Definition of the Definite Integral

by Sophia

WHAT'S COVERED

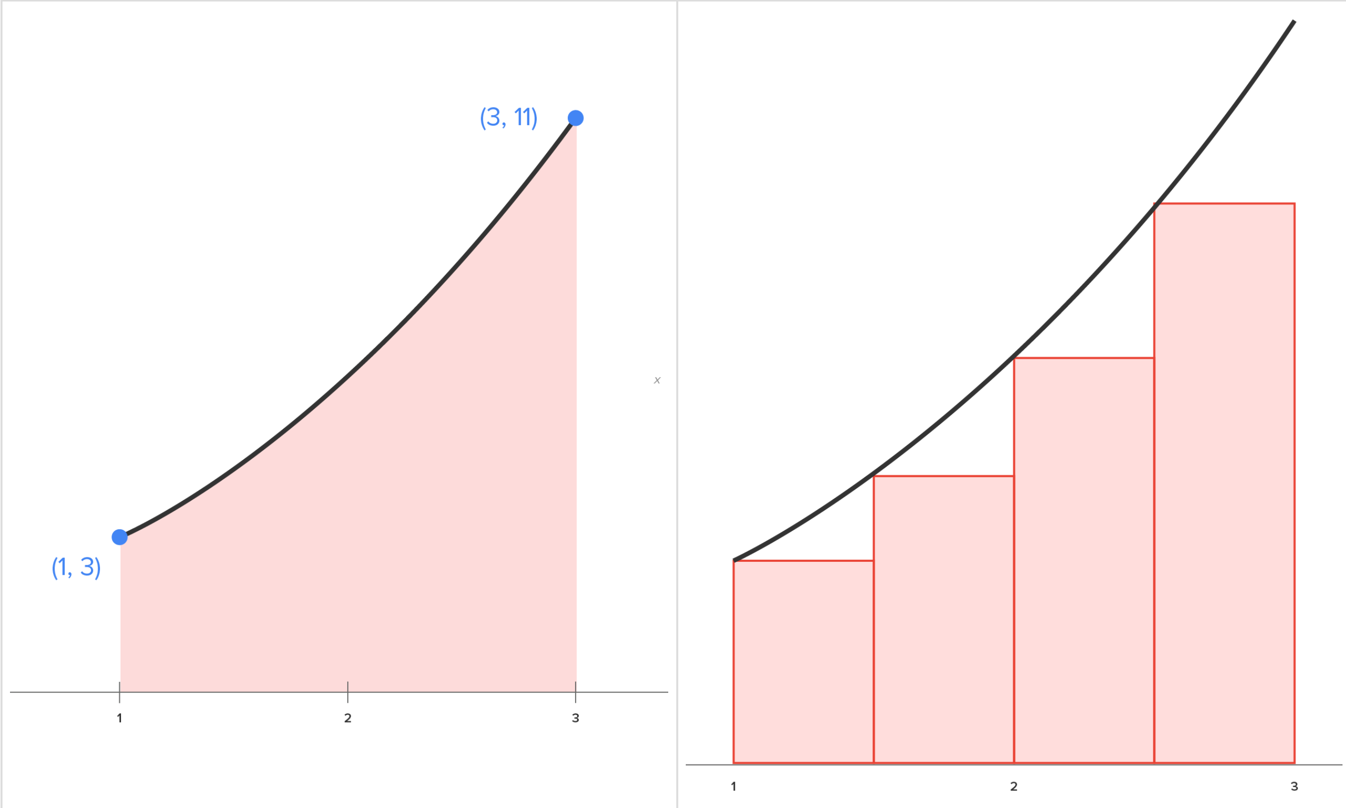
In this lesson, you will connect Riemann sums and the definition of the definite integral. This is the key step in moving into integral calculus. Specifically, this lesson will cover:

