Continuous Functions

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WHAT'S COVERED

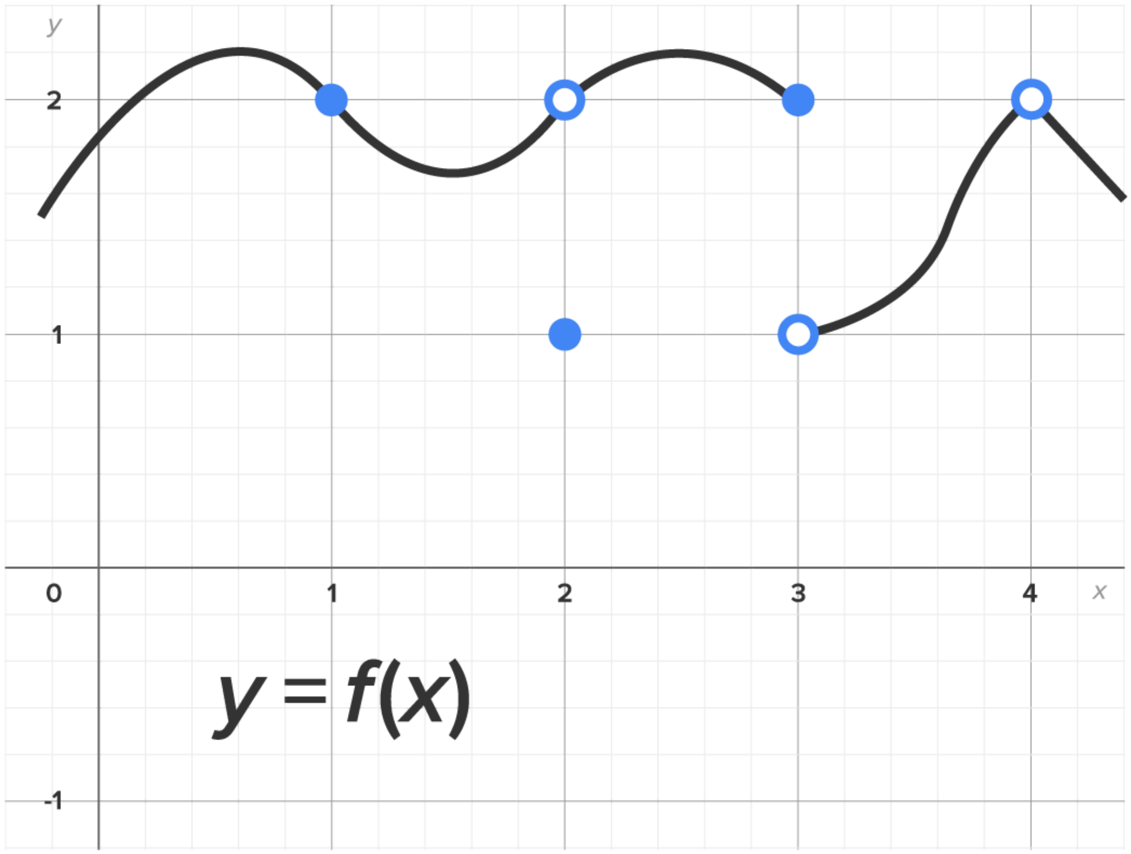
In this lesson, you will learn what it means for a function to be continuous, including how limits are used in relation to continuity. Specifically, this lesson will cover:

1. The Definition of Continuity

A function is called **continuous** at a point where there is no break in the graph *at that point*.

That is, $\lim_{x \to a} f(x) = f(a)$.

Consider the graph of $y = f(x)$ shown below. We will examine the continuity of $f(x)$ when $x = 1, 2, 3,$ and 4.



