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Experiment No. : 05

Power Spectrum Estimation

Generation of the random sequence $x(n)$ of a known PSD:

Step: 1. Generate a white Gaussian random sequence $r_w(n)$ of zero mean and variance σ_r^2.

2. Pass the sequence through a digital filter

$$H(z)=\frac{1}{(1-0.9z^{-1})(1-0.9jz^{-1})(1+0.9jz^{-1})}$$ to create an output sequence $x(n)$.

Now, the PSD of the sequence $x(n)$ is to be estimated from a finite length of the sequence $x(n)$ (say, N = 128).

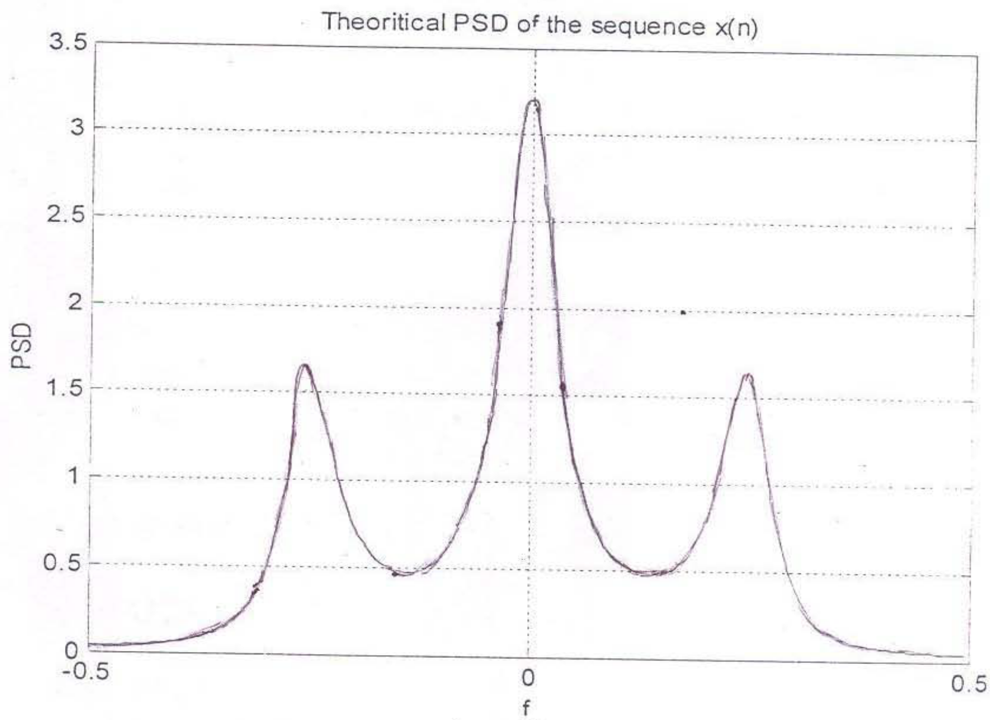

Fig.1 $\left|H\left(e^{j2\pi f}\right)\right|^2\sigma_r^2$

A. The Welch Nonparametric method: Averaging Modified Periodogram

Theory: The PSD of a random sequence $x(n)$ is defined as,

$$P_{xx}(f)=\lim_{M\to\infty}\left[\frac{1}{2M+1}E\left\{\left|\sum_{n=-M}^{M}x(n)e^{-j2\pi fn}\right|^2\right\}\right]$$

The PSD $P_{xx}(f)$ is estimated in classical *periodogram* method as,

$$\hat{P}_{xx}(f)=\frac{1}{M}\left|\sum_{n=0}^{M-1}x(n)e^{-j2\pi fn}\right|^2$$

It can be shown that the *periodogram* estimate is asymptotically unbiased however its variance does *not* decay to zero as $M\to\infty$. This drawback can be recovered by segmenting the data $x(n)$ and calculating the *average* of the *periodogram* of each segment. The Welch method introduces two further refinements namely, data windowing and overlapping data segments as described below.

