Solving y' = f(x)

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WHAT'S COVERED

In this lesson, you will solve differential equations of the form $y' = f(x)$. Specifically, this lesson will cover:

1. What Is a Differential Equation?

A **differential equation** is an equation that contains derivatives of some function *y*. The solution of the differential equation is the function *y* that satisfies the equation.

Examples of differential equations include:

- $y' = 2x + 5$
- $y'' + 4y' + 4y = 5\sin t$

In this course, we will solve differential equations of the form $y' = f(x)$.

TERM TO KNOW

Differential Equation

An equation that contains derivatives of some function *y*.

2. Solving a Differential Equation

2a. Finding the General Solution

Consider the differential equation $y' = f(x)$. The solution can be written $y = \int f(x)dx$. This tells us to take the antiderivative of $f(x)$ to solve this type of differential equation.

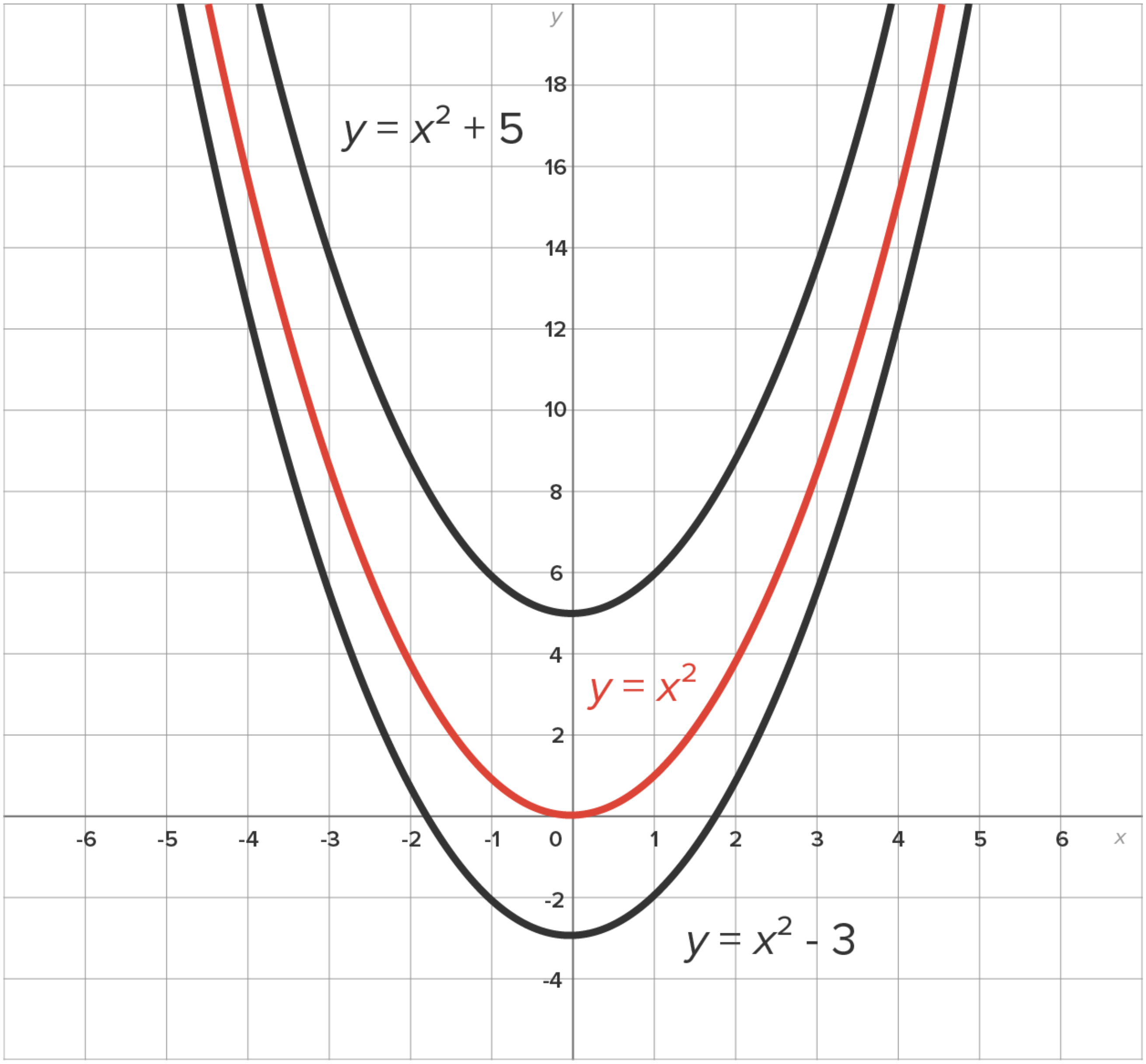
Now, recall that we often use $F(x)$ to represent the antiderivative of $f(x)$. That is, $F(x) = \int f(x)dx$.

