Putting It All Together: Sketching a Graph

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WHAT'S COVERED

In this lesson, you will use properties of $f(x)$, $f'(x)$, and $f''(x)$, and limits to sketch the graph of a function. Specifically, this lesson will cover:

1. Graphing Functions: A General Strategy

Calculus is necessary to get a big-picture understanding of what the graph of a function $y = f(x)$ looks like. Below are all the important characteristics of a function, grouped by which stem from $f(x)$, $f'(x)$, or $f''(x)$.

$f(x)$	$f'(x)$	$f''(x)$
Domain Asymptotes Intercepts	Critical Numbers Increasing/Decreasing Local Max/Min	Concavity Inflection Points

In addition, it's important to know the shape of the graph in between key points. For example, the table below shows that there are two ways for a function to increase: when it's concave up and concave down (upper left and lower right).

Increasing and Concave Down	Decreasing and Concave Down
	
Decreasing and Concave Up	Increasing and Concave Up

To graph a function, we will follow this process:

STEP BY STEP

1. Find $f'(x)$ and $f''(x)$.
2. From the original function, $f(x)$:
   a. Find the domain of the function.
   b. Determine if there are any vertical, horizontal, slant, or nonlinear asymptotes (recall that these require limits).
   c. Find coordinates of all x- and y-intercepts.
3. Find all critical numbers of $f'(x)$. Recall that these are values of x for which $f'(x)=0$ or undefined. Remember also that all critical numbers must also be in the domain of $f(x)$.
4. Find all values of x for which $f''(x)=0$ or undefined.
5. Form a combined sign graph for $f'(x)$ and $f''(x)$. This can be used to determine where the function is increasing, decreasing, concave up, concave down, and any combination of them.
6. Identify any local extreme points and inflection points.
7. Using all the information from 1-6, graph the function by using all key points and characteristics you find.

2. Graphing Functions: Examples

EXAMPLE Use the techniques from this unit to sketch the graph of $f(x)=x^4-18x^2+32$.

1. Find the first and second derivative:

$$f'(x)=4x^3-36x$$
$$f''(x)=12x^2-36$$

2. Information from $f(x)$:

a. The domain of the function is all real numbers.

b. Since $f(x)$ is a polynomial, there are no asymptotes.

c. The y-intercept is $(0, 32)$. To find the x-intercepts, set $f(x)=0$ and solve.

$$x^4-18x^2+32=0$$
$$(x^2-16)(x^2-2)=0$$
$$x^2-16=0 \text{ or } x^2-2=0$$
$$x^2=16 \text{ or } x^2=2$$
$$x=\pm 4,\ x=\pm\sqrt{2}$$

Thus, the graph of $f(x)$ has 4 x-intercepts: $(\pm 4, 0)$, $(\pm\sqrt{2}, 0)$

3. Find critical numbers:

Since $f'(x)$ is a polynomial, it is never undefined. Therefore, set $f'(x)=0$ and solve:

$$4x^3-36x=0$$
$$4x(x^2-9)=0$$
$$4x(x+3)(x-3)=0$$
$$4x=0 \text{ or } x+3=0 \text{ or } x-3=0$$
$$x=0,\ -3,\ 3$$

4. Find all values for which $f''(x)=0$ (note: $f''(x)$ is never undefined).

$$12x^2-36=0$$
$$12x^2=36$$
$$x^2=3$$
$$x=\pm\sqrt{3}$$

5. Create a combined sign graph for $f'(x)$ and $f''(x)$. Note that the key numbers are $x=-3$, 0, and 3 from $f'(x)$ and $x=\pm\sqrt{3}$ from the second derivative. This is a total of 5 numbers, therefore we have six intervals. Here is the sign graph:

