Applications of Rates of Change

by Sophia

WHAT'S COVERED

In this lesson, you will examine some real-world applications in which derivatives are required to describe certain ideas. Specifically, this lesson will cover:

1. Vertical Motion

In challenge 3.2, we discussed velocity and acceleration of an object that is traveling vertically. Recall the following, which are measured after t seconds of travel:

- $h(t) =$ the height of the object
- $v(t) = h'(t) =$ the velocity of the object
- $a(t) = h''(t) =$ the acceleration of the object

TRY IT

A tennis ball is launched off of a building that is 60 feet high at a velocity of 30 feet per second. Its height after t seconds is $h(t) = -16t^2 + 30t + 60$.

Determine if each question can be answered with or without calculus. Then, answer each question.

How high is the object after 1 second? +

No calculus required.

Answer: $h(1) = 74$ feet

What is the velocity of the object after 1 second? +

Calculus is required.

$v(t) = h'(t) = -32t + 30$

$v(1) = h'(1) = -32(1) + 30 = -2$ ft/sec

When does the object strike the ground? +

No calculus required.

Set $h(t) = 0$.

$h(t) = -16t^2 + 30t + 60$

Answer: $t \approx 3.089$ seconds

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2. Growth and Decay

The function $f(t) = Ae^{kt}$ is used to model the exponential growth or decay of a substance.

- If $k > 0$, it models growth.
- If $k < 0$, it models decay.

Naturally, $f'(t)$ is used to find instantaneous rates of growth (or decay) at specific values of t.

TRY IT

The amount of bacteria present in a medium after t hours is given by $A(t) = 50e^{0.1t}$.

How many bacteria are present at the beginning? +

$A(0) = 50e^{0.1(0)} = 50e^0 = 50$ bacteria

How many bacteria are present after 7 hours? +

$A(7) = 50e^{0.1(7)} = 50e^{0.7} \approx 100.68$, so about 101 bacteria

At what rate is the amount of bacteria changing after 8 hours? +

We want to find $A'(8)$. First, find $A'(t)$:

$A(t)=50e^{0.1t}$	Start with the original function.
$A'(t)=D[50e^{0.1t}]=50D[e^{0.1t}]$	Use the constant multiple rule.
$A'(t)=50e^{0.1t}\cdot 0.1$	$D[e^{u}]=e^{u}\cdot u'$
$A'(t)=5e^{0.1t}$	Simplify.

Then, the rate of change is $A'(8)=5e^{0.1(8)}=5e^{0.8}\approx 11.13$ bacteria per hour.

3. Applications to Business

3a. Total Cost and Marginal Cost

In business, particularly in manufacturing, there are two functions that are analyzed the most often to help minimize costs:

- The total cost of producing *x* items, usually represented by $C(x)$.
- The average cost of producing *x* items, usually represented by $AC(x)$.

Note: $AC(x)=\frac{C(x)}{x}$.

The **fixed cost** is the cost before any items are produced. This is also called overhead.

Given a specific production level, it is useful to have an estimate for the cost of producing the *next* item.

EXAMPLE Given $C(x)=0.2x^2+14x+500$:

- The total cost of producing 9 units is $C(9)=0.2(9)^2+14(9)+500=\$642.20$.
- The total cost of producing 10 units is $C(10)=0.2(10)^2+14(10)+500=\$660$.

Then, the cost of producing the 10th item is \$660 - \$642.20 = \$17.80.

Thus, given a production level of 9 units, the cost of producing the next item is \$17.80.

TRY IT

Consider the cost function $C(x) = 0.2x^2 + 14x + 500$.

Find the cost of producing the 7th item. Then, compare it to the cost of producing the 10th item in the previous example. What do you notice? +

The cost to produce the 7th item is $C(7) - C(6)$.

