Determining Differentiability Graphically

by Sophia

WHAT'S COVERED

In this lesson, you will look at what causes a function to not be differentiable and use graphical reasoning to determine differentiability, which is more straightforward than the analytical approach. Specifically, this lesson will cover:

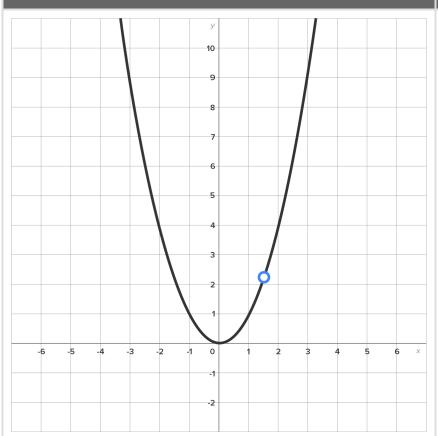
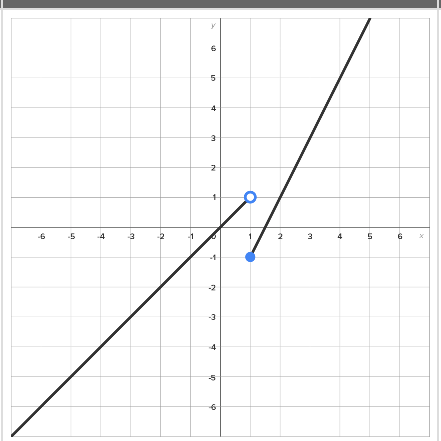
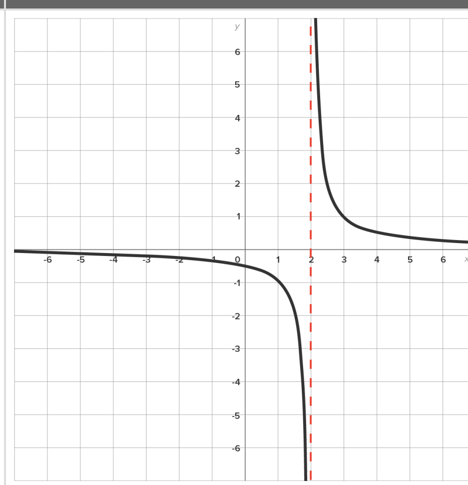
1. Discontinuities

As discussed in the previous section, a function is not differentiable at $x = a$ if it is discontinuous at $x = a$.

BIG IDEA

Therefore, if there is a break in the graph when $x = a$, the function is not differentiable at $x = a$.

There are three types of discontinuity:

