Concavity

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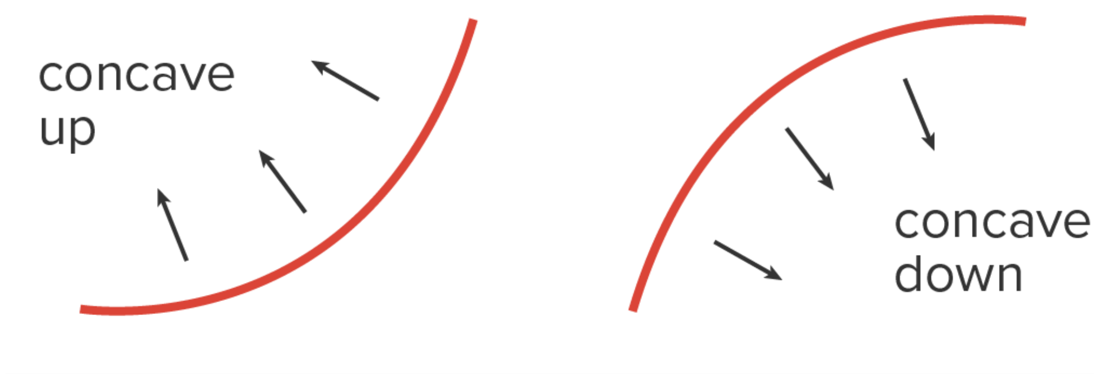
WHAT'S COVERED

In this lesson, you will learn how to use the second derivative to determine the direction that the graph of a function opens, also known as its concavity. Specifically, this lesson will cover:

1. Defining Concavity
2. Determining Where a Function Is Concave Up/Concave Down

1. Defining Concavity

Concavity refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward. A graph is **concave up** on an interval if it opens upward on that interval. A graph is **concave down** on an interval if it opens downward on that interval.



WATCH

This video shows how concavity relates to how slopes of tangent lines change.

BIG IDEA

Based on the video, we make the following observations:

- If $f''(x) > 0$ on an interval, then the graph of $f(x)$ is concave up on the same interval.
- If $f''(x) < 0$ on an interval, then the graph of $f(x)$ is concave down on the same interval.

HINT

Remember that a function can change between positive and negative when it is either equal to 0 or when it is undefined. Therefore, to determine where the graph of the function is concave up or concave down, find all values where $f''(x) = 0$ or $f''(x)$ is undefined. Then, make a sign graph similar to what you did for the first derivative test.

TERMS TO KNOW

Concavity

Refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward.

Concave Up

When a graph opens upward on an interval.

Concave Down

When a graph opens downward on an interval.

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2. Determining Where a Function Is Concave Up/Concave Down

EXAMPLE Determine the interval(s) over which the graph of $f(x) = x^3 - 3x^2 + 5$ is concave up or concave down. Since concavity is determined from the second derivative, we start there.

$f(x) = x^3 - 3x^2 + 5$	Start with the original function.
$f'(x) = 3x^2 - 6x$	Take the first derivative.
$f''(x) = 6x - 6$	Take the second derivative.

Since $f''(x)$ is never undefined, we set it to 0 and solve:

$6x - 6 = 0$	The second derivative is set to 0.
$6x = 6$	Add 6 to both sides.

$x = 1$ Divide both sides by 6.

Thus, $f(x)$ could be changing concavity when $x = 1$. This means that at any x-value on the interval $(-\infty, 1)$, the concavity is the same. The same can be said for the interval $(1, \infty)$.

Now, select one number (called a test value) inside each interval to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 1)$	$(1, \infty)$
Test Value	0	2
Value of $f''(x) = 6x - 6$	-6	6
Behavior of $f(x)$	Concave down	Concave up

Therefore, the graph of $f(x)$ is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

EXAMPLE Determine the interval(s) over which the graph of $f(x) = 5x^2 - 18x^{5/3}$ is concave up or concave down. Note that the domain of $f(x)$ is all real numbers.

