Finding Maximums and Minimums of a Function

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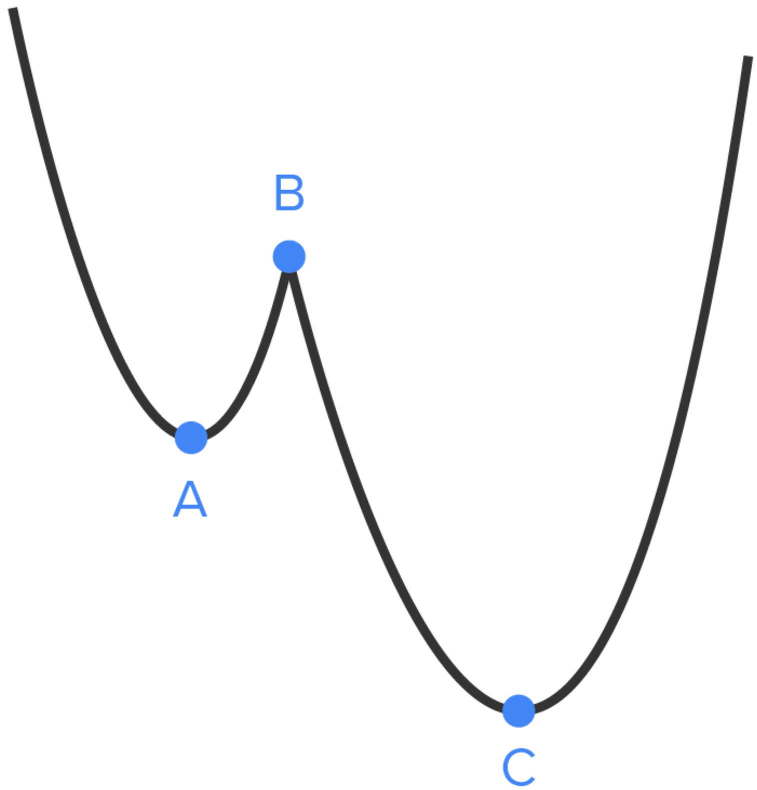
WHAT'S COVERED

In this lesson, you will use derivatives and critical numbers to find local maximum and local minimum values. Specifically, this lesson will cover:

1. The Relationship Between Critical Numbers and Local Extrema
2. Finding Local Extrema

1. The Relationship Between Critical Numbers and Local Extrema

Consider the graph of a function $y = f(x)$, shown here:



- At point *A*, $f'(x) = 0$.
- At point *B*, $f'(x)$ is undefined.
- At point *C*, $f'(x) = 0$.

As discussed in the previous tutorial, values of x in the domain of $f(x)$ where $f'(x) = 0$ or $f'(x)$ is undefined are called critical numbers.
Therefore, critical numbers can tell us where local maximum or minimum values could occur.

However, the only way to find out is through further analysis, which will be covered in a future challenge.

STEP BY STEP

To identify relative extrema:

1. Find all critical values.
2. Use a graph of the function to determine which critical numbers correspond to which relative extreme points.

Now that we know the connection between critical numbers and extrema, let's look at a few examples.