1. The Definition of the Definite Integral

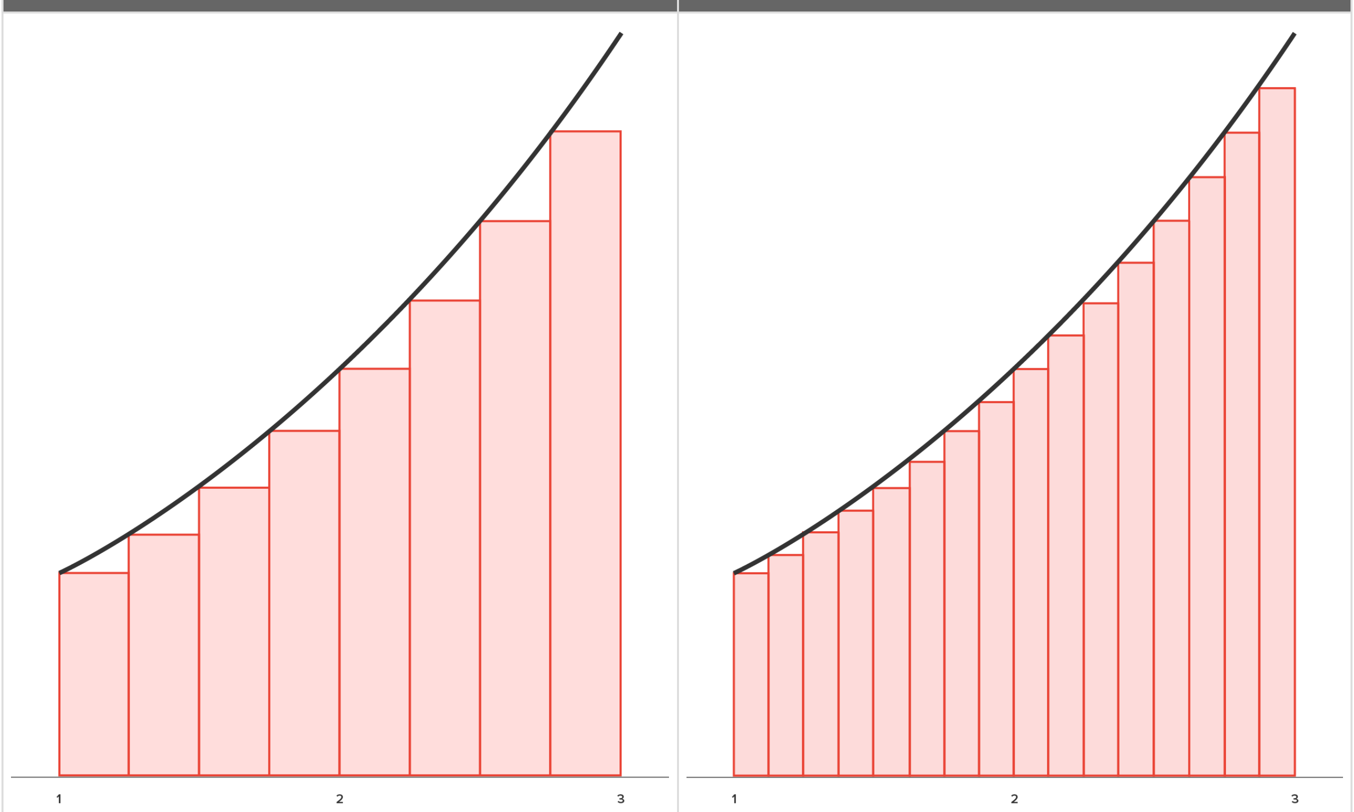
Consider the area of the region bounded by $y = x^2 + 2$ between the x-axis, $x = 1$, and $x = 3$.

Shown below is the actual region as well as the region approximated by 4, 8, and 16 rectangles; all have equal width.

Actual Region	4 Rectangles



8 Rectangles	16 Rectangles



When the subintervals have equal width, we notice the following as the number of rectangles (and subintervals) increases:

1. The width of each rectangle (and subinterval) decreases.

2. The sum of the areas of the rectangles gets closer to the actual area under the curve.

When calculating a Riemann sum for a function $f(x)$ on $[a, b]$, we will only use rectangles that have equal width. That is, when *n* subintervals are used, the width of each subinterval is $\Delta x = \frac{b-a}{n}$. This also means that Riemann sums from this point forward will be written as:

$$\sum_{k=1}^{n} f(c_k)\Delta x$$

Assume (for now) that $f(x)$ is nonnegative. As the number of rectangles gets larger, which means that $n \to \infty$, this quantity will get closer to the actual area as long as $\Delta x \to 0$ for all *k*.