Given Point	Continuity of $f(x)$ at the Given Point
$x = 1$	The graph of $f(x)$ is continuous when $x = 1$ since there are no breaks in the graph at that point. Looking just before $x = 1$, the graph passes through the point $(1, f(1))$ and continues to "flow" afterwards.
$x = 2$	The graph of $f(x)$ is NOT continuous when $x = 2$. There is a hole in the graph when $x = 2$, meaning there is a break in the graph.
$x = 3$	The graph is NOT continuous when $x = 3$. There is a break in the graph.
$x = 4$	The graph is NOT continuous when $x = 4$. There is a hole in the graph.

Now, considering these 4 points, let's examine the limits at these points and the values of $f(x)$ at these points as well as whether or not the function is continuous at these points:

x-value	$\lim\limits_{x \to a} f(x)$	$f(a)$	Continuous?
$x = 1$	$\lim\limits_{x \to 1} f(x) = 2$	$f(1) = 2$	Yes
$x = 2$	$\lim\limits_{x \to 2} f(x) = 2$	$f(2) = 1$	No
$x = 3$	$\lim\limits_{x \to 3} f(x)$ does not exist (the left-hand and right-hand limits are	$f(3) = 2$	No

	not equal).		
$x = 4$	$\lim\limits_{x \to 4} f(x) = 2$	$f(4)$ is not defined.	No

From this table, we can conclude the following:

- A function $f(x)$ is continuous at $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$. That is, $\lim\limits_{x \to a} f(x)$ exists and is equal to the value of $f(a)$.
- A function $f(x)$ is not continuous at $x = a$ if any of the following occur:
  - $\lim\limits_{x \to a} f(x)$ does not exist.
  - $f(a)$ is undefined.
  - $\lim\limits_{x \to a} f(x)$ exists, but is not equal to $f(a)$.

TERM TO KNOW

Continuous Function

A function that has no breaks in the graph. That is, $\lim\limits_{x \to a} f(x) = f(a)$.

2. Determining if a Function Is Continuous at *x* = *a*

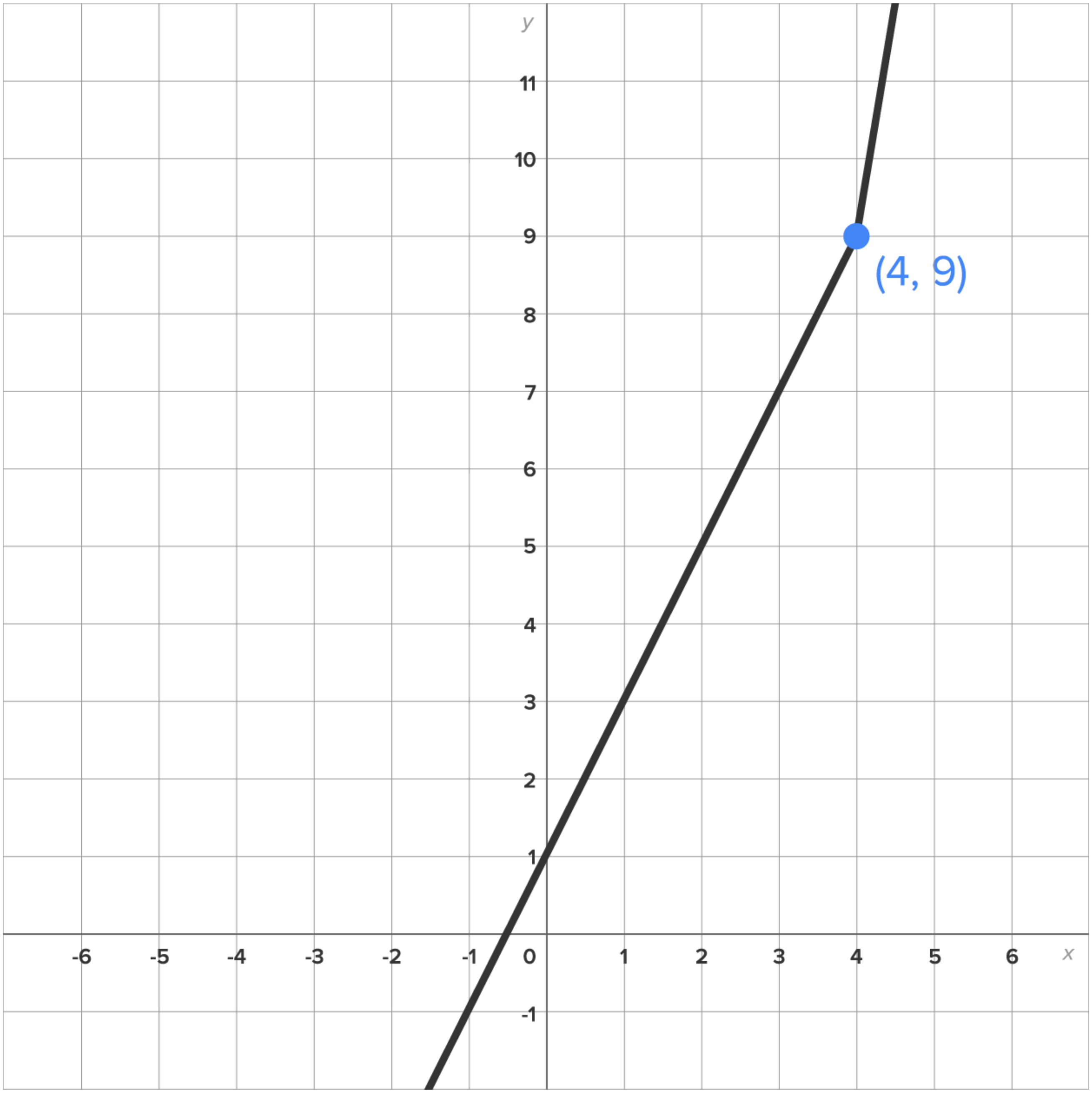
To determine if a function is continuous at $x = a$, we need to compare the values of $\lim\limits_{x \to a} f(x)$ and $f(a)$. While computing $f(a)$ is straightforward, computing $\lim\limits_{x \to a} f(x)$ requires more care, and sometimes requires one-sided limits.

