Area

by Sophia

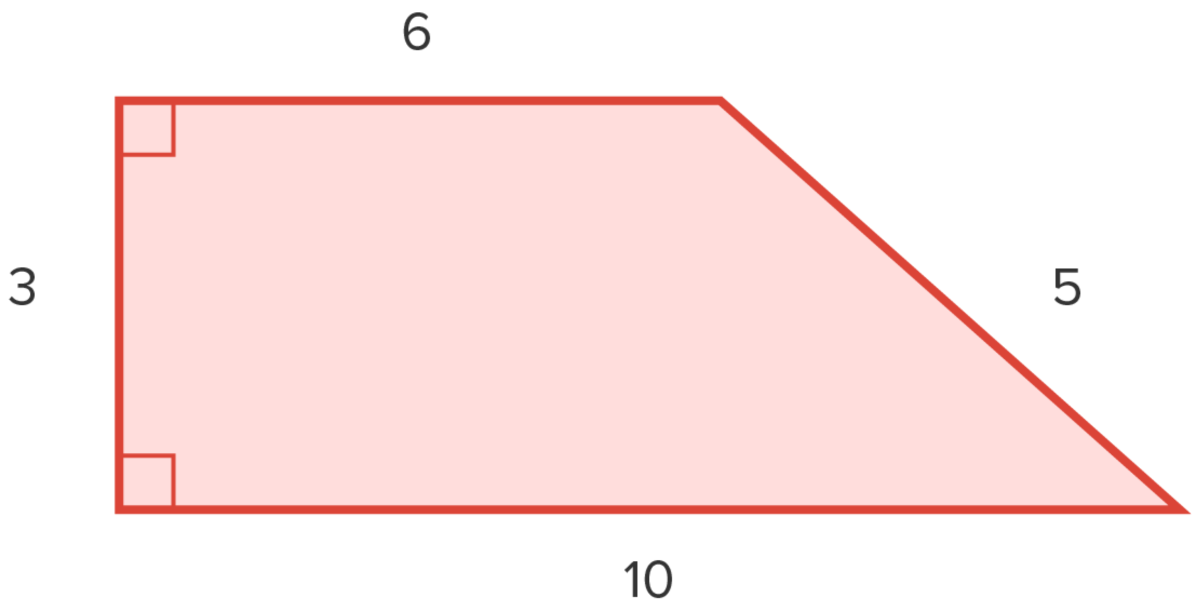
WHAT'S COVERED

In this lesson, you will find areas by using formulas and approximation methods. So far, we have been using differential calculus to solve problems by finding rates of change. We are now moving into integral calculus, in which areas are used as a visual to solve problems. In this section, we will start by finding areas that involve combining simpler areas. Specifically, this lesson will cover:

1. Finding the Area by Using Geometric Formulas

Before we get into the connection between areas and calculus, let's get some practice finding areas of some shapes using basic geometric formulas.

Consider the figure shown here.



There are three ways to find the area.

- Method I: This is a trapezoid, with area formula $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height and b_1 and b_2 are the lengths of the parallel bases.

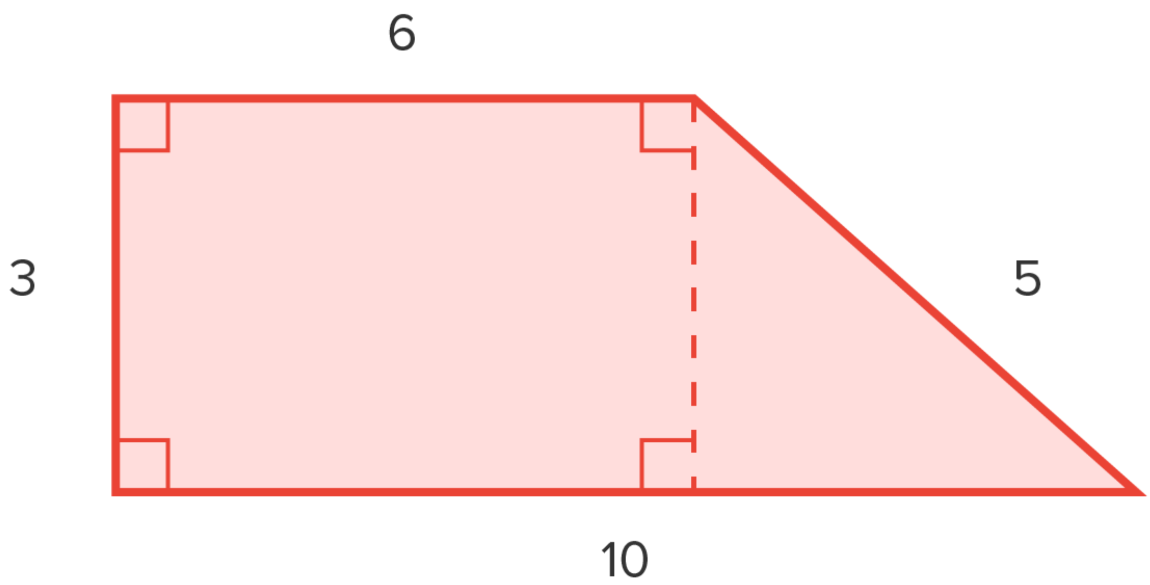
  Then, $A = \frac{1}{2}(3)(10 + 6) = 24 \text{ units}^2$.

- Method II: Split the trapezoid into a rectangle and a triangle, as shown in the figure below.

  The area of the rectangle is $3(6) = 18 \text{ units}^2$.

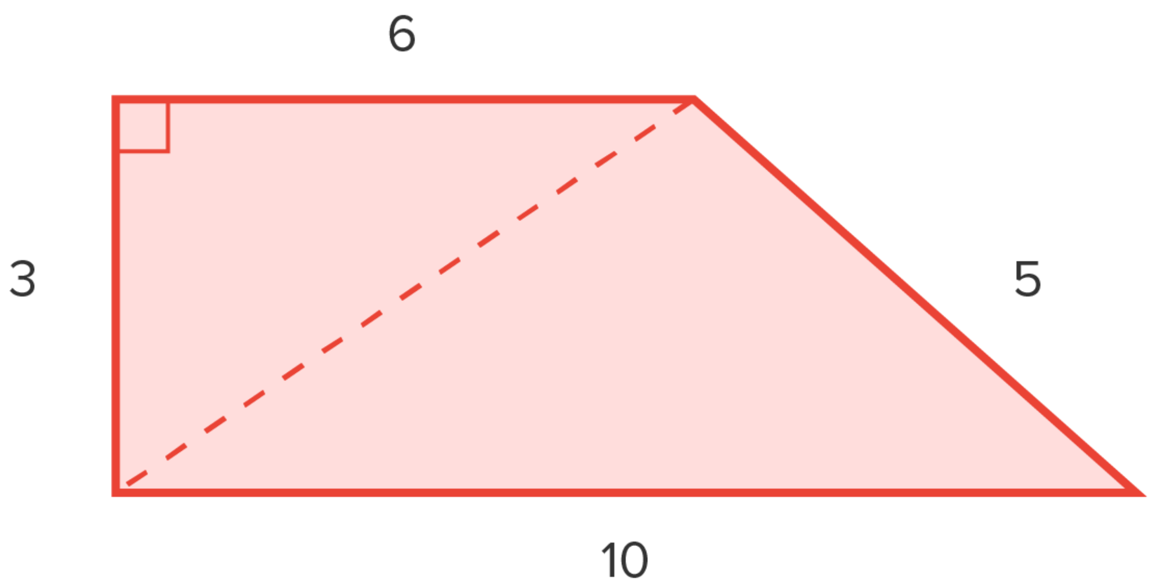
  The triangle has base 4 (found by 10 – 6) and height 3.

