Using Tables to Find Antiderivatives

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WHAT'S COVERED

In this lesson, you will use a table of integrals to find antiderivatives and solve problems that cannot be solved only using the formulas and techniques learned thus far. Specifically, this lesson will cover:

1. Using a Table to Find Antiderivatives
2. Using a Table to Solve Applications Involving Definite Integrals

1. Using a Table to Find Antiderivatives

Given any function, we have the necessary tools to find its derivative.

But to find antiderivatives of many functions, new techniques are required. Since these techniques are not covered in this course, we will make use of the table of integrals as referenced below.

Table of Integration Formulas

In all integrals, assume that *k*, *n*, *a*, and *b* represent real numbers.

Part One: Integrals Involving Powers

1. $\int k\,\mathrm{d}x = kx + C$

2. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$

3. $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$

4. $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C,\ n \neq -1$

5. $\int (ax+b)^{-1}dx = \int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

6. $\int \frac{1}{x(ax+b)}dx = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$

7. $\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}[\ln|x+a| - \ln|x+b|] + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$ if $a \neq b$

8. $\int \frac{1}{(x+a)(x+a)}dx = \int \frac{1}{(x+a)^2}dx = \frac{-1}{x+a} + C$

9. $\int x(ax+b)^n dx = \frac{1}{a^2}(ax+b)^{n+1}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C,\ n \neq -1, -2$

10. $\int x(ax+b)^{-1}dx = \int \frac{x}{ax+b}dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$

11. $\int x(ax+b)^{-2}dx = \int \frac{x}{(ax+b)^2}dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

Part Two: Integrals Involving Trigonometric Functions

12. $\int \sin(ax)\,dx = \frac{-1}{a}\cos(ax) + C$

13. $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$

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14. $\int \tan(ax)\,dx = \int \frac{\sin(ax)}{\cos(ax)}\,dx = \frac{-1}{a}\ln\left|\cos(ax)\right| + C = \frac{1}{a}\ln\left|\sec(ax)\right| + C$

15. $\int \cot(ax)\,dx = \int \frac{\cos(ax)}{\sin(ax)}\,dx = \frac{1}{a}\ln\left|\sin(ax)\right| + C$

16. $\int \sec(ax)\,dx = \frac{1}{a}\ln\left|\sec(ax) + \tan(ax)\right| + C$

17. $\int \csc(ax)\,dx = \frac{-1}{a}\ln\left|\csc(ax) + \cot(ax)\right| + C$

18. $\int \sin^2(ax)\,dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax) + C = \frac{1}{2}x - \frac{1}{2a}\sin(ax)\cos(ax) + C$

19. $\int \cos^2(ax)\,dx = \frac{1}{2}x + \frac{1}{4a}\sin(2ax) + C = \frac{1}{2}x + \frac{1}{2a}\sin(ax)\cos(ax) + C$

20. $\int \tan^2(ax)\,dx = \frac{1}{a}\tan(ax) - x + C$

21. $\int \cot^2(ax)\,dx = \frac{-1}{a}\cot(ax) - x + C$

22. $\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + C$

23. $\int \csc^2(ax)\,dx = \frac{-1}{a}\cot(ax) + C$

24. $\int \sin^3(ax)\,dx = \frac{-1}{3a}\sin^2(ax)\cos(ax) - \frac{2}{3a}\cos(ax) + C$

25. $\int \cos^3(ax)\,dx = \frac{1}{3a}\cos^2(ax)\sin(ax) + \frac{2}{3a}\sin(ax) + C$

26. $\int \tan^3(ax)\,dx = \frac{1}{2a}\tan^2(ax) + \frac{1}{a}\ln\left|\cos(ax)\right| + C$

27. $\int \cot^3(ax)\,dx = \frac{-1}{2a}\cot^2(ax) - \frac{1}{a}\ln\left|\sin(ax)\right| + C$

28. $\int \sec^3(ax)\,dx = \frac{1}{2a}\sec(ax)\tan(ax) + \frac{1}{2a}\ln\left|\sec(ax) + \tan(ax)\right| + C$

29. $\int \csc^3(ax)\,dx = \frac{-1}{2a}\csc(ax)\cot(ax) - \frac{1}{2a}\ln\left|\csc(ax) + \cot(ax)\right| + C$

30. $\int \sin(ax)\sin(bx)\,dx = \frac{1}{2(a-b)}\sin((a-b)x) - \frac{1}{2(a+b)}\sin((a+b)x) + C,\ |a| \neq |b|$

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31. $\int \cos(ax)\cos(bx)\,dx = \frac{1}{2(a-b)}\sin((a-b)x) + \frac{1}{2(a+b)}\sin((a+b)x) + C,\ |a| \neq |b|$

32. $\int \sin(ax)\cos(bx)\,dx = \frac{-1}{2(a-b)}\cos((a-b)x) - \frac{1}{2(a+b)}\cos((a+b)x) + C,\ |a| \neq |b|$

Part Three: Integrals Involving Exponential & Logarithmic Functions

33. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$

34. $\int b^{ax}dx = \frac{1}{a\ln b}b^{ax} + C$

35. $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax} + C$

36. $\int x^2e^{ax}dx = \frac{1}{a}x^2e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax} + C$

37. $\int \ln x\,dx = x\ln x - x + C$

38. $\int x^n \ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C,\ n \neq -1$

39. $\int x^{-1}\ln x\,dx = \int \ln x \cdot \frac{1}{x}dx = \frac{1}{2}(\ln x)^2 + C$

40. $\int \frac{1}{x\ln x}dx = \ln|\ln x| + C$

41. $\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)] + C$

42. $\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)] + C$

Part Four: Integrals Involving $a^2 \pm x^2$ and $x^2 \pm a^2$

43. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$

44. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

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45. $\int \frac{1}{|x|\sqrt{x^2 - a^2}}dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$

46. $\int \frac{1}{a^2 - x^2}dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

47. $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

48. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

49. $\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{1}{8}x\left(2x^2 \pm a^2\right)\sqrt{x^2 \pm a^2} - \frac{1}{8}a^4\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

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EXAMPLE Assuming a is a constant, use formula #44, which is $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$, to find $\int \frac{dx}{x^2+81}$.

Using the formula as a model, we see that $a=9$. Then, $\int \frac{dx}{x^2+81} = \frac{1}{9}\arctan\left(\frac{x}{9}\right)+C$.

Here is another example, this time using logarithmic functions.

EXAMPLE Find the indefinite integral: $\int x^4 \ln x dx$

According to formula #38 in the integral table, $\int x^n \ln x dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1}+C,\ n \neq -1$.

Therefore, use this formula with $n=4$.

Then, $\int x^4 \ln x dx = \frac{x^{4+1}}{(4+1)^2}\{(4+1)\ln x - 1\}+C = \frac{x^5}{25}\{5\ln x - 1\}+C$.

TRY IT

Consider $\int \frac{dt}{100-t^2}$.

Find the antiderivative by using formula #46 in the table. +

Formula #46 states that $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C$

In this case, $a=10$, so we have:

$$\int \frac{1}{10^2-t^2}dt = \frac{1}{2(10)}\ln\left|\frac{t+10}{t-10}\right|+C = \frac{1}{20}\ln\left|\frac{t+10}{t-10}\right|+C$$

EXAMPLE Use an appropriate formula to find the indefinite integral: $\int \sin^3(2x)dx$

According to formula #24, $\int \sin^3(ax)dx = \frac{-\sin^2(ax)\cos(ax)}{3a} - \frac{2}{3a}\cos(ax)+C$.

With $a=2$, we have: $\int \sin^3(2x)dx = \frac{-\sin^2(2x)\cos(2x)}{3(2)} - \frac{2}{3(2)}\cos(2x)+C = \frac{-\sin^2(2x)\cos(2x)}{6} - \frac{1}{3}\cos(2x)+C$

2. Using a Table to Solve Applications Involving Definite Integrals

Since the key step in evaluating $\int_a^b f(x)dx$ is finding the antiderivative of $f(x)$, we can solve area and distance problems using the tables of integrals when necessary.

EXAMPLE Evaluate the definite integral: $\int_0^\pi 4\cos^2(5x)dx$

According to formula #19 in the table, $\int \cos^2(ax)dx = \frac{1}{2}x + \frac{1}{2a}\sin(ax)\cos(ax) + C$.

In our integral, $a = 5$, which means $\int \cos^2(5x)dx = \frac{1}{2}x + \frac{1}{10}\sin(5x)\cos(5x) + C$.

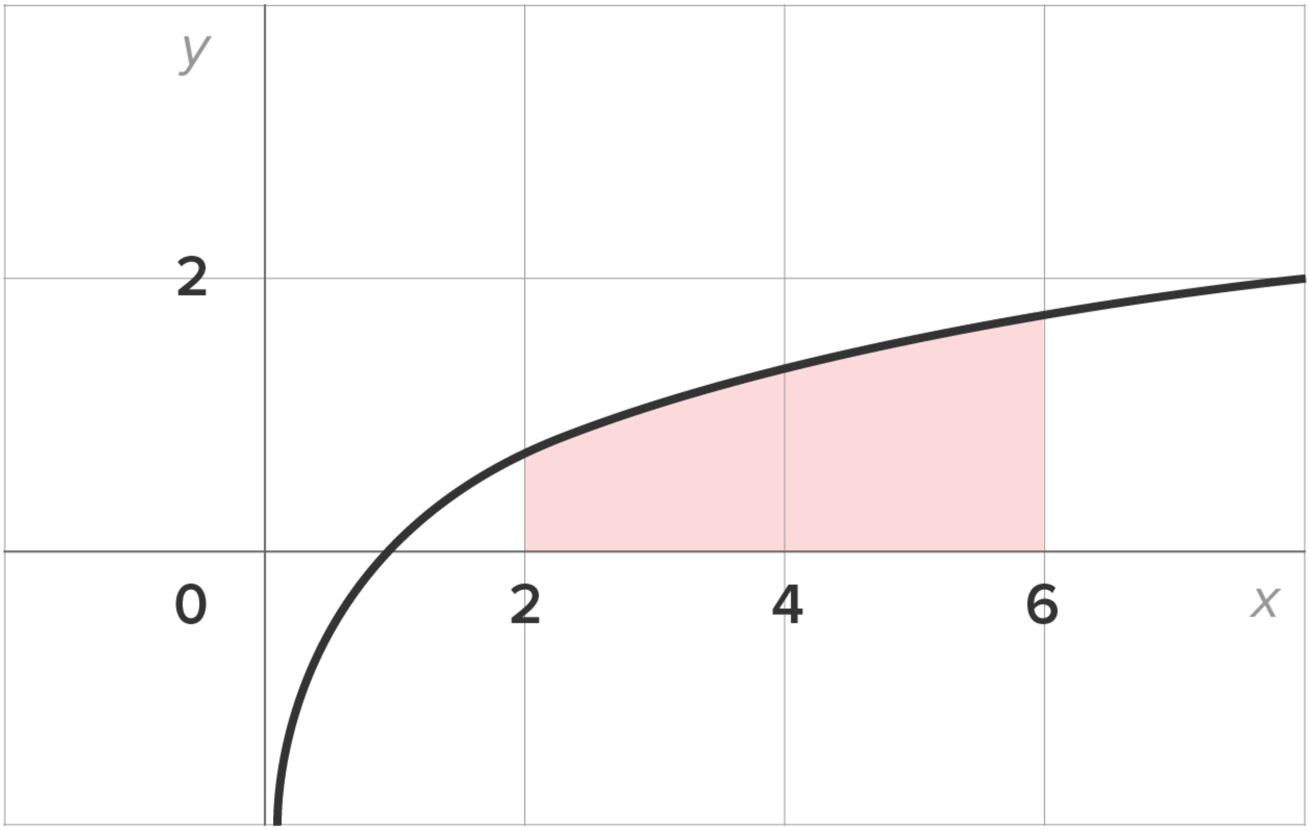
Then, $\int 4\cos^2(5x)dx = 4\left[\frac{1}{2}x + \frac{1}{10}\sin(5x)\cos(5x)\right] + C = 2x + \frac{2}{5}\sin(5x)\cos(5x) + C$.