Since concavity is determined from the second derivative, we start there.

$f(x) = 5x^2 - 18x^{5/3}$ Start with the original function.

$f'(x) = 10x - 18 \cdot \frac{5}{3}x^{2/3}$ Take the first derivative.

$= 10x - 30x^{2/3}$

$f''(x) = 10 - 30\left(\frac{2}{3}\right)x^{-1/3}$ Take the second derivative.

$= 10 - 20x^{-1/3}$

$= 10 - \frac{20}{x^{1/3}}$

Note that $f''(x)$ is undefined when $x = 0$.

To find other possible transition points, set $f''(x) = 0$ and solve:

$10 - \frac{20}{x^{1/3}} = 0$ The second derivative is set to 0.

$10x^{1/3} - 20 = 0$ Multiply everything by $x^{1/3}$.

$$10x^{1/3} = 20 \quad \text{Add 20 to both sides.}$$

$$x^{1/3} = 2 \quad \text{Divide both sides by 10.}$$

$$x = 8 \quad \text{Cube both sides.}$$

Thus, $f(x)$ could be changing concavity when $x = 0$ or $x = 8$. This means that at any x-value on the interval $(-\infty, 0)$, the concavity is the same. The same can be said for the intervals $(0, 8)$ and $(8, \infty)$.

Now, select one number (called a test value) inside each interval to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, 8)$	$(8, \infty)$
Test Value	-1	1	27
Value of $f''(x) = 10 - \frac{20}{x^{1/3}}$	30	-10	$\frac{10}{3}$
Behavior of $f(x)$	Concave up	Concave down	Concave up

Thus, the graph of $f(x)$ is concave up on $(-\infty, 0) \cup (8, \infty)$ and concave down on the interval $(0, 8)$.

WATCH

In this video, we'll determine the intervals over which the function $f(x) = \ln(x^2 + 1)$ is concave up or concave down.

Here is a problem for you to try, step by step. This will also help you review some algebra skills.

TRY IT

Consider the function $f(x) = -x^5 + 5x^4 + 13x - 12$.

Find the first and second derivatives of *f*(*x*). +

$f'(x) = -5x^4 + 20x^3 + 13$ and $f''(x) = -20x^3 + 60x^2$

Find all values of *x* for which *f*(*x*)=0. +

$$-20x^3 + 60x^2 = 0 \quad \text{Set } f''(x) = 0.$$

$$-20x^2(x - 3) = 0 \quad \text{Remove common factor of } -20x^3.$$

$-20x^2 = 0$ or $x - 3 = 0$ Set each factor equal to 0.

$x = 0$ or $x = 3$ Solve each equation.

Make a sign graph of the second derivative, using appropriate test values. +

Test values were chosen to ease calculations. For each interval, any value could be used. For example, a test value of "-5" could have have been used on the interval $(-\infty, 0)$.

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
Test Value	-1	1	4
Value of $f''(x) = -20x^3 + 60x^2$	80	40	-320
Behavior of $f(x)$	Concave up	Concave up	Concave down

Find all intervals over which *f*(*x*) is concave up. +

According to the sign graph, $f(x)$ is concave up on the intervals $(-\infty, 0)$ and $(0, 3)$. Since $f(x)$ is defined when $x = 0$, we say that $f(x)$ is concave up on the interval $(-\infty, 3)$.

Find all intervals over which *f*(*x*) is concave down. +

According to the sign graph, $f(x)$ is concave down on the interval $(3, \infty)$.

SUMMARY

In this lesson, you learned that **concavity is defined** as the direction in which a graph opens, noting that a graph is concave up if it opens upward on an interval and concave down if it opens downward on an interval. You also learned that you can **determine where a function is concave up/concave down** by using the second derivative of the function $f(x)$.

TERMS TO KNOW

Concave Down
When a graph opens downward on an interval.

Concave Up
When a graph opens upward on an interval.

Concavity
Refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward.