Limits with Variable Bases and Exponents

by Sophia

WHAT'S COVERED

In this lesson, you will learn strategies to evaluate indeterminate forms that have both variable bases and exponents. Specifically, this lesson will cover:

1. The Strategy for Evaluating Limits With Variable Bases and Exponents
2. Evaluating Limits With Variable Bases and Exponents

1. The Strategy for Evaluating Limits With Variable Bases and Exponents

Consider a function that has the form $y = f(x)^{g(x)}$. Of all the possible behaviors of $f(x)$ and $g(x)$ that could occur in a limit, there are three situations that lead to indeterminate forms.

Form	Explanation
0^0	The base and exponent both approach 0.
∞^0	The base grows without bound and at the same time, the exponent approaches 0.
1^∞	When the base approaches 1 and at the same time, the exponent increases without bound.

Since L'Hopital's rule can only be applied to limits with indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, limits with the indeterminate forms 0^0, ∞^0, or 1^∞ will need to be manipulated in order to use L'Hopital's rule.

To see how to start, consider the identity $a = e^{\ln a}$, which is valid as long as $a > 0$.

Replacing a with $f(x)^{g(x)}$, we can write $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}}$.

By the property of logarithms, we know that $\ln(f(x)^{g(x)}) = g(x) \cdot \ln f(x)$, which allows us to write $f(x)^{g(x)} = e^{g(x) \cdot \ln f(x)}$.

This also means that $\lim\limits_{x \to a} f(x)^{g(x)} = \lim\limits_{x \to a} e^{g(x) \cdot \ln f(x)}$.

The limit on the right-hand side suggests that we can focus on the exponent $g(x) \cdot \ln f(x)$, which is a product, something that we have already handled using L'Hopital's rule.

BIG IDEA

If $\lim\limits_{x \to a} g(x) \cdot \ln f(x) = L$, then the limit we seek is $\lim\limits_{x \to a} f(x)^{g(x)} = \lim\limits_{x \to a} e^{g(x) \cdot \ln f(x)} = e^{L}$.

To summarize, these steps will help to evaluate limits with indeterminate forms 0^0, ∞^0, or 1^{∞}.

STEP BY STEP

To evaluate a limit with an indeterminate form 0^0, 1^{∞}, or ∞^0:

1. Let $y = f(x)^{g(x)}$. Then, $\ln y = g(x) \cdot \ln f(x)$.
2. Find $\lim\limits_{x \to a} \ln y$.
3. Assuming that $\lim\limits_{x \to a} \ln y = L$, we know $\lim\limits_{x \to a} y = e^{L}$, where $y = f(x)^{g(x)}$.

Let's see how this methodology is applied to specific examples.

---

2. Evaluating Limits With Variable Bases and Exponents

Now that we have a strategy, let's evaluate a few limits that have one of these indeterminate forms.

EXAMPLE Evaluate the following limit: $\lim\limits_{x \to 0^+} x^x$

Note that this is a limit of the form 0^0, which will use our new strategy:

1. Take the natural logarithm of x^x: $\ln x^x = x \ln x$
2. Now find the limit:

$\lim\limits_{x \to 0^+} x^x$	Start with the limit that needs to be evaluated.
$\lim\limits_{x \to 0^+} x \ln x$	Evaluate the limit of the natural logarithm of the function. This has the form $0 \cdot (-\infty)$, which is another indeterminate form.
$= \lim\limits_{x \to 0^+} \dfrac{\ln x}{\left(\dfrac{1}{x}\right)}$	The strategy here is to rewrite as either $\dfrac{x}{\left(\dfrac{1}{\ln x}\right)}$ or $\dfrac{\ln x}{\left(\dfrac{1}{x}\right)}$. The latter is preferable.

$$= \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$$

The limit has the form $\frac{\infty}{\infty}$ and both numerator and denominator are differentiable, so L'Hopital's rule can be used.
$D\left[\frac{1}{x}\right] = D\left[x^{-1}\right] = -x^{-2} = \frac{-1}{x^2}$, $D[\ln x] = \frac{1}{x}$

$$= \lim_{x \to 0^+} (-x)$$

Simplify $\frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \frac{1}{x} \cdot \frac{x^2}{-1} = -x$.

$$= 0$$

Use direct substitution.

3. Then, the limit of the original function is $e^0 = 1$.

Thus, $\lim_{x \to 0^+} x^x = 1$.

WATCH

In this video, we will evaluate the limit $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$.

EXAMPLE Evaluate the limit $\lim_{x \to 0^+} (3x + 1)^{2\csc x}$.

Note that this is a limit of the form 1^{∞} since $3x + 1 \to 1$ and $2\csc x \to \infty$ as $x \to 0^+$.

Since the base and exponent are both variable, we use our new strategy:

1. Take the natural logarithm of $(\cos x)^{1/x}$:

$$\ln(3x + 1)^{2\csc x} = 2\csc x \ln(3x + 1) = \frac{2\ln(3x + 1)}{\sin x}$$

Note: $\csc x = \frac{1}{\sin x}$

2. Evaluate $\lim_{x \to 0^+} \frac{2\ln(3x + 1)}{\sin x}$

$$\lim_{x \to 0^+} \frac{2\ln(3x + 1)}{\sin x}$$

This is the limit we are evaluating.

$$= \lim_{x \to 0^+} \frac{2 \cdot \frac{1}{3x + 1} \cdot 3}{\cos x}$$

The limit has indeterminate form 0/0, which means L'Hopital's rule can be applied.

$$D[2\ln(3x+1)] = \frac{2}{3x+1}(3)$$

$$D[\sin x] = \cos x$$

$$= \lim_{x \to 0^+} \frac{6\left(\frac{1}{3x+1}\right)}{\cos x} \quad \text{Simplify.}$$

$$= \frac{6(1)}{1} = 6 \quad \text{By direct substitution, } \frac{1}{3(0)+1} = 1 \text{ and } \cos 0 = 1.$$

3. Since we applied the natural logarithm to evaluate the limit, we undo this operation by treating the limit value as an exponent on base e. The value of the original limit is e^6.

Thus, $\lim\limits_{x \to 0^+} (3x+1)^{2\csc x} = e^6$.

TRY IT

Consider the following limit: $\lim\limits_{x \to \infty} x^{3/x}$

Evaluate the limit. +

First, note that the limit has the form ∞^0, which is indeterminate.

The strategy is to analyze the natural logarithm of the expression. Note that $\ln(x^{3/x}) = \frac{3}{x}\ln x$, which can be written as a single fraction $\frac{3\ln x}{x}$.

Therefore, we can evaluate $\lim\limits_{x \to \infty} \frac{3\ln x}{x}$. Since both the numerator and denominator tend to ∞ as $x \to \infty$, L-Hopital's Rule applies, which states that $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \frac{f'(x)}{g'(x)}$.

Then, we have:

$$\lim_{x \to \infty} \frac{3\ln x}{x} = \lim_{x \to \infty} \frac{3\left(\frac{1}{x}\right)}{1} = \lim_{x \to \infty} \frac{3}{x} = 0$$

Since this is the limit of the natural logarithm of the original expression, the original limit has value

$e^0 = 1$.

SUMMARY

In this lesson, you learned **the strategy for evaluating limits with variable bases and exponents**. For instance, when evaluating $\lim_{x \to a} f(x)^{g(x)}$ and the limit results in one of the indeterminate forms (0^0, 1^∞, and ∞^0), the limit will need to be manipulated using logarithms in order to use L'Hopital's rule.