The Area Between Two Curves That Intertwine

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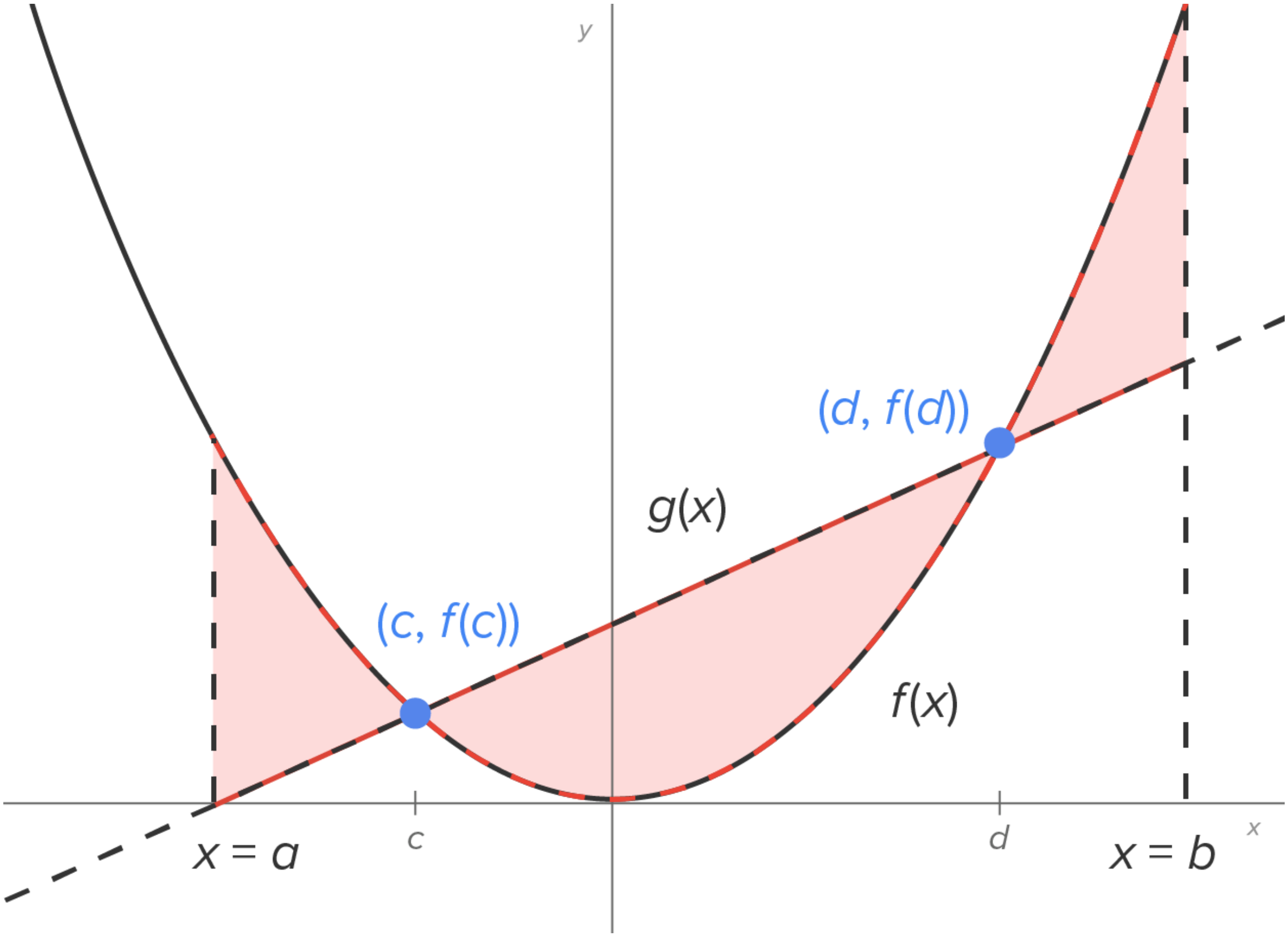
WHAT'S COVERED

In this lesson, you will learn how to find areas of regions when the curves intertwine. Specifically, this lesson will cover:

1. Introduction: A Strategy

Consider the region shown in the figure below.



