Extreme Value Theorem - Endpoint Extremes

by Sophia

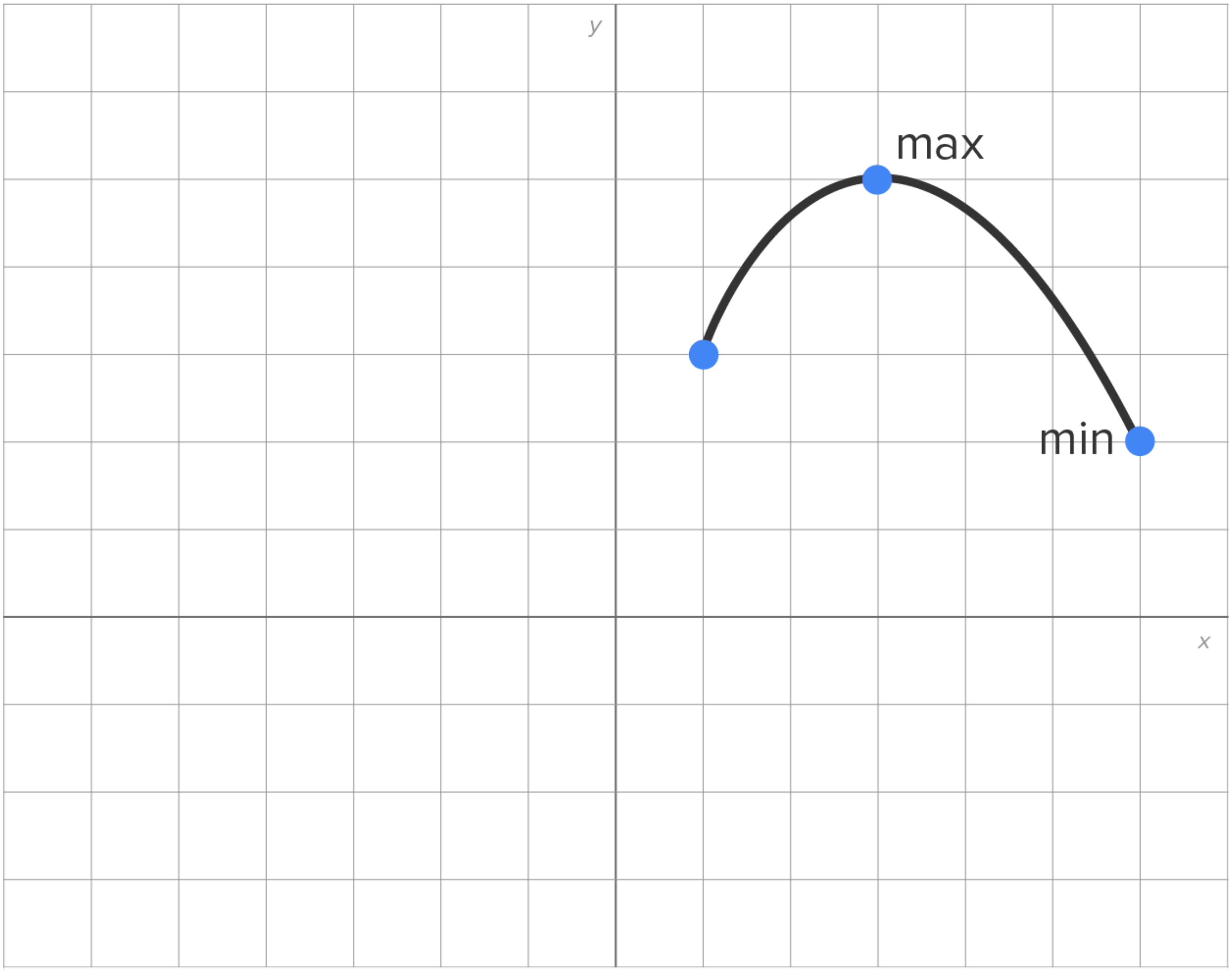
WHAT'S COVERED

In this lesson, you will use critical numbers and endpoint analysis to determine the maximum and minimum values of a continuous function on some closed interval. Specifically, this lesson will cover:

1. The Extreme Value Theorem

If a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is guaranteed to have global maximum and global minimum values on the interval $[a, b]$. This is known as the **extreme value theorem**.