  The area of the triangle is $\frac{1}{2}(4)(3) = 6 \text{ units}^2$.



The combined area is $18 + 6 = 24 \text{ units}^2$.

- Method III: Split the trapezoid into two triangles, as shown in the figure below.



The area of the triangle on the left is $A = \frac{1}{2}(6)(3) = 9 \text{ units}^2$.

The area of the triangle on the right is $A=\frac{1}{2}(10)(3)=15\text{ units}^2$.

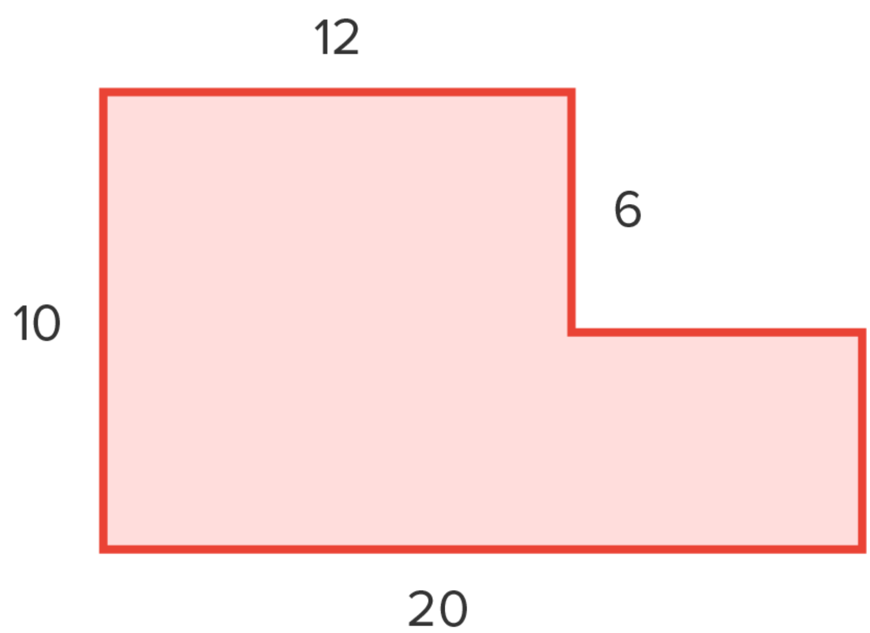
Once again, the combined area is $9+15=24\text{ units}^2$.

HINT

Even though the trapezoid formula was the most straightforward method to find the area, most people are more comfortable with rectangles and triangles.

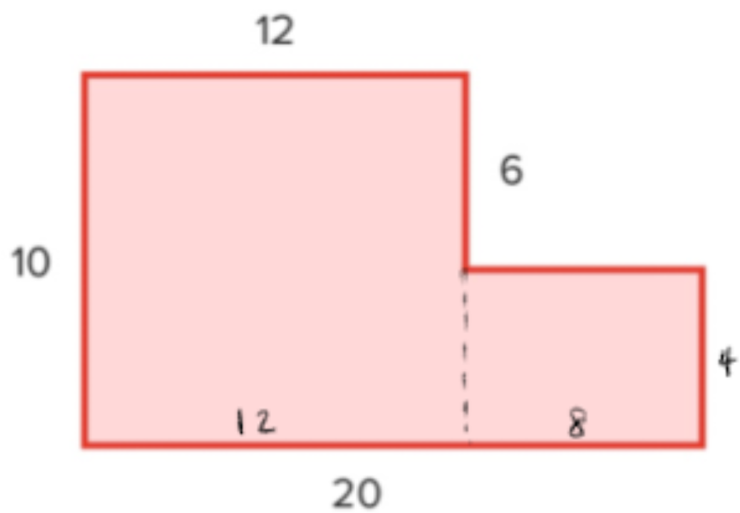
TRY IT

Consider the shape below:



Find the area of this shape. +

Split the figure into two rectangles and label as follows:



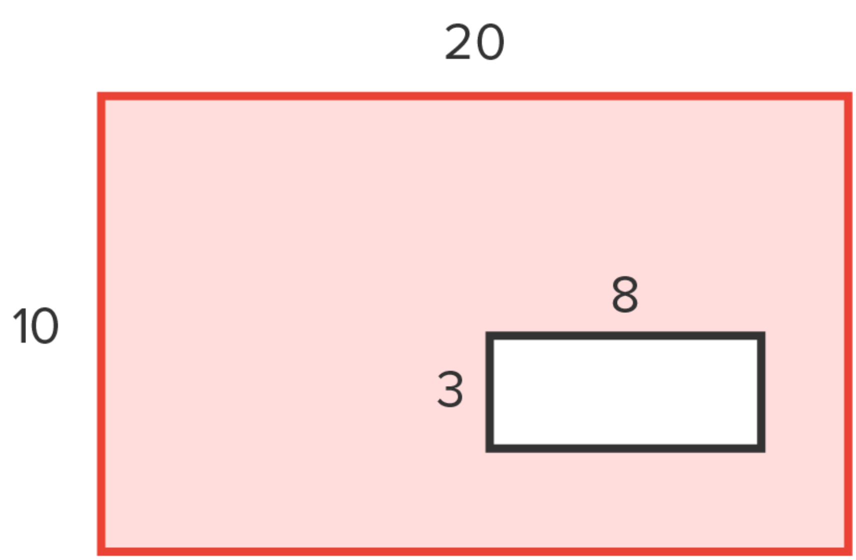
The area of the left-hand rectangle is $10(12) = 120 \text{ units}^2$.