Hole in the Graph	Graph Jumps	Vertical Asymptote
		

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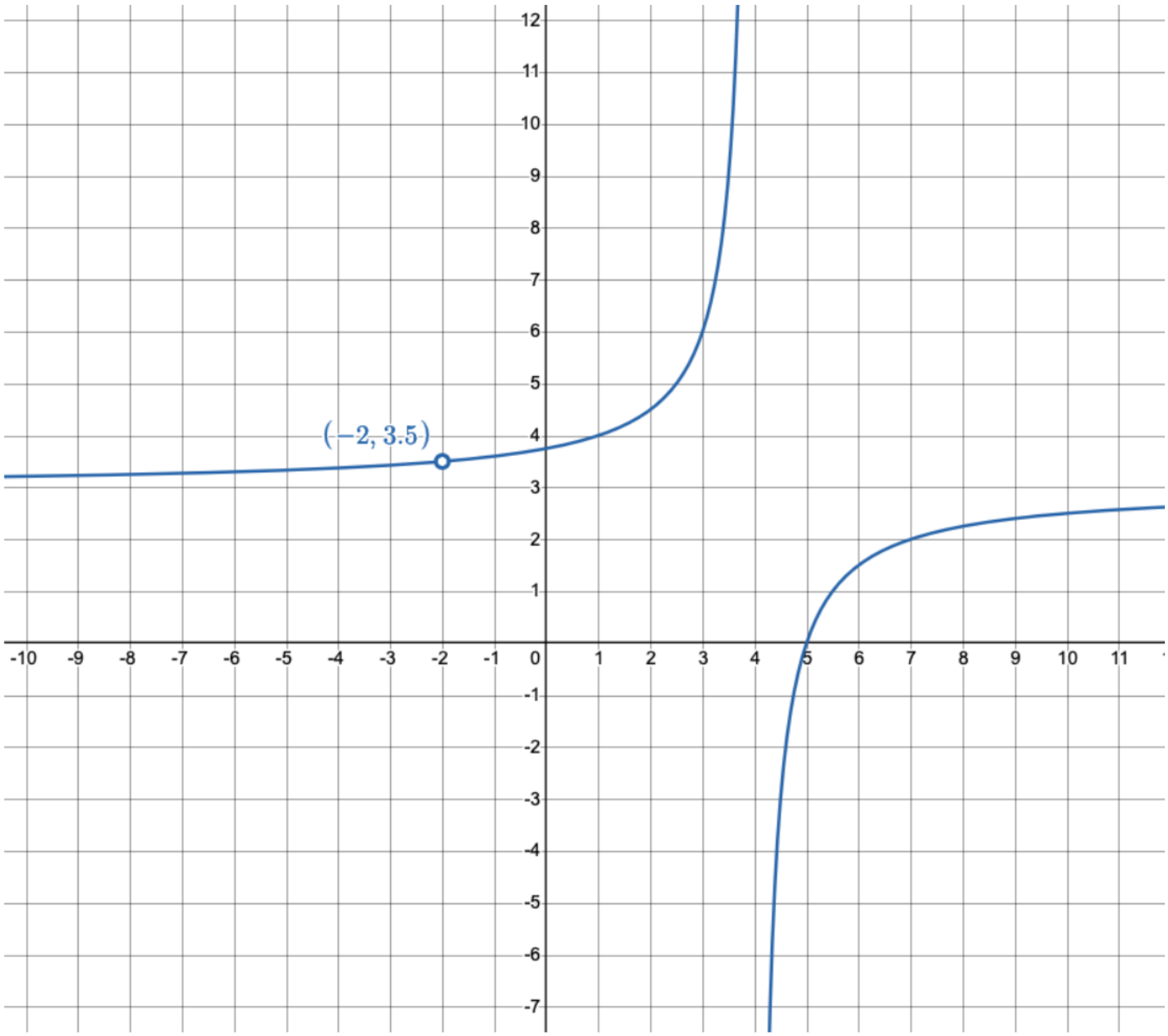
Since the continuity requirement isn't met at any discontinuity, it follows that a function is not differentiable at any x-value where $f(x)$ is discontinuous.

EXAMPLE Consider the function $f(x)=\frac{3(x-5)(x+2)}{(x+2)(x-4)}$, whose graph is shown below.



The graph has a hole at the point $(-2, 3.5)$ and a vertical asymptote at $x=4$.

Therefore, $f(x)$ is not differentiable when $x=-2$ and when $x=4$.

2. Vertical Tangents

When a tangent line is vertical, its slope is undefined. Since the derivative is the slope of a tangent line, a function is not differentiable at any point where there is a vertical tangent line.

