The Average Value of a Continuous Function on a Closed Interval

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WHAT'S COVERED

In this lesson, you will learn about the average value of a continuous function over an interval $[a, b]$. Recall that the average of a set of numbers is the sum of the numbers, divided by the number of numbers. This takes on a different meaning for continuous functions. Specifically, this lesson will cover:

1. The Idea Behind Average Value
2. Computing the Average Value of a Continuous Function

1. The Idea Behind Average Value

When finding the average of a set of numbers, you add up all the numbers, then divide by how many numbers there are.

> EXAMPLE Given the numbers 81, 85, 89, and 71, the average of these four numbers is $\frac{81+85+89+71}{4}=81.5$.

Now consider a function $y=f(x)$ on some interval $[a, b]$. Break up the interval $[a, b]$ into *n* equal subintervals. Then, select a value of *x* from each subinterval. Call these values $x_1, x_2, \ldots, x_n$.

Then, the average of these values is $\frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}=\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{1}{n}\right]$.

The summation resembles a Riemann sum, but the Δx term is missing inside the summation. Recall that $\Delta x=\frac{b-a}{n}$.

We can multiply the summation by $\frac{b-a}{b-a}$ as follows:

$$\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{b-a}{b-a}\,\frac{1}{n}\right]$$

$$=\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{b-a}{n}\,\frac{1}{b-a}\right]$$

We replace $\frac{b-a}{n}$ with Δx:

$$=\sum_{k=1}^{n}\left[f(x_k)\cdot\Delta x\cdot\frac{1}{b-a}\right]$$

Since $\frac{1}{b-a}$ is a constant, it can be factored out and written in front of the summation:

$$\frac{1}{b-a}\sum_{k=1}^{n}[f(x_k)\cdot\Delta x]$$

Recall that the summation $\sum_{k=1}^{n}[f(x_k)\cdot\Delta x]$ approaches the value of $\int_a^b f(x)dx$ as $n\to\infty$ as long as $f(x)$ is integrable on $[a, b]$. Since we are assuming $f(x)$ is continuous on $[a, b]$, $f(x)$ is also integrable on $[a, b]$. Note that the summation for the average value is the Riemann sum for $f(x)$ but multiplied by $\frac{1}{b-a}$.

This leads to an integral formula to find the average value of a continuous function $f(x)$ on an interval $[a, b]$.

FORMULA TO KNOW

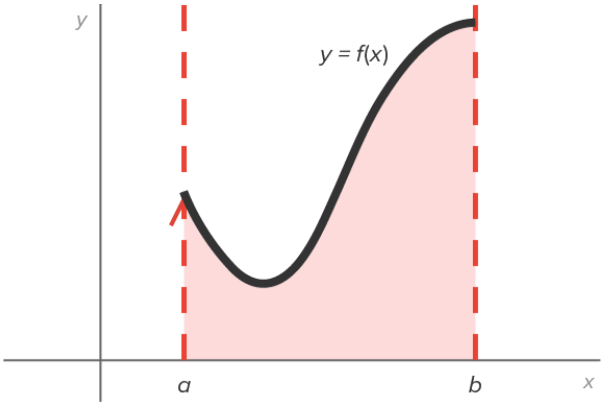
Average Value of a Function

If $f(x)$ is continuous on the closed interval $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is

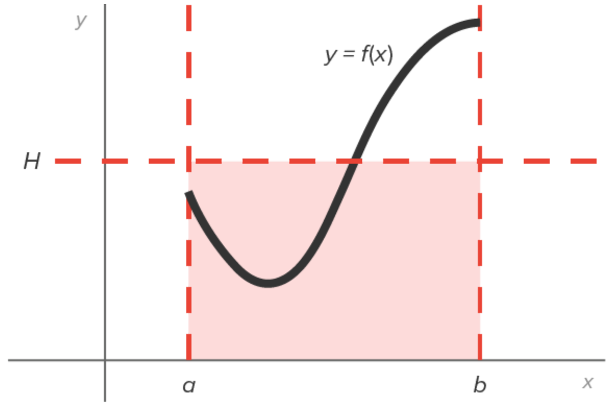
$$\frac{1}{b-a}\int_a^b f(x)dx.$$

BIG IDEA

For a geometric interpretation of average value, let $H=$ the average value of a nonnegative function $f(x)$ on $[a, b]$. The figure below shows an illustration of this.



