Assignment 3

04/02/2021

1. **Determine the coefficients of FIR filter using rectangular window with following**

   $$H_d(e^{jw}) = \begin{cases} e^{-j3w} & -\pi/4 \le w \le +\pi/4 \\ 0 & \pi/4 \le w \le \pi \end{cases}$$

   Length of the filter is 7.

Ans1. Taking, I.F.T of $H_d(e^{jw})$, $h_d(n) = \frac{1}{2\Pi}\int_{-\Pi}^{\Pi}\left(H_d(e^{j\omega})\cdot e^{j\omega n}\right)d\omega$

$$h_d(n) = \frac{\sin\frac{\pi(n-3)}{4}}{\pi(n-3)}$$

- Considering a rectangular window,

$$W_r(n) = \begin{cases} 1 & n=0 \text{ to } 6 \\ 0 & \text{otherwise.} \end{cases}$$

Response of filter is $h(n)=h_d(n).w(n)$.

To determine the coefficients, find the value of $h_d(n)$ for n in the range 0 to 6. (As $w_r(n)$ is 1).

The values are 0.075, 0.159, 0.225, 0.25, 0.225, 0.159, 0.075.

2. **Let $h_d[n]$, $-\infty<n<\infty$, denotes the impulse response samples of a zero-phase filter with frequency response $H_d(\exp\{j\omega\})$. It is known that the frequency response $H_t(\exp\{j\omega\})$ of the zero-phase FIR filter $h_t[n]$, $-M \le n \le M$, obtained by multiplying $h_d[n]$ with a rectangular window $w_R[n]$, $-M \le n \le M$, has the least integral-squared error Φ_R defined in the following equation.**

   $$\Phi_R = \sum_{n=-\infty}^{\infty}\left|h_t[n]-h_d[n]\right|^2$$

   Let Φ_Hann denote the integral-squared error if a length 2M+1 Hann window is used to develop FIR filter. Determine an expression for the excess error

   $$\Phi\_excess = \Phi\_R - \Phi\_Hann$$

Ans2.

$$\Phi_R = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_t\left(e^{j\omega}\right) - H_d\left(e^{j\omega}\right)\right|^2 d\omega\text{, where } H_t\left(e^{j\omega}\right) = \sum_{n=-M}^{M} h_t[n]e^{-j\omega n}.$$

Using Parseval's relation, we can write $\Phi_R = \sum_{n=-\infty}^{\infty}\left|h_t[n] - h_d[n]\right|^2$

$$= \sum_{n=-M}^{M}\left|h_t[n] - h_d[n]\right|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n].$$

Now, $\Phi_{Haan} = \sum_{n=-\infty}^{\infty}\left|h_d[n]\cdot w_{Hann}[n] - h_d[n]\right|^2$

$$= \sum_{n=-M}^{M}\left|h_d[n]\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi n}{2M+1}\right)\right) - h_d[n]\right|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]$$

Hence, $\Phi_{Excess} = \Phi_R - \Phi_{Haan}$

$$= \sum_{n=-M}^{M}\left|h_d[n]\cdot w_R[n] - h_d[n]\right|^2 - \sum_{n=-M}^{M}\left|h_d[n]\cdot\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi n}{2M+1}\right)\right) - h_d[n]\right|^2$$

$$= -\sum_{n=-M}^{M}\left|\frac{h_d[n]}{2}\cos\left(\frac{2\pi n}{2M+1}\right) - \frac{h_d[n]}{2}\right|^2 = \frac{1}{2}(1+2M)\left|\cos\left(\frac{2\pi M}{2M+1}\right) - 1\right|^2.$$

3. **Repeat above problem if a Hamming window is used instead.**

Ans3.

$$\Phi_R = \sum_{n=-\infty}^{\infty}\left|h_t[n] - h_d[n]\right|^2 \text{ and } \Phi_{Hamm} = \sum_{n=-\infty}^{\infty}\left|h_d[n]\cdot w_{Hamm}[n] - h_d[n]\right|^2.$$

$$\text{Therefore, } \Phi_{Excess} = \Phi_R - \Phi_{Hamm} = -\sum_{n=-M}^{M}\left|h_d[n]\left(0.46\cos\left(\frac{2\pi n}{2M+1}\right) - 0.46\right)\right|^2$$

$$= -\sum_{n=-M}^{M}\left|0.46 h_d[n]\left(\cos\left(\frac{2\pi n}{2M+1}\right) - 1\right)\right|^2 = 0.46(2M+1)\left|\cos\left(\frac{2\pi M}{2M+1}\right) - 1\right|^2.$$

4. **Determine the coefficients {h(n)} of a linear-phase FIR filter of length M =15 which has a symmetric unit sample response and a frequency response that satisfies the condition**

$$H_r\left(\frac{2\pi k}{15}\right) = \begin{cases} 1, & k = 0, 1, 2, 3 \\ 0 & k = 4, 5, 6, 7 \end{cases}$$

Ans4.

$$M = 15. H_r(\frac{2\pi k}{15}) = \begin{cases} 1, & k = 0,1,2,3 \\ 0, & k = 4,5,6,7 \end{cases}$$

$$\begin{aligned} H_r(w) &= h(\frac{M-1}{2}) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n) cosw(\frac{M-1}{2} - n) \\ h(n) &= h(M-1-n) \\ h(n) &= h(14-n) \\ H_r(w) &= h(7) + 2\sum_{n=0}^{6} h(n) cosw(7-n) \end{aligned}$$

Solving the above eqn yields,

$$\begin{aligned} h(n) = \{&0.3189, 0.0341, -0.1079, -0.0365, 0.0667, 0.0412, -0.0498, 0.4667 \\ &0.4667, -0.0498, 0.0412, 0.0667, -0.0365, -0.1079, 0.0341, 0.3189\} \end{aligned}$$

5. Design a 5-tap FIR band reject filter with a lower cutoff frequency of 2,000 Hz, an upper cutoff frequency of 2,400 Hz, and a sampling rate of 8,000 Hz using the Hamming window method. Determine the transfer function.