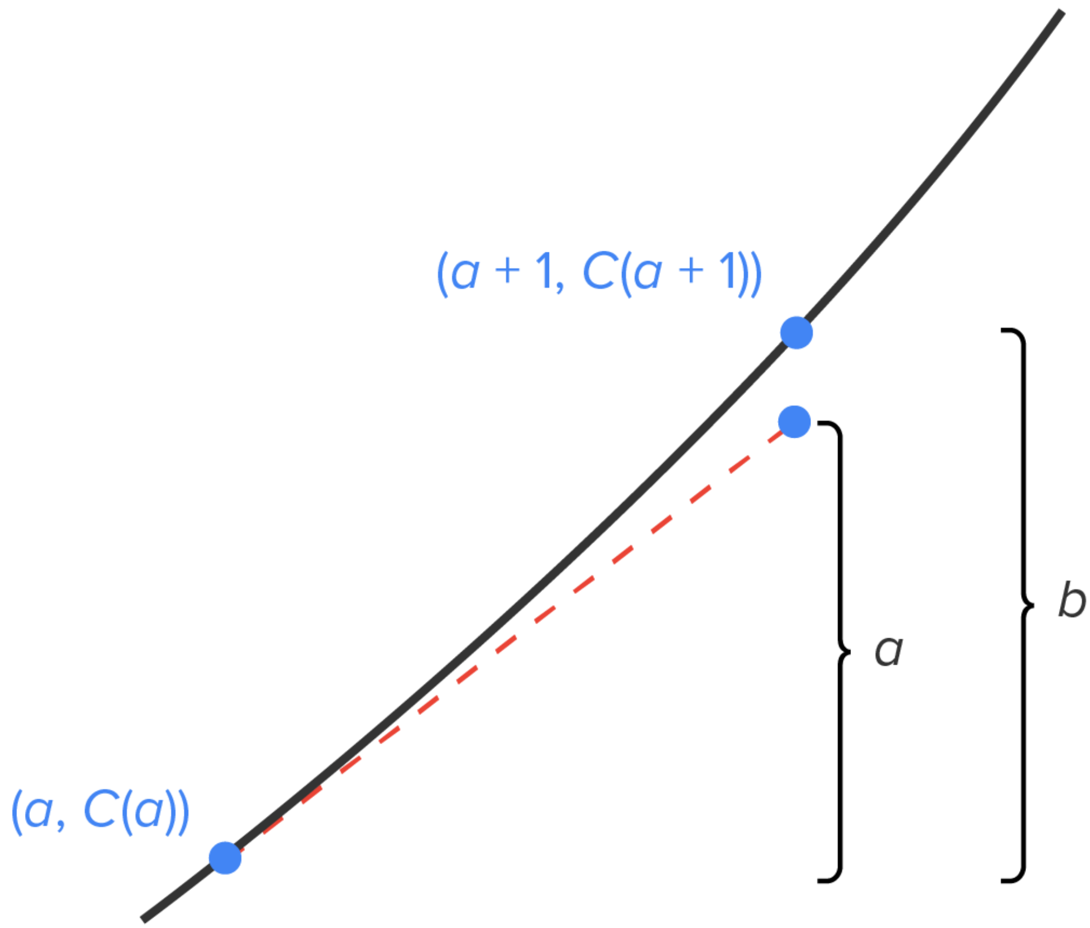
$$C(7) = 0.2(7)^2 + 14(7) + 500 = \$607.80$$

$$C(6) = 0.2(6)^2 + 14(6) + 500 = \$591.20$$

Then, $C(7) - C(6) = \$607.80 - \$591.20 = \$16.60$. This result suggests that the cost to produce each individual unit is different.

Another way to approximate the cost of producing the next unit is to use the derivative of the cost function, which is called the **marginal cost function**. This makes sense since the rate of change in the cost would be measured in dollars per unit.

Here is a graph to show the reasoning, where the solid graph is the cost function, and the dotted graph is the tangent line:



As we can see, the tangent line at $(x, C(x))$ is a good approximation for the cost (and as a result, the change in cost) between production levels of x and $x + 1$. Therefore, the marginal cost function at x approximates the cost of the $(x + 1)$st unit.

EXAMPLE Using $C(x) = 0.2x^2 + 14x + 500$, approximate the cost of producing the 10th and 7th units by using the marginal cost.

First, the marginal cost is the derivative of cost: $C'(x) = 0.4x + 14$

The approximate cost of producing the 10th unit is $C'(9) = 0.4(9) + 14 = 17.6$.
The approximate cost of producing the 7th unit is $C'(6) = 0.4(6) + 14 = 16.4$.

Thus, the 10th unit costs approximately $17.60 and the 7th costs approximately $16.40. Compared to previous results ($17.80 and $16.60 respectively), the derivative provides a good estimate.

