Chapter 2
Sets, Relations and Functions

Key Topics
Sets
Set Operations
Russell's Paradox
Relations
Composition of Relations
Reflexive, Symmetric and Transitive Relations
Functions
Partial and Total Functions
Injective, Surjective and Transitive Functions

2.1 Introduction

This chapter provides an introduction to the fundamental building blocks in mathematics such as sets, relations and functions. Sets are collections of well-defined objects, relations indicate relationships between members of two sets A and B and functions are a special type of relation where there is exactly or at most[1] one relationship for each element $a \in A$ with an element in B.

A set is a collection of well-defined objects that contains no duplicates. The term "well defined" means that for a given value it is possible to determine whether or not it is a member of the set. There are many examples of sets such as the set of natural numbers $\mathbb{N}$, the set of integer numbers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$. The set of natural numbers $\mathbb{N}$ is an infinite set consisting of the numbers $\{1, 2, \dots\}$. Venn diagrams may be used to represent sets pictorially.

A binary relation $R(A, B)$ where A and B are sets is a subset of the Cartesian product $(A \times B)$ of A and B. The domain of the relation is A and the co-domain of the relation is B. The notation aRb signifies that there is a relation between a and b and that

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[1] We distinguish between total and partial functions. A total function $f : A \rightarrow B$ is defined for every element in A whereas a partial function may be undefined for one or more values in A.

$(a, b) \in R$. An n-ary relation $R (A_1, A_2, \ldots, A_n)$ is a subset of $(A_1 \times A_2 \times \cdots \times A_n)$. However, an n-ary relation may also be regarded as a binary relation $R(A, B)$ with $A = A_1 \times A_2 \times \cdots \times A_{n-1}$ and $B = A_n$.

Functions may be total or partial. A total function $f : A \to B$ is a special relation such that for each element $a \in A$ there is exactly one element $b \in B$. This is written as $f(a) = b$. A partial function differs from a total function in that the function may be undefined for one or more values of A. The domain of a function (denoted by **dom** f) is the set of values in A for which the function is defined. The domain of the function is A provided that f is a total function. The co-domain of the function is B.

2.2 Set Theory

A set is a fundamental building block in mathematics, and it is defined as a collection of well-defined objects. The elements in a set are of the same kind, and they are distinct with no repetition of the same element in the set[2]. Most sets encountered in computer science are finite as computers can only deal with finite entities. Venn diagrams[3] are often employed to give a pictorial representation of a set and may be used to illustrate various set operations such as set union, intersection and set difference.

There are many well-known examples of sets including the set of natural numbers denoted by $\mathbb{N}$, the set of integers denoted by $\mathbb{Z}$, the set of rational numbers is denoted by $\mathbb{Q}$, the set of real numbers denoted by $\mathbb{R}$ and the set of complex numbers denoted by $\mathbb{C}$.

Example 2.1 The following are examples of sets:

- The books on the shelves in a library.
- The books currently overdue from the library.
- The customers of a bank.
- The bank accounts in a bank.
- The set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots \}$.
- The integer numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- The non-negative integers $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$.
- The set of prime numbers $= \{2, 3, 5, 7, 11, 13, 17, \ldots\}$.
- The rational numbers is the set of quotients of integers

$$\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \text{ and } q \neq 0\}.$$

A finite set may be defined by listing all of its elements. For example, the set $A = \{2, 4, 6, 8, 10\}$ is the set of all even natural numbers less than or equal to 10. The

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[2] There are mathematical objects known as multi-sets or bags that allow duplication of elements. For example, a bag of marbles may contain three green marbles, two blue and one red marble.

[3] The British logician, John Venn, invented the Venn diagram. It provides a visual representation of a set and the various set theoretical operations. Their use is limited to the representation of two or three sets as they become cumbersome with a larger number of sets.

order in which the elements are listed is not relevant: i.e., the set $\{2, 4, 6, 8, 10\}$ is the same as the set $\{8, 4, 2, 10, 6\}$.


Sets may be defined by using a predicate to constrain set membership. For example, the set $S = \{n : \mathbb{N} : n \leq 10 \wedge n \bmod 2 = 0\}$ also represents the set $\{2, 4, 6, 8, 10\}$. That is, the use of a predicate allows a new set to be created from an existing set by using the predicate to restrict membership of the set. The set of even natural numbers may be defined by a predicate over the set of natural numbers that restricts membership to the even numbers. It is defined by:

$$\text{Evens} = \{x | x \in \mathbb{N} \wedge even(x)\}.$$

In this example, $even(x)$ is a predicate that is true if x is even and false otherwise. In general, $A = \{x \in E | P(x)\}$ denotes a set A formed from a set E using the predicate P to restrict membership of A to those elements of E for which the predicate is true.

The elements of a finite set S are denoted by $\{x_1, x_2, \ldots, x_n\}$. The expression $x \in S$ denotes set membership and indicates that the element x is a member of the set S. The expression $x \notin S$ indicates that x is not a member of the set S.

