

Derivatives of Inverse Trigonometric Functions

by Sophia



WHAT'S COVERED

In this lesson, you will learn and use rules to differentiate the inverse trigonometric functions. Specifically, this lesson will cover:

1. Derivatives of the Inverse Trigonometric Functions
2. Derivatives of Functions That Involve Inverse Trigonometric Functions

1. Derivatives of the Inverse Trigonometric Functions

Consider the function $y = \sin^{-1}x$, which is also written $x = \sin y$. To find $\frac{dy}{dx}$, we will use the equation $x = \sin y$ and find the derivative implicitly.

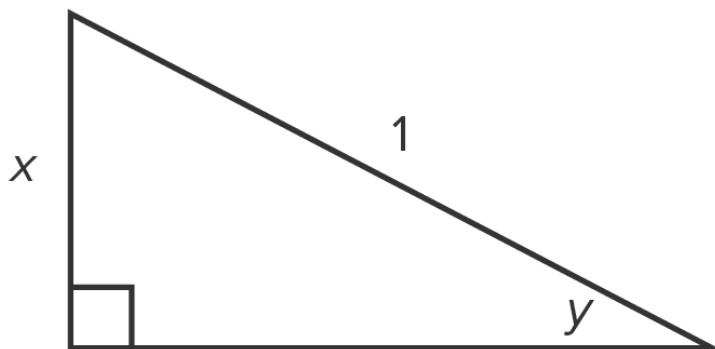
$$x = \sin y \quad \text{Start with the original equation.}$$

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y] \quad \text{Set up the derivative on each side.}$$

$$1 = \cos y \frac{dy}{dx} \quad \text{Take the derivative of each side.}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad \text{Solve for } \frac{dy}{dx}.$$

At this point, it would appear that we are done, but the goal is to get an expression in terms of x alone, instead of a function of y .



To do so, let's use a right triangle with angle y . Since $x = \sin y$, this means the side opposite y is x and the hypotenuse is 1.

By using the Pythagorean theorem, the length of the adjacent side is $\sqrt{1-x^2}$.

$$\text{Then, } \cos y = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

$$\text{Thus, } \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

$$\text{In summary, } D[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}.$$

Through similar reasoning, the derivatives of all six inverse trigonometric functions are shown below. Note that each formula has the basic version (with x as the variable) and the chain rule version (with u as the variable, where u represents a function of x .)



FORMULA TO KNOW

Derivative of the Inverse Sine Function

$$\frac{d}{dx}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\sin^{-1}u] = \frac{u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Cosine Function

$$\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\cos^{-1}u] = \frac{-u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Tangent Function

$$\frac{d}{dx}[\tan^{-1}x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\tan^{-1}u] = \frac{u'}{1+u^2}$$

Derivative of the Inverse Cotangent Function

$$\frac{d}{dx}[\cot^{-1}x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\cot^{-1}u] = \frac{-u'}{1+u^2}$$

Derivative of the Inverse Secant Function

$$\frac{d}{dx}[\sec^{-1}x] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\sec^{-1}u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

Derivative of the Inverse Cosecant Function

$$\frac{d}{dx}[\csc^{-1}x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\csc^{-1}u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

2. Derivatives of Functions That Involve Inverse Trigonometric Functions

With our new derivative rules, we can now find derivatives of functions that contain inverse trigonometric functions.

⇒ **EXAMPLE** Find the derivative of $y = \tan^{-1}(2x)$.

$y = \tan^{-1}(2x)$ Start with the original equation.

$$\frac{dy}{dx} = \frac{2}{1+(2x)^2} \quad \frac{dy}{dx} = \frac{u'}{1+u^2}, u = 2x, u' = 2$$

$$\frac{dy}{dx} = \frac{2}{1+4x^2} \quad \text{Simplify.}$$

$$\text{Thus, } \frac{dy}{dx} = \frac{2}{1+4x^2}.$$

⇒ **EXAMPLE** Consider the function $f(x) = x^2 \cdot \sin^{-1}x$. Find its derivative.

$$f(x) = x^2 \sin^{-1}x \quad \text{Start with the original equation.}$$

$$f'(x) = 2x \sin^{-1}x + x^2 \frac{1}{\sqrt{1-x^2}} \quad \text{Use the product rule with } x^2 \text{ and } \sin^{-1}x.$$

$$f'(x) = 2x \sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}} \quad \text{Simplify.}$$

$$\text{Thus, } f'(x) = 2x \sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}}.$$



TRY IT

Consider the function $f(x) = \cos^{-1}(x^3)$.

Find the derivative.

+

$$\text{The derivative of } y = \cos^{-1}x \text{ is } \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}.$$

$$\text{Therefore, } f'(x) = \frac{-1}{\sqrt{1-(x^3)^2}} \cdot D[x^3] = \frac{-3x^2}{\sqrt{1-x^6}}$$



WATCH

Find the derivative of $y = 2x^3 \arctan(5x^2 + 3)$.

Naturally, we can apply what we know about inverse trigonometric functions to applications such as finding the slope of the tangent line.

⇒ **EXAMPLE** Compute the slope of the line tangent to the function $y = \sec^{-1}(x^2 + 1)$ when $x = -1$. First, find the derivative of $y = \sec^{-1}(x^2 + 1)$.

$$y = \sec^{-1}(x^2 + 1) \quad \text{Start with the original equation.}$$

$$\frac{dy}{dx} = \frac{2x}{|x^2+1|\sqrt{(x^2+1)^2-1}}$$

$$\frac{dy}{dx} = \frac{u'}{|u|\sqrt{u^2-1}}, u = x^2+1, u' = 2x$$

$$\frac{dy}{dx} = \frac{2x}{|x^2+1|\sqrt{x^4+2x^2}}$$

$$\text{Simplify } (x^2+1)^2-1 = x^4+2x^2+1-1 = x^4+2x^2.$$

$$m_{\tan} = -\frac{\sqrt{3}}{3}$$

Substitute -1 for x to get $-\frac{1}{\sqrt{3}}$, then rationalize the denominator.

Thus, the slope of the tangent line is $-\frac{1}{\sqrt{3}}$, which after rationalizing the denominator, is $-\frac{\sqrt{3}}{3}$.



SUMMARY

In this lesson, you learned that by knowing the **derivative rules for the inverse trigonometric functions**, you can now find **derivatives of functions that involve inverse trigonometric functions**, thus expanding on the types of functions you are able to analyze for slope and rates of change, etc.

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