Saddlepoint approximation of the Rough Heston model

Master's Thesis submitted to

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Contents

Zusammenfassung

In dieser Masterarbeit führen wir die asymptotische Analyse von dem Rough Heston Modell durch. Wir bedienen uns der Tatsache, dass die charakteristische Funktion im Rough Heston Modell aus der Lösung einer Volterra-Integralgleichung gewonnen werden kann. Wir untersuchen die asymptotische Entwicklung der Lösung der genannten Integralgleichung, um das asymptotische Verhalten der Randdichte für extreme Werte des Preises herzuleiten. Darauffolgend nutzen wir die Asymptotik der Randdichte, um auf das Verhalten der Preise europäischer Call-Optionen und der implizierten Black–Scholes Volatität für extreme Werte von Strike zu schließen. Das resultierende Verhalten zeigt eine Verbesserung der Näherung der implizierten Volatilität für extreme Werte von Strike. Das resultierende Verhalten zeigt eine Verbesserung der Näherung der implizierten Volatilität für extreme Werte vom Strike.

1 Introduction

In the past decade the financial industry underwent a major paradigm shift. In response to the subprime mortgage crisis of the late 2000s and resulting regulatory constraints, focus of the researchers and practitioners has moved from the asset pricing perspective to the issues concerned with risk management [BW14]. This developement has increased the demand for more accurate models. In this regard we consider the novel Rough Heston model introduced in [ER16]. The Rough Heston model is the "rough" modification of the classical Heston model, meaning that the variance process is driven by the fractional Brownian motion with Hurst index $H < \frac{1}{2}$. The model features realistic asset price dynamics and allows for refined modelling of the volatility smile.

In this thesis we set out the objective to derive a refined asymptotic expansion of the marginal density $D_T(x)$ in the Rough Heston model. We study the asymptotic behavior of $D_T(x)$ at extreme values of the spot ($\log x \to \pm\infty$). Subsequently, we aim to translate the results onto the wing behavior of other model outputs of interest, such as call prices and implied volatility surface.

We largely follow the approach from [Fri+10], where the density asymptotics has been studied in the case of the classical Heston model. We briefly outline the method employed in [Fri+10]. In the classical Heston model logarithm of the moment-generating function of the log-spot $m(s,T) = \log \mathbb{E}[e^{s \log S_T}]$ for a fixed moment $s \in \mathbb{R}$ can be obtained from the solution of an ordinary differential equation of Riccati type. Moreover, for a particular subset of the parameter space of the classical Heston model the function $m(s,T)$ explodes in finite time. Thus one can define critical moments

$$\begin{aligned} s_+(T) &= \sup \left\{ u \in \mathbb{R} : \mathbb{E}\left[e^{u \log S_T}\right] < \infty \right\} < +\infty \\ s_-(T) &= \inf \left\{ u \in \mathbb{R} : \mathbb{E}\left[e^{u \log S_T}\right] < \infty \right\} > -\infty. \end{aligned} \tag{1.1}$$

Based on the above, the asymptotic expansion of $m(s,T)$ near the blow-up is obtained in [Fri+10] by applying ODE techniques to the Riccati equation. Thereafter, one recovers asymptotic expansion of the density with the saddlepoint method applied to the Mellin transform of $m(s,T)$.

At the core of the analysis is finitude of the moment explosion times (1.1) in the classical Heston model. Moreover, authors in [Fri+10] point out that the analysis is solely based on affine principles associated with the Heston model and thus lends itself to other models which exhibit similar structure. Observing that the Rough Heston model exhibits affine structure akin to the classical Heston model, it seems natural to generalize the ansatz to the Rough Heston model, which incorporates the classical Heston model as a special case.

However, analysis of the Rough Heston model poses technical challenges. The moment-generating function in the Rough Heston model is characterized by a solution of a *fractional* Riccati equation, as opposed to the *ordinary* differential equation in the classical case. One of the limitations of the fractional calculus is absence of an elementary chain rule. It renders ODE techniques from [Fri+10] not applicable and raises demand for a different asymptotic expansion approach. Thus we incorporate an ansatz inspired by [RO96] which entails passage to the Volterra integral equation. One then recovers asymptotic expansion of the solution near the blow-up from poles in the Mellin domain. Noting from [GGP18] that finite moments in the Rough Heston model behave in a way similar to the classical Heston case, we then implement the saddlepoint method for the Rough Heston density along the lines of [Fri+10].

The rest of the thesis is structured as follows. In Section 2 we briefly summarize some mathematical tools which will prove useful in our analysis. In Section 3 we introduce the Rough Heston model and outline its defining features. In Section 4 we derive asymptotics of the Rough Heston density. In Section 5 we apply the asymptotic expansion of the density to the call price and implied volatility asymptotics. In Section 6 we showcase the results with a numerical illustration. In Section 7 we state concluding remarks.

2 Preliminaries

In this section we overview some mathematical tools which we will utilize in what follows. We outline basics of the fractional calculus, the Parseval formula for the Mellin transforms, the Cauchy integral formula and relations which hold for the Gaussian hypergeometric function. Other structures and methods not revisited in this section will be defined at the place of their direct application.

2.1 Fractional calculus

A key step in the passage from the classical Heston model to the Rough Heston model is substitution of the classical derivative operator in the Riccati equation with the fractional one. In this regard we review some fundamentals of fractional calculus.

Definition 2.1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function defined on the positive real numbers. The *Riemann–Liouville fractional integral operator* $I^\alpha f(t)$ and *fractional derivative operator* $D^\alpha f(t)$ of order $\alpha \in (0, 1]$ are given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_0^t (t-s)^{-\alpha} f(s) ds.$$

As manifested in the following identities, fractional integration and fractional differentiation act in a predictable way on the power function.

Proposition 2.2. *For $\alpha \in (0, 1]$ it holds*

$$\begin{aligned} I^\alpha t^\nu &= t^{\nu+\alpha} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} \qquad &&\textit{for } \nu > -1 \\ D^\alpha t^\nu &= t^{\nu-\alpha} \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} \qquad &&\textit{for } \nu > -1+\alpha \end{aligned} \tag{2.1}$$

In classical calculus one way to look at the exponential function $f(x) = C \exp(ax)$ with $a, C \in \mathbb{R}$ is by saying that it is the function whose derivative is proportional to the function itself. The fractional equivalent of this statement involves special functions called Mittag–Leffler functions.

Definition 2.3. For $z \in \mathbb{C}$ and parameters $\alpha, \beta \in \mathbb{C}$ the Mittag–Leffler function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

Note that $E_{1,1} \equiv \exp$.

Finally, we present the variation of the constant formula for the fractional derivative operator. The proof of the following proposition can be found in [Pod98].

Proposition 2.4. *Let h be a continuous function, $\alpha \in (0, 1]$ and $\gamma \in \mathbb{R}$. Then the initial value problem*

$$D^\alpha C(t) = \gamma C(t) + h(t)$$

$$I^{1-\alpha} C(0) = 0$$

has a unique continuous solution

$$C(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\gamma(t-s)^\alpha\right) h(s) ds.$$

Additionally, for constant $h(s) \equiv \theta$, $\gamma < 0$ and $\alpha \in [0, 1]$ the solution satisfies

$$0 \leq I^\alpha C(t) \leq K$$

for some positive constant K.

2.2 Mellin transform

In our analysis we will encounter the Mellin transform in two contexts. First, we will apply the Parseval formula for the Mellin transforms to the fractional Riccati equation in order to analyze its blow-up behavior. Subsequently, we will recover the Rough Heston density from the moment-generating function of the log-spot via the Mellin inversion formula. We start with the definition of the Mellin transform.

Definition 2.5. For a complex number $z \in \mathbb{C}$ the *Mellin transform* of a function $f : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$M[f(x); z] := \int_0^\infty x^{z-1} f(x) dx. \tag{2.2}$$

It is apparent from the definition of the Mellin transform that it generally does not exist for all values of $z \in \mathbb{C}$. This motivates the following definition.

Definition 2.6. The open strip $\{z \in \mathbb{C} : \alpha < \Re(z) < \beta\}$ where

$$\alpha = \inf\{u \in \mathbb{R} : f = O(t^{-u}) \text{ as } t \to 0\}$$
$$\beta = \sup\{u \in \mathbb{R} : f = O(t^{-u}) \text{ as } t \to +\infty\}$$

is called the *fundamental strip* or *strip of analyticity* of $M[f(x); z]$.

The definition implies that strip of analyticity of the Mellin transform $M[f(x); z]$ is characterized by the asymptotic behavior of the function f at the limits $t \to 0$ and $t \to +\infty$. Another simple consequence from the definition of the Mellin transform is its scale invariance.

Proposition 2.7. *For $\eta \in \mathbb{R}$ the Mellin transform satisfies*

$$M[f(\eta x); z] = \eta^{-z} M[f(x); z]. \tag{2.3}$$

We next introduce the Mellin inversion formula.