A set S is a subset of a set T (denoted $S \subseteq T$) if whenever $s \in S$ then $s \in T$, and in this case the set T is said to be a superset of S (denoted $T \supseteq S$). Two sets S and T are said to be equal if they contain identical elements: i.e., $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$. A set S is a proper subset of a set T (denoted $S \subset T$) if $S \subseteq T$ and $S \neq T$. That is, every element of S is an element of T and there is at least one element in T that is not an element of S. In this case, T is a proper superset of S (denoted $T \supset S$).
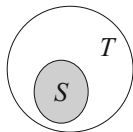
The empty set (denoted by $\varnothing$ or $\{\}$) represents the set that has no elements. Clearly $\varnothing$ is a subset of every set. The singleton set containing just one element x is denoted by $\{x\}$, and clearly $x \in \{x\}$ and $x \neq \{x\}$. Clearly, $y \in \{x\}$ if and only if $x = y$.

Example 2.2

(i) $\{1, 2\} \subseteq \{1, 2, 3\}$.
(ii) $\varnothing \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

The cardinality (or size) of a finite set S defines the number of elements present in the set. It is denoted by $|S|$. The cardinality of an infinite[4] set S is written as $|S| = \infty$.

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[4] The natural numbers, integers and rational numbers are countable sets whereas the real and complex numbers are uncountable sets.

Example 2.3

(i)   Given $A = \{2, 4, 5, 8, 10\}$ then $|A| = 5$.
(ii) Given $A = \{x \in \mathbb{Z} : x^2 = 9\}$ then $|A| = 2$.
(iii) Given $A = \{x \in \mathbb{Z} : x^2 = -9\}$ then $|A| = 0$.

2.2.1 Set Theoretical Operations

Several set theoretical operations are considered in this section. These include the
Cartesian product operation, the set union operation, the set intersection operation,
the set difference operation and the symmetric difference operation.

Cartesian Product The Cartesian product allows a new set to be created from
existing sets. The Cartesian[5] product of two sets S and T (denoted $S \times T$) is the set
of ordered pairs $\{(s, t) | s \in S, t \in T\}$. Clearly, $S \times T \neq T \times S$ and so the Cartesian
product of two sets is not commutative. Two ordered pairs (s_1, t_1) and (s_2, t_2) are
considered equal if and only if $s_1 = s_2$ and $t_1 = t_2$.

The Cartesian product may be extended to that of n sets $S_1, S_2, \ldots, S_n$. The Carte-
sian product $S_1 \times S_2 \times \cdots \times S_n$ is the set of ordered tuples $\{(s_1, s_2, \ldots, s_n) | s_1 \in S_1,
s_2 \in S_2, \ldots, s_n \in S_n\}$. Two ordered n-tuples $(s_1, s_2, \ldots, s_n)$ and $(s_1', s_2', \ldots, s_n')$ are
considered equal if and only if $s_1 = s_1', s_2 = s_2', \ldots, s_n = s_n'$.

The Cartesian product may also be applied to a single set S to create ordered
n-tuples of S: i.e., $S^n = S \times S \times \cdots \times S$ (n times).

Power Set The power set of a set A (denoted $\mathbb{P}A$) denotes the set of all subsets of A.
For example, the power set of the set $A = \{1, 2, 3\}$ has eight elements and is given
by:

$$\mathbb{P}A = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

There are $2^3 = 8$ elements in the power set of $A = \{1, 2, 3\}$ and the cardinality of A
is 3. In general, there are $2^{|A|}$ elements in the power set of A.

Theorem 2.1 (Cardinality of Power Set of A) There are $2^{|A|}$ elements in the power
set of A.

Proof Let $|A| = n$ then the subsets of A include subsets of size $0, 1, \ldots, n$. There
are $\binom{n}{k}$ subsets of A of size k. Therefore, the total number of subsets of A is the total
number of subsets of size $0, 1, 2, \ldots$ up to n. That is,

$$|\mathbb{P}A| = \sum_{k=0}^{n} \binom{n}{k}.$$

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[5] Cartesian product is named after René Descartes who was a famous 17th French mathematician
and philosopher. He invented the Cartesian coordinates system that links geometry and algebra,
and allows geometric shapes to be defined by algebraic equations.

The binomial theorem states that:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Therefore, putting $x = 1$ we get that:

$$2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k = |\mathbb{P}A\,|.$$

Union and Intersection Operations   The union of two sets A and B is denoted by $A \cup B$. It results in a set that contains all of the members of A and of B and is defined by:

$$A \cup B = \{r | r \in A \text{ or } r \in B\}\,.$$

For example, suppose $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$. Set union is a commutative operation: i.e., $A \cup B = B \cup A$. Venn Diagrams are used to illustrate these operations pictorially.
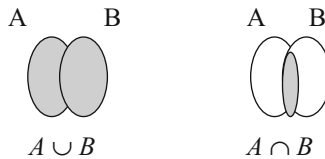
The intersection of two sets A and B is denoted by $A \cap B$. It results in a set containing the elements that A and B have in common and is defined by:

$$A \cap B = \{r | r \in A \text{ and } r \in B\}\,.$$

Suppose $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cap B = \{2, 3\}$. Set intersection is a commutative operation: i.e., $A \cap B = B \cap A$.