Here is an illustration of the extreme value theorem:

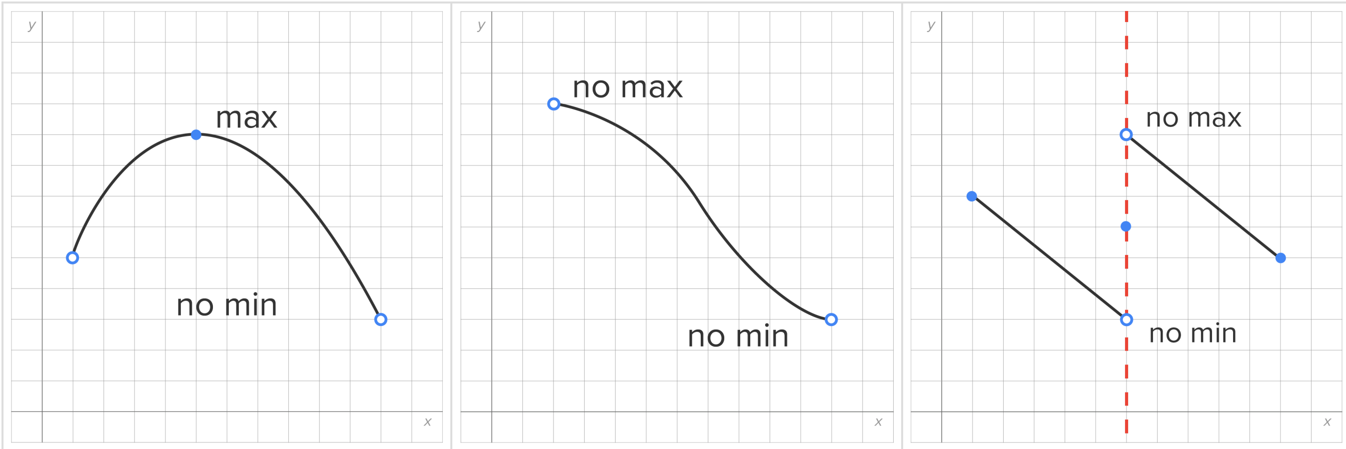


f continuous closed interval

- The function is continuous on the interval $[a, b]$.
- The maximum point occurs inside the interval.
- The minimum occurs at an endpoint.

The following are examples of situations in which one of the criteria is violated.

f Continuous Open Interval	f Continuous Open Interval	f Not Continuous Closed Interval



TERM TO KNOW

Extreme Value Theorem

If $f(x)$ is a continuous function on some closed interval $[a, b]$, then $f(x)$ has global maximum and global minimum values on the interval $[a, b]$.

2. Finding Extreme Values of a Continuous Function on a Closed Interval

As a result of the theorem, here is what we need to do in order to find the global minimum and maximum values of $f(x)$ on a closed interval $[a, b]$.

1. Find all critical numbers of $f(x)$ that are in the interval $[a, b]$.
2. Evaluate $f(x)$ at each endpoint and each critical number. The largest value of *f* is the global maximum and the smallest value of *f* is the global minimum.

EXAMPLE Find the global maximum and minimum points of the function $f(x) = x^3 - 6x^2 + 5$ on the interval $[-1, 3]$.

First, find the critical numbers.

$f(x) = x^3 - 6x^2 + 5$	Start with the original function.
$f'(x) = 3x^2 - 12x$	Take the derivative.
$3x^2 - 12x = 0$ $3x(x-4) = 0$ $x = 0, x = 4$	Since $f'(x)$ is a polynomial, it is never undefined. Set $f'(x) = 0$ and solve for *x*.

Therefore, the critical numbers are $x = 0$ and $x = 4$.

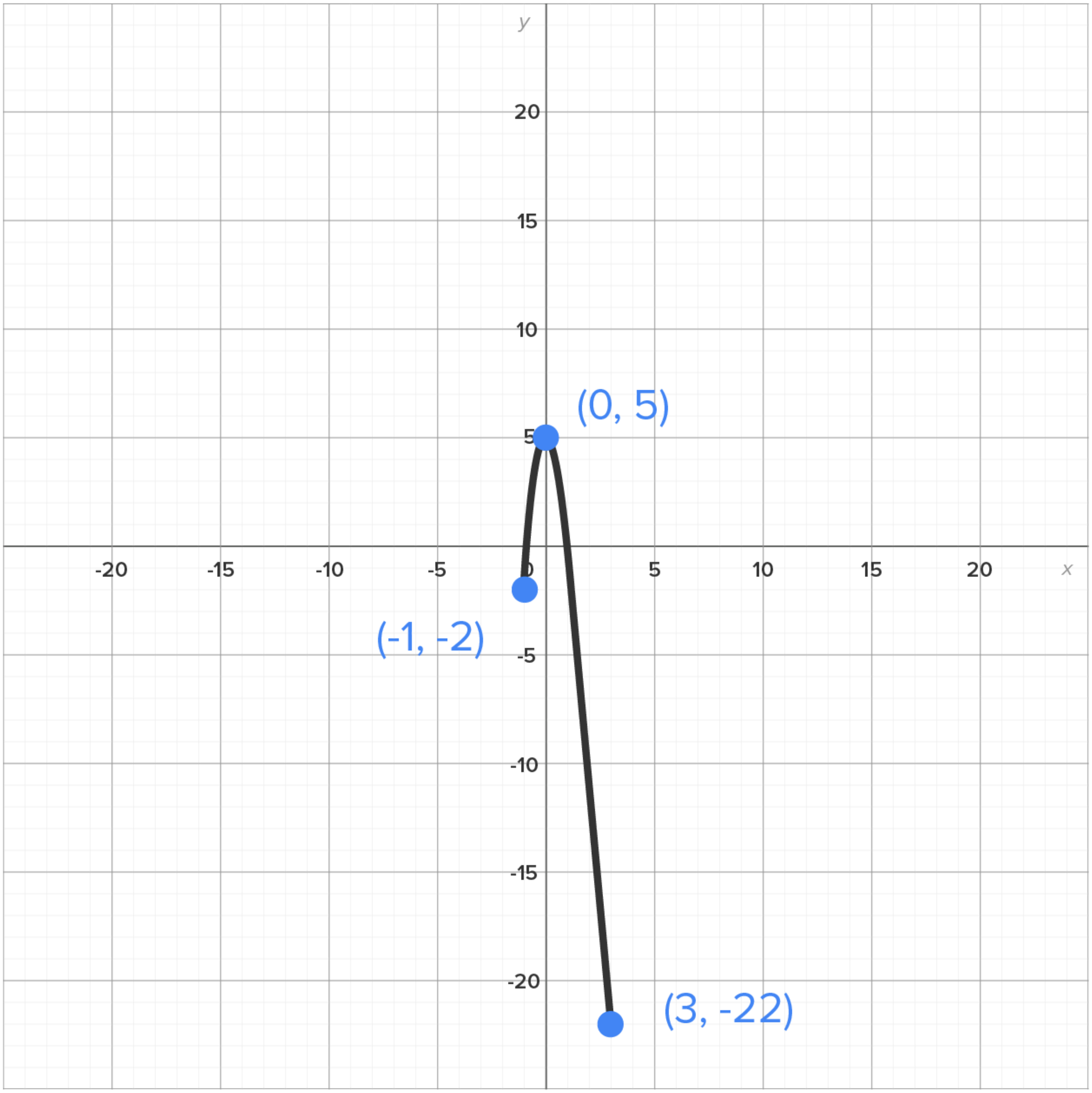
However, since only the closed interval $[-1, 3]$ is considered, the critical value $x = 4$ is not used.

