Higher-Order Derivatives

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WHAT'S COVERED

In this lesson, you will take all the rules you have learned about derivatives and apply them to higher-order derivatives. There are situations in which the change in the rate of change is important and useful. Specifically, this lesson will cover:

1. **Higher-Order Derivatives: Definitions and Notation**
2. **Finding Higher-Order Derivatives**
3. **Interpreting Higher-Order Derivatives**

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1. Higher-Order Derivatives: Definitions and Notation

When you take the derivative of a function, you actually are taking the first derivative of a function. That said, the second derivative of a function is the derivative of the first derivative. Then, the third derivative is the derivative of the second derivative, and so on.

Here is a table that shows past notations used with first derivatives, as well as the corresponding notation used for higher-order derivatives.

Function Representation	y-notation		$f(x)$-notation	
1st Derivative	y'	$\dfrac{dy}{dx}$	$f'(x)$	$D[f(x)]$
2nd Derivative	y''	$\dfrac{d^2y}{dx^2}$	$f''(x)$	$D^2[f(x)]$
3rd Derivative	y'''	$\dfrac{d^3y}{dx^3}$	$f'''(x)$	$D^3[f(x)]$
4th Derivative	$y^{(4)}$	$\dfrac{d^4y}{dx^4}$	$f^{(4)}(x)$	$D^4[f(x)]$

n^{th} Derivative	$y^{(n)}$	$\dfrac{d^n y}{dx^n}$	$f^{(n)}(x)$	$D^n[f(x)]$

Note that in the "prime" notations, the order of the derivative is written as a number enclosed in parentheses. Anything more than three prime symbols would be difficult to read.

2. Finding Higher-Order Derivatives

To find a second derivative, we need to find the first derivative, then differentiate a second time.

⮞ EXAMPLE Find the second derivative of the function $f(x) = 3x^4 - 9x^3 + 3x^2 + 8x - 12$.

First derivative:

$f'(x) = 3(4x^3) - 9(3x^2) + 3(2x) + 8(1) - 0$ Apply the sum/difference and constant multiple rules.

$f'(x) = 12x^3 - 27x^2 + 6x + 8$ Simplify.

Second derivative:

$f''(x) = D[f'(x)] = D[12x^3 - 27x^2 + 6x + 8]$ Substitute the first derivative.

$f''(x) = 12(3x^2) - 27(2x) + 6 + 0$ Apply the sum/difference and constant multiple rules.

$f''(x) = 36x^2 - 54x + 6$ Simplify.

Thus, $f''(x) = 36x^2 - 54x + 6$.

⮞ EXAMPLE Find the 5th derivative of the function $y = 4\cos x$.

Note that in each step, the constant multiple rule is used.

$y = 4\cos x$ Start with the original function, $f(x)$.

$y' = 4(-\sin x) = -4\sin x$ Find the first derivative.

$y'' = -4(\cos x) = -4\cos x$ Find the second derivative.

$y''' = -4(-\sin x) = 4\sin x$ Find the third derivative.

$y^{(4)} = 4(\cos x) = 4\cos x$ Find the fourth derivative (notice this is the same as $f(x)$).

$y^{(5)} = 4(-\sin x) = -4\sin x$ Lastly, find the fifth derivative (notice this is the same as $f'(x)$).

Thus, $y^{(5)} = 4(-\sin x) = -4\sin x$.

📝 **TRY IT**

Consider the function $y = 10x^2 + 2\sin x$.

> ### Find the 6th derivative of the above formula.   +

To find $y^{(6)}$, we take the derivative of y six times:

$$y' = 20x + 2\cos x$$
$$y'' = 20 + 2(-\sin x) = 20 - 2\sin x$$
$$y''' = -2\cos x$$
$$y^{(4)} = -2(-\sin x) = 2\sin x$$
$$y^{(5)} = 2\cos x$$
$$y^{(6)} = 2(-\sin x) = -2\sin x$$

⤷ **EXAMPLE** Find the third derivative of $f(x) = 4x^2 - \dfrac{4}{x^3}$.

First, rewrite using negative exponents to make use of the power rule: $f(x) = 4x^2 - \dfrac{4}{x^3} = 4x^2 - 4x^{-3}$

Now, take the appropriate derivatives:

$$f'(x) = 4(2x) - 4(-3x^{-4})$$ Apply the sum/difference and power rules.

$$f'(x) = 8x + 12x^{-4}$$ Simplify.

Since we are finding more derivatives, there is no need to rewrite with positive exponents just yet. We will save this for when all derivatives are taken.

$$f''(x) = 8 + 12(-4x^{-5})$$ Apply the sum/difference and power rules.

$$f''(x) = 8 - 48x^{-5}$$ Simplify.

And now the third derivative:

$$f'''(x) = 0 - 48\left(-5x^{-6}\right)$$ Apply the sum/difference and power rules.

$$f'''(x) = 240x^{-6}$$ Simplify.

$$f'''(x) = \frac{240}{x^6}$$ Rewrite with positive exponents.

Thus, $f'''(x) = \frac{240}{x^6}$.

↪ EXAMPLE Find the 4th derivative of $f(x) = \frac{3}{x} - 6\sin x$.

First, write $\frac{3}{x}$ as $3x^{-1}$ to make use of the power rule.