The area of the right-hand rectangle is $8(4) = 32 \text{ units}^2$.

The total area is $120 + 32 = 152 \text{ units}^2$.

Let’s look at an example where there is a “hole” in the shape.

EXAMPLE Find the area of the shape (represented by the shaded region), assuming each side is measured in inches.



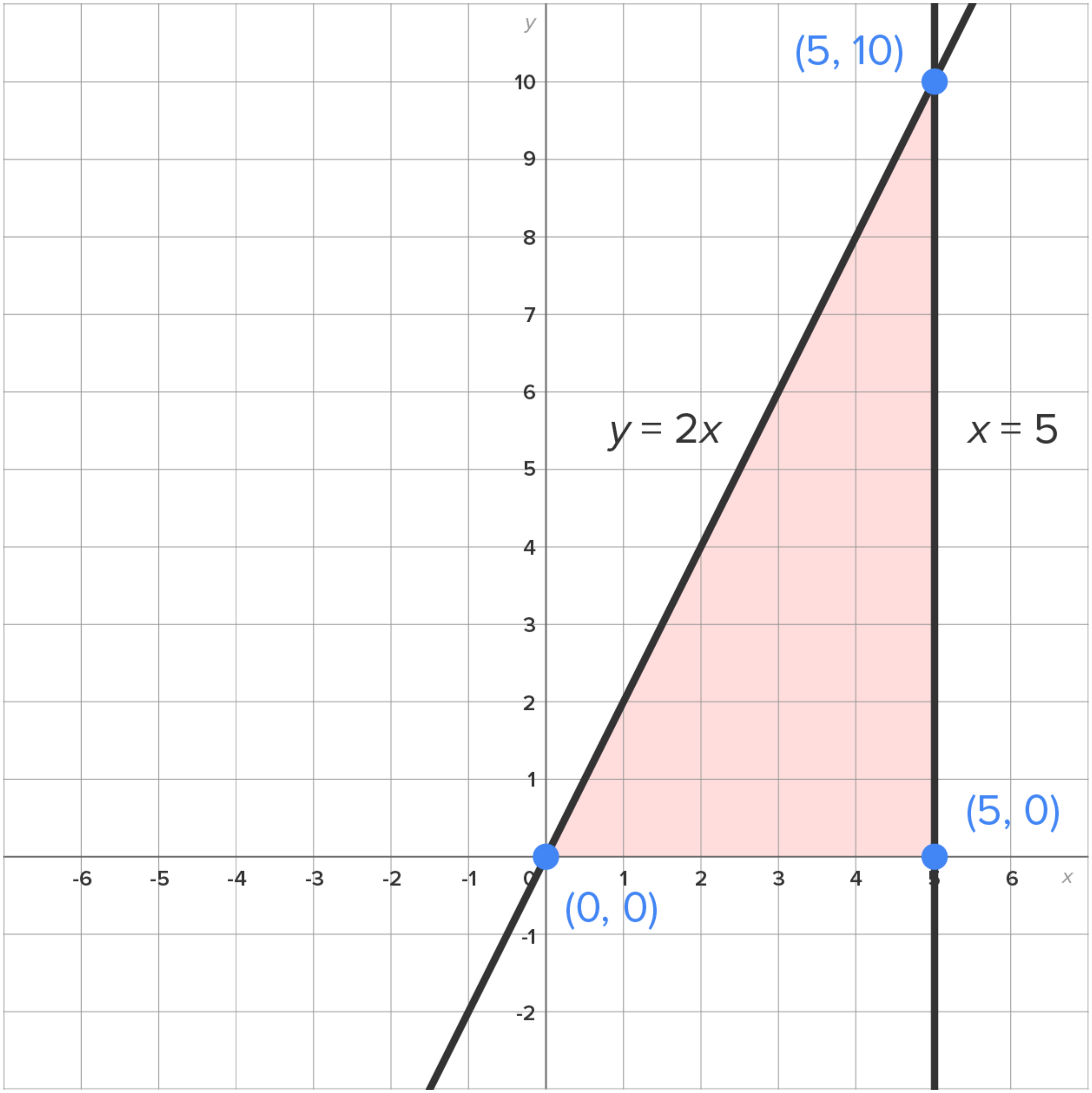
The outer rectangle has area $10(20) = 200 \text{ in}^2$. The hole, also in the shape of a rectangle, has area 24 units^2.

Then, the area of the shape is the difference between the areas: $200 - 24 = 176 \text{ in}^2$.

We can also find areas when certain graphs are used, since they are familiar shapes. Before diving in, here is a summary of equations that, when graphed, could form areas we know from formulas:

Forms	Equation
Horizontal line	$y = b$
Slanted line	$y = mx + b$
Circle with radius *r*	$x^2 + y^2 = r^2$
Semicircle with radius *r*	$y = \sqrt{r^2 - x^2}$

