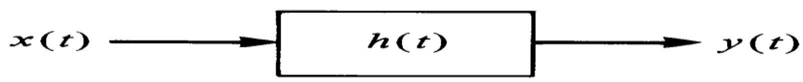
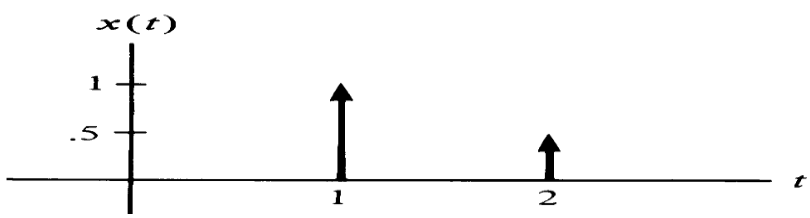
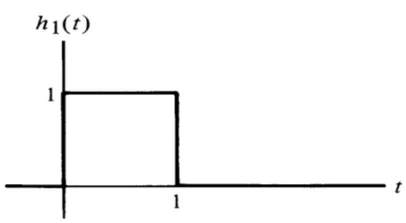
# ASSIGNMENT 5 SOLUTIONS

Q1) Consider the signal, x(t) = $\delta$(t-1) and 0.5 $\delta$(t-2). The signal x(t) is interpolated with different h(t) (h1(t), h2(t) and h3(t)) as shown below. Sketch y(t) for the following:

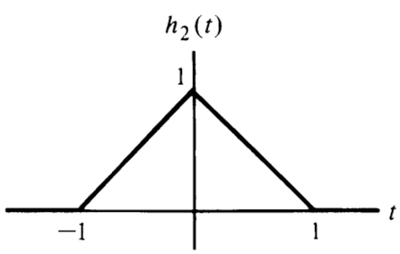



a)




b)

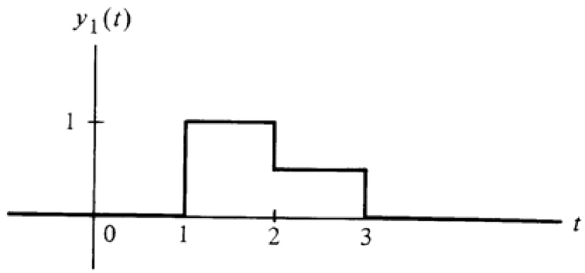



c)

$$h(t) = \frac{\sin(\pi t)}{\pi t}$$

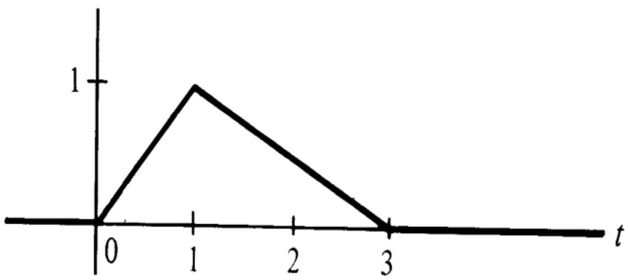
**Ans 1)**

a)



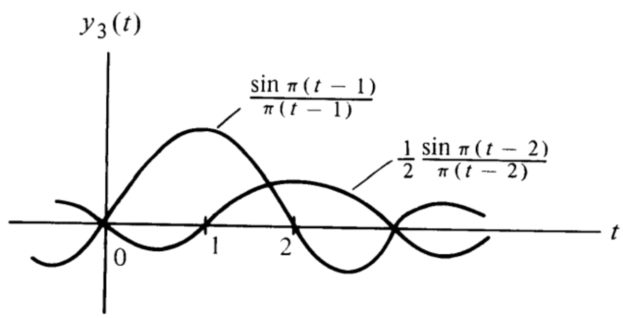
$y_1(t)$
1
0
1
2
3
t


b)



1
0
1
2
3
t


C)