Step:

1. Divide $x(n)$ into L (Typically, L = 8) overlapping blocks, each block of length M with D samples common between two successive blocks as,

   $$x_i(n) = x(n+iD) \quad n = 0,1,\ldots,M-1$$
   $$i = 0,1,\ldots,L-1$$

   For the sake of simplicity you may however take $D = 0$ first, which means no overlapping. Later, you can observe the effect of overlapping by choosing different values of D (Typically, 50% overlapping is used).
2. Obtain the estimated PSD of block i as,

   $$\hat{P}_{xx}^{(i)}(f) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} x_i(n)w(n)e^{-j2\pi fn}\right|^2 \quad i = 0,1,\ldots,L-1$$

   where $w(n)$ is the window function of length M (usually, a Hamming window) and U is a normalization factor for the power in the window function defined as,

   $$U = \frac{1}{M}\sum_{n=0}^{M-1} w^2(n)$$

- Note that the DTFT at step 4 is to be computed by FFT algorithm and can be obtained only at discrete frequencies. You may zero pad each block to N_0 .In that case the frequencies at which the PSD is obtained is $f = k/N_0$ $k = 0,1,\ldots,(N_0-1)$.

3. Obtain Welch spectrum estimate by the average of these modified periodogram, that is,

   $$P_{xx}^{W}(f) = \frac{1}{L}\sum_{i=0}^{L-1} \tilde{P}_{xx}^{(i)}(f)$$

4. Plot the estimated PSD and compare it with the known PSD $\left|H\left(e^{j2\pi f}\right)\right|^2 \sigma_r^2$ (fig.1).
5. Try with different transfer function $H(z)$.

B. Parametric methods for Power Spectrum Estimation: The Yule-Walker AR model

Theory: The modern parametric method for PSD estimation exploits the prior knowledge available regarding the source of data generation. It assumes a model for that source and estimates the parameters of that particular model. No windowing occurs in this method. This is particularly useful when only a short data segment is available. The simplest model that is used in practice is a pth-order AR model described by the transfer function

$$H(z) = \frac{1}{1+\sum_{k=1}^{p} a_k z^{-k}}$$

The data sequence is assumed to be generated by passing a zero mean white random sequence to the above filter. The procedure to estimate the AR-model parameters is described below.

Step.

1. Obtain the autocorrelation estimate of the sequence $x(n)$ given by,

$$r_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-m-1} x(n)x(n+m) \qquad m \geq 0$$

2. Find out the estimated AR model parameters $a_1, a_2, \ldots, a_p$ as,

$$\begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(p-1) \\ r_{xx}(1) & r_{xx}(0) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ r_{xx}(p-1) & \cdots & \cdots & r_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_p \end{bmatrix} = -\begin{bmatrix} r_{xx}(1) \\ r_{xx}(2) \\ \cdot \\ r_{xx}(p) \end{bmatrix}$$

3. Obtain the estimated variance $\hat{\sigma}_{rp}^2$ as,

$$\hat{\sigma}_{rp}^2 = r_{xx}(0) + \sum_{k=1}^{p} a_k r_{xx}(k)$$

4. Construct the estimated AR model $\hat{H}(z)$ by,

$$\hat{H}(z) = \frac{1}{1 + \sum_{k=1}^{p} a_k z^{-k}}$$

5. The estimated PSD $P_{xx}^{YW}(f)$ by Yule-Walker method is given by,

$$P_{xx}^{YW}(f) = \left|\hat{H}\left(e^{j2\pi f}\right)\right|^2$$

$$= \frac{\hat{\sigma}_{rp}^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi fk}\right|^2}$$

6. Plot the estimated PSD and compare it with the known PSD $\left|H\left(e^{j2\pi f}\right)\right|^2 \sigma_r^2$ (Fig.1).

- MATLAB functions that you may require: `fft,filter,xcorr,freqz`
- **Don't use the functions:** **`periodogram,pwelch,pyulear,aryule`**