The Differential of f

by Sophia

WHAT'S COVERED

In this lesson, you will express linear approximations in terms of differentials. Specifically, this lesson will cover:

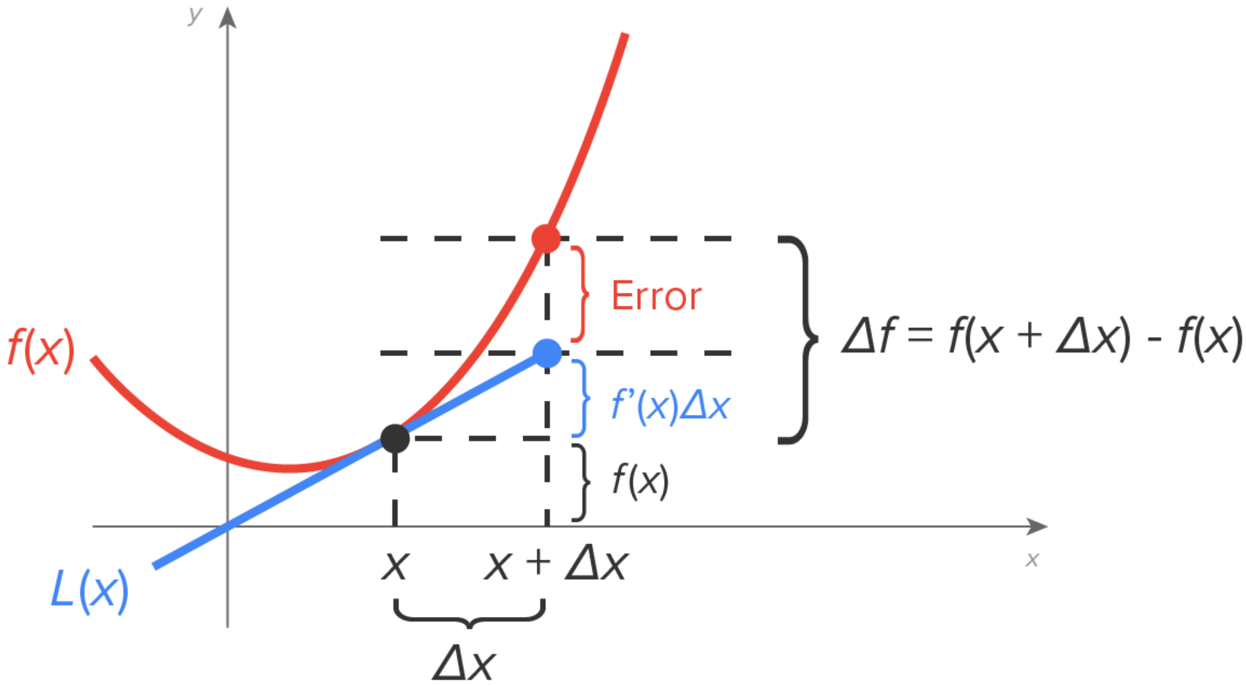
1. Defining the Differential of *f*

From when we first discussed rates of change and the derivative, recall the following quantities:

- $\Delta x =$ change in *x* (horizontal change)
- $\Delta y =$ change in *y* (vertical change)

When $y=f(x)$, Δy can be replaced with Δf to show that this is the change in function *f*. Another goal in linear approximation is to find the change in *f* for a corresponding change in *x*.

Consider the image below:



Let $\Delta x =$ the horizontal change in x-values. This was "$x - a$" in the linear approximation formula.

Let $\Delta f =$ the actual change in *f* when moving from *x* to $x + \Delta x$. Then, $\Delta f = f(x + \Delta x) - f(x)$.

Now, let $A =$ the approximate change in *f* along the tangent line, which can be found as follows:

- Slope $= \dfrac{rise}{run} = f'(x) = \dfrac{A}{\Delta x}$
- Then, solving for *A*, we get $A = f'(x) \cdot \Delta x$.

Since *A* is approximating Δf, we can also say that $\Delta f \approx f'(x) \cdot \Delta x$.
This means that the change in *f* (when moving from *x* to $x + \Delta x$) can be approximated by multiplying the slope $f'(x)$ by Δx, the change in *x*.

This leads to the definition of the **differential of *f*.**

FORMULA TO KNOW

Differential of *f*

$df = f'(x)dx$ for any choice of *x* and any real number *dx*.

When $y = f(x)$, we can also write $dy = f'(x)dx$.

HINT

The differential uses the derivative at an x-value to give the approximate change in *f* when *x* changes to $x + \Delta x$.

When approximating the change in *y*, we use $dx = \Delta x$, but *dy* is an approximation of Δy.

2. Calculating the Differential of *f*

EXAMPLE Given $f(x) = 4x^2 + 7x$, find the differential *df*.

Since $f'(x) = 8x + 7$, the differential is $df = (8x + 7)dx$.

EXAMPLE Given $y = \ln(x^2 + 3)$, find the differential *dy*.

Since $\frac{dy}{dx} = \frac{1}{x^2+3} \cdot 2x = \frac{2x}{x^2+3}$, the differential is $dy = \frac{2x}{x^2+3}dx$.

TRY IT

Let $f(x) = x^2 \sin(2x)$.

Find the differential *df*. +

To find the differential, we need to find the derivative of *f*, which requires the product rule:

$$\begin{aligned} f'(x) &= \sin(2x) \cdot D[x^2] + x^2 \cdot D[\sin(2x)] \\ &= \sin(2x) \cdot 2x + x^2(\cos(2x) \cdot D[2x]) \\ &= \sin(2x) \cdot 2x + x^2(\cos(2x) \cdot 2) \\ &= 2x\sin(2x) + 2x^2\cos(2x) \end{aligned}$$

Then, the differential of *f* is $df = f'(x)dx = (2x\sin(2x) + 2x^2\cos(2x))dx$.

WATCH

Here is a video in which we find the differential *dy* of $y = e^{-4x}\cos(7x)$.

SUMMARY

In this lesson, you learned how to **define and calculate the differential of *f***, which is an approximation for the change in *f* when *x* changes by *dx* units.

FORMULAS TO KNOW

Differential of f

$df = f'(x)dx$ for any choice of *x* and any real number *dx*.

When $y = f(x)$, we can also write $dy = f'(x)dx$.