for Dirac basis(with + - - -):

$$\begin{pmatrix} p^0 - m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -p^0 - m \end{pmatrix}$$

mode:

$$\begin{pmatrix} p^0 + m \\ p\cdot\sigma \end{pmatrix}, \begin{pmatrix} -p\cdot\sigma \\ -p^0 + m \end{pmatrix}$$

it could be written as:

$$(\not{p} + m)\begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

for $\not{p} + m$:

$$\begin{pmatrix} p^0 + m & -p\cdot\sigma \\ p\cdot\sigma & -p^0 + m \end{pmatrix}$$

but, plz notice that, those solution will be linear dependent.

it's obvious when u change the frame to the rest frame, if $p^0$ is positive, then the $\not{p} + m$ just have one non-zero element:

$$\begin{pmatrix} 2m & 0 \\ 0 & 0 \end{pmatrix}$$

so it's some kind of projection operator,

for convention, if u decided the frequency to be positive, we pick up the two solution represented by $\phi$, if it's negative, we choose the $\chi$, and the basis is:

$$\begin{pmatrix} p\cdot\sigma \\ p^0 + m \end{pmatrix}$$

where all the p is positive.

for srenicki chiral basis(with - + + +):

$$\begin{pmatrix} m & -p_0 + \vec{p}\cdot\vec{\sigma} \\ -p_0 - \vec{p}\cdot\vec{\sigma} & m \end{pmatrix}$$

mode:

$$\begin{pmatrix} m \\ p_0 + p\cdot\sigma \end{pmatrix}, \begin{pmatrix} -p_0 + p\cdot\sigma \\ -m \end{pmatrix}$$

it could be expressed as:

$$\begin{pmatrix} -m & -p_0 + p\cdot\sigma \\ -p_0 - p\cdot\sigma & -m \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

which is just $\not{p} - m$

For dirac basis: plz pick solution

$$u = (\not{p} + m)\begin{pmatrix} \phi^{(\pm)} \\ 0 \end{pmatrix}, \ v = (-\not{p} + m)\begin{pmatrix} 0 \\ \phi^{(\pm)} \end{pmatrix}$$

with normalization $1/\sqrt{p^0+m}$ and $1/\sqrt{-p^0+m}$

for chiral basis: plz pick solution

$$u = (\not{p} - m)\begin{pmatrix} \phi^{(\pm)} \\ \phi^{(\pm)} \end{pmatrix},\ v = -(\not{p} + m)\begin{pmatrix} \pm\phi^{(\pm)} \\ \mp\phi^{(\pm)} \end{pmatrix} = (\not{p} + m)\begin{pmatrix} \pm\phi^{(\mp)} \\ \mp\phi^{(\mp)} \end{pmatrix}$$

with normalization $-1/\sqrt{m}$ and $1/\sqrt{m}$

derivation: 1. First we assume that we have eignstate with eignvalue p:

$$i\partial_\mu \psi = p_\mu \psi$$

Those eignstate would be a complete basis for solution, which is our assumption, and it's ok to be degenerate.

The Dirac equation $(i\not{\partial}_\mu - m)\psi = 0$ in Dirac basis have matrix form:

$$\begin{pmatrix} p^0 - m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -p^0 - m \end{pmatrix}_{4\times 4} \psi = 0$$

Notice that we could use the exact formula : $(\not{p}_\mu + m)(\not{p}_\mu - m)\psi = (p^2 - m^2)\psi = 0$ , which means the eignvalue p that satisfy the Dirac equation must satisfy KG condition $(p^2 - m^2 = 0)$ too.

Then, we conclude that: $(\not{p}_\mu - m)(\not{p}_\mu + m) = p^2 - m^2 = 0$ for every possible eignvalue p that represented a eignstate solution $\psi$.

Although

$$\not{p}_\mu + m = \begin{pmatrix} p^0 + m & -p\cdot\sigma \\ p\cdot\sigma & -p^0 + m \end{pmatrix}$$

matrix have determinant 0, which is obvious that in the rest framework with positive $p^0$, it's $\begin{pmatrix} 2m & 0 \\ 0 & 0 \end{pmatrix}$, and with negative $p^0$, it's $\begin{pmatrix} 0 & 0 \\ 0 & 2m \end{pmatrix}$, same for $\not{p}_\mu - m$, we could use the matrix $\not{p}_\mu + m$ to construct our solution.

For positive $p^0$,

obviously $\begin{pmatrix} p^0 + m \\ p\cdot\sigma \end{pmatrix}_{4\times 2} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} p^0 + m \\ p\cdot\sigma \end{pmatrix}_{4\times 2} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will be two linear independent solution. We take the notation $\begin{pmatrix} (p^0+m)_{2\times 2}\ \phi^{(\pm)} \\ p\cdot\sigma_{2\times 2}\ \phi^{(\pm)} \end{pmatrix}$.

Because the oder of matrix $\not{p}_\mu - m$ is 2 , obvious when we take the rest frame, that it's explictly, $\begin{pmatrix} 0 & 0 \\ 0 & -2m_{2\times 2} \end{pmatrix}$, and the dimension of spinor $\psi$ is 4, it's apparent that this is a complete basis.

