Derivative of y = a^x

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WHAT'S COVERED

In this lesson, you will find derivatives of exponential functions with any base. For example, $f(x)=2^x$, $g(x)=3^{-x^2}$, and $A(t)=\left(\frac{1}{2}\right)^{t/300}$. Sometimes it is more convenient to model situations with bases other than *e*, so it is important that we learn about the derivatives of $y=a^x$ and $y=a^u$. Specifically, this lesson will cover:

1. Derivatives of $y = a^x$ and Combinations of Functions With $y = a^x$

WATCH

Please view this video to see how we arrive at the derivative formula for $f(x)=a^x$, where $a>0$.

So, we can say the derivative of a^x can be expressed with the following formula:

FORMULA TO KNOW

The Derivative of a^x

$D[a^x]=a^x\cdot\ln a$

For instance, this means that $D[3^x]=3^x\cdot\ln 3$ and $D\left[\left(\frac{1}{2}\right)^x\right]=\left(\frac{1}{2}\right)^x\cdot\ln\left(\frac{1}{2}\right)$.

Let's look at a few examples where $f(x)=a^x$ is combined with other functions.

EXAMPLE Consider the function $f(x)=x\cdot 10^x$. Find its derivative.

$f(x) = x \cdot 10^x$	Start with the original function.
$f'(x) = D[x] \cdot 10^x + x \cdot D\left[10^x\right]$	Use the product rule.
$f'(x) = (1) \cdot 10^x + x \cdot (10^x \cdot \ln 10)$	$D[x] = 1$, $D\left[10^x\right] = 10^x \cdot \ln 10$
$f'(x) = 10^x + x \cdot 10^x \cdot \ln 10$	Remove extra grouping symbols.

Thus, $f'(x) = 10^x + x10^x \ln 10$.

This could also be rewritten by factoring out 10^x: $f'(x) = 10^x(1 + x\ln 10)$

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2. Derivatives of $y = a^u$ and Combinations of Functions With $y = a^u$, Where u Is a Function of x

As a result of the chain rule, we have the following derivative formula:

FORMULA TO KNOW

The Derivative of a^u, Where u Is a Function of x

$D\left[a^u\right] = (a^u \cdot \ln a) \cdot u'$

EXAMPLE Consider the function $f(x) = 3^{-x^2}$. Find its derivative.

$f(x) = 3^{-x^2}$	Start with the original function.
$f'(x) = \left(3^{-x^2} \cdot \ln 3\right) \cdot (-2x)$	$D\left[3^u\right] = (3^u \cdot \ln 3) \cdot u'$ Here, $u = -x^2$.
$f'(x) = -2x3^{-x^2}\ln 3$	Write "-2*x*" in front and remove unnecessary grouping symbols.

Thus, $f'(x) = -2x3^{-x^2}\ln 3$.

TRY IT

Consider the function $f(x) = \sqrt{5^x + 2}$.

Find its derivative.

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First, rewrite as $f(x) = (5^x + 2)^{1/2}$.

By the chain rule:

$$\begin{aligned} f'(x) &= \frac{1}{2}(5^x + 2)^{-1/2} \cdot D[5^x + 2] \\ &= \frac{1}{2}(5^x + 2)^{-1/2}(5^x \ln 5 + 0) \\ &= \frac{1}{2}(5^x + 2)^{-1/2}(5^x \ln 5) \end{aligned}$$

Rewrite in terms of nonnegative exponents:

$$f'(x) = \frac{1}{2(5^x + 2)^{1/2}} \cdot 5^x \ln 5$$

Write as a single fraction. While you can also leave the power as ½, you could also write as a square root. Here is the result:

$$f'(x) = \frac{5^x \ln 5}{2\sqrt{5^x + 2}}$$

EXAMPLE Find the derivative of $f(x) = \tan(2^{3x-1} + 5)$.

$f(x) = \tan(2^{3x-1} + 5)$	Start with the original function.
$f'(x) = \sec^2(2^{3x-1} + 5) \cdot D[2^{3x-1} + 5]$	$D[\tan u] = \sec^2 u \cdot u'$
$= \sec^2(2^{3x-1} + 5) \cdot (2^{3x-1} \ln 2) \cdot D[3x - 1]$	$D[2^u] = 2^u \cdot \ln 2$ Also, apply the sum/difference rules and $D[5] = 0$.
$= \sec^2(2^{3x-1} + 5) \cdot (2^{3x-1} \ln 2) \cdot 3$	$D[3x - 1] = 3$
$= 3\sec^2(2^{3x-1} + 5) \cdot (2^{3x-1} \ln 2)$	Rewrite with the "3" in front.

In conclusion, $f'(x) = 3\sec^2(2^{3x-1} + 5) \cdot (2^{3x-1} \ln 2)$.

EXAMPLE A drug has a half-life of 6 hours, which means that after 6 hours in the bloodstream, half of the original amount remains. When 40mg of this drug is introduced into the bloodstream, the amount remaining after t hours is $A(t) = 40\left(\frac{1}{2}\right)^{t/6}$.

At what rate is the amount of drug in the bloodstream changing after 8 hours?

In this problem, we want to find $A'(8)$. So, let's first find $A'(t)$.

$A(t)=40\left(\frac{1}{2}\right)^{t/6}$	Start with the original function.
$A'(t)=40\left[\left(\frac{1}{2}\right)^{t/6}\cdot\ln\left(\frac{1}{2}\right)\right]\cdot\frac{1}{6}$	$D[a^u]=a^u\cdot\ln a\cdot u'$ $u=\frac{t}{6}=\frac{1}{6}t,\ u'=\frac{1}{6}$
$A'(t)=\frac{20}{3}\cdot\left(\frac{1}{2}\right)^{t/6}\cdot\ln\left(\frac{1}{2}\right)$	$40\left(\frac{1}{6}\right)=\frac{40}{6}=\frac{20}{3}$ Remove extra symbols.

Then, $A'(8)=\frac{20}{3}\cdot\left(\frac{1}{2}\right)^{8/6}\cdot\ln\left(\frac{1}{2}\right)\approx-1.83384$. This means that the amount of drug in the bloodstream is decreasing at a rate of about 1.83 mg/hr.

SUMMARY

In this lesson, you explored finding the **derivatives of** $y=a^x$ **and combinations of functions with** $y=a^x$**.** You also learned how to find the **derivatives of the general exponential function** $y=a^u$**, where *u* is a function of *x* (and related combinations of functions)**, which allows you to explore even more functions and applications. Remember that the derivative rule for $y=a^u$ is very similar to that of $y=e^u$ where *u* is a function of *x*, but with an extra factor of $\ln a$.

FORMULAS TO KNOW

The Derivative of a^x

$D[a^x]=a^x\cdot\ln a$

The Derivative of a^u, Where u Is a Function of x

$D[a^u]=(a^u\cdot\ln a)\cdot u'$

