Derivatives of Natural Logarithmic Functions

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WHAT'S COVERED

In this lesson, you will learn how to differentiate logarithmic functions. Recall that a logarithmic function is the inverse of an exponential function. Thus, in any situation in which the rate of change of an exponential function is desired, it makes sense to also discuss the rates of change of logarithmic functions. Specifically, this lesson will cover:

1. The Derivative of $f(x) = \ln x$ and Functions Involving $\ln x$
2. The Derivative of $f(u) = \ln u$ and Functions Involving $\ln u$, Where u Is a Function of x
3. Using Properties of Logarithms Before Differentiating

1. The Derivative of $f(x) = \ln x$ and Functions Involving $\ln x$

WATCH

Please view this video to see how to derive a formula for the derivative of $f(x) = \ln x$.

So, we can say the derivative of the natural log function can be expressed with the following formula:

FORMULA TO KNOW

Derivative of the Natural Logarithmic Function

$$D[\ln x] = \frac{1}{x}$$

With this new derivative rule, let's compute a few derivatives.

EXAMPLE Consider the function $f(x) = x^2 \ln x$.

$f(x) = x^2 \ln x$	Start with the original function.
$f'(x) = D[x^2] \cdot \ln x + x^2 \cdot D[\ln x]$	Use the product rule.
$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$	$D[x^2] = 2x$, $D[\ln x] = \frac{1}{x}$
$f'(x) = 2x \ln x + x$	Simplify $x^2 \cdot \frac{1}{x} = x$ and remove extra symbols.

Thus, $f'(x) = 2x \ln x + x$.

TRY IT

Consider the function $f(x) = \frac{\ln x}{x}$.

Find its derivative. +

The quotient rule is required to find $f'(x)$.

$$f'(x) = \frac{x \cdot D[\ln x] - \ln x \cdot D[x]}{x^2}$$
$$= \frac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2}$$
$$= \frac{1 - \ln x}{x^2}$$

HINT

Similar to trigonometric functions, powers of natural logarithmic functions are sometimes written with the power after the "ln". For example, $\ln^4 x$ means $(\ln x)^4$.

EXAMPLE Consider the function $f(x) = \ln^3 x$. Find its derivative.

$f(x) = \ln^3 x = (\ln x)^3$	Start with the original function. Rewrite in a more recognizable form.
$f'(x) = 3(\ln x)^2 \cdot \frac{1}{x}$	$D[u^3] = 3u^2 \cdot u'$ (Apply the chain rule.)
$f'(x) = \frac{3(\ln x)^2}{x}$	Combine as a single fraction.

Thus, $f'(x) = \frac{3(\ln x)^2}{x}$. It is also acceptable to write $f'(x) = \frac{3\ln^2 x}{x}$.

2. The Derivative of $f(u) = \ln u$ and Functions Involving $\ln u$, Where u Is a Function of x

In step with the chain rule, and the fact that $D[\ln x] = \frac{1}{x}$, we have the following rule for the derivative of $\ln u$:

FORMULA TO KNOW

Derivative of $\ln u$, Where u Is a Function of x

$$D[\ln u] = \frac{1}{u} \cdot u'$$

EXAMPLE Consider the function $f(x) = \ln(x^2 + 1)$. Find its derivative.

$f(x) = \ln(x^2 + 1)$	Start with the original function.
$f'(x) = \frac{1}{x^2+1} \cdot 2x$	$D[\ln u] = \frac{1}{u} \cdot u'$
$f'(x) = \frac{2x}{x^2+1}$	Rewrite as a single fraction.

Thus, $f'(x) = \frac{2x}{x^2+1}$.

HINT

The rule $D[\ln u] = \frac{1}{u} \cdot u'$ can also be written as $D[\ln u] = \frac{u'}{u}$.

EXAMPLE Consider the function $f(x) = \ln(\cos x)$. Find its derivative.

$f(x) = \ln(\cos x)$	Start with the original function.
$f'(x) = \frac{-\sin x}{\cos x}$	$D[\ln u] = \frac{u'}{u}$
$f'(x) = -\tan x$	Use this trigonometric identity: $\frac{\sin x}{\cos x} = \tan x$

Thus, $f'(x) = -\tan x$.

TRY IT

Consider the function $f(x) = \ln(2 + \sin x)$.

Find its derivative. +

By the chain rule:

$$\begin{aligned} f'(x) &= \frac{1}{2+\sin x} \cdot D[2+\sin x] \\ &= \frac{1}{2+\sin x} \cdot (0+\cos x) \\ &= \frac{1}{2+\sin x} \cdot \cos x \\ &= \frac{\cos x}{2+\sin x} \end{aligned}$$

WATCH

The video below illustrates how to find the derivative of $f(x) = x \cdot \ln(x^3+1)$, which requires a combination of the product and chain rules.