Proposition 2.8. *Consider the Mellin transform $M[f(x); z]$ of a piecewise continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ with the fundamental strip $\alpha < \Re(z) < \beta$. Assume it additionally satisfies*

- *$M[f(x); z]$ is absolutely convergent in the fundamental strip;*

- *For all $\alpha < \Re(z) < \beta$ the Mellin transform $M[f(x); z]$ converges uniformly to zero as $\Im(z) \to \pm\infty$.*

Then for any $\gamma \in \mathbb{R}$ with $\alpha < \gamma < \beta$ the Mellin inversion formula *holds*

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} M[f(\cdot); z] x^{-z} \mathrm{d}z. \tag{2.4}$$

In conclusion, we state the Parseval formula for the Mellin transforms reminiscent of the Plancherel theorem for the Fourier transforms.

Proposition 2.9. *Assume that $f, g : \mathbb{R}_+ \to \mathbb{R}$ are functions such that their Mellin transforms exist and*

$$\int_0^\infty f(t) g(t) dt < \infty.$$

For $\gamma \in \mathbb{R}$ lying in the fundamental strip of both Mellin transform $M[g(\cdot); z]$ and $M[f(\cdot); 1-z]$ the Parseval formula *for Mellin transforms reads*

$$\int_0^\infty f(t) g(t) dt = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} M[g(\cdot); z] M[f(\cdot); 1-z] dz. \tag{2.5}$$

The Mellin inversion formula (2.4) requires contour integration in the complex plane. Shifting the integration contour in the complex plane might entail computation of the contour integral around the poles. This can be done with the Cauchy integral formula.

Proposition 2.10. *Let f be a holomorphic function and $a \in \mathbb{C}$. Consider a contour γ which encloses a. Then the* Cauchy integral formula *holds*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}} dz. \tag{2.6}$$

2.3 Gaussian hypergeometric function

We cite some results on the Gaussian hypergeometric function ${}_2F_1$ from [Olv+10], which will arise in the analysis of the fractional integral. We start with the integral definition of the function ${}_2F_1$.

Definition 2.11. Let $a, b, c, z \in \mathbb{C}$ such that in addition $\Re(c) > \Re(b) > 0$ and denote by B the Euler beta function. The *Gaussian hypergeometric function* ${}_2F_1$ is given by

$$\frac{1}{\mathrm{B}(b, c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx = {}_2F_1(a, b; c; z). \tag{2.7}$$

Alternatively the function ${}_2F_1$ can be given in terms of the power series.

Proposition 2.12. *The power series for the function* ${}_2F_1$ *reads*

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\ (q)_n &= \begin{cases} 1 & n = 0 \\ q(q+1)\cdots(q+n-1) & n > 0 \end{cases} \end{aligned} \tag{2.8}$$

We will also make use of th2e following relation for the function ${}_2F_1$.

Proposition 2.13. *The hypergeometric function* ${}_2F_1$ *satisfies the relation*

$$\begin{aligned} {}_2F_1(a, b, c; z) = {} & \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-z) \\ & + \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z). \end{aligned} \tag{2.9}$$

3 Rough Heston model

The celebrated Black–Scholes model is the simplest asset pricing model that laid the foundation for option pricing theory [BS73]. In the Black–Scholes model the spot process S_t under the risk-neutral measure follows log-normal dynamics

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t, \tag{3.1}$$

where r is the risk-free rate, σ is the annualized constant volatility of the returns and W_t is the standard Brownian motion. Despite its omnipresence in the financial world, the Black–Scholes model is not consistent with the market. Contrary to the assumptions of the Black–Scholes model, volatility surfaces implied from the market prices of liquid vanilla instruments are not flat and exhibit volatility smile [DK94].

In order to capture the volatility smile one can expand the Black–Scholes model to general Markovian setting and introduce dependence of the volatility σ in (3.1) on time t and the asset price process S_t. This leads to the class of *local volatility models* introduced in [Dup94], where the volatility process reads $\sigma(t, S_t)$. While such models manage to accommodate the entire volatility smile and are actively used in the valuations of exotic derivatives, dynamic evolution of the market smile in local volatility models is inconsistent with the market. When the price of the spot increases, local volatility models predict that the market smile shifts to lower prices and vice versa [Hag+02]. This contradicts the typical market behavior: the smiles move in the same direction as the underlying. This discord between the market and the model leads to incorrect hedges derived from the local volatility models.

In addition, empirical studies of asset price dynamics in the past decades accumulated statistical evidence for a number of universal properties associated with asset returns that used to be part of the market folklore [Con01]. These properties are commonly referred to as "stylized facts" and are expected to be reproduced by any reasonable asset pricing model. Some of the most prominent of them are

(i) *Fat tails.* The tails of the marginal returns distribution are heavier when compared to normal distribution [Man67].

(ii) *Volatility clustering.* Tendency of large price fluctuations to be followed by large price fluctuations of either sign [Man67].

(iii) *Leverage effect.* Negative relation between the movements of volatility and spot [Chr82].

(iv) *Time-varying volatility* [Con01].

A natural way to accommodate the above features while retaining Markoviantity is equipment of the model with another stochastic factor. Models in which the volatility process σ_t admits its own source of randomness form a class of *stochastic volatility models*. One of the most popular stochastic volatility models is the renowned Heston stochastic volatility model [Hes93]. The Heston model is a stochastic volatility model where the variance process follows mean-reverting CIR process. Specifically, dynamics in the Heston model is given by

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dW_t \\ dV_t &= \lambda\left(\bar{v} - V_t\right)dt + \xi\sqrt{V_t}dB_t \\ \langle dW_t, dB_t\rangle &= \rho dt. \end{aligned} \tag{3.2}$$

Table 1 summarizes parametrization of the Heston model (3.2).

Parameter	Interpretation	Constraint
v_0	initial variance	$v_0 > 0$
$\bar{v}$	long-term mean variance	$\bar{v} > 0$
λ	mean-reversion speed	$\lambda > 0$
ξ	volatility of volatility	$\xi > 0$
ρ	spot-variance correlation	$-1 \leq \rho \leq 1$

Table 1: Parameters of the Heston model.

A defining feature of the Heston model is its *affine structure*: in the original paper [Hes93] the moment-generating function is obtained from (3.2) by considering affine ansatz for the Itô differential of $\mathbb{E}\left[e^{s\log S_T}|\mathcal{F}_t\right]$ for the filtration $\mathcal{F}_t = \sigma\left(W_s, B_s; s \le t\right)$. Ultimately, one obtains the moment-generating function $\mathbb{E}\left[e^{s\log S_T}\right]$ from solution ψ of the ordinary differential equation with boundary condition

$$m(s,t) = \log \mathbb{E}\left[e^{s\log S_t}\right] = \bar{v}\lambda I^1\psi(s,t) + v_0\psi(s,t) \tag{3.3}$$

$$\begin{aligned}
\dot{\psi}(s,t) &= R(s,\psi(s,t)) \\
\psi(\cdot,0) &= 0 \\
R(s,w) &= \frac{1}{2}\left(s^2 - s\right) + (\rho\xi s - \lambda)w + \frac{1}{2}\xi^2 w^2.
\end{aligned} \tag{3.4}$$

There are several reasons why the Heston model has become a mainstay of the financial industry:

- *Conformance with the stylized facts from low-frequency price data.* The Heston model exhibits leverage effect, time-varying volatility and fat tails [DY02].
- *Reasonable shapes for the generated implied volatility surfaces.* Volatility of volatility ξ controls convexity of the smile, spot-variance correlation ρ controls the skewness, baseline variance level v_0 fixes at-the-money volatility level [FJL12].
- *Consistent dynamics of the implied volatility.* Volatility smile moves in the same direction as the asset price [Gat12].
- *Tractability of the model.* The ordinary differential equation of Riccati type (3.4) is explicitly solvable. This leads to explicit formula for the characteristic function [Hes93]. The closed-form characteristic function in turn induces explicit pricing formulae with transform methods and instantaneous model calibration with fast Fourier transform algorithm à la Carr–Madan [Lee04b; CM01].

However, the Heston model, along with other classical stochastic volatility models driven by the standard Brownian motion, fails to capture risk of large price movements for short maturities [Jab18]. Thus the generated European option prices are not consistent with observed market prices. This can be remediated by considering the novel class of "rough" volatility models, where the volatility is driven by the *fractional* Brownian motion.

3.1 Rough Heston model

The key ingridient in rough volatility models is fractional Brownian motion which drives the volatility process. We briefly introduce the fractional Brownian motion and address without proof regularity of its paths. In the following we work on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, \mathbb{F})$ large enough to support the model in consideration.

Definition 3.1. *Fractional Brownian motion* W_t^H with Hurst parameter $H \in (0,1)$ is a stochastic process with the following features:

- *Self-similarity.* For $a \in \mathbb{R}_+$ holds $\left(W_{at}^H\right) \stackrel{law}{=} a^H \left(W_t^H\right)$.
- *Stationary increments.* $\left(W_{t+h}^H - W_t^H\right) \stackrel{law}{=} \left(W_h^H\right)$.
- Marginal distribution of W_1^H is a Gaussian random variable with $E\left[W_1^H\right] = 0$ and $E\left[(W_1^H)^2\right] = 1$.