EXAMPLE Consider the function $f(x) = \begin{cases} 2x+1 & \text{if } x < 4 \\ (x-1)^2 & \text{if } x \geq 4 \end{cases}$. Determine if $f(x)$ is continuous at $x = 4$.

First, check to see if $\lim\limits_{x \to 4} f(x)$ exists. Since $f(x)$ changes definition when $x = 4$, we need to consider the one-sided limits:

- Left-sided limit: $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (2x+1) = 2(4) + 1 = 9$
- Right-sided limit: $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} (x-1)^2 = (4-1)^2 = 9$
- Conclusion: $\lim\limits_{x \to 4} f(x) = 9$, which means it exists and is equal to 9.

From looking at the function definition, $f(4) = (4-1)^2 = 9$. Thus, the limit and the function value are the same; therefore the function is continuous at $x = 4$. Here is the graph of $f(x)$ to help visualize this:



EXAMPLE Consider the function $f(x)=\begin{cases} \dfrac{x^2-x-12}{x^2-16} & if\ x \neq 4 \\ 5 & if\ x=4 \end{cases}$. Determine if $f(x)$ is continuous at $x=4$.

First, evaluate $\lim_{x \to 4} f(x)$. Since $f(x)=\begin{cases} \dfrac{x^2-x-12}{x^2-16} & if\ x \neq 4 \\ 5 & if\ x=4 \end{cases}$ is defined on both sides of $x=4$, there is no need to compute one-sided limits.