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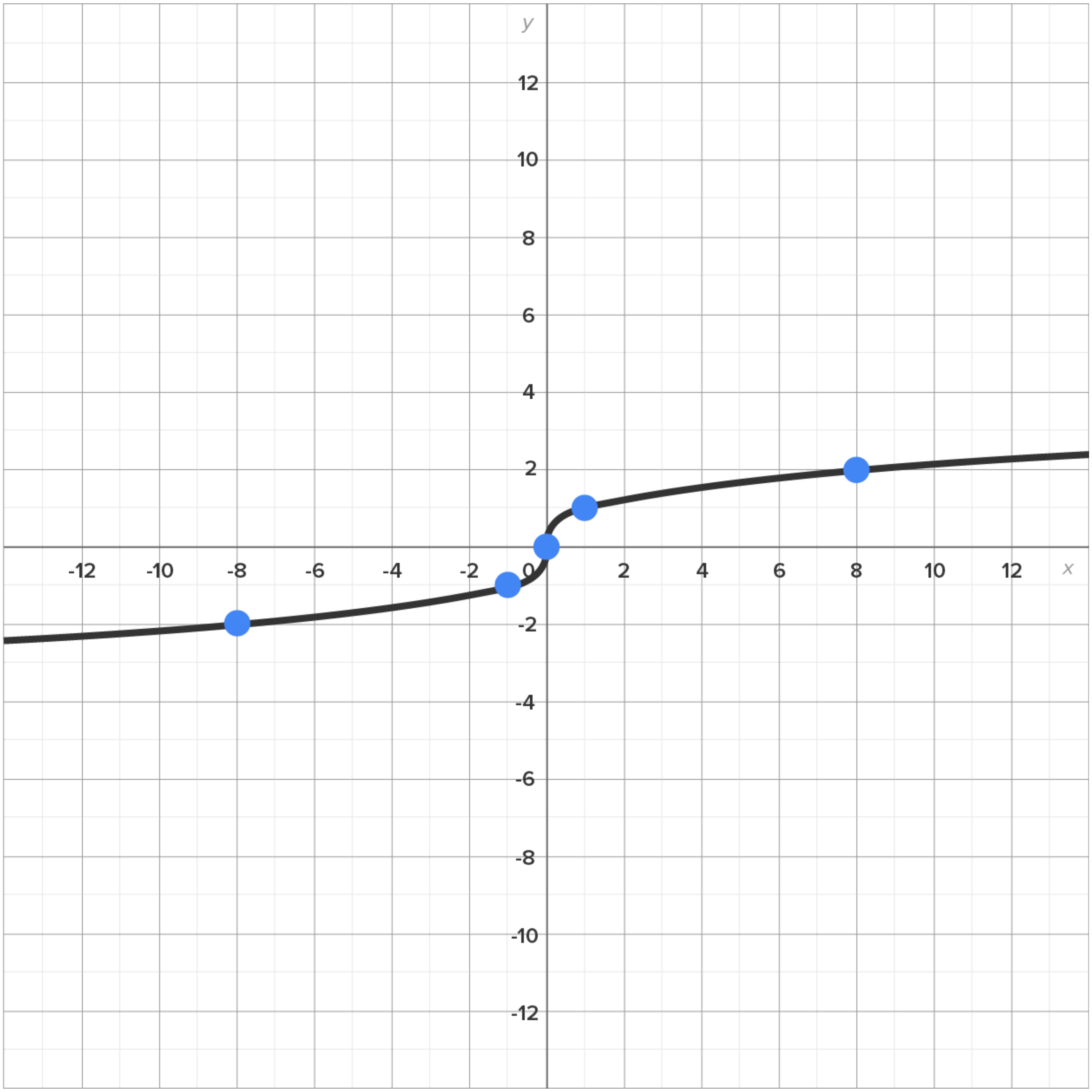
EXAMPLE Consider the graph of $y = \sqrt[3]{x}$, shown below:



Note that when $x = 0$, the tangent line appears to be vertical. In fact, we can show this by finding $f'(x)$, which was calculated in 3.2.1: $f'(x) = \dfrac{1}{3x^{2/3}}$, which is undefined when $x = 0$, indicating an undefined slope (vertical line). Recall that in order for a function to be differentiable at a point, the derivative has to be defined at that point. Therefore, a vertical tangent line at a point is an indication that the function is not differentiable at that point.