We pick the normalized type of this solution as our positive frequency basis:

$$u = (\sqrt{p^0+m})^{-1}(\not{p} + m)\begin{pmatrix} \phi^{\pm} \\ 0 \end{pmatrix} \tag{15}$$

The dagger of u will become:

$$u^\dagger = (\sqrt{p^0+m})^{-1}\,(\phi^{\pm}, 0)\begin{pmatrix} p^0 + m & p\cdot\sigma \\ -p\cdot\sigma & -p^0 + m \end{pmatrix}$$

Thus,

$$\sum_{\pm} uu^\dagger = (p^0{+}m)^{-1} \begin{pmatrix} p^0+m & -p\cdot\sigma \\ p\cdot\sigma & -p^0+m \end{pmatrix} \sum_{\pm} \begin{pmatrix} \phi^\pm \\ 0 \end{pmatrix} (\phi^\pm, 0) \begin{pmatrix} p^0+m & p\cdot\sigma \\ -p\cdot\sigma & -p^0+m \end{pmatrix}$$

$$= (p^0+m)^{-1} \begin{pmatrix} p^0+m & -p\cdot\sigma \\ p\cdot\sigma & -p^0+m \end{pmatrix} \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p^0+m & p\cdot\sigma \\ -p\cdot\sigma & -p^0+m \end{pmatrix}$$

$$= (p^0+m)^{-1} \begin{pmatrix} p^0+m & -p\cdot\sigma \\ p\cdot\sigma & -p^0+m \end{pmatrix} \begin{pmatrix} p^0+m & p\cdot\sigma \\ 0 & 0 \end{pmatrix}$$

$$= (p^0+m)^{-1} \begin{pmatrix} (p^0+m)(p^0+m) & (p^0+m)p\cdot\sigma \\ (p^0+m)p\cdot\sigma & p\cdot\sigma\ p\cdot\sigma \end{pmatrix}$$

Notice that $(p\cdot\sigma)^2 = p^2 = (p^0)^2 - m^2 = (p^0+m)(p^0-m)$

$$\sum_{\pm} uu^\dagger = \begin{pmatrix} p^0+m & p\cdot\sigma \\ p\cdot\sigma & p^0-m \end{pmatrix} = (\not{p}+m)\gamma^0 \tag{16}$$

Where $\gamma^0 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}$

By the definition: $\bar{u} = u^\dagger\gamma^0$

We simply get:

$$\sum_{\pm} u\bar{u} = (\not{p}+m)\gamma^0\gamma^0 = \not{p}+m \tag{17}$$

2. Follow previous derivation, now we assume the eignvalue $p^0$ is negative, so the matrix $\not{p}_\mu - m$ is the transformation of $\begin{pmatrix} -2m_{2\times 2} & 0 \\ 0 & 0 \end{pmatrix}$.

We construct the solution basis from matrix $\not{p}_\mu + m$ too, but for convention, we take $k = -p$, and use k to represent solution.

$$\not{p}_\mu + m = -\not{k}_\mu + m = \begin{pmatrix} -k^0+m & k\cdot\sigma \\ -k\cdot\sigma & k^0+m \end{pmatrix}$$

We pick up the solution

$$\begin{pmatrix} (k\cdot\sigma)\chi^\pm \\ (k^0+m)\chi^\pm \end{pmatrix}$$

Where $\chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

It's obvious that they are linear independent, and because the order of $\not{k}_\mu + m$ is 2, and the dimension of spinor is 4, this basis is complete.

We take the notation:

$$v = (\sqrt{-p^0+m})^{-1}(-\not{p}+m) \begin{pmatrix} 0 \\ \chi^\pm \end{pmatrix} \tag{23}$$

The dagger of v will become:

$$v^\dagger = (\sqrt{p^0+m})^{-1}\left(0, \chi^\pm\right)\begin{pmatrix} -p^0+m & -p\cdot\sigma \\ p\cdot\sigma & p^0+m \end{pmatrix}$$

Thus,

$$\sum_\pm vv^\dagger = (p^0+m)^{-1}\begin{pmatrix} -p^0+m & p\cdot\sigma \\ -p\cdot\sigma & p^0+m \end{pmatrix}\sum_\pm \begin{pmatrix} 0 \\ \chi^\pm \end{pmatrix}\left(0, \chi^\pm\right)\begin{pmatrix} -p^0+m & -p\cdot\sigma \\ p\cdot\sigma & p^0+m \end{pmatrix}$$

$$= (p^0+m)^{-1}\begin{pmatrix} -p^0+m & p\cdot\sigma \\ -p\cdot\sigma & p^0+m \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1_{2\times 2} \end{pmatrix}\begin{pmatrix} -p^0+m & -p\cdot\sigma \\ p\cdot\sigma & p^0+m \end{pmatrix}$$

$$= (p^0+m)^{-1}\begin{pmatrix} -p^0+m & p\cdot\sigma \\ -p\cdot\sigma & p^0+m \end{pmatrix}\begin{pmatrix} 0 & 0 \\ p\cdot\sigma & p^0+m \end{pmatrix}$$

$$= (p^0+m)^{-1}\begin{pmatrix} (p\cdot\sigma)(p\cdot\sigma) & (p^0+m)(p\cdot\sigma) \\ (p^0+m)(p\cdot\sigma) & (p^0+m)^2 \end{pmatrix}$$

Notice that $(p\cdot\sigma)^2 = p^2 = (p^0)^2 - m^2 = (p^0+m)(p^0-m)$

$$\sum_\pm vv^\dagger = \begin{pmatrix} p^0-m & p\cdot\sigma \\ p\cdot\sigma & p^0+m \end{pmatrix} = -(-\not{p}+m)\gamma^0 \tag{24}$$

Where $\gamma^0 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}$

By the definition: $\bar{v} = v^\dagger\gamma^0$

We simply get:

$$\sum_\pm v\bar{v} = -(-\not{p}+m)\gamma^0\gamma^0 = -(-\not{p}+m) \tag{25}$$