The graph of $f(x)$ has a parabolic shape, and the graph of $g(x)$ is linear. Observe the following:

- On the interval $[a, c]$, the graph of $f(x)$ is above the graph of $g(x)$.
- On the interval $[c, d]$, the graph of $g(x)$ is above the graph of $f(x)$.
- On the interval $[d, b]$, the graph of $f(x)$ is above the graph of $g(x)$.

Recall from the last tutorial that $\int_a^b (f(x)-g(x))dx$ is the area of the region between two graphs as long as $f(x) \geq g(x)$, meaning that the graph of $f(x)$ is at least as high as the graph of $g(x)$ on $[a, b]$.
Considering the region again, it stands to reason that the integral expression for the total area is:

$$\int_a^c (f(x)-g(x))dx + \int_c^d (g(x)-f(x))dx + \int_d^b (f(x)-g(x))dx$$

This is our strategy for calculating the area of a region between two curves when the curves intertwine.

BIG IDEA

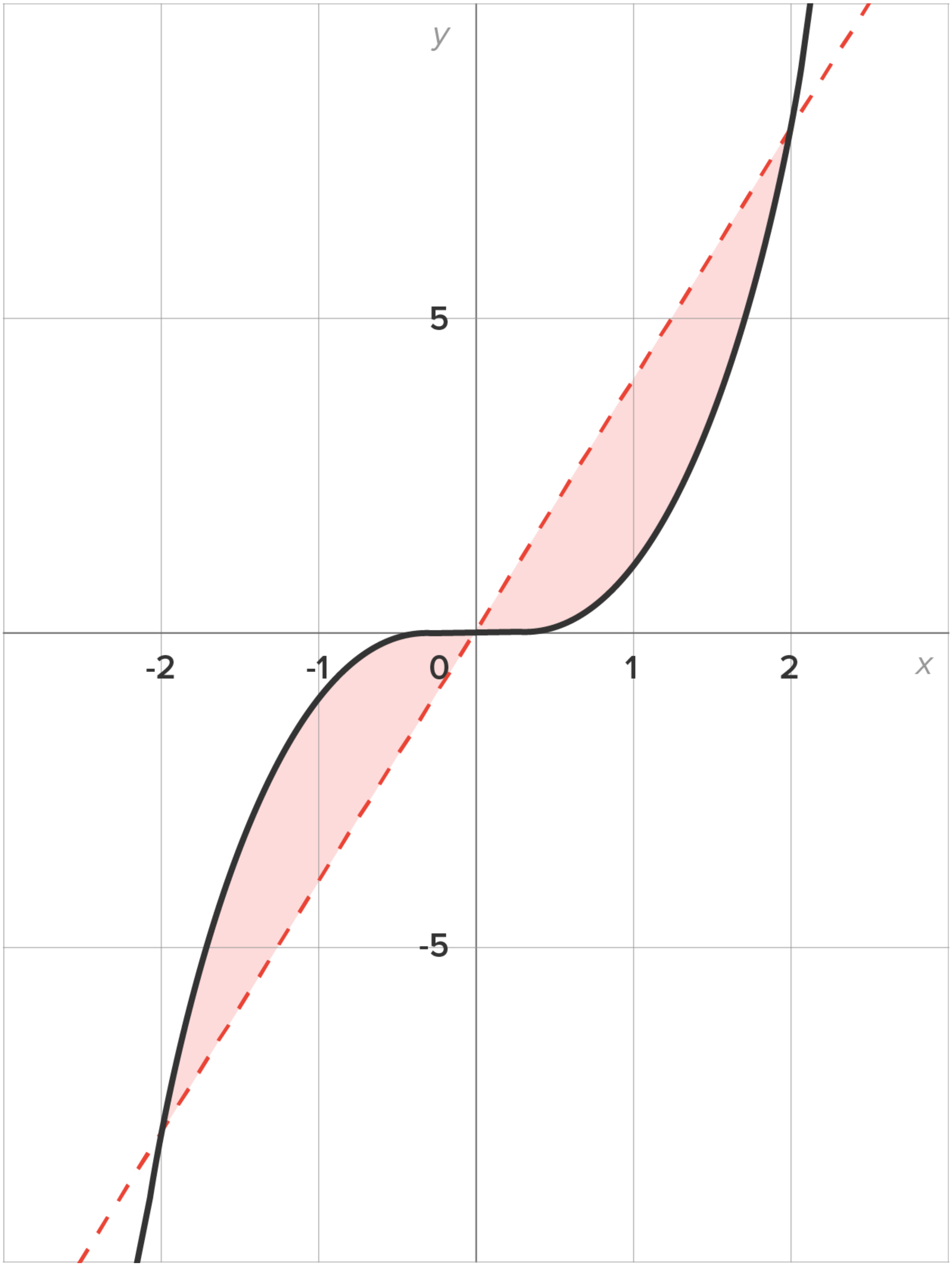
When finding the area of the region between two curves $y=f(x)$ and $y=g(x)$, keep the following in mind:

