The Product Rule

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to find derivatives of functions that are products of other functions. For example, $f(x)=x^2\sin x$ is the product of x^2 and $\sin x$. Specifically, this lesson will cover:

1. The Product Rule

Suppose you want to find $D[f(x)\cdot g(x)]$.

At first glance, one might (incorrectly) think that $D[f(x)\cdot g(x)]=D[f(x)]\cdot D[g(x)]$. In other words, is the derivative of the product equal to the product of the derivatives?

Let's test this out. Let $f(x)=x^2$ and $g(x)=x^3$. Then, $f(x)\cdot g(x)=x^2\cdot x^3=x^5$.

Now, let's find $D[f(x)\cdot g(x)]$ and $D[f(x)]\cdot D[g(x)]$ to see if they are equal:

- $D[f(x)\cdot g(x)]=D\left[x^5\right]=5x^4$
- $D\left[x^3\right]\cdot D\left[x^2\right]=3x^2\cdot 2x=6x^3$

Oh no! These expressions are not equal!
In general, we can conclude that $D[f(x)\cdot g(x)]\neq f'(x)\cdot g'(x)$.

So, what is the correct product rule?

FORMULA TO KNOW

Product Rule for Derivatives

$$D[f(x) \cdot g(x)] = D[f(x)] \cdot g(x) + f(x) \cdot D[g(x)]$$

Using alternate notation: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

In words: "The derivative of a product of two functions is the derivative of the first times the second, plus the first times the derivative of the second."

EXAMPLE Find the derivative of $f(x) = x^2 \sin x$.

$f'(x) = D[x^2] \cdot \sin x + x^2 \cdot D[\sin x]$	Apply the product rule.
$f'(x) = 2x \cdot \sin x + x^2 \cdot \cos x$	Use known derivatives.
$f'(x) = 2x \sin x + x^2 \cos x$	Remove unnecessary symbols.

Thus, $f'(x) = 2x \sin x + x^2 \cos x$.

EXAMPLE Find the derivative of $f(x) = \cos^2 x$.

First, notice that $\cos^2 x$ can be rewritten as $\cos x \cdot \cos x$, so the product rule can be used to find its derivative.

$f'(x) = D[\cos x \cdot \cos x] = D[\cos x] \cdot \cos x + \cos x \cdot D[\cos x]$	Apply the product rule.
$f'(x) = (-\sin x)\cos x + \cos x(-\sin x)$	$D[\cos x] = -\sin x$
$f'(x) = -\sin x \cos x - \sin x \cos x$	Remove the grouping.
$f'(x) = -2\sin x \cos x$	Combine like terms.

Thus, $f'(x) = -2\sin x \cos x$.

Let's now look at one with a constant multiple.

EXAMPLE Find the derivative of $f(t) = 4\sin t \cos t$.

This appears as a product of three functions, but the "4" is a constant. Let's write $f(t) = 4\sin t \cos t = (4\sin t)(\cos t)$. Then:

$f'(t) = D[4\sin t] \cdot \cos t + 4\sin t \cdot D[\cos t]$	Apply the product rule.
$f'(t) = 4\cos t \cdot \cos t + 4\sin t \cdot (-\sin t)$	Use the derivative rules for $\sin t$, $\cos t$, and the constant multiple.
$f'(t) = 4\cos^2 t - 4\sin^2 t$	Condense into exponential notation.

Thus, $f'(t) = 4\cos^2 t - 4\sin^2 t$.

Now, here are two examples for you to try.

TRY IT

Consider the function $g(x) = 4x^5 \cos x$.

Find the derivative of *g*(*x*). +

The function $g(x)$ is the product of $4x^5$ and $\cos x$.

$g'(x) = \cos x \cdot D[4x^5] + 4x^5 \cdot D[\cos x]$	Set up the product rule.
$g'(x) = \cos x \cdot 20x^4 + 4x^5(-\sin x)$	Find each derivative.
$g'(x) = 20x^4 \cos x - 4x^5 \sin x$	Simplify.

TRY IT

Consider the function $f(t) = 4\sin^2 t$.

Find the derivative of *f*(*t*). +

First, think of $f(t) = 4\sin t \sin t$.

This means that $f(t)$ is the product of $4\sin t$ and $\sin t$.

$f'(t) = \sin t \cdot D[4\sin t] + 4\sin t \cdot D[\sin t]$	Set up the product rule.
$f'(t) = \sin t(4\cos t) + 4\sin t(\cos t)$	Find each derivative.
$f'(t) = 4\sin t \cos t + 4\sin t \cos t$	Simplify.
$f'(t) = 8\sin t \cos t$	Combine like terms.

