Second Shape Theorem

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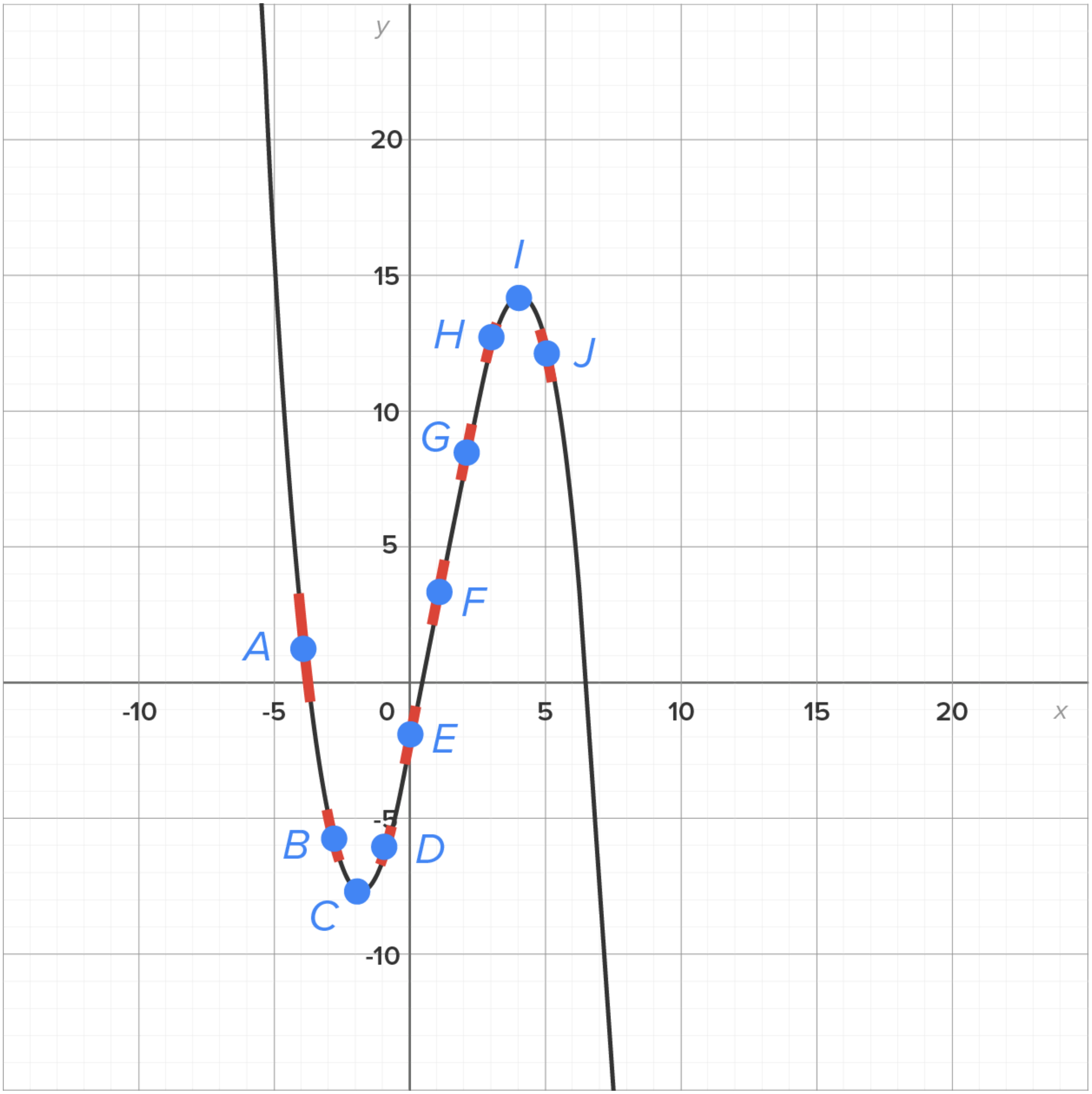
WHAT'S COVERED

In this lesson, you will sketch the shape of a function $f(x)$ given values of its derivative, $f'(x)$. Specifically, this lesson will cover:

1. Using Values of $f'(x)$ to Sketch the Shape of the Graph of $f(x)$
2. Using $f'(x)$ to Detect Local Maximum and Minimum Values

1. Using Values of $f'(x)$ to Sketch the Shape of the Graph of $f(x)$

Consider the graph below of a function $y=f(x)$, which is the same as the graph in the last tutorial.