Union and intersection are binary operations but may be extended to more generalized union and intersection operations. For example,

$$\bigcup_{i=1}^{n} A_i \text{ denotes the union of } n \text{ sets,}$$

$$\bigcap_{i=1}^{n} A_i \text{ denotes the intersection of } n \text{ sets.}$$

Set Difference Operations   The set difference operation $A \backslash B$ yields the elements in A that are not in B. It is defined by:
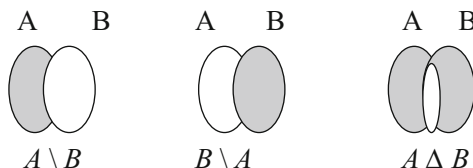
$$A \backslash B = \{a | a \in A \text{ and } a \notin B\}\,.$$

For A and B defined as $A = \{1, 2\}$ and $B = \{2, 3\}$ we have $A \backslash B = \{1\}$ and $B \backslash A = \{3\}$. Clearly, set difference is not commutative: i.e., $A \backslash B \neq B \backslash A$. Clearly, $A \backslash A = \emptyset$ and $A \backslash \emptyset = A$.

The symmetric difference of two sets A and B is denoted by $A \triangle B$ and is defined by:

$$A \triangle B = A \backslash B \cup B \backslash A.$$

The symmetric difference operation is commutative: i.e., $A \triangle B = B \triangle A$. Venn diagrams are used to illustrate these operations pictorially.

The complement of a set A (with respect to the universal set U) is the elements in the universal set that are not in A. It is denoted by A^c (or A') and is defined as:

$$A^c = \{u | u \in U \text{ and } u \notin A\} = U \backslash A.$$

The complement of the set A is illustrated by the shaded area below:
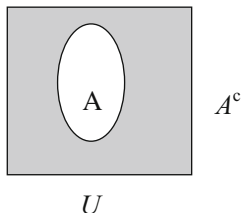
2.2.2   Properties of Set Theoretical Operations

The set union and set intersection properties are commutative and associative. The properties are listed in Table 2.1.

We give a proof of the distributive property.

Proof of Properties (Distributive Property) To show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Suppose $x \in A \cap (B \cup C)$ then

$$x \in A \wedge x \in (B \cup C),$$
$$\Rightarrow x \in A \wedge (x \in B \vee x \in C),$$

Table 2.1 Properties of set operations

Property	Description
Commutative	Union and intersection operations are commutative: i.e., $S \cup T = T \cup S$ $S \cap T = T \cap S$
Associative	Union and intersection operations are associative: i.e., $R \cup (S \cup T) = (R \cup S) \cup T$ $R \cap (S \cap T) = (R \cap S) \cap T$
Identity	The identity under set union is Ø and the identity under intersection is U. $S \cup \varnothing = \varnothing \cup S = S$ $S \cap U = U \cap S = S$
Distributive	The union operator distributes over the intersection operator and vice versa. $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$
DeMorgan's[a] law	The complement of $S \cup T$ is given by: $(S \cup T)^c = S^c \cap T^c$. The complement of $S \cap T$ is given by: $(S \cap T)^c = S^c \cup T^c$

[a]De Morgan's law is named after Augustus De Morgan, a nineteenth century English mathematician who was a contemporary of George Boole.

$$\Rightarrow (x \in A \land x \in B) \lor (x \in A \land x \in C),$$

$$\Rightarrow x \in (A \cap B) \lor x \in (A \cap C),$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C).$$

Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
 Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
 Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2.2.3 Russell's Paradox

Bertrand Russell was a famous British logician, mathematician and philosopher. He was the co-author with Alfred Whitehead of *Principia Mathematica*, which aimed to derive all of the truths of mathematics from logic. Russell's paradox was discovered by Bertrand Russell in 1901, and showed that the system of set theory being proposed by Frege contained a contradiction (Fig. 2.1).

Question 2.1 (Posed by Russell to Frege) Is the set of all sets that do not contain themselves as members a set?

Russell's Paradox   Let $A = \{S$ a set and $S \notin S\}$. Is $A \in A$? Then $A \in A \Rightarrow A \notin A$ and vice versa. Therefore, a contradiction arises in either case and there is no such set A.

Two ways of avoiding the paradox were developed in 1908, and these were Russell's theory of types and Zermelo set theory. Russell's theory of types was a response to the paradox that argued that the set of all sets is ill formed. Russell developed a hierarchy of sets with individual elements at the lowest level, sets of elements at the next level, sets of sets of elements at the next level and so on. It is then prohibited for a set to contain members of different types.

A set of elements has a different type from its elements, and one cannot speak of the set of all sets that do not contain themselves as members as these are of different types. The other way of avoiding the paradox was Zermelo's axiomatization of set theory.

Remark   Russell's paradox may also be illustrated by the story of a town that has exactly one barber who is male. *The barber shaves all and only those men in town who do not shave themselves*. The question is who shaves the barber.

If the barber does not shave himself then according to the rule he is shaved by the barber (i.e., himself). If he shaves himself then according to the rule he is not shaved by the barber (i.e., himself).

The paradox occurs due to self-reference in the statement and a logical examination shows that the statement is a contradiction.

