FUZZY OPTIMIZATION

Most technical fields, including all those in engineering, involve some form of optimization that is required in the process of design. Since design is an open-ended problem with many solutions, the quest is to find the "best" solution according to some criterion. In fact, almost any optimization process involves trade-offs between costs and benefits because finding optimum solutions is analogous to creating designs – there can be many solutions, but only a few might be optimum, or useful, particularly where there is a generally nonlinear relationship between performance and cost. Optimization, in its most general form, involves finding the most optimum solution from a family of reasonable solutions according to an optimization criterion. For all but a few trivial problems, finding the global optimum (the best optimum solution) can never be guaranteed. Hence, optimization in the last three decades has focused on methods to achieve the best solution per unit computational cost.

In cases where resources are unlimited and the problem can be described analytically and there are no constraints, solutions found by exhaustive search [Akai, 1994] can guarantee global optimality. In effect, this global optimum is found by setting all the derivatives of the criterion function to zero, and the coordinates of the stationary point that satisfy the resulting simultaneous equations represent the solution. Unfortunately, even if a problem can be described analytically there are seldom situations with unlimited search resources. If the optimization problem also requires the simultaneous satisfaction of several constraints and the solution is known to exist on a boundary, then constraint boundary search methods such as Lagrangian multipliers are useful [deNeufville, 1990]. In situations where the optimum is not known to be located on a boundary, methods such as the steepest gradient, Newton–Raphson, and penalty function have been used [Akai, 1994], and some very promising methods have used genetic algorithms [Goldberg, 1989].

For functions with a single variable, search methods such as Golden section and Fibonacci are quite fast and accurate. For multivariate situations, search strategies such as parallel tangents and steepest gradients have been useful in some situations. But most of these classical methods of optimization [Vanderplaats, 1984] suffer from one or more disadvantages: the problem of finding higher order derivatives of a process, the issue of describing the problem as an analytic function, the problem of combinatorial explosion when dealing with many variables, the problem of slow convergence for small spatial or temporal step sizes, and the problem of overshoot for step sizes too large. In many situations, the precision of the optimization approach is greater than the original data describing the problem, so there is an *impedance mismatch* in terms of resolution between the required precision and the inherent precision of the problem itself.

In the typical scenario of an optimization problem, fast methods with poorer convergence behavior are used first to get the process near a solution point, such as a Newton method, then slower but more accurate methods, such as gradient schemes, are used to converge to a solution. Some current successful optimization approaches are now based on this hybrid idea: fast, approximate methods first, slower and more precise methods second. Fuzzy optimization methods have been proposed as the first steps in hybrid optimization schemes. One of these methods will be introduced here. More methods can be found in Sakawa [1993].

One-dimensional Optimization

Classical optimization for a one-dimensional (one independent variable) relationship can be formulated as follows. Suppose we wish to find the optimum solution, x^*, which maximizes the objective function $y = f(x)$, subject to the constraints

$$g_i(x) \leq 0, \qquad i = 1, m \tag{14.1}$$

Each of the constraint functions $g_i(x)$ can be aggregated as the intersection of all the constraints. If we let $\mathrm{C}_i = \{x \mid g_i(x) \leq 0\}$, then

$$\mathrm{C} = \mathrm{C}_1 \cap \mathrm{C}_2 \cap \cdots \cap \mathrm{C}_m = \{x \mid g_1(x) \leq 0,\ g_2(x) \leq 0, \ldots, g_m(x) \leq 0\} \tag{14.2}$$

which is the feasible domain described by the constraints C_i. Thus, the solution is