3. Sharp Corners

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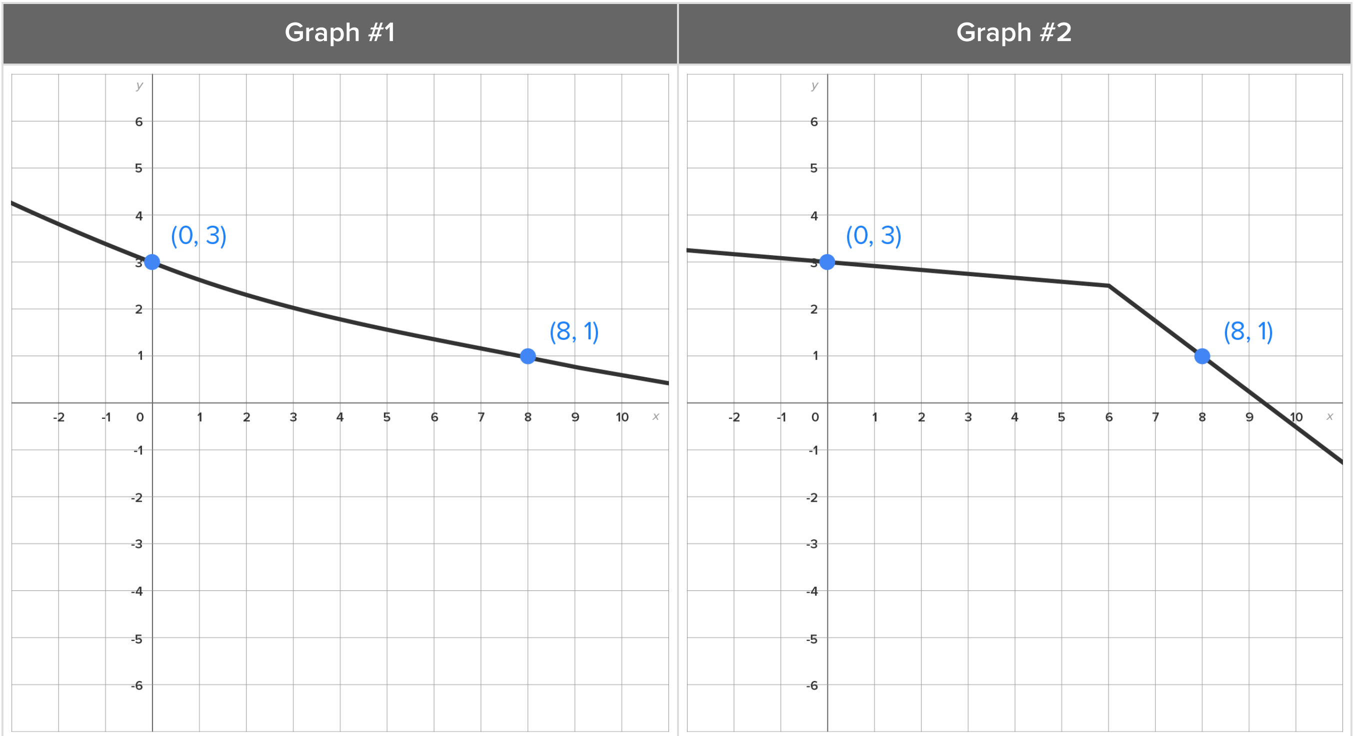
Suppose you want to drive along a road from (0, 3) to (8, 1). Which graph provides a smoother ride?



Hopefully, you said the first one. The second graph shows a sudden transition (change in slope) when $x = 6$ (at the sharp corner), while the first graph changes smoothly from start to finish.

BIG IDEA

When the slope suddenly changes at $x = a$, then we say $f(x)$ is not differentiable when $x = a$. This is sometimes referred to as a sharp corner.

Thus, the function represented in the second graph is not differentiable when $x = 6$.

4. Cusps

We already have seen that a function is not differentiable at $x = a$ if there is a corner point at $x = a$. If the corner point happens to also have undefined slope, then that corner point is called a **cusp**. A cusp is a special type of corner point in that the slope of the tangent line at the cusp is undefined (vertical tangent line).

