Differentiability

by Sophia

WHAT'S COVERED

In this lesson, you will investigate the differentiability of a function by using analytical techniques, which include a determination of continuity. Specifically, this lesson will cover:

1. Defining Differentiability

Differentiability is an important concept in calculus since it pertains to the “smoothness” of a curve. A function $y=f(x)$ is said to be **differentiable** at $x=a$ if $f(x)$ is continuous at $x=a$ and $f'(a)$ is defined.

TERM TO KNOW

Differentiable

A function $y=f(x)$ is said to be differentiable at $x=a$ if $f(x)$ is continuous at $x=a$ and $f'(a)$ is defined.

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2. Determining Differentiability at *x* = *a* Analytically

The following statements are equivalent:

- If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.
- If $f(x)$ is not continuous at $x=a$, then $f(x)$ is not differentiable at $x=a$.

How to interpret these statements:

- If $f(x)$ is not continuous at $x=a$, then it is never differentiable at $x=a$.
- If $f(x)$ is differentiable at $x=a$, then it is always continuous at $x=a$.

Note: This means that if $f(x)$ is continuous at $x=a$, $f(x)$ may or may not be differentiable at $x=a$.

HINT

Recall that the definition of continuity of $f(x)$ at $x=a$ is $\lim_{x \to a} f(x) = f(a)$.

2a. Continuous but Not Differentiable

Here is an example of a function that is continuous but not differentiable at a point.

EXAMPLE Determine if $f(x)=\sqrt[3]{x}$ is differentiable at $x=0$. First, check for continuity at $x=0$:

$$f(0)=\sqrt[3]{0}=0$$

$$\lim_{x \to 0} \sqrt[3]{x}=0$$

Since $\sqrt[3]{x}$ is defined for positive and negative real numbers, there is no need to use one-sided limits. Since the limit and $f(0)$ are equal, $f(x)$ is continuous at $x=0$.

Now, let’s check the derivative. Note that $f(x)=\sqrt[3]{x}=x^{1/3}$. Then, $f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$.

Since $f'(0)$ is undefined (0 in the denominator), $f(x)$ is not differentiable at $x=0$.

Note: In this case, this means that the slope of the tangent line is undefined, and that the tangent line is vertical.

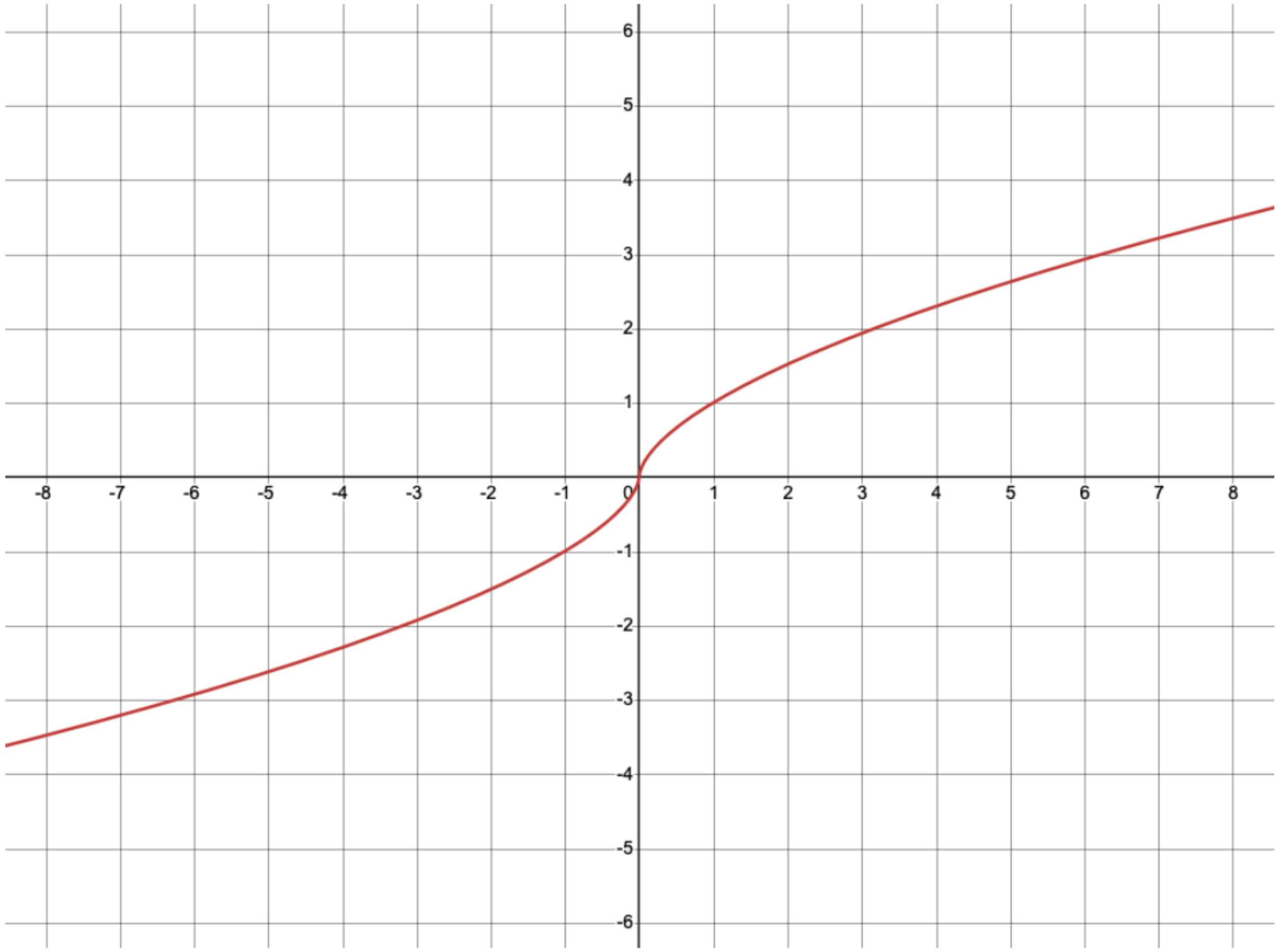
EXAMPLE Consider the function $f(x)=\sqrt[5]{x^3}$. Determine if $f(x)$ is differentiable at $x=0$. If not, identify the reason why.