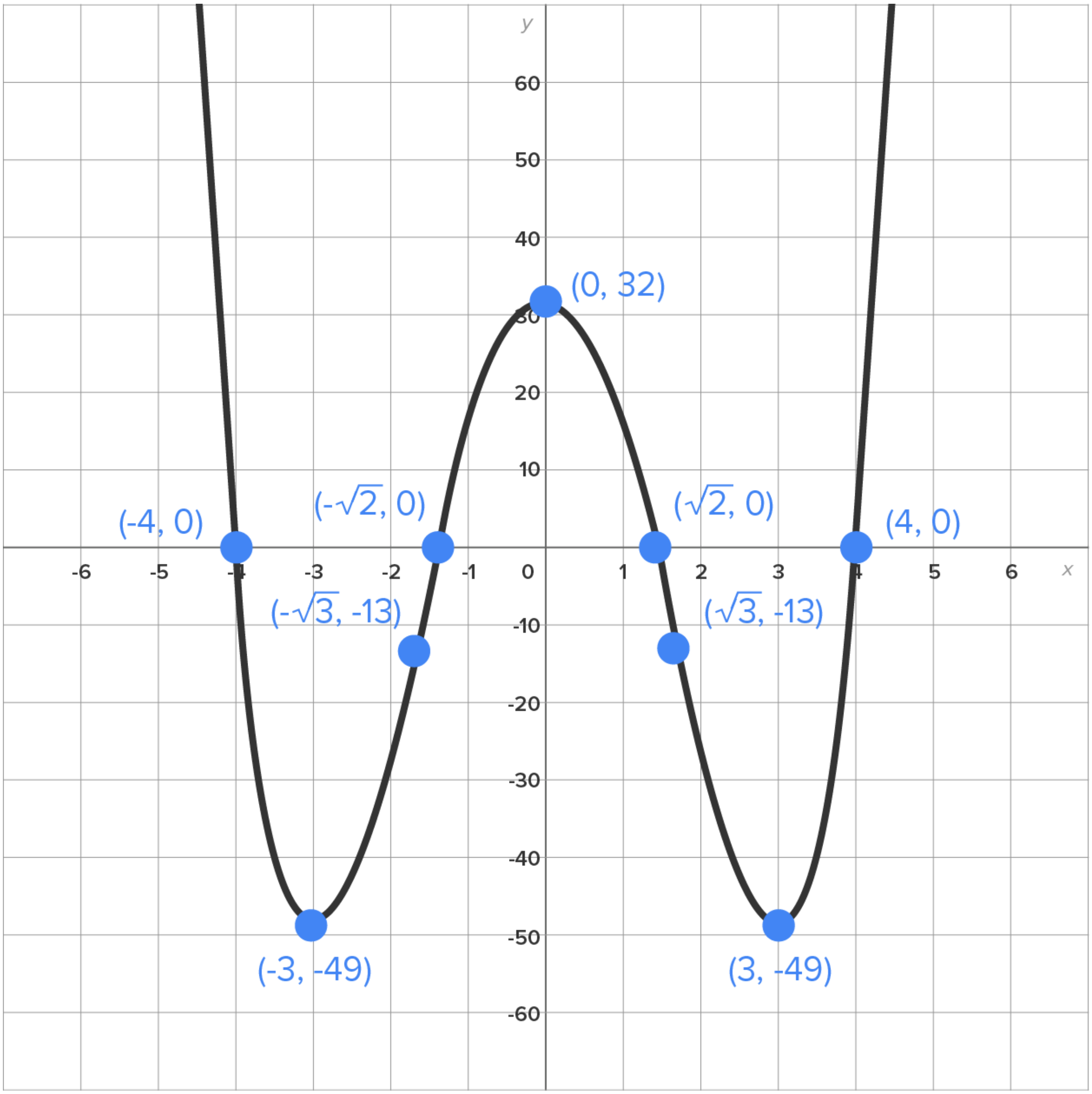
Interval	$(-\infty, -3)$	$(-3, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, 3)$	$(3, \infty)$
Test Value	-4	-2	-1	1	2	4
Value of $f'(x)$	-112	40	32	-32	-40	112

Sign of $f'(x)$	-	+	+	-	-	+
Behavior of $f(x)$	Dec.	Inc.	Inc.	Dec.	Dec.	Inc.
Value of $f''(x)$	156	12	-24	-24	12	156
Sign of $f''(x)$	+	+	-	-	+	+
Behavior of $f(x)$	Concave Up	Concave Up	Concave Down	Concave Down	Concave Up	Concave Up
Shape of Graph						

6. From the sign graph, we can see the following:

- Local minimum at $(-3, f(-3)) = (-3, -49)$
- Inflection point at $\left(-\sqrt{3}, f\left(-\sqrt{3}\right)\right) = \left(-\sqrt{3}, -13\right)$
- Local maximum at $(0, f(0)) = (0, 32)$
- Inflection point at $\left(\sqrt{3}, f\left(\sqrt{3}\right)\right) = \left(\sqrt{3}, -13\right)$
- Local minimum at $(3, f(3)) = (3, -49)$

7. Pulling this information together, the graph of the function with all important points labeled is shown in the figure.