This all considered, the function $y = F(x) + C$ is called the **general solution** of the differential equation $y' = f(x)$. The general solution is actually a family of solutions since the equation represents a set of functions that differ only by the value of *C*, the arbitrary constant.

EXAMPLE Consider the differential equation $y' = 2x$, which we know has solution $y = x^2 + C$.

This means that $y = x^2$, $y = x^2 - 3$, and $y = x^2 + 5$ are solutions to the differential equation, to name a few.

In fact, if you were to graph each solution, they only differ by a vertical shift.



BIG IDEA

To solve a differential equation of the form $y' = f(x)$, find the antiderivative of $f(x)$, which is called $F(x) = \int f(x)dx$. Then, the general solution is $y = F(x) + C$.

EXAMPLE Solve $y' = \cos(2x)$.

This means that y is the antiderivative of $\cos(2x)$, written $y = \int \cos(2x)dx$.

Let's go through the antiderivative process.

$\int \cos(2x)dx$	Start with the original expression.
$= \int \cos u \cdot \frac{1}{2} du$	Let $u = 2x$. Then, the differential is $du = 2dx$. Then, $dx = \frac{1}{2} du$.
$= \frac{1}{2} \int \cos u du$	Move the constant $\frac{1}{2}$ outside the integral sign.
$= \frac{1}{2} \sin u + C$	Use the antiderivative rule for $\cos u$.
$= \frac{1}{2} \sin(2x) + C$	Back-substitute $u = 2x$.

Thus, the solution to the differential equation $y' = \cos(2x)$ is $y = \frac{1}{2}\sin(2x) + C$.

TRY IT

Consider the differential equation $y' = x^2 - 4x + 7$.

Find the solution. +

The solution, *y*, is found by finding the antiderivative of *y'*.

First, find the antiderivative and simplify:

$$\int (x^2 - 4x + 7)dx = \frac{1}{3}x^3 - 4\left(\frac{1}{2}\right)x^2 + 7x + C$$
$$= \frac{1}{3}x^3 - 2x^2 + 7x + C$$

Then, the solution to the differential equation is $y = \frac{1}{3}x^3 - 2x^2 + 7x + C$.

TRY IT

Consider the differential equation $y' = \cos(3x) + 3e^x - 9x^2$.

Find the solution. +

The solution, *y*, is found by finding the antiderivative of *y'*.

Start by finding $\int(\cos(3x)+3e^x-9x^2)dx$.

The first function, $\cos(3x)$, requires a u-substitution, so let's split up the integrals.

$$\int\cos(3x)dx+\int 3e^x dx-\int 9x^2 dx$$

To find $\int\cos(3x)dx$, let $u=3x$, then $du=3dx$. Since there is no extra factor of 3, rewrite this equation as $\frac{1}{3}du=dx$.

Then, $\int\cos(3x)dx=\int\cos u\cdot\frac{1}{3}du=\frac{1}{3}\sin u+C$. Replacing *u* with 3*x*, this becomes $\frac{1}{3}\sin(3x)+C$.