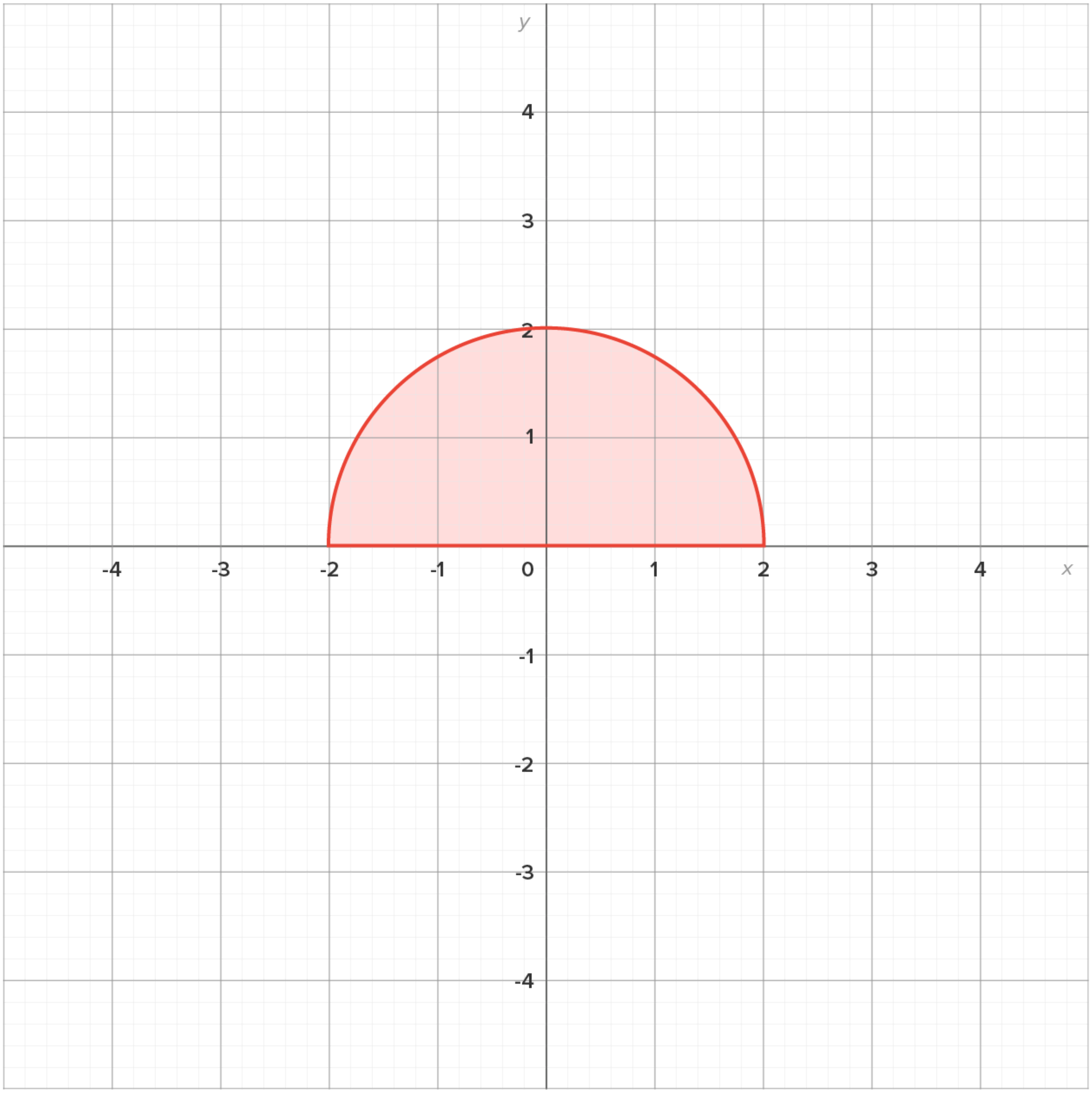
EXAMPLE Find the area of the region bounded by the x-axis, the line $x = 5$, and the line $y = 2x$. The graph of the region is shown in the figure below.



The region is triangular, with base 5 and height 10.

Thus, the area of the region is $A=\frac{1}{2}(5)(10)=25\text{ units}^2$.

EXAMPLE The figure below shows the graph of the region between $f(x)=\sqrt{4-x^2}$ and the x-axis. What is the area of this region?

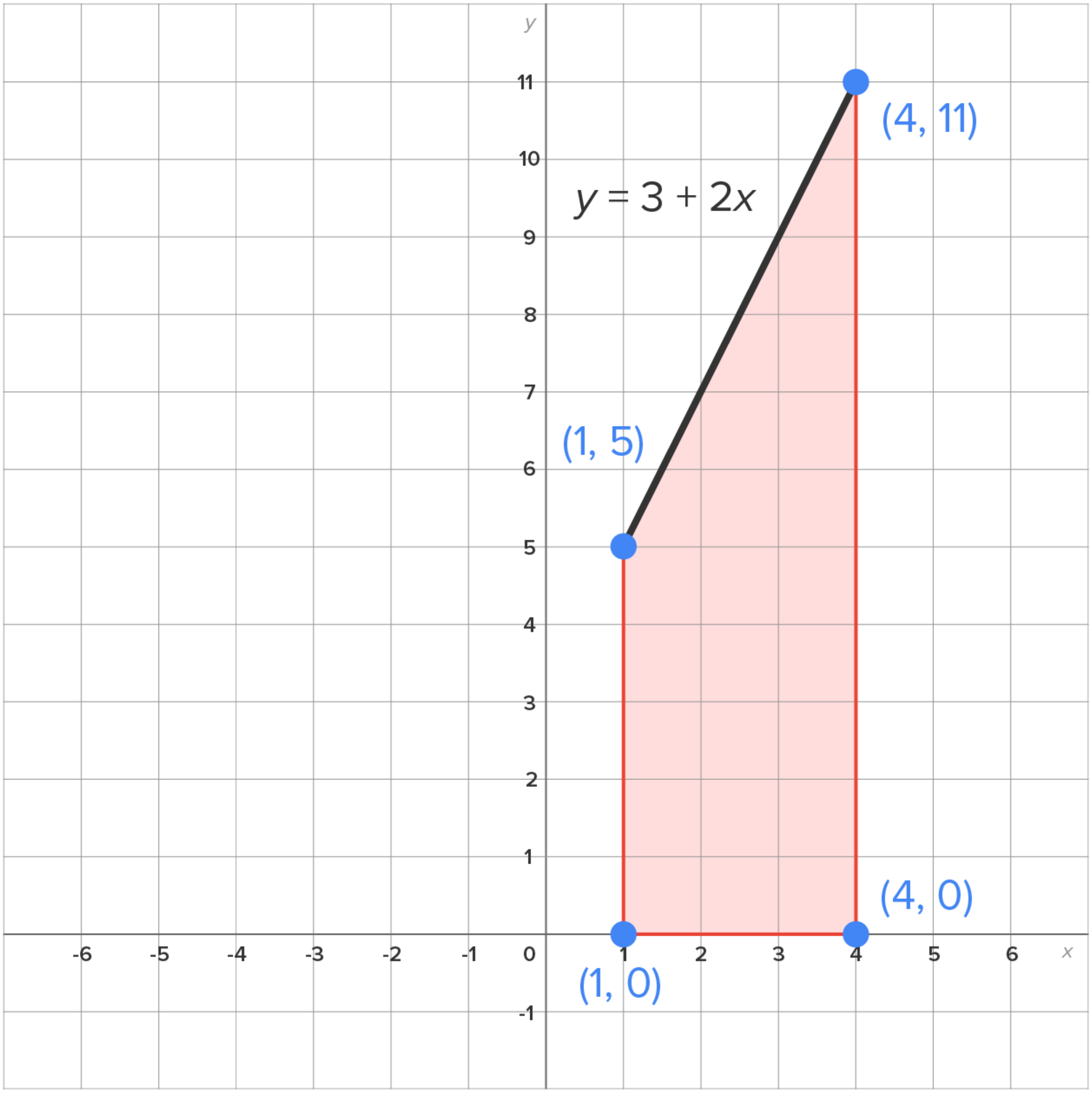


This region is a semicircle whose radius is 2. Recall the area of a circle is $A = \pi r^2$.

The area of the region is $\frac{1}{2} \cdot \pi(2)^2 = 2\pi$ units^2.

TRY IT

Consider the graph below:



Find the area of the region in the graph.