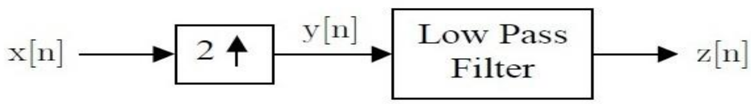
$y_3(t)$
$\frac{\sin \pi(t-1)}{\pi(t-1)}$
$\frac{1}{2}\frac{\sin \pi(t-2)}{\pi(t-2)}$
0
1
2
t

Q2) The following system shows an interpolator with discrete input x[n]. Assume that the low pass filter has frequency response $H(e^{j\omega})=2\mathrm{rect}(\omega/\pi)$ for $|\omega|<\pi$. Compute the z[n] for the following inputs.

a. $x[n]=\delta[n-1]$

b. $x[n]= 1$

c. $x[n]= \cos(n\pi/4)$




Ans 2)

a)

$$x[n] = \delta[n-1]$$
$$X(e^{jw}) = e^{-jw}\ (1) \ \text{ for } \ w \in (-\pi, \pi)$$
$$Y(e^{jw}) = X(e^{j2w}) = e^{-j2w}\ (1) \ \text{ for } \ w \in (-\pi, \pi)$$
$$Z(e^{jw}) = Y(e^{jw}) \times 2\mathrm{rect}(\frac{w}{\pi}) = e^{-j2w}\ (2\mathrm{rect}(\frac{w}{\pi})) \ \text{ for } \ w \in (-\pi, \pi)$$
$$z[n] = 2\,\frac{\sin(\frac{\pi}{2}(n-2))}{\pi(n-2)}$$

Note that, since it is up-sampled by a factor of 2, z[n] is shifted 2 to the right , even though the input x[n] is shifted 1 to the right .

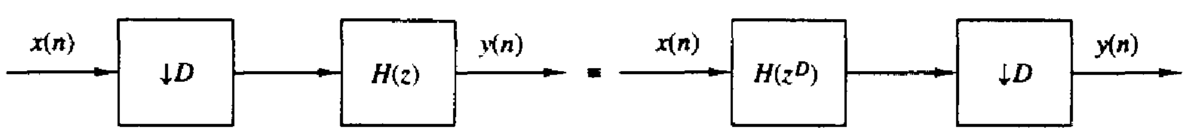
b)

$$x[n] = 1$$
$$X(e^{jw}) = 2\pi\delta(w) \ \text{for} \ w \in (-\pi, \pi)$$
$$Y(e^{jw}) = X(e^{j2w}) = 2\pi\delta(2w) = \pi\delta(w) \ \text{for} \ w \in (-\pi, \pi), \ \text{since} \ k\delta(kw) = \delta(w), \ \forall \ k \neq 0$$
$$Z(e^{jw}) = Y(e^{jw}) \times 2\mathrm{rect}(\frac{w}{\pi}) = \pi\delta(w)(2\mathrm{rect}(\frac{w}{\pi})) = 2\pi\delta(w) \ \text{for} \ w \in (-\pi, \pi)$$
$$z[n] = 1$$

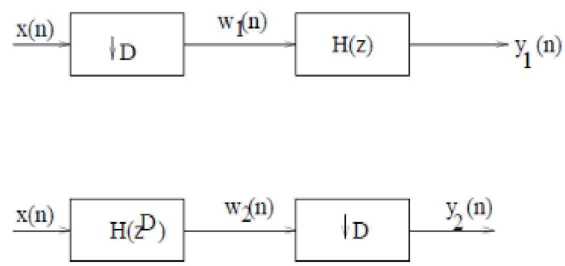
c)