It also follows that $\int 3e^x dx=3e^x+C$.

Additionally, $\int 9x^2 dx=9\left(\frac{1}{3}\right)x^3+C=3x^3+C$.

Putting this all together, $\int(\cos(3x)+3e^x-9x^2)dx=\frac{1}{3}\sin(3x)+3e^x-3x^3+C$.

Thus, the solution to the differential equation is $y=\frac{1}{3}\sin(3x)+3e^x-3x^3+C$.

➡ EXAMPLE Solve $y'=\frac{3}{2x+1}$.

This means $y=\int\frac{3}{2x+1}dx$.

Again, let's go through the antiderivative process.

$y=\int\frac{3}{2x+1}dx$	Start with the original expression.
$=\int\frac{1}{u}\cdot\frac{3}{2}du$	Let $u=2x+1$. Then, the differential is $du=2dx$. Then, $dx=\frac{1}{2}du$.

$$= \frac{3}{2}\int \frac{1}{u}du$$ Move the constant $\frac{3}{2}$ outside the integral sign.

$$= \frac{3}{2}\ln|u| + C$$ Use the antiderivative rule for $\frac{1}{u}$.

$$= \frac{3}{2}\ln|2x+1| + C$$ Back-substitute $u = 2x + 1$.

Thus, the solution to the differential equation $y' = \frac{3}{2x+1}$ is $y = \frac{3}{2}\ln|2x+1| + C$.

TERM TO KNOW

General Solution

The general solution of a differential equation is a function of the form $y = F(x) + C$ that satisfies a differential equation regardless of the value of *C*.

2b. Finding a Particular Solution Based on Initial Conditions

When obtaining the solution to the differential equation, sometimes we are given a value of *y* when *x* is some number in the domain. This is called an **initial condition**.

For example, if the graph is to pass through the point $(1, 5)$, then the initial condition is written $y(1) = 5$.

EXAMPLE Solve $y' = 2x$, given that the solution passes through the point $(3, 20)$.

We know $\int 2x dx = x^2 + C$. Thus, the solution to the differential equation is $y = x^2 + C$. Use this equation to find the solution that passes through $(3, 20)$.

$$y = x^2 + C$$ Use this equation to find the solution.

$$20 = 3^2 + C$$ Replace *x* with 3 and *y* with 20.

$$11 = C$$ Simplify.

Substituting this answer back into $y = x^2 + C$, the solution to the differential equation is $y = x^2 + 11$.

BIG IDEA

The initial condition is used to find the value of *C*, the constant of integration.

A **particular solution** is the solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

EXAMPLE Solve $y' = e^{-3x} + x + \sin x$, given that the solution passes through the point $(0, 4)$.

First, find the family of solutions.

$y' = e^{-3x} + x + \sin x$	Start with the original expression.
$y = \int \left(e^{-3x} + x + \sin x\right) dx$	If $y' = f(x)$, then $y = \int f(x) dx$.
$y = \int e^{-3x} dx + \int x dx + \int \sin x dx$	Use the sum of antiderivatives property.
$y = \frac{-1}{3} e^{-3x} + \frac{1}{2} x^2 - \cos x + C$	Apply antiderivative formulas.
$4 = \frac{-1}{3} e^{(3 \cdot 0)} + \frac{1}{2}(0)^2 - \cos 0 + C$	Apply the initial condition and replace x with 0 and y with 4.
$C = \frac{16}{3}$	Solve for C.

Thus, the particular solution is $y = \frac{-1}{3} e^{-3x} + \frac{1}{2} x^2 - \cos x + \frac{16}{3}$.

TRY IT

Consider the differential equation $y' = 2\cos(4x) - 12\sin(6x)$ with $y(0) = -1$.

Solve the differential equation. +

The solution, y, is found by computing $\int (2\cos(4x) - 12\sin(6x)) dx$. Since each of these terms requires u-substitution, we'll find the antiderivatives separately.