(1)



(2)

- The graph in (1) is the region bounded by the graph of $f(x)$ and the x-axis on $[a, b]$.
- The graph in (2) is the rectangle with an area equal to $\int_a^b f(x)dx$. Note that the base is $b-a$, and its height is *H*, where *H* is the average value of $f(x)$ on $[a, b]$.

The area of the rectangle with height *H* and width $b-a$ is equal to the area of the region bounded by the graph of $f(x)$ and the x-axis on $[a, b]$.

2. Computing the Average Value of a Continuous Function

Now that we have a formula for average value, let's compute and interpret average values.

EXAMPLE Find the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

From the formula, this is equal to $\frac{1}{\pi - 0}\int_0^\pi \sin x dx = \frac{1}{\pi}\int_0^\pi \sin x dx$.

Now, we evaluate the definite integral:

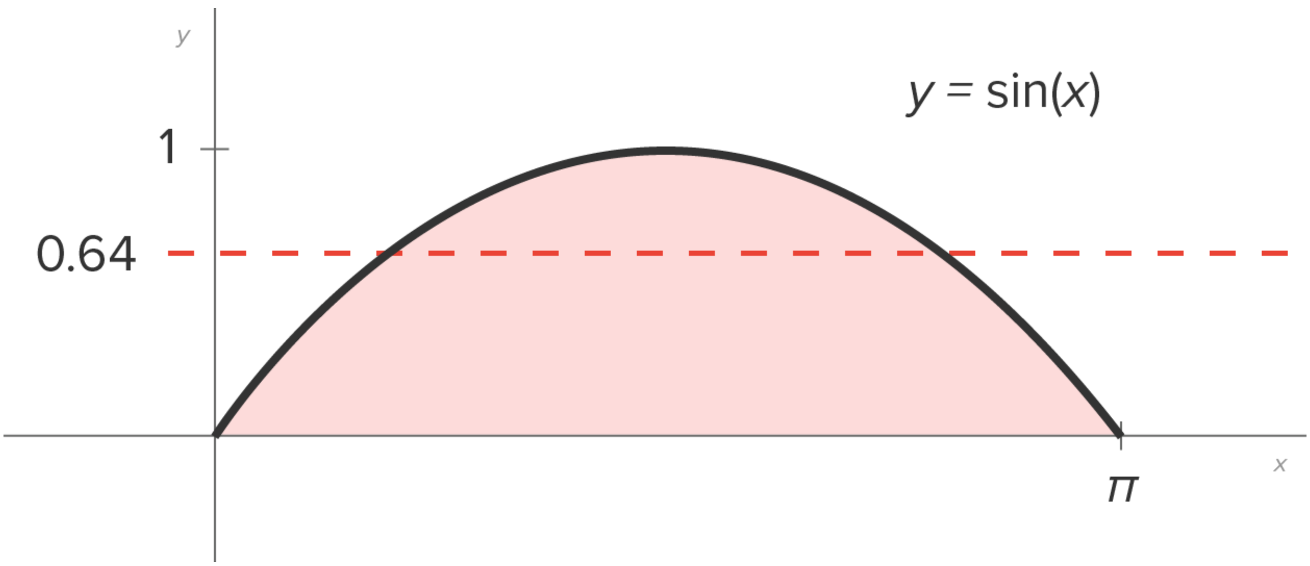
$\frac{1}{\pi}\int_0^\pi \sin x dx$	Start with the original expression.
$= \frac{1}{\pi}(-\cos x)\Big\vert_0^\pi$	Apply the fundamental theorem of calculus.
$= \frac{1}{\pi}(-\cos\pi) - \frac{1}{\pi}(-\cos 0)$	Substitute the upper and lower endpoints.

$$= \frac{1}{\pi} + \frac{1}{\pi}$$ Evaluate the parentheses.

$$= \frac{2}{\pi}$$ Simplify.

The average value of $f(x) = \sin x$ on the interval $[0, \pi]$ is equal to $\frac{2}{\pi}$.