WATCH

In this video, we will use the techniques from this unit to sketch the graph of $f(x) = x - 6\sqrt{x-1}$. (Note: This video is over 10 minutes long.)

WATCH

In this video, we will use the techniques from this unit to sketch the graph of $f(x) = 10x^3 - 3x^5$.

Let's review one last example, involving a function that has asymptotes.

➦ EXAMPLE Use the techniques from this unit to sketch the graph of $f(x)=x^2+\frac{8}{x}$.

1. Find the first and second derivative. To prepare for this, rewrite $f(x)=x^2+8x^{-1}$.

$f'(x)=2x-8x^{-2}$

$f''(x)=2+16x^{-3}$

2. Information from $f(x)$:

a. Domain: $(-\infty, 0)\cup(0, \infty)$

b. Asymptotes:

Vertical asymptote: $x=0$

Nonlinear Asymptote: $y=x^2$ (Since $\frac{8}{x}\rightarrow 0$ as $x\rightarrow\infty$)

c. Intercepts:

There is no y-intercept since $x=0$ is not in the domain of $f(x)$.

To find x-intercepts, set $x^2+\frac{8}{x}=0$ and solve:

$x^3+8=0$	Multiply both sides by *x*.
$x^3=-8$	Isolate x^3 to one side.
$x=-2$	Take the cube root of both sides.

Thus, there is an x-intercept at $(-2, 0)$.

3. Find critical numbers:

Earlier, we calculated $f'(x)=2x-8x^{-2}=2x-\frac{8}{x^2}$. $f'(x)$ is undefined when $x=0$, which is not in the domain of *f*. Therefore, 0 is not a critical number.

To find other critical numbers, solve $2x-\frac{8}{x^2}=0$.

$2x^3-8=0$	Multiply both sides by x^2.
$2x^3=8$	Add 8 to both sides.
$x^3=4$	Divide both sides by 2.
$x=\sqrt[3]{4}\approx 1.59$	Take the cube root of both sides.

4. Find all values for which $f''(x)=0$ or undefined.

Earlier, we computed $f''(x)=2+16x^{-3}=2+\frac{16}{x^3}$. $f''(x)$ is undefined when $x=0$, but $f(x)$ could still change concavity there.

To find possible inflection points, set $2+\frac{16}{x^3}=0$ and solve:

$2x^3+16=0$	Multiply both sides by x^3.
$x^3=-8$	Subtract 16 from both sides, then divide both sides by 2.
$x=-2$	Take the cube root of both sides.

5. Create a combined sign graph for $f'(x)$ and $f''(x)$. Note that the key numbers are $x=0$ and $\sqrt[3]{4}$ from $f'(x)$ and $x=-2$ from the second derivative. This is a total of 3 numbers, therefore we have four intervals. Here is the sign graph:

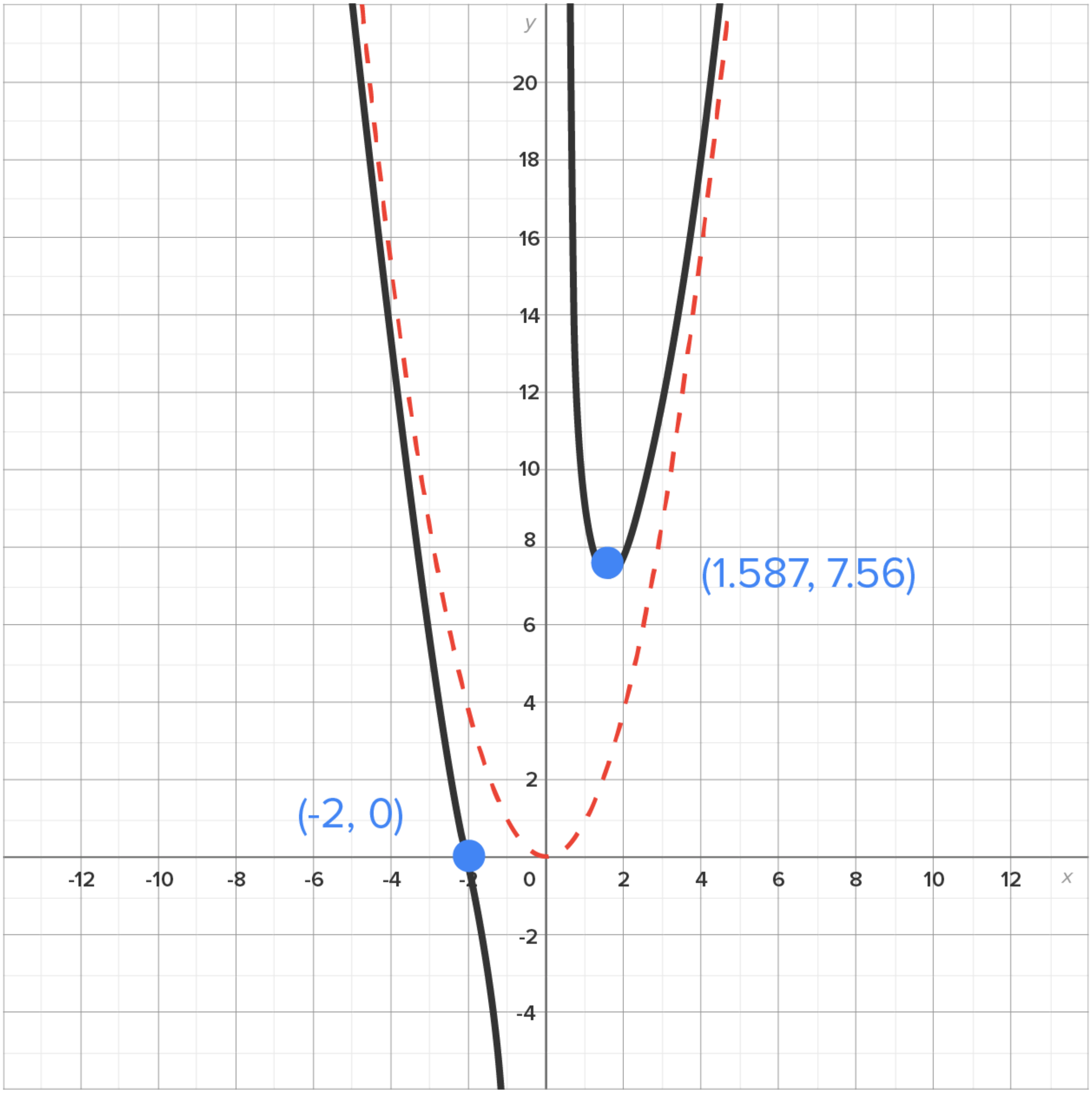
Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \sqrt[3]{4})$	$(\sqrt[3]{4}, \infty)$
Test Value	-3	-1	1	2
Value of $f'(x)$	$-\frac{62}{9}$	-10	-6	2
Sign of $f'(x)$	-	-	-	+
Behavior of $f(x)$	Dec.	Dec.	Dec.	Inc.
Value of $f''(x)$	$\frac{70}{27}$	-14	18	4
Sign of $f''(x)$	+	-	+	+
Behavior of $f(x)$	Concave Up	Concave Down	Concave Up	Concave Up

Shape of Graph				

6. From the sign graph, we can see the following:

- Local minimum at $\left(\sqrt[3]{4}, f\left(\sqrt[3]{4}\right)\right) \approx \left(\sqrt[3]{4}, 7.56\right)$
- Inflection point at $(-2, f(-2)) = (-2, 0)$
- Note: Even though $f(x)$ changes concavity at $x = 0$, there is no inflection point there since $f(x)$ is not defined when $x = 0$.

7. Pulling this information together, the graph of the function with all important points labeled is shown in the figure.



SUMMARY

In this lesson, you learned **a general strategy** of using limits and properties of $f(x)$, $f'(x)$, and $f''(x)$ together to **graph a function** $y = f(x)$, followed by several **examples** of applying these techniques to **graph functions**.