EXAMPLE Consider the graph of $y = f(x)$ shown below.

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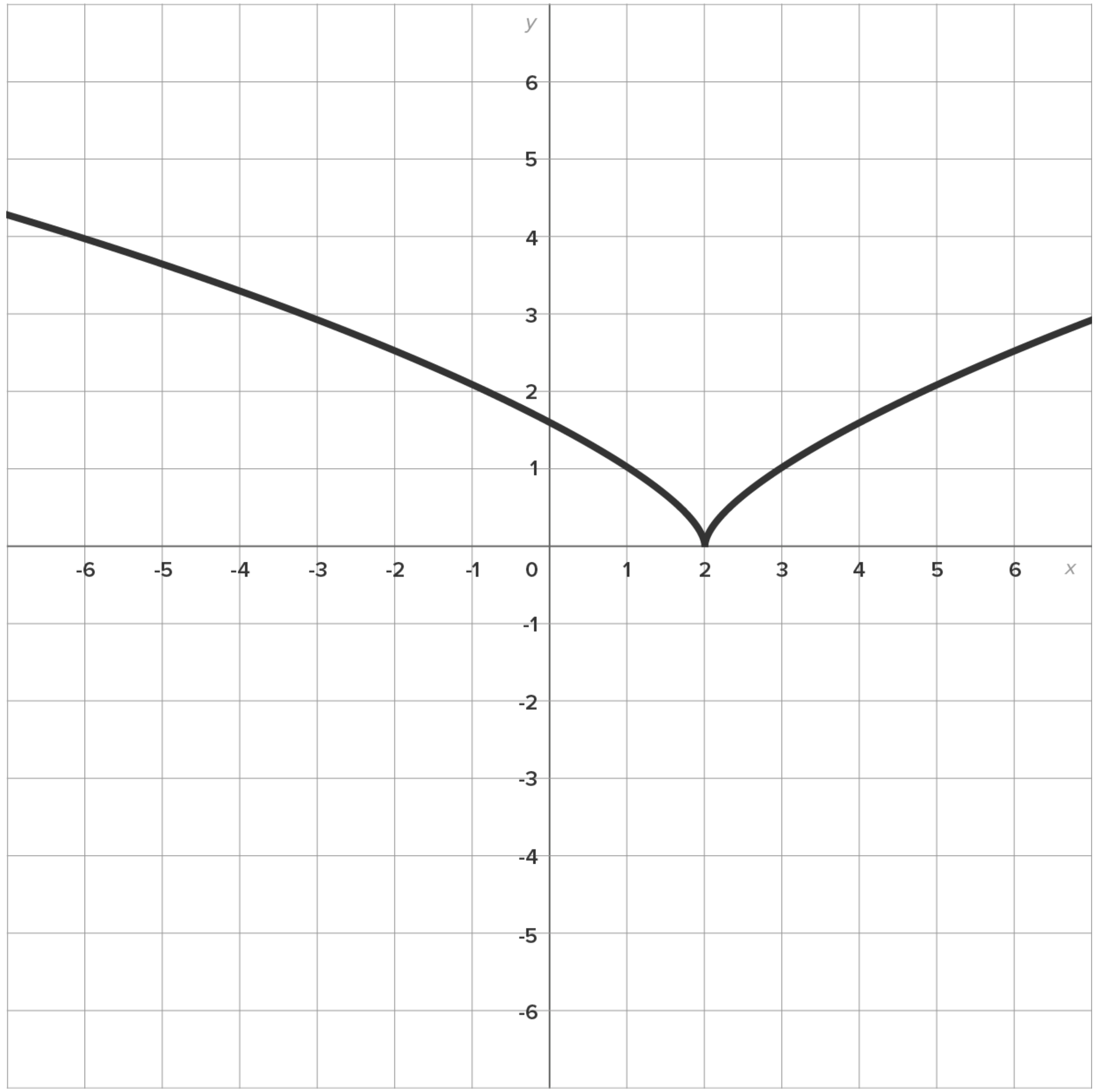

This graph shows a sharp corner at $x = 2$.

