The Intuitive Approach

by Sophia

WHAT'S COVERED

In this lesson, you will demonstrate the definition of a limit by finding the value of δ that corresponds to a given ε for a specific limit. Specifically, this lesson will cover:

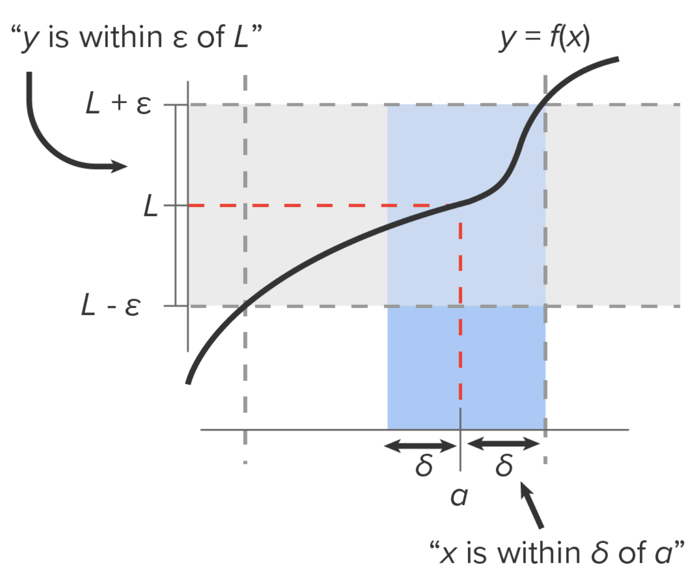
1. Finding the Value of δ That Corresponds to a Given Value of ε for a Linear Function

Recall a general limit statement: $\lim_{x \to a} f(x) = L$

Based on methods we talked about in this course so far, the general idea is that the value of $f(x)$ gets closer to L as x gets closer to a.

We now take a more analytical approach to establishing limits. Consult the figure on the right:

- The symbol ε is the Greek letter epsilon.
- The symbol δ is the Greek letter delta.



The idea illustrated here is that if the value of $f(x)$ is within ε units of the limit L, then there is a corresponding value of δ such that x is within δ units of a.
Written as distances, we have the following:

- $f(x)$ is within ε units of the limit *L*: $|f(x)-L|<\varepsilon$
- *x* is within δ units of a: $|x-a|<\delta$

These ideas are used to establish the **Formal Definition of a Limit**, which states:
$\lim_{x \to a} f(x)=L$ means that for every given $\epsilon>0$, there exists $\delta>0$ so that:

- If *x* is within δ units of a (and $x \neq a$), then $f(x)$ is within ε units of *L*.
- This translates to $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$.

The goal in this part of the challenge will be to find the value of δ for a given value of ε.

HINT

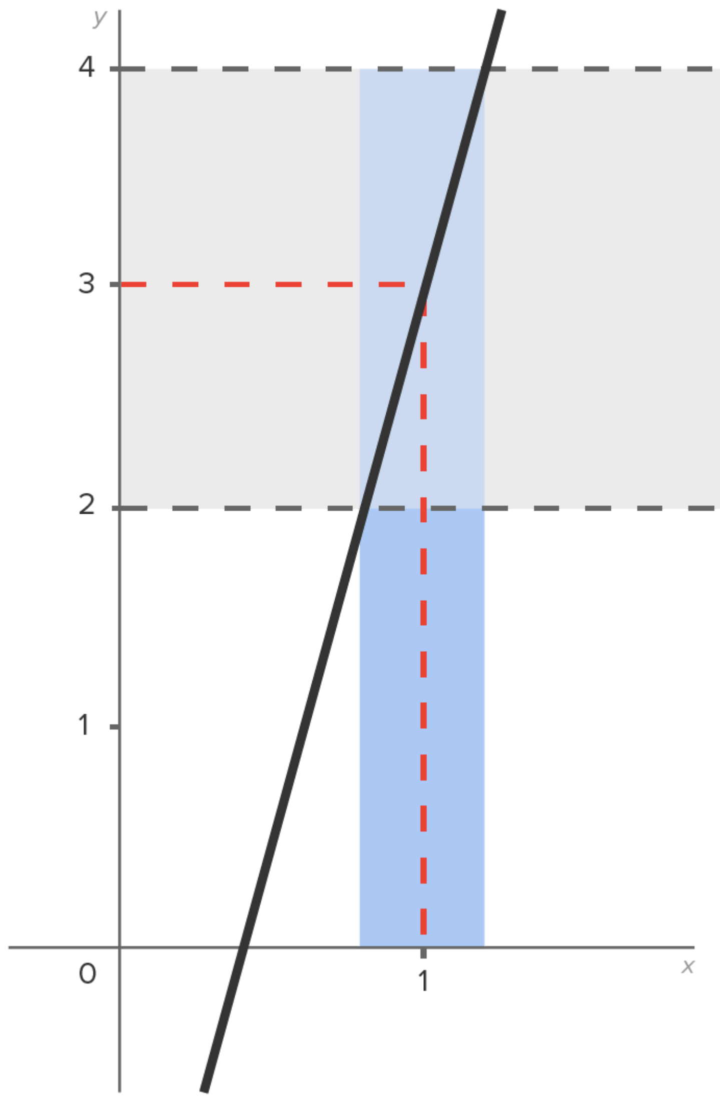
You may recall from algebra that $|x|<a$ is equivalent to saying $-a<x<a$ for any positive number a.