Ans5.

$2N + 1 = 5$ which leads that $N = 2$, $h(n)$ is given by

$$h(n) = \begin{cases} \dfrac{\pi - \omega_H + \omega_L}{\pi}, & n = 0 \\ \dfrac{-\sin(\omega_H n)}{\pi n} + \dfrac{\sin(\omega_L n)}{\pi n} & n \neq 0 \end{cases}$$

Then the five samples of $h(n)$ shifted by 2 is equal to

$$h(2) = 0.9,$$
$$h(1) = h(3) = 0.01558$$
$$h(0) = h(4) = 0.09355$$

For Hamming window with $M = 5$

$$w(n) = 0.54 - 0.46\cos\left(\frac{n\pi}{2}\right)$$

and for $n = 0, 1, 2, 3, 4$

$$w_{ham}(2) = 1,$$
$$w_{ham}(1) = w_{ham}(3) = 0.54,$$
$$w_{ham}(0) = w_{ham}(4) = 0.08,$$

Then the windowed impulse response coefficients are

$$h_w(2) = 0.9,$$
$$h_w(1) = h_w(3) = 0.00841$$
$$h_w(0) = h_w(4) = 0.00748$$

Convert to z-domain

$$H(z) = 0.00748 + 0.00841z^{-1} + 0.9z^{-2} + 0.00841z^{-3} + 0.00748z^{-4}$$

6. **What are the filter coefficients for a 3-tap FIR low pass filter with a cutoff frequency of 800 Hz and a sampling rate of 8,000 Hz using Hamming window? Also determine the transfer function and difference equation of the designed FIR system.**

Ans6.

$$\omega_c = 2\pi f_c T_s = 2\pi \times 800/8000 = 0.2\pi$$

The coefficients of the lowpass filter

$$h(0) = 0.2,\ h(-1) = h(1) = 0.1871$$

For Hamming window with M = 3

$$w(n) = 0.54 + 0.46\cos(n\pi)$$

and for n = 0 ,1, 2

$$w_{ham}(0) = 1,\ w_{ham}(1) = w_{ham}(2) = 0.08$$

Then the windowed impulse response coefficients are

$$h_w(0) = 0.2,\ h_w(1) = h_w(2) = 0.01497$$

convert to z-domain

$$H(z) = 0.01497 + 0.2z^{-1} + 0.1497z^{-2}$$

Then the difference equation is

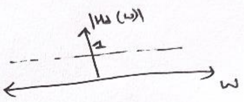
$$y(n) = 0.01497x(n) + 0.2x(n-1) + 0.01497x(n-2)$$

7. **A Hilbert Transform is a filter with frequency response $H_d(w)$=-j sign(w)**
   A] Roughly plot the magnitude and phase plot of the filter.
   B] Determine $h_d(n)$.

Ans7.

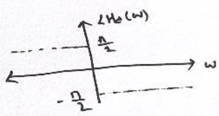
1. Given, $H_d(\omega) = -j\ \text{sign}(\omega)$

$\therefore\ |H_d(\omega)| = 1 \quad \forall\ \omega$



$\angle H_d(\omega) = -\frac{\pi}{2}$ if $\omega > 0$

$= \frac{\pi}{2}$ if $\omega < 0$



$$\rightarrow h_d(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} -j\ \text{sign}(\omega)\, e^{j\omega m}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} j e^{j\omega m}\, d\omega + \frac{1}{2\pi}\int_{0}^{\pi} -j e^{j\omega m}\, d\omega$$

$$= \frac{1}{2\pi} \times \frac{j}{jm}\left[e^{j\omega m}\right]_{-\pi}^{0} + \frac{1}{2\pi} \times \frac{-j}{jm}\left[e^{j\omega m}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi m}\left[1 - e^{-jm\pi}\right] + \frac{1}{2\pi m}\left[1 - e^{jm\pi}\right] = \frac{1}{2\pi m}\left[1 - e^{-jm\pi} + 1 - e^{jm\pi}\right] = \frac{1}{2\pi m} \times 2\left[1 - \cos m\pi\right]$$

$$= \frac{1}{m\pi} \times 2\sin^2\left(\frac{m\pi}{2}\right)$$

$$\boxed{\begin{aligned} h_d(m) &= \frac{1}{m\pi} \times 2\sin^2\left(\frac{m\pi}{2}\right) \quad \text{when } m \neq 0 \\ &= 0 \quad (m = 0) \end{aligned}}$$

8. **A Digital Filter is defined by the difference equation y[n] = 0.99*y[n-1] + x[n].**
   A] Determine the filter transfer function.
   B] What is filter impulse function?
   C] Is it LPF or HPF? (hint: check pole-zero plot)
   Ans8.

3. $y(n) = .99\, y(n-1) + x(n)$

$\therefore\ Y(z) = .99 z^{-1} Y(z) + X(z)$

or, $Y(z)(1 - .99 z^{-1}) = X(z)$

or, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - .99 z^{-1}}$

$\therefore\ \boxed{h(n) = (.99)^n u(n)}$ [imposing causality for practical implementation]

$\rightarrow \boxed{H(z) = \frac{1}{1 - .99 z^{-1}}}$

$= \frac{z}{z - .99}$

↳ Pole at $z = .99$

or, $z \approx 1$

↓

taking into Fourier domain by $z = e^{j\omega}$

↓

$\boxed{\text{Pole is close to } \omega = 0}$

↓

large values near low freauencies

↓

LPF