Rewritten, we have $f(x) = 3x^{-1} - 6\sin x$.

First derivative:

$$f'(x) = 3(-1)x^{-2} - 6(\cos x)$$ Apply the power rule and sum/difference rules. Note that $D[\sin x] = \cos x$.

$$= -3x^{-2} - 6\cos x$$ Simplify.

At this point, we won't rewrite yet since we have more derivatives to take.

Second derivative:

$$f''(x) = -3(-2)x^{-3} - 6(-\sin x)$$ Apply the power rule and sum/difference rules. Note that $D[\cos x] = -\sin x$.

$$= 6x^{-3} + 6\sin x$$ Simplify.

Third derivative:

$$f'''(x) = 6(-3)x^{-4} + 6(\cos x)$$ Apply the power rule and sum/difference rules. Note that $D[\sin x] = \cos x$.

$$= -18x^{-4} + 6\cos x$$ Simplify.

And finally, the 4th derivative. Note the use of the notation $f^{(4)}(x)$.

$$f^{(4)}(x) = -18(-4)x^{-5} + 6(-\sin x)$$ Apply the power rule and sum/difference rules. Note that $D[\cos x] = -\sin x$.

$$= 72x^{-5} - 6\sin x$$ Simplify.

Since this is the last derivative we need to take, we can rewrite the negative exponent.

In conclusion, $f^{(4)}(x) = \dfrac{72}{x^5} - 6\sin x$.

Sometimes different rules are needed to find higher-order derivatives.

↪ EXAMPLE   Find the second derivative of the function $y = 5\tan x$.

First derivative: $\dfrac{dy}{dx} = D[5\tan x] = 5\sec^2 x$

Second derivative:

$$\frac{d^2y}{dx^2} = D[5\sec^2 x] = 5D[\sec^2 x]$$ Apply the constant multiple rule.

$$\frac{d^2y}{dx^2} = 5(2\sec x \cdot D[\sec x])$$ Apply the power rule.

$$\frac{d^2y}{dx^2} = 5(2\sec x \cdot \sec x \tan x)$$ $D[\sec x] = \sec x \tan x$

$$\frac{d^2y}{dx^2} = 10\sec^2 x \tan x$$ Perform multiplications.

Thus, $\dfrac{d^2y}{dx^2} = 10\sec^2 x \tan x$.

☑   **TRY IT**

Consider the function $f(x) = 2\sec x$.

Find the second derivative.   +

To find $f''(x)$, first find $f'(x)$, then take its derivative.

First, $f'(x) = 2(\sec x \tan x) = 2\sec x \tan x$.

Now, we need to find the derivative of $2\sec x \tan x$, which is the product of $2\sec x$ and $\tan x$.

Then:

$$f''(x) = D[2\sec x] \cdot \tan x + D[\tan x] \cdot 2\sec x$$
$$= 2\sec x \tan x(\tan x) + \sec^2 x(2\sec x)$$
$$= 2\sec x \tan^2 x + 2\sec^3 x$$

3. Interpreting Higher-Order Derivatives

Recall that $f'(x)$ is the (instantaneous) rate of change of $f(x)$. Then:

- $f''(x)$ is the rate of change of $f'(x)$.
- $f'''(x)$ is the rate of change of $f''(x)$.
- And so on...

The interpretation of higher-order derivatives depends on the meaning of the original function. For example, if $h(t)$ measures the distance that an object has traveled t seconds after being set into motion, then $h'(t)$ is the change in distance with respect to time. This is called the **velocity** of the object.
Furthermore, we are also interested in how the velocity changes, which is $h''(t)$. This is called the **acceleration** of the object. The rate of change in acceleration, called **jerk**, is $h'''(t)$.

⤷ EXAMPLE A tennis ball is launched off the top of a building. Its height (in feet) after t seconds is modeled by the function $h(t) = -16t^2 + 20t + 40$. Let's find the height, velocity, and acceleration at the precise moment the object was in the air for 2 seconds.

Remember units:
- $h(t)$ = height after t seconds
- $h'(t)$ = velocity = change in height per second (feet/sec)
- $h''(t)$ = acceleration = change in velocity per second (ft/sec)/sec

Height: $h(2) = -16(2)^2 + 20(2) + 40 = 16$ feet

Velocity: $h'(t) = -32t + 20$
Then, $h'(2) = -32(2) + 20 = -44$ feet per second

Acceleration: $h''(t) = -32$
This means that $h''(t) = -32$ regardless of t. Therefore, $h''(2) = -32$ feet per second per second, meaning that the velocity is changing by -32 feet per second each second.

▶ WATCH

In this video, we will examine the rate of change in the derivative (slope) of $f(x) = \sin^2 x$ at $x = 0$.

📋 SUMMARY

In this lesson, you learned the **definitions and notation of higher-order derivatives**, understanding that **finding higher-order derivatives** involves using rules of derivatives repeatedly. You learned that the **interpretation of higher-order derivatives** depends on the meaning of the original function. For example, if $h(t)$ measures the distance that an object has traveled t seconds after being set into motion, then the first derivative, $h'(t)$, is the velocity of the object; the second derivative, $h''(t)$, is the acceleration of the object; and the third derivative, $h'''(t)$, is the rate of change in acceleration, or jerk. We will explore the geometrical meanings of the first and second derivatives in unit 4.

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