+

The figure is in the shape of a trapezoid. The parallel bases have lengths 5 and 11; while the height (which is along the x-axis) has length 3.

The area of the trapezoid is $\frac{h}{2}(b_1 + b_2) = \frac{3}{2}(5 + 11) = 24 \text{ units}^2$.

HINT

Since you will be using areas to solve problems in this unit, refer to this sheet which contains formulas for areas of various shapes (circles, trapezoids, etc.). This is also available as a PDF file at the end of this tutorial.

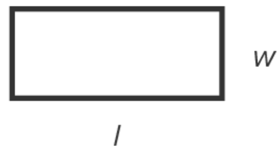
Geometric Formulas

Square

$P = 4s$

$A = s^2$

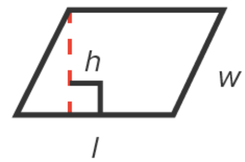
Rectangle



$P = 2l + 2w$

$A = lw$

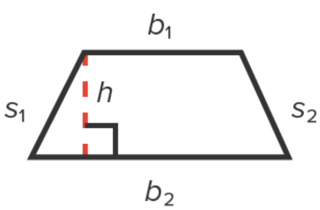
Parallelogram



$P = 2l + 2w$

$A = lh$

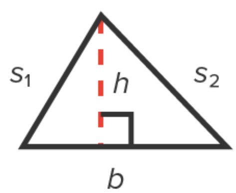
Trapezoid



$P = s_1 + s_2 + b_1 + b_2$

$A = ½h(b_1 + b_2)$

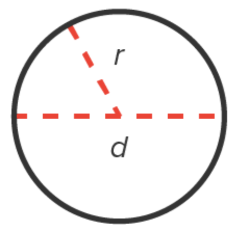
Triangle



$P = s_1 + s_2 + b$

$A = ½bh$

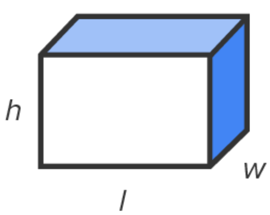
Circle



$C = 2\pi r$ or $C = \pi d$

$A = \pi r^2$

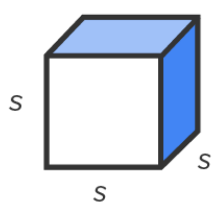
Rectangular Solid



$S = 2lh + 2wh + 2wl$

$V = lwh$

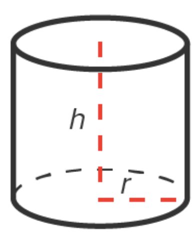
Cube



$S = 6s^2$

$V = s^3$

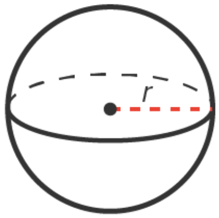
Right Circular Cylinder



$S = 2\pi rh + 2\pi r^2$

$V = \pi r^2 h$

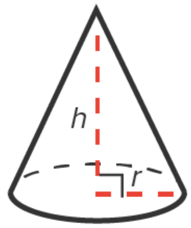
Sphere



$S = 4\pi r^2$

$V = {}^4/_3\pi r^3$

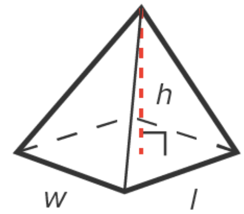
Right Circular Cone



$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$