- If $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is $\int_a^b [f(x) - g(x)]dx$.
- If $f(x) \leq g(x)$ on $[a, b]$, then the area between the curves is $\int_a^b [g(x) - f(x)]dx$.

That is, the integrand is "upper – lower."

2. Finding Areas of Regions Between Two Curves That Intertwine

EXAMPLE Find the total area bounded by the graphs of $y = x^3$ and $y = 4x$. The graph of the region is shown in the figure below.



First, find the points where the graphs intersect:

$$x^3 = 4x$$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x(x+2)(x-2) = 0$$

$$x = 0,\ x = -2,\ x = 2$$

- On the interval $[-2, 0]$, the graph of $y = x^3$ is above the graph of $y = 4x$.
- On the interval $[2, 0]$, the graph of $y = 4x$ is above the graph of $y = x^3$.

Thus, the total area is $\int_{-2}^{0}(x^3 - 4x)dx + \int_{0}^{2}(4x - x^3)dx$.

Now evaluate each integral, starting with $\int_{-2}^{0}(x^3 - 4x)dx$.

$\int_{-2}^{0}(x^3 - 4x)dx$	Start with the first integral.
$= \left(\frac{1}{4}x^4 - 2x^2\right)\Big\vert_{-2}^{0}$	Apply the fundamental theorem of calculus. Note: $\int 4x dx = 4\left(\frac{1}{2}x^2\right) = 2x^2$
$= \left[\frac{1}{4}(0)^4 - 2(0)^2\right] - \left[\frac{1}{4}(-2)^4 - 2(-2)^2\right]$	Substitute the upper and lower endpoints.
$= 0 - [-4]$	Evaluate the parentheses.
$= 4$	Simplify.

The area of this part of the region is 4 units^2.

Now evaluate the second integral, $\int_{0}^{2}(4x - x^3)dx$.

$\int_{0}^{2}(4x - x^3)dx$	Start with the second integral.
$= \left(2x^2 - \frac{1}{4}x^4\right)\Big\vert_{0}^{2}$	Apply the fundamental theorem of calculus. Note: $\int 4x dx = 4\left(\frac{1}{2}x^2\right) = 2x^2$

$$= \left[2(2)^2 - \frac{1}{4}(2)^4\right] - \left[2(0)^2 - \frac{1}{4}(0)^4\right]$$ Substitute the upper and lower endpoints.

$$= 4 - 0$$ Evaluate the parentheses.

$$= 4$$ Simplify.

The area of this part of the region is 4 units^2.

Thus, the total area bounded by the graphs of $y = x^3$ and $y = 4x$ is $4 + 4 = 8 \text{ units}^2$.