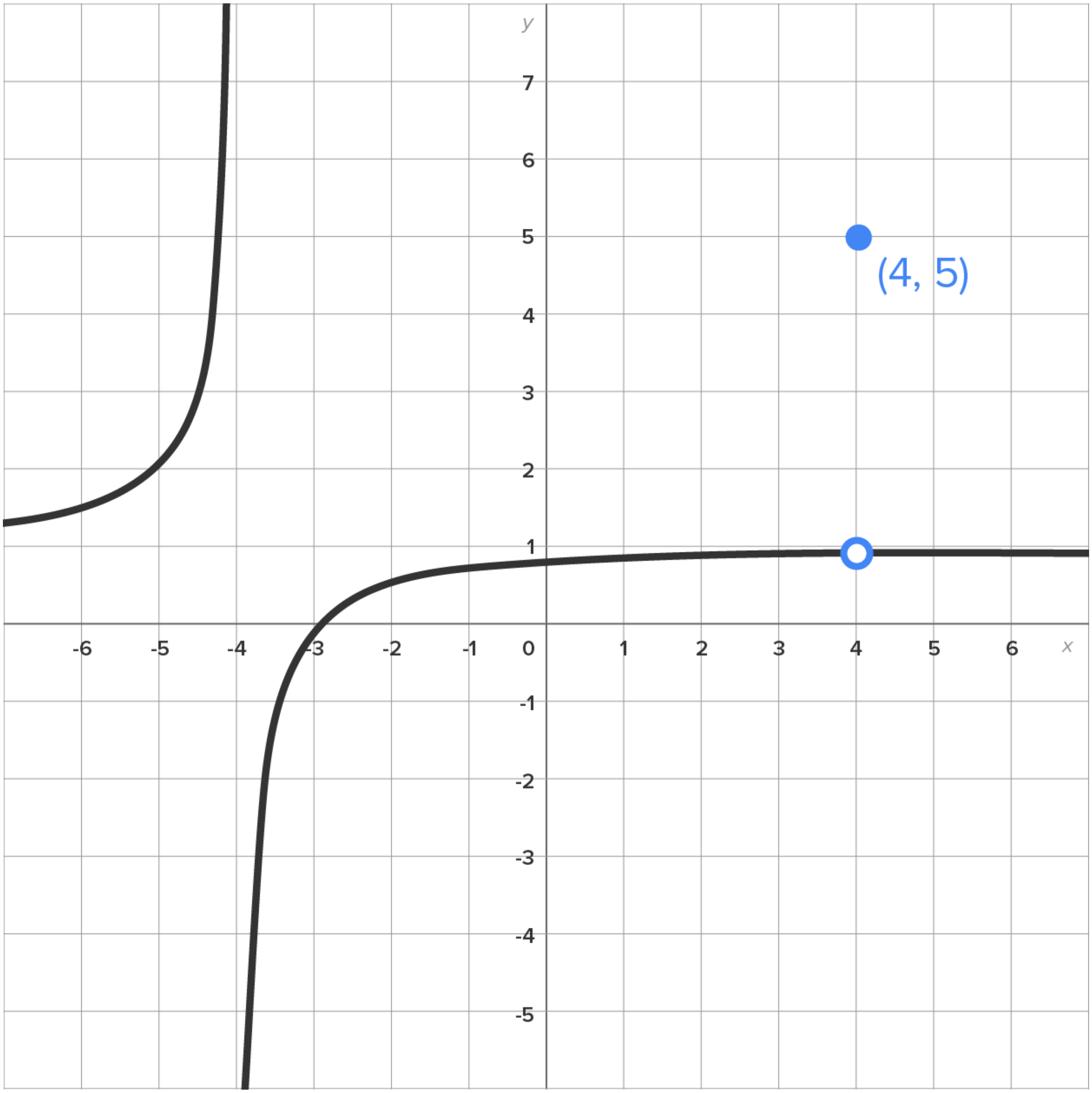
To evaluate the limit, first consider direct substitution:

$$\lim_{x \to 4} \frac{x^2 - x - 12}{x^2 - 16} = \frac{4^2 - 4 - 12}{4^2 - 16} = \frac{0}{0}$$

While this is undefined, notice that both the numerator and denominator are 0. This means that the expression can be manipulated in order to evaluate the limit.