When $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x$ exists and has the same value regardless of the values of c_k used in each subinterval, then $f(x)$ is **integrable** on $[a, b]$.

If we call *A* the definite integral of $f(x)$ on $[a, b]$, then $A = \int_a^b f(x)dx$.

In this notation,

- The numbers *a* and *b* represent the lower and upper limits of integration.
- The function $f(x)$ is called the integrand.
- *x* is called the variable of integration.
- The differential *dx* tells us that the definite integral is computed by letting values of *x* increase from *a* to *b*.

BIG IDEA

For a non-negative function $f(x)$, the value of the Riemann sum approaches the definite integral as $n \to \infty$. Then, the quantity $\int_a^b f(x)dx$ is the area between the graph of a nonnegative function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

TERM TO KNOW

Integrable

If the value of $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$ exists and is equal to *A* regardless of the values of c_k used in each subinterval, then we say that $f(x)$ is integrable on the interval $[a, b]$.

2. Definite Integrals and Riemann Sums

Using the definition of a definite integral, we can write Riemann sums as definite integrals and vice versa.

EXAMPLE Write $\int_0^4 2x\,dx$ as a Riemann sum.

Recall the Riemann sum for a function $f(x)$ is $\sum_{k=1}^{n} f(c_k)\Delta x$.

Since $f(x) = 2x$, the definite integral is the value of the Riemann sum as $n \to \infty$: $\lim_{n \to \infty} \sum_{k=1}^{n} 2c_k \Delta x$

Now, let's take a Riemann sum and write it as a definite integral.

EXAMPLE A function $f(x)$ on the interval $[-4, 4]$ has the Riemann sum $\sum_{k=1}^{n} \frac{1}{1+c_k^2}\Delta x$.

The definite integral is the value of the Riemann sum as $n \to \infty$, and is written $\int_{-4}^{4} \frac{1}{1+x^2}dx$.

TRY IT

Given below are (a) the limit of a Riemann sum and (b) a definite integral.

(a) $\lim_{n \to \infty} \sum_{k=1}^{n} \sin(c_k)\Delta x,\ 0 \le x \le \pi$

(b) $\int_0^9 \sqrt{x+16}\,dx$

Write the definite integral that corresponds to the Riemann sum in (a). +

$\int_0^{\pi} \sin x\,dx$

Write the limit of the Riemann sum that corresponds to the definite integral in (b). +

$\lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{c_k + 16}\,\Delta x,\ 0 \le x \le 9$

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3. Using Riemann Sums to Evaluate Definite Integrals

We learned that $f(x)$ is integrable on $[a, b]$ if $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$ exists and is equal to the same value for any choice of $c_1, c_2, \ldots, c_n$, each of which is in their respective subintervals.

By using the formulas for sigma notation combined with the limit definition, we can evaluate some definite integrals of functions for which we don't know the area of the corresponding region.

Here is how:

STEP BY STEP

1. Since each subinterval has equal width, we know $\Delta x = \frac{b-a}{n}$.
2. Select c_k to be the right-hand endpoint of the interval. Then, $c_1 = a + \Delta x,\ c_2 = a + 2\Delta x,\ c_3 = a + 3\Delta x,$ ... which means $c_k = a + k\Delta x$.
3. Substitute c_k into the function.
4. Evaluate the sum (using formulas), then compute the limit.

EXAMPLE Use a Riemann sum to evaluate $\int_0^4 2x\,dx$.

1. Find the width of each subinterval: $\Delta x = \frac{4-0}{n} = \frac{4}{n}$
2. Find the right-hand endpoints: $c_k = a + k\Delta x = 0 + k\left(\frac{4}{n}\right) = \frac{4k}{n}$
3. Evaluate the function at each c_k: $f(c_k) = 2\left(\frac{4k}{n}\right) = \frac{8k}{n}$
4. Then, $\int_0^4 2x\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{8k}{n}\right)\left(\frac{4}{n}\right)$.

