Solution of assignment 2

1.

$$1 \longleftrightarrow 2\pi \sum_{k} \delta(\omega+2\pi k)$$

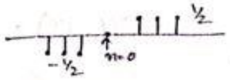
$$\delta[n] \longleftrightarrow 1$$

$$u_e[n] = \frac{1}{2} + \frac{1}{2}\delta[n]$$

$$U_e(e^{j\omega}) = \frac{1}{2}\left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega+2\pi k)\right] + \frac{1}{2}$$

$$\boxed{U_e(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega+2\pi k) + \frac{1}{2}}$$

$$u_o[n] = \frac{1}{2}\left[u[n] - u[-n]\right]$$



$$u_o[n] = u[n] - \frac{1}{2} - \frac{1}{2}\delta[n]$$

$$u_o[n-1] = u[n-1] - \frac{1}{2} - \frac{1}{2}\delta[n-1]$$

$$u_o[n] - u_o[n-1] = u[n] - u[n-1] - \frac{1}{2}\left[\delta[n] - \delta[n-1]\right]$$

$$= \delta[n] - \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$$

$$= \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$$

$$U_o(e^{j\omega})\left[1 - e^{-j\omega}\right] = \frac{1}{2} + \frac{1}{2}e^{-j\omega}$$

$$\boxed{U_o(e^{j\omega}) = \frac{1}{2}\frac{\left[1+e^{-j\omega}\right]}{1-e^{-j\omega}}}$$

$$U(e^{j\omega}) = \mathcal{F}\left[u[n]\right]$$

$$= \pi \sum_{k=-\infty}^{\infty} \delta(\omega+2\pi k) + \frac{1}{2}\left[1 + \frac{1+e^{-j\omega}}{1-e^{-j\omega}}\right]$$

$$\boxed{U(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega+2\pi k) + \frac{1}{1-e^{-j\omega}}}$$

2.

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - x[n-1]$$

After taking fourier transform on both sides,

$$Y[\omega]\left(1 + \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega}\right) = X[\omega]\left(1 + e^{-j\omega}\right)$$

System response,

$$h[n] \underset{F}{\Leftrightarrow} H[\omega] = \frac{Y[\omega]}{X[\omega]} = \frac{(1-e^{-j\omega})}{(1\div\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{2}{(1\div\frac{1}{2}e^{-j\omega})} - \frac{1}{(1-\frac{1}{4}e^{-j\omega})}$$

Now taking inverse DTFT,

$$h[n] = 2\left(\frac{-1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

At $\omega = 0$

$$H[0] = 0;$$

At $\omega = \pi/4$

$$H[\pi/4] = 0.65e^{j1.22};$$

At $\omega = -\pi/4$

$$H[-\pi/4] = H^*[\pi/4] = 0.65e^{-j1.22};$$

At $\omega = 9\pi/4$.

$$H[9\pi/4] = H\ \ [\pi/4] = 0.65e^{j1.22};$$

3.

. From Discrete Fourier transform, we have,

$$X(\omega) = \sum_n x(n)e^{-j\omega n}$$

$$X(0) = \sum_n x(n)$$

$$\left.\frac{d(X(\omega))}{d\omega}\right|_{\omega=0} = -j\sum_n nx(n)e^{-j\omega n}\Big|_{\omega=0}$$

$$= -j\sum_n nx(n)$$

Therefore,

$$c = \frac{j\left.\frac{dX(\omega)}{d\omega}\right|_{\omega=0}}{X(0)}$$

4.

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

$$y(n) = x^2(n)$$

$$= \left(\frac{1}{4}\right)^{2n} u^2(n)$$

$$= \left[\left(\frac{1}{4}\right)^2\right]^n u(n)$$

$$= \left(\frac{1}{16}\right)^n u(n)$$

$$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{16} e^{-j\omega}}$$

$$Y(e^{j0}) = \frac{1}{1 - \frac{1}{16}} = \frac{16}{15}$$

5.

(a) $X(0) = \sum_{n} x(n) = -3+4-5+4-3$

$= -3$

(b) $\int_{-\pi}^{\pi} X(\omega)\, d\omega = 2\pi\, x(0) = 2\pi \times -5$

$= -10\pi$

$\left[\because x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, d\omega \right]$

(c) $X(\pi) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\pi} = \sum_{n} (-1)^n x(n)$

$= -3-4-5-4-3 = -19$

(d) $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 2\pi \sum_{n} |x(n)|^2$

$= 2\pi\, (9+16+25+9+16)$

$= 2\pi\, (75) = 150\pi$

6. $$H(e^{jw}) = \frac{1}{1-\alpha e^{-jw}}$$

$$X(e^{jw}) = \frac{1}{1-\beta e^{-jw}}$$

$$Y(e^{jw}) = H(e^{jw})X(e^{jw}) = \frac{1}{(1-\alpha e^{-jw})(1-\beta e^{-jw})}$$

7. $$\begin{aligned} R(e^{j\omega}) &= \sum_{n=0}^{M} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} \\ &= e^{-j\frac{M}{2}\omega}\left(\frac{e^{j\frac{M+1}{2}\omega}-e^{-j\frac{M+1}{2}\omega}}{e^{j\omega}-e^{-j\omega}}\right) \\ &= e^{-j\frac{M}{2}\omega}\left(\frac{\sin(\frac{M+1}{2}\omega)}{\sin(\omega/2)}\right) \end{aligned}$$

8. Let $x[n]$ be real & have a Fourier transform $X(e^{j\omega})$. Find $y[n]$ such that

$$Y(e^{j\omega}) = X(e^{j3\omega}).$$

$$X(e^{j\omega 3}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j3n\omega}$$

we put $3n = r$

$$= \sum_{\substack{\cancel{-\infty} \\ r=0,\pm3,\pm6}}^{\infty} x[r/3]\, e^{-jr\omega}$$

$$X(e^{j\omega 3}) = \sum_{r=0,\pm3,\pm6} x\left[\frac{r}{3}\right] e^{-jr\omega}$$

$$\therefore \boxed{y[n] = \begin{cases} x\left[\frac{n}{3}\right] & , n = 0, \pm3, \pm6 \ldots \\ 0 & , \text{otherwise} \end{cases}}$$