$\lim_{x \to 4} \frac{x^2 - x - 12}{x^2 - 16} = \lim_{x \to 4} \frac{(x-4)(x+3)}{(x+4)(x-4)}$	Factor the expression.
$= \lim_{x \to 4} \frac{x+3}{x+4}$	Remove the common factor of $x-4$.
$= \frac{4+3}{4+4}$	Direct substitution works since the denominator is not 0.
$= \frac{7}{8}$	Simplify.

However, $f(4) = 5$. Since the limit and the function value are different, this function is not continuous at $x = 4$. Here is a graph to help visualize this:



TRY IT

Consider the function: $f(x) = \begin{cases} 3x+4 & if\ x<1 \\ \sqrt{x+8} & if\ x \geq 1 \end{cases}$

Determine if $f(x)$ is continuous when $x = 1$. +

Find the limit as $x \to 1$ from each side:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x+4) = 3(1)+4 = 7$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x+8} = \sqrt{1+8} = \sqrt{9} = 3$$

Since the one-sided limits are not equal, $f(x)$ is not continuous at $x = 1$.

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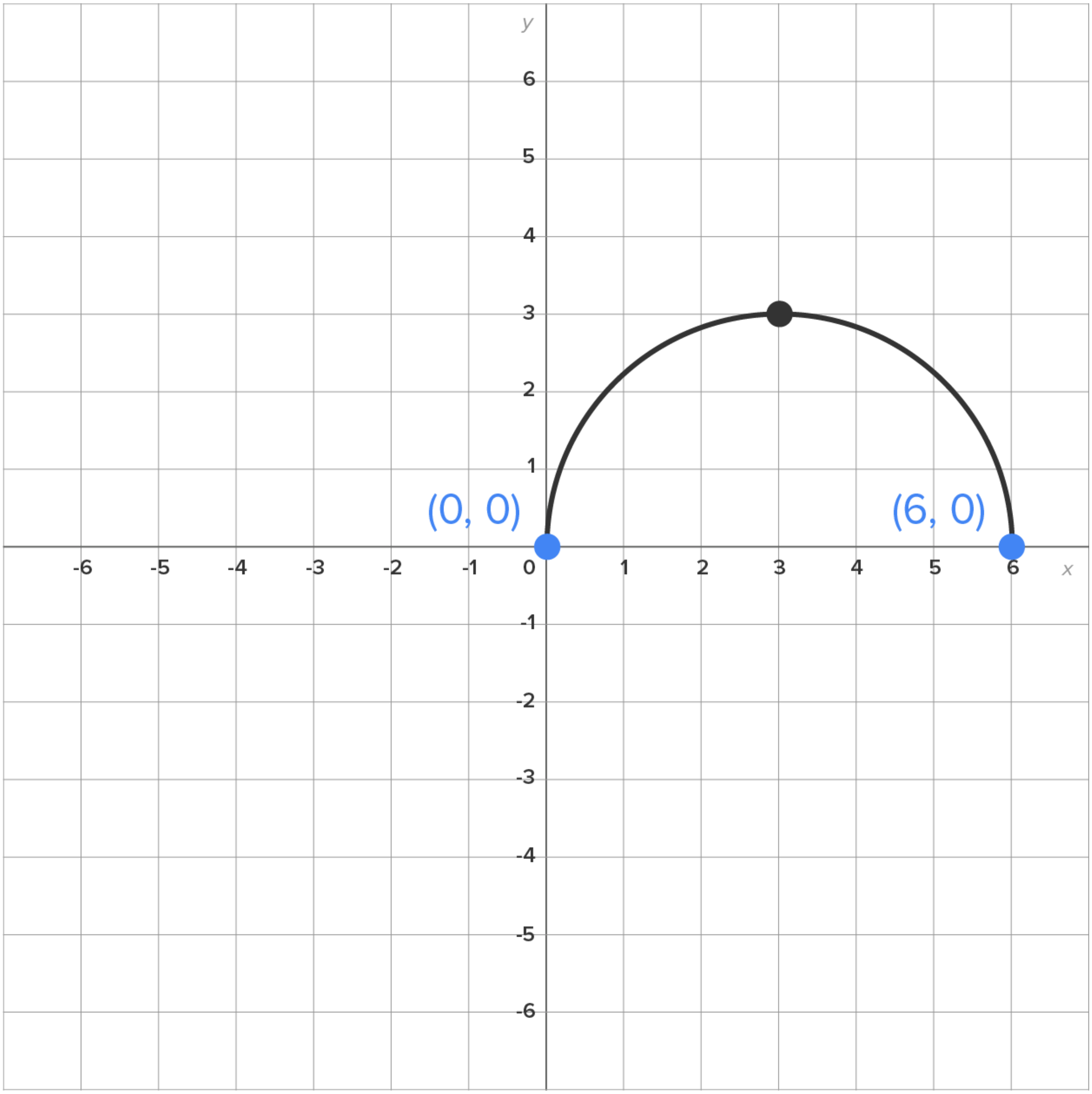
3. Determining Intervals Over Which a Function Is Continuous