2.3   Relations

A binary relation $R(A, B)$ where A and B are sets is a subset of $A \times B$: i.e., $R \subseteq A \times B$. The notation aRb signifies that $(a, b) \in R$.

A binary relation $R(A, A)$ is a relation between A and A. This type of relation may always be composed with itself, and its inverse is also a binary relation on A. The identity relation on A is defined by $a\, i_A a$ for all $a \in A$.

Example 2.4 There are many examples of relations:

(i) The relation tall defined on a set of students in a class where $(a, b) \in R$ if the height of a is greater than the height of b.

(ii) The relation between A and B where $A = \{0, 1, 2\}$ and $B = \{3, 4, 5\}$ with R given by:

$$R = \{(0, 3), (0, 4), (1, 4)\}.$$

(iii) The relation less than ($<$) between $\mathbb{R}$ and $\mathbb{R}$ is given by:

$$\{(x, y) \in \mathbb{R}^2 : x < y\}.$$

(iv) A bank may represent the relationship between the set of accounts and the set of customers by a relation. The implementation of a bank account could be a positive integer with at most eight decimal digits.

The relationship between accounts and customers may be done with a relation $R \subseteq A \times B$, with the set A chosen to be the set of natural numbers, and the set B chosen to be the set of all human beings alive or dead. The set A could also be chosen to be $A = \{n \in \mathbb{N} : n < 10^8\}$.

A relation $R(A, B)$ may be represented pictorially. This is referred to as the graph of the relation and is illustrated in the diagram below. An arrow from x to y is drawn if (x, y) is in the relation. Thus for the height relation R given by $\{(a, p), (a, r), (b, q)\}$ an arrow is drawn from a to p, from a to r and from b to q to indicate that (a, p), (a, r) and (b, q) are in the relation R.
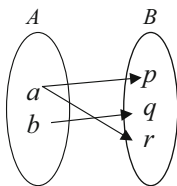
The pictorial representation of the relation makes it easy to see that the height of a is greater than the height of p and r, and that the height of b is greater than the height of q.

An n-ary relation $R\,(A_1, A_2, \ldots, A_n)$ is a subset of $(A_1 \times A_2 \times \cdots \times A_n)$. However, an n-ary relation may also be regarded as a binary relation $R(A, B)$ with $A = A_1 \times A_2 \times \cdots \times A_{n-1}$ and $B = A_n$.

Fig. 2.2 Reflexive relation
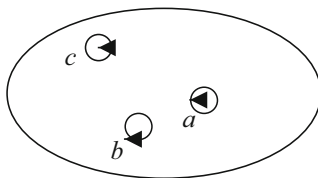
Fig. 2.3 Symmetric relation
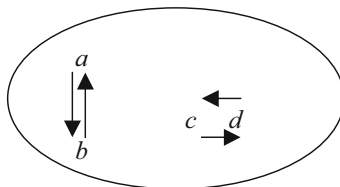
2.3.1   Reflexive, Symmetric and Transitive Relations

There are various types of relations on a set A including reflexive, symmetric and transitive relations.

(i)   A relation on a set A is *reflexive* if $(a, a) \in R$ for all $a \in A$.
(ii)   A relation R is *symmetric* if whenever $(a, b) \in R$ then $(b, a) \in R$.
(iii) A relation is *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

A relation that is reflexive, symmetric and transitive is termed an *equivalence relation*. An equivalence relation gives a partition of the set A.

Example 2.5 (Reflexive Relation) A relation is reflexive if each element possesses an edge looping around on itself. The relation below is reflexive since we have a loop (a, a) for each $a \in A$ (Fig. 2.2).

Example 2.6 (Symmetric Relation) The graph of a symmetric relation will show for every arrow from a to b an opposite arrow from b to a. The relation in Fig. 2.3 is symmetric: i.e., whenever $(a, b) \in R$ then $(b, a) \in R$ (Fig. 2.3).

Example 2.7 (Transitive Relation) The graph of a transitive relation will show that whenever there is an arrow from a to b and an arrow from b to c that there is an arrow from a to c. The relation in Fig. 2.4 is transitive: i.e., whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ (Fig. 2.4).

Example 2.8 (Equivalence Relation) The relation on the set of integers $\mathbb{Z}$ defined by $(a, b) \in R$ if $a - b = 2k$ for some $k \in \mathbb{Z}$ is an equvalence relation, and partitions the set integers into two equivalence classes: i.e., the even and odd integers.

Domain and Range of Relation The domain of a relation R (A, B) is given by $\{a \in A | \exists b \in B \text{ and } (a, b) \in R\}$. It is denoted by **dom** R. The domain of the relation $R = \{(a, p), (a, r), (b, q)\}$ is $\{a, b\}$.

Fig. 2.4 Transitive relation
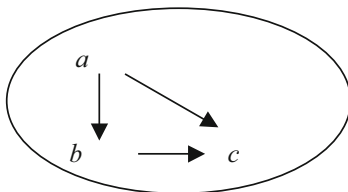
Fig. 2.5 Partitions of A
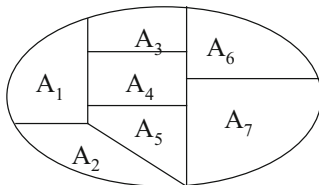
The range of a relation R (A, B) is given by $\{b \in B | \exists a \in A \text{ and } (a, b) \in R\}$. It is denoted by **rng** R. The range of the relation $R = \{(a, p), (a, r), (b, q)\}$ is $\{p, q, r\}$.