Notice that the graph of $f(x)$ is increasing at every point where its tangent line has a positive slope, and the graph of $f(x)$ is decreasing at every point where its tangent line has a negative slope. This means we can extend the ideas of increasing and decreasing from points to intervals.

BIG IDEA

If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that same interval.

If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that same interval.

Based on this fact, we can use the derivative to determine entire intervals over which a function $f(x)$ is increasing or decreasing.

WATCH

In the following video, we'll find all intervals over which $f(x) = -x^3 + 3x^2 + 24x + 10$ is increasing or decreasing. One of the main objectives of this video is to show how the information gets organized, so viewing this video is important!

Now that you see how to organize the information, let's look at more examples.

EXAMPLE Let $f(x) = \frac{x}{x^2+1}$. Determine the intervals over which $f(x)$ is increasing or decreasing. Since this information comes from the first derivative, we start there. Finding the derivative of this function requires the quotient rule.

$f(x) = \frac{x}{x^2+1}$	Start with the original function; the domain is all real numbers.
$f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2}$	Apply the quotient rule.
$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$	Simplify.

$f'(x)$ can only change its sign when $f'(x) = 0$ or is undefined. It is never undefined since the expression in the denominator is never zero.

Now, set $f'(x) = 0$ and solve.

$\frac{1-x^2}{(x^2+1)^2} = 0$	Set $f'(x) = 0$. Since $f'(x)$ is never undefined, this case is not considered.
$1 - x^2 = 0$	Multiply both sides by $(x^2+1)^2$.
$1 = x^2$	Solve for *x*.
$x = \pm 1$	

This means that the real number line is broken into three intervals:

$(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$

Now, select one number (called a test value) inside each interval to determine the sign of $f'(x)$ on that interval:

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test Value	-2	0	2

Value of $f'(x) = \frac{1-x^2}{(x^2+1)^2}$	$\frac{-3}{25}$	1	$\frac{-3}{25}$
Behavior of $f(x)$	Decreasing	Increasing	Decreasing

Thus, $f(x)$ is increasing on the interval $(-1, 1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$.

Let's look at an exponential function now.

➯ EXAMPLE Let $f(x) = e^{-x^2}$. Determine the intervals over which $f(x)$ is increasing or decreasing. Since this information comes from the first derivative, we start there. This derivative requires the chain rule.

$f(x) = e^{-x^2}$	Start with the original function.
$f'(x) = -2x \cdot e^{-x^2}$	Apply the chain rule.

$f'(x)$ can only change its sign when $f'(x) = 0$ or is undefined. It is never undefined since there are no domain restrictions.

Now, set $f'(x) = 0$ and solve.

$-2x \cdot e^{-x^2} = 0$	The derivative is set to 0.
$-2x = 0$ or $e^{-x^2} = 0$	Set each factor equal to 0.
$x = 0$	$-2x = 0$ implies $x = 0$. $e^{-x^2} = 0$ has no solution.

This means that the real number line is broken into two intervals:

$(-\infty, 0)$, $(0, \infty)$

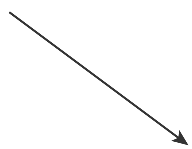
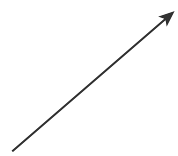
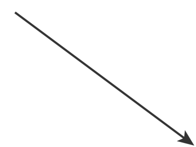
Now, select one number (called a test value) inside each interval to determine the sign of $f'(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	-1	1
Value of $f'(x) = -2xe^{-x^2}$	$2e^{-1}$	$-2e^{-1}$
Behavior of $f(x)$	Increasing	Decreasing

Thus, $f(x)$ is increasing on the interval $(-\infty, 0)$ and decreasing on $(0, \infty)$.