For a function to be continuous on an interval of values, it has to be continuous at every point contained in the interval.

> EXAMPLE $f(x) = x^2 - 4x + 5$ is continuous at every real number. Thus, we say that $f(x)$ is continuous on the interval $(-\infty, \infty)$.

> EXAMPLE $f(x) = \frac{2}{x-1}$ is continuous at every value except $x = 1$. We can say that $f(x)$ is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$. This can also be written as $(-\infty, 1) \cup (1, \infty)$.

It is also possible to define continuity at an endpoint. For example, consider $f(x) = \sqrt{6x - x^2}$, whose graph is shown below. Note that the domain of this function is $[0, 6]$.



This means that defining continuity at $x=0$ and $x=6$ takes a bit more care.

Consider the endpoint $x=0$. It can only be approached from the right. Looking at the graph, observe that $\lim_{x \to 0^+} f(x) = 0$ and $f(0)=0$.

Consider the endpoint $x=6$. It can only be approached from the left. Looking at the graph, observe that $\lim_{x \to 6^-} f(x) = 0$ and $f(6)=0$.

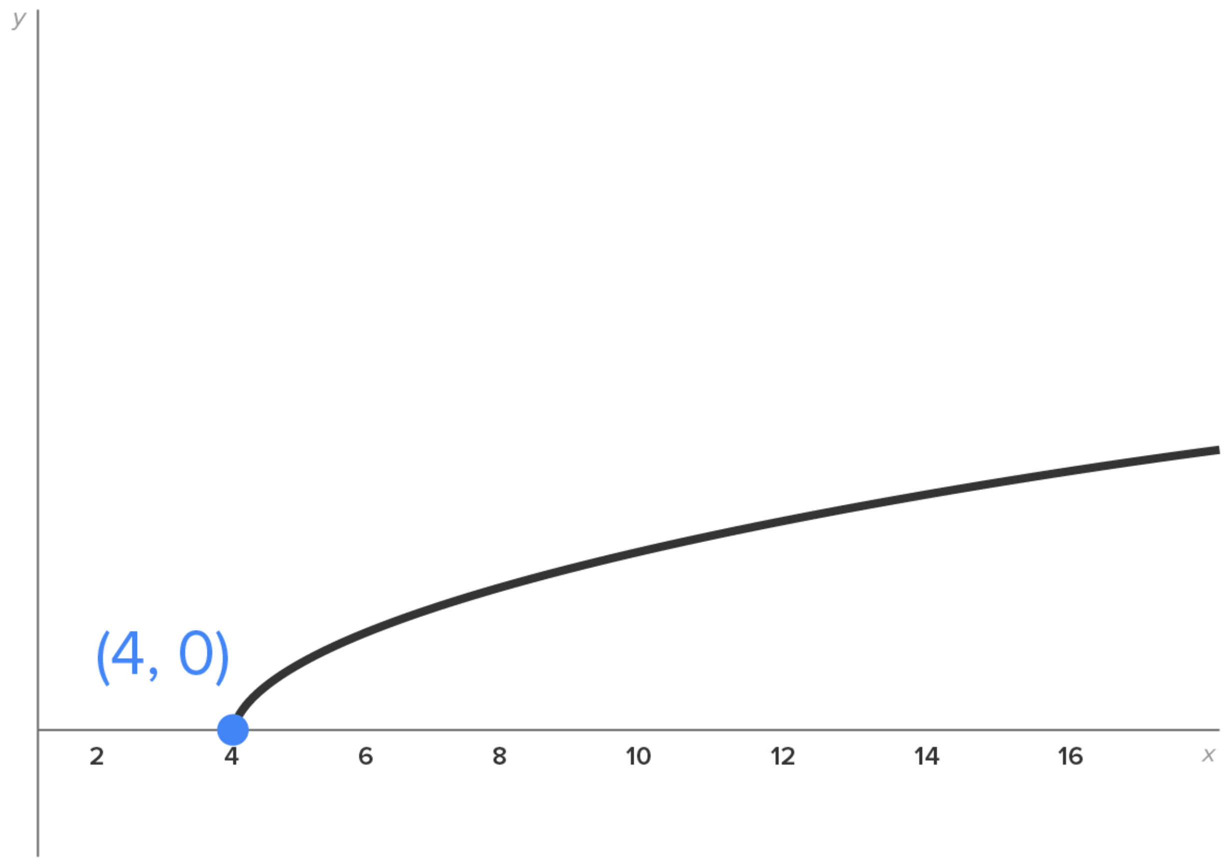
BIG IDEA

A function is **continuous from the left** at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

A function is **continuous from the right** at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Thus, in the previous problem, we can say that $f(x)$ is continuous from the left at $x = 6$ and continuous from the right at $x = 0$. This enables us to say that $f(x)$ is continuous for all values on the interval $[0, 6]$.

EXAMPLE Determine the interval(s) over which $f(x) = \sqrt{x-4}$ is continuous. The graph is shown below.



Note that the domain of $f(x)$ is $[4, \infty)$. It follows that $f(x)$ is continuous on the interval $[4, \infty)$, noting that it is continuous from the right at $x = 4$.

TRY IT