$$f(x^*) = \max_{x \in \mathrm{C}}\{f(x)\} \tag{14.3}$$

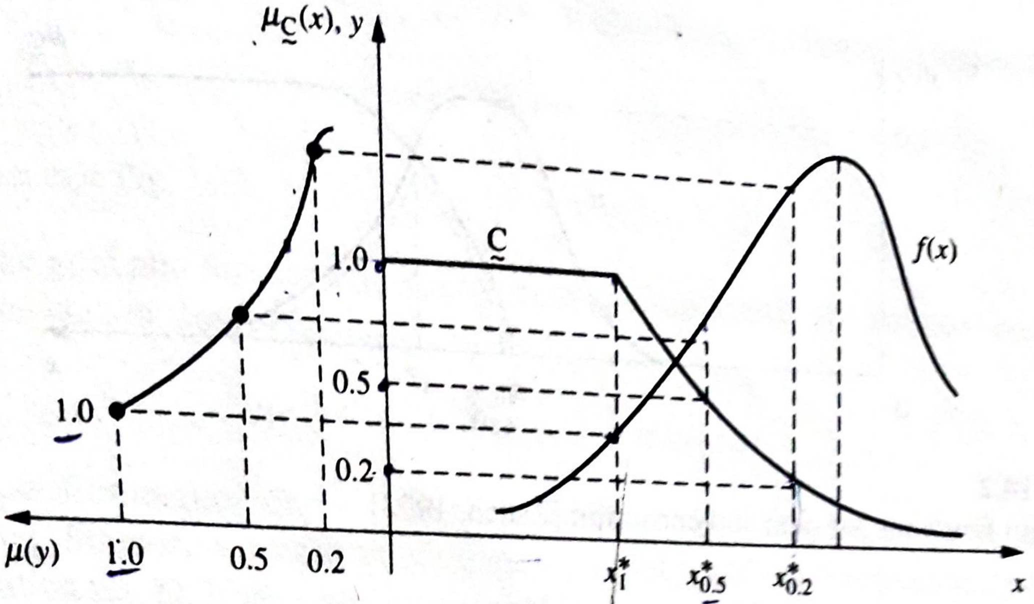

FIGURE 14.1
Function to be optimized, $f(x)$, and fuzzy constraint, $\underset{\sim}{C}$.

In a real environment, the constraints might not be so crisp, and we could have fuzzy feasible domains (see Fig. 14.1) such as "x could exceed x_0 a little bit." If we use λ-cuts on the fuzzy constraints $\underset{\sim}{C}$, fuzzy optimization is reduced to the classical case. Obviously, the optimum solution x^* is a function of the threshold level λ, as given in Eq. (14.4):

$$f(x_\lambda^*) = \max_{x \in C_\lambda}\{f(x)\} \tag{14.4}$$

Sometimes, the goal and the constraint are more or less contradictory, and some trade-off between them is appropriate. This can be done by converting the objective function $y = f(x)$ into a pseudogoal $\underset{\sim}{G}$ [Zadeh, 1972] with membership function

$$\mu_{\underset{\sim}{G}}(x) = \frac{f(x) - m}{M - m} \tag{14.5}$$

where $m = \inf_{x \in X} f(x)$

$M = \sup_{x \in X} f(x)$

Then the fuzzy solution set $\underset{\sim}{D}$ is defined by the intersection

$$\underset{\sim}{D} = \underset{\sim}{C} \cap \underset{\sim}{G} \tag{14.6}$$

membership is described by

$$\mu_{\underset{\sim}{D}}(x) = \min\{\mu_{\underset{\sim}{C}}(x), \mu_{\underset{\sim}{G}}(x)\} \tag{14.7}$$

and the optimum solution will be x^* with the condition

$$\mu_{\underset{\sim}{D}}(x^*) \geq \mu_{\underset{\sim}{D}}(x) \qquad \text{for all } x \in X \tag{14.8}$$

where $\mu_{\underset{\sim}{C}}(x), \underset{\sim}{C}, x$ should be substantially greater than x_0. Figure 14.2 shows this situation.