This means that $|f(x)-L|<\varepsilon$ can be rewritten $-\varepsilon<f(x)-L<\varepsilon$ and $|x-a|<\delta$ can be rewritten as $-\delta<x-a<\delta$.

These ideas are useful in determining the value of δ for a given ε.

EXAMPLE Consider the limit statement: $\lim_{x \to 1}(5x-2)=3$. What value of δ is required when $\varepsilon=1$?

Consider the picture shown below (the slanted line is the graph of $f(x)=5x-2$):



Remember that $\varepsilon = 1$ means that we desire $f(x)$ to be within 1 unit of 3 (the limit). This means $|f(x) - 3| < 1$. Let's solve this:

$\lvert 5x - 2 - 3\rvert < 1$	Replace $f(x)$ with $5x - 2$.
$\lvert 5x - 5\rvert < 1$	Simplify the expression.
$-1 < 5x - 5 < 1$	$\lvert x\rvert < a$ means $-a < x < a$.
$4 < 5x < 6$	Add 5 to all three parts.
$0.8 < x < 1.2$	Divide all three parts by 5.

Thus, $|f(x) - 3| < 1$ implies that $0.8 < x < 1.2$.

So, what is the value of δ?

Recall that the goal is to find δ so that $|x-a|<\delta$. In this problem, $a=1$, so this can be written as $|x-1|<\delta$.

Recall from algebra that this means $-\delta<x-1<\delta$. Thus, it helps to get an inequality with $x-1$ in the middle. Then the left and right parts of the inequality give information as to what δ is.

We left off with $0.8<x<1.2$. To get $x-1$ in the middle, subtract 1 from all parts of the inequality. This gives $-0.2<x-1<0.2$. Thus, $\delta=0.2$.

In summary, we state the following: If *x* is within 0.2 units of 1, then $f(x)$ is within 1 unit of 3.

While a graph is helpful, let's try one now without the graph.

EXAMPLE Consider the limit statement: $\lim_{x\to 3}(4x-5)=7$. Find the corresponding values of δ when ε = 0.5, 0.1, and 0.01.

For ε = 0.5, this means we want $|4x-5-7|<0.5$. Now solve:

$\|4x-12\|<0.5$	Simplify.
$-0.5<4x-12<0.5$	$\|x\|<a$ means $-a<x<a$.
$11.5<4x<12.5$	Add 12 to all three parts.
$2.875<x<3.125$	Divide all three parts by 4.
$-0.125<x-3<0.125$	Subtract 3 from all three parts to get $x-3$ in the middle.

Thus, δ = 0.125.

For ε = 0.1, this means we want $|4x-5-7|<0.1$. Now solve:

$\|4x-12\|<0.1$	Simplify.
$-0.1<4x-12<0.1$	$\|x\|<a$ means $-a<x<a$.
$11.9<4x<12.1$	Add 12 to all three parts.
$2.975<x<3.025$	Divide all three parts by 4.
$-0.025<x-3<0.025$	Subtract 3 from all three parts to get $x-3$ in the middle.

Thus, δ = 0.025.

For ε = 0.01, this means we want $|4x-5-7|<0.01$. Now solve:

$\|4x-12\|<0.01$	Simplify.

$-0.01 < 4x - 12 < 0.01$	$\lvert x \rvert < a$ means $-a < x < a$.
$11.99 < 4x < 12.01$	Add 12 to all three parts.
$2.9975 < x < 3.0025$	Divide all three parts by 4.
$-0.0025 < x - 3 < 0.0025$	Subtract 3 from all three parts to get $x - 3$ in the middle.

Thus, δ $=$ 0.0025.

HINT

Note that as the value of ε gets smaller, so does δ. This is the essence of a limit. As one distance gets smaller, the other does as well.

BIG IDEA

As the chosen values of ε get closer to 0, the corresponding value of δ also gets closer to 0.

When $f(x)$ is a linear function, finding the value of δ is fairly straightforward since the final inequality always has the form $-\delta < x - a < \delta$.

When $f(x)$ is a nonlinear function, this may not be the case, which means we have to think more critically to get the appropriate value of δ.

TERM TO KNOW

Formal Definition of a Limit

$\lim_{x \to a} f(x) = L$ means that for every given ε > 0, there exists δ > 0 so that:

- If *x* is within δ units of *a* (and $x \neq a$), then $f(x)$ is within ε units of *L*.
- This translates to $\lvert f(x) - L \rvert < \varepsilon$ whenever $0 < \lvert x - a \rvert < \delta$.

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2. Finding the Value of δ That Corresponds to a Given Value of ε for a Nonlinear Function

The following are inequalities that may be useful. In each case, assume that *c* and *d* are nonnegative numbers.