Notice that as *x* gets closer to 2 from either side, the slope gets steeper and steeper until becoming vertical. This implies that the derivative at $x = 2$ is undefined, meaning that this function is not differentiable when $x = 2$.

Since the tangent line is vertical at this point, we call this point a cusp.

THINK ABOUT IT

Imagine you are riding on a roller coaster. Obviously you would want the track to be continuous, but on a track full of corner points, the wheels would hit them, resulting in a very bumpy, loud, and dangerous ride.

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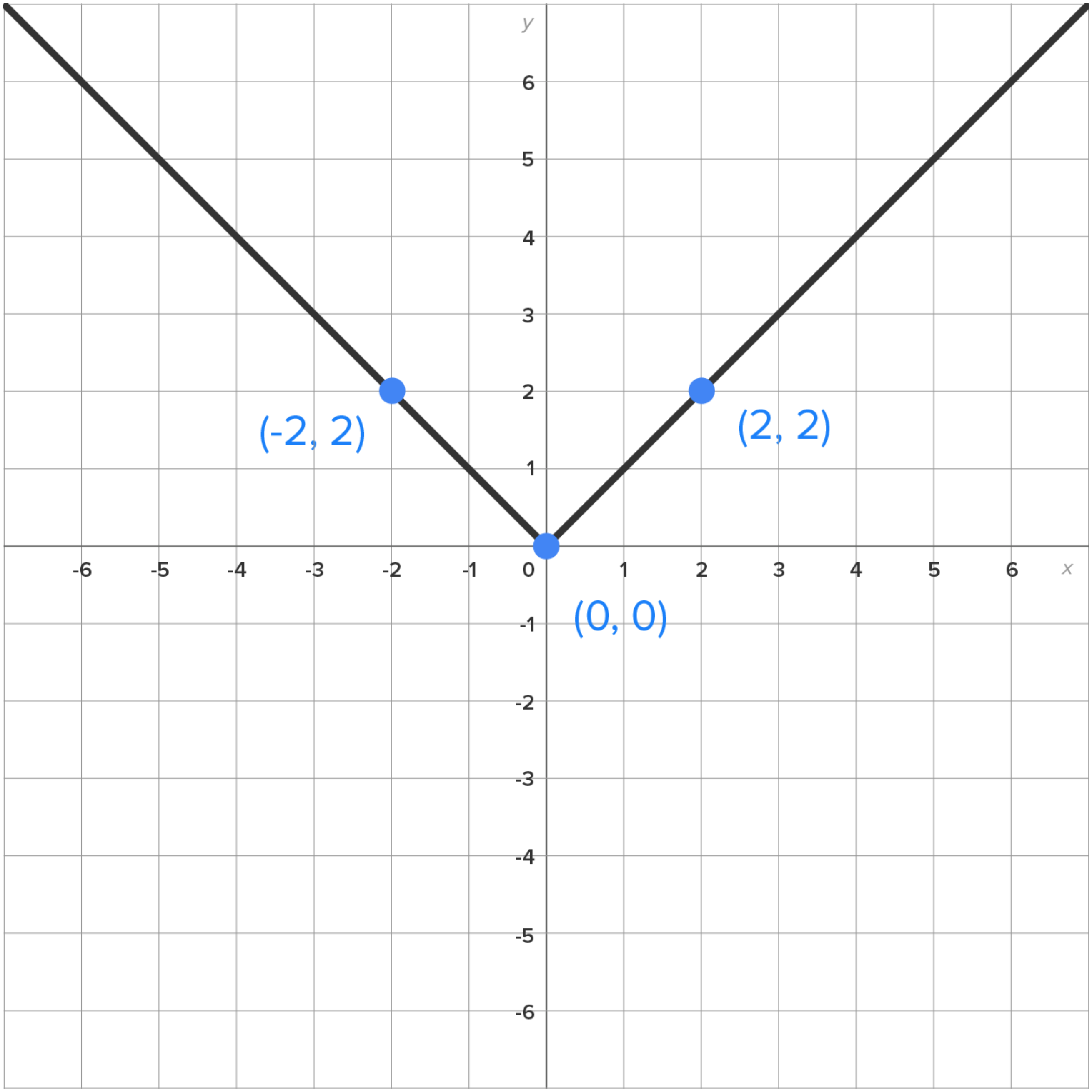
EXAMPLE Consider the graph of $f(x) = |x|$.



