Derivatives of Non-Natural Logarithmic Functions

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WHAT'S COVERED

In this lesson, you will find derivatives involving logarithmic functions that deal with bases other than e. Since there were reasons to use exponential functions with bases other than e, it makes sense to discuss the corresponding logarithmic functions. Specifically, this lesson will cover:

1. The Derivative of $y = \log_a x$ and $y = \log_a u$, Where u Is a Function of x

Consider the function $y = \log_a x$, where a is any positive number except 1.

If we apply the change of base formula, we have $\log_a x = \frac{\ln x}{\ln a} = \left(\frac{1}{\ln a}\right) \cdot \ln x$.

Then, $D[\log_a x] = D\left[\left(\frac{1}{\ln a}\right) \cdot \ln x\right] = \left(\frac{1}{\ln a}\right) \cdot \frac{1}{x} = \frac{1}{x \cdot \ln a}$.

So, we can say the derivative of a logarithm function with base a can be expressed with the following formula:

FORMULA TO KNOW

Derivative of a Logarithm Function, Base *a*

$$D[\log_a x] = \frac{1}{x \cdot \ln a}$$

When x is replaced with u (a function of x), the chain rule is used.

FORMULA TO KNOW

Derivative of a Composite Logarithm Function, Base *a*

$$D\left[\log_a u\right] = \frac{1}{u \cdot \ln a} \cdot u' = \frac{u'}{u \cdot \ln a}$$

HINT

Notice that this derivative formula is the same as the one for ln*x*, but there is also a factor of ln*a* in the denominator.

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2. Derivatives of Functions Involving $\log_a x$ and $\log_a u$, Where *u* Is a Function of *x*

Since the derivatives of $\log_a x$ and ln*x* are similar, we'll look at various functions.

EXAMPLE Consider the function $f(x) = \log_2(x^3 + 5x)$. Find its derivative.

$f(x) = \log_2(x^3 + 5x)$	Start with the original function.
$f'(x) = \frac{1}{(x^3 + 5x)\ln 2} \cdot (3x^2 + 5)$	$D\left[\log_a u\right] = \frac{1}{u \cdot \ln a} \cdot u'$
$f'(x) = \frac{3x^2 + 5}{(x^3 + 5x)\ln 2}$	Write as a single fraction.

Thus, $f'(x) = \frac{3x^2 + 5}{(x^3 + 5x)\ln 2}$.

TRY IT

Consider the function $f(x) = x^2 \log_3(2x + 1)$.

Find its derivative. +

The product rule is required to find $f'(x)$.

$$\begin{aligned} f'(x) &= \log_3(2x+1)\cdot D[x^2] + x^2\cdot D[\log_3(2x+1)] \\ &= \log_3(2x+1)\cdot 2x + x^2\cdot \frac{1}{(2x+1)\ln 3}\cdot D[2x+1] \\ &= \log_3(2x+1)\cdot 2x + x^2\cdot \frac{1}{(2x+1)\ln 3}\cdot 2 \\ &= 2x\log_3(2x+1) + \frac{2x^2}{(2x+1)\ln 3} \end{aligned}$$

WATCH

The following video illustrates the use of properties of logarithms to find the derivative of $f(x)=\log\left(\frac{x\cdot \sin x}{3x^2+1}\right)$.

As you have seen, derivatives of non-natural logarithmic functions have several small parts to them. While paying attention to detail is important, it's especially important with this topic, since even incorrect answers can look correct at a quick glance.

For example, the derivative of $f(x)=\log_3(x^2+1)$ is $f'(x)=\frac{2x}{(\ln 3)(x^2+1)}$. However, it's possible to mistakenly write this as $f'(x)=\frac{2x}{3(x^2+1)}$, which looks similar but is a much different expression.

SUMMARY

In this lesson, you learned to find **the derivatives of** $y=\log_a x$ **and** $y=\log_a u$**, where** ***u*** **is a function of** ***x*****,** and **derivatives of functions involving** $\log_a x$ **and** $\log_a u$**, where** ***u*** **is a function of** ***x*****.** Remember that the derivative of the "base *a*" logarithmic function is very similar to that of the natural logarithmic function with a factor of $\ln a$ in the denominator. With this function added to your toolbox, you now have the ability to find derivatives of any combination of polynomial, trigonometric, exponential, and logarithmic functions.

FORMULAS TO KNOW

Derivative of a Composite Logarithm Function, Base a

$$D[\log_a u] = \frac{1}{u\cdot \ln a}\cdot u' = \frac{u'}{u\cdot \ln a}$$

Derivative of a Logarithm Function, Base a

$$D[\log_a x] = \frac{1}{x \cdot \ln a}$$