The Algorithm for Newton's Method

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WHAT'S COVERED

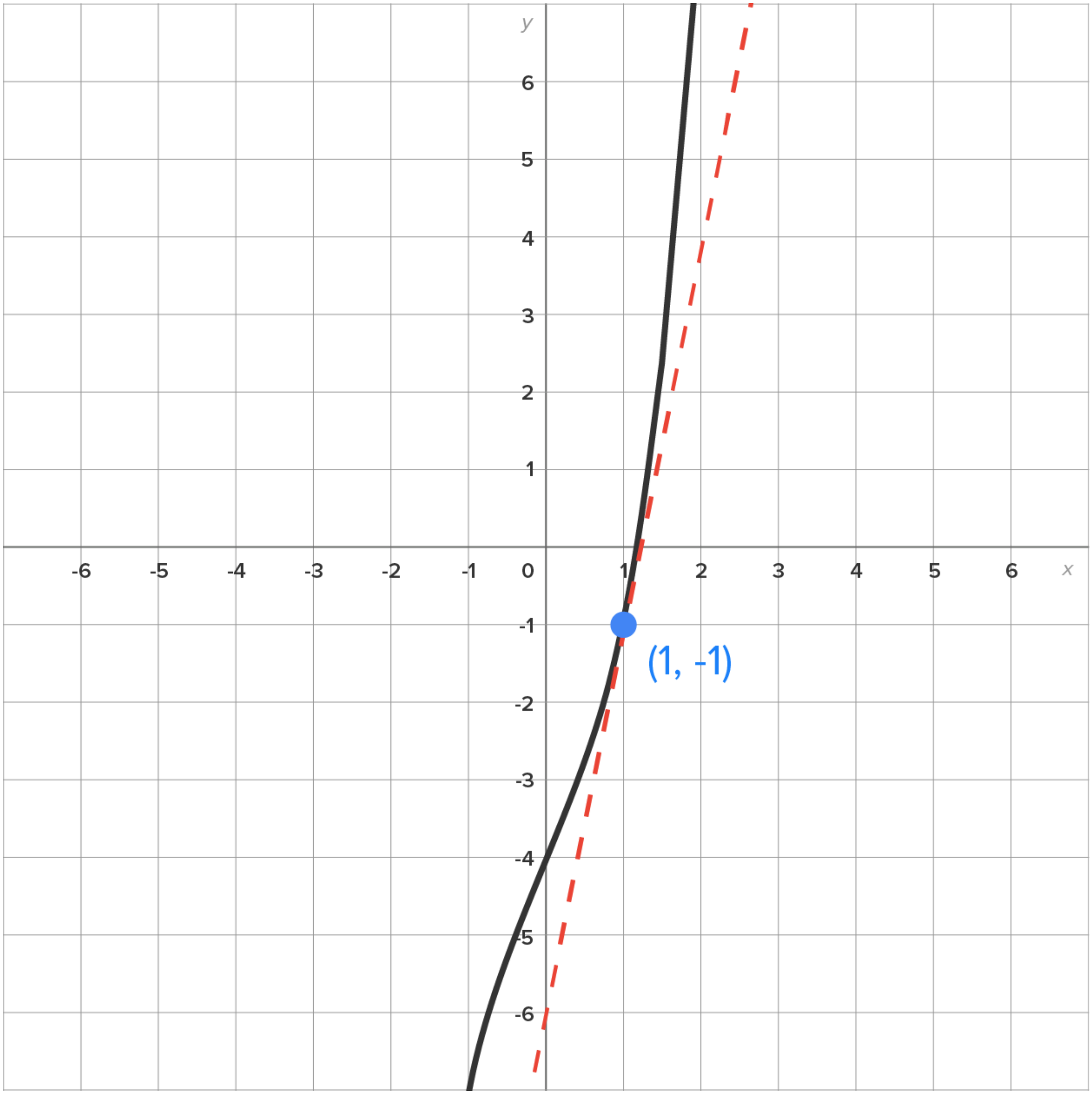
As you have seen in this challenge, the tangent line to a function at $x = a$ provides a good estimate to $f(x)$ near $x = a$. Another way to use the tangent line is to find its x-intercept to approximate the x-intercept of $f(x)$. In this lesson, you will learn Newton's method, which uses successive tangent lines to approximate an x-intercept. Most graphing utilities use Newton's method to locate x-intercepts and points of intersections of graphs. Specifically, this lesson will cover:

1. The Idea Behind Newton's Method
2. Applying Newton's Method
   - 2a. Newton's Method: The Algorithm
   - 2b. Approximating x-Intercepts with Newton's Method

1. The Idea Behind Newton's Method

The goal of Newton's method is to use tangent lines to approximate an x-intercept of the graph of $y = f(x)$. In other words, the goal is to solve the equation $f(x) = 0$.

Consider the function $f(x) = x^3 + 2x - 4$.



Now, consider the picture shown above, which has two graphs:

- The solid curve is the graph of $f(x)$.
- The dashed line is the tangent line at $x = 1$ (this corresponds to our "guess").

To start the process for Newton's method, we're going to "guess" $x = 1$ as the x-intercept.
Notice that the x-intercept of $f(x)$ is very close to the x-intercept of the tangent line. The advantage of using the tangent line is that it is much easier to solve a linear equation than it is a cubic equation.

First step: Find the equation of the tangent line at $x = 1$.

Given $f(x)=x^3+2x-4$, the derivative is $f'(x)=3x^2+2$. Then, the slope of the tangent line is $f'(1)=3(1)^2+2=5$.

Then, the equation of the tangent line is:

$$y=f(1)+f'(1)(x-1)$$
$$y=-1+5(x-1)$$
$$y=-1+5x-5$$
$$y=5x-6$$

Then, the x-intercept of the tangent line is found by letting $y=0$ and solving for *x*:

$$0=5x-6$$
$$6=5x$$
$$\frac{6}{5}=x \text{ (or 1.2 in decimal form)}$$

Thus, our approximation for the x-intercept is $(1.2, 0)$.
So, where would we go from here?

We now have a new "guess" for the x-intercept of the graph of $f(x)$. To continue with this process, find the equation of the tangent line to $f(x)$ at $x=1.2$, then find its x-intercept. We'll formalize this process and then complete this problem in the next part of this challenge.

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2. Applying Newton's Method

Consider a function $y=f(x)$ and let x_0 be the first guess for its x-intercept.

Write the equation of the tangent line at $x=x_0$: $y=f(x_0)+f'(x_0)(x-x_0)$.

Find the x-intercept of the tangent line, which means $y=0$:

$0=f(x_0)+f'(x_0)(x-x_0)$	Replace *y* with 0.
$-f(x_0)=f'(x_0)(x-x_0)$	Subtract $f(x_0)$ from both sides.
$-\frac{f(x_0)}{f'(x_0)}=x-x_0$	Divide both sides by $f'(x_0)$.
$x_0-\frac{f(x_0)}{f'(x_0)}=x$	Add x_0 to both sides.

Now, this x-intercept is the next guess for the intercept, which under normal conditions, is a closer estimate than x_0. Since this process will continue, let's call the x-intercept of the tangent line x_1. Then, $x_1=x_0-\frac{f(x_0)}{f'(x_0)}$.

Now, suppose we want to continue this process:

- Find the equation of the tangent line at $x = x_1$.
- Find the x-intercept of the tangent line and call it x_2. Then, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

If we continue this process, we get a sequence of estimates x_0, x_1, x_2, ... for the estimates of the x-intercept that get closer to some number (which would be the actual x-intercept). Performing these iterations is what is known as Newton's method.

