Definition of Derivative

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WHAT'S COVERED

In this lesson, you will algebraically compute the derivative function by using the limit definition of the derivative. Specifically, this lesson will cover:

1. Finding Derivatives by Limit Definition

In the previous part of this challenge, we estimated the derivative of a function graphically at specific points on the graph of $y=f(x)$. We also noted that the derivative is itself a function since its value changes as *x* changes (and there is only one slope possible for a given x-value). Thus, it is important to be able to reliably compute the derivative function for a given function, which will be first done through limits and algebra.

The derivative of $f(x)$ is another function written $f'(x)$ and called "*f* prime of *x*." It represents the slope of the tangent line to $f(x)$ at a specific value of *x*.

Recall in earlier challenges that the slope of the secant line between the points $(x, f(x))$ and $(x+h, f(x+h))$ is given by the expression $\frac{f(x+h)-f(x)}{h}$. This can also be expressed as $\frac{\Delta f}{\Delta x}$ or $\frac{\Delta y}{\Delta x}$ to emphasize that this is the ratio of the change in *y* to the change in *x*. Then, the slope of the tangent line is found by letting *h* get closer to 0.

FORMULA TO KNOW

Limit Definition of Derivative

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The notation $f'(x)$ is used to emphasize that $f'(x)$ is related to the function $f(x)$. There are also other notations that are used in conjunction with taking the derivative $f(x)$.

Derivative Notation	Description
$f'(x)$	This is the most common notation and is called "*f* prime of *x*."
$D[f(x)]$	The "D" emphasizes that the derivative operation is being applied to $f(x)$.
$\frac{d}{dx}[f(x)]$	The "$\frac{d}{dx}$" emphasizes that the derivative operation is being applied to $f(x)$. The "*dx*" means that the derivative is taken with respect to *x*.
$\frac{df}{dx}$	This notation helps to visualize that the derivative is $\lim_{h \to 0} \frac{\Delta f}{\Delta x}$.
$\frac{dy}{dx}$	Similar to $\frac{df}{dx}$, this notation helps to visualize that the derivative is $\lim_{h \to 0} \frac{\Delta y}{\Delta x}$.
y'	This is called "y prime," an alternate notation to $\frac{dy}{dx}$.

HINT

Any letter could be used for the independent variable, and any name can be used for a function. For example, given $y = g(t)$, we could write $\frac{dy}{dt}$, $g'(t)$, etc.

There are several notations used for the derivative, so you may see each of them used throughout the rest of this course.

Let's now use the limit definition to find the derivative of a popular function, $f(x) = x^2$.

EXAMPLE Use the limit definition of derivative to find $f'(x)$ when $f(x) = x^2$. The limit definition is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

First, compute $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$.

Then, evaluate the limit:

$f'(x) = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$	Replace $f(x+h)$ with $x^2 + 2hx + h^2$ and $f(x)$ with x^2.
$= \lim_{h \to 0} \frac{2hx + h^2}{h}$	Combine like terms in the numerator.
$= \lim_{h \to 0} \left(\frac{2hx}{h} + \frac{h^2}{h} \right)$	Separate the fractions.
$= \lim_{h \to 0} (2x + h)$	Remove the common factor of *h* in each fraction.
$= 2x$	Substitute 0 for *h*.

Thus, if $f(x)=x^2$, then $f'(x)=2x$. This means that the slope of the tangent line to the curve $f(x)=x^2$ is $m_{\tan}=2x$.

Let's check this with the results we got in the previous challenge. By using the graph, we estimated the slope of the tangent line at $x=1$ to be 2.

Since $f'(x)=2x$, evaluate this function when $x=1$ to get the slope. Since $f'(1)=2(1)=2$, this tells us that the slope of the tangent line is 2.

From the "Try It" in tutorial 3.1.1, you (hopefully) estimated the slope to be 2 when $x=1$. Substituting into the derivative, we have $f'(1)=2(1)=2$, which checks.

EXAMPLE Find the derivative of $f(x)=2x^3$ by using the limit definition. As we know, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

First, note that $f(x+h)=2(x+h)^3=2(x^3+3hx^2+3h^2x+h^3)=2x^3+6hx^2+6h^2x+2h^3$.

Now, let's replace the function notations with expressions and evaluate the limit:

$f'(x)=\lim\limits_{h\to 0}\dfrac{2x^3+6hx^2+6h^2x+2h^3-2x^3}{h}$	Replace $f(x+h)$ with $2x^3+6hx^2+6h^2x+2h^3$ and $f(x)$ with $2x^3$.
$=\lim\limits_{h\to 0}\dfrac{6hx^2+6h^2x+2h^3}{h}$	Combine like terms in the numerator.
$=\lim\limits_{h\to 0}\left(\dfrac{6hx^2}{h}+\dfrac{6h^2x}{h}+\dfrac{2h^3}{h}\right)$	Separate the fractions.
$=\lim\limits_{h\to 0}(6x^2+6hx+2h^2)$	Remove the common factor of h in each fraction.
$=6x^2$	Substitute 0 for h.

Thus, if $f(x)=2x^3$, then $f'(x)=6x^2$. This means that the slope of the tangent line to the curve $f(x)=2x^3$ is $m_{\tan}=6x^2$.

Having the function for the derivative is an advantage since estimating the slope at a specific point can be difficult. In the previous example, we found that $f'(x)=6x^2$ when $f(x)=2x^3$. Let's say we want the slope of the tangent line to $f(x)$ when $x=2$. According to the derivative function, $f'(2)=6(2)^2=24$.

A slope of 24 might be difficult to obtain visually without a very carefully drawn graph. This is why having a function is important.