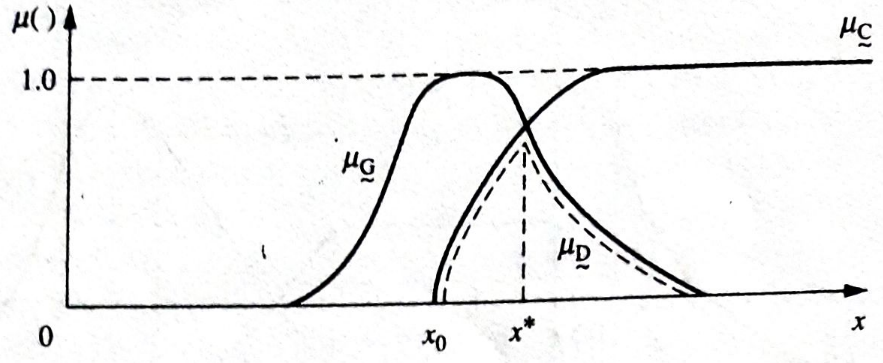

FIGURE 14.2
Membership functions for goal and constraint [Zadeh, 1972].

Example 14.1. Suppose we have a deterministic function given by

$$f(x) = xe^{(1-x/5)}$$

for the region $0 \leq x \leq 5$, and a fuzzy constraint given by

$$\mu_{\underset{\sim}{C}}(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ \dfrac{1}{1+(x-1)^2}, & x > 1 \end{cases}$$

Both of these functions are illustrated in Fig. 14.3. We want to determine the solution set $\underset{\sim}{D}$ and the optimum solution x^*, i.e., find $f(x^*) = y^*$. In this case we have $M = \sup[f(x)] = 5$ and $m = \inf[f(x)] = 0$; hence Eq. (14.5) becomes

$$\mu_{\underset{\sim}{G}}(x) = \frac{f(x) - 0}{5 - 0} = \frac{x}{5}e^{(1-x/5)}$$

which is also shown in Fig. 14.3. The solution set membership function, using Eq. (14.7), then becomes

$$\mu_{\underset{\sim}{D}}(x) = \begin{cases} \dfrac{x}{5}e^{(1-x/5)}, & 0 \leq x \leq x^* \\ \dfrac{1}{1+(x-1)^2}, & x > x^* \end{cases}$$

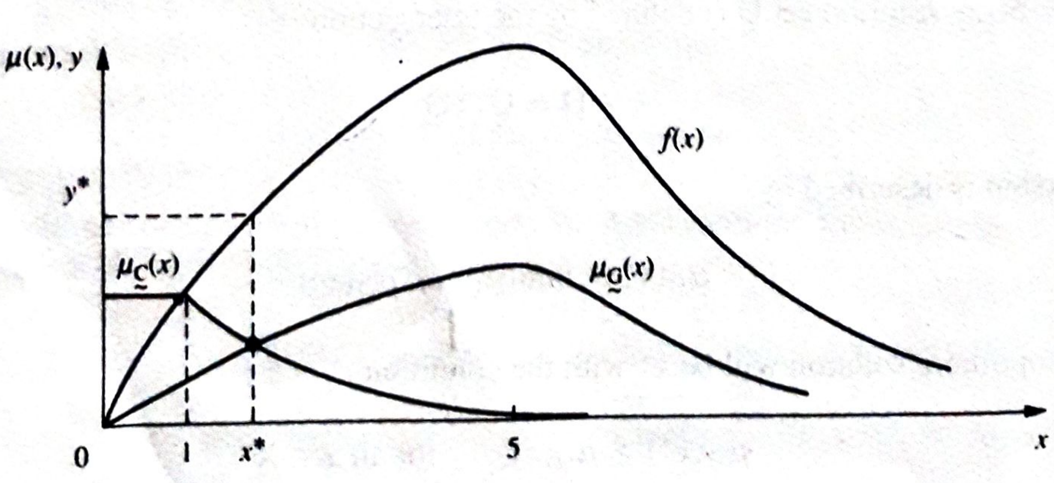

FIGURE 14.3
Problem domain for Example 14.1.