Let W_t be the standard Brownian motion and Γ the gamma function. The *Mandelbrot–van Ness representation* of the fractional Brownian motion is given by

$$W_t^H = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{0} \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right) dW_s + \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-\frac{1}{2}} dW_s.$$

The *Riemann–Liouville representation* of the fractional Brownian motion is given by

$$W_t^H = \sqrt{2H} \int_0^t (t-s)^{H-\frac{1}{2}} dW_t.$$

Remark 3.2. For all $\varepsilon > 0$, paths of W^H are $\mathbb{P}-$a.s. $(H-\varepsilon)$–Hölder continuous.

Remark 3.3. For $H = \frac{1}{2}$ we recover the standard Brownian motion. Paths of standard Brownian motion exhibit Hölder regularity of $\frac{1}{2} - \varepsilon$ for all $\varepsilon > 0$.

Statistical analysis of ultra-high-frequency time series data performed in the seminal work [GJR14] concluded that paths of the log-volatility of major market indices exhibit Hölder regularity lower than that of the standard Brownian motion. Specifically, the realized log-volatility can be accurately fitted by the fractional Brownian motion with Hurst index 0.1. Moreover, volatility models driven by the fractional Brownian motion also generate reasonable shapes of the implied volatility surfaces for short maturities. These considerations motivate introduction of the rough counterpart of the Heston model to allow for the fractional volatility driver.

Definition 3.4. Let (W_t, B_t) be the standard two-dimensional Brownian motion. For $\alpha \in \left(\frac{1}{2}, 1\right)$ we define the *Rough Heston model* (S_t, V_t) where the spot process S_t and variance process V_t obey the equations

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dW_t \\ V_t &= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\left(\bar{v} - V_s\right)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\xi\sqrt{V_s}dB_s \\ \langle dW_t, dB_t\rangle &= \rho dt. \end{aligned} \tag{3.5}$$

The interpretation of the parameters is equivalent with the classical Heston model in Table 1. Additional parameter α corresponds to the *roughness* of the paths.

Remark 3.5. For $\alpha = 1$, we recover the Heston model (3.2). From now on we refer to the Heston model defined in (3.2) as the *classical Heston model* in order to differentiate between the two.

Virtually all stylized facts attributed to the behavior of volatility are recovered with the fractional driver [ER16]. Moreover, rough volatility models provide a remarkable fit of the implied volatility surface for short maturities [BFG15].

However, working with the fractional Brownian motion entails technical challenges. First, the resulting variance process in (3.5) is not a semimartingale, since it has unbounded quadratic variation, which renders Itô calculus not applicable [ER16]. Moreover, the resulting process depends of the past realization of the Brownian motion and hence it is not Markovian.

3.2 Characteristic function of the log-stock price

As mentioned above, a major feature of the classical Heston model is the explicit characteristic function. It turns out that the Rough Heston model exhibits similar affine structure. However, one cannot derive it by means of Itô calculus akin to the classical Heston model. The variance process in the Rough Heston model is not a semimartingale, which renders the Itô's lemma not applicable. Instead, the equation for the logarithm of moment-generating function in the Rough Heston model was established in [ER16] by means of a sequence of high-frequency models based on Hawkes processes.

Definition 3.6. A *two-dimensional Hawkes process* is a point process $\left(N_t^+, N_t^-\right)_{t\geq 0}$ with intensity given by

$$\begin{pmatrix} \lambda_t^+ \\ \lambda_t^- \end{pmatrix} = \begin{pmatrix} \mu^+ \\ \mu^- \end{pmatrix} + \int_0^t \begin{pmatrix} \varphi_1(t-s) & \varphi_3(t-s) \\ \varphi_2(t-s) & \varphi_4(t-s) \end{pmatrix} \cdot \begin{pmatrix} dN_s^+ \\ dN_s^- \end{pmatrix}, \tag{3.6}$$

where $\mu^+, \mu^- \in \mathbb{R}$ correspond to baseline intensity level and φ_i for $i \in \{1,2,3,4\}$ are measurable non-negative deterministic functions.

Note that Hawkes process is not Markovian, since the intensity level at time t depends on the entire trajectory $(N_s^+, N_s^-)_{0\leq s\leq t}$. In the language of branching processes such process is called *self-exciting*. The microstructural model of the asset price in [ER16] is then simply given by

$$P_t = N_t^+ - N_t^- \tag{3.7}$$

where N_t^+ corresponds to upward price movements and N_t^- corresponds to downward price movements.

A crucial role in the construction plays the kernel matrix $\left(\begin{smallmatrix} \varphi_1 & \varphi_3 \\ \varphi_2 & \varphi_4 \end{smallmatrix}\right)$ in the definition of the intensity of the Hawkes process in (3.6). The kernel matrix quantifies degree of endogeneity of the market and is thus sometimes dubbed *memory kernel function*. High degree of the market endogeneity is manifested in the L^1-norm of the largest eigenvalue of the kernel matrix being close to unity. Liquid markets are highly *endogenous*, meaning that most transactions have no economic rationale but are rather executed algorithmically in reaction to other large institutional orders or *metaorders* [FS12]. The metaorders are subjected to optimal execution algorithms and broken down in smaller orders.

Depending on the postulated endogeneity of the market, with appropriate scaling one obtains the Heston model [JR13] or the Rough Heston model [ER16] as the scaling limit of the high-frequency models (3.7). The characteristic function of the Rough Heston model is then obtained from the limiting behavior of the characteristic function of the Hawkes process. We cite the correspoding result for the Rough Heston model from [ER16].

Proposition 3.7. *The logarithm of moment-generating function of the log-spot* $m(s,t)$ *in the Rough Heston model* (3.5) *is characterized by the solution of the fractional Riccati equation:*

$$m(s,t) = \log \mathbb{E}\left[e^{s \log S_t}\right] = \bar{v}\lambda I^1 \psi(s,t) + v_0 I^{1-\alpha}\psi(s,t) \tag{3.8}$$

$$\begin{aligned} D^\alpha \psi(s,t) &= R(s, \psi(s,t)) \\ \psi(\cdot, 0) &= 0 \\ R(s,w) &= \frac{1}{2}\left(s^2 - s\right) + (\rho\xi s - \lambda)w + \frac{1}{2}\xi^2 w^2. \end{aligned} \tag{3.9}$$

Note that the equation (3.9) is almost identical with the classical Heston case in (3.4), but now the ordinary differential operator is replaced with the fractional one.

At last, we cite a result from [KST06] which establishes one-to-one correspondence between solutions of fractional Riccati equations and Volterra integral equations. Working with integral equations instead will prove beneficial due to application of integral transforms. Heuristically equivalence of (3.9) with the corresponding integral equation can be seen by applying the fractional integration operator $I^\alpha(\cdot)$ to both sides of (3.8).

Proposition 3.8. *Let* $\alpha \in (0,1)$, $T > 0$ *and suppose that* $\psi \in C[0,T]$ *and* $H \in C(\mathbb{R})$*. Then* ψ *satisfies the fractional differential equation*

$$\begin{aligned} D_t^\alpha \psi(t) &= H(\psi(t)) \\ I_t^{1-\alpha}\psi(0) &= 0 \end{aligned}$$

if and only if it satisfies the Volterra integral equation

$$\psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H(\psi(s)) ds.$$

Thus Proposition 3.8 allows us to work with Volterra integral equation.

Proposition 3.9. *The logarithm of moment-generating function of the log-spot* $m(s,t)$ *in the Rough Heston model* (3.5) *is characterized by the solution of the Volterra integral equation:*

$$\begin{aligned} \psi(s,t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} R(s, \psi(s,\tau)) d\tau \\ \psi(\cdot, 0) &= 0 \\ R(s,w) &= \frac{1}{2}\left(s^2 - s\right) + (\rho\xi s - \lambda) w + \frac{1}{2}\xi^2 w^2. \end{aligned} \tag{3.10}$$

3.3 Moment explosions

As discussed above, one of the "stylized facts" associated with asset returns is fat tails of the marginal distributions. This leads to the insight that moments of the asset price process in realistic models may be infinite.

Definition 3.10. For $s \in \mathbb{R}$ we define the *critical time*

$$T^*(s) := \sup\left\{t \geq 0 : \mathbb{E}\left[e^{s \log S_t}\right] < \infty\right\}. \tag{3.11}$$

For $T > 0$ we define the *critical moments*

$$\begin{aligned} s_+(T) &:= \sup\left\{u \in \mathbb{R} : \mathbb{E}\left[e^{u \log S_T}\right] < \infty\right\} \\ s_-(T) &:= \inf\left\{u \in \mathbb{R} : \mathbb{E}\left[e^{u \log S_T}\right] < \infty\right\} \end{aligned} \tag{3.12}$$

with conventions $\sup\{\varnothing\} = +\infty$ and $\inf\{\varnothing\} = -\infty$.

We overview some facts on the critical time and critical moments from [GGP18].