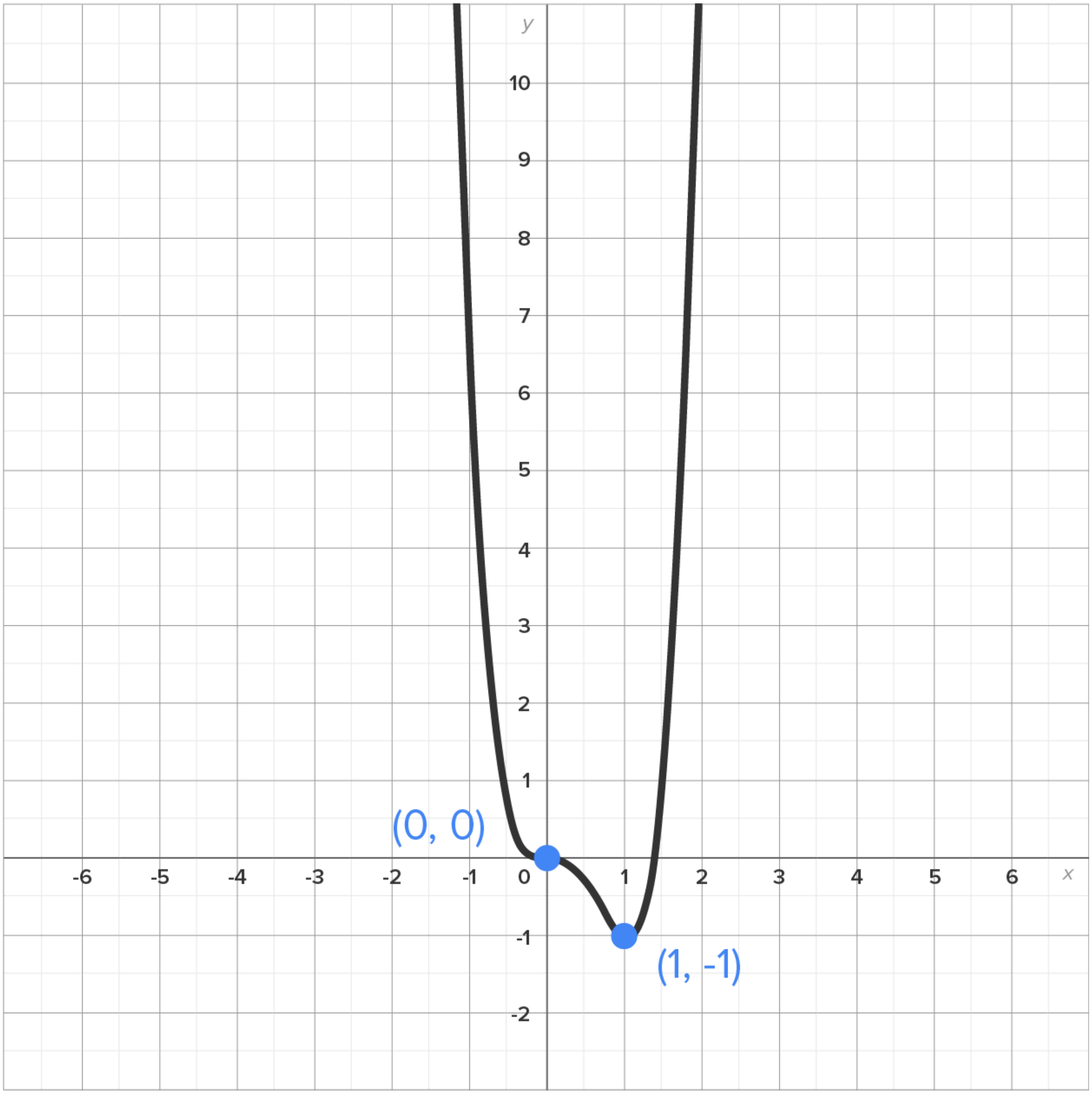
2. Finding Local Extrema

EXAMPLE Consider the function $f(x) = 3x^4 - 4x^3$. First, find all critical numbers:

$f(x) = 3x^4 - 4x^3$	Start with the original function; the domain is all real numbers.
$f'(x) = 12x^3 - 12x^2$	Take the derivative.
$12x^3 - 12x^2 = 0$	Since $f'(x)$ is a polynomial, it is never undefined. Set $f'(x) = 0$ and solve.
$12x^2(x-1) = 0$	Factor.
$12x^2 = 0,\ x - 1 = 0$	Set each factor equal to 0.
$x = 0,\ x = 1$	Solve.

Thus, the critical numbers are $x = 0$ and $x = 1$.

Now, the graph of $f(x)$ is shown.



The point (0, 0) is neither a local maximum nor a local minimum, while a local minimum (also a global minimum) occurs at (1, -1).

TRY IT

Consider the function $f(x) = -\frac{1}{2}x^4 + 9x^2 + 10$.

Find all critical numbers of *f*, then determine the local minimum and maximum points by using a graph. +

The critical numbers are the values of x for which $f'(x) = 0$ or possibly undefined.

First, find $f'(x) = -2x^3 + 18x$.

Since this is a polynomial, there is no possibility of $f'(x)$ being undefined.

Setting equal to 0 and solve:

$$-2x^3 + 18x = 0$$
$$-2x(x^2 - 9) = 0$$
$$-2x(x+3)(x-3) = 0$$
$$-2x = 0,\ \ x+3 = 0,\ \ x-3 = 0$$
$$x = 0,\ -3,\ 3$$

Therefore, the critical numbers are $x = 0$, -3, and 3.

Substituting each value into $f(x)$, we have the following:

$$f(0) = -\frac{1}{2}(0)^4 + 9(0)^2 + 10 = 10$$
$$f(-3) = -\frac{1}{2}(-3)^4 + 9(-3)^2 + 10 = 50.5$$
$$f(3) = -\frac{1}{2}(3)^4 + 9(3)^2 + 10 = 50.5$$

By examining a graph, you see that the local minimum is located at $(0, 10)$ and the local maximum points are located at $(-3, 50.5)$ and $(3, 50.5)$.

Here is another example that requires us to pay attention to many details.

➦ EXAMPLE Find all local minimum and maximum values of the function $f(x) = \sqrt{x^2 + 1} - \frac{3}{5}x$.

First, we find the critical numbers. Note that the domain of $f(x)$ is all real numbers.

