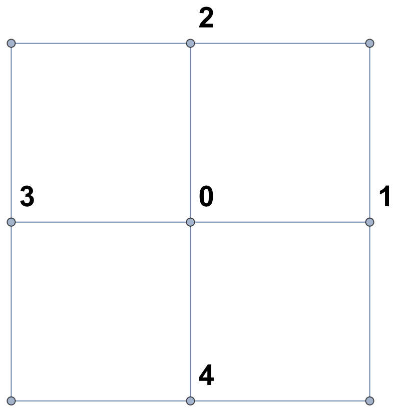
应用随机游走方法和有限差分方法求解泊松方程

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = q(x, y) & (x, y) \in \mathrm{Inter}(\mathbb{D}) \\ \phi(s) = F(s) & s \in \Gamma \end{cases}$$



其中 $\mathbb{D}$ 是一个有限区域，$\mathrm{Inter}(\mathbb{D})$ 是区域 $\mathbb{D}$ 的内部，Γ 是区域 $\mathbb{D}$ 的边界

将区域 $\mathbb{D}$ 用几个正方形均匀网格划分，用差分代替微分

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \to \frac{\left(\frac{\phi_1-\phi_0}{h}\right) - \left(\frac{\phi_0-\phi_3}{h}\right)}{h} + \frac{\left(\frac{\phi_2-\phi_0}{h}\right) - \left(\frac{\phi_0-\phi_4}{h}\right)}{h} = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\,\phi_0}{h^2}$$

得到差分方程

$$\begin{cases} \phi_0 = \frac{1}{4}\left(\phi_1 + \phi_2 + \phi_3 + \phi_4 - h^2 q_0\right) & 0 \text{ 是正则内点} \\ \phi(s) = F(s) & s \in \Gamma \end{cases}$$

该差分方程的解并不严格等于原偏微分方程的解，其误差称为离散化误差，当 $h \to 0$ 时，该误差也趋向 0

下面利用随机游走的方法求解该差分方程

设粒子从区域 $\mathbb{D}$ 网格中某内点 x_0 开始做随机游走，经过 n 步随机游走后粒子位于 x_n
若 x_n 是内点，则 x_{n+1} 在 X_n 的四个邻近点中等概率随机选取
若 x_n 是边界点，则 $x_{n+1} = x_n$
随机变量 X_n 表示粒子经过 n 次随机游走后的位置

定义函数

$$Q(x) = \begin{cases} q(x) & x \in \mathrm{Inter}(\mathbb{D}) \\ 0 & x \in \Gamma \end{cases}$$

记 $\phi(X_n) = \phi_n$，$q(X_n) = q_n$，$Q(X_n) = Q_n$，注意 ϕ_n 和 Q_n 都是随机变量
$\mathbb{E}(X)$ 表示随机变量 X 的期望，$\mathbb{E}|_\alpha(X)$ 表示在 α 条件下随机变量 X 的期望

求 ϕ 在某内点 x_0 处的函数值 $\phi(x_0) = \mathbb{E}(\phi_0)$

x_0 是内点，由差分方程易知

$$\phi(x_0) = \mathbb{E}(\phi_1) - \frac{h^2}{4} q(x_0) = \mathbb{E}(\phi_1) - \frac{h^2}{4} Q(x_0)$$

考虑 ϕ_n，如果 x是内点

$$\phi(x) = \mathbb{E}|_{X_n=x}(\phi_{n+1}) - \frac{h^2}{4} q(x) = \mathbb{E}|_{X_n=x_n}(\phi_{n+1}) - \frac{h^2}{4} Q(x)$$

如果 x是边界点

$$\phi(x) = F(x) = \mathbb{E}|_{X_n=x}(\phi_{n+1}) = \mathbb{E}|_{X_n=x}(\phi_{n+1}) - \frac{h^2}{4}Q(x)$$

所以

$$\phi(x) = \mathbb{E}|_{X_n=x}(\phi_{n+1}) - \frac{h^2}{4}Q(x)$$

$$\begin{aligned}
&\mathbb{E}(\phi_n) \\
&= \sum_x (P(X_n = x)\,\phi(x)) \\
&= \sum_x \left(P(X_n = x)\left(\mathbb{E}|_{X_n=x}(\phi_{n+1}) - \frac{h^2}{4}Q(x)\right)\right) \\
&= \sum_x \left(P(X_n = x)\left(\sum_y (P(X_{n+1} = y \mid X_n = x)\,\phi(y)) - \frac{h^2}{4}Q(x)\right)\right) \\
&= \sum_x \sum_y (P(X_n = x)\,P(X_{n+1} = y \mid X_n = x)\,\phi(y)) - \sum_x \left(P(X_n = x)\,\frac{h^2}{4}Q(x)\right) \\
&= \sum_y (P(X_{n+1} = y)\,\phi(y)) - \frac{h^2}{4}\mathbb{E}\left(Q(X_n)\right) \\
&= \mathbb{E}(\phi_{n+1}) - \frac{h^2}{4}\mathbb{E}\left(Q_n\right)
\end{aligned}$$

所以

$$\begin{aligned}
&\phi(x_0) \\
&= \mathbb{E}(\phi_1) - \frac{h^2}{4}Q(x_0) \\
&= \mathbb{E}(\phi_1) - \frac{h^2}{4}\mathbb{E}\left(Q_0\right) \\
&= \mathbb{E}(\phi_2) - \frac{h^2}{4}\mathbb{E}\left(Q_1\right) - \frac{h^2}{4}\mathbb{E}\left(Q_0\right) \\
&\ldots \\
&= \mathbb{E}(\phi_n) - \frac{h^2}{4}\mathbb{E}\left(Q_{n-1}\right) - \ldots - \frac{h^2}{4}\mathbb{E}\left(Q_0\right) \\
&= \mathbb{E}\left(\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i\right) \\
&= \lim_{n\to\infty}\mathbb{E}\left(\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i\right)
\end{aligned}$$

假设

$$\lim_{n\to\infty}\left(P(X_n \notin \Gamma)\,\mathbb{E}|_{X_n\notin\Gamma}\left(\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i\right)\right) = 0$$

则

$$
\begin{aligned}
&\phi(x_0) \\
&= \lim_{n\to\infty} \mathbb{E}\left(\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i\right) \\
&= \lim_{n\to\infty} \left(P(X_n \in \Gamma)\,\mathbb{E}|_{X_n\in\Gamma}\left(\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i\right) + P(X_n \notin \Gamma)\,\mathbb{E}|_{X_n\notin\Gamma}\left(\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i\right)\right) \\
&= \lim_{n\to\infty} \left(\mathbb{E}|_{X_n\in\Gamma}\left(\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i\right)\right)
\end{aligned}
$$

对随机游走序列采样，取足够大的 n，在 $X_n \in \Gamma$ 条件下应用舍选法，用

$$\phi_n - \frac{h^2}{4}\sum_{i=0}^{n-1} Q_i$$

的样本平均估计待求函数值