Proposition 3.11. *Let S_t be a positive martingale. Then the following holds:*

(i) $T^*(s)$ *increases for* $u \leq 0$ *and decreases for* $u \geq 0$.

(ii) $s_+(T)$ *decreases and* $s_-(T)$ *increases.*

(iii) If the increase (decrease) in the above statements is strict, one function is the inverse of the other one.

Finite critical moments are interesting for multiple reasons. First, they allow to identify convergence rate of some numerical procedures and identify the models that would assign infinite prices to particular financial products [AP05]. In addition, they are important for the optimal choice of a good damping parameter in option pricing with transform methods to avoid highly oscillatory integrands, as will become apparent in Section 6.1.2. We in turn follow the suit of [Fri+10] and employ critical moments to study the asymptotic behavior of the implied variance.

A comprehensive study of the moment explosions in the Rough Heston model has been performed in [GGP18]. We start by citing a statement relating finitude of the explosion times in the Rough Heston model and classical Heston model.

Proposition 3.12. *The moment* $\mathbb{E}\left[S_t^s\right]$ *for* $s \in \mathbb{R}$ *explodes in the Rough Heston model* (3.5) *in finite time if and only if the moment explodes in finite time for the corresponding configuration of the classical Heston model with* $\alpha = 1$.

Note that while finitude of the critical time in the Rough Heston model is equivalent to the finitude of the critical time in the corresponding classical Heston model with $\alpha = 1$, the actual values of the critical time will be different in both models. The following statement from [GGP18] characterizes the region of the parameter space in which a particular moment $s \in \mathbb{R}$ explodes.

Proposition 3.13. *The moment $s \in \mathbb{R}$ in the Rough Heston model* (3.5) *explodes in finite time if and only if both of the following conditions are satisfied:*

(i) $\frac{1}{2}s(s-1) > 0$;

(ii) $\frac{1}{2}(\rho\xi s - \lambda) \geq 0$ `or` $\left\{\frac{1}{2}(\rho\xi s - \lambda) \leq 0 \text{ and } \frac{1}{4}(\rho\xi s - \lambda)^2 - \frac{1}{4}\xi^2 s(s-1) < 0\right\}$.

In our analysis, we are interested in the inverse function of the critical time – the critical moment. Finitude of the critical moments in the Rough Heston model has been also addressed in [GGP18].

Proposition 3.14. *In the Rough Heston model the critical moments defined in* (3.12) *are finite for every $T > 0$.*

It follows that while there may be moments $s \in \mathbb{R}$ such that the critical time $T^*(s)$ is infinite, unless conditions from Proposition 3.12 hold, the critical moment $s_+(T)$ is finite for every time $T > 0$.

At last, we address calculation of the critical time and critical moments in the Rough Heston model. In the classical Heston model, owing to the closed-form characteristic function, the critical time can be calculated explicitly. The critical moments are then obtained by a root-finding procedure [AP05]. Meanwhile, the Rough Heston model does not enjoy the same liberty. However, for specific configurations of the Rough Heston model the critical time can be calculated iteratively [GGP18]. We thus compute the critical time and its differentials numerically. The critical moments are then obtained similarly by means of a root-finding procedure.

Figure 1 illustrates critical moments $s_+(T)$ in the classical Heston model and Rough Heston model with $\alpha = 0.9$ and $\alpha = 0.8$, respectively. Note that critical moments in the Rough Heston model are decreasing with decreasing value of α. This signifies fatter tails in the rougher models with lower values of α. Note also that the difference of the critical moments between the models is more pronounced for smaller time pegs, while critical moments for larger time pegs are almost equal. This manifests the ability of the Rough Heston model to capture more accurately the tail risk for short maturities.

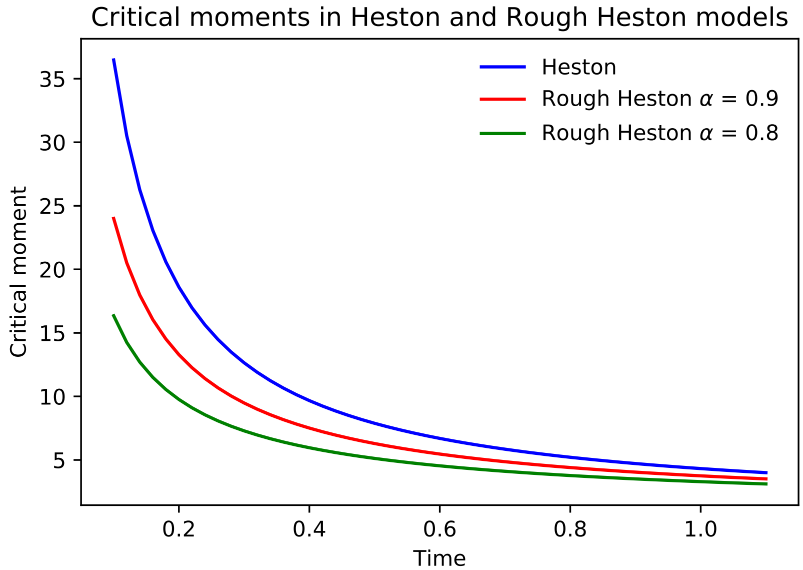

Figure 1: Critical moments for the classical Heston model and the Rough Heston model with $\alpha = 0.9$ and $\alpha = 0.8$, respectively.

4 Expansion of the Rough Heston density

In this section we aim to obtain asymptotic expansion of the Rough Heston density for extreme values of the log-spot $\log x \to \pm\infty$ at a fixed timescale $T > 0$. We start by studying the blow-up behavior of the solution ψ of the fractional Riccati equation in (3.9) inspired by the procedure from in [RO96]. It entails techniques from [BH75] involving the Mellin transform of the integral equation. Proposition 4.1 implies that for a fixed $s \in \mathbb{R}$ the explosive solutions $\psi(s,t)$ of the fractional Riccati equation in (3.8) behave near the explosion time $T^*(s)$ as a power law $\psi(s,t) \sim (T^*(s) - t)^{-\alpha}$. Proposition 4.1 relies on assumptions imposed on the solution ψ of the equation (3.9) which we summarize in the Remark 4.2.

Thereafter, we investigate the asymptotic expansion of the moment-generating function in the proximity of the critical moment. Proposition 4.5 warrants singular behavior of order $m(s,t) \sim (s_+ - s)^{1-2\alpha}$ for the logarithm of the moment-generating function in terms of the critical moment s_+. Proposition 4.5 relies on assumptions concerning the differentiability of the critical time in the Rough Heston model.

At last, we proceed along the lines of [Fri+10] to obtain the asymptotic expansion of the density with the saddlepoint method. After locating the approximate saddlepoint and establishing necessary tail estimates, the refined asymptotic expansion of the density is given in Proposition 4.14.

We will focus on the right-wing asymptotics ($\log x \to +\infty$). The analysis of the left-wing asymptotics ($\log x \to -\infty$) is completely analogous to the right-wing case. The corresponding results on the left-wing asymptotics are presented in Corollary 4.15 at the end of the section.

4.1 Asymptotic expansion of the fractional Riccati equation

The fractional Riccati equation (3.9) is not solvable explicitly. As mentioned before, ODE techniques from [Fri+10] used to obtain asymptotic expansion of the equation near blow-up are not applicable due to absence of an elementary chain rule for the fractional derivative. Hence we need to employ an alternative approach to expand the solution of (3.9).

Note that asymptotic expansion of the equation (3.9) near $t = 0$ has been obtained in [GGP18] by means of a fractional power series ansatz

$$\psi(s, t) \sim \sum_{n=1}^{\infty} a_n(s) t^{\alpha n}. \tag{4.1}$$

Applying $D^\alpha(\cdot)$ to both sides of (4.1) yields some recursive relation between coefficients $a_n(s)$ which results in the asymptotic expansion of ψ near the origin. However, we are interested in the asymptotic expansion of the equation (3.9) near the blow-up $t \to T^*(s)$. In order to figure out the degree of the highest-order term in the asymptotic expansion of ψ, we conjecture that

$$\psi(s, t) \sim \Theta \cdot (T^*(s) - t)^u \tag{4.2}$$

for some $u, \Theta \in \mathbb{R}$. Plugging (4.2) into (3.9) and using rules for fractional differentiation of power laws (2.1) we obtain

$$\Theta \cdot \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} (T^*(s) - t)^{u-\alpha} \sim \Theta^2 \cdot \frac{1}{2} \xi^2 \cdot (T^*(s) - t)^{2u}.$$

Matching the exponents suggests that $u = -\alpha$ and $\Theta = \frac{\Gamma(2\alpha)}{\frac{1}{2}\xi^2 \Gamma(\alpha)}$. Therefore, the order of the explosion must be

$$\psi(s, t) \sim \frac{\Gamma(2\alpha)}{\frac{1}{2}\xi^2 \Gamma(\alpha)} \cdot (T^*(s) - t)^{-\alpha}.$$

The exponent $-\alpha$ in the explosive term and the constant $\frac{\Gamma(2\alpha)}{\frac{1}{2}\xi^2 \Gamma(\alpha)}$ are conformal with [RO96]. It also agrees with the explosion order of the ordinary Riccati differential equation for $\alpha = 1$ obtained in [Fri+10].