- If $c < -x < d$, then $-d < x < -c$.
- If $c < x^2 < d$, then $\sqrt{c} < x < \sqrt{d}$ (assuming *x* is positive).
- If $c < \sqrt{x} < d$, then $c^2 < x < d^2$.
- If $c < \frac{1}{x} < d$, then $\frac{1}{d} < x < \frac{1}{c}$.

EXAMPLE Consider the limit statement: $\lim_{x \to 64} \sqrt{x} = 8$. Let's find the corresponding value of δ when $\varepsilon = 2$.

We want $\left|\sqrt{x} - 8\right| < 2$.

$-2 < \sqrt{x} - 8 < 2$	$\lvert x \rvert < a$ means $-a < x < a$.
$6 < \sqrt{x} < 10$	Add 8 to all three parts.
$36 < x < 100$	Square all parts of the inequality.
$-28 < x - 64 < 36$	Subtract 64 from all three parts to get $x - 64$ in the middle.

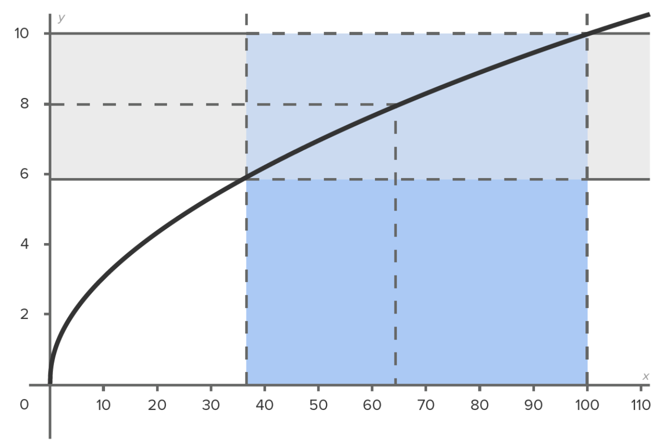
Notice that this inequality is not "balanced." This makes it unclear what to select for δ. Is the answer 28 or 36? Remember what we are trying to say:

In order for $f(x)$ to be within 2 units of 8, *x* has to be within ______ units of 64.

Consider the graph shown to the right:

- The horizontal band shows that $6 < y < 10$.
- The vertical band shows that $36 < x < 100$.

The intersection is the "area of interest" for the limit.



If we move 28 units away from $x = 64$ in either direction, we stay inside the vertical band, which guarantees that $f(x)$ is within 2 units of the limit.

If we move 36 units away from $x = 64$ in either direction, we could fall outside the vertical band on the left-hand side, which does not guarantee that $f(x)$ is within 2 units of the limit.

To guarantee that $f(x)$ is within 2 units of the limit (8), *x* needs to be within 28 units of 64. Thus, when $\varepsilon = 2$, $\delta = 28$.

WATCH

The following video provides an example for a linear function and a radical function.

TRY IT

Consider the limit statement $\lim_{x \to 3} x^2 = 9$.

Find the value of δ that corresponds to ε = 0.5. Round δ to the nearest hundredth. +

We want $8.5 < x^2 < 9.5$, which means $2.915 < x < 3.082$, which in turn means $-0.085 < x - 3 < 0.082$. To find δ, compare $|-0.085|$ and $|0.082|$ since we are examining the distances between *x* and 3. Since $0.082 < 0.085$, use $\delta = 0.082$.

EXAMPLE Consider the limit statement: $\lim_{x \to 5} \frac{1}{2x} = \frac{1}{10}$. Let's find δ when $\varepsilon = 0.05$.

Start with $\left|\frac{1}{2x} - \frac{1}{10}\right| < 0.05$.

$-0.05 < \frac{1}{2x} - \frac{1}{10} < 0.05$	$\lvert x \rvert < a$ means $-a < x < a$.
$\frac{1}{20} < \frac{1}{2x} < \frac{3}{20}$	Add $\frac{1}{10}$, and convert all to fractions.
$\frac{20}{3} < 2x < 20$	$c < \frac{1}{x} < d$ means $\frac{1}{d} < x < \frac{1}{c}$.
$\frac{10}{3} < x < 10$	Divide by 2.
$-\frac{5}{3} < x - 5 < 5$	Subtract 5.

It follows that $\delta = \frac{5}{3}$ since $\left|-\frac{5}{3}\right| = \frac{5}{3}$ and $\frac{5}{3}$ is smaller than 5.

SUMMARY

In this lesson, you learned that by using the formal definition of a limit, you can observe the relationship between ε and δ, which emphasizes the idea of "$f(x)$ getting closer to the limit as *x* gets closer to *a*." In this challenge, the goal was to **find the value of δ that corresponds to a given value of ε for a linear function and a nonlinear function**, and we observed that one getting smaller causes the other to get smaller. For linear functions, identifying δ is rather straightforward, but for nonlinear functions, more critical thinking is required to find the appropriate value of δ.

TERMS TO KNOW

Formal Definition of a Limit

$\lim_{x \to a} f(x) = L$ means that for every given ε > 0, there exists δ > 0 so that:

- If *x* is within δ units of *a* (and $x \neq a$), then $f(x)$ is within ε units of *L*.
- This translates to $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.