Comparing Limits of Functions: Squeeze Theorem

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WHAT'S COVERED

In this lesson, you will evaluate more difficult limits by comparing them to other known limits. Specifically, this lesson will cover:

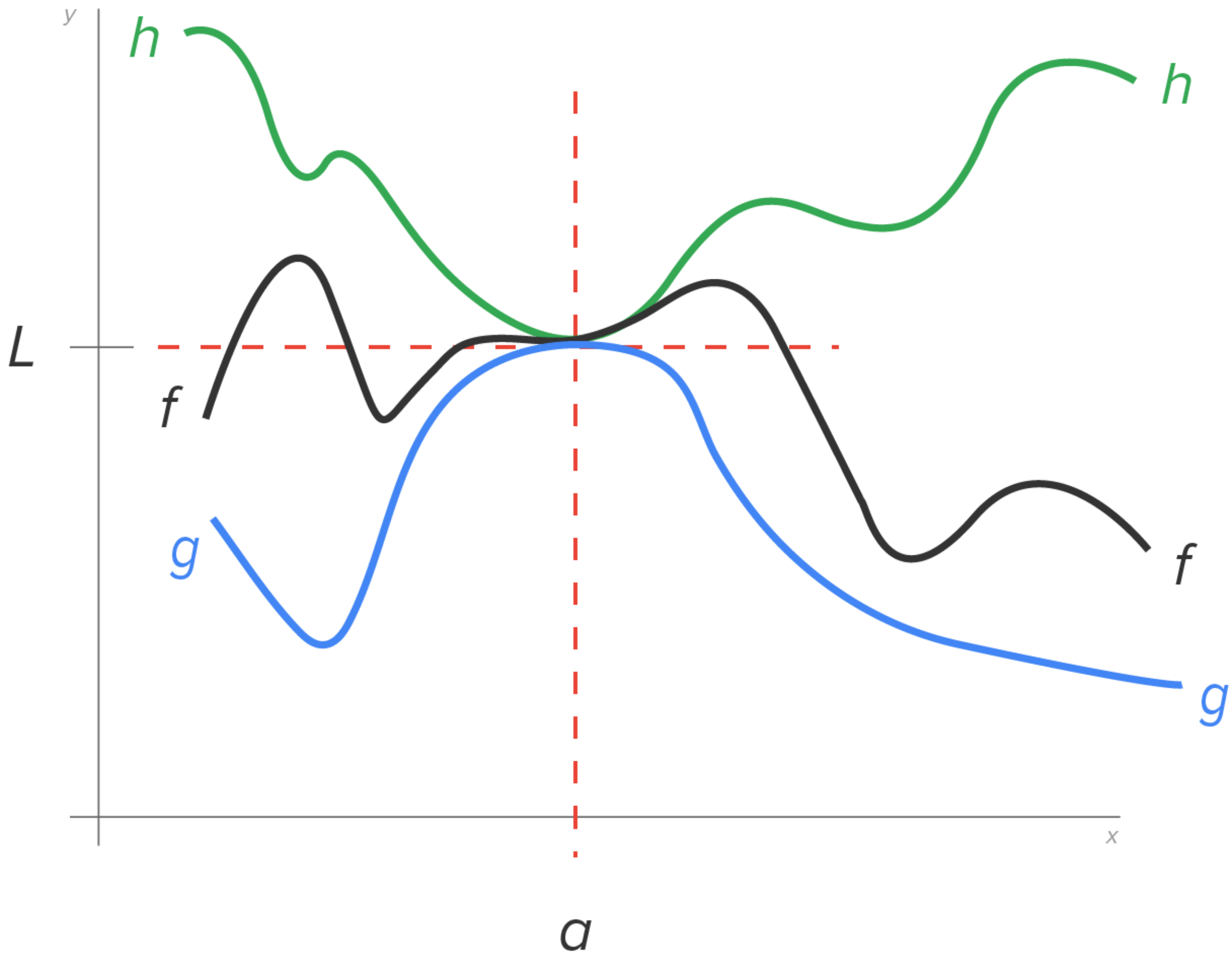
1. Defining the Squeeze Theorem

The squeeze theorem is a theorem that uses limit values and states the following:

KEY CONCEPT

Suppose that $g(x) \le f(x) \le h(x)$ for all values of x near $x = a$, as shown in the figure below.

If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.



2. Evaluating Limits by Using the Squeeze Theorem

You can evaluate limits by using the squeeze theorem.

EXAMPLE Consider the limit $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$. Note that direct substitution does not work since the function is undefined when $x = 0$.

