# The Formal Definition of a Limit

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WHAT'S COVERED

In this lesson, you will establish a relationship between ε and δ to formally prove the value of a limit. Specifically, this lesson will cover:



# 1. The Formal Definition of a Limit

This process is much like what we did in the last challenge, except this time we will have symbols instead of numbers. Since this process is more complicated with nonlinear functions, we will be focusing on linear functions only.

The formal definition of a limit is as follows:

KEY CONCEPT

$\lim_{x \to a} f(x) = L$ means for every given $\epsilon > 0$, there is a $\delta > 0$ so that if *x* is within δ units of $a$ (and $x \neq a$), then $f(x)$ is within ε units of *L*.

Equivalently, we can say $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

HINT

The use of the lower case Greek letters ε (epsilon) and δ (delta) in the definition is standard, and this definition is sometimes called the "epsilon-delta" definition of limit.

# 2. Proving Limits (Finding δ in Terms of ε)

In order to prove a limit, we need to accomplish the following:

- Find an expression for δ in terms of ε, starting with $|f(x) - L| < \varepsilon$.
- For the expression you found, show that $0 < |x - a| < \delta$ leads to $|f(x) - L| < \varepsilon$.

EXAMPLE Prove the limit: $\lim_{x \to 4} (2x - 1) = 7$

Start with $|2x - 1 - 7| < \varepsilon$. Then, convert to an inequality with $x - 4$ in the middle (just as before).

| | |
|---|---|
| $\|2x - 8\| < \varepsilon$ | Simplify the expression. |
| $-\varepsilon < 2x - 8 < \varepsilon$ | $\|x\| < a$ means $-a < x < a$. |
| $-\varepsilon + 8 < 2x < \varepsilon + 8$ | Add 8 to all parts. |
| $-\frac{\varepsilon}{2} + 4 < x < \frac{\varepsilon}{2} + 4$ | Divide all parts by 2. |
| $-\frac{\varepsilon}{2} < x - 4 < \frac{\varepsilon}{2}$ | Subtract 4 from all parts to get $x - 4$ in the middle. |

Thus, it appears that $\delta = \frac{\varepsilon}{2}$. Note also that the last inequality can be written $|x - 4| < \frac{\varepsilon}{2}$.

Now, to prove the limit, we basically just reverse the steps we did. While this is redundant, it is important since we need to establish that $-\frac{\varepsilon}{2} < x - 4 < \frac{\varepsilon}{2}$ directly leads to a statement that $f(x)$ is within ε units of 7.

The goal is to convert this into an inequality with $2x - 1$ in the middle.

Start with $-\frac{\varepsilon}{2} < x - 4 < \frac{\varepsilon}{2}$.

| | |
|---|---|
| $-\frac{\varepsilon}{2} + 4 < x < \frac{\varepsilon}{2} + 4$ | Add 4 to all parts. |
| $-\varepsilon + 8 < 2x < \varepsilon + 8$ | Multiply all parts by 2. |
| $-\varepsilon + 7 < 2x - 1 < \varepsilon + 7$ | Subtract 1 from all parts to get $2x - 1$ in the middle. |

This inequality is enough justification that $f(x) = 2x - 1$ is within ε units of 7. This completes the proof.

Note: With $\delta = \frac{\varepsilon}{2}$, note that as ε gets closer to 0, so does δ. This is crucial for proving a limit. This is the mathematical version of saying "as *x* gets closer to $a$, $f(x)$ gets closer to *L*".

EXAMPLE Prove the limit: $\lim_{x \to 9} (50 - 4x) = 14$

First, find an expression for δ in terms of ε:

| | |
|---|---|
| $\|50 - 4x - 14\| < \varepsilon$ | Start with the inequality. |

| | |
|---|---|
| $\lvert 36-4x \rvert < \varepsilon$ | Simplify the expression. |
| $-\varepsilon < 36-4x < \varepsilon$ | $\lvert x \rvert < a$ means $-a < x < a$. |
| $-\varepsilon-36 < -4x < \varepsilon-36$ | Subtract 36 from all parts. |
| $\frac{\varepsilon}{4}+9 > x > -\frac{\varepsilon}{4}+9$ | Divide all parts by -4 (note: inequalities change direction). |
| $-\frac{\varepsilon}{4}+9 < x < \frac{\varepsilon}{4}+9$ | Rewrite the inequality. |
| $-\frac{\varepsilon}{4} < x-9 < \frac{\varepsilon}{4}$ | Subtract 9 to get $x-9$ in the middle. |

At this point, it appears that $\delta = \frac{\varepsilon}{4}$. Now, reverse the steps.

| | |
|---|---|
| $-\frac{\varepsilon}{4} < x-9 < \frac{\varepsilon}{4}$ | Start with the inequality. |
| $-\frac{\varepsilon}{4}+9 < x < \frac{\varepsilon}{4}+9$ | Add 9 to all parts. |
| $\varepsilon-36 > -4x > -\varepsilon-36$ | Multiply all parts by -4 (inequalities change direction). |
| $-\varepsilon-36 < -4x < \varepsilon-36$ | Rewrite the inequality. |
| $-\varepsilon+14 < -4x+50 < \varepsilon+14$ | Add 50 to all parts. |

At this point, we see that $f(x)$ is within ε units of 14. This completes the proof.

WATCH

The following video walks you through an example of finding δ to prove a limit for a linear function.

TRY IT

Consider $\lim_{x \to 3} (6x-11) = 7$.

**Prove this limit.** +

Let $f(x) = 6x - 11$, $a = 3$, and $L = 7$.

Start with $\lvert f(x) - L \rvert < \epsilon$, which in this case is $\lvert 6x - 11 - 7 \rvert < \epsilon$.

The goal is to arrive at an inequality with $x-3$ in the center. Here are the steps:

| | |
|---|---|
| $\lvert 6x - 18 \rvert < \epsilon$ | Simplify inside the absolute value. |

| | |
|---|---|
| $-\epsilon < 6x - 18 < \epsilon$ | $\lvert x \rvert < a$ means $-a < x < a$. |
| $-\epsilon + 18 < 6x < \epsilon + 18$ | Add 18 to all parts of the inequality. |
| $\frac{-\epsilon + 18}{6} < x < \frac{\epsilon + 18}{6}$ | Divide all parts by 6. |
| $\frac{-\epsilon}{6} + 3 < x < \frac{\epsilon}{6} + 3$ | Perform the divisions. |
| $\frac{-\epsilon}{6} < x - 3 < \frac{\epsilon}{6}$ | Subtract 3 from all parts. |

Note that the final inequality can be written $\lvert x - 3 \rvert < \frac{\epsilon}{6}$.

This means that $\delta = \frac{\epsilon}{6}$.

Now, to prove the limit, reverse the steps.

The goal is to get an inequality with $6x - 11$ in the middle. This is because $f(x) = 6x - 11$.

| | |
|---|---|
| $\lvert x - 3 \rvert < \frac{\epsilon}{6}$ | Start here. |
| $-\frac{\epsilon}{6} < x - 3 < \frac{\epsilon}{6}$ | $\lvert x \rvert < a$ means $-a < x < a$. |
| $-\frac{\epsilon}{6} + 3 < x < \frac{\epsilon}{6} + 3$ | Add 3 to all parts of the inequality. |
| $-\epsilon + 18 < 6x < \epsilon + 18$ | Multiply all parts of the inequality by 6. |
| $-\epsilon + 7 < 6x - 11 < \epsilon + 7$ | Subtract 11 from all parts of the inequality. |

The final inequality shows that $f(x)$ is within $\varepsilon$ units of 7, which completes the proof.

## SUMMARY

In this lesson, you learned that by using **the formal definition of a limit**, you can establish the true meaning of what it means for a limit to exist, by comparing δ and ε. You also learned that when **proving limits (finding δ in terms of ε)**, it crucial to note that as ε gets closer to 0, δ also gets closer to 0, and is the mathematical expression of “as *x* gets closer to *a*, $f(x)$ gets closer to *L*.”