Digital Signal Processing

Assignment-4 (Q&A)

1. What is the time domain signal for $X[z] = \frac{1}{1-\frac{1}{4}z^{-2}}, |z| > \frac{1}{4}$?

Ans.

1. $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ (if $|x| < 1$)

∴ Here, $\frac{1}{1-\frac{1}{4}z^{-2}} = 1 + \frac{1}{4}z^{-2} + \left(\frac{1}{4}z^{-2}\right)^2 + \cdots$ ($\because |z| > \frac{1}{4}$)

So, $X(z) = \sum_{k=0}^{\infty} \left(\frac{1}{4} z^{-2}\right)^k$

∴ $x(n) = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \delta(n-2k)$

$= \left(\frac{1}{4}\right)^{n/2}$ for n even and $n \geq 0$

$= 0$ for n odd

$= \begin{cases} 2^{-n} & \text{for } n \text{ even and } n \geq 0 \\ 0 & \text{for } n \text{ odd} \end{cases}$

2. Determine the z-transform and ROC of the following

$$x(n) = \begin{cases} (1/2)^n & n \geq 5 \\ 0 & n \leq 4 \end{cases}$$

Ans.

$$
\begin{aligned}
X(z) &= \sum_{n} x(n) z^{-n} \\
&= \sum_{n=5}^{\infty} (\frac{1}{2})^{n} z^{-n} \\
&= \sum_{n=5}^{\infty} (\frac{1}{2z})^{n} \\
&= \sum_{m=0}^{\infty} (\frac{1}{2} z^{-1})^{m+5} \\
&= (\frac{z^{-1}}{2})^{5} \frac{1}{1-\frac{1}{2}z^{-1}} \\
&= (\frac{1}{32}) \frac{z^{-5}}{1-\frac{1}{2}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{2}
\end{aligned}
$$

3. Find the inverse z-transform and ROC of the given expression:

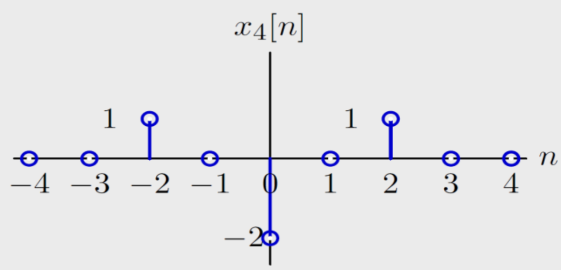
$$X(z) = \left(\frac{1-z^2}{z}\right)^2$$

Ans.

$$X_4(z) = \frac{1-2z^2+z^4}{z^2} = z^{-2} - 2 + z^2$$

This sum converges for all z. Therefore there is a single region of convergence $0 < z < \infty$.

$$x_4[n] = \delta[n-2] - 2\delta[n] + \delta[n+2]$$



4. Let, x[n]=δ[n-2]+δ[n+2].Find the unilateral z transform of x[n]?

Ans.

we need $X^{+}(z) = \sum_{m=0}^{\infty} x(m) z^{-m}$

$= \delta(m-2) z^{-m}$

$= z^{-2}$

5. Find the z-transform of x(n)= $(1/5)^n$ u(n-5)

Ans. We know Z.T. of $a^n u(n)$ is z/(z-a).

From the time shifting property of z-transform,
Z.T.[x(n-k)] is $z^{-k}X(z)$

Therefore, Z.T.[$\left(\frac{1}{5}\right)^{(n-5)} u(n-5)$] is $z^{-5} \cdot \frac{z}{z-\frac{1}{5}}$

Therefore, Z.T.[$\left(\frac{1}{5}\right)^{n} u(n-5)$]=Z.T.[0.2^5. $\left(\frac{1}{5}\right)^{(n-5)} u(n-5)$]= 0.2^5. $z^{-5} \cdot \frac{z}{z-\frac{1}{5}}$

6. Prove that Z.T. of x(n)=n is $z/(z-1)^2$
And Find

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

Ans.

We know that $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z[n] = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n} \quad = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^2} + \ldots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= \frac{1}{z}\left[\left(1-\frac{1}{z}\right)^{-2}\right] \quad \left[\because (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\right]$$

$$= \frac{1}{z}\left[\left(\frac{z-1}{z}\right)^{-2}\right] \quad = \frac{1}{z}\left[\frac{z}{z-1}\right]^2 \quad = \frac{z}{(z-1)^2}.$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]=Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$=Z^{-1}\left[\frac{z}{z-a}\right]*Z^{-1}\left[\cdot\frac{z}{z-b}\right]$$

$$=a^n*b^n$$

$$=\sum_{n=0}^{n}a^m b^{n-m}=b^n\sum_{m=0}^{n}\left(\frac{a}{b}\right)^m$$

$$=b^n\frac{\left(\frac{a}{b}\right)^{n+1}-1}{\frac{a}{b}-1}\text{being a G.P}$$

$$=\frac{a^{n+1}-b^{n+1}}{a-b}$$

7.

