Implicit Differentiation

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WHAT'S COVERED

In this lesson, you will apply techniques of derivatives when the equation defines *y* implicitly. So far, we know how to find derivatives when *y* is explicitly a function of *x*, meaning $y=f(x)$. The equation $x^2+2y^2=22$ is an example of an equation where *y* is defined implicitly, meaning *y* is not isolated to one side. The equation still defines a curve, so it makes sense to discuss the derivative and slopes of tangent lines, etc. Specifically, this lesson will cover:

1. Implicit Differentiation
2. Slopes and Equations of Tangent Lines

1. Implicit Differentiation

If *y* is some function of *x*, we know that the derivative of *y* is $\frac{dy}{dx}$.

Then, by the chain rule, we know the following:

$$\frac{d}{dx}[y^2]=2yD[y]=2y\frac{dy}{dx}$$

$$\frac{d}{dx}[\sin y]=\cos yD[y]=\cos y\frac{dy}{dx}$$

$$\frac{d}{dx}[\ln y]=\frac{1}{y}D[y]=\frac{1}{y}\frac{dy}{dx}$$

Now, consider the equation $x^2+2y^2=22$, where *y* is some function of *x*. If we take the derivative of both sides of the equation with respect to *x*, we get:

$\frac{d}{dx}[x^2]+\frac{d}{dx}[2y^2]=\frac{d}{dx}[22]$ Use the sum/difference rules.

$$2x+4y\frac{dy}{dx}=0 \quad D[x^2]=2x,\ D[2y^2]=4y\frac{dy}{dx},\ D[22]=0$$

At this point, notice that $\frac{dy}{dx}$ is a quantity in the equation. In order to get an expression for $\frac{dy}{dx}$, we solve for it as if it were a variable.

$2x+4y\frac{dy}{dx}=0$	Start where we left off.
$4y\frac{dy}{dx}=-2x$	Subtract $2x$ from both sides.
$\frac{dy}{dx}=-\frac{2x}{4y}$	Divide both sides by $4y$.
$\frac{dy}{dx}=-\frac{x}{2y}$	Simplify the fraction to its lowest terms.

This means that $\frac{dy}{dx}=-\frac{x}{2y}$. Note that the expression is written in terms of both x and y. This is very common with implicit differentiation.

STEP BY STEP

To find $\frac{dy}{dx}$ implicitly, perform these steps to the equation.

1. Differentiate both sides with respect to x.
2. Collect all terms with $\frac{dy}{dx}$ to one side.
3. Solve for $\frac{dy}{dx}$.

EXAMPLE Now, let's look at another example. Given $2x^2+3xy+4y^2=100$, compute $\frac{dy}{dx}$.

$2x^2+3xy+4y^2=100$	Start with the original relation.
$\frac{d}{dx}[2x^2]+\frac{d}{dx}[3xy]+\frac{d}{dx}[4y^2]=\frac{d}{dx}[100]$	Apply the derivative to each term (use the sum/difference rule).
$4x+3(y)+3x\frac{dy}{dx}+8y\frac{dy}{dx}=0$	$D[2x^2]=4x$ $D[3xy]=D[3x\cdot y]=(3)y+3x\frac{dy}{dx}$ (product rule) $D[4y^2]=8y\frac{dy}{dx}$
$3x\frac{dy}{dx}+8y\frac{dy}{dx}=-4x-3y$	Subtract $4x$ and $3y$ from both sides.
$(3x+8y)\frac{dy}{dx}=-4x-3y$	Factor out $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{-4x - 3y}{3x + 8y}$ Divide both sides by $3x + 8y$.

Thus, $\frac{dy}{dx} = \frac{-4x - 3y}{3x + 8y}$.

TRY IT

Consider the equation $10x^2y^2 + 4x^3 - 3y^5 = 11$.

