The Chain Rule

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WHAT'S COVERED

In this lesson, you will learn how to find derivatives of general composite functions by using the chain rule. In a previous tutorial, you learned how to find derivatives of $y=(f(x))^n$, which is a composite function, but this doesn't cover all composite functions. For example, we are still unable to find the derivative of functions such as $f(x)=\sin(3x)$ and $y=\tan(x^2)$. In this tutorial, we'll learn the techniques necessary to find derivatives of said functions. Specifically, this lesson will cover:

1. Motivation for the Chain Rule

1a. Composite Functions

Recall that a composite function has the form $y=f(g(x))$. We call $g(x)$ the "inner function" since it is plugged into the function *f*.

To find derivatives of composite functions, it will be helpful to first identify the inner function. Given $y=f(g(x))$, let $u=g(x)$. Then $y=f(u)$, a less complicated function. Here are a few examples:

Function	Inner Function	Written in Terms of *u*
$y=\sqrt{2x^2+5x}$	$u=2x^2+5x$	$y=\sqrt{u}$
$y=\sin(3x)$	$u=3x$	$y=\sin u$

$y=\frac{3}{(2+\sin x)^4}$	$u=2+\sin x$	$y=\frac{3}{u^4}$

1b. Examining Rates of Change

The chain rule is a derivative technique that uses several rates of change in one problem. Here is a real-life consideration:

A factory can produce 30 units of a certain item per hour at a cost of $20 per item. What is the cost per hour of producing the items?

If you think the answer is $600, great! That is correct. But let's look at this more closely so that we can understand the rates of change.

Let x = the number of hours, u = the number of units produced, and C, the cost.

We can translate the given information into rates of change:

- The factory can produce 30 units per hour: Slope $=\frac{\Delta u}{\Delta x}=30$
- The units cost $20 each to produce: Slope $=\frac{\Delta C}{\Delta u}=20$

Then, the cost per hour is $\frac{\Delta C}{\Delta x}$, which can be written as $\frac{\Delta C}{\Delta u}\cdot\frac{\Delta u}{\Delta x}=(20)(30)=\$600$ per hour.

Generally speaking, let's say that C is a function of u, and u is a function of x. Then, C is also a function of x, and $\frac{\Delta C}{\Delta x}=\frac{\Delta C}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$.

While not a proof, this idea can be extended to derivatives (instantaneous rates of change). That is, given C is a function of u and u is a function of x, $\frac{dC}{dx}=\frac{dC}{du}\cdot\frac{du}{dx}$. This derivative rule applies to composite functions and is called the chain rule.

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2. Applying the Chain Rule

2a. Basic Functions

The chain rule can be expressed with the following formula:

FORMULA TO KNOW

Chain Rule

Suppose $y=f(u)$, a composite function, where u is a function of x.

Then, $\frac{dy}{dx}=\frac{df}{du}\cdot\frac{du}{dx}$.

Using “prime” notation, we can write $y' = f'(u) \cdot u'$.

Using “D” notation, we can write $D[y] = f'(u) \cdot D[u]$.

BIG IDEA

With the chain rule in mind, we can write the derivative of each basic function.

- $D[u^n] = n \cdot u^{n-1} \cdot D[u]$
- $D[\sin u] = \cos u \cdot D[u]$
- $D[\cos u] = -\sin u \cdot D[u]$
- $D[\tan u] = \sec^2 u \cdot D[u]$
- $D[\cot u] = -\csc^2 u \cdot D[u]$
- $D[\sec u] = \sec u \tan u \cdot D[u]$
- $D[\csc u] = -\csc u \cot u \cdot D[u]$

HINT

$D[u^n] = n \cdot u^{n-1} \cdot D[u]$ is the power rule from **The General Power Rule for Functions**.

EXAMPLE Consider the function $y = \sin(3x)$. Find its derivative.

$y = \sin u$	Let $u = 3x$, the inner function.
$\frac{dy}{dx} = \cos u \cdot D[u]$	Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\frac{dy}{dx} = \cos u \cdot 3$	$D[u] = D[3x] = 3$
$\frac{dy}{dx} = 3\cos 3x$	Rewrite “3” in front and replace u with $3x$.

Thus, $\frac{dy}{dx} = 3\cos 3x$.

In challenge 3.2, you took derivatives of functions of the form $y = [f(x)]^n$, which is a specific form of the chain rule ($y = u^n$ where $u = f(x)$). Here is a reminder of a power function that requires the chain rule.

EXAMPLE Consider the function $y = \sqrt[3]{4x^2 + \sin x}$. Find its derivative.

$y = \sqrt[3]{u} = u^{1/3}$	Let $u = 4x^2 + \sin x$, the inner function.

$$\frac{dy}{dx} = \frac{1}{3}u^{-2/3} \cdot D[u]$$ Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{3}u^{-2/3} \cdot (8x + \cos x)$$ $D[u] = D\left[4x^2 + \sin x\right] = 8x + \cos x$

$$\frac{dy}{dx} = \frac{1}{3}(4x^2 + \sin x)^{-2/3}(8x + \cos x)$$ Replace u with $4x^2 + \sin x$.