TERMS TO KNOW

Fixed Cost (or Overhead)

The costs that are incurred before any items are produced. Mathematically, it is the total cost of producing 0 items.

Marginal Cost Function

The derivative (rate of change) of the cost function. Given a production level *x*, it approximates the cost of the next item.

3b. Average Cost

The average cost function, $AC(x)$, is the total cost to produce *x* items, divided by the number of items.

FORMULA TO KNOW

Average Cost Function

$$AC(x) = \frac{C(x)}{x}$$

Since a total cost function includes a fixed cost, It appears that the first few units cost more to make, since the fixed costs affect the total costs more.

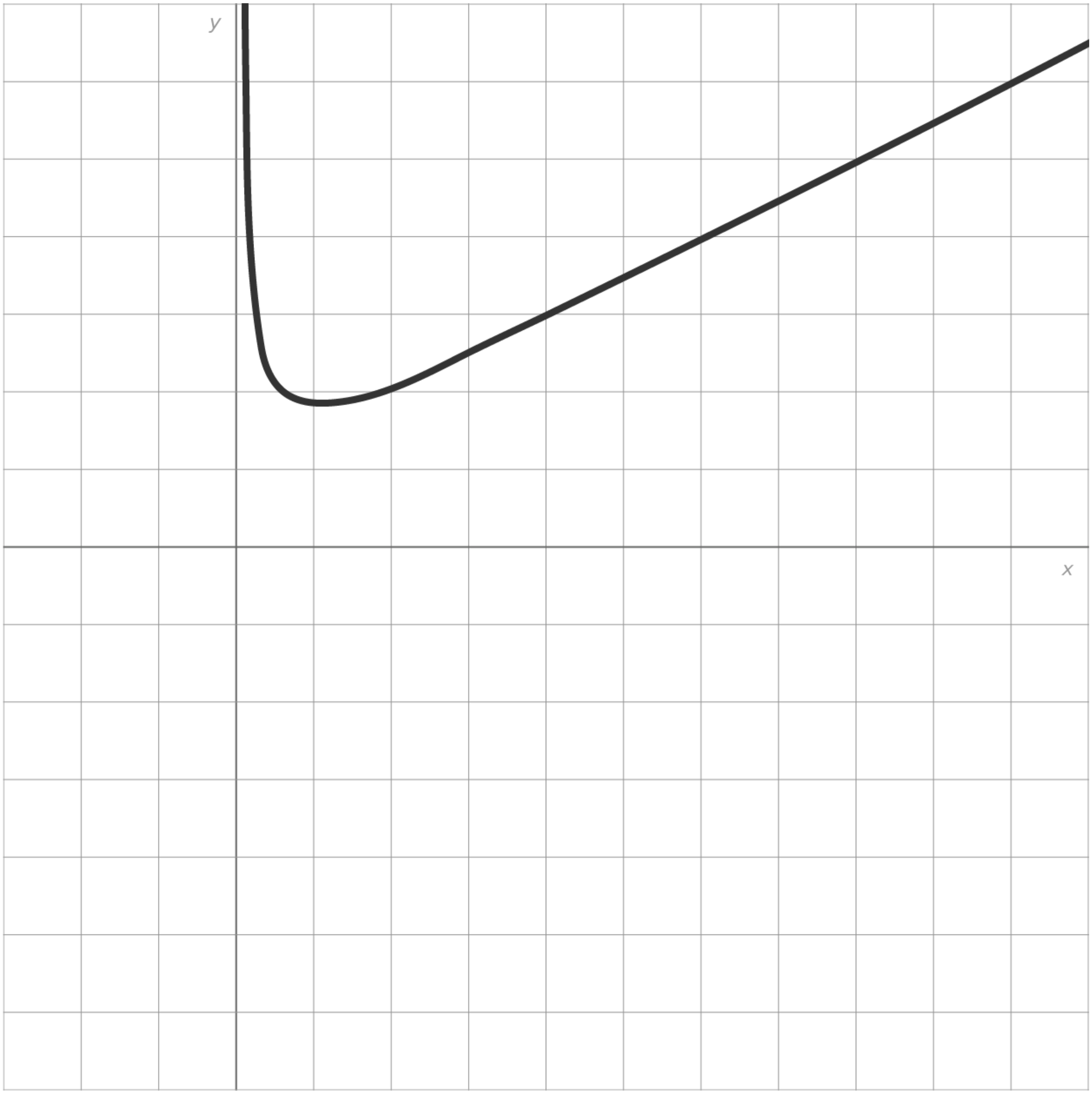
For example, consider the total cost of producing *x* items, $C(x) = 0.5x^2 + 40x + 600$.