Find the derivative implicitly. +

Take the derivative of both sides with respect to *x*:

$$\frac{d}{dx}[10x^2y^2] + \frac{d}{dx}[4x^3] - \frac{d}{dx}[3y^5] = \frac{d}{dx}[11]$$

The first term requires the product rule:

$$\begin{aligned}\frac{d}{dx}[10x^2y^2] &= y^2 \cdot D[10x^2] + 10x^2 \cdot D[y^2] \\ &= y^2(20x) + 10x^2 \cdot 2y \cdot \frac{dy}{dx} \\ &= 20y^2 + 20x^2y\frac{dy}{dx}\end{aligned}$$

The second term is a simple power rule:

$$\frac{d}{dx}[4x^3] = 12x^2$$

The third term also requires the power rule, but is a function of *y*:

$$\frac{d}{dx}[3y^5] = 15y^4 \cdot \frac{dy}{dx}$$

The derivative of the right-hand side is 0 since 11 is a constant.

Putting this all together, we have $20y^2 + 20x^2y\frac{dy}{dx} + 12x^2 - 15y^4\frac{dy}{dx} = 0$.

Next, isolate all $\frac{dy}{dx}$ terms to one side of the equation. In this case, move the $\frac{dy}{dx}$ terms to the right-hand side.

$$20y^2 + 12x^2 = 15y^4\frac{dy}{dx} - 20x^2y\frac{dy}{dx}$$

Now, factor out $\frac{dy}{dx}$ from the right side:

$$20y^2 + 12x^2 = \frac{dy}{dx}(15y^4 - 20x^2y)$$

Lastly, divide both sides by $15y^4 - 20x^2y$ to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{20y^2 + 12x^2}{15y^4 - 20x^2y}$$

WATCH

Here is a video in which we find $\frac{dy}{dx}$ of $\cos(xy) = -\frac{1}{2} + e^y$.

Once we know the derivative, it is possible to find the slope of the tangent line, then the equation of the tangent line.

Since the implicit derivatives use the notation $\frac{dy}{dx}$ for the derivative, we need a way to show that we are evaluating the derivative at a point.

BIG IDEA

The notation $\left.\frac{dy}{dx}\right|_{(a,\, b)}$ means to evaluate $\frac{dy}{dx}$ when $x = a$ and $y = b$.

Now, we are ready to find slopes of tangent lines with implicit functions.

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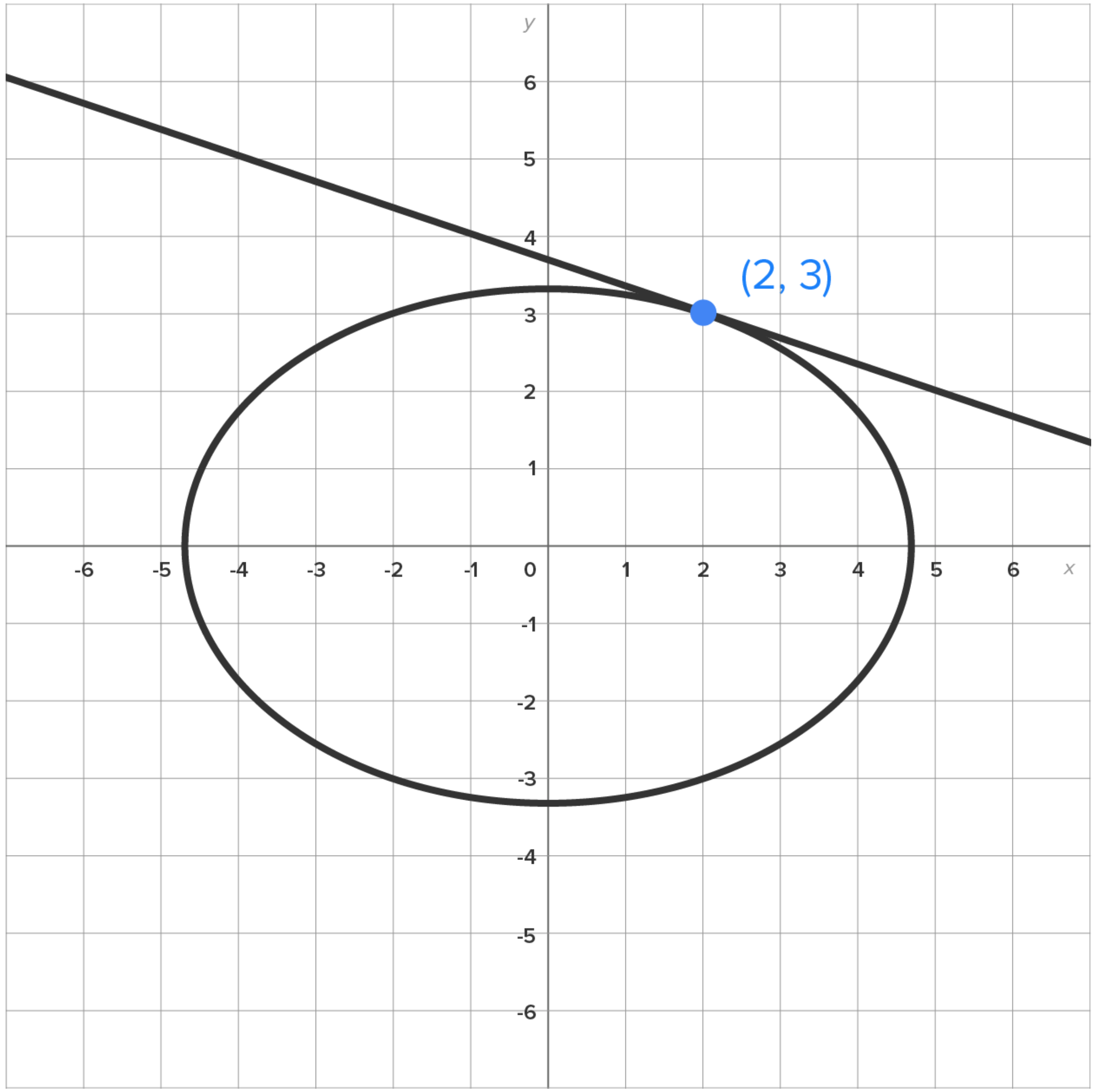
2. Slopes and Equations of Tangent Lines