To use the fundamental theorem of calculus, use $C = 0$. This leads to:

$$\int_0^\pi 4\cos^2(5x)dx = \left(2x + \frac{2}{5}\sin(5x)\cos(5x)\right)\Big|_0^\pi$$
$$= 2\pi + \frac{2}{5}\sin(5\pi)\cos(5\pi) - 2(0) + \frac{2}{5}\sin(5\cdot 0)\cos(5\cdot 0)$$
$$= 2\pi$$

Thus, $\int_0^\pi 4\cos^2(5x)dx = 2\pi$.

EXAMPLE Find the exact area of the region between the graphs of $y = \ln x$ and the x-axis between $x = 2$ and $x = 6$. The region is in the figure below.



Since the region is entirely above the x-axis on the interval $[2, 6]$, the area is given by the definite integral $\int_2^6 \ln x dx$.

Formula #37 in the table states that $\int \ln x dx = x\ln x - x + C$. It follows that:

$$\int_2^6 \ln x dx = (x\ln x - x)\Big|_2^6$$
$$= 6\ln 6 - 6 - (2\ln 2 - 2)$$
$$= 6\ln 6 - 2\ln 2 - 4$$

TRY IT

Consider the region bounded by the graphs of $f(x) = x(2x+1)^4$ and the x-axis between $x = 0$ and $x = 3$.

Set up and evaluate a definite integral that gives the area of the described region. +

Since the graph of $f(x) = x(2x+1)^4$ is above the x-axis on the interval $[0, 3]$, the definite integral used to compute the area is $\int_0^3 x(2x+1)^4 dx$.

This integral resembles formula #9 from the table, which states:

$$\int x(ax+b)^n dx = \frac{1}{a^2}(ax+b)^{n+1}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C, n \neq -1, -2$$

For our integral, $a=2$, $b=1$, and $n=4$, which means:

$$\int x(2x+1)^4 dx = \frac{1}{2^2}(2x+1)^{4+1}\left[\frac{2x+1}{4+2} - \frac{1}{4+1}\right] + C$$

Or in simpler form:

$$\int x(2x+1)^4 dx = \frac{1}{4}(2x+1)^5\left(\frac{2x+1}{6} - \frac{1}{5}\right) + C$$

Then, the value of the definite integral is:

$$\frac{1}{4}(2x+1)^5\left(\frac{2x+1}{6} - \frac{1}{5}\right)\Bigg|_0^3$$

Substituting $x=3$ and $x=0$, then subtracting, we have:

$$\frac{1}{4}(2(3)+1)^5\left(\frac{2(3)+1}{6} - \frac{1}{5}\right) - \frac{1}{4}(2(0)+1)^5\left(\frac{2(0)+1}{6} - \frac{1}{5}\right)$$
$$= \frac{1}{4}(7)^5\left(\frac{7}{6} - \frac{1}{5}\right) - \frac{1}{4}(1)^5\left(\frac{1}{6} - \frac{1}{5}\right)$$
$$= \frac{243{,}701}{60} \text{ square units}$$

This is approximately 4061.68 square units.

SUMMARY

In this lesson, you learned how to **use a table to find antiderivatives**, as well as **to solve applications involving definite integrals**. Having a table of integrals is a great tool when solving problems in which antiderivatives are required, but as you can tell, these would be very difficult (and not realistic) to memorize. Through more study of antiderivatives, you can learn the techniques that are required to arrive at these antiderivatives.