Changing the Variable: u-Substitution with Exponential Functions

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WHAT'S COVERED

In this lesson, you will find antiderivatives of composite functions that involve exponential functions. Specifically, this lesson will cover:

1. Indefinite Integrals Where the Inner Function Is Exponential
2. Indefinite Integrals Where the Outer Function Is Exponential

1. Indefinite Integrals Where the Inner Function Is Exponential

Even though we talked about exponential functions with base "*a*" in the past, we will focus on exponential functions with base *e* for the sake of finding antiderivatives.

Recall the derivative formula: $D\left[e^{u}\right]=e^{u}\cdot u'$

These will be useful in making substitutions where the inside function is exponential.

EXAMPLE Find the indefinite integral: $\int e^{x}\sin(e^{x})dx$

Note that e^{x} is the inner function, and it is also a factor in the integrand. This means a *u*-substitution will work.

$\int e^{x}\sin(e^{x})dx$	Start with the original expression.
$=\int \sin u\,du$	Make the substitution: $u=e^{x}$ Find the differential: $du=e^{x}dx$

Replace the inner e^x with u and dx with $e^u du$.

$= -\cos u + C$ Use the antiderivative rule for $\sin u$.

$= -\cos(e^x) + C$ Back-substitute $u = e^x$.

Thus, $\int e^x \sin(e^x)dx = -\cos(e^x) + C$.

HINT

When making the substitution $u = e^x$, it can be difficult to know where the substitutions go in the integral.

Remember, the goal is to make $\int (expression)dx$ look like $\int (expression)du$. Therefore, the term with the *du* should always be outside the function, and *u* should always go inside the function.

Here is one to try on your own.

TRY IT

Consider $\int e^x \sqrt{e^x + 4}\,dx$.

Find the indefinite integral. +

Since $e^x + 4$ appears to be the "inner function", let $u = e^x + 4$ then $du = e^x dx$. Then, replace $e^x + 4$ with *u* and $e^x dx$ with *du* in the integral.

After making these replacements, the integral becomes $\int \sqrt{u}\,du = \int u^{1/2} du$.

Next, find the antiderivative, simplify, then replace *u* with $e^x + 4$.

$$\int u^{1/2} du = \frac{1}{\left(\frac{3}{2}\right)} u^{3/2} du + C = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(e^x + 4)^{3/2} + C$$

Thus, $\int e^x \sqrt{e^x + 4}\,dx = \frac{2}{3}(e^x + 4)^{3/2} + C$.

Here is an example with a more complicated substitution.

EXAMPLE Find the indefinite integral: $\int \frac{e^{4x}}{e^{4x}+20}dx$

$\int \frac{e^{4x}}{e^{4x}+20}dx$	Start with the original expression.
$=\int \frac{1}{u}\cdot\frac{1}{4}du$	Make the substitution: $u=e^{4x}+20$ Find the differential: $du=4e^{4x}dx$ Solve for $e^{4x}dx$: $e^{4x}dx=\frac{1}{4}du$ Then, make the following replacements: $e^{4x}+20\to u$, $e^{4x}dx\to\frac{1}{4}du$
$=\frac{1}{4}\int \frac{1}{u}du$	Move the constant $\frac{1}{4}$ outside the integral sign.
$=\frac{1}{4}\ln\lvert u\rvert+C$	Use the antiderivative rule for $\frac{1}{u}$.
$=\frac{1}{4}\ln\lvert e^{4x}+20\rvert+C$	Back-substitute $u=e^{4x}+20$.

Thus, $\int \frac{e^{4x}}{e^{4x}+20}dx=\frac{1}{4}\ln\lvert e^{4x}+20\rvert+C$. It's worth noting that since $e^{4x}+20$ is positive for all real numbers *x*, the antiderivative can be written without the use of absolute value. That is, $\int \frac{e^{4x}}{e^{4x}+20}dx=\frac{1}{4}\ln(e^{4x}+20)+C$.

TRY IT

Consider $\int \frac{3e^{8x}}{(e^{8x}+15)^3}dx$.

Find the indefinite integral. +

Since the inner function is $e^{8x}+15$, let $u=e^{8x}+15$, then $du=8e^{8x}dx$ (chain rule).

Since there is no extra factor of 8 in the integral, rewrite the "du" equation as $\frac{1}{8}du=e^{8x}dx$.

Next, replace $e^{8x}+15$ with *u* and $e^{8x}dx$ with $\frac{1}{8}du$.

The integral becomes $\int \frac{3\left(\frac{1}{8}\right)du}{u^3}=\int \frac{3}{8}u^{-3}du$.