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{(4x^2 + \sin x)^{2/3}}(8x + \cos x)$$ Rewrite with positive exponents.

$$\frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}}$$ Combine fractions.

Thus, $\frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}}$.

EXAMPLE Consider the function $y = \tan(x^2 + 1)$. Find its derivative.

$$y = \tan u$$ Let $u = x^2 + 1$, the inner function.

$$\frac{dy}{dx} = \sec^2 u \cdot D[u]$$ Apply the chain rule: $\frac{dy}{dx} = f'(u) \cdot D[u]$

$$\frac{dy}{dx} = (\sec^2 u) \cdot 2x$$ $D[u] = D\left[x^2 + 1\right] = 2x$ (parentheses added to show separation)

$$\frac{dy}{dx} = 2x \cdot \sec^2(x^2 + 1)$$ Rewrite "$2x$" in front and replace u with $x^2 + 1$.

Thus, $\frac{dy}{dx} = 2x \cdot \sec^2(x^2 + 1)$.

TRY IT

Consider the function $y = \cos(2x^3)$.

Find its derivative. +

This function is a composite function, start by letting $u = 2x^3$. Then, $u' = 6x^2$.

Then, $y = \cos u$.

By the chain rule, $\frac{dy}{dx} = -\sin u \cdot u'$.

Replacing u and u', we have $\frac{dy}{dx} = -\sin(2x^3) \cdot (6x^2)$.

Rearranging factors, $\frac{dy}{dx} = -6x^2 \cdot \sin(2x^3)$.

WATCH

Here is a video in which we find the derivative of $y = \sec(3x+1)$.

2b. Applying the Chain Rule Twice

When the inner function is itself a composite function, the chain rule is applied more than once. Rather than assign a letter to each inside function, we'll present another way to organize this.

EXAMPLE Consider the function $y = \sin^2 3x$. Find its derivative.

$y = (\sin(3x))^2$	Rewrite as a quantity squared.
$\frac{dy}{dx} = 2(\sin(3x)) \cdot D[\sin(3x)]$	Apply the chain rule: $D[u^2] = 2u \cdot u'$
$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot D[3x]$	Apply the chain rule again: $D[\sin u] = \cos u \cdot u'$
$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot 3$	$D[3x] = 3$
$\frac{dy}{dx} = 6\sin 3x \cos 3x$	$2 \cdot 3 = 6$; omit unnecessary parentheses.

Thus, $\frac{dy}{dx} = 6\sin 3x \cos 3x$.

TRY IT

Consider the function $y = \tan^4 2x$.

Find its derivative. +

First, rewrite the function as $y = (\tan 2x)^4$.

Then, let $u = \tan 2x$.

The function becomes $y = u^4$, which means $\frac{dy}{dx} = 4u^3 \cdot u'$.

With $u = \tan 2x$, its derivative is:

$$
\begin{aligned}
u' &= \sec^2 2x \cdot D[2x] \\
&= \sec^2 2x \cdot 2 \\
&= 2\sec^2 2x
\end{aligned}
$$

Putting all the pieces together:

$$
\begin{aligned}
\frac{dy}{dx} &= 4u^3 \cdot u' \\
&= 4(\tan 2x)^3 (2\sec^2 2x) \\
&= 8\tan^3 2x \sec^2 2x
\end{aligned}
$$

2c. Combining the Chain Rule With Other Rules

Sometimes sums, differences, products, and quotients contain composite functions. The key is to approach the derivative carefully.

EXAMPLE Consider the function $y = x\sin 4x$. Find its derivative.

$y = x\sin 4x$	Start with the given function.
$\frac{dy}{dx} = D[x]\cdot\sin 4x + x\cdot D[\sin 4x]$	Apply the product rule.
$\frac{dy}{dx} = 1\cdot\sin 4x + x\cdot\cos 4x\cdot D[4x]$	$D[x] = 1$, $D[\sin u] = \cos u \cdot D[u]$
$\frac{dy}{dx} = 1\cdot\sin 4x + x\cdot\cos 4x\cdot(4)$	$D[4x] = 4$
$\frac{dy}{dx} = \sin 4x + 4x\cos 4x$	Omit unnecessary grouping symbols.

Thus, $\frac{dy}{dx} = \sin 4x + 4x\cos 4x$.

SUMMARY

In this lesson, you began by understanding the **motivation for the chain rule**, a derivative technique used to compute derivatives of **composite functions**, namely that it expands your ability to find **rates of change**. You learned how to **apply the chain rule**, exploring the formula and the derivative of each **basic function**. You also learned that when the inner function is itself a composite function, you need to **apply the chain rule twice**. Lastly, you examined an example of finding a derivative by **combining the chain rule with other rules**, such as the product rule.

FORMULAS TO KNOW

Chain Rule

Suppose $y=f(u)$, a composite function, where *u* is a function of *x*.

Then, $\frac{dy}{dx}=\frac{df}{du}\cdot\frac{du}{dx}$.

Using “prime” notation, we can write $\frac{dy}{dx}=f'(u)\cdot u'$.

Using “D” notation, we can write $\frac{dy}{dx}=f'(u)\cdot D[u]$.