$$x[n] = \cos(\frac{\pi}{4}n)$$
$$X(e^{jw}) = \pi[\delta(w - \frac{\pi}{4}) + \delta(w + \frac{\pi}{4})] \ \text{for} \ w \in (-\pi, \pi)$$
$$Y(e^{jw}) = X(e^{j2w}) = \pi[\delta(2w - \frac{\pi}{4}) + \delta(2w + \frac{\pi}{4})]$$
$$= \pi[\delta(2(w - \frac{\pi}{8})) + \delta(2(w + \frac{\pi}{8}))] = \frac{\pi}{2}[\delta(w - \frac{\pi}{8}) + \delta(w + \frac{\pi}{8})] \ \text{for} \ w \in (-\pi, \pi)$$
$$\text{since} \ k\delta(kw) = \delta(w), \ \forall \ k \neq 0$$
$$Z(e^{jw}) = Y(e^{jw}) \times 2\mathrm{rect}(\frac{w}{\pi}) = \frac{\pi}{2}[\delta(w - \frac{\pi}{8}) + \delta(w + \frac{\pi}{8})](2\mathrm{rect}(\frac{w}{\pi}))$$
$$= \pi[\delta(w - \frac{\pi}{8}) + \delta(w + \frac{\pi}{8})] \ \text{for} \ w \in (-\pi, \pi)$$
$$z[n] = \cos(\frac{\pi}{8}n)$$

Q3) Prove the equivalence of the two decimator configuration.



Ans 3)




$$
\begin{aligned}
y_1(n) &= h(n) * w_1(n) \\
&= h(n) * x(nD) \\
&= \sum_{k=0}^{\infty} h(k)x[(n-k)D] \\
H(z^D) &= \ldots h(0)z^0 + h(1)z^D + h(2)z^{2D} + \ldots \\
H(z^D) &\leftrightarrow \tilde{h}(n) \\
&= \left\{ h_0, \underbrace{0,\ldots,0}_{D-1}, H_1, \underbrace{0,\ldots,0}_{D-1}, h(2), \ldots \right\} \\
\text{so } w_2(n) &= \sum_{k=0}^{nD-1} \tilde{h}(k)x(n-k) \\
&= \sum_{k=0}^{n} \tilde{h}(kD)x(n-kD) \\
&= \sum_{k=0}^{n} h(k)x(n-kD) \\
y_2(n) &= w_2(nD) \\
&= \sum_{k=0}^{n} h(k)x(nD-kD) \\
&= \sum_{k=0}^{n} h(k)x[(n-k)D] \\
\text{So } y_1(n) &= y_2(n)
\end{aligned}
$$

Q4)

Consider an arbitrary digital filter with transfer function

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

**(a)** Perform a two-component polyphase decomposition of $H(z)$ by grouping the even-numbered samples $h_0(n) = h(2n)$ and the odd-numbered samples $h_1(n) =$

$h(2n+1)$. Thus show that $H(z)$ can be expressed as

$$H(z) = H_0(z^2) + z^{-1} H_1(z^2)$$

and determine $H_0(z)$ and $H_1(z)$.

**(b)** Generalize the result in part (a) by showing that $H(z)$ can be decomposed into an $D$-component polyphase filter structure with transfer function

$$H(z) = \sum_{k=0}^{D-1} z^{-k} H_k(z^D)$$

Determine $H_k(z)$.

Ans 4)

(a)

$$
\begin{aligned}
H(z) &= \sum_n h(2n) z^{-2n} + \sum_n h(2n+1) z^{-2n-1} \\
&= \sum_n h(2n)(z^2)^{-n} + z^{-1} \sum_n h(2n+1)(z^2)^{-n} \\
&= H_0(z^2) + z^{-1} H_1(z^2) \\
\text{Therefore } H_0(z) &= \sum_n h(2n) z^{-n} \\
H_1(z) &= \sum_n h(2n+1) z^{-n}
\end{aligned}
$$

(b)

$$
\begin{aligned}
H(z) &= \sum_n h(nD) z^{-nD} + \sum_n h(nD+1) z^{-nD-1} + \ldots \\
&\quad + \sum_n h(nD+D-1) z^{-nD-D+1} \\
&= \sum_{k=0}^{D-1} z^{-k} \sum_n h(nD+k)(z^D)^{-n} \\
\text{Therefore } H_k(z) &= \sum_n h(nD+k) z^{-n}
\end{aligned}
$$

Q5)

Let $x_c(t)$ be a real-valued continuous-time signal with highest frequency $2\pi(250)$ radians/second. Furthermore, let $y_c(t) = x_c(t - 1/1000)$.