Now, evaluate $f(x)$ at the endpoints, $x = -1$ and $x = 3$, and the remaining critical number, $x = 0$.

x	$f(x)$	Result
-1	$(-1)^3 - 6(-1)^2 + 5 = -2$	Neither a Global Maximum or Global Minimum
0	$(0)^3 - 6(0)^2 + 5 = 5$	Global Maximum
3	$3^3 - 6(3)^2 + 5 = -22$	Global Minimum

In conclusion, the global maximum occurs at the point (0, 5) and the global minimum occurs at the point (3, -22). In order words, the global maximum value is 5 and occurs when $x = 0$; and the global minimum value is -22 and occurs when $x = 3$.

The graph of the function on $[-1, 3]$ is shown below, which confirms the results.



WATCH

In this video, we'll find the global minimum and maximum values of $f(x) = 10\sqrt{x} - x$ on the interval $[16, 64]$.

When finding critical numbers, it's important to consider only those that are in the interval $[a, b]$. Here is an example that helps illustrate this.

EXAMPLE Find all global extreme values of $f(x) = -\frac{1}{4}x^4 + \frac{2}{3}x^3$ on the interval $[1, 4]$.

To start, we find the critical numbers.

$f(x) = -\frac{1}{4}x^4 + \frac{2}{3}x^3$	Start with the original function.
$f'(x) = -x^3 + 2x^2$	Use the power rule to find the derivative.
$-x^3 + 2x^2 = 0$	Set $f'(x) = 0$.
$-x^2(x-2) = 0$	Factor out $-x^2$.
$-x^2 = 0$ or $x - 2 = 0$	Set each factor equal to 0.
$x = 0$ or $x = 2$	Solve each equation.

Thus, the critical numbers are $x = 0$ and $x = 2$.

Considering the interval we are interested in, $[1, 4]$, notice that $x = 0$ is not contained in this interval. This means that $x = 0$ is not considered in any further analysis.

To determine the values of the local minimum and maximum values, we consider $x = 2$ along with the endpoints of the interval, $x = 1$ and $x = 4$.

Here is a table of values for $f(x) = -\frac{1}{4}x^4 + \frac{2}{3}x^3$. Note the approximations are also provided to make comparisons easier.

x	1	2	4
f(*x*)	$\frac{5}{12} \approx 0.433...$	$\frac{4}{3} \approx 1.333...$	$-\frac{64}{3} \approx -21.333...$

The global minimum value is $-\frac{64}{3}$ when $x = 4$, and the global maximum value is $\frac{4}{3}$ when $x = 2$.

SUMMARY

In this lesson, you learned that when $f(x)$ is continuous on a closed interval, **the extreme value theorem** guarantees a global minimum value and a global maximum value at some location within the closed interval. Then, you applied this theorem to **find extreme values of a continuous function on a closed interval**, by first finding all critical numbers of $f(x)$ that are in the interval $[a, b]$, then evaluating $f(x)$ at each endpoint and each critical number. This concept is going to be very useful once we use derivatives to solve optimization problems.

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TERMS TO KNOW

Extreme Value Theorem

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