Next, simplify the sum and calculate the limit.

$\lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{8k}{n}\right)\left(\frac{4}{n}\right)$	Calculate the limit.
$= \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{32k}{n^2}\right)$	Simplify.
$= \lim_{n \to \infty} \frac{32}{n^2} \sum_{k=1}^{n} k$	$\frac{32}{n^2}$ is a constant factor since k is the index of summation. Therefore, it can be factored out.
$= \lim_{n \to \infty} \left(\frac{32}{n^2} \cdot \frac{n(n+1)}{2}\right)$	Apply the summation formula: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$= \lim_{n \to \infty} \frac{16n^2 + 16n}{n^2} \quad \text{Simplify.}$$

$$= \lim_{n \to \infty} \left(16 + \frac{16}{n}\right) \quad \text{Divide each term by } n^2.$$

$$= 16 \quad \text{Evaluate the limit.}$$

Regardless of the values of c_k used (left endpoints, etc.), this holds true.

Thus, $\int_0^4 2x\,dx = 16$.

WATCH

In this video, we'll use the Riemann sum to evaluate $\int_1^3 (x^2 + 2)dx$.

TRY IT

Consider the integral $\int_2^6 (10 - x)dx$.

Use the Riemann sum to evaluate this integral. +

Since the integral is taken from $x = 2$ to $x = 6$, we need to set up the values of c_k.

When using n subintervals, the width of each rectangle is $\Delta x = \frac{6-2}{n} = \frac{4}{n}$.

Then, $c_k = a + k\Delta x = 2 + k\left(\frac{4}{n}\right) = 2 + \frac{4k}{n}$

Then, the Riemann Sum for n rectangles is:

$$\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} \left[10 - \left(2 + \frac{4k}{n}\right)\right]\left(\frac{4}{n}\right) = \sum_{k=1}^{n} \left[8 - \frac{4k}{n}\right]\left(\frac{4}{n}\right)$$

Multiplying and simplifying, we have the following (note that n is a constant since the summation depends on k).

$$= \sum_{k=1}^{n} \left(\frac{32}{n} - \frac{16k}{n^2}\right) = \sum_{k=1}^{n} \frac{32}{n} - \frac{16}{n^2} \sum_{k=1}^{n} k$$

Evaluating each sum, we have:

$$= n\left(\frac{32}{n}\right) - \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) = 32 - \frac{8(n+1)}{n}$$

Lastly, the value of the definite integral is the limit of this expression as $n \to \infty$, we have:

$$\lim_{n \to \infty}\left[32 - \frac{8(n+1)}{n}\right] = \lim_{n \to \infty} 32 - \lim_{n \to \infty} \frac{8(n+1)}{n} = 32 - 8 = 24$$

Thus, the value of the definite integral is 24.