Inverse of a Relation Suppose $R \subseteq A \times B$ is a relation between A and B then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by:

$$b R^{-1} a \text{ if and only if } a R b.$$

That is,

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}.$$

Example 2.9 Let R be the relation between $\mathbb{Z}$ and $\mathbb{Z}^+$ defined by mRn if and only if $m^2 = n$. Then $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z}^+ : m^2 = n\}$ and $R^{-1} = \{(n, m) \in \mathbb{Z}^+ \times \mathbb{Z} : m^2 = n\}$.
  For example, $-3 \ R \ 9$, $-4 \ R \ 16$, $0 \ R \ 0$, $16 \ R^{-1} \ 4$, $9 \ R^{-1} \ 3$, etc.

Partitions and Equivalence Relations An equivalence relation on A leads to a partition of A, and vice versa for every partition of A there is a corresponding equivalence relation.
  Let A be a finite set and let $A_1, A_2, \ldots, A_n$ be subsets of A such $A_i \neq \emptyset$ for all i, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A = \cup_i^n \quad A_i = A_1 \cup A_2 \cup \cdots \cup A_n$. The sets A_i partition the set A and these sets are called the classes of the partition (Fig. 2.5).

Theorem 2.2 An equivalence relation on A gives rise to a partition of A where the equivalence classes are given by $\text{Class}(a) = \{x | x \in A \text{ and } (a, x) \in R\}$. Similarly, a partition gives rise to an equivalence relation R, where $(a, b) \in R$ if and only if a and b are in the same partition.

Proof Clearly, $a \in \text{Class}(a)$ since R is reflexive and clearly the union of the equivalence classes is A. Next, we show that two equivalence classes are either equal or disjoint.

Suppose Class$(a) \cap$ Class$(b) \neq \emptyset$. Let $x \in$ Class$(a) \cap$ Class(b) and so (a, x) and $(b, x) \in R$. By the symmetric property $(x, b) \in R$ and since R is transitive from (a, x) and (x, b) in R we deduce that $(a, b) \in R$. Therefore $b \in$ Class(a). Suppose y is an arbitrary member of Class (b) then $(b, y) \in R$ therefore from (a, b) and (b, y) in R we deduce that (a, y) is in R. Therefore since y was an arbitrary member of Class(a) we deduce that Class$(b) \subseteq$ Class(a). Similarly, Class$(a) \subseteq$ Class(b) and so Class$(a) =$ Class(b).

This proves the first part of the theorem and for the second part we define a relation R such that $(a, b) \in R$ if a and b are in the same partition. It is clear that this is an equivalence relation.

2.3.2   Composition of Relations

The composition of two relations $R_1(A, B)$ and $R_2(B, C)$ is given by $R_2 \circ R_1$ where $(a, c) \in R_2 \circ R_1$ if and only there exists $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. The composition of relations is associative: i.e.,

$$(R_3 \circ R_2) \circ R_1 = R_3 \circ (R_2 \circ R_1).$$

Example 2.10 (Composition of Relations) Consider a library that maintains two files. The first file maintains the serial number s of each book with details of the author a of the book. This may be represented by the relation $R_1 = s R_1 a$. The second file maintains the library card number c of its borrowers and the serial number s of any books that they have borrowed. This may be represented by the relation $R_2 = c R_2 s$.

The library wishes to issue a reminder to its borrowers of the authors of all books currently on loan to them. This may be determined by the composition of $R_1 \circ R_2$: i.e., $c R_1 \circ R_2 a$ if there is book with serial number s such that $s R_1 a$ and $c R_2 s$.

Example 2.11 (Composition of Relations) Consider sets $A = \{a, b, c\}$, $B = \{d, e, f\}$, $C = \{g, h, i\}$ and relations $R(A, B) = \{(a, d), (a, f), (b, d), (c, e)\}$ and $S(B, C) = \{(d, h), (d, i), (e, g), (e, h)\}$. Then we graph these relations and show how to determine the composition pictorially (Fig. 2.6).

$S \circ R$ is determined by choosing $x \in A$ and $y \in C$ and checking if there is a route from x to y in the graph. If so, we join x to y in $S \circ R$. For example, if we consider a and h we see that there is a path from a to d and from d to h and therefore (a, h) is in the composition of S and R.

$S \circ R$
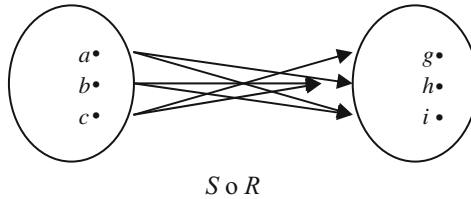
Fig. 2.6 Composition of relations
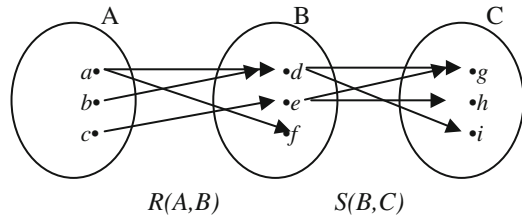
The union of two relations $R_1(A, B)$ and $R_2(A, B)$ is meaningful (as these are both subsets of $A \times B$). The union $R_1 \cup R_2$ is defined as $(a, b) \in R_1 \cup R_2$ if and only if $(a, b) \in R_1$ or $(a, b) \in R_2$.