2. Using $f'(x)$ to Detect Local Maximum and Minimum Values

Consider the function $f(x) = \frac{x}{x^2+1}$. In part 1, we obtained the following sign graph for $f'(x)$. Notice that the bottom row has been added to show the direction the graph is moving when increasing or decreasing.

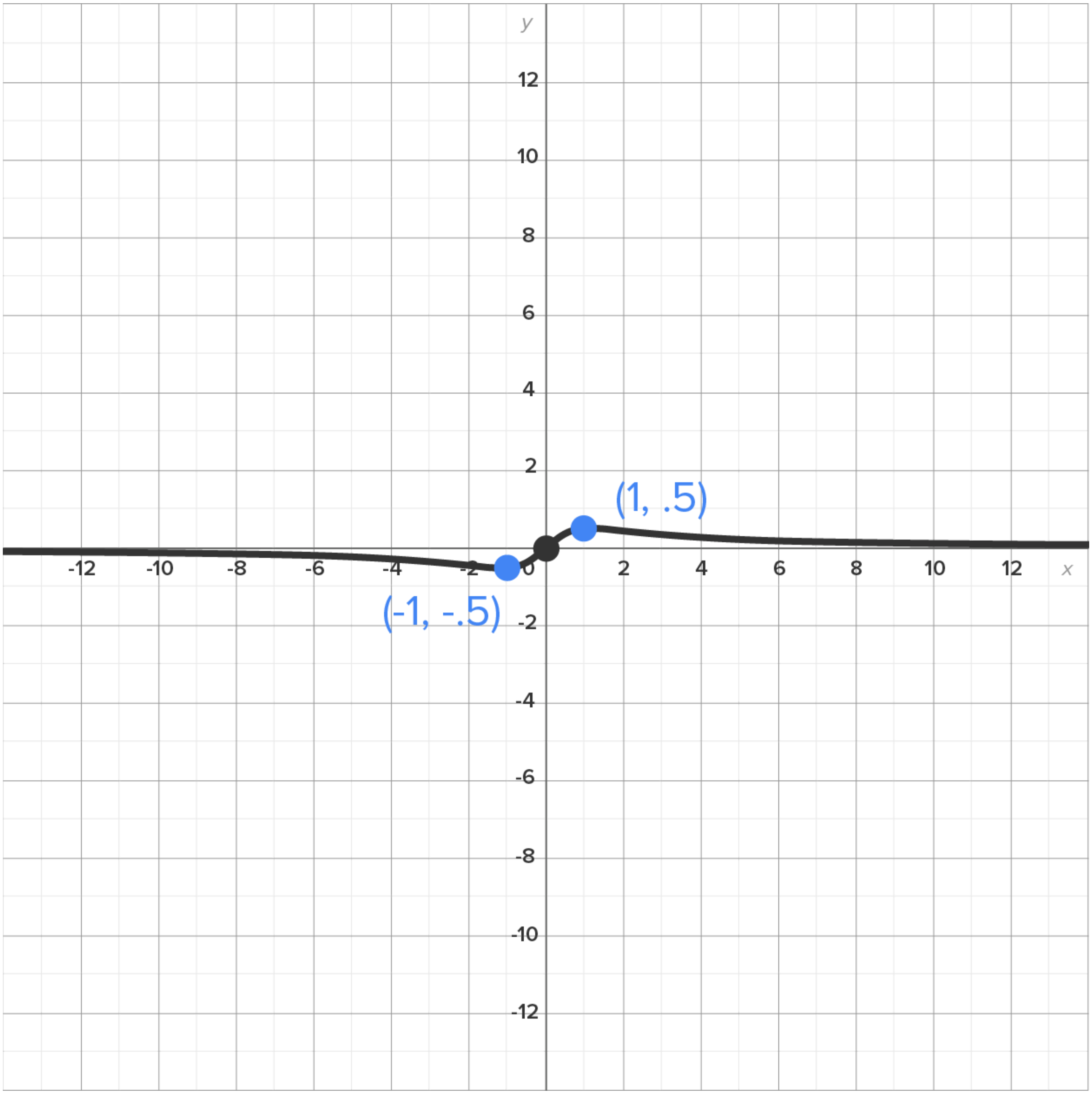
Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test Value	-2	0	2
Value of $f'(x) = \frac{1-x^2}{(x^2+1)^2}$	$\frac{-3}{25}$	1	$\frac{-3}{25}$
Behavior of $f(x)$	Decreasing	Increasing	Decreasing
Direction			