First, write $f(x)=x^{3/5}$.

Then, $f'(x)=\frac{3}{5}x^{-2/5}=\frac{3}{5x^{2/5}}$

Note that $f'(0)$ is undefined since the denominator is 0 when $x=0$. Thus, $f'(x)$ is undefined when $x=0$.

To help investigate this further, consider the graph of $f(x)$.



Note that $f(x)$ is continuous for all real numbers, including $x=0$. The graph however has a vertical tangent line at $x=0$, which is confirmed by $f'(x)$ being undefined. Thus, $f(x)$ is not differentiable at $x=0$ because there is a vertical tangent line at $(0, 0)$.

2b. Differentiable for All Real Numbers

Here is an example of a function that is differentiable for all real numbers.

EXAMPLE Show that $f(x)=x^3$ is differentiable for all real numbers.

Check continuity: Since $f(x)$ is a polynomial function, it is continuous for all real numbers (this was established earlier in the course).

Check the derivative: $f'(x)=3x^2$, which is defined for all real numbers.

Thus, $f(x)=x^3$ is differentiable for all real numbers.

2c. Not Continuous

Here is an example of a function that is not continuous at a point, which means that it is also not differentiable at the point.

EXAMPLE Consider the function $f(x) = \frac{2}{x-1}$.

Since $f(x)$ is not continuous at $x = 1$, it is also not differentiable at $x = 1$.

TRY IT

Consider the following functions and x-values.

Function	Given x-value	Differentiable (Yes or No)?
$f(x) = \cos x$	$x = 0$	?
$g(x) = \sqrt{x}$	$x = 4$	?
$h(x) = \frac{x}{2x-1}$	$x = \frac{1}{2}$	?

Determine if each function is differentiable at the given x-value in the table. +

Function	Given x-value	Differentiable (Yes or No)?
$f(x) = \cos x$	$x = 0$	Yes
$g(x) = \sqrt{x}$	$x = 4$	Yes
$h(x) = \frac{x}{2x-1}$	$x = \frac{1}{2}$	No (not continuous, therefore not differentiable at $x = \frac{1}{2}$)

SUMMARY

In this lesson, you explored the first of two ways to **define differentiability**, noting that if a function is to be differentiable at $x = a$, it must be continuous at $x = a$ and $f'(a)$ needs to be defined. You learned how to **determine differentiability at $x = a$ analytically**, exploring examples of functions that are **continuous but not differentiable**, **differentiable for all real numbers**, and also **not continuous** and therefore not differentiable at the points of discontinuity.

TERMS TO KNOW

Differentiable

A function $y=f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.