Earlier in this challenge, we computed $\frac{dy}{dx}$ for the curve $x^2 + 2y^2 = 22$.

Shown in the graph below is the curve (the ellipse), and its tangent line at the point (2, 3).



The derivative formula we calculated earlier is $\frac{dy}{dx} = -\frac{x}{2y}$.

Then, the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(2,\ 3)} = -\frac{2}{2(3)} = -\frac{1}{3}$.

To write the equation of the tangent line, we normally need $f(a)$ and $f'(a)$. Since y is defined implicitly, we do not have the "*f*" notation. That being the case, we'll make use of the point-slope form of a line.

Now, let's find the equation of the tangent line.

$y - y_1 = m(x - x_1)$ Use the point-slope form.

$y - 3 = -\frac{1}{3}(x - 2)$ The line passes through (2, 3) and has slope $-\frac{1}{3}$.

$y - 3 = -\frac{1}{3}x + \frac{2}{3}$ Distribute $-\frac{1}{3}$.

$y = -\frac{1}{3}x + \frac{11}{3}$ Add 3 to both sides.

The equation of the tangent line is $y = -\frac{1}{3}x + \frac{11}{3}$.

TRY IT

Consider the curve $2x^2 + 3xy + 4y^2 = 100$ with $\frac{dy}{dx} = \frac{-4x - 3y}{3x + 8y}$.

Write the equation of the line tangent to this curve at the point (0, 5). +

First, find the slope of the tangent line by substituting $(0, 5)$ into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-4(0) - 3(5)}{3(0) + 8(5)} = \frac{-15}{40} = -\frac{3}{8}$$

Since the y-intercept is given, the equation of the line can be found by replacing $m = -\frac{3}{8}$ and $b = 5$ in the equation $y = mx + b$.

The equation of the tangent line is $y = -\frac{3}{8}x + 5$.

WATCH

Watch this video to see an example of writing an equation of a tangent line to $x^2 + 2xy + 4y^2 = 12$ at the point (2, 1).

SUMMARY

In this lesson, you learned that through **implicit differentiation**, it is possible to find the derivative of a mathematical relation that is not explicitly solved for *y*. You also learned that in an equation where *y* is defined implicitly, when asked to write the equation of a tangent line, you will be given a point on the curve; therefore, you can use the **point-slope form to write the equation of the tangent line**.