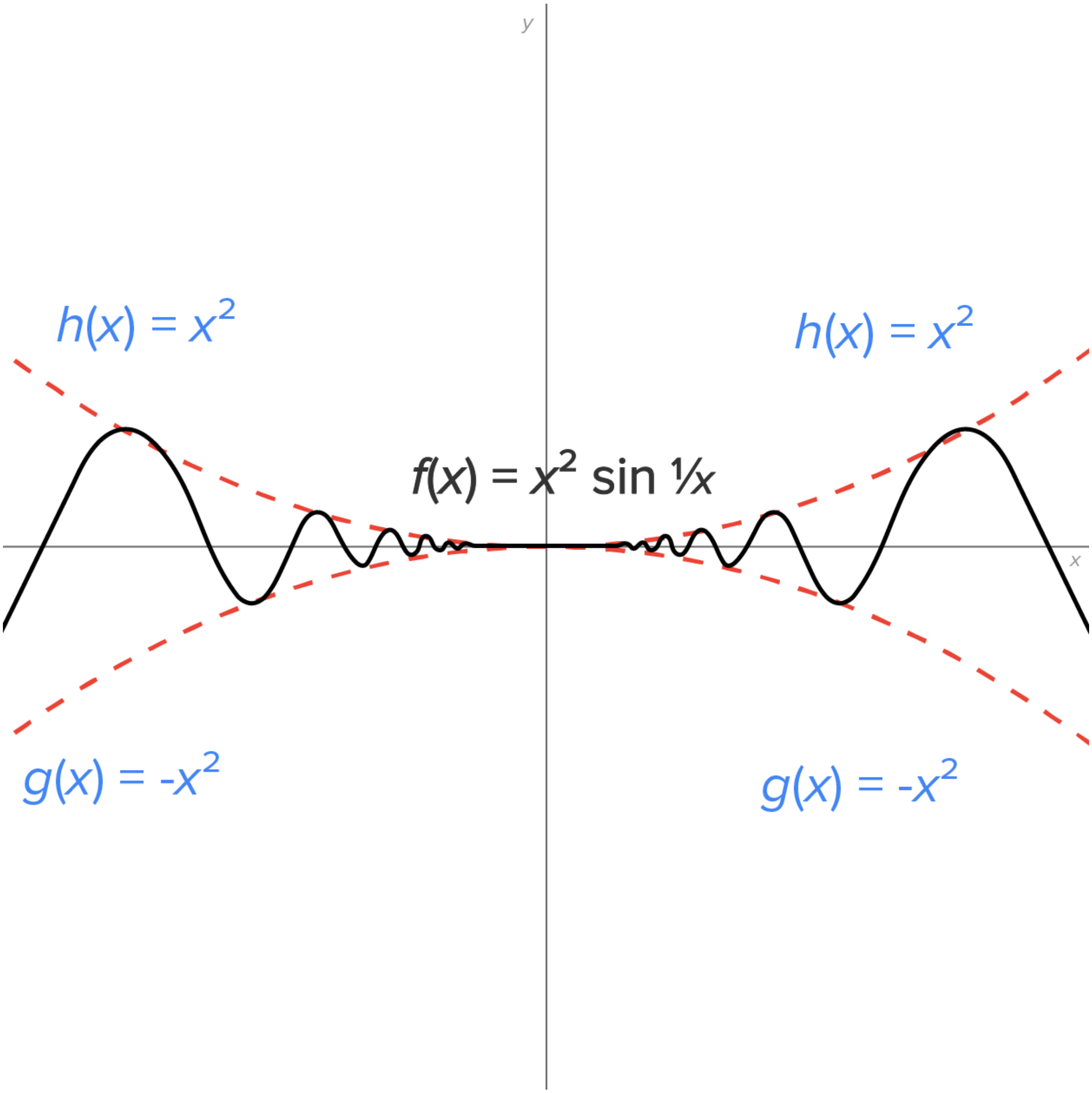
Recall that the range of the sine function is $[-1, 1]$. This means for any choice of angle θ, $-1 \le \sin\theta \le 1$. This also means that $-1 \le \sin\left(\frac{1}{x}\right) \le 1\ x \ne 0$.

Now, multiply all three parts of the inequality by x^2. Since $x^2 > 0$, the direction of the inequalities is preserved:

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2 \; x \neq 0$$

Let $g(x) = -x^2$, $h(x) = x^2$, and $f(x) = x^2 \sin\left(\frac{1}{x}\right)$. Since $\lim_{x \to 0} (-x^2) = 0$ and $\lim_{x \to 0} x^2 = 0$, it follows by the squeeze theorem that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Here is a graph that helps to describe the situation. As you can see, the graph of $f(x)$ is always between the graphs of $g(x)$ and $h(x)$.



EXAMPLE Suppose $4x - 3 \le f(x) \le x^2 + 1$ for all x near $x = 2$, except possibly at $x = 2$. Let's evaluate $\lim_{x \to 2} f(x)$.

Since $\lim_{x \to 2}(4x - 3) = 4(2) - 3 = 5$ and $\lim_{x \to 2}(x^2 + 1) = 2^2 + 1 = 5$, it follows by the squeeze theorem that $\lim_{x \to 2} f(x) = 5$.

TRY IT

Consider the fact that $\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}$ near $x = 0$. Suppose you want to find $\lim_{x \to 0} \frac{\sin x}{x}$.

Evaluate this limit.

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Note that $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} \frac{1}{\cos x} = 1$.

Since $\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}$, it follows that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

SUMMARY

In this lesson, you learned the **definition of the squeeze theorem**, which lets us find the limit of a function as *x* approaches *a* whose function values are between two other functions on both sides of *a*, and where the limits of the two other functions are the same as *x* approaches *a*. You learned that you can use the **squeeze theorem to evaluate limits** that are particularly difficult, with functions that have function values between two functions with known and equal limit values.