THINK ABOUT IT

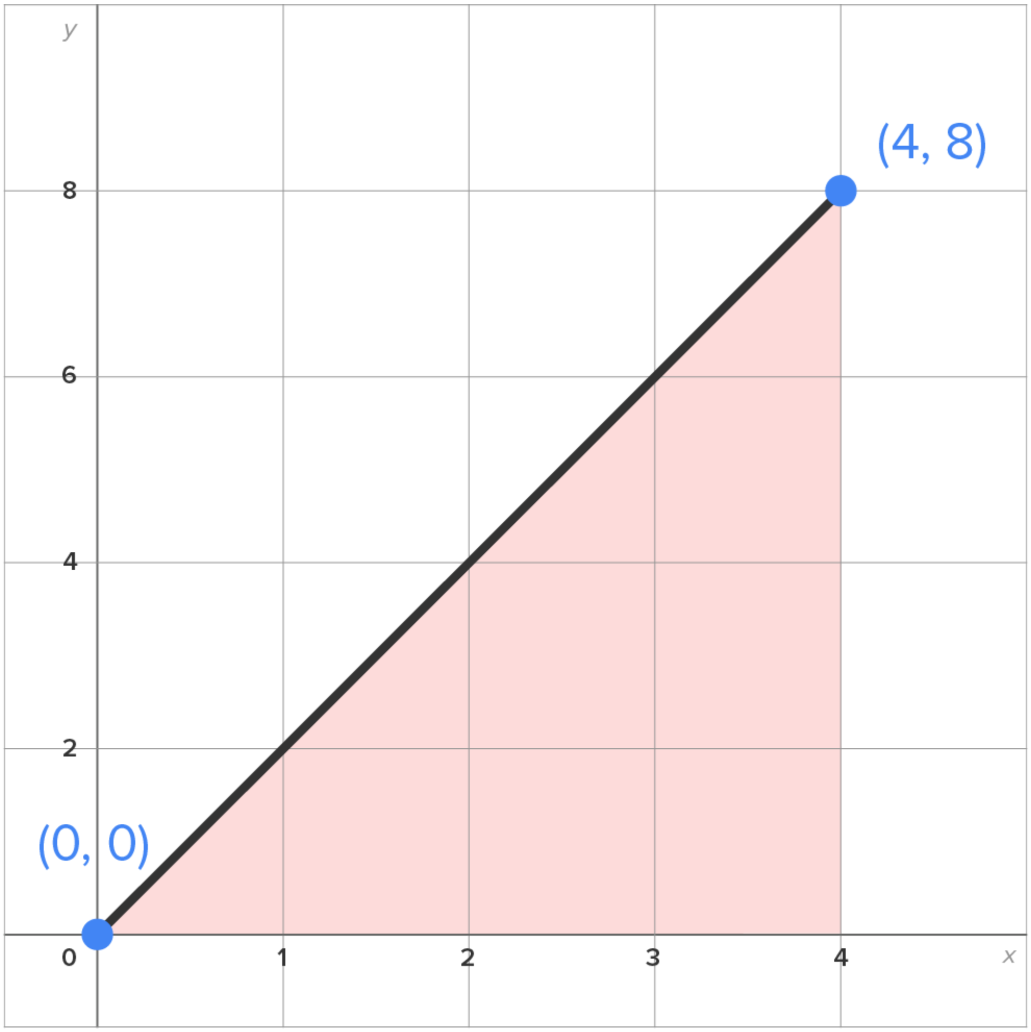
Fortunately, this is not the only method to evaluate definite integrals. These samples were chosen based on known summation formulas.

For example, consider the definite integral $\int_0^{\pi} \sin x dx$. The corresponding limit of a Riemann sum is $\lim_{n \to \infty} \sum_{k=1}^{n} \left[\sin\left(\frac{k\pi}{n}\right) \cdot \frac{\pi}{n}\right]$, which has no known summation formula. We will learn how to find the value of the definite integral without summations in a future challenge.

4. Using Area to Evaluate Riemann Sums and Definite Integrals

EXAMPLE Evaluate the definite integral: $\int_0^4 2x dx$

The figure shows the region bounded by the graph of $f(x) = 2x$ and the x-axis on the interval $[0, 4]$.

