion: $\lim_{x \to 5} \frac{x^2-6x+5}{x^2-25} = \frac{2}{5}$

WATCH

The following video walks you through the process of evaluating $\lim_{x \to -2} \frac{x^3+8}{x^2-x-6}$ by simplifying the rational expression.

EXAMPLE Evaluate the limit: $\lim_{x \to 6} \frac{\left(\frac{1}{x} - \frac{1}{6}\right)}{x-6}$

Attempting direct substitution, notice that the numerator and denominator are both 0. This means we should try to simplify:

$\lim_{x \to 6} \frac{\left(\frac{1}{x} - \frac{1}{6}\right)}{x-6}$	Start with the original limit.
$= \lim_{x \to 6} \frac{\left(\frac{1}{x} - \frac{1}{6}\right)}{x-6} \cdot \frac{6x}{6x}$	Since this is a complex fraction, multiply the numerator and denominator by the LCD, which is $6x$.
$= \lim_{x \to 6} \frac{6-x}{6x(x-6)}$	Distribute in the numerator and simplify. Leave the denominator in factored form.
$= \lim_{x \to 6} \frac{-(x-6)}{6x(x-6)}$	Factor out the negative factor in the numerator.
$= \lim_{x \to 6} \frac{-1}{6x}$	Remove the common factor.
$= -\frac{1}{36}$	Substitute $x = 6$.

TRY IT

Consider the function $f(x) = \frac{14x - 42}{x^2 - 8x + 15}$.

Evaluate the limit as x approaches 3. +

The limit is equal to -7. Here is why:

When substituting $x = 3$, the numerator and denominator are both 0. This means that we can likely simplify the expression to evaluate the limit.

$\lim_{x \to 3} \frac{14x - 42}{x^2 - 8x + 15}$	Original expression
$= \lim_{x \to 3} \frac{14(x-3)}{(x-3)(x-5)}$	Factor the numerator and denominator.
$= \lim_{x \to 3} \frac{14}{x-5}$	Remove common factor of $x-3$.

$$= \frac{14}{3-5}$$ Use direct substitution.

$$= \frac{14}{-2} = -7$$ Simplify and reduce.

In conclusion, $\lim_{x \to 3} \frac{14x - 42}{x^2 - 8x + 15} = -7$.

SUMMARY

In this lesson, you learned that you can **evaluate limits** algebraically using two methods: **the direct substitution method** and by **simplifying the expression** first. Note that when $f(a)$ is undefined, direct substitution doesn't work. When using direct substitution with $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$, the numerator and denominator are both 0, which is a signal that the expression can be simplified. Both of these methods are useful since they require algebraic facts and will give exact answers rather than tables or graphs that often give approximations.