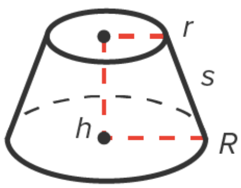
$V = ⅓\pi r^2 h$

Square or Rectangular Pyramid



$V = ⅓lwh$

Right Circular Cone Frustum



$S = \pi s(R + r) + \pi r^2 + \pi R^2$

$V = \dfrac{\pi(r^2 + rR + R^2)h}{3}$

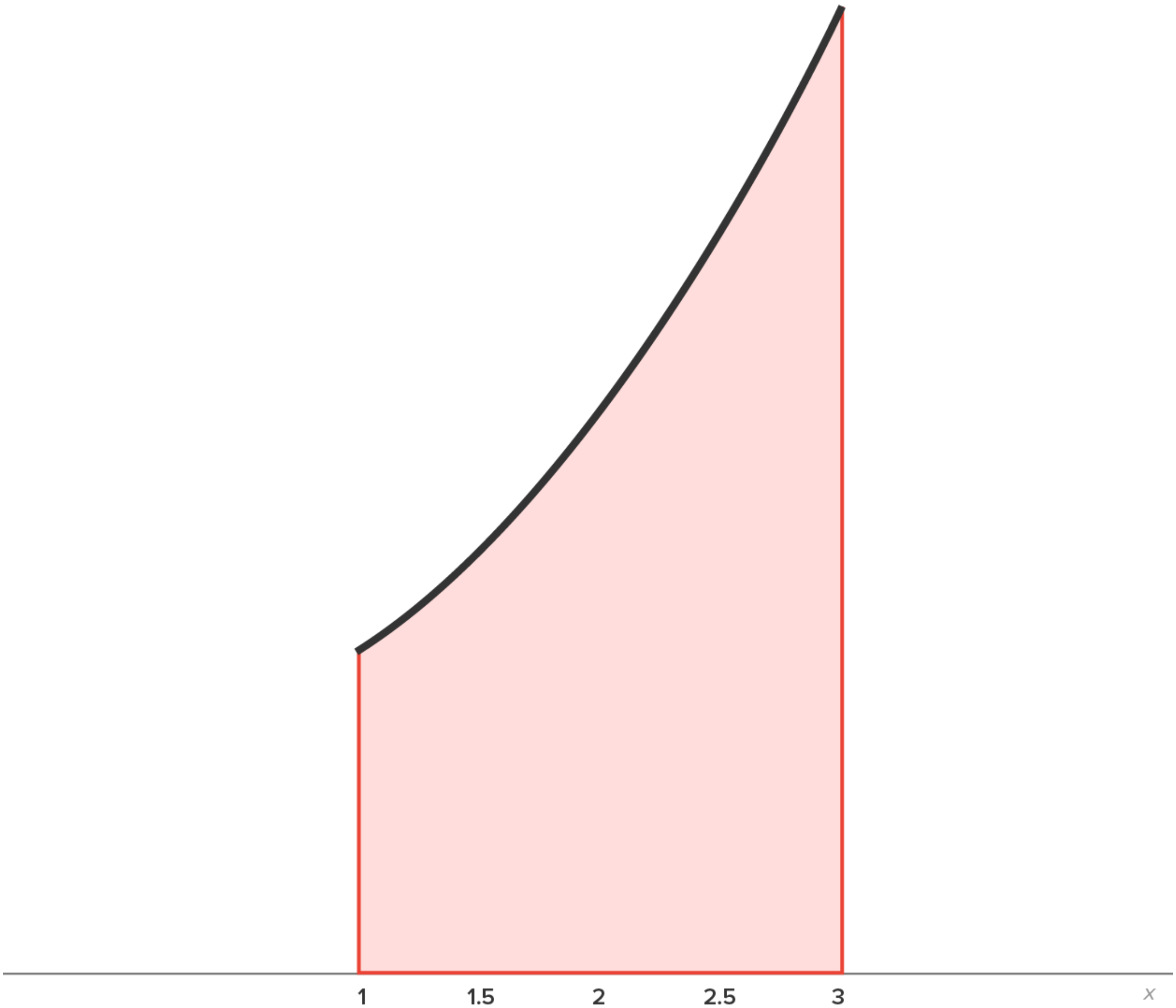
Geometric Symbols

A = Area	S = Surface Area
P = Perimeter	C = Circumference
V = Volume	π = Pi Constant

2. Approximating Areas by Using Rectangles and Graphs

There are some regions, particularly those which come from a graph, which cannot be found using formulas.

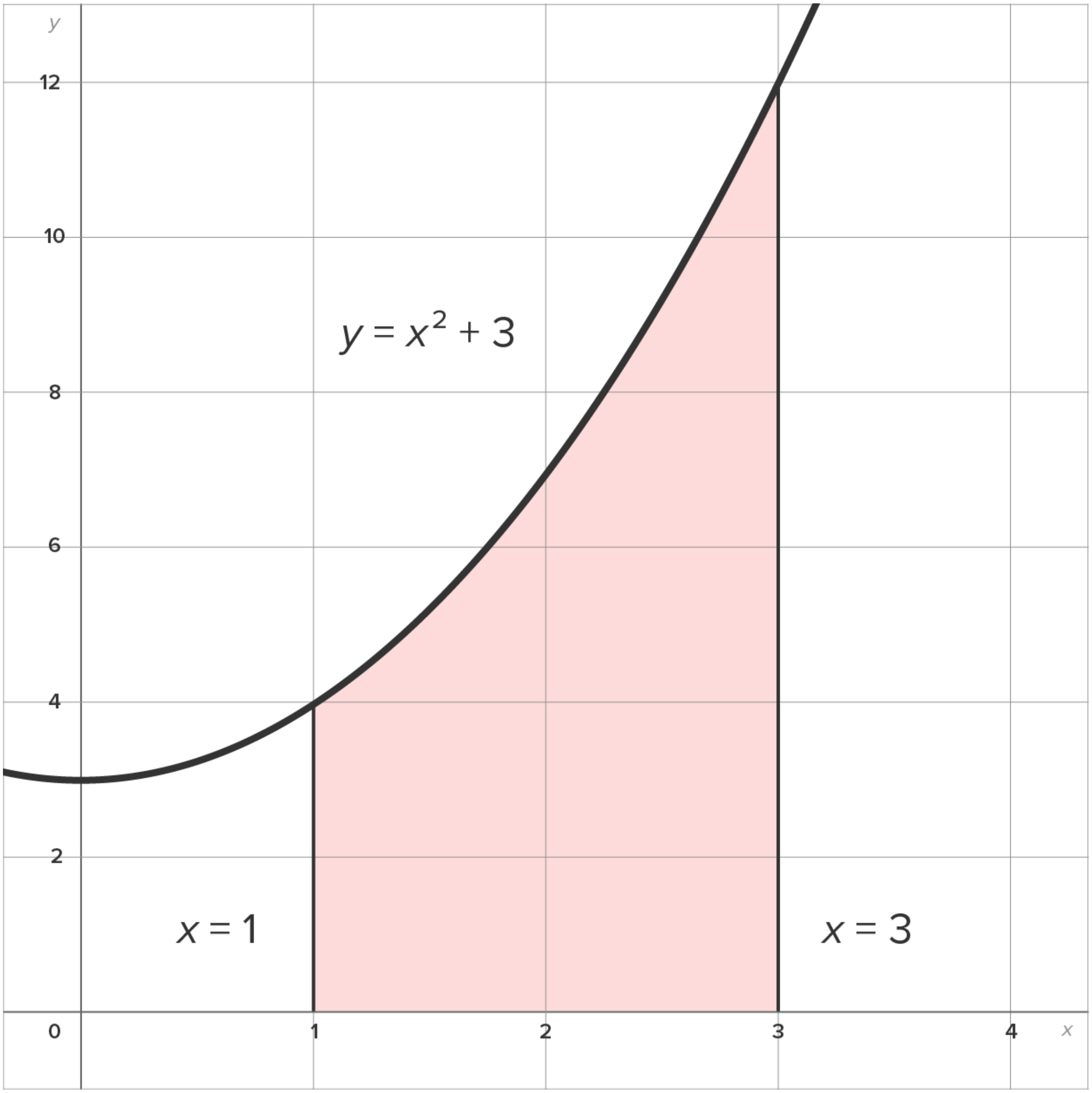
For example, consider the region bounded by the graphs of $y = x^2 + 3$, the x-axis, $x = 1$, and $x = 3$. The region is shown below.



A major focus of integral calculus is being able to find areas of regions like this. For now, we need to come up with a way to estimate the area.

The most convenient way is to use rectangles whose bases are along the x-axis. This is illustrated in the next few examples.

> EXAMPLE Approximate the area of the region bounded by $y = x^2 + 3$, the x-axis, $x = 1$, and $x = 3$, as shown in the graph below:



To find this area, first find the combined area of the rectangles, as shown in each figure.

Graph	Description

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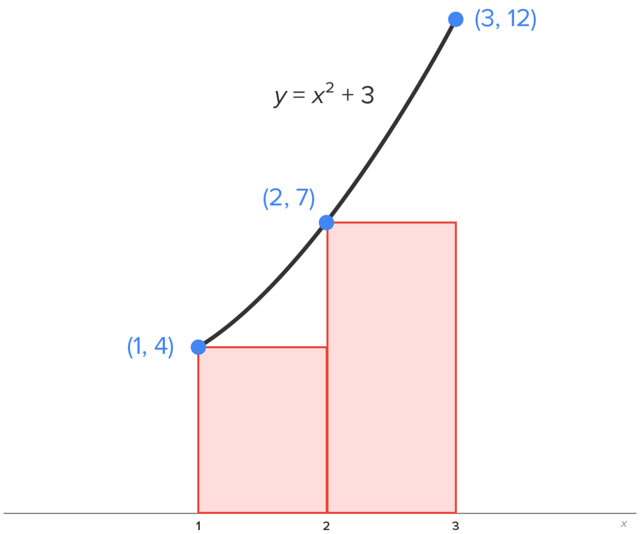

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The rectangles used here are **inscribed**, meaning the largest possible rectangle drawn within the region. Notice how one corner of each rectangle is also on the curve.