Similarly, the intersection of R_1 and R_2 ($R_1 \cap R_2$) is meaningful and is defined as $(a, b) \in R_1 \cap R_2$ if and only if $(a, b) \in R_1$ and $(a, b) \in R_2$. The relation R_1 is a subset of $R_2 (R_1 \subseteq R_2)$ if whenever $(a, b) \in R_1$ then $(a, b) \in R_2$.

The inverse of the relation R was discussed earlier and is given by the relation R^{-1} where $R^{-1} = \{(b, a) | (a, b) \in R\}$.

The composition of R and R^{-1} yields: $R^{-1} \circ R = \{(a, a) | a \in \text{dom } R\} = i_A$ and $R \circ R^{-1} = \{(b, b) | b \in \mathbf{dom}\ R^{-1}\} = i_B$.

2.3.3 Binary Relations

A binary relation R on A is a relation between A and A, and a binary relation can always be composed with itself. Its inverse is a binary relation on the same set. The following are all relations on A:

$$R^2 = R \circ R,$$

$$R^3 = (R \circ R) \circ R,$$

$$R^0 = i_A \text{ (identity relation)},$$

$$R^{-2} = R^{-1} \circ R^{-1}.$$

Example 2.12 Let R be the binary relation on the set of all people P such that $(a, b) \in R$ if a is a parent of b. Then the relation R^n is interpreted as:

R is the parent relationship.
R^2 is the grandparent relationship.
R^3 is the great grandparent relationship.
R^{-1} is the child relationship.
R^{-2} is the grandchild relationship.
R^{-3} is the great grandchild relationship.

This can be generalized to a relation R^n on A where $R^n = R \circ R \circ \cdots \circ R$ (n-times). The transitive closure of the relation R on A is given by:

$$R^* = \cup_{i=0}^{\infty} R^i = R^0 \cup R^1 \cup R^2 \cup \ldots R^n \cup \ldots,$$

where R^0 is the reflexive relation containing only each element in the domain of R: i.e., $R^0 = i_A = \{(a, a) | a \in \textbf{dom } R\}$.

The positive transitive closure is similar to the transitive closure except that it does not contain R^0. It is given by:

$$R^+ = \cup^\infty_{i=1} R^i = R^1 \cup R^2 \cup \ldots R^n \cup \ldots.$$

$a\, R^+\, b$ if and only if $a\, Rn\, b$ for some $n > 0$: *i.e.*, there exists $c_1, c_2, \ldots c_n \in A$ such that:

$$a R c_1, c_1 R c_2, \ldots, c_n R b.$$

Parnas[6] introduced the concept of the limited domain relation (LD-relation), and a LD relation L consists of an ordered pair (R_L, C_L) where R_L is a relation and C_L is a subset of Dom R_L. The relation R_L is on a set U and C_L is termed the competence set of the LD relation L. A description of LD relations and a discussion of their properties are in Chap. 2 of [Par:01].

The importance of LD relations is that they may be used to describe program execution. The relation component of the LD relation L describes a set of states such that if execution starts in state x it may terminate in state y. The set U is the set of states. The competence set of L is such that if execution starts in a state that is in the competence set then it is guaranteed to terminate.

2.4   Functions

A function $f : A \rightarrow B$ is a special relation such that for each element $a \in A$ there is exactly (or at most)[7] one element $b \in B$. This is written as $f(a) = b$.
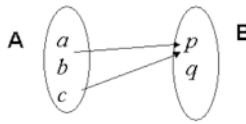

A function is a relation but not every relation is a function. For example, the relation in the diagram below is not a funcion since there are two arrows from the element $a \in A$.

---

[6] Parnas made important contributions to software engineering in the 1970s. He invented information hiding which is used in object-oriented design.

[7] We distinguish between total and partial functions. A total function $f : A \rightarrow B$ is defined for all elements in A whereas a partial function may be undefined for one or more elements in A.

Fig. 2.7 Domain and range of
a partial function
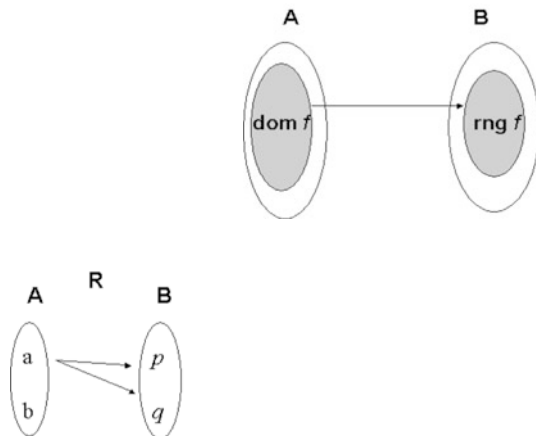
The domain of the function (denoted by **dom** f) is the set of values in A for which
the function is defined. The domain of the function is A provided that f is a total
function. The co-domain of the function is B. The range of the function (denoted **rng**
f) is a subset of the co-domain and consists of:

$$\textbf{rng } f = \{r | r \in B \text{ such that } f(a) = r \text{ for some } a \in A\}.$$

Functions may be partial or total. A *partial function* (or partial mapping) may be
undefined for some values of A, and patial functions arise regularly in the computing
field. *Total functions* are defined for every value in A and many functions encountered
in mathematics are total (Fig. 2.7).