To find $\int 2\cos(4x) dx$:

Let $u = 4x$, then $du = 4dx$ and $\frac{1}{4} du = dx$. Next, replace $4x$ with u and dx with $\frac{1}{4} du$, then find the antiderivative:

$$\begin{aligned}\int 2\cos u \cdot \frac{1}{4} du &= \int \frac{1}{2} \cos u du \\ &= \frac{1}{2} \sin u + C \\ &= \frac{1}{2} \sin(4x) + C\end{aligned}$$

To find $\int 12\sin(6x) dx$:

Let $u=6x$, then $du=6dx$ and $\frac{1}{6}du=dx$. Next, replace $6x$ with u and dx with $\frac{1}{6}du$, then find the antiderivative:

$$\begin{aligned}\int 12\sin u\cdot\frac{1}{6}du &= \int 2\sin u\,du\\ &= -2\cos u + C\\ &= -2\cos(6x)+C\end{aligned}$$

Now, we can put the solution together. Note that only one "C" is needed.

$$\begin{aligned}y &= \int(2\cos(4x)-12\sin(6x))dx\\ &= \frac{1}{2}\sin(4x)-(-2\cos(6x))+C\\ &= \frac{1}{2}\sin(4x)+2\cos(6x)+C\end{aligned}$$

Now, we need to address the initial condition that was given, $y(0)=-1$. Recall that this information will lead us to the value of C.

Substituting 0 in for *x* in our equation, we have:

$$\begin{aligned}y(0) &= \frac{1}{2}\sin(4\cdot 0)+2\cos(6\cdot 0)+C\\ -1 &= \frac{1}{2}(0)+2(1)+C\\ -1 &= 2+C\\ -3 &= C\end{aligned}$$

Lastly, substituting $C=-3$ into our expression for *y*, our solution is $y=\frac{1}{2}\sin(4x)+2\cos(6x)-3$.

WATCH

In this video, we'll solve a differential equation: $\frac{dy}{dx}=\frac{x}{\sqrt{x^2+16}}$, $y(3)=1$

TERMS TO KNOW

Initial Condition

From a differential equation, a point that the solution's graph passes through.

Particular Solution

The solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

3. Applications

3a. Finding Height Given Velocity and Initial Position

EXAMPLE Suppose the velocity of an object that is falling from a height of 400 feet is given by $v(t)=120-120e^{-0.4t}$, where t is measured in seconds, and $v(t)$ is measured in meters per second.

What is the object's height above the ground after t seconds, given that the object's initial position was 400 feet above the ground?

We know that the height, $h(t)$, is the antiderivative of the velocity $v(t)$. Let's start there.

$v(t)=120-120e^{-0.4t}$	Start with the original expression.
$h(t)=\int(120-120e^{-0.4t})dt$	We can say $h(t)=\int v(t)dt$.
$h(t)=120t+300e^{-0.4t}+C$	Integrate the right-hand side.
$400=120(0)+300e^{(-0.4\cdot 0)}+C$	We were also told that the initial position was 400 feet away in the positive direction. This means $h(0)=400$.
$400=300+C$	Simplify.
$100=C$	Solve for C.

Thus, the object's height function is $h(t)=120t+300e^{-0.4t}+100$.

3b. Finding Velocity and Position Given Acceleration

If $s(t)$ represents a function's position at time t (usually measured in seconds), recall the following:

- If $v(t)$ represents its velocity at time t, then $v(t)=s'(t)$.
- If $a(t)$ represents its acceleration at time t, then $a(t)=v'(t)=s''(t)$.
- It follows that $v(t)=\int a(t)dt$ and $s(t)=\int v(t)dt$.

EXAMPLE A tennis ball is dropped (no starting velocity) from a height of 400 feet at time $t=0$, where t is measured in seconds. This means that $v(0)=0$ and $s(0)=400$.

If the tennis ball's acceleration (due to gravity) is $-32\text{ ft}/\text{s}^2$, find its velocity and position as functions of time.

Since we are given acceleration, we find the velocity function, $v(t)$, first.