Notice that the graph is continuous everywhere (no breaks), but there is a sharp turn when $x = 0$. What could this mean? Let's explore this.

- To the left of (0, 0), the graph has slope -1.
- To the right of (0, 0), the graph has slope 1.
- At (0, 0), the slope changes directly from -1 to 1.

Thus, $f(x) = |x|$ is not differentiable at $x = 0$.

TRY IT

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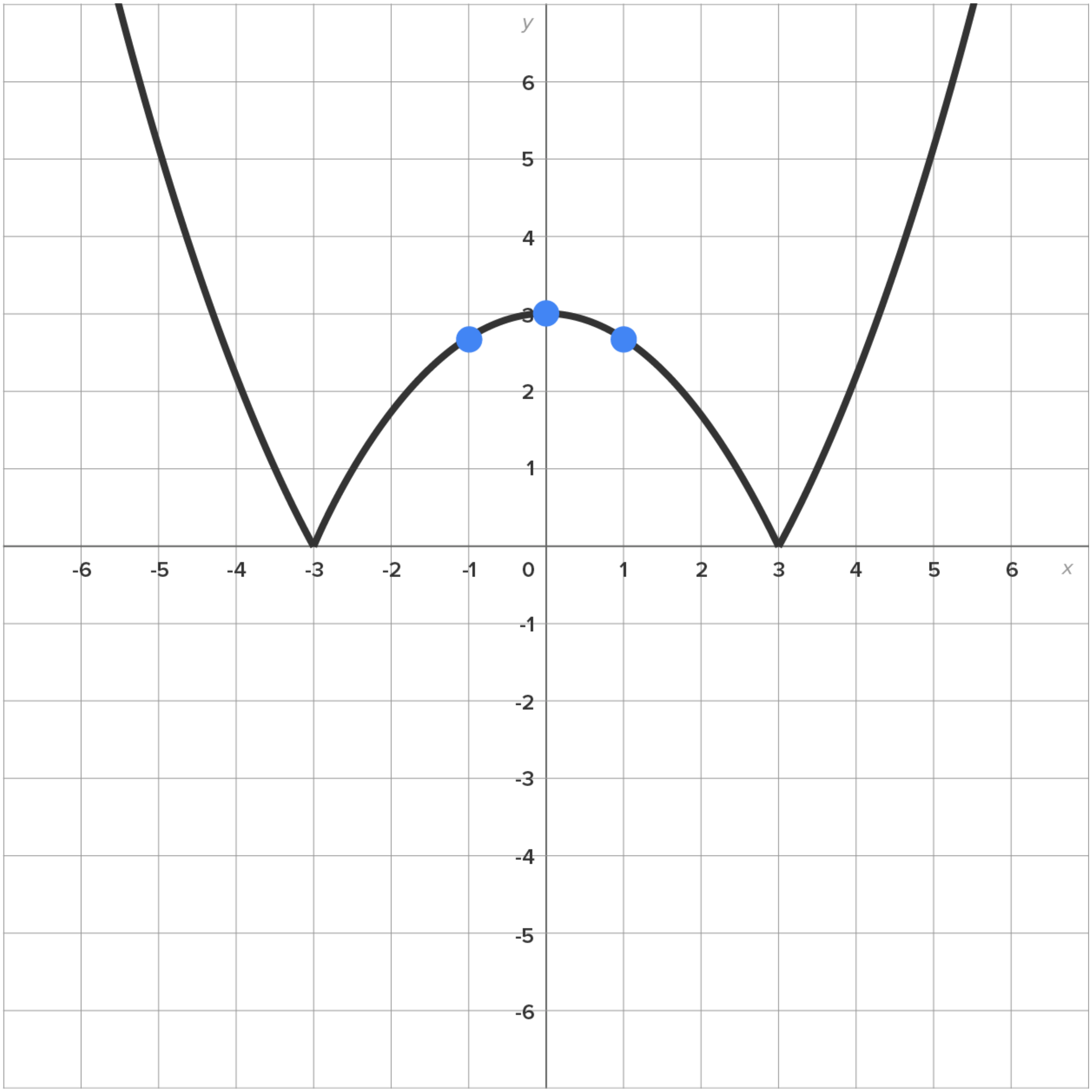
Shown below is the graph of $f(x) = |x^2 - 9|$.