**(a)** If $x[n] = x_c(n/500)$, is it theoretically possible to recover $x_c(t)$ from $x[n]$? Justify your answer.

**(b)** If $y[n] = y_c(n/500)$, is it theoretically possible to recover $y_c(t)$ from $y[n]$? Justify your answer.

**(c)** Is it possible to obtain $y[n]$ from $x[n]$ using the system in Figure 1? If so, determine $H_1(e^{j\omega})$.

**(d)** It is also possible to obtain $y[n]$ from $x[n]$ without any upsampling or downsampling using a single LTI system with frequency response $H_2(e^{j\omega})$. Determine $H_2(e^{j\omega})$.

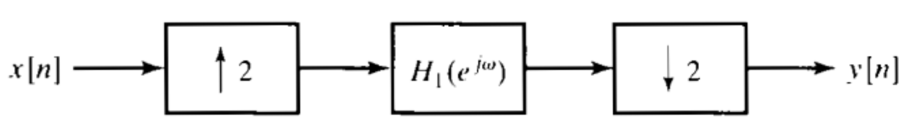



**Figure 1**

Ans 5)

A1. (a) The Nyquist criterion states that $x_c(t)$ can be recovered as long as

$$\frac{2\pi}{T} \geq 2 \times 2\pi(250) \Rightarrow T \leq \frac{1}{500}$$

In this case, $T = \frac{1}{500}$, So, Nyquist Criterion is satisfied and $x_c(t)$ can be recovered.

(b) Yes, A delay in time does not change the bandwidth of the signal. Hence $y_c(t)$ has the same bandwidth and same Nyquist sampling rate as $x_c(t)$.

(c) Consider the following expressions for $X(e^{j\omega})$ and $Y(e^{j\omega})$:

$$X(e^{j\omega}) = \frac{1}{T} X_c(j\Omega)\Big|_{\Omega = \frac{\omega}{T}} = \frac{1}{500} X_c(j\,500\,\omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} Y_c(j\Omega)\Big|_{\Omega = \frac{\omega}{T}}$$

$$= \frac{1}{T} e^{-j\frac{\Omega}{1000}} X_c(j\Omega)\Big|_{\Omega = \frac{\omega}{T}}$$

$$= \frac{1}{500} e^{-j\frac{\omega}{2}} X_c(j\,500\,\omega)$$

$$= e^{-\frac{j\omega}{2}} X(e^{j\omega})$$

Hence, we let $H(e^{j\omega}) = \begin{cases} 2e^{-j\omega}, & |\omega| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$

Then, in the following figure,

$$R(e^{j\omega}) = X(e^{j2\omega})$$

$$W(e^{j\omega}) = \begin{cases} 2e^{-j\omega} X(e^{j2\omega}), & |\omega| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = e^{-j\omega/2} X(e^{j\omega})$$

$$x[n] \longrightarrow \boxed{\uparrow 2} \xrightarrow{r[n]} \boxed{H} \xrightarrow{w[n]} \boxed{\downarrow 2} \longrightarrow y[n]$$

(d) Yes, from all the above analyses,

$$H_2(e^{j\omega}) = e^{-j\omega/2}$$

Q6)

Consider the system shown in Figure 2. For each of the following input signals $x[n]$, indicate whether the output $x_r[n] = x[n]$.

**(a)** $x[n] = \cos(\pi n/4)$

**(b)** $x[n] = \cos(\pi n/2)$

**(c)**

$$x[n] = \left[\frac{\sin(\pi n/8)}{\pi n}\right]^2$$

*Hint:* Use the modulation property of the Fourier transform to find $X(e^{j\omega})$.

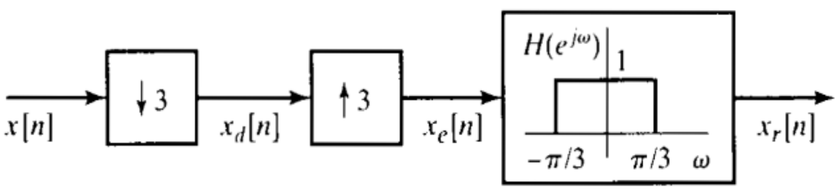



**Figure 2**

Ans 6)

**A2.** The output $x_r[n] = x[n]$ if no aliasing occurs as result of downsampling. That is,

$X(e^{j\omega}) = 0$ for $\pi/3 \leq |\omega| \leq \pi$

(a) $x[n] = \cos(\pi n/4)$.