Consider the following table:

Function	Continuous Interval
$f(x) = 3x - x^4$	?
$g(x) = \dfrac{x}{x+4}$	?

$h(x) = \sqrt{2x-1}$	?

Determine the interval(s) over which each function is continuous. +

Function	Continuous Interval
$f(x) = 3x - x^4$	$(-\infty, \infty)$
$g(x) = \frac{x}{x+4}$	$(-\infty, -4) \cup (-4, \infty)$
$h(x) = \sqrt{2x-1}$	$\left[\frac{1}{2}, \infty\right)$

TERMS TO KNOW

Continuous From the Left

A function is continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

Continuous From the Right

A function is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

SUMMARY

In this lesson, you learned **the definition of continuity**, understanding that when given a graph, continuity is determined by locations where the graph has no breaks, jumps, or holes. A continuous function has no breaks in the graph; that is, $\lim_{x \to a} f(x) = f(a)$. You learned that you can use limits to **determine if a function is continuous at** $x = a$ (a specific point) by comparing the values of $\lim_{x \to a} f(x)$ and $f(a)$. It's important to note that while computing $f(a)$ is straightforward, computing $\lim_{x \to a} f(x)$ requires more care, and sometimes requires one-sided limits. Lastly, you learned that by examining the domain of a function, you can use it to **determine the intervals over which a function is continuous**, noting that the function has to be continuous at every point contained in the interval in order to say the function is continuous on the interval.

TERMS TO KNOW

Continuous From the Left

A function is continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

Continuous From the Right

A function is continuous from the left at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Continuous Function

A function that has no breaks in the graph. That is, $\lim_{x \to a} f(x) = f(a)$.