Next, find the antiderivative, simplify, then replace u with $e^{8x}+15$.

$$\begin{aligned}\int \frac{3}{8}u^{-3}du &= \frac{3}{8}\left(\frac{1}{-2}\right)u^{-2}+C\\ &= -\frac{3}{16}u^{-2}+C\\ &= -\frac{3}{16u^2}+C\\ &= -\frac{3}{16(e^{8x}+15)^2}+C\end{aligned}$$

Thus, $\int \frac{3e^{8x}}{(e^{8x}+15)^3}dx = -\frac{3}{16(e^{8x}+15)^2}+C$.

2. Indefinite Integrals Where the Outer Function Is Exponential

Now we will move on to antiderivatives where the exponential is the outer function. Let’s look at an example to see how this is different:

EXAMPLE Find the indefinite integral: $\int e^{3x}dx$

Note: the inner function is “$3x$” since we know the antiderivative of e^u. This is where we start:

$\int e^{3x}dx$	Start with the original expression.
$=\int e^u \cdot \frac{1}{3}du$	Make the substitution: $u=3x$ Find the differential: $du=3dx$ Solve for dx: $dx=\frac{1}{3}du$
$=\frac{1}{3}\int e^u du$	Move the constant $\frac{1}{3}$ outside the integral sign.
$=\frac{1}{3}e^u+C$	Use the antiderivative rule for e^u.
$=\frac{1}{3}e^{3x}+C$	Back-substitute $u=3x$.

Thus, $\int e^{3x}dx = \frac{1}{3}e^{3x} + C$.

This result is used so often that it might be handy to remember this formula:

FORMULA TO KNOW

Antiderivative of e^{kx}, Where k is a Constant

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$$

TRY IT

Consider $\int e^{-x}dx$.

Find the indefinite integral. +

Use the formula $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$ with $k = -1$.

Then, $\int e^{-x}dx = \frac{1}{-1}e^{-x} + C = -e^{-x} + C$.

Here is another more complicated example.

EXAMPLE Find the indefinite integral: $\int 12x^2 e^{-6x^3}dx$

Note: the inner function is "$-6x^3$" since we know the antiderivative of e^u. This is where we start:

$\int 12x^2 e^{-6x^3}dx$	Start with the original expression.
$= \int e^u \cdot \left(-\frac{2}{3}\right)du$	Make the substitution: $u = -6x^3$ Find the differential: $du = -18x^2dx$ Solve for x^2dx: $x^2dx = \frac{-1}{18}du$ Then, make the following replacements: $-6x^3 \rightarrow u$, $x^2dx \rightarrow \frac{-1}{18}du$ Note: $12\left(\frac{-1}{18}du\right) = \frac{-2}{3}du$
$= -\frac{2}{3}\int e^u du$	Move the constant $-\frac{2}{3}$ outside the integral sign.
$= \frac{-2}{3}e^u + C$	Use the antiderivative rule for e^u.

$= \frac{-2}{3} e^{-6x^3} + C$ Back-substitute $u = -6x^3$.

Thus, $\int 12x^2 e^{-6x^3} dx = \frac{-2}{3} e^{-6x^3} + C$.

TRY IT

Consider $\int e^{-2\cos x} (\sin x) dx$.

Find the indefinite integral. +

Since $-2\cos x$ is inside another function, let $u = -2\cos x$, then $du = -2(-\sin x)dx = 2\sin x dx$. Since there is no extra factor of 2 in the integrand, rewrite the "du" equation as $\frac{1}{2} du = \sin x dx$.

Replace $-2\cos x$ with *u* and $\sin x dx$ with $\frac{1}{2} du$.

The new integral is $\int e^u \cdot \frac{1}{2} du$, or $\int \frac{1}{2} e^u du$.

Next, find the antiderivative, then simplify, then replace *u* with $-2\cos x$.

$$\int \frac{1}{2} e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{-2\cos x} + C$$

Thus, $\int e^{-2\cos x} (\sin x) dx = \frac{1}{2} e^{-2\cos x} + C$.

SUMMARY

In this lesson, you learned how to use *u*-substitution to find **indefinite integrals where the inner function is exponential** and **the outer function is exponential**. With exponential functions added, this expands your capabilities for finding more antiderivatives. You now have a sizable toolbox from which to apply antiderivatives, which is what we'll do in the next tutorial and in Challenge 5.4.

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FORMULAS TO KNOW

Antiderivative of e^{kx}, Where k is a Constant

$$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$$