The Mean Value Theorem for Integrals

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WHAT'S COVERED

In this lesson, you will connect the mean value theorem to integrals. Specifically, this lesson will cover:

1. The Mean Value Theorem for Integrals

Similar to the mean value theorem for derivatives, we can establish a theorem for integrals. If $f(x)$ is continuous on $[a, b]$, then at some point *c* in $[a, b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

In other words, there is at least one value of *c* in the interval $[a, b]$ such that $f(c) =$ the average value of $f(x)$ on $[a, b]$.

TERM TO KNOW

The Mean Value Theorem for Integrals

If $f(x)$ is continuous on $[a, b]$, then at some point *c* in $[a, b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

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2. Finding the Value of *c* Guaranteed by the Mean Value Theorem for Integrals

Let's look at a few examples to help illustrate the mean value theorem for integrals.

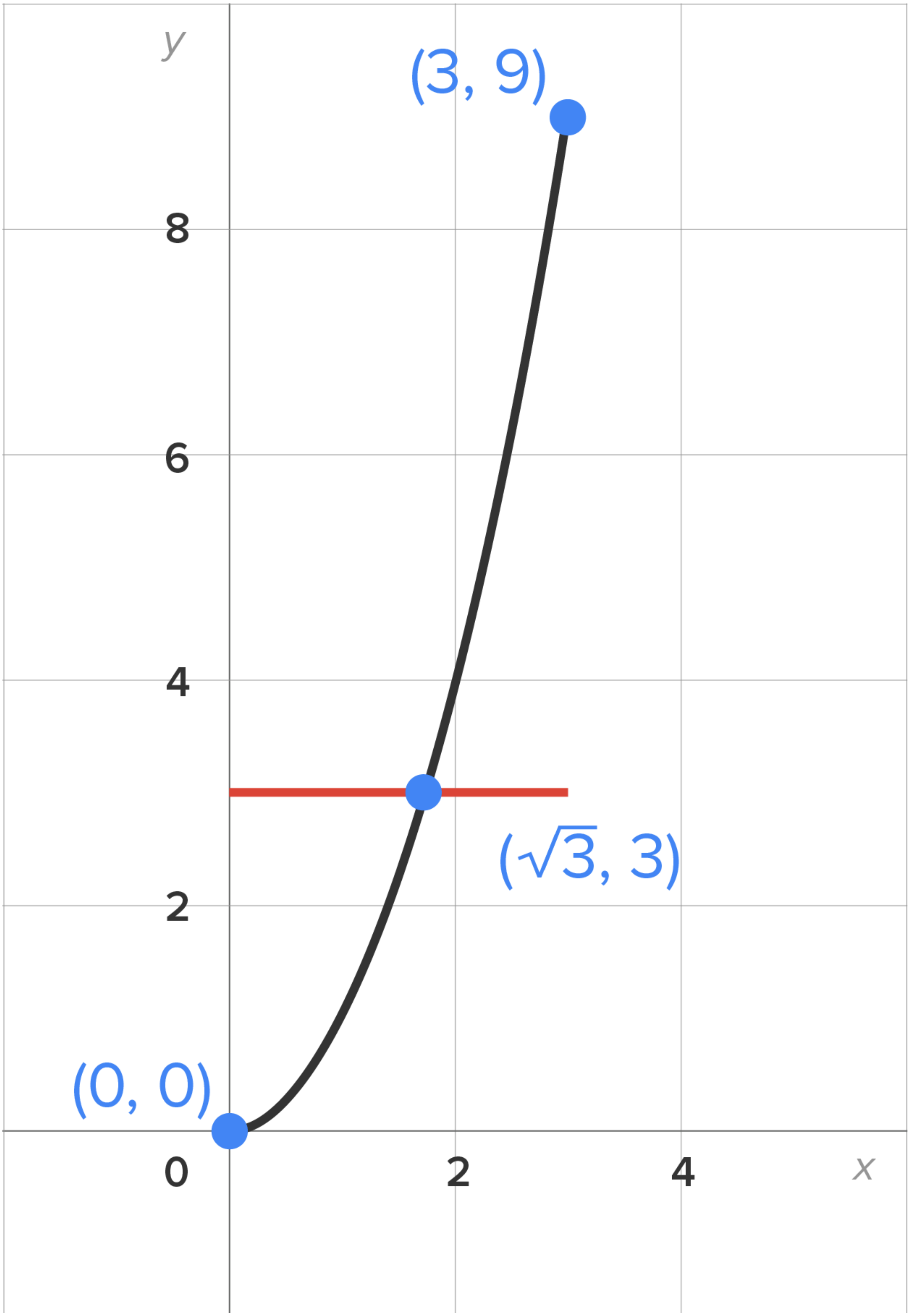
EXAMPLE Consider the function $f(x)=x^2$ on the interval $[0, 3]$.