- Each rectangle is 1 unit wide.
- The rectangle on the left has a height of 4 units.
- The area of the first rectangle is $(1)(4) = 4 \text{ units}^2$.
- The rectangle on the right has a height of 7 units.
- The area of the second rectangle is $(1)(7) = 7 \text{ units}^2$.

The combined area is 11 units^2. We know this is an underestimate of the actual area since the rectangles are inscribed.

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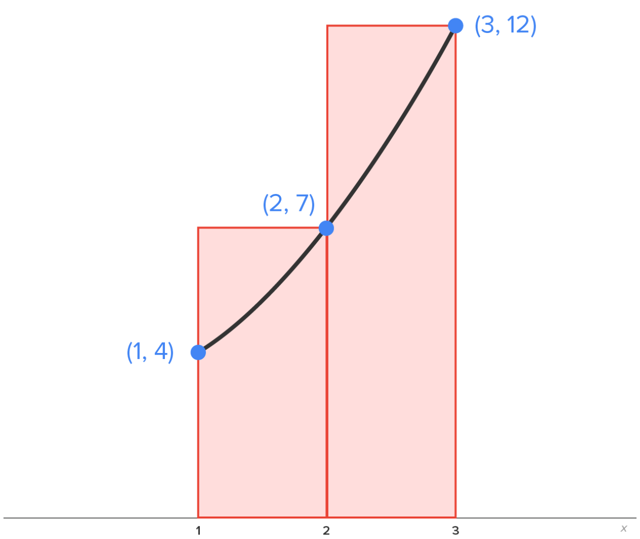

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The rectangles used here are **circumscribed**, meaning drawn in such a way that the rectangle completely encloses the region, but is as small as possible. Notice how one corner of each rectangle is also on the curve.

- Each rectangle is 1 unit wide.
- The rectangle on the left has a height of 7 units.
- The area of the first rectangle is $(1)(7) = 7 \text{ units}^2$.
- The rectangle on the right has a height of 12 units.
- The area of the second rectangle is $(1)(12) = 12 \text{ units}^2$.

The combined area is 19 units^2. We know this is an overestimate of the actual area since the rectangles are circumscribed.

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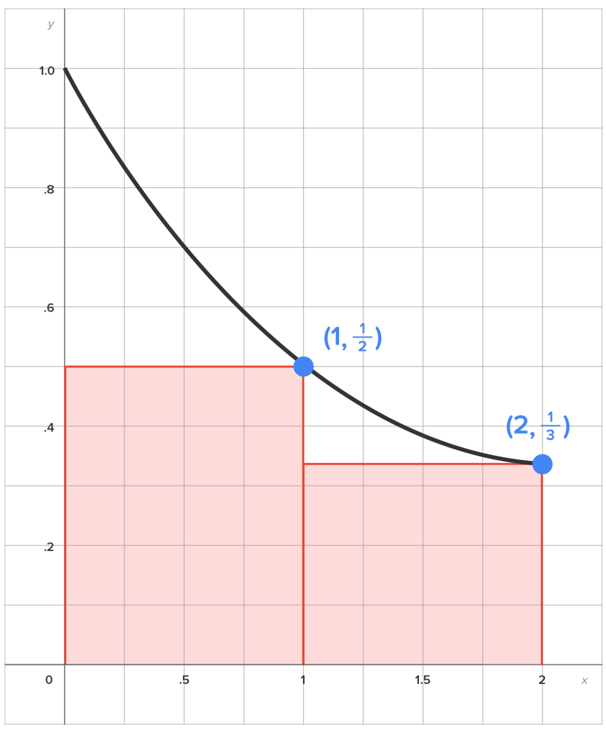
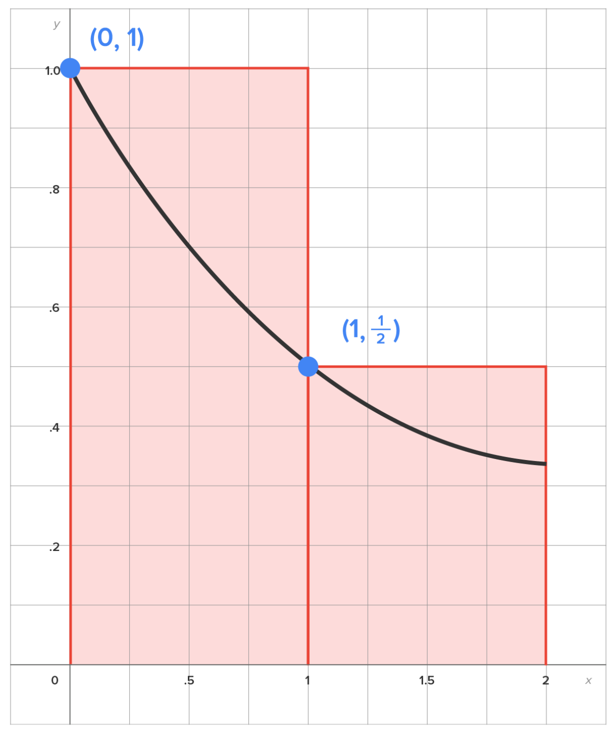
Since one estimate is an underestimate and one is an overestimate, one way to get a better approximation is to average them.

Using this logic, an estimate for the actual area is $\frac{11+19}{2} = 15 \text{ units}^2$.

TRY IT

Approximate the area of the region bounded by $y = \dfrac{1}{x+1}$, the x-axis, $x = 0$, and $x = 2$ by finding the combined area of the rectangles, as shown in each figure. Then, find the average of the estimates.