Find all values of x for which $f(x)$ is not differentiable.

+

Since there are sharp turns at $(-3, 0)$ and $(3, 0)$, the function is not differentiable at $x = 3$ and $x = -3$.

TRY IT

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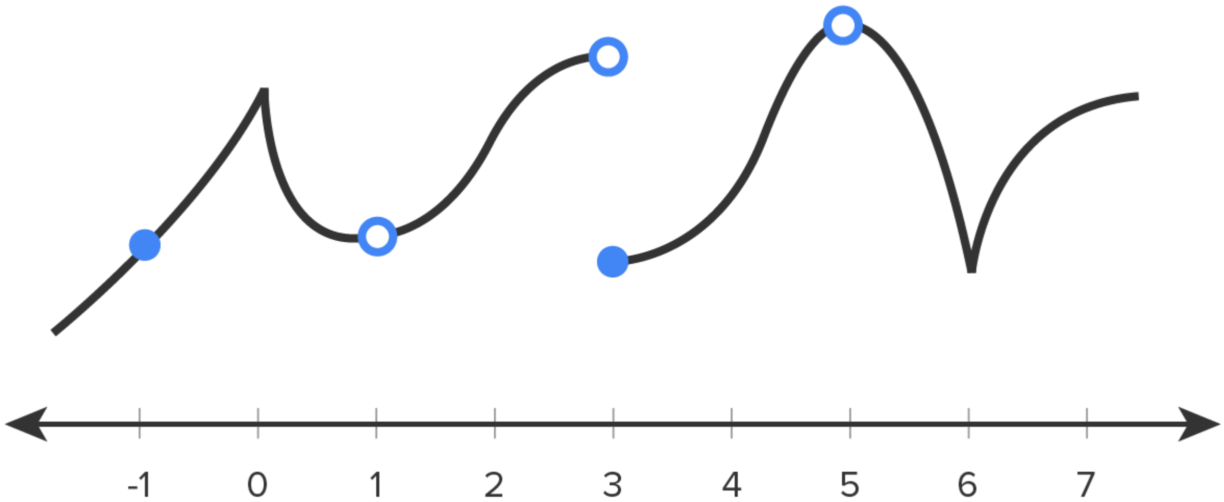

Using the graph, determine the values of x for which $f(x)$ is not differentiable. +

$x = 0$ (cusp), 1 (hole), 3 (jump), 5 (hole), 6 (cusp)

TERM TO KNOW

Cusp

A pointed end where two parts of a curve meet at a vertical tangent.

SUMMARY

In this lesson, you learned that there are several graphical properties that indicate that a function is not differentiable: **discontinuities**, **vertical tangents**, **sharp corners**, and **cusps**. These are relatively easy to spot on a graph and therefore make the work of determining differentiability simpler. As you also saw, differentiability is necessary in circumstances in which smooth transitions are important, such as in the track of a roller coaster.

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TERMS TO KNOW

Cusp

A pointed end where two parts of a curve meet at a vertical tangent.

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