Example 2.13 (Functions) Functions are an essential part of mathematics and com-
puter science, and there are many well-known functions such as the trigonometric
functions $\sin(x)$, $\cos(x)$ and $\tan(x)$; the logarithmic function $\ln(x)$; the exponential
functions e^x and polynomial functions.

(i)   Consider the partial function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \frac{1}{x} \quad \text{(where } x \neq 0\text{)}.$$

This partial function is defined everywhere except for $x = 0$.
(ii) Consider the function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = x^2.$$

Then this function is defined for all $x \in \mathbb{R}$.
   Partial functions often arise in computing as a program may be undefined or fail
to terminate for several values of its arguments (e.g., infinite loops). Care is required
to ensure that the partial function is defined for the argument to which it is to be
applied.

Consider a program P that has one natural number as its input and which for some input values will never terminate. Suppose that if it terminates it prints a single real result and halts. Then P can be regarded as a partial mapping from $\mathbb{N}$ to $\mathbb{R}$,

$$P : \mathbb{N} \to \mathbb{R}.$$

Example 2.14 How many total functions $f : A \to B$ are there from A to B (where A and B are finite sets)?

Each element of A maps to any element of B, i.e. there are $|B|$ choices for each element $a \in A$. Since there are $|A|$ elements in A the number of total functions is given by:

$\quad\quad |B||B|\dots|B| \quad\quad\quad\quad\quad\quad (|A| \text{times})$

$\quad\quad = |B|^{|A|} \quad\quad\quad\quad\quad\quad\quad\quad \text{total functions between } A \text{ and } B.$

Example 2.15 How many partial functions $f : A \to B$ are there from A to B (where A and B are finite sets)?

Each element of A may map to any element of B or to no element of B (as it may not appear). In other words, there are $|B| + 1$ choices for each element of A. Since there are $|A|$ elements in A the number of distinct partial functions between A and B is given by:

$\quad\quad (|B| + |1|)(|B| + |1|)\dots(|B| + |1|) \quad\quad\quad (|A| \text{times})$
$\quad\quad = (|B| + |1|)^{|A|}.$

Two partial functions f and g are equal if:

1. $\operatorname{dom} f = \operatorname{dom} g$.
2. $f(a) = g(a)$ for all $a \in \operatorname{dom} f$.

A function f is less defined than a function $g (f \subseteq g)$ if the domain of f is a subset of the domain of g, and the functions agree for every value on the domain of f:

1. $\operatorname{dom} f \subseteq \operatorname{dom} g$.
2. $f(a) = g(a)$ for all $a \in \operatorname{dom} f$.

The composition of functions is similar to the composition of relations. Suppose $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$ is a function, and this is written as $g \circ f(x)$ or $g(f(x))$ for $x \in A$.

The composition of functions is not commutative and this can be seen by an example. Consider the function $f : R \to R$ such that $f(x) = x^2$ and the function $g : R \to R$ such that $g(x) = x + 2$. Then

$$g \circ f(x) = g(x^2) = x^2 + 2,$$
$$f \circ g(x) = f(x + 2) = (x + 2)^2 = x^2 + 4x + 4.$$

Clearly, $g \circ f(x) \neq f \circ g(x)$ and so composition of functions is not commutative. The composition of functions is associative, as the composition of relations is associative and every function is a relation. For $f : A \to B$, $g : B \to C$, and $h : C \to D$ we have:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Fig. 2.8 Injective and
surjective functions

1-1, Not Onto   Onto, Not 1-1
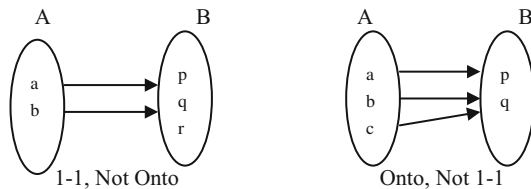
Fig. 2.9 Bijective function.
(One to one and onto)
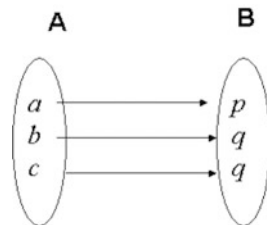
A function $f : A \rightarrow B$ is *injective* (*one to one*) if:

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

For example, consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = x^2$. Then $f(3) = f(-3) = 9$ and so this function is not one to one.

A function $f : A \rightarrow B$ is *surjective* (*onto*) if given any $b \in B$ there exists an $a \in A$ such that $f(a) = b$. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = x + 1$. Clearly, given any $r \in \mathbb{R}$ then $f(r - 1) = r$ and so f is onto (Fig. 2.8).