Figure 1	Figure 2
	

What is the area of Figure 1? +

$$\text{Area} = 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) = \frac{5}{6}\ \text{units}^2$$

What is the area of Figure 2? +

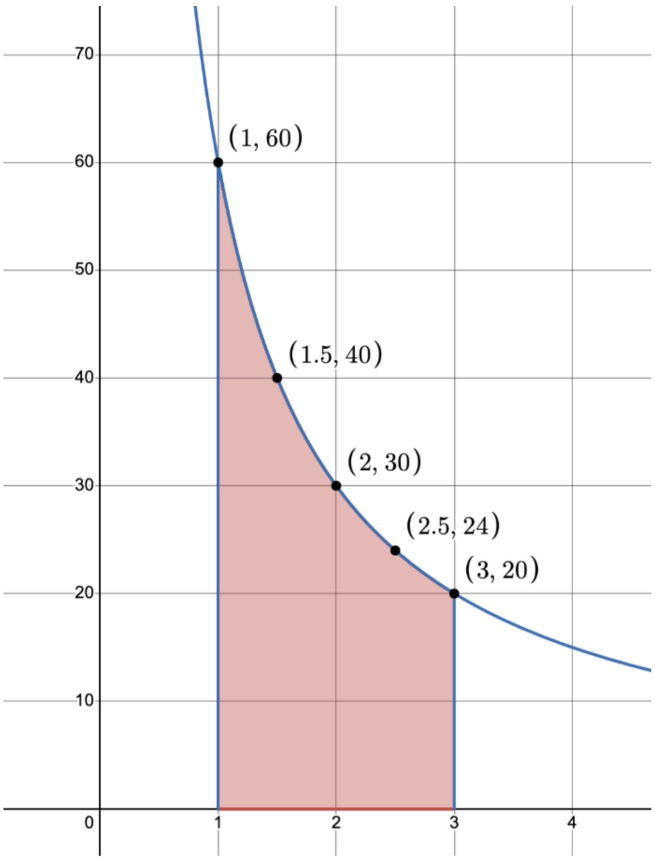
$$\text{Area} = 1(1) + 1\left(\frac{1}{2}\right) = \frac{3}{2}\ \text{units}^2$$

What is the average area? +

$$\text{Average} = \frac{\frac{5}{6} + \frac{3}{2}}{2} = \frac{7}{6} \text{ units}^2$$

Here is a final example that uses more rectangles.

➢ EXAMPLE Suppose we wish to estimate the area between the x-axis and the graph of $y = \frac{60}{x}$ on the interval $[1, 3]$. The graph of the region is shown below.



To estimate the area, we'll find the combined area of the rectangles, as shown in each figure.

Graph	Description

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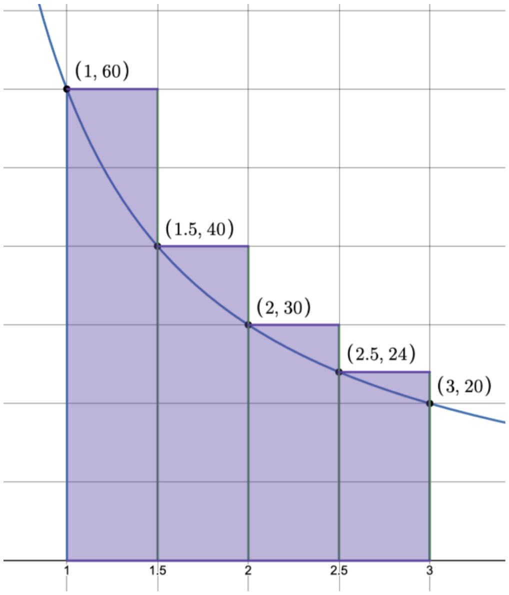
<td>
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<td>
The rectangles used here are circumscribed, meaning that the rectangle encloses the region, but is the smallest rectangle possible. Notice how one corner of each rectangle is also on the curve.
Note that each rectangle is ½ unit wide.
Going from left to right, find the area of each rectangle:

1st rectangle: $\text{Area} = (1/2)(60) = 30 \text{ units}^2$
2nd rectangle: $\text{Area} = (1/2)(40) = 20 \text{ units}^2$
3rd rectangle: $\text{Area} = (1/2)(30) = 15 \text{ units}^2$
4th rectangle: $\text{Area} = (1/2)(24) = 12 \text{ units}^2$

The combined area is 77 units^2. We know this is an overestimate of the actual area since the rectangles encompass the region and have additional area.
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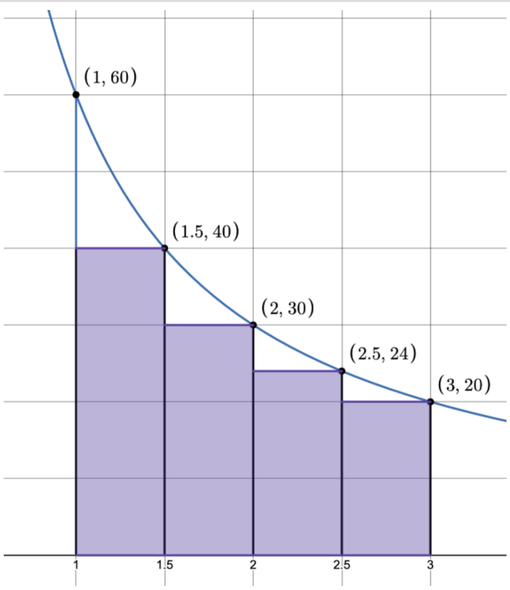
<td>
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<td>
The rectangles used here are inscribed, meaning that the rectangle is within the region, but is the largest rectangle possible. Notice how one corner of each rectangle is also on the curve.
Note that each rectangle is ½ unit wide.
Going from left to right, find the area of each rectangle:

1st rectangle: $\text{Area} = (1/2)(40) = 20 \text{ units}^2$
2nd rectangle: $\text{Area} = (1/2)(30) = 15 \text{ units}^2$
3rd rectangle: $\text{Area} = (1/2)(24) = 12 \text{ units}^2$
4th rectangle: $\text{Area} = (1/2)(20) = 10 \text{ units}^2$

The combined area is 57 units^2. We know this is an underestimate of the actual area since the rectangles are all within the region.
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Since 77 is an overestimate and 57 is an underestimate, the average is likely a better estimate of the actual area.

Using this logic, our estimate for the actual area is $\frac{77+57}{2} = 67 \text{ units}^2$.

Note: Later in this course, we will earn techniques to find the exact area. In this case, the actual area is 65.1967 units^2.

TERMS TO KNOW

Inscribed (Rectangles)
A rectangle is inscribed inside a region if it is the largest rectangle that stays inside the region.

Circumscribed (Rectangles)
A rectangle is circumscribed outside a region if it is the smallest rectangle that encompasses the region.

SUMMARY

In this lesson, you learned that **area can be found using basic geometric formulas**, by combining areas, or by subtracting areas. For example, when finding the area of a trapezoid, you could use the trapezoid area formula, or you could split the trapezoid into a rectangle and a triangle and combine their respective areas. You also learned that when there is no area formula available, you can **approximate areas by using rectangles and graphs**.

TERMS TO KNOW

Circumscribed (Rectangles)
A rectangle is circumscribed outside a region if it is the smallest rectangle that encompasses the region.

Inscribed (Rectangles)
A rectangle is inscribed inside a region if it is the largest rectangle that stays inside the region.