2a. Newton's Method: The Algorithm

Suppose the goal is to find an approximation to an x-intercept of a function $y = f(x)$, which is equivalent to finding a solution to $f(x) = 0$. Starting with an initial guess at $x = x_0$, the sequence of guesses x_1, x_2, x_3, ... is generated by the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

The process stops when one of two things occurs:

- Two consecutive x-values are "close enough" together.
- The x-values are jumping around to the point where they aren't getting closer to a common number.

FORMULA TO KNOW

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

2b. Approximating x-Intercepts with Newton's Method

EXAMPLE Let's pick back up with the function $f(x) = x^3 + 2x - 4$. When we left off, we had $x_0 = 1$ and $x_1 = 1.2$. Let's perform two more iterations of Newton's Method to get a better approximation of the x-intercept. To use Newton's method, it is best to organize the information into a table:

Note: $f(x) = x^3 + 2x - 4$ and $f'(x) = 3x^2 + 2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	-1	5	1.2
1	1.2	0.128	6.32	1.179746835
2	1.179746835	0.001468379	6.175407787	1.179509057
3	1.179509057	0.0000002	6.173724847	1.179509025

The last two estimates are identical to 6 decimal places, so we conclude that the x-intercept to six decimal places of $f(x)$ is (1.179509, 0). This also means that the equation $x^3+2x-4=0$ has the solution $x \approx 1.179509$.

In this next example, we'll see how we can apply Newton's Method to approximating a square root of a number.

EXAMPLE We're going to use Newton's Method a little differently to approximate the value of $\sqrt{13}$, which is a solution to the equation $x^2-13=0$.

Therefore, we will apply Newton's Method to the function $f(x)=x^2-13$.

First, find the derivative of $f(x)$, which is $f'(x)=2x$.

Next, we know from Newton's formula that $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

Instead of making a table of values, we will form Newton's formula for this choice of $f(x)$. Using $f(x)=x^2-13$ and $f'(x)=2x$, we have:

$$x_{n+1}=x_n-\frac{x_n^2-13}{2x_n}$$

This is the formula we will use to find our approximations to the solution to $f(x)=0$. Recall that we need a starting value. In this case, we'll use $x_0=4$ since we know that $\sqrt{13}$ is between 3 and 4; likely closer to 4.

By the formula:

$$x_1=x_0-\frac{x_0^2-13}{2x_0}=4-\frac{4^2-13}{2(4)}=4-\frac{3}{8}=3.625$$

Applying the formula again to find x_2, we have:

$$x_2=x_1-\frac{x_1^2-13}{2x_1}=3.625-\frac{3.625^2-13}{2(3.625)}\approx 3.605603448$$

(Note: Remember that it is important to carry as many decimal places as possible from one step to the next to ensure accuracy. Whatever the desired accuracy is, it should be applied to the final answer only.)

Applying the formula one last time to find x_3, we have:

$$x_3 = x_2 - \frac{x_2^2 - 13}{2x_2} = 3.605603448 - \frac{3.605603448^2 - 13}{2(3.605603448)} \approx 3.605551276$$

Thus, after three iterations of Newton's Method, the (positive) solution to $x^2 - 13 = 0$ is approximately 3.605551276.

Using your calculator, you would see that $\sqrt{13} \approx 3.605551275$, which is very close!

WATCH

Use Newton's method to find the approximate solution to $x - \cos x = 0$.

SUMMARY

In this lesson, you learned **the idea behind Newton's method**, which is to use tangent lines to approximate an x-intercept of the graph of $y = f(x)$. Newton's method is a very straightforward approximation method designed to solve equations of the form $f(x) = 0$ (equivalent to finding the x-intercepts of the graph of $y = f(x)$). You learned how to **apply Newton's method** using its **algorithm**, by starting with an initial guess at $x = x_0$, then generating a sequence of guesses x_1, x_2, x_3, ... to arrive at a close **approximation of the x-intercept**.

FORMULAS TO KNOW

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.