The General Power Rule for Functions

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WHAT'S COVERED

In this lesson, you will expand upon your derivative knowledge even further by examining powers of functions whose derivatives we know. For example, $f(x) = (3x+1)^5$ and $y = \sin^4 x$. This idea will also help in finding the derivatives of some other commonly used functions. Specifically, this lesson will cover:

1. Derivatives of Functions of the Form $y = [f(x)]^n$

Derivatives of powers of a function have several uses, as we will see once we get to applications of derivatives. To establish a pattern for this type of derivative, we'll consider the functions $y = f^2$, $y = f^3$, and $y = f^4$, where f is being used to represent some function $f(x)$.

First, consider the function $y = f^2 = f \cdot f$.

By the product rule, we have:

$$\begin{aligned} y' = D[f^2] &= D[f] \cdot f + f \cdot D[f] \\ &= f' \cdot f + f \cdot f' \\ &= 2f \cdot f' \end{aligned}$$

Now consider the function $y = f^3 = f^2 \cdot f$.

By the product rule again, we have:

$y' = D[f^3] = D[f^2] \cdot f + f^2 \cdot D[f]$	Apply the product rule.
$= (2f \cdot f') \cdot f + f^2 \cdot f'$	Replace $D[f^2]$ with $= 2f \cdot f'$.

$= 2f^2 \cdot f' + f^2 \cdot f'$ Combine $f \cdot f = f^2$.

$= 3f^2 \cdot f'$ Combine like terms.

Next, consider $y = f^4 = f^3 \cdot f$.

$D[f^4] = D[f^3] \cdot f + f^3 \cdot D[f]$ Apply the product rule.

$= (3f^2 \cdot f') \cdot f + f^3 \cdot f'$ Replace $D[f^3]$ with $= 3f^2 \cdot f'$.

$= 3f^3 \cdot f' + f^3 \cdot f'$ Combine $f^2 \cdot f = f^3$.

$= 4f^3 \cdot f'$ Combine like terms.

By looking at this pattern, it seems as though the derivative of f^n is $n \cdot f^{n-1}$ (looks like the power rule), but then also multiplied by f'.

FORMULA TO KNOW

General Power Rule for Derivatives of Functions

If $f(x)$ is some function, then $D[[f(x)]^n] = n \cdot [f(x)]^{n-1} \cdot f'(x)$.

EXAMPLE Earlier, we found the derivative of $f(x) = \cos^2 x$ by using the product rule. Let's use the power rule and compare.

First, note that this can be written as $f(x) = (\cos x)^2$.

By the power rule, we have the following:

$f'(x) = 2(\cos x) \cdot D[\cos x]$ Apply the power rule.

$= 2(\cos x)(-\sin x)$ $D[\cos x] = -\sin x$

$= -2\sin x \cos x$ Combine and eliminate parentheses.

This matches the answer obtained in challenge 3.2.4.

EXAMPLE Find the derivative of the function $f(x) = (5x + 1)^{10}$.

By the power rule, we have the following:

$f'(x) = 10(5x + 1)^9 \cdot D[5x + 1]$ Apply the power rule.

$= 10(5x + 1)^9(5)$ $D[5x + 1] = 5$

$= 50(5x+1)^9$ Combine $10 \cdot 5$.

A common mistake to make here is to multiply $50(5x+1)$ to get $250x + 50$, and subsequently $(250x+50)^9$. This is not correct since the $(5x+1)$ is raised to the 9th power and the 50 is not; therefore, they cannot be combined this way. The final answer is $f'(x) = 50(5x+1)^9$.

TRY IT

Consider the function $y = (x^2 - 9x + 20)^4$.

Find the derivative. +

The function is in the form $y = [f(x)]^4$, where $f(x) = x^2 - 9x + 20$.

Then, by the general power rule, the derivative of y is $\frac{dy}{dx} = 4[f(x)]^3 \cdot f'(x)$.

Since $f'(x) = 2x - 9$, we have $\frac{dy}{dx} = 4(x^2 - 9x + 20)^3(2x - 9)$.

Rearranging factors, this can also be written $\frac{dy}{dx} = 4(2x - 9)(x^2 - 9x + 20)^3$.

Remember the other expressions that can be written as powers of *x*.

EXAMPLE Find the derivative of the function $f(x) = \sqrt{3x^2 + 8}$.

Remember that $\sqrt{u} = u^{1/2}$. Then the power rule can be used.