WATCH

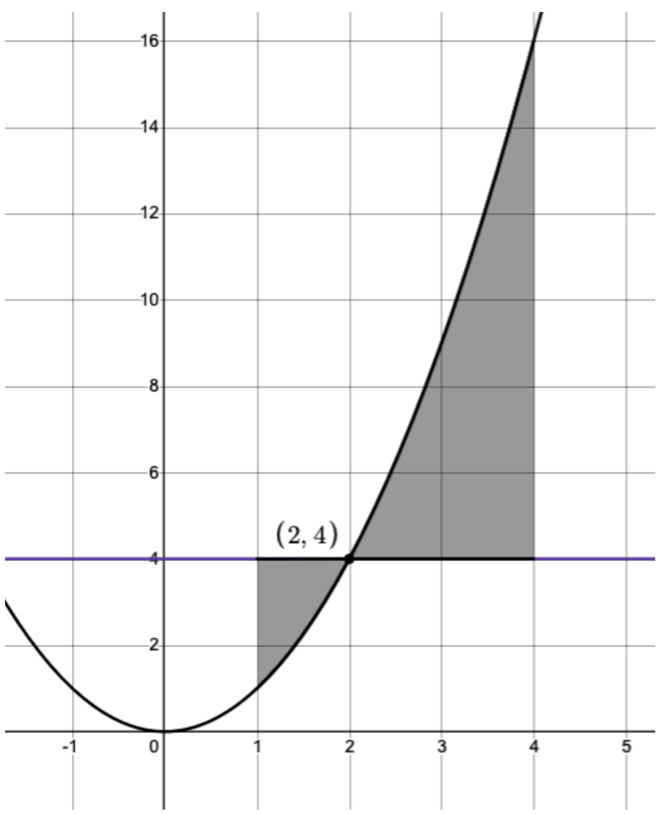
Here is another example of intertwining curves. In the video, we will find the total area between $f(x) = \frac{9}{x}$ and $g(x) = x$ over the interval $[1, 4]$.

TRY IT

Consider the region bounded by the graphs of $y = x^2$ and $y = 4$ between $x = 1$ and $x = 4$.

Find the exact area of the region. +

The graph of the region is shown below.



Between $x=1$ and $x=2$, the graph of $y=4$ is above the graph of x^2. The integral used to find the area is $\int_1^2 (4-x^2)dx$.

Evaluate the integral:

$=4x-\frac{1}{3}x^3\Big\vert_1^2$	$\int 4dx = 4x$ $\int x^2 dx = \frac{1}{3}x^3$
$=\left(4(2)-\frac{1}{3}(2)^3\right)-\left(4(1)-\frac{1}{3}(1)^3\right)$	Evaluate when $x=2$ and $x=1$, then subtract.
$=\frac{16}{3}-\frac{11}{3}$	Simplify within each group of parentheses.
$=\frac{5}{3}$	Simplify.

The area of the first region is $\frac{5}{3}$ square units.

Between $x=2$ and $x=4$, the graph of $y=4$ is below the graph of x^2. The integral used to find the area is $\int_2^4 (x^2-4)dx$.

Evaluate the integral:

$=\frac{1}{3}x^3-4x\Big\vert_2^4$	$\int x^2 dx = \frac{1}{3}x^3$ $\int 4dx = 4x$
$=\left(\frac{1}{3}(4)^3-4(4)\right)-\left(\frac{1}{3}(2)^3-4(2)\right)$	Evaluate when $x=4$ and $x=2$, then subtract.
$=\frac{16}{3}-\left(-\frac{16}{3}\right)$	Simplify within each group of parentheses.
$=\frac{32}{3}$	Simplify.

The area of the first region is $\frac{32}{3}$ square units.

In conclusion, the total area between the graphs of $y=4$ and $y=x^2$ between $x=1$ and $x=4$ is equal to $\frac{5}{3}+\frac{32}{3}=\frac{37}{3}$ square units.

SUMMARY

In this lesson, you learned the **strategy** for **finding the area of regions between two curves that intertwine**, which is a natural extension of the area computations that you investigated in the last tutorial. The main idea is to make sure that the integrand is “upper function – lower function” to ensure that the definite integral yields a positive result (area).