This means that when $x = -1$, the graph is transitioning from decreasing to increasing; and when $x = 1$, the graph is transitioning from increasing to decreasing. Since $f(x)$ is continuous at $x = -1$ and $x = 1$, this means that there is a local minimum at $x = -1$ and a local maximum at $x = 1$.

To find the coordinates of these points on the graph, substitute each value into $f(x)$:

- Local minimum: $(-1, f(-1))$, or $\left(-1, \frac{-1}{2}\right)$
- Local maximum: $(1, f(1))$, or $\left(1, \frac{1}{2}\right)$

The graph of $f(x)$ is shown here for reference:



Given a function $f(x)$, the first derivative, $f'(x)$, can be used to locate maximum and minimum values. To do so, use the same sign graph you used to determine where the function is increasing or decreasing. Then, observe the critical numbers where $f(x)$ transitions between increasing and decreasing. When $f(x)$ is continuous at these critical numbers, there is either a local minimum or local maximum point.

Performing this analysis to locate local minimum and maximum points is called the **first derivative test**.

EXAMPLE Use the first derivative test to determine all local minimum and maximum points of the function $f(x) = x^4 - 12x^3$.

First, find all critical numbers:

$f(x)=x^4-12x^3$	Start with the original function; the domain is all real numbers.
$f'(x)=4x^3-36x^2$	Take the derivative.
$4x^3-36x^2=0$	Set $f'(x)=0$. Since $f'(x)$ is a polynomial, there is no value of *x* for which $f'(x)$ is undefined, so this case is not considered.
$4x^2(x-9)=0$	Solve by factoring.
$4x^2=0,\ x-9=0$	
$x=0,\ x=9$	

Thus, the critical numbers are $x=0$ and $x=9$. Now we move to the first derivative test, which means making a sign graph, determining the intervals of increase and decrease, then observing which critical numbers produce a local maximum or local minimum.

Interval	$(-\infty, 0)$	$(0, 9)$	$(9, \infty)$
Test Value	-1	1	10
Value of $f'(x)=4x^3-36x^2$	-40	-32	400
Behavior of $f(x)$	Decreasing	Decreasing	Increasing
Direction	$\searrow$	$\searrow$	$\nearrow$

At $x=0$, $f(x)$ does not change direction; it is decreasing on both sides. There is no local extreme value at $x=0$.

At $x=9$, the $f(x)$ transitions from decreasing to increasing, indicating that there is a local minimum value when $x=9$.

Thus, there is a local minimum at $(9, f(9))$, or $(9, -2187)$.

BIG IDEA

There is no guarantee that a local extreme value occurs at a critical number. This is why we perform the first derivative test.

TRY IT

Consider the function $f(x) = xe^{-x}$.

Use the first derivative test to determine the local extrema of the function. +

First we find $f'(x)$ to find all critical numbers.

$$\begin{aligned} f'(x) &= e^{-x} \cdot D[x] + x \cdot D\left[e^{-x}\right] \\ &= e^{-x}(1) + x(e^{-x}) \cdot D[-x] \\ &= e^{-x} + xe^{-x}(-1) \\ &= e^{-x} - xe^{-x} \end{aligned}$$

Now, set equal to 0 to solve (note that $f'(x)$ is never undefined).

$$e^{-x} - xe^{-x} = 0$$

$$e^{-x}(1 - x) = 0$$

$$e^{-x} = 0, \ \ 1 - x = 0$$

$$x = 1$$

Note: There is no value of *x* for which $e^{-x} = 0$.

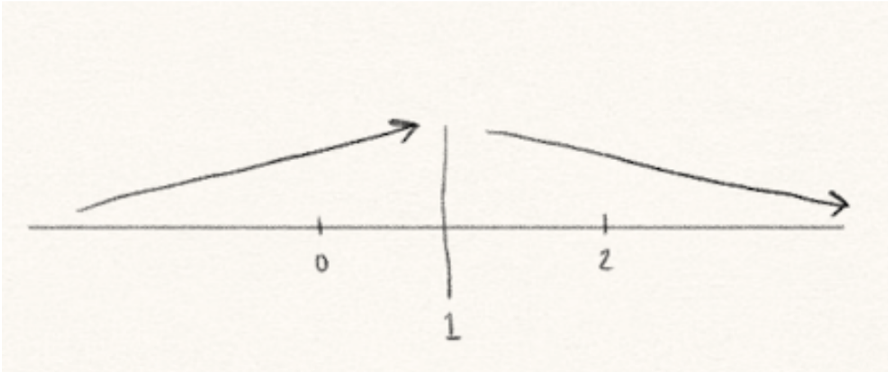
Now, we perform the first derivative test.

Since $x = 1$ is a critical number, choose convenient values above and below to determine if $f'(x)$ is positive or negative. In this case, we'll choose $x = 0$ and $x = 2$.

$f'(0) = e^0 - 0e^0 = 1$, which means $f(x)$ is increasing.

$f'(2) = e^{-2} - 2e^{-2} \approx -0.135$, which means $f(x)$ is decreasing.

The number line is as follows:



This means that there is a local maximum at $(1, f(1))$.

Since $f(1) = 1e^{-1} = \frac{1}{e}$, the location of the local maximum is the point $\left(1, \frac{1}{e}\right)$.

Let’s now look at a function whose derivative is not defined for all real numbers.

EXAMPLE Use the first derivative test to determine all local minimum and maximum points of the function $f(x) = x - 3\sqrt[3]{x}$. Note that the domain of $f(x)$ is the set of all real numbers, or $(-\infty, \infty)$.

First, find all critical numbers:

$f(x) = x - 3x^{1/3}$	Rewrite the radical using exponents to set up the differentiation.
$f'(x) = 1 - x^{-2/3}$	Take the derivative.
$f'(x) = 1 - \frac{1}{x^{2/3}}$	Rewrite in terms of positive exponents.

Since $f'(x)$ has a variable denominator, we need to check for values of *x* for which $f'(x)$ is undefined as well as where $f'(x) = 0$.

$f'(x)$ is undefined when the denominator is 0, meaning $x^{2/3} = 0$, which means when $x = 0$.

Checking the original function, $f(0) = 0$, which is defined. Therefore, $x = 0$ is a critical number of $f(x)$.

Next, find all values of *x* for which $f'(x) = 0$.

$1 - \frac{1}{x^{2/3}} = 0$	Set $f'(x) = 0$.
$1 = \frac{1}{x^{2/3}}$	Add $\frac{1}{x^{2/3}}$ to both sides.

$x^{2/3} = 1$	Multiply both sides by $x^{2/3}$.
$\sqrt[3]{x^2} = 1$	Rewrite $x^{2/3}$ as a radical.
$x^2 = 1$	Cube both sides.
$x = \pm 1$	Apply the square root property to both sides. When doing so, always remember the "±".

Thus, $x = -1$ and $x = 1$ are also critical numbers.

Now we move to the first derivative test, which means making a sign graph, determining the intervals of increase and decrease, then observing which critical numbers produce a local maximum or local minimum.

Here is a table to help organize this information. Note the test values of -8, -1/8, 1/8, and 8 are used since the expression for $f'(x)$ contains cube roots.

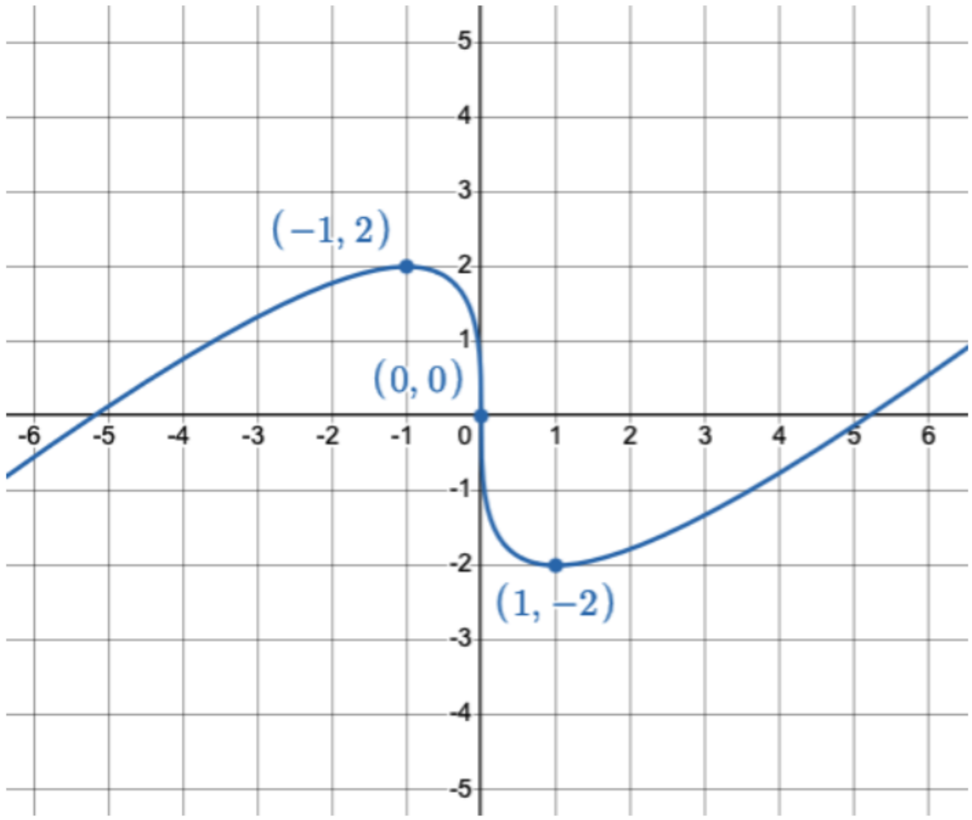
Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Test Value	-8	$-\frac{1}{8}$	$\frac{1}{8}$	8
Value of $f'(x) = 1 - x^{2/3}$	$\frac{3}{4}$	-3	-3	$\frac{3}{4}$
Behavior of $f(x)$	Increasing	Decreasing	Decreasing	Increasing
Direction	↗	↘	↘	↗

At $x = -1$, the fiunction transitions from increasing to decreasing, which means that $f(x)$ has a local maximum at $(-1, f(-1))$, or $(-1, 2)$.

At $x = 1$, the function transitions from decreasing to increasing, which means that $f(x)$ has a local minimum at $(1, f(1))$, or $(1, -2)$.

At $x = 0$, $f(x)$ is not changing direction since it is decreasing on both sides of $x = 0$. This means that there is neither a minimum nor a maximum point at $(0, f(0))$, or $(0, 0)$.

The graph of $f(x)$ is shown below, which confirms these results.



TERM TO KNOW

First Derivative Test

Used to identify possible local maximum and minimum points.

SUMMARY

In this lesson, you learned that the derivative, $f'(x)$, can be used to provide information about where $f(x)$ is increasing or decreasing. In other words, you can **use values of $f'(x)$ to sketch the shape of the graph of $f(x)$**. You also learned that you can apply these ideas further, **using $f'(x)$ to detect local maximum and minimum values** by performing an analysis called the first derivative test.

TERMS TO KNOW

First Derivative Test

Used to identify possible local maximum and minimum points.