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Related Rates Problems Using Proportional Reasoning and Trigonometry

by Sophia

WHAT'S COVERED

In this lesson, you will continue to explore related rates problems that involve proportional reason (such as similar triangles) and trigonometry. For example, we may want to determine how an angle of inclination changes for a camera that is following a rocket that is taking off. Specifically, this lesson will cover:

1. Related Rates Problems Involving Proportions

We will now look at problems where we need to use proportional reasoning (similar triangles, etc.) to get a relationship between the variables.

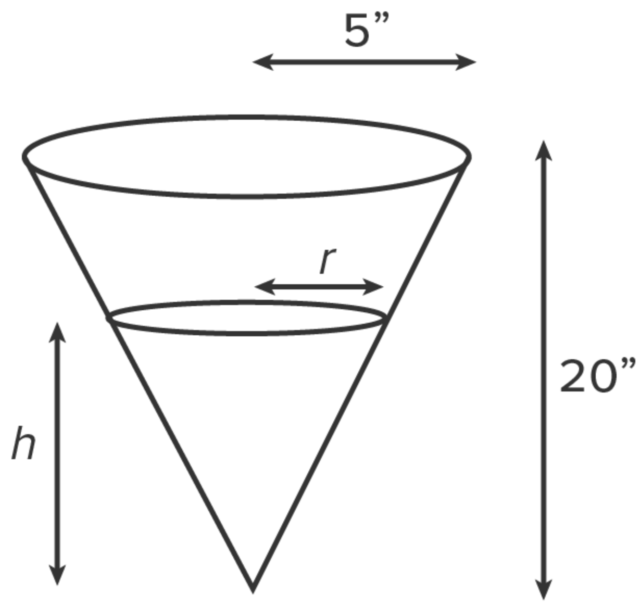
EXAMPLE Water is filling a cone-shaped container at a rate of $30\pi \text{ in}^3/\text{min}$. The container is 10” wide at the top and 20” deep. At what rate is the height of the water changing when the water is 10 inches deep?

This information means we are given $\frac{dV}{dt}=30\pi$, and we want $\frac{dh}{dt}$ when $h=10$.

From the geometry formulas, we know that $V=\frac{\pi}{3}r^2h$, where:

- r = the radius of the water in the cone
- h = the height of the water in the cone

Notice that there isn’t any information given about the radius of the conical shape. However, we do have the information we need to solve this problem.



Since the cone is 10” wide across at the top, its radius is 5”.

Notice that the water in the cone forms a smaller version of the conical container, which means that the height and radius are in proportion to each other. In the full cone, the height is 4 times the radius. To figure out the relationship, we know that $\frac{h}{r} = \frac{20}{5}$.

Now we have to decide whether to solve for *h* or *r*.

Since there is no information given about the radius in this problem, we want to replace *r* in the formula. This means we want to solve for *r*. Solving $\frac{h}{r} = \frac{20}{5}$ gives $5h = 20r$, or $r = \frac{1}{4}h$.

Now, we can write a volume formula in terms of only *h*:

$$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi}{3}\cdot\frac{h^2}{16}\cdot h = \frac{\pi}{48}h^3$$
$$V = \frac{\pi}{48}h^3$$

Since rates are involved, find the derivative with respect to time:

$V = \frac{\pi}{48}h^3$	Start with the original equation.
$\frac{dV}{dt} = \frac{\pi}{48}(3h^2)\cdot\frac{dh}{dt}$	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.

$$\frac{dV}{dt} = \frac{\pi}{16}h^2 \cdot \frac{dh}{dt} \quad \text{Simplify.}$$

$$30\pi = \frac{\pi}{16}(10)^2 \cdot \frac{dh}{dt} \quad \frac{dV}{dt} = 30\pi,\ h = 10,\ \frac{dh}{dt} = ?$$

$$30\pi = \frac{25\pi}{4} \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = 30\pi \cdot \frac{4}{25\pi} = \frac{24}{5}\ \text{in/min}$$

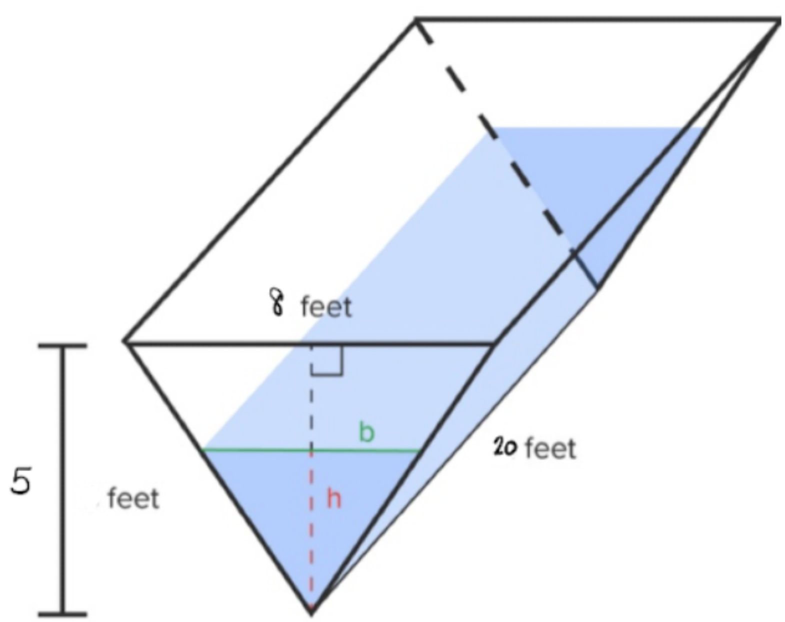
In conclusion, the height is increasing at a rate of 4.8 inches per minute when the water is 10 inches deep.