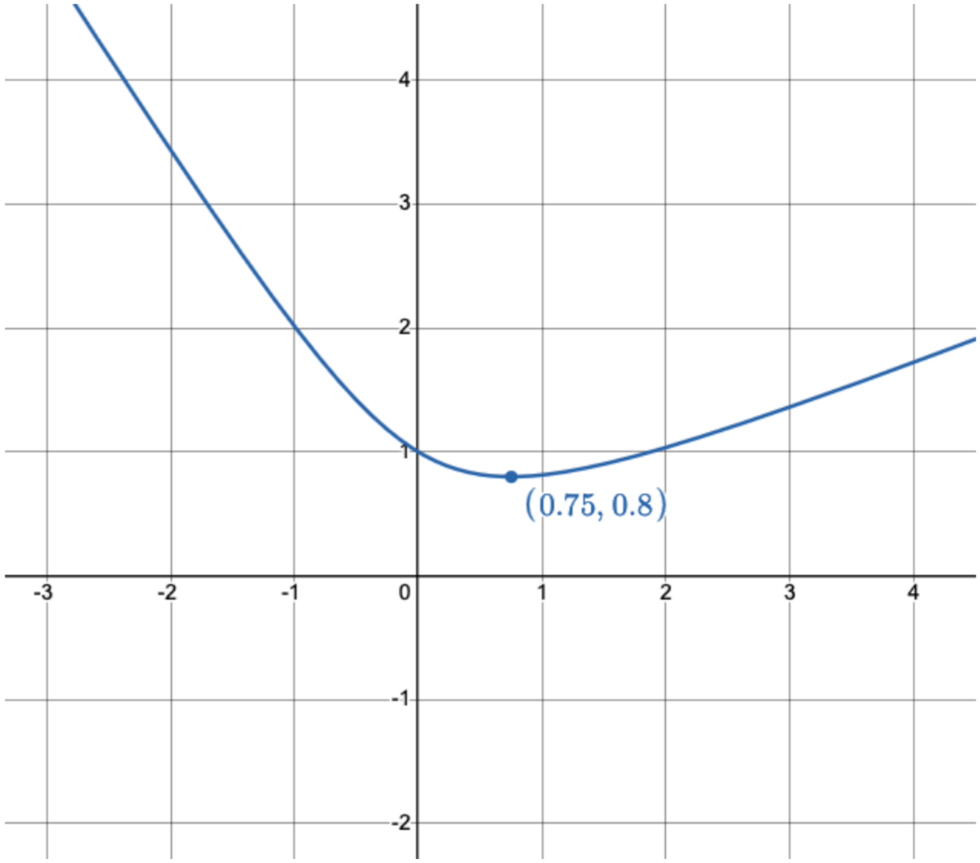
$f(x) = (x^2 + 1)^{1/2} - \frac{3}{5}x$ Rewrite $f(x)$ using exponents to set up differentiation.

$f(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x) - \frac{3}{5}$	Find the derivative. Note that $D\left[(x^2+1)^{1/2}\right] = \frac{1}{2}(x^2+1)^{-1/2} \cdot D\left[x^2\right]$.
$f(x) = \frac{x}{(x^2+1)^{1/2}} - \frac{3}{5}$	Simplify the first term, then write $(x^2+1)^{-1/2}$ in terms of positive exponents.
$f(x) = \frac{x}{\sqrt{x^2+1}} - \frac{3}{5}$	Rewrite $(x^2+1)^{1/2}$ as $\sqrt{x^2+1}$.
$\frac{x}{\sqrt{x^2+1}} - \frac{3}{5} = 0$	Set $f'(x) = 0$.
$\frac{x}{\sqrt{x^2+1}} = \frac{3}{5}$	Add $\frac{3}{5}$ to both sides.
$5x = 3\sqrt{x^2+1}$	Cross multiply.
$(5x)^2 = \left(3\sqrt{x^2+1}\right)^2$	Since a variable is under a square root, square both sides.
$25x^2 = 9(x^2+1)$	Simplify.
$25x^2 = 9x^2 + 9$	Distribute on the right-hand side.
$x^2 = \frac{9}{16}$	Subtract $9x^2$ from both sides, then divide both sides by 16.
$x = \pm\sqrt{\frac{9}{16}} = \pm\frac{3}{4}$	Apply the square root principle. Remember that this yields both a positive and a negative solution!

You may recall that when solving equations by squaring both sides, it's possible to get extraneous solutions. While $x = \frac{3}{4}$ solves the original equation $\frac{x}{\sqrt{x^2+1}} = \frac{3}{5}$, $x = -\frac{3}{4}$ does not. To see this, substitute $x = -\frac{3}{4}$ into the equation – you will get $-\frac{3}{5} = \frac{3}{5}$, which is not true.

Therefore, the only critical number is $x = \frac{3}{4}$.

The graph of $f(x)$ is shown below.



As you can see, the point $(0.75, 0.8)$ is a low point on the graph, therefore there is a local minimum value of 0.8 when $x = 0.75$.

SUMMARY

In this lesson, you learned about **the relationship between critical numbers and local extrema**, which is that critical numbers help locate the coordinates of local maximum and minimum points. Understanding this connection between critical numbers and extrema, you practiced **finding local extrema** of a function by first determining all critical numbers. As you saw with one example, a critical number at $x = c$ doesn't automatically imply that there is a local maximum or minimum at $x = c$.