In the following, we aim to formalize these asymptotics using method from [RO96] by rescaling the equation (3.9) and applying the Parseval formula for the Mellin transform to the right-hand side of (3.10). The terms in the asymptotic expansion of (3.10) result from poles in the Mellin domain. The analysis relies on analytical assumptions imposed on ψ, which will emerge in the proof of Proposition 4.1. The assumptions are additionally summarized after the proof in Remark 4.2.

Proposition 4.1. *Let ψ be a solution the Volterra integral equation* (3.10).

- *Assume that the function ψ admits for some $\Theta \in \mathbb{R}$ an asymptotic expansion*
$$\psi(s,t) \sim \Theta(T^*(s) - t)^{-\alpha}. \tag{4.3}$$
- *Assume additionally that the asymptotic estimate* (4.16) *in the following proof holds.*

Then the asymptotic expansion of ψ is given by

$$\psi(s,t) = \Theta(T^*(s) - t)^{-\alpha} + O(1) \tag{4.4}$$
$$\Theta = \frac{\Gamma(2\alpha)}{\frac{1}{2}\xi^2\Gamma(\alpha)}.$$

Proof. **Rescaling the equation.** Let $\eta_0 := \frac{1}{T^*(s)}$. We start by rescaling the equation in terms of the new variable $\eta := \frac{1}{T^*(s)-t} - \eta_0$ such that $\eta \in [0,\infty)$ and $\eta \to \infty$ as $t \uparrow T^*(s)$. Consider the rescaled function $w(\eta)$ which satisfies due to (3.10)

$$\begin{aligned} w(\eta) &:= \psi\left(s, T^*(s) - \frac{1}{\eta + \eta_0}\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T^*(s) - \frac{1}{\eta+\eta_0}} \left(T^*(s) - \frac{1}{\eta+\eta_0} - \tau\right)^{\alpha-1} R(s,\psi(s,\tau))d\tau. \end{aligned} \tag{4.5}$$

Note that as intended we have for $w(0) = \psi(s,0)$ and

$$T^*(s) - \frac{1}{\eta + \eta_0} \to T^*(s) \qquad \text{as } \eta \to \infty.$$

Substitution $\tau = T^*(s) - \frac{1}{\zeta+\eta_0}$ in (4.5) allows to reformulate (3.10) in terms of $w(\eta)$ as

$$w(\eta) = \int_0^\eta k\{(\eta - \zeta)[(\eta+\eta_0)(\zeta+\eta_0)]^{-1}\} \cdot (\zeta+\eta_0)^{-2} R(s, w(\zeta))d\zeta \tag{4.6}$$

where $k(u) := \frac{1}{\Gamma(\alpha)} u^{\alpha-1}$ is kernel of the integral in (4.6). We next substitute $\zeta = \eta\omega$ to remove η from the limits of integration and obtain

$$\begin{aligned} w(\eta) &= \int_0^1 k\{\eta(1-\omega)[(\eta\omega+\eta_0)(\eta+\eta_0)]^{-1}\} \cdot (\eta\omega+\eta_0)^{-2} R(s, w(\eta\omega)) d(\eta\omega) \\ &= \eta \cdot \left(\frac{\eta}{\eta+\eta_0}\right)^{\alpha-1} \int_0^\infty K(\omega) F(\eta\omega) d\omega \\ &\sim \eta \cdot \int_0^\infty K(\omega) F(\eta\omega) d\omega \end{aligned} \tag{4.7}$$

where the kernel and non-linearity are given in terms of the Heaviside step function $\theta(x) = \mathbb{1}_{\{x \geq 0\}}$ by

$$\begin{aligned} K(\omega) &:= \frac{1}{\Gamma(\alpha)} (1-\omega)^{\alpha-1} \theta(1-\omega) \\ F(\eta\omega) &:= (\eta\omega+\eta_0)^{-1-\alpha} R(s, w(\eta\omega)). \end{aligned} \tag{4.8}$$

Parseval formula. The reformulation (4.7) in terms of Mellin convolution allows us to utilize Mellin transform technique along the lines of [BH75]. Specifically, we employ the Parseval formula for Mellin transform (2.5). Recalling scale invariance property of the Mellin transform (2.3) we can write

$$w(\eta) \sim \frac{\eta}{2\pi i} \int_{r-i\infty}^{r+i\infty} \eta^{-z} \mathrm{M}[K(\omega); 1-z] \mathrm{M}[F(\omega); z] dz =: \frac{\eta}{2\pi i} \int_{r-i\infty}^{r+i\infty} \eta^{-z} G(z) dz, \tag{4.9}$$

where the vertical path of integration lies within the fundamental strip of both Mellin transforms in the integrand. The following idea is to shift the contour to the right in (4.9) and collect terms in the asymptotic expansion resulting from residues near the poles of the integrand:

$$\int_{r-i\infty}^{r+i\infty} M[K(\tau); 1-z] M[F(\eta\tau); z] dz = -\sum_{r<\Re(z)<R} \operatorname{res}\left\{\eta^{-z} G(z)\right\} + \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \eta^{-z} G(z) dz \tag{4.10}$$

In order to shift the contour to the right, one needs to ensure the following conditions listed in [BH75].

(A) The function $G(z)$ has to be meromorphically continued to the right half-space. With no further information, it is not known to be a meromorphic function in the entire region.

(B) Assuming that the analytic continuation can be accomplished, one needs to justify the displacement of the contour of integration to establish validity of (4.10).

(C) Finally, in order to ensure asymptotic nature of the expansion, one must show that the error estimate holds

$$\frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty}\eta^{-z}G(z)dz = O(\eta^{-R}).$$

Mellin transform of the kernel. Mellin transform of the kernel K is given by

$$\mathrm{M}[K(\omega);1-z] = \frac{1}{\Gamma(\alpha)}\int_0^1 \omega^{1-z}(1-\omega)^{\alpha-1}d\omega = \frac{\Gamma(1-z)}{\Gamma(1+\alpha-z)}. \tag{4.11}$$

$\Gamma(1-z)$ has poles at $z=n\in\mathbb{N}$ with residues $\frac{(-1)^n}{n!}$. Therefore, $\frac{\Gamma(1-z)}{\Gamma(1+\alpha-z)}$ is analytic for $\Re(z)<1$. The asymptotic (4.9) now reads

$$w(\eta)\sim\frac{\eta}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{\Gamma(1-z)}{\Gamma(1+\alpha-z)}M[F(\eta\omega);z]dz, \qquad \eta\to\infty. \tag{4.12}$$

Alternatively, we can write the pole expansion of the Mellin transform of the kernel as

$$M[K(\omega);1-z] = -\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\alpha-n)n!}\left(\frac{1}{z-(n+1)}\right).$$

Mellin transform of the non-linearity. Due to our assumption (4.3) we have

$$w(\eta)\sim\Theta\eta^{\alpha} \qquad \text{as } \eta\to+\infty. \tag{4.13}$$

Plugging the ansatz into the definition (4.8) of the non-linearity F we obtain

$$F(\eta\omega) \sim (\eta\omega)^{-1-\alpha}\left(\frac{1}{2}\xi^2\Theta^2(\eta\omega)^{2\alpha}\right) = \frac{1}{2}\xi^2\Theta^2(\eta\omega)^{-1+\alpha}. \tag{4.14}$$

The asymptotics of the Mellin transform of the highest-order term then reads

$$M\left[\frac{1}{2}\xi^2 d_1^2(\eta\omega)^{-1+\alpha}; z\right] \sim \frac{1}{2}\xi^2 d_1^2\left(\frac{\eta^{-z}}{z-(1-\alpha)}\right)$$

which exists if and only if $z < 1-\alpha$. More generally, powers of $\eta\omega$ in the expansion of $F(\eta\omega)$ given in (4.14) correspond to poles in the Mellin domain. The poles corresponding to powers in the asymptotic expansion of $w(\eta)$ of order lower than $1-\alpha$ would be located on the real line right from $1-\alpha$. The residues result from the coefficients near the poles. We can therefore write

$$M[F(\eta\omega); z] \sim \eta^{-z}\frac{\frac{1}{2}\xi^2\Theta^2}{z-(1-\alpha)} \tag{4.15}$$

with the remaining poles residing on the real line to the right from $1-\alpha$. Thus $M[F(\eta\omega); z]$ is analytic for $\Re(z) < 1-\alpha$. The leading order contribution comes from the pole implied by (4.15).

Fundamental strip. For the fundamental strips of the factors in the integrand of (4.9) we have

- $M[K(\omega); 1-z]$ is analytic for $\Re(z) < 1$;
- $M[F(\eta\omega); z]$ is analytic for $\Re(z) < 1-\alpha$.

Hence the entire integrand in (4.9) is analytic for $\Re(z) < 1-\alpha$. Thus the integration contour in (4.12) lies within $\Re(z) < 1-\alpha$, to the left of all the poles.