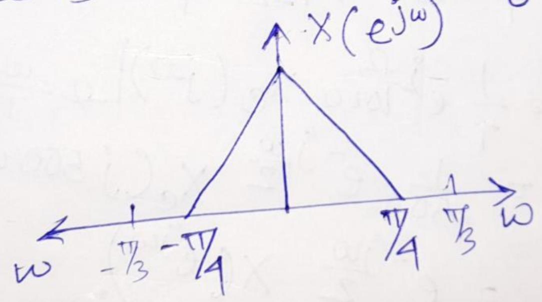
$X(e^{j\omega})$ has impulses at $\omega = \pm\pi/4$, so there is no aliasing. $x_r[n] = x[n]$.

(b) $x[n] = \cos(\pi n/2)$.

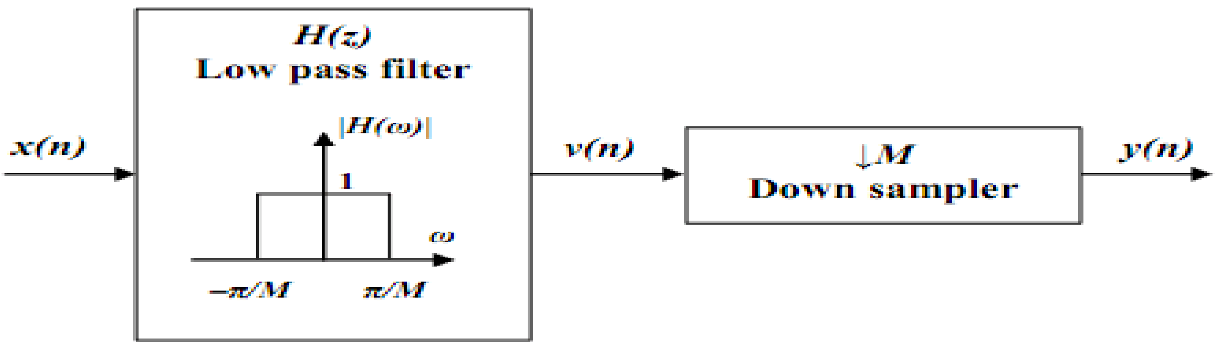
$X(e^{j\omega})$ has impulses at $\omega = \pm\pi/2$, so there is aliasing. $x_r[n] \neq x[n]$.

(c) A sketch of $X(e^{j\omega})$ is shown below.

There is also no aliasing and $x_r[n] = x[n]$.

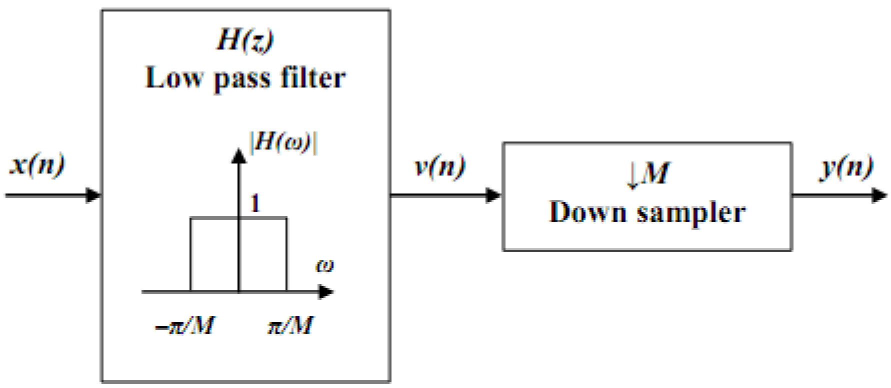



Q7) Which process has a block diagram as shown in the figure below? Give justification.




Ans 7)

The figure below shown represents Decimation process.




Explanation: The block diagram shown in the figure is of sampling rate conversion by decimation.