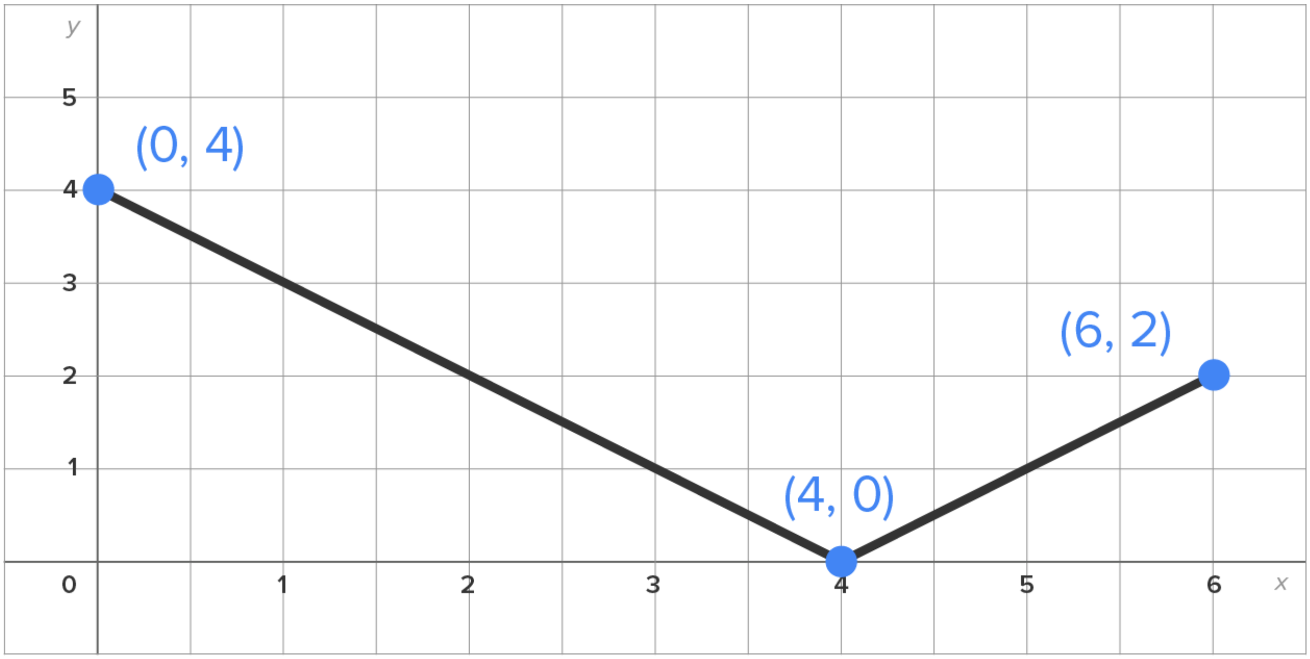


The region is the shape of a triangle with base 4 and height 8, which has area $A = \frac{1}{2}(4)(8) = 16$ square units.

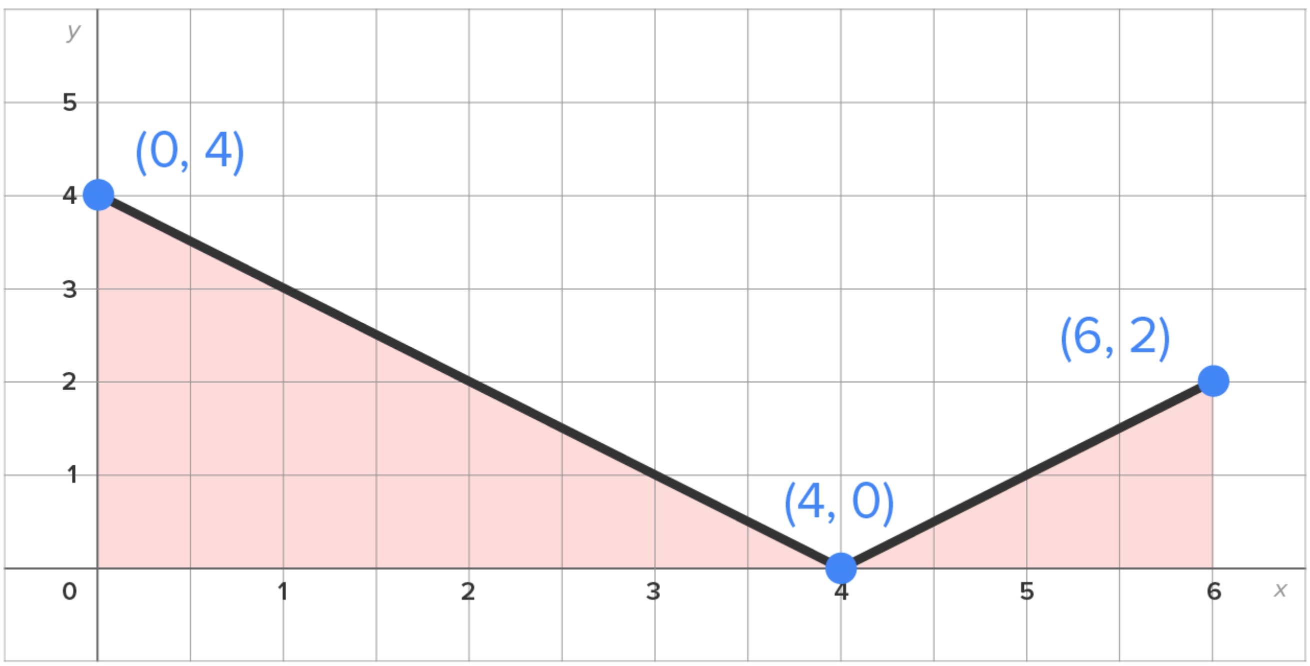
Then, $\int_0^4 2x\,dx = 16$.