Shifting the contour and coefficient matching. Shifting the contour to the right allows us to collect the expansion terms. However, note that in order to justify the asymptotic expansion, one should ensure that conditions (A)–(C) hold.

Thanks to the asymptotics in (4.13), we have according to Lemma 4.3.3 in [BH75] that $M[f;z]$ can be analytically continued as a meromorphic function to $\Re(z) > 1-\alpha$ which justifies (A). Additionally, Lemma 4.3.3 in [BH75] warrants that $M[f;z] \to 0$ as $\Im(z) \to \pm\infty$ which justifies (B).

However, from the asymptotic expansion (4.13) one cannot immediately deduce an error estimate $O(\eta^{-R})$ for some value $R > 1-\alpha$. In order to proceed we make an additional assumption that there are no more diverging terms in the asymptotic expansion of $w(\eta)$. This would entail that (C) holds with $R = 1$. Specifically, we assume that

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \eta^{-z} G(z) dz = O\left(\eta^{-1}\right). \tag{4.16}$$

With this additional assumption displacement of the contour to the right past the pole $z = 1-\alpha$ is justified. Therefore, for a contour γ oriented counterclockwise and enclosing $z = 1-\alpha$ we get with the Cauchy integral formula (2.6)

$$\frac{\eta}{2\pi i}\int_\gamma \frac{\Gamma(1-z)}{\Gamma(1+\alpha-z)} M[F(\eta\tau);z]dz = \eta\cdot\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\cdot \mathrm{Res}(M[F(\eta\tau);z], 1-\alpha) = \frac{\frac{1}{2}\xi^2\Gamma(\alpha)}{\Gamma(2\alpha)}\Theta^2\eta^\alpha.$$

Matching the coefficients from our ansatz (4.13) leads to

$$\Theta\eta^\alpha = \frac{1}{2}\xi^2\Theta^2\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\eta^\alpha \qquad \Longleftrightarrow \qquad \Theta = \frac{\Gamma(2\alpha)}{\frac{1}{2}\xi^2\Gamma(\alpha)}.$$

We ultimately obtain

$$w(\eta) \sim \frac{\Gamma(2\alpha)}{\frac{1}{2}\xi^2\Gamma(\alpha)}\eta^\alpha.$$

Recalling the definition (4.5) of $w(\eta)$ we obtain the asymptotic expansion of the solution of the original equation near the blow-up $t \to T^*(s)$

$$\psi(s,t) \sim \frac{\Gamma(2\alpha)}{\frac{1}{2}\xi^2\Gamma(\alpha)}(T^*(s)-t)^{-\alpha}. \tag{4.17}$$

□

Remark 4.2. We emphasize again that the Proposition 4.1 relies on two assumptions:

- Assumption (4.3) concerning blow-up behavior of the solution based on heuristic analysis at the beginning of the section.
- Assumption (4.16) concerning the error estimate, which $O(1)$.

Note that both of these assumptions hold in the case of the ordinary Riccati equation $\alpha = 1$ resulting from [Fri+10].

4.2 Behavior of the moment-generating function near blow-up

Turning back to the formula (3.8), note that for the logarithm of moment-generating function we need to consider asymptotics of $I^1\psi$ and $I^{1-\alpha}\psi$.

Proposition 4.3. *Consider the function $m(s,t)$ defined in* (3.8)*. Then as $t \uparrow T^*(s)$, the logarithm of moment-generating function $m(s,t)$ has the asymptotic expansion*

$$m(s,t) = \Lambda_0(T^*(s)-t)^{1-2\alpha} + \Lambda_1 + \Lambda_2(T^*(s)-t)^{1-\alpha} + O(T^*(s)-t)^{2-2\alpha} \tag{4.18}$$

with constants

$$\Lambda_0 = v_0\Theta\frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)}$$

$$\begin{aligned}\Lambda_1 = v_0\Theta\left(T^*(s)^{1-2\alpha}\right)&\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)\Gamma(2-2\alpha)} + \bar{v}\lambda\Theta T^*(s)^{(1-\alpha)}\\ &+ \bar{v}\lambda\int_0^{T^*(s)}\left[\psi(s,\tau) - \Theta(T^*(s)-\tau)^{-\alpha}\right]d\tau\\ &+ \frac{v_0}{\Gamma(1-\alpha)}\int_0^{T^*(s)}(t-\tau)^{-\alpha}\left[\psi(s,\tau)-\Theta(T^*(s)-\tau)^{-\alpha}\right]d\tau\end{aligned}$$

$$\Lambda_2 = \bar{v}\lambda\Theta.$$

Remark 4.4. Note that Λ_1 is a indeed a finite constant since $\psi(s,\tau) = \Theta(T^*(s)-\tau)^{-\alpha} + O(1)$ due to (4.17). We thus subtract the singularity in the integrands and integrate the remaining $O(1)$ part over finite interval. Since $\alpha < 1$, integrability of Λ_1 follows from Hölder inequality.

Proof. **Expansion of $I^1\psi$.** Recall from (4.17) that $\psi(s,\tau) = \Theta(T^*(s)-\tau)^{-\alpha} + O(1)$. We thus subtract the diverging term $\Theta(T^*(s)-\tau)^{-\alpha}$ from ψ in the integrand to obtain

$$
\begin{aligned}
I^1\psi(s,t) &= \int_0^t \psi(s,\tau)d\tau \\
&= \int_0^t \left[\psi(s,\tau) - \Theta(T^*(s)-\tau)^{-\alpha} + \Theta(T^*(s)-\tau)^{-\alpha}\right] d\tau \\
&= \int_0^{T^*(s)} \left[\psi(s,\tau) - \Theta(T^*(s)-\tau)^{-\alpha}\right] d\tau \\
&\qquad - \int_t^{T^*(s)} \left[\psi(s,\tau) - \Theta(T^*(s)-\tau)^{-\alpha}\right] d\tau \\
&\qquad + \Theta T^*(s)^{(1-\alpha)} - \Theta(T^*(s)-t)^{1-\alpha} \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned}
\tag{4.19}
$$

Since $\psi(s,\tau) = \Theta(T^*(s)-\tau)^{-\alpha} + O(1)$, we have $A_1 = O(1)$ and $A_2 = O(T^*(s)-t)$. Moreover, we have $A_3 = O(1)$ and just $A_4 = -\Theta(T^*(s)-t)^{1-\alpha}$. In particular, due to $1-\alpha > 0$ the entire integral $I^1\psi(s,t)$ is $O(1)$ and provides no contribution to the explosion of the logarithm of moment-generating function.

Expansion of $I^{1-\alpha}\psi$. Analogically for $I^{1-\alpha}\psi(s,t)$ we obtain

$$
\begin{aligned}
I^{1-\alpha}\psi(s,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\psi(s,\tau)d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^{T^*(s)} (t-\tau)^{-\alpha}\left[\psi(s,\tau)-\Theta(T^*(s)-\tau)^{-\alpha}\right]d\tau \\
&\quad - \frac{1}{\Gamma(1-\alpha)}\int_t^{T^*(s)} (t-\tau)^{-\alpha}\left[\psi(s,\tau)-\Theta(T^*(s)-\tau)^{-\alpha}\right]d\tau \\
&\quad + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\left[\Theta(T^*(s)-\tau)^{-\alpha}\right]d\tau \\
&=: B_1+B_2+B_3.
\end{aligned}
\tag{4.20}
$$

Again since $\psi(s,\tau)=\Theta(T^*(s)-\tau)^{-\alpha}+O(1)$ and $\alpha<1$, we can apply Hölder inequality to obtain $B_1=O(1)$ and $B_2=O(T^*(s)-t)$.

On the other hand, the term B_3 is explosive. Let us write

$$
I := \int_0^t (t-\tau)^{-\alpha}(T^*(s)-\tau)^{-\alpha}d\tau.
$$

Using the substitution $\tau =: t-\frac{u}{T^*(s)-t}$ we obtain for $1/2<\alpha<1$

$$
\begin{aligned}
I \sim \int_{-\infty}^t (t-\tau)^{-\alpha}(T^*(s)-\tau)^{-\alpha}d\tau &= (T^*(s)-t)^{1-2\alpha}\int_0^\infty u^{-\alpha}(1+u)^{-\alpha}du \\
&= \frac{\Gamma(2\alpha-1)\Gamma(1-\alpha)}{\Gamma(\alpha)}\cdot(T^*(s)-t)^{1-2\alpha}
\end{aligned}
\tag{4.21}
$$