A function is *bijective* if it is one to one and onto. That is, there is a one to one correspondence between the elements in A and B, and for each $b \in B$ there is a unique $a \in A$ such that $f(a) = b$ (Fig. 2.9).

The inverse of a relation was discussed earlier and the relational inverse of a function $f : A \rightarrow B$ clearly exists. The relational inverse of the function may or may not be a function.

However, if the relational inverse is a function it is denoted by $f^{-1} : B \rightarrow A$. A total function has an inverse if and only if it is bijective whereas a partial function has an inverse if and only if it is injective.

The identity function $1_A : A \rightarrow A$ is a function such that $1_A(a) = a$ for all $a \in A$. Clearly, when the inverse of the function exists then we have that $f^{-1} \circ f = 1_A$ and $f^{-} \circ f^{-1} = 1_B$.

Theorem 2.3 A total function has an inverse if and only if it is bijective.

Proof Suppose $f : A \rightarrow B$ has an inverse f^{-1}. Then we show that f is bijective.

We first show that f is one to one. Suppose $f(x_1) = f(x_2)$ then

$$f^{-1}(f(x_1)) = f^{-1}(f(x_2)),$$
$$\Rightarrow f^{-1} \circ f(x_1) = f^{-1} \circ f(x_2),$$

$$\Rightarrow 1_A(x_1) = 1_A(x_2),$$
$$\Rightarrow x_1 = x_2.$$

Next we first show that f is onto. Let $b \in B$ and let $a = f^{-1}(b)$ then

$$f(a) = f(f^{-1}(b)) = b \text{ and so } f \text{ is surjective.}$$

The second part of the proof is concerned with showing that if $f : A \to B$ is bijective then it has an inverse f^{-1}. Clearly, since f is bijective we have that for each $a \in A$ there exists a unique $b \in B$ such that $f(a) = b$.

Define $g: B \to A$ by letting $g(b)$ be the unique a in A such that $f(a) = b$. Then we have that:

$$gof(a) = g(b) = a \text{ and } fog(b) = f(a) = b.$$

Therefore, g is the inverse of f.

2.5   Review Questions

1. What is a set? A relation? A function?
2. Explain the difference between a partial and a total function.
3. Explain the difference between a relation and a function.
4. Determine $A \times B$ where $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.
5. Determine the symmetric difference $A \triangle B$ where $A = \{a, b, c, d\}$ and $B = \{c, d, e\}$.
6. What is the graph of the relation $\leq$ on the set $A = \{2, 3, 4\}$.
7. What is the composition of S and R (i.e., $S \circ R$), where R is a relation between A and B, and S is a relation between B and C. The sets A, B, C are defined as $A = \{a, b, c, d\}$, $B = \{e, f, g\}$, $C = \{h, i, j, k\}$ and $R = \{(a, e), (b, e), (b, g), (c, e), (d, f)\}$ with $S = \{(e, h), (e, k), (f, j), (f, k), (g, h)\}$.
8. What is the domain and range of the relation R where $R = \{(a, p), (a, r), (b, q)\}$.
9. Determine the inverse relation R^{-1} where $R = \{(a, 2), (a, 5), (b, 3), (b, 4), (c, 1)\}$.
10. Determine the inverse of the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x-2}{x-3}(x \neq 3)$ and $f(3) = 1$.
11. Give examples of injective, surjective and bijective functions.
12. Let $n \geq 2$ be a fixed integer. Consider the relation $\equiv$ defined by $\{(p, q) : p, q \in \mathbb{Z}, n|(q-p)\}$
    a. Show $\equiv$ is an equivalence relation.
    b. What are the equivalence classes of this relation?

2.6   **Summary**

This chapter provided an introduction to set theory, relations and functions. Sets are collections of well-defined objects, a relation between A and B indicates relationships between members of the sets A and B and functions are a special type of relation where there is at most one relationship for each element $a \in A$ with an element in B.

A set is a collection of well-defined objects that contains no duplicates. There are many examples of sets such as the set of natural numbers $\mathbb{N}$, the integer numbers $\mathbb{Z}$ and so on.

The Cartesian product allows a new set to be created from existing sets. The Cartesian product of two sets S and T (denoted $S \times T$) is the set of ordered pairs $\{(s, t) | s \in S, t \in T\}$.

A binary relation $R(A, B)$ is a subset of the Cartesian product $(A \times B)$ of A and B where A and B are sets. The domain of the relation is A and the co-domain of the relation is B. The notation aRb signifies that there is a relation between a and b and that $(a, b) \in R$. An n-ary relation $R(A_1, A_2, \ldots, A_n)$ is a subset of $(A_1 \times A_2 \times \cdots \times A_n)$.

A total function $f: A \rightarrow B$ is a special relation such that for each element $a \in A$ there is exactly one element $b \in B$. This is written as $f(a) = b$. A function is a relation but not every relation is a function.

The domain of the function (denoted by **dom** f) is the set of values in A for which the function is defined. The domain of the function is A provided that f is a total function. The co-domain of the function is B.