3. Using Properties of Logarithms Before Differentiating

EXAMPLE Consider the function $f(x) = \ln(x \cdot e^{-2x})$. Find the derivative of this function.

$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot D\left[x \cdot e^{-2x}\right]$	$D[\ln u] = \frac{1}{u} \cdot u'$
$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot \left[D[x] \cdot e^{-2x} + x \cdot D\left[e^{-2x}\right]\right]$	Use the product rule.
$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot \left[1 \cdot e^{-2x} + x \cdot e^{-2x}(-2)\right]$	$D[x] = 1$, $D\left[e^{u}\right] = e^{u} \cdot u'$
$f'(x) = \frac{1}{x \cdot e^{-2x}} \cdot \left[e^{-2x} - 2x \cdot e^{-2x}\right]$	Simplify and remove unnecessary symbols.
$f'(x) = \frac{e^{-2x}}{x \cdot e^{-2x}} - \frac{2xe^{-2x}}{x \cdot e^{-2x}}$	Distribute.
$f'(x) = \frac{1}{x} - 2$	Remove the common factors.

Thus, $f'(x) = \frac{1}{x} - 2$.

This process was quite cumbersome. However, if we use the properties of logarithms that we reviewed in Unit 1, this can be made simpler.

FORMULA TO KNOW

Product Property

$\ln(ab) = \ln a + \ln b$

Quotient Property

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Power Property

$\ln(a^b) = b \cdot \ln a$

EXAMPLE Consider the function $f(x) = \ln(x \cdot e^{-2x})$. Find its derivative by first using logarithm properties.

Since, $\ln(x \cdot e^{-2x})$ is the logarithm of a product, use properties of logarithms to rewrite:

$f(x) = \ln(x \cdot e^{-2x})$	Start with the original function.
$f(x) = \ln x + \ln(e^{-2x})$	$\ln(ab) = \ln a + \ln b$
$f(x) = \ln x + (-2x)\ln e$	$\ln(a^b) = b \cdot \ln a$
$f(x) = \ln x - 2x$	$\ln e = 1,\ -2x(1) = -2x$

So, in expanded (and simpler) form, $f(x) = \ln x - 2x$.

Then, $f'(x) = D[\ln x] - D[2x] = \frac{1}{x} - 2$.

HINT

To find the derivative of $\ln u$, where u is a product, quotient, or power (or any combination of them), use logarithm properties before finding the derivative. This results in simpler derivatives.

EXAMPLE Find the derivative of $f(x) = \ln\left(\frac{x\cos x}{3x^2+8}\right)$.

Notice that the argument of the “ln” function is a combination of a product and a quotient. According to the Hint, it’s suggested that we expand this first using the properties of logarithms.

$f(x) = \ln\left(\frac{x\cos x}{3x^2+8}\right)$	Start with the function in its original form.
$= \ln(x\cos x) - \ln(3x^2+8)$	Use the property $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
$= \ln x + \ln(\cos x) - \ln(3x^2+8)$	Use the property $\ln(xy) = \ln x + \ln y$.

At this point, $f(x)$ cannot be expanded any further. Now we can find $f'(x)$ more efficiently.

$f(x) = \ln x + \ln(\cos x) - \ln(3x^2 + 8)$	Start with the function in expanded form.
$f'(x) = \frac{1}{x} + \frac{1}{\cos x}(-\sin x) - \frac{1}{3x^2+8}(6x)$	Use the derivative formula $D[\ln u] = \frac{1}{u} \cdot u'$ in all three terms.
$= \frac{1}{x} - \frac{\sin x}{\cos x} - \frac{6x}{3x^2+8}$	Rewrite each term as a single fraction.
$= \frac{1}{x} - \tan x - \frac{6x}{3x^2+8}$	Rewrite $\frac{\sin x}{\cos x} = \tan x$.

Thus, $f'(x) = \frac{1}{x} - \tan x - \frac{6x}{3x^2+8}$.

WATCH

In this video, we'll use properties of logarithms to find the derivative of $f(x) = \ln\left(\frac{x}{\sqrt{2x+1}}\right)$.

SUMMARY

In this lesson, you learned how to find **the derivative of a natural logarithmic function** (represented by $f(x) = \ln x$) and, given the chain rule, **the derivative of** $f(u) = \ln u$. These are the latest additions to our library of derivatives, and you've seen through examples and videos the different way the natural logarithmic function can be combined with other functions. You also learned that since logarithms have special properties, it is more advantageous to **use properties of logarithms before differentiating** functions that involve products, quotients, and powers.

FORMULAS TO KNOW

Derivative of lnu, Where u Is a Function of x

$$D[\ln u] = \frac{1}{u} \cdot u'$$

Derivative of the Natural Logarithmic Function

$$D[\ln x] = \frac{1}{x}$$

Power Property

$\ln(a^b) = b \cdot \ln a$

Product Property

$\ln(ab) = \ln a + \ln b$

Quotient Property

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$