The average value of $f(x)$ on $[0, 3]$ is $\frac{1}{3}\int_0^3 x^2 dx$. Evaluating, we have:

$$\frac{1}{3}\int_0^3 x^2 dx=\frac{1}{3}\cdot\frac{1}{3}x^3\Big|_0^3=\frac{1}{9}(3)^3-\frac{1}{9}(0)^3=3$$

To find the value of c, set $f(c)=3$. This means $c^2=3$, which means $c=\pm\sqrt{3}$. Since $-\sqrt{3}$ is not in the interval $[0, 3]$, the value of c guaranteed by the theorem is $c=\sqrt{3}$.

Here is the graph of $f(x)=x^2$ on the interval $[0, 3]$ along with the line $y=3$ (the average value). Note that they intersect at the point $(\sqrt{3}, 3)$.


y
(3, 9)
8
6
4
(√3, 3)
2
(0, 0)
0
2
4
x

WATCH

Check out this video to see the example to find the average value and the value of *c* guaranteed by the mean value theorem for $f(x) = 2x^2 - x$ on $[-1, 3]$.

TRY IT

Consider the function $f(x) = \frac{16}{x^3}$.

Find the average value of *f*(*x*) on the interval [1, 2]. +

The average value is found by calculating $\frac{1}{2-1}\int_1^2 \frac{16}{x^3}dx = \int_1^2 \frac{16}{x^3}dx$.

Now, evaluate the integral:

$= \int 16x^{-3}dx$	Rewrite in terms of negative exponents so the power rule can be used.
$= -8x^{-2}\Big\|_1^2$	Find the antiderivative: $\int 16x^{-3}dx = 16\left(\frac{1}{-2}\right)x^{-2} = -8x^{-2}$
$= (-8(2)^{-2}) - (-8(1)^{-2})$	Substitute $x = 2$ and $x = 1$, then subtract.
$= -8\left(\frac{1}{4}\right) + 8(1)$	Evaluate the exponential terms: $2^{-2} = \frac{1}{4}$ and $1^{-1} = 1$.
$= 6$	Simplify.

Therefore, the average value of $f(x)$ on $[1, 2]$ is equal to 6.

Find the value of *c* guaranteed by the mean value theorem for integrals. +

Since the average value is 6, we seek the value of *c* so that $f(c) = 6$.

This means we need to solve the equation $\frac{16}{c^3} = 6$.

$16 = 6c^3$	Multiply both sides by c^3.

$$\frac{8}{3} = c^3$$ Divide both sides by 6, then reduce the fraction.

$$\sqrt[3]{\frac{8}{3}} = c$$ Apply the cube root to both sides.

The approximate value of *c* is 1.39, which is between $x = 1$ and $x = 2$. Therefore, this is the value of *c* that is guaranteed by the mean value theorem for integrals.

HINT

Remember to check that each value of *c* is in the interval $[a, b]$.

- Those that are in $[a, b]$ are guaranteed by the mean value theorem for integrals.
- Those that are not in the interval are not guaranteed by the theorem.

SUMMARY

In this lesson, you learned that through **the mean value theorem for integrals**, you are able to guarantee that there is some input value (*c*) of a function $f(x)$ on $[a, b]$ in which $f(x)$ is equal to its average value on $[a, b]$. Next, you practiced **finding the value of *c* guaranteed by the mean value theorem for integrals**.

TERMS TO KNOW

The Mean Value Theorem for Integrals
If $f(x)$ is continuous on $[a, b]$, then at some point *c* in $[a, b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$