WATCH

This is an example of related rates of distance from a flagpole.

TRY IT

A trough is 20 feet long and its ends are in the form of inverted isosceles triangles having an altitude of 5 feet and a base of 8 feet. A figure is shown here.



Notice that *b* and *h* in the figure are in proportion to each other. Write an equation for *b* in terms of *h*. +

The sides *b* and *h* are in proportion to the sides 8 and 5, respectively.

This means that $\frac{b}{h} = \frac{8}{5}$. Solving for *b* gives $b = \frac{8}{5}h$, or $b = 1.6h$.

Write a formula for the volume of the water in the trough in terms of *h*. +

The volume of the trough is the area of the triangular base, times its length.

The triangular base has area $\frac{1}{2}bh$, and the length of the trough is 20 feet regardless of how much water is in the trough.

Therefore, the volume is $V=(20)\left(\frac{1}{2}bh\right)=10bh$.

Now, replace b with $1.6h$: $V=10(1.6h)(h)=16h^2$.

Find the derivative of each side with respect to *t* to get an equation that relates the rates of change. +

$$\frac{dV}{dt}=32h\cdot\frac{dh}{dt}$$

Water is being pumped in at a rate of 3 cubic feet per minute. How fast is the height of the water changing when the water is 4 feet deep? +

First, substitute all known quantities. We have $\frac{dV}{dt}=3$ and we want to find $\frac{dh}{dt}$ when $h=4$.

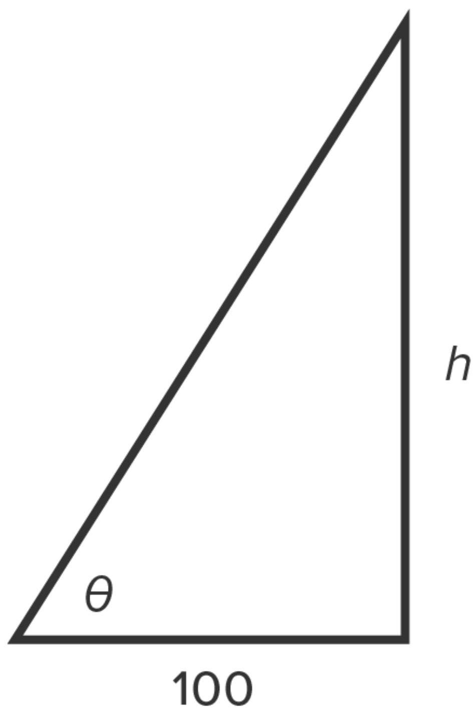
$\frac{dV}{dt}=32h\cdot\frac{dh}{dt}$	Use this equation as a starting point.
$3=32(4)\cdot\frac{dh}{dt}$	Replace the known quantities.
$\frac{3}{128}=\frac{dh}{dt}$	Simplify the right-hand side, then divide both sides by 128 to solve for $\frac{dh}{dt}$.

Conclusion: The height is changing at a rate of $\frac{3}{128}$ feet per minute.

2. Related Rates Problems Involving Trigonometry

Related rates can also help answer questions about angles of inclination.

A camera is on ground level 100 feet from a rocket's launchpad and inclines upward as a rocket takes off vertically at a rate of 250 ft/s. At what rate is the angle of inclination changing when the rocket is 1000 feet off the ground?



Notice that the base is always 100 feet. This is not changing. Since the height of the rocket changes, the vertical side is a variable. Since the angle is changing, it is labeled with a variable (θ) as well.

To relate all the relevant quantities, we need to use a trigonometric function. Since the opposite and adjacent sides to angle θ are labeled, tangent is the best choice.

The equation is $\tan\theta = \frac{h}{100}$. Solving for θ, we have $\theta = \tan^{-1}\left(\frac{h}{100}\right)$.

We were given $\frac{dh}{dt} = 250$ and we want to find $\frac{d\theta}{dt}$ when $h = 1000$ feet.

Now, we take the derivative:

$\theta = \tan^{-1}\left(\frac{h}{100}\right)$ Start with the original equation.

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{h}{100}\right)^2} \cdot \frac{d}{dt}\left[\frac{h}{100}\right]$$

Take the derivative of both sides, remembering that each variable is being differentiated implicitly.

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{h}{100}\right)^2} \cdot \frac{1}{100} \cdot \frac{dh}{dt}$$

Distribute.

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{1000}{100}\right)^2} \cdot \frac{1}{100} \cdot 250$$

Substitute quantities.

$$\frac{d\theta}{dt} = \frac{1}{101} \cdot \frac{1}{100} \cdot 250 \approx 0.02475 \text{ radians/sec}$$

Simplify.

The angle is increasing at a rate of 0.02475 radians per second. For reference, this is about $1.42°$/second.

SUMMARY

In this lesson, you learned how to solve **related rates problems involving proportions and trigonometry** (which can help answer questions about angles of inclination). In these problems, equations were derived using mathematical facts more so than a standard formula. Once the equation is determined, the procedure for finding the related rates is exactly the same: take the derivative, substitute what it is known, then solve for the unknown.