Critical Numbers

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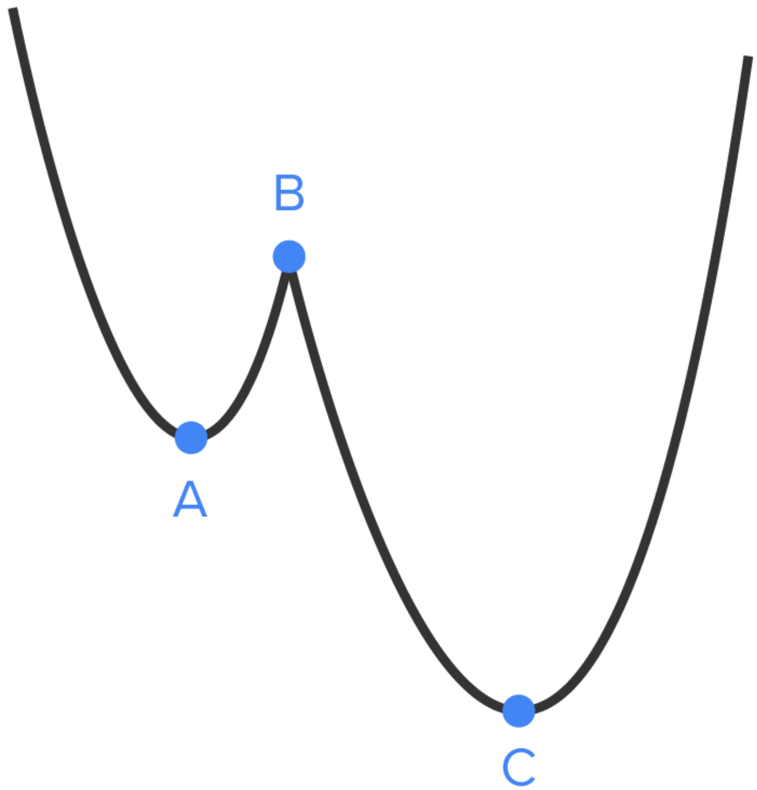
WHAT'S COVERED

In this lesson, you will now take a more analytical approach to locating maximum and minimum values of a function by finding critical values. Specifically, this lesson will cover:

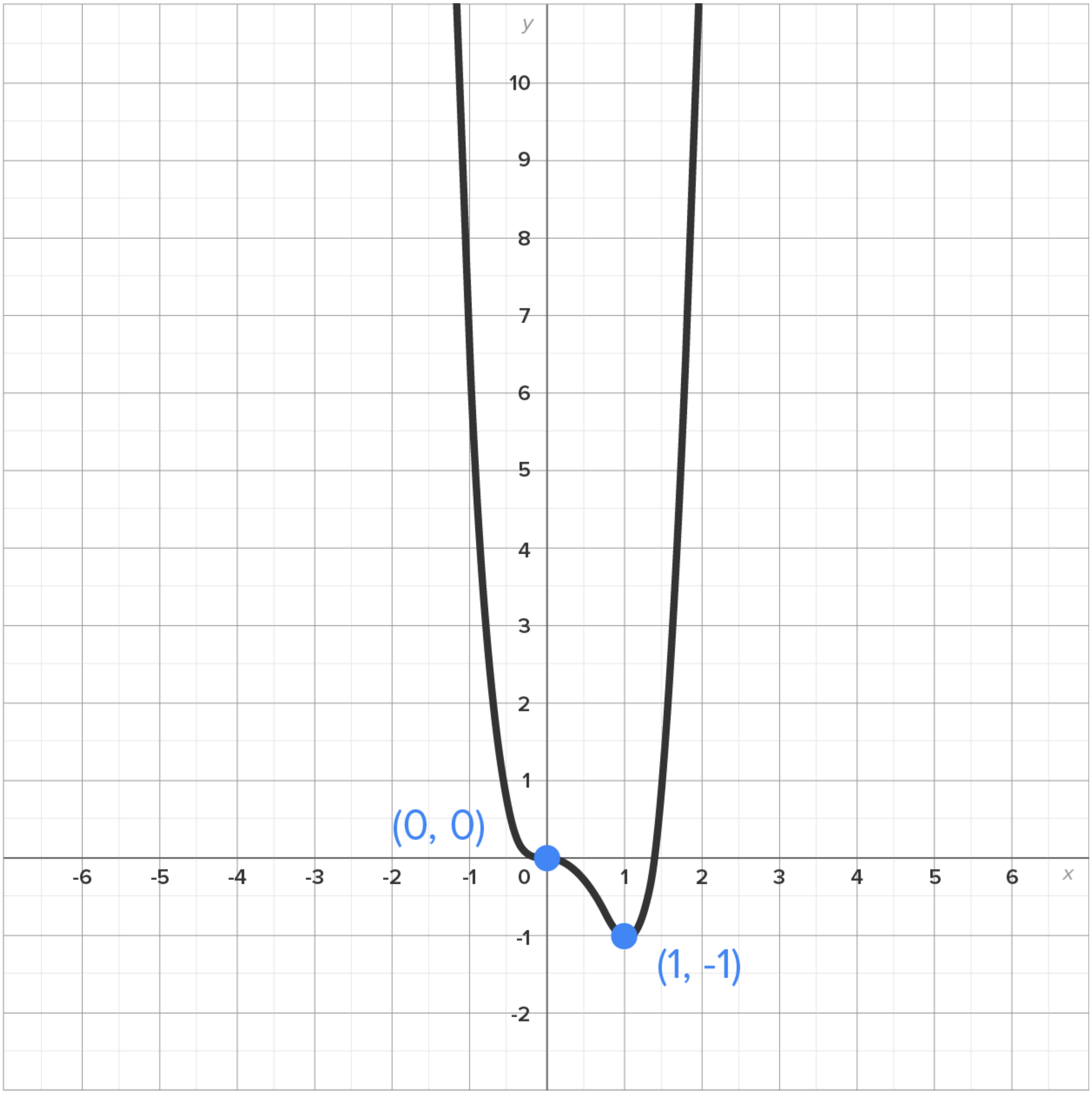
1. Defining Critical Numbers of a Function