Alternatively, the integral has a representation in terms of the Gaussian hypergeometric function ${}_2F_1$ defined in (2.7). The representation can be used to obtain terms of higher order. Recalling the identities from Section 2.3 we get

$$
\begin{aligned}
I &= t^{1-\alpha} \cdot T^*(s)^{-\alpha} \int_0^1 (1-s)^{-\alpha} \left(1 - \frac{t}{T^*(s)} s\right)^{-\alpha} ds \\
&\overset{(2.7)}{=} t^{1-\alpha} \cdot T^*(s)^{-\alpha} \cdot B(1, 1-\alpha) \cdot {}_2F_1 \left(\alpha, 1, 2-\alpha, \frac{t}{T}^*(s)\right) \\
&\overset{(2.9)}{=} t^{1-\alpha} \cdot T^*(s)^{-\alpha} \cdot B(1, 1-\alpha) \cdot \frac{\Gamma(2\alpha-1)\Gamma(2-\alpha)}{\Gamma(\alpha)\Gamma(1)} \cdot \left(\frac{T^*(s)-t}{T^*(s)}\right)^{1-2\alpha} \\
&\quad \cdot {}_2F_1 \left(2-2\alpha, 1-\alpha, 2-2\alpha, \frac{T^*(s)-t}{T}\right) \\
&\quad + t^{1-\alpha} \cdot T^*(s)^{-\alpha} \cdot B(1, 1-\alpha) \cdot \frac{\Gamma(1-2\alpha)\Gamma(2-\alpha)}{\Gamma(2-2\alpha)\Gamma(1-\alpha)} \cdot {}_2F_1 \left(\alpha, 1, 2\alpha, \frac{T^*(s)-t}{T^*(s)}\right) \\
&\sim \frac{\Gamma(1-\alpha)\Gamma(2\alpha-1)}{\Gamma(\alpha)} \cdot {}_2F_1 \left(2-2\alpha, 1-\alpha, 2-2\alpha, \frac{T^*(s)-t}{T^*(s)}\right) \cdot (T^*(s)-t)^{1-2\alpha} \\
&\quad + T^*(s)^{1-2\alpha} \cdot \frac{\Gamma(1-2\alpha)}{\Gamma(2-2\alpha)} \cdot {}_2F_1 \left(\alpha, 1, 2\alpha, \frac{T^*(s)-t}{T}\right) \\
&\overset{(2.8)}{=} \frac{\Gamma(2\alpha-1)\Gamma(1-\alpha)}{\Gamma(\alpha)} \cdot (T^*(s)-t)^{1-2\alpha} + (T^*(s)^{1-2\alpha}) \cdot \frac{\Gamma(1-2\alpha)}{\Gamma(2-2\alpha)} + O(T^*(s)-t)^{2-2\alpha}.
\end{aligned}
$$

Note that the constant next to the leading order term is the same as in the shorter calculation (4.21) above. We thus obtain

$$
B_3 = \Theta \cdot \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \cdot (T^*(s)-t)^{1-2\alpha} + \Theta \cdot (T^*(s)^{1-2\alpha}) \cdot \frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)\Gamma(2-2\alpha)} + O(T^*(s)-t)^{2-2\alpha}. \tag{4.22}
$$

Note that since $1 - 2\alpha < 0$, the expression B_3 diverges as $t \uparrow T^*(s)$.

Plugging the asymptotic expansions of the integral (4.19) and the fractional integral (4.20) into the definition (3.8) of the Rough Heston logarithm of moment-generating function, we obtain the asymptotic expansion (4.18) of the Rough Heston logarithm of moment-generating function.

□

Thus we have obtained asymptotic expansion of $m(s,t)$ in terms of proximity to the explosion time $T^*(s) - t$. Let us fix the timescale $T > 0$ and denote by s_+ the critical moment $s_+(T)$ corresponding to the time $T = T^*(s_+)$. We can write any smaller time $t < T^*(s_+)$ as critical time for some smaller moment $t = T^*(s)$. Then our asymptotic parameter is proximity $T^*(s_+) - T^*(s)$. Next we want to consider asymptotic parameter $s_+ - s$ and express the expansion of the logarithm of moment-generating function in terms of proximity to the critical moment s_+.

As suggested in Section 3.3, the Rough Heston model does not possess an explicit form of the critical time. However, it is known from Proposition 3.11 that $T^*(s)$ is strictly increasing for positive s and strictly decreasing for negative s. We also saw in Section 3.3 that pattern of the critical moment in the Rough Heston model is similar to the critical moment in the classical Heston model. Hence it is plausible to assume that the critical time $T^*(s)$ is a twice differentiable function of the moment s.

Proposition 4.5. *Assume that the critical time T^* defined in* (3.11) *is a twice continuously differentiable of the moment s. Denote by $\sigma_+ := -T^*(s_+)'$ and $\kappa := T^*(s_+)''$ the* critical slope *and* critical curvature*, respectively.*

Then the Rough Heston logarithm of moment-generating function $m(s,t)$ defined in (3.8) *satisfies the asymptotic expansion*

$$\begin{aligned} m(s,t) = \Lambda_0 \sigma_+^{1-2\alpha}(s_+ - s)^{1-2\alpha} + \Lambda_1 + \Lambda_2 \sigma_+^{1-\alpha}(s_+ - s)^{1-\alpha} \\ + \Lambda_0 \frac{\kappa(1-2\alpha)}{2\sigma_+^{-2\alpha}}(s_+ - s)^{2-2\alpha} + O(s_+ - s). \end{aligned} \tag{4.23}$$

Proof. Consider the Taylor expansion

$$T^*(s) = T^*(s_+) + (T^*(s_+))'(s - s_+) + \frac{1}{2}(T^*(s_+))''(s - s_+)^2 + O(s - s_+)^3.$$

We thus obtain

$$
\begin{aligned}
(T^*(s_+)-T^*(s))^{1-2\alpha} &= \left[\sigma_+(s_+-s)+\frac{\kappa}{2}(s_+-s)^2+O(s_+-s)^3\right]^{1-2\alpha}\\
&= (\sigma_+(s_+-s))^{1-2\alpha}\left[1+\frac{\kappa}{2\sigma_+}(s_+-s)+O(s_+-s)^2\right]^{1-2\alpha}\\
&= \sigma_+^{1-2\alpha}(s_+-s)^{1-2\alpha}+\frac{\kappa(1-2\alpha)}{2\sigma_+^{-2\alpha}}(s_+-s)^{2-2\alpha}+O(s_+-s)^{3-2\alpha}
\end{aligned}
$$

where we used $(1+ax)^{1-2\alpha}=1+a(1-2\alpha)x+O(x^2)$. Analogously we obtain

$$
\begin{aligned}
(T^*(s_+)-T^*(s))^{1-\alpha} &= \left[\sigma_+(s_+-s)+\frac{\kappa}{2}(s_+-s)^2+O(s_+-s)^3\right]^{1-\alpha}\\
&= \sigma_+^{1-\alpha}(s_+-s)^{1-\alpha}+\frac{\kappa(1-\alpha)}{2\sigma_+^{-\alpha}}(s_+-s)^{2-\alpha}+O(s_+-s)^{3-\alpha}.
\end{aligned}
$$

We can now use (4.18) to express the expansion in terms of the critical moment s_+

$$
\begin{aligned}
m(s,t) = \Lambda_0\sigma_+^{1-2\alpha}(s_+-s)^{1-2\alpha}+\Lambda_1+\Lambda_2\sigma_+^{1-\alpha}(s_+-s)^{1-\alpha}\\
+\Lambda_0\frac{\kappa(1-2\alpha)}{2\sigma_+^{-2\alpha}}(s_+-s)^{2-2\alpha}+O(s_+-s).
\end{aligned}
\tag{4.24}
$$

□

4.3 Asymptotics of the Rough Heston density

In this section we denote by $D_T(x)$ density of the Rough Heston model at a fixed timescale $T>0$ and aim to obtain asymptotics of the density as $\log x\to\pm\infty$. We proceed along the lines of [Fri+10].

4.3.1 Local expansion around the saddlepoint

Recall that the Rough Heston model is characterized in the equation (3.8) by the moment-generating function of the log-spot. If we denote by D_T the density of the spot in the Rough Heston model, we can express the moment-generating function as

$$\exp(m(s,T)) = \mathbb{E}\left[e^{s\log S_T}\right] = \mathbb{E}\left[S_T^s\right] = \int_{-\infty}^{+\infty} x^s D_T(x)dx = M\left[D_T(x); s+1\right].$$

Remark 4.6. In the Mellin domain, we denote the variable u instead of s and work with $u_+ = s_+ + 1$ (due to definition of the Mellin transform the variable is shifted by one).

The density can be recovered thusly from the moment-generating function of the log-spot via the Mellin inversion formula (2.4)

$$\begin{aligned} D_T(x) &= \frac{1}{2\pi i}\int_{\hat{u}_+-i\infty}^{\hat{u}_++i\infty} e^{-u\log x + m(u-1,T)}du \\ &= \frac{1}{2\pi i}\int_{\hat{u}_+-i\infty}^{\hat{u}_++i\infty} e^{-u\log x + \bar{v}\lambda I^1\psi(u-1,T) + v_0 I^{1-\alpha}\psi(u-1,T)}du. \end{aligned} \tag{4.25}$$

where $\hat{u}_+$ is some real number such that $\hat{u}_+ \in (s_-(T)+1, s_+(T)+1)$. However, we recall from Section 2.2 that in order to apply Mellin inversion formula, we need to warrant decay of the integrand

$$e^{m(u-1,T)} \to 0 \qquad \text{as } \Im(u) \to \infty.$$

The following lemma establishes exponential decay of the integrand above. The lemma covers the decay of the right tail $\Im(u) > 0$, the decay of the left tail is treated analogously. The lemma will be revisited in the context of asymptotic expansion of the density (4.25).