In the table, we have the number of items, the total cost of *x* items, and the average cost.

x = # items	Total Cost, *C*(*x*)	Average Cost, $\frac{C(x)}{x}$
1	$640.50	$640.50
2	$682	$341
3	$724.50	$241.50
4	$768	$192

As you can see, as *x* increases, the average cost function drops sharply as *x* increases, but this isn't the whole story.

A typical average cost curve looks like the graph provided.



To the left of the minimum point, the average cost falls sharply, meaning that its derivative is negative.

To the right of the minimum point, the graph rises fairly steadily, meaning that its derivative is positive.

Knowing the rate of change of the average cost can help determine if a production strategy is cost effective when compared to revenue. For example, if the average cost to produce each unit is $50, but the slope is negative, this means that the average cost per unit could be made smaller if production is increased. This is a wide idea, assuming that the additional items could be sold.

EXAMPLE Given a total cost function $C(x)=0.2x^2+14x+500$, find the rate at which the average cost is changing after 10 units are produced.

First, write the average cost function:

$AC(x)=\frac{C(x)}{x}=\frac{0.2x^2+14x+500}{x}$	Start with the average cost function.
$AC(x)=\frac{0.2x^2}{x}+\frac{14x}{x}+\frac{500}{x}$	Divide each term by *x*.
$AC(x)=0.2x+14+500x^{-1}$	Perform simplifications; write the last term to prepare for the derivative.

Now, find the derivative:

$AC'(x)=0.2(1)+0+500(-1)x^{-2}$	Use the constant multiple rule.
$AC'(x)=0.2-\frac{500}{x^2}$	Simplify and rewrite with positive exponents.

Then, the rate of change at a production level of 10 units is $AC'(10)=0.2-\frac{500}{10^2}=0.2-5=-4.80$.

Thus, the average cost is declining at a rate of $4.80 per item, per item.

TRY IT

The total cost of producing *x* items is $C(x)=4x^2+60x+1200$.

At what rate is the average cost changing at a production level of 20 units? +

The average cost is:

$$\begin{aligned} AC(x)&=\frac{C(x)}{x}=\frac{4x^2+60x+1200}{x}\\ &=\frac{4x^2}{x}+\frac{60x}{x}+\frac{1200}{x}\\ &=4x+60+\frac{1200}{x}\end{aligned}$$

To find the rate of change, we need to find $AC'(x)$.

Before finding the derivative, we need to rewrite the last term as a power:

$$AC(x)=4x+60+1200x^{-1}$$

Now, find the derivative:

$$AC'(x) = 4 + 0 - 1200x^{-2} = 4 - \frac{1200}{x^2}$$

Then, the rate of change at a production level of 20 units is:

$$AC'(20) = 4 - \frac{1200}{20^2} = 1$$

Since the average cost has units “dollars per unit”, the rate of change has units “dollars per unit, per unit”. Thus, the rate of change in the average coast at a production level of 20 units is $1 per unit, per unit.

A question that is left for the next unit is, “What is the minimum average cost, and what is the production level required to achieve it?”

SUMMARY

In this lesson, you learned how derivatives are used in a wide range of applications, including **vertical motion**, exponential **growth and decay**, and some **applications to business**. You learned that in business, particularly in manufacturing, the two functions that are analyzed most often to help minimize costs are **total cost** and **average cost**. Given a specific production level, it is also useful to have an estimate for the cost of producing the next item, which can be determined by finding the **marginal cost**, or derivative of the cost. As you progress through the course, you will learn more applications of the derivative.

TERMS TO KNOW

Fixed Cost (or Overhead)

The costs that are incurred before any items are produced. Mathematically, it is the total cost of producing 0 items.

Marginal Cost Function

The derivative (rate of change) of the cost function. Given a production level *x*, it approximates the cost of the next item.

FORMULAS TO KNOW

Average Cost Function

$$AC(x) = \frac{C(x)}{x}$$