Consider the graph below, which shows the graph of a function $y = f(x)$.



- At point A, there is a local minimum, and the derivative is 0.
- At point B, there is a local maximum, and the derivative is undefined.
- At point C, there is a local minimum, and the derivative is 0.

The x-coordinates where the derivative is 0 or undefined can lead to finding maximum or minimum values. Since this is important, these values of x are called **critical numbers**.
It is not always the case that a critical number leads to a maximum or minimum value. Consider this graph:



Note how $f'(x) = 0$ at both $x = 0$ and $x = 1$, but there is no minimum or maximum when $x = 0$ (but there is a local and global minimum when $x = 1$).

Thus, the critical numbers give information about where the minimum and maximum points could be, but further analysis will be required to determine the exact behavior at each critical number.

TERM TO KNOW

Critical Number

A value of c in the domain of $f(x)$ for which $f'(c) = 0$ or $f'(c)$ is undefined, provided that $f(c)$ is defined.

2. Finding Critical Numbers of a Function

Now that we know what critical numbers are, let's get some practice finding the critical numbers of a few functions.

HINT

When your goal is to find critical numbers, first note the domain of the function. When $f'(x)$ is undefined, this will help you determine which x-values are critical numbers and which x-values are not.

EXAMPLE Consider the function $f(x) = -x^3 + 12x + 10$. Find the critical numbers of $f(x)$.

$f(x) = -x^3 + 12x + 10$	Start with the original function; the domain is all real numbers.
$f'(x) = -3x^2 + 12$	Find the derivative of *f*.
$-3x^2 + 12 = 0$	Since $f'(x)$ is a polynomial, it is never undefined. Set $f'(x)$ equal to 0 to find critical numbers.
$-3(x^2 - 4) = 0$	Factor out -3.
$-3(x+2)(x-2) = 0$	Factor $x^2 - 4$.
$x + 2 = 0$ or $x - 2 = 0$	Set each factor equal to 0.
$x = -2, x = 2$	Solve.

Conclusion: the critical numbers of $f(x)$ are $x = -2$ and $x = 2$.

EXAMPLE Find the critical numbers of $f(x) = 10\sqrt{x} - x$.

$f(x) = 10\sqrt{x} - x$	Start with the original function. Note that the domain is $[0, \infty)$.
$f(x) = 10x^{1/2} - x$	Rewrite as a power to take the derivative.
$f'(x) = 5x^{-1/2} - 1$	Find the derivative of *f*.
$f'(x) = \frac{5}{x^{1/2}} - 1$	Rewrite using positive exponents.
$f'(x) = \frac{5}{\sqrt{x}} - 1$	Then, rewrite as a radical.
$\frac{5}{\sqrt{x}} - 1 = 0$	The derivative is undefined when $x = 0$. Since $f(0)$ is defined, $x = 0$ is a critical number. Set equal to 0 to find other critical numbers.
$\frac{5}{\sqrt{x}} = 1$	Add 1 to both sides.

$5 = \sqrt{x}$	Multiply both sides by $\sqrt{x}$.
$25 = x$	Square both sides.

Thus, the critical numbers are $x = 0$ and $x = 25$.

TRY IT

Consider the function $f(x) = \frac{1}{4}x - \ln x$.

Find all critical numbers of the function. +

The critical numbers are the values of *x* for which $f'(x) = 0$ or possibly undefined.

$$f'(x) = \frac{1}{4} - \frac{1}{x}$$

Note that $f'(0)$ is undefined, but $f(0)$ is also undefined, so $x = 0$ is not a critical number.

Setting $f'(x) = 0$, we have $\frac{1}{4} - \frac{1}{x} = 0$.

Add $\frac{1}{x}$ to both sides: $\frac{1}{4} = \frac{1}{x}$

Cross multiply: $x = 4$

Therefore, the critical number is $x = 4$.

Here is another example, but this time with a function where the quotient rule is required to find the derivative.

EXAMPLE Find all critical numbers of $f(x) = \frac{x^2 - 4x + 13}{x - 2}$.

$f(x) = \frac{x^2 - 4x + 13}{x - 2}$	Start with the original function.
$f'(x) = \frac{(x-2)\cdot(2x-4) - (x^2 - 4x + 13)\cdot(1)}{(x-2)^2}$	Since f(x) is a quotient of two functions, use the quotient rule to find its derivative. Note: $D[x^2 - 4x + 13] = 2x - 4$ and $D[x - 2] = 1$
$f'(x) = \frac{2x^2 - 8x + 8 - x^2 + 4x - 13}{(x-2)^2}$	Multiply out $(2x - 4)(x - 2)$ and distribute $-(x^2 - 4x + 13)$.

$$f'(x) = \frac{x^2 - 4x - 5}{(x-2)^2}$$ Combine like terms in the numerator.

At this point, we must consider values of *x* where $f'(x)$ is undefined or when $f'(x) = 0$.

We see that $f'(x)$ is undefined when the denominator $(x-2)^2 = 0$, which means $x = 2$. However, the original function is also undefined when $x = 2$, meaning that $x = 2$ is not a critical number.

Setting $f'(x) = 0$, we have the following:

$\frac{x^2 - 4x - 5}{(x-2)^2} = 0$	Set $f'(x) = 0$.
$x^2 - 4x - 5 = 0$	For a fraction to be equal to 0, its numerator must be equal to 0.
$(x-5)(x+1) = 0$	Factor the numerator.
$x - 5 = 0$ or $x + 1 = 0$	Set each factor equal to 0.
$x = 5$ or $x = -1$	Solve each equation.

Therefore, the critical numbers of $f(x)$ are $x = 5$ and $x = -1$.

Note: To solve $x^2 - 4x - 5$, you may choose to use the quadratic formula instead. Factoring was used since it worked out nicely.

SUMMARY

In this lesson, you learned how to **define critical numbers of a function**, which are values of *x* in the domain of the function where the derivative is either 0 or undefined, that give information where the local minimum and maximum points could be. Keep in mind, however, that further analysis is needed to determine the exact behavior at each critical number. Then, you used this newly acquired knowledge to practice **finding the critical numbers of a few functions**, noting that it is important to pay attention to the domain of the function, which will help you determine which x-values are critical numbers and which are not.

TERMS TO KNOW

Critical Number

A value of *c* in the domain of $f(x)$ for which $f'(c) = 0$ or $f'(c)$ is undefined, provided that $f(c)$ is defined.