To see the geometric interpretation, here is the graph of the region bounded by $f(x) = \sin x$ and the x-axis on the interval $[0, \pi]$ and the rectangle whose height is the average value and whose width is π. Note: $\frac{2}{\pi} \approx 0.64$



WATCH

Find the average value of $f(x) = \frac{15x}{x^2+1}$ on the interval $[0, 5]$.

TRY IT

Consider the function $f(x) = x^2 + 2$.

Find the average value of *f*(*x*) on the interval [0, 4]. +

The average value of $f(x) = x^2 + 2$ on the interval $[0, 4]$ is found by calculating

$$\frac{1}{4-0}\int_0^4 (x^2+2)dx = \frac{1}{4}\int_0^4 (x^2+2)dx.$$

Evaluate the integral:

$$= \frac{1}{4}\left(\frac{1}{3}x^3 - 2x\right)\Big|_0^4$$ $\int x^2 dx = \frac{1}{3}x^3$

$\int 2 dx = 2x$

$$= \frac{1}{4}\left[\left(\frac{1}{3}(4)^3 - 2(4)\right) - \left(\frac{1}{3}(0)^3 - 2(0)\right)\right]$$ Evaluate when $x = 4$ and $x = 0$, then subtract.

$$= \frac{1}{4}\left[\left(\frac{88}{3}\right) - 0\right]$$ Simplify within each group of parentheses.

$$= \frac{22}{3}$$ Simplify.

The average value of $f(x) = x^2 + 2$ on the interval $[0, 4]$ is equal to $\frac{22}{3}$.

TRY IT

Consider the function $f(x) = \frac{4}{x^2}$.

Find the average value of *f*(*x*) on the interval [1, 2]. +

The average value of $f(x) = \frac{4}{x^2}$ on the interval $[1, 2]$ is found by calculating $\frac{1}{2-1}\int_1^2 \frac{4}{x^2} dx$.

Evaluate the integral:

$$= \int_1^2 4x^{-2} dx$$ $\frac{1}{2-1} = \frac{1}{1} = 1$, so there is no need to have a constant outside the integral.

Also, rewrite $\frac{4}{x^2}$ as $4x^{-2}$ so that the power rule can be used.

$$= -4x^{-1}\Big|_1^2$$ Find the antiderivative using the power rule.

$\int 4x^{-2} dx = 4\left(\frac{1}{-1}\right)x^{-1} = -4x^{-1}$

$$= (-4(2)^{-1}) - (-4(1)^{-1})$$ Substitute $x = 2$ and $x = 1$, then subtract.

$= (-2) - (-4)$ Simplify within each group of parentheses. $2^{-1} = \frac{1}{2}$ and $1^{-1} = 1$.

$= 2$ Simplify.

The average value of $f(x) = \frac{4}{x^2}$ on the interval $[1, 2]$ is equal to 2.

In each case, the units of $f(x)$ and the units of the average value of $f(x)$ are the same. So if $f(x)$ was measured in feet, then the average value would also be measured in feet.