Let's look at an example of a continuous piecewise function.

EXAMPLE Consider the graph of $f(x)$ shown in the figure. Use it to evaluate $\int_0^6 f(x)\,dx$.



The region is shown in the figure below and is composed of two triangles.

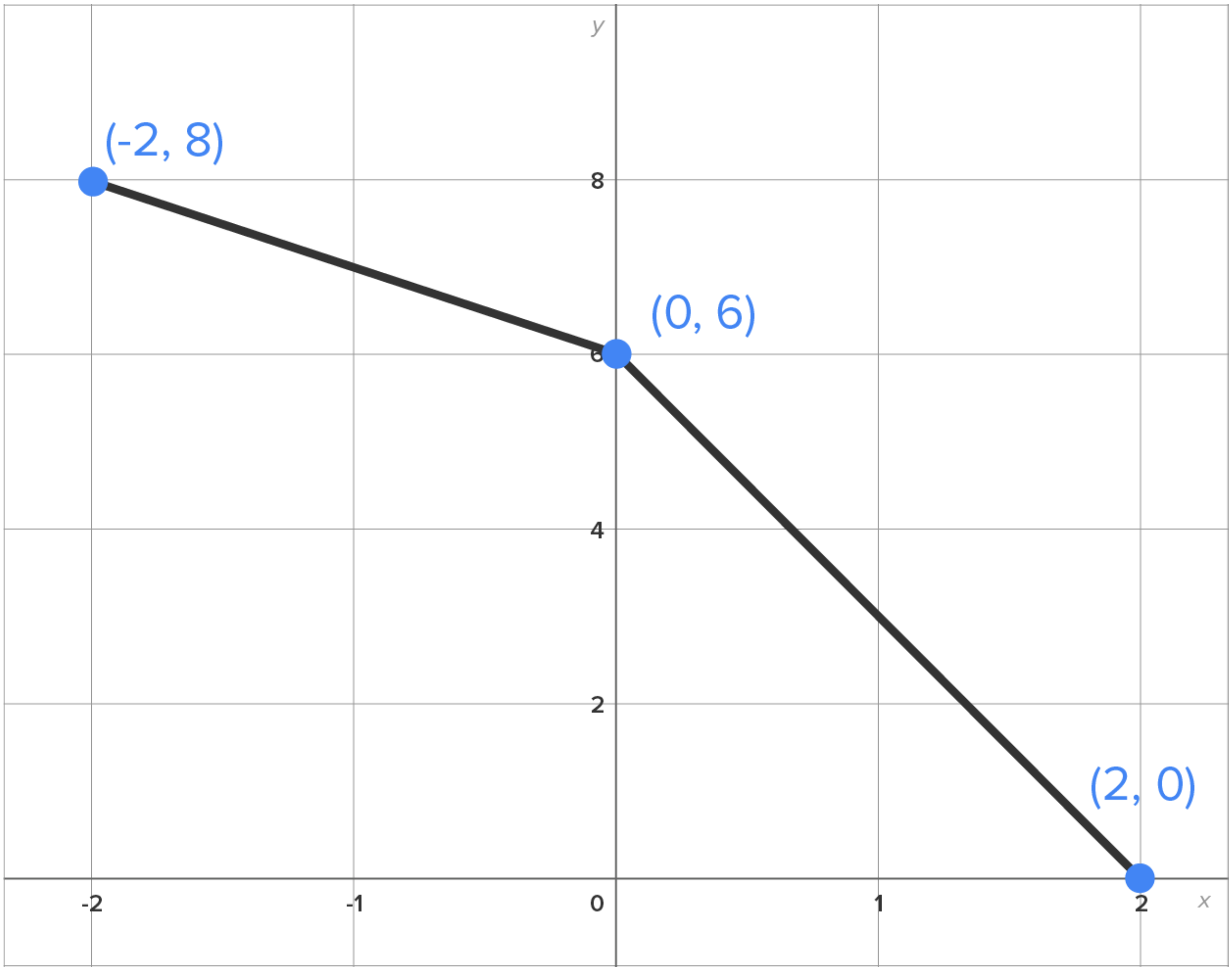


The triangle on $[0, 4]$ has area $\frac{1}{2}(4)(4) = 8 \text{ units}^2$ and the triangle on $[4, 6]$ has area $\frac{1}{2}(2)(2) = 2 \text{ units}^2$.

The total area is 10 units^2, which means that $\int_0^6 f(x)dx = 10$.

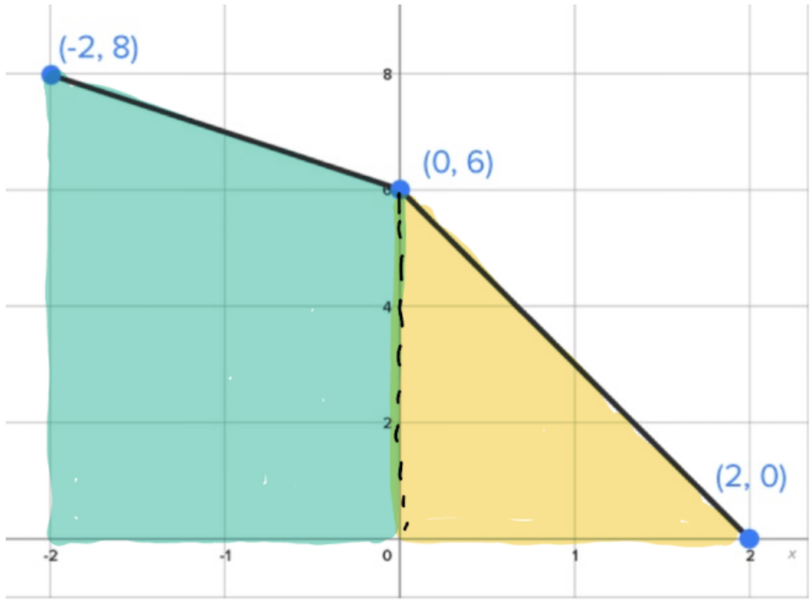
TRY IT

Consider the graph of $f(x)$ shown in the figure that can be used to evaluate $\int_{-2}^{2} f(x)dx$.



Evaluate this integral. +

The value of the definite integral is the total area between the graph and the x-axis.



From $x = -2$ to $x = 0$, the region is a trapezoid with bases 8 and 6, and height 2. The area of this region is $A_1 = \frac{2}{2}(8+6) = 14 \text{ units}^2$.

From $x = 0$ to $x = 2$, the region is a triangle with base 2 and height 6. The area of this region is $A_2 = \frac{1}{2}(2)(6) = 6 \text{ units}^2$.

Then, the value of $\int_{-2}^{2} f(x)dx = A_1 + A_2 = 14 + 6 = 20 \text{ units}^2$.

As it turns out, $f(x)$ doesn't have to be continuous in order to be integrable. Here is an example that illustrates this.

WATCH

In this video, we'll use a graph of some function $y = f(x)$ shown to evaluate $\int_0^{10} f(x)dx$.

SUMMARY

In this lesson, you learned **the definition of the definite integral**, understanding that for a non-negative function $f(x)$, the value of the **Riemann sum approaches the definite integral** as $n \to \infty$. You also learned that by using the formulas for sigma notation combined with the limit definition, you can **evaluate definite integrals** of functions for which you don't know the area of the corresponding region, by **using Riemann sums** to visualize how the area between $f(x)$ and the x-axis on $[a, b]$ is obtained.

Finally, through a series of examples **using area to evaluate Riemann sums and definite integrals**, you have seen that $f(x)$ does not have to be continuous to be integrable.

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TERMS TO KNOW

Integrable

If the value of $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$ exists and is equal to *A* regardless of the values of c_k used in each subinterval, then we say that $f(x)$ is integrable on the interval $[a, b]$.