$v(t) = \int a(t)dt$	Given the acceleration, we can find the velocity first.
$v(t) = \int (-32)dt$	Plug in the known value of acceleration, -32.
$v(t) = -32t + C$	Evaluate.
$0 = -32(0) + C$	Knowing that $v(0) = 0$, substitute to find *C*.
$0 = C$	Simplify.

Thus, $v(t) = -32t$.

Now repeat the process to find the position function, $s(t)$.

$s(t) = \int v(t)dt$	Given the velocity, we can find the position.
$s(t) = \int (-32t)dt$	Plug in the known velocity function, $v(t) = -32t$.
$s(t) = -32\left(\frac{1}{2}\right)t^2 + C$	Evaluate.
$s(t) = -16t^2 + C$	Simplify.
$400 = -16(0)^2 + C$	Knowing that $s(0) = 400$, substitute to find *C*.
$400 = C$	Simplify.

Then, $s(t) = -16t^2 + 400$.

TRY IT

At time $t = 0$, a tomato is launched with an upward velocity of 20 feet per second from a height of 300 feet, where *t* is measured in seconds. This means that $v(0) = 20$ and $s(0) = 200$. Assume the tomato's acceleration due to gravity is $-32 \text{ ft}/\text{s}^2$.

Find the velocity of the tomato as a function of time. +

We know that $v(t) = \int a(t)dt$.

Given $a(t) = -32$, we have $v(t) = \int -32\,dt = -32t + C$.

We are also given that $v(0)=20$. Substituting into our expression for $v(t)$, we can use this information to find *C*. According to the velocity function we found:

$$v(0) = -32(0) + C$$
$$20 = 0 + C$$
$$20 = C$$

Substituting $C=20$ into our expression for $v(t)$, we have $v(t) = -32t + 20$.

Find the position of the tomato as a function of time. +

We know that $s(t) = \int v(t)dt$.

Given $v(t) = -32t + 20$, we have:

$$\begin{aligned} s(t) &= \int(-32t+20)dt \\ &= -32t\left(\frac{1}{2}t^2\right) + 20t + C \\ &= -16t + 20t + C \end{aligned}$$

We were given that $s(0)=200$. This information is used to find *C*. According to the position function we found:

$$s(0) = -16(0)^2 + 20(0) + C$$
$$200 = C$$

Substituting into the equation for $s(t)$, we have $s(t) = -16t^2 + 20t + 200$.

DID YOU KNOW

In the metric system, the basic unit of distance is meters. The acceleration due to gravity is approximately $-9.8\ \text{m}/\text{s}^2$. Therefore, we can find $v(t)$ and $s(t)$ for a moving object when the distance is measured in meters rather than feet.

BIG IDEA

If an object moves with acceleration *a* (constant) with initial velocity v_0 and has initial position s_0, we know the following:

- Its velocity after *t* seconds is $v(t) = at + v_0$.
- Its position at time *t* is $s(t) = \frac{1}{2}at^2 + v_0t + s_0$.

SUMMARY

In this lesson, you began by defining **a differential equation**, which is an equation that contains derivatives of some function *y*. You learned that when **solving a differential equation** of the form $y' = f(x)$, there are two ways to express the solution. When there is no initial condition, **finding the general solution** of the differential equation has the form $y = F(x) + C$, where *C* is any constant. This produces a family of solutions that are vertical shifts of one another. When **finding a particular solution based on an initial condition** that is given, you can find the specific curve from the same family of solutions that satisfied the initial condition.

Extending this idea to other **applications**, you are able to **find the height** of an object in motion **given velocity and initial position** as well as **find the velocity and position** of an object in motion, **given its acceleration** at any time *t*.

TERMS TO KNOW

Differential Equation
An equation that contains derivatives of some function *y*.

General Solution
The general solution of a differential equation is a function of the form $y = F(x) + C$ that satisfies a differential equation regardless of the value of *C*.

Initial Condition
From a differential equation, a point that the solution's graph passes through.

Particular Solution
The solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.