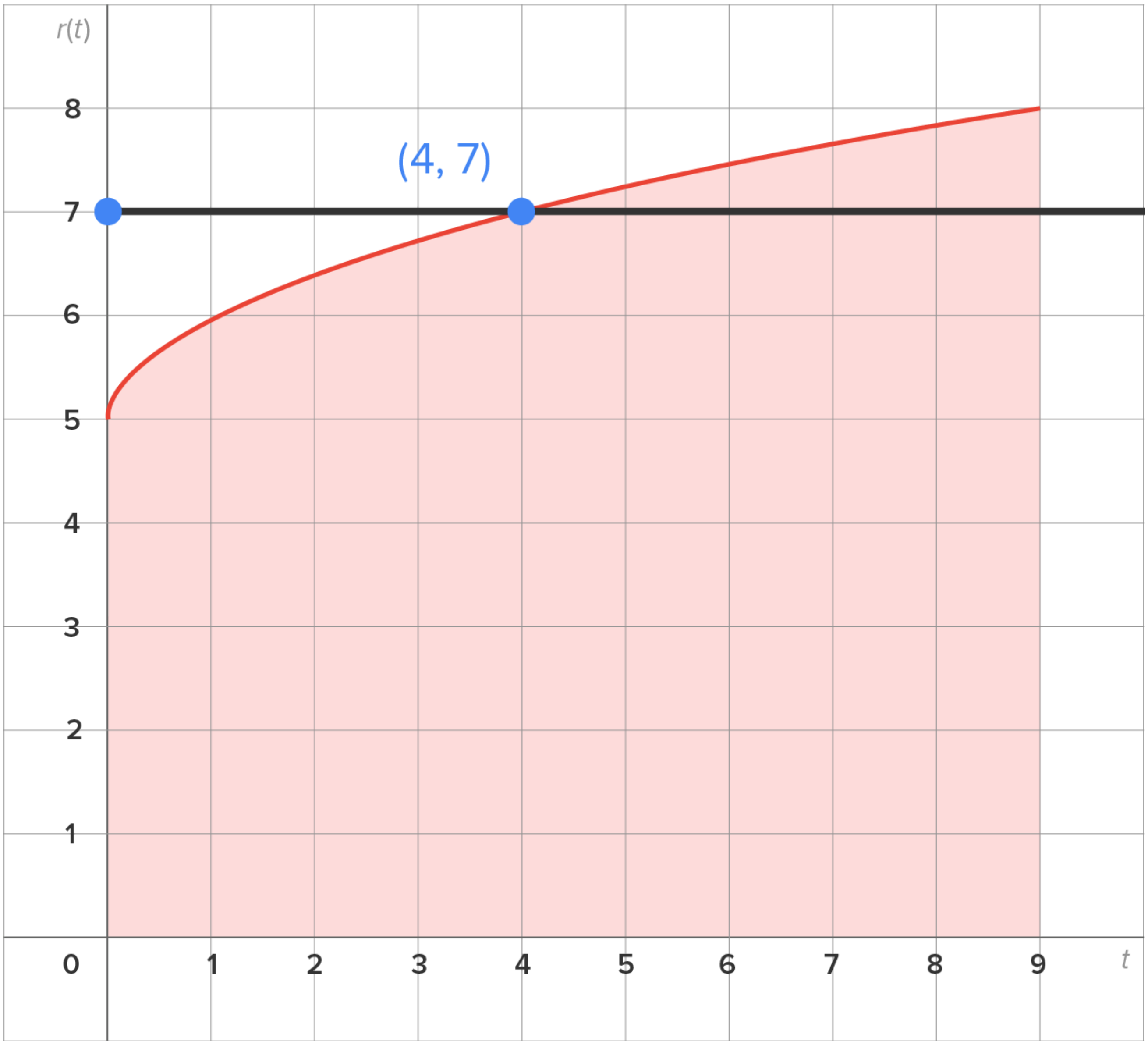
➦ EXAMPLE During a 9-hour workday, the production rate at time t hours is $r(t) = 5 + \sqrt{t}$ cars per hour. What is the average hourly production rate?

We seek the average value of $r(t)$ over the interval $[0, 9]$.

$\text{Average Value} = \frac{1}{9-0}\int_0^9 (5 + t^{1/2})dt$	Start with the original expression. Rewrite $\sqrt{t} = t^{1/2}$ to be able to use the power rule.
$= \frac{1}{9}\left(5t + \frac{2}{3}t^{3/2}\right)\Big\|_0^9$	Apply the fundamental theorem of calculus.
$= \frac{1}{9}\left[5(9) + \frac{2}{3}(9)^{3/2}\right] - \frac{1}{9}\left[5(0) + \frac{2}{3}(0)^{3/2}\right]$	Substitute the upper and lower endpoints.
$= \frac{1}{9}(45 + 18) - \frac{1}{9}(0)$	Evaluate.
$= 7$	Simplify.

The average rate of production is 7 cars per hour.

Shown in the figure is the region between $r(t) = 5 + \sqrt{t}$ and the t-axis, as well as the horizontal line $r(t) = 7$. Note that the area between $r(t)$ and the t-axis is equal to the area of the rectangle with the same base (9) and height 7 (the average value).



SUMMARY

In this lesson, you began by understanding **the idea behind average value**, following the path from the formula to find the average of a set of numbers to an integral formula to find the average value of a continuous function $f(x)$ on an interval $[a, b]$. You also learned how the fundamental theorem of calculus can be used to **compute the average value of a continuous function** $f(x)$ on an interval $[a, b]$.

FORMULAS TO KNOW

Average Value of a Function

If $f(x)$ is continuous on the closed interval $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.