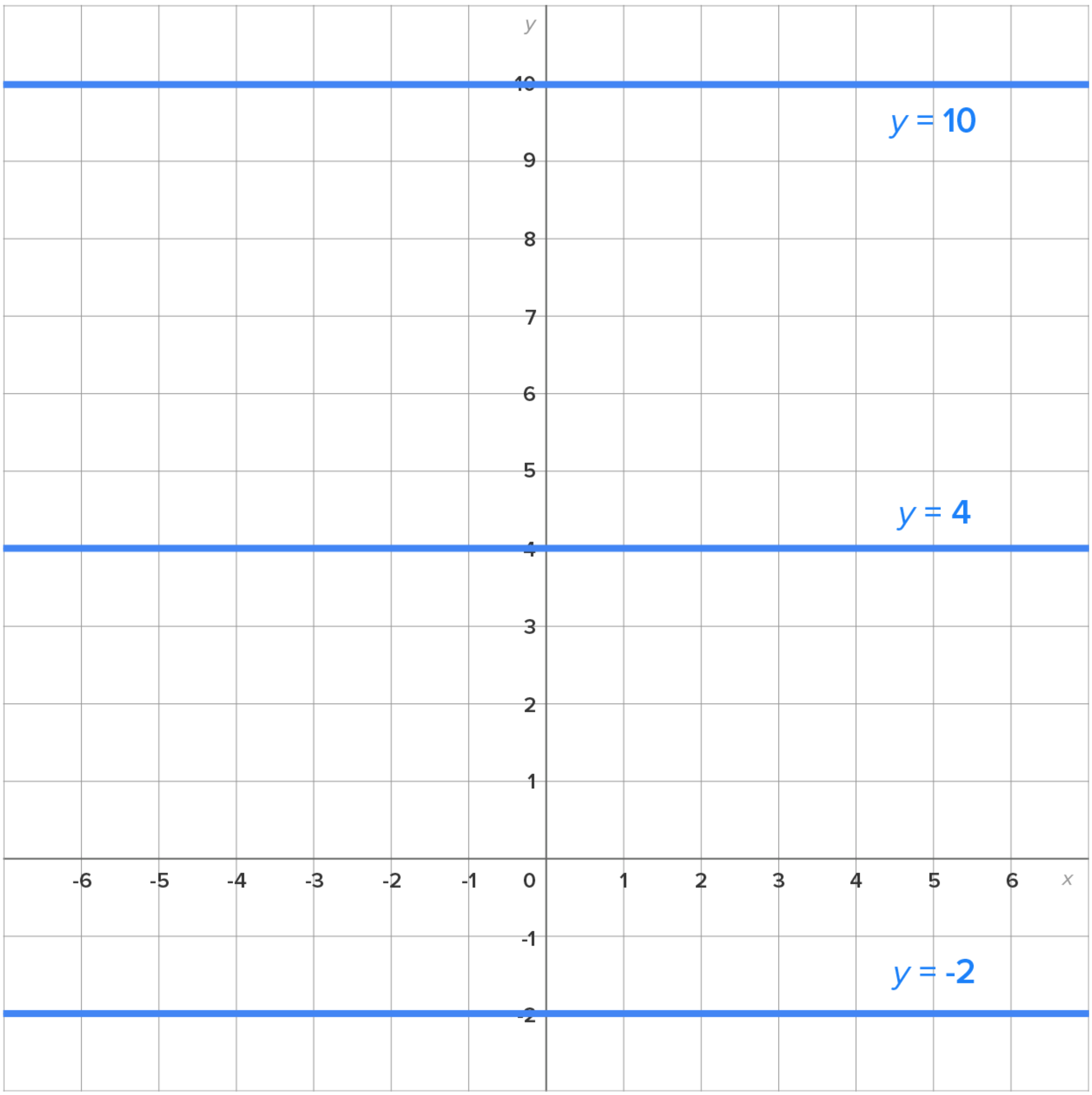
WATCH

The following video illustrates how to use the limit definition to find the derivative of $f(x) = \frac{4}{x+3}$.

Now, let's establish some general rules for derivatives of functions that have certain forms.

2. The Derivative of a Constant Function *f*(*x*) = *k*

If you graph the function $f(x) = k$ for any constant *k*, you would notice they all have something in common: their slopes are 0. Shown below are the graphs of *y* = -2, *y* = 4, and *y* = 10.



When speaking of the derivative of each function, it stands to reason that the derivative of each function is 0. Let's show this using the limit definition:

Let $f(x)=k$. Now, evaluate the limit:

$f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$	Apply the limit definition of a derivative.
$=\lim\limits_{h\to 0}\dfrac{k-k}{h}$	Since $f(x)=k$, a constant, it follows that $f(x+h)=k$. Replace both $f(x)$ and $f(x+h)$ with k.
$=\lim\limits_{h\to 0}\dfrac{0}{h}$	Simplify the numerator.
$=\lim\limits_{h\to 0} 0$	The limit implies that $h\neq 0$, therefore $\dfrac{0}{h}=0$.
$=0$	The limit of a constant is the constant.

BIG IDEA

Thus, when $f(x)=k$, $f'(x)=0$. We often say "the derivative of a constant is 0."

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3. The Derivative of a Linear Function *f*(*x*) = *mx* + *b*

Since the slope of a linear function $f(x)=mx+b$ is m, it follows that the derivative of a linear function should also be m (since this is also the slope of a tangent line).

Let's use the limit definition of derivative to establish this:

Let $f(x)=mx+b$. Then, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

$=\lim\limits_{h\to 0}\dfrac{mx+mh+b-(mx+b)}{h}$	$f(x+h)=m(x+h)+b=mx+mh+b$ Also replace $f(x)$ with $mx+b$.
$=\lim\limits_{h\to 0}\dfrac{mh}{h}$	Simplify the numerator.
$=\lim\limits_{h\to 0} m$	Remove the common factor of h.
$=m$	The limit of a constant is the constant.

Thus, when $f(x)=mx+b$ (a linear function), its derivative is $f'(x)=m$.

TRY IT

Find each derivative function:

D [9] +

Since 9 is a constant expression, $D[9] = 0$.

D [2*x* + 9] +

Since $2x + 9$ is a linear expression, it follows that $D[2x + 9] = 2$ (which is its slope).

D [17 - 4.2*x*] +

Since $17 - 4.2x$ is a linear expression, it follows that $D[17 - 4.2x] = -4.2$ (which is its slope).

SUMMARY

In this lesson, you learned that the limit definition of the derivative produces the derivative function analytically (as opposed to estimating the derivative at specific points graphically). You used this knowledge to **find derivatives by the limit definition**. You learned that the derivative function can be used to easily calculate the slopes of tangent lines for given values of *x*. You also learned special derivative rules that can be used to find **the derivative of a constant function** $f(x) = k$ and **the derivative of a linear function** $f(x) = mx + b$.

FORMULAS TO KNOW

Limit Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$