Consider $G(z)=\frac{P(z)}{Q(z)}$, let ρ_l be the residue of $G(z)$ at a simple pole $z=\lambda_l$, show that

$$\rho_l=-\lambda_l\frac{P(z)}{Q'(z)}\quad at\quad z=\lambda_l$$

where

$$Q'(z)=\frac{dQ(z)}{dz^{-1}}$$

Ans.

dz^{-1}

Soln

$$G(z) = \frac{P(z)}{(1-\lambda_\ell z^{-1})Q_1(z)}$$

$$\ell_\ell = (1-\lambda_\ell z^{-1})\,G(z)\Big|_{z=\lambda_\ell}$$

$$\ell_\ell = \frac{P(z)}{Q_1(z)}\Big|_{z=\lambda_\ell}$$

$$Q(z) = (1-\lambda_\ell z^{-1})\,Q_1(z)$$

$$Q'(z) \triangleq \frac{dQ(z)}{dz^{-1}} = -\lambda_\ell Q_1(z) + (1-\lambda_\ell z^{-1})\frac{dQ_1(z)}{dz^{-1}}$$

$$Q'(\lambda_\ell) = -\lambda_\ell\, Q_1(\lambda_\ell) + 0$$

$$\boxed{\ell_\ell = -\lambda_\ell\frac{P(\lambda_\ell)}{Q'(\lambda_\ell)} \cancel{-\lambda_\ell}} = -\lambda_\ell\frac{P(\lambda_\ell)}{Q'(\lambda_\ell)}$$

8.

Consider an arbitrary digital filter with transfer function

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

Perform a two-component polyphase decomposition of $H(z)$ by grouping the even-numbered samples $h_0(n) = h(2n)$ and the odd-numbered samples $h_1(n) = h(2n+1)$. Thus show that H(z) can be expressed as,

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2)$$

Ans.

$$H(z) = \sum_n h(2n)z^{-2n} + \sum_n h(2n+1)z^{-2n-1}$$

$$= \sum_n h(2n)(z^2)^{-n} + z^{-1}\sum_n h(2n+1)(z^2)^{-n}$$

$$\therefore H(z) = H_0(z^2) + z^{-1}H_1(z^2)$$

9.

Consider

$$G(z) = \frac{p_0 + p_1 z^{-1} .... + p_M z^{-M}}{d_0 + d_1 z^{-1} .... + d_N z^{-N}} \quad where \quad M < N$$

if $G(z)$ has only simple poles, show that

$$\frac{p_0}{d_0} = sum \quad of \quad the \quad residues \quad in \quad the \quad PFE \quad of \quad G(z)$$

Note: Partial Fraction Expansion(PFE)

Ans.

Sol) $G(z)$

$$G(z) = \sum_{i=1}^{N} \frac{A_i}{(1 - q_i z^{-1})} \rightarrow \because \text{ simple } \cancel{\text{pole}} \text{ poles}$$

$$\boxed{G(\infty) = \sum_{i=1}^{N} A_i = \frac{P_0}{d_0}}$$

sum of the residues.

if $M = N$

$$G(\infty) = \frac{P_N}{d_N} + \frac{P_0'}{d_0}, \quad \frac{P_0'}{d_0} = \sum_{i=1}^{N-1} A_i$$

if $M > N$

$$G(z) = (a_0 + b_0 z^{-1} + --) + \frac{n_0 + n_1 z^{-1} + ...}{d_0 + d_1 z^{-1} + --}$$

$$\text{Sum of residues} = \frac{n_0}{d_0}$$

10.

Determine the causal signal $x(n)$ having the z-transform

$$X(z) = \frac{1}{(1 - 2z^{-1})(1 - z^{-1})^2}$$

Ans.

$$\begin{aligned}
X(z) &= \frac{1}{(1-2z^{-1})(1-z^{-1})^2} \\
&= \frac{A}{(1-2z^{-1})} + \frac{B}{(1-z^{-1})} + \frac{Cz^{-1}}{(1-z^{-1})^2} \\
A &= 4, B = -3, C = -1 \\
\text{Hence,} x(n) &= \left[4(2)^n - 3 - n\right] u(n)
\end{aligned}$$