$f(x) = \sqrt{3x^2+8} = (3x^2+8)^{1/2}$	Rewrite the radical using a power.
$f'(x) = \frac{1}{2}(3x^2+8)^{-1/2} \cdot D[3x^2+8]$	Use the power rule for derivatives.
$f'(x) = \frac{1}{2}(3x^2+8)^{-1/2} \cdot 6x$	$D[3x^2+8] = 6x$
$f'(x) = 3x(3x^2+8)^{-1/2}$	$\frac{1}{2} \cdot 6x = 3x$
$f'(x) = \frac{3x}{(3x^2+8)^{1/2}}$	Rewrite with nonnegative exponents.

Thus, $f'(x) = \frac{3x}{(3x^2+8)^{1/2}}$, which could also be written $f'(x) = \frac{3x}{\sqrt{3x^2+8}}$ if radical notation is desired.

EXAMPLE Find the derivative of the function $f(x) = \frac{1}{(5x+\cos x)^3}$.

$f(x) = \frac{1}{(5x+\cos x)^3} = (5x+\cos x)^{-3}$	Rewrite so that the power rule can be used.
$f'(x) = -3(5x+\cos x)^{-4} \cdot D[5x+\cos x]$	Apply the power rule.
$f'(x) = -3(5x+\cos x)^{-4} \cdot (5-\sin x)$	$D[5x+\cos x] = 5+(-\sin x) = 5-\sin x$
$f'(x) = -3(5-\sin x)(5x+\cos x)^{-4}$	Rearrange the factors.
$f'(x) = \frac{-3(5-\sin x)}{(5x+\cos x)^4}$	Rewrite with nonnegative exponents.

Thus, $f'(x) = \frac{-3(5-\sin x)}{(5x+\cos x)^4}$.

TRY IT

Consider the function $g(x) = \sqrt[3]{6x^4+5}$.

Find the derivative. +

First, rewrite as $g(x) = (6x^4+5)^{1/3}$.

The function is in the form $y = [f(x)]^{1/3}$, where $f(x) = 6x^4+5$.

Then, by the general power rule, $g'(x) = \frac{1}{3}[f(x)]^{-2/3} \cdot f'(x)$.

Since $f'(x) = 24x^3$, we have $g'(x) = \frac{1}{3}(6x^4+5)^{-2/3} \cdot 24x^3$.

Rearranging factors, this becomes $g'(x) = \frac{1}{3}(24x^3)(6x^4+5)^{-2/3}$, which simplifies to

$g'(x) = 8x^3(6x^4+5)^{-2/3}$.

Finally, writing in terms of nonnegative exponents: $g'(x) = \dfrac{8x^3}{(6x^4+5)^{2/3}}$

EXAMPLE The distance (measured in feet) from a moving camera to an object positioned at the point (1, 4) is given by the function $f(t) = \sqrt{2t^2 - 2t + 1}$, where t is measured in seconds. At what rate is the distance changing after 3 seconds?

Mathematically speaking, we want to compute $f'(3)$.

To find the derivative, we first need to rewrite $f(t)$:

$f(t) = \sqrt{2t^2 - 2t + 1} = (2t^2 - 2t + 1)^{1/2}$	Write the radical as $\frac{1}{2}$ power.
$f'(t) = \frac{1}{2}(2t^2 - 2t + 1)^{-1/2} \cdot D[2t^2 - 2t + 1]$	Apply the power rule.
$f'(t) = \frac{1}{2}(2t^2 - 2t + 1)^{-1/2} \cdot (4t - 2)$	$D[2t^2 - 2t + 1] = 4t - 2$
$f'(t) = \frac{1}{2}(4t - 2) \cdot (2t^2 - 2t + 1)^{-1/2}$	Rearrange the terms.
$f'(t) = (2t - 1) \cdot (2t^2 - 2t + 1)^{-1/2}$	Distribute $\frac{1}{2}(4t - 2) = 2t - 1$
$f'(t) = \dfrac{2t - 1}{(2t^2 - 2t + 1)^{1/2}}$	Rewrite with nonnegative exponents.

Now, we desire the rate of change when $t = 3$, so we substitute 3.

$$f'(3) = \frac{2(3) - 1}{(2(3)^2 - 2(3) + 1)^{1/2}} = \frac{5}{(13)^{1/2}} \approx 1.39 \text{ feet per second}$$

2. Combining Derivative Rules

Now that we are building up our derivative rules, we can find derivatives of more complex functions.

EXAMPLE Find the derivative of the function $f(x) = 4x\sqrt{2x + 1}$.