BIG IDEA

What we are starting to see is that finding derivatives involves building blocks rather than memorization. For example, the derivative of $f(x) = x^3 \sin x$ is not worth memorizing, but knowing the product rule, $D[x^3]$, and $D[\sin x]$, one can pretty quickly find $f'(x)$.

Now, let's look at an example where the product rule could be used, but isn't required.

EXAMPLE Find the derivative of $f(x)=(2x^2+5)(x-8)$.

This is clearly a product, so the product rule can be used to find the derivative:

$f'(x)=D[2x^2+5]\cdot(x-8)+(2x^2+5)\cdot D[x-8]$	Apply the product rule.
$f'(x)=(4x)(x-8)+(2x^2+5)(1)$	Take the derivatives.
$f'(x)=4x^2-32x+2x^2+5$	Distribute.
$f'(x)=6x^2-32x+5$	Combine like terms.

However, notice that $(2x^2+5)(x-8)$ can be multiplied before differentiation:

$$f(x)=(2x^2+5)(x-8)$$

$$f(x)=2x^2(x)-8(2x^2)+5x-40$$

$$f(x)=2x^3-16x^2+5x-40$$

Written this way, the derivative can be found with the sum/difference and constant multiple rules.

$f(x)=2x^3-16x^2+5x-40$	Start with the original evaluated function.
$f'(x)=D[2x^3]-D[16x^2]+D[5x]-D[40]$	Use the sum/difference properties.
$f'(x)=2(3x^2)-16(2x)+5(1)-0$	Apply the constant multiple and power rules.
$f'(x)=6x^2-32x+5$	Simplify.

As you can see, the results are identical. It looks like simplifying the expression first enables us to use basic rules rather than the product rule.

BIG IDEA

When you need to find the derivative of a product of two functions, check first to see if the function can be manipulated/simplified first. You could avoid having to use the product rule in exchange for an easier rule.

TRY IT

Consider the function $f(x)=(x^2-4)(2x^2+3)$.

Find the derivative. +

Note that $f(x)$ can be rewritten using algebra.

$$
\begin{aligned}
&(x^2-4)(2x^2+3)\\
&=x^2(2x^2)+3x^2-4(2x^2)-4(3)\\
&=2x^4-5x^2-12
\end{aligned}
$$

Now, we can find the derivative by using the power rule (and avoiding the product rule).

$$
\begin{aligned}
f'(x)&=4(2x^3)-5(2x)-0\\
&=8x^3-10x
\end{aligned}
$$

2. Combining the Product Rule With Other Rules

As we learn more derivative rules, functions can get more complex. Here is an example of one such function.

EXAMPLE Find the derivative of $f(x)=x\sin x+\cos x$.

Since $f(x)$ is a sum of two functions, the sum/difference rule should be applied first.

$f'(x)=D[x\sin x]+D[\cos x]$	Apply the sum/difference rule.
$f'(x)=D[x]\cdot\sin x+x\cdot D[\sin x]+D[\cos x]$	Apply the product rule.
$f'(x)=1\cdot\sin x+x\cdot(\cos x)+(-\sin x)$	$D[x]=1$, $D[\sin x]=\cos x$, $D[\cos x]=-\sin x$
$f'(x)=\sin x+x\cos x-\sin x$	Simplify and remove excess symbols.
$f'(x)=x\cos x$	Combine like terms.

Thus, $f'(x)=x\cos x$.

The product rule can be extended to three or more functions (three is usually the most, though, as far as being optimistic).

WATCH

In the video below, we'll find the derivative of $f(x)=x\sin x\cos x$. Note: The instructor makes a distribution error in this video at 2:48. The final answer should be sin(x)**cos(x)** + x$\cos^2$(x) – x$\sin^2$(x)

SUMMARY

In this lesson, you learned that with the knowledge of **the product rule**, you are able to find derivatives of even more combinations of functions, namely the product of two (or more) functions. It is important to note, however, that before automatically turning to the product rule to find the derivative of a product of two functions, always check first to see if the function can be manipulated or simplified first, allowing you to exchange the product rule for an easier rule. You also learned that as functions get increasingly complex, you may need to **combine the product rule with other rules** to find their derivatives.

FORMULAS TO KNOW

Product Rule for Derivatives

$D[f(x) \cdot g(x)] = D[f(x)] \cdot g(x) + f(x) \cdot D[g(x)]$

Using alternate notation: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$