At this point, we are conditioned to write radicals as fractional powers (to use the power rule).

$f(x) = 4x\sqrt{2x + 1} = 4x(2x + 1)^{1/2}$	Rewrite the square root as $\frac{1}{2}$ power.

$f'(x) = D[4x]\cdot(2x+1)^{1/2} + 4x\cdot D\left[(2x+1)^{1/2}\right]$	Apply the product rule.
$f'(x) = 4\cdot(2x+1)^{1/2} + 4x\cdot\frac{1}{2}(2x+1)^{-1/2}(2)$	$D[4x] = 4$, $D\left[(2x+1)^{1/2}\right] = \frac{1}{2}(2x+1)^{-1/2}(2)$
$f'(x) = 4(2x+1)^{1/2} + 4x(2x+1)^{-1/2}$	$\frac{1}{2}\cdot 2 = 1$; remove excess symbols.
$f'(x) = 4(2x+1)^{1/2} + \frac{4x}{(2x+1)^{1/2}}$	Rewrite with positive exponents.

At this point, $f'(x)$ is reasonably simplified. Thus, $f'(x) = 4(2x+1)^{1/2} + \frac{4x}{(2x+1)^{1/2}}$.

It is possible to go further by forming a common denominator and combining the fractions. Let's see how this plays out:

$f'(x) = \frac{4(2x+1)^{1/2}}{1}\cdot\frac{(2x+1)^{1/2}}{(2x+1)^{1/2}} + \frac{4x}{(2x+1)^{1/2}}$	The common denominator is $(2x+1)^{1/2}$. Write $4(2x+1)^{1/2}$ over 1 so it "looks" like a fraction.
$f'(x) = \frac{4(2x+1)}{(2x+1)^{1/2}} + \frac{4x}{(2x+1)^{1/2}}$	Perform multiplication. $(2x+1)^{1/2}\cdot(2x+1)^{1/2} = (2x+1)^1 = 2x+1$
$f'(x) = \frac{8x+4}{(2x+1)^{1/2}} + \frac{4x}{(2x+1)^{1/2}}$	Distribute $4(2x+1) = 8x+4$.
$f'(x) = \frac{12x+4}{(2x+1)^{1/2}}$	Combine the numerators.

As you can see, the expression simplified nicely to one single fraction. That said, writing $f'(x) = 4(2x+1)^{1/2} + \frac{4x}{(2x+1)^{1/2}}$ is equally acceptable.

WATCH

Sometimes factoring is very useful in obtaining a nicer form of the derivative. In the following video, we'll take the derivative of $f(x) = (4x-1)^3(2x+5)^4$ and write it in factored form.

TRY IT

Consider the function $f(x) = (x+1)^4(2x+1)^3$.

Find the derivative and write your final answer in factored form. +

Note that $f(x)$ is a product of two functions: $(x+1)^4$ and $(2x+1)^3$.

Then:

$$
\begin{aligned}
f'(x) &= (2x+1)^3 \cdot D\left[(x+1)^4\right] + (x+1)^4 \cdot D\left[(2x+1)^3\right] \\
&= (2x+1)^3 \cdot 4(x+1)^3(1) + (x+1)^4 \cdot 3(2x+1)^2(2) \\
&= 4(2x+1)^3(x+1)^3 + 6(x+1)^4(2x+1)^2
\end{aligned}
$$

Now, look at the two large "terms" $4(2x+1)^3(x+1)^3$ and $6(x+1)^4(2x+1)^2$. The terms have a common factor of $2(x+1)^3(2x+1)^2$, so we factor this out:

$$f'(x) = 2(x+1)^3(2x+1)^2[2(2x+1) + 3(x+1)]$$

The quantity inside the brackets simplifies as follows:

$$
\begin{aligned}
&2(2x+1) + 3(x+1) \\
&= 4x + 2 + 3x + 3 \\
&= 7x + 5
\end{aligned}
$$

Thus, the final factored form is $f'(x) = 2(x+1)^3(2x+1)^2(7x+5)$.

SUMMARY

In this lesson, you learned how to apply the general power rule for **derivatives of functions, such as the form $y = [f(x)]^n$**. As you develop your repertoire of derivative formulas, you are able to **combine derivative rules** to find derivatives of more complex functions, such as the ones explored in this unit.

FORMULAS TO KNOW

General Power Rule for Derivatives of Functions

If $f(x)$ is some function, then $D\left[[f(x)]^n\right] = n \cdot [f(x)]^{n-1} \cdot f'(x)$.