Lemma 4.7. *Let $t > 0$ and $1 \le u_1 \le \Re(u) \le u_2$ and $\rho \in (-1,1)$. Consider the right tail $\Im(u) > 0$. Then the moment-generating function in the Rough Heston model defined in* (3.8) *satisfies*

$$\left| e^{\bar{v}\lambda I^1\psi(u,t)+v_0 I^{1-\alpha}\psi(u,t)} \right| = O\left(e^{-K\Im(u)}\right).$$

where the constant C depends on t, u_1, u_2 and v_0.

Proof. We write $\psi = f + ig$ and investigate only the real part f for the absolute value above since

$$\left| e^{\bar{v}\lambda I^1\psi(u,t)+v_0 I^{1-\alpha}\psi(u,t)} \right| = e^{\bar{v}\lambda I^1 f(u,t)+v_0 I^{1-\alpha} f(u,t)}.$$

We write $u = x + iy$ and $\gamma = -(\rho\xi u - \lambda)$. Due to (3.8) the real and imaginary parts of ψ satisfy the equations

$$D^\alpha f = \frac{1}{2}\left(x^2 - y^2 - x\right) + \frac{\xi^2}{2}\left(f^2 - g^2\right) - \gamma f - y\rho\xi g \qquad f(u,0) = 0 \tag{4.26}$$

$$D^\alpha g = \frac{1}{2}(2\xi y - y) + \xi^2 fg - \gamma g + y\rho\xi f \qquad g(u,0) = 0 \tag{4.27}$$

We focus on the equation (4.26) for f. First we aim to eliminate g from the equation. By completing the square we obtain for any $0 < \varepsilon < \frac{1-\rho^2}{2}$

$$\begin{aligned} D^\alpha f &= \frac{1}{2}\left(x^2 - y^2 - x\right) + \frac{\xi^2}{2} f(u,t)^2 - \gamma f(u,t) - \frac{1}{2}(\xi g(u,t) + y\rho)^2 + \frac{(y\rho)^2}{2} \\ &\leq \frac{1}{2}\left(x^2 - x\right) + y^2\left(\frac{\rho^2 - 1}{2}\right) - \gamma f(u,t) + \frac{\xi^2}{2} f(u,t)^2 \\ &\leq -\varepsilon y^2 - \gamma f(u,t) + \frac{\xi^2}{2} f(u,t)^2 =: V(y, f(u,t)) \end{aligned}$$

eventually for some large $y > y_0$, where y_0 depends on u_1, u_2 and ρ.

We now want to come up with some function F such that

$$\begin{aligned} V(y, F(y,t)) &\leq D^\alpha F(y,t) \\ F(y,0) &= f(s,0) = 0 \end{aligned} \tag{4.28}$$

which would thusly control the function f. Define $\theta \in \mathbb{R}$ and positive function $C(t)$

$$-\frac{1}{4} = -\varepsilon + \frac{\xi^2}{2}\theta^2 t^2 \tag{4.29}$$

$$C(t) := \theta \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\gamma(t-s)^{\alpha}\right) ds. \tag{4.30}$$

$$F(y,t) := -yC(t) \tag{4.31}$$

From Proposition 2.4 we know that $C(t)$ is the variation of constant solution additionally satisfying

$$D^{\alpha}C(t) = -\gamma C(t) + \theta \tag{4.32}$$

$$0 \leq C(t) \leq \theta t. \tag{4.33}$$

We use our definition of F to obtain

$$\begin{aligned}
V(y, F(y,t)) &= -\varepsilon y^2 - \gamma F(y,t) + \frac{\xi^2}{2} F(y,t)^2 \\
&\overset{(4.31)}{=} -\varepsilon y^2 + \gamma y C(t) + \frac{\xi^2}{2} y^2 C(t)^2 \\
&\overset{(4.29)}{\leq} \left(-\varepsilon + \frac{\xi^2}{2} t^2 \theta^2\right) y^2 + \gamma y C(t) \\
&\overset{(4.29)}{=} -\frac{1}{4} y^2 + \gamma y C(t) \\
&\overset{(4.32)}{=} \left(-\frac{1}{4} y^2 + \theta y\right) - D^{\alpha}C(t) y \\
&\leq -D^{\alpha}C(t) y \\
&\overset{(4.31)}{=} D^{\alpha}F(y,t)
\end{aligned}$$

for some large $y > y_1 > y_0$ where y_1 depends only θ, and thus only on ξ and t. In summation, we have defined a function $F(y,t) = -yC(t)$ with positive $C(t)$ satisfying (4.28).

Now define $h(t) := f(s,t) - F(s,t)$. Since the function V is locally Lipschitz, we obtain

$$\begin{aligned} D^\alpha h(t) &\leq V(s, f(s,t)) - V(s, F(y,t)) \\ &\leq L|f(s,t) - F(y,t)| \\ &= Lh(t). \end{aligned}$$

Finally, applying I^α to both sides of the equation, with Grönwall's lemma we obtain

$$h(t) \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \qquad \Rightarrow \qquad h(t) \leq 0.$$

Therefore, we can write $f(s,t) \leq F(y,t)$ and use the bounds from the Proposition 2.4 to obtain

$$\begin{aligned} \left| e^{\bar{v}\lambda I^1 \psi(u,t) + v_0 I^{1-\alpha}\psi(u,t)} \right| &= \exp\left(\bar{v}\lambda I^1 f(u,t) + v_0 I^{1-\alpha} f(u,t)\right) \\ &\leq \exp\left(\bar{v}\lambda I^1 F(y,t) + v_0 I^{1-\alpha} F(y,t)\right) \\ &= \exp\left(-y(\bar{v}\lambda I^1 C(t) + v_0 I^{1-\alpha} C(t))\right) \\ &= \exp(-K\Im(u)) \end{aligned}$$

$\square$

Our goal now is to establish asymptotics of the density $D_T(x)$ as $\log x \to \infty$. Thus we are looking for the asymptotic expansion of the integral (4.25) with the asymptotic parameter given by the log-spot which we denote $k := \log x$.

We want to apply the saddlepoint method to ease the calculation of the integral. The saddlepoint method entails shifting the integration contour so that derivative of the integrand vanishes. The idea of the saddlepoint method is that for the shifted contour the integral is concentrated in the vicinity of the approximate saddlepoint $\hat{u}_+ \in \mathbb{R}$. If we can control the tails of the integral, the asymptotic expansion will be given by the values of the integrand at $\hat{u}_+$. Note, however, that we only have asymptotic expansion of the function $m(u-1, T)$ at hand, hence the derivative and the exact saddlepoint cannot be calculated explicitly. However, we can use the asymptotic expansion of

$m(u-1,T)$ to calculate the approximate saddlepoint $\hat{u}_+$ such that derivative of the integrand vanishes asymptotically.

Proposition 4.8. *Approximate saddlepoint $\hat{u}_+$ for the integral* (4.25) *as $k \to +\infty$ satisfies*

$$u_+ - \hat{u}_+ \sim \beta_+ k^{-\frac{1}{2\alpha}} \tag{4.34}$$

with $\beta_+ = \left(\Lambda_0(2\alpha-1)\sigma_+^{1-2\alpha}\right)^{\frac{1}{2\alpha}}$.

Proof. We start by considering the saddlepoint equation. We are looking for $\hat{u}_+$ where the derivative of the integrand vanishes:

$$\frac{\partial}{\partial u}\left\{m(u-1,t) - ku\right\} = 0. \tag{4.35}$$

This expression can be rewritten

$$\frac{\partial}{\partial u} m(u-1,t) = k. \tag{4.36}$$

Using (4.24) we arrive at

$$\begin{aligned} m(u-1,t) &\sim \Lambda_0(\sigma_+(u_+ - u))^{1-2\alpha} \\ \frac{\partial}{\partial u} m(u-1,t) &\sim -\Lambda_0(1-2\alpha)\sigma_+^{1-2\alpha}(u_+ - u)^{-2\alpha} \\ u_+ - \hat{u}_+ &\sim \beta_+ k^{-\frac{1}{2\alpha}}. \end{aligned}$$

□

Remark 4.9. Let us briefly elaborate on how moment explosions emerge in the study of the model asymptotics. Recall that the saddlepoint equation (4.36) reads

$$\frac{\partial}{\partial u} m(u-1,T) = k.$$

This means that in order to apply saddlepoint method, we are looking for such $\hat{u}_+ = \hat{u}_+(k)$ that the derivative $\frac{\partial}{\partial u} m(u-1,T)\big|_{u=\hat{u}_+(k)}$ explodes as $k \to \pm\infty$. For the right-wing asymptotic $k \to +\infty$ the derivative must satisfy $\frac{\partial}{\partial u} m(u-1,T) \to +\infty$, which happens as m approaches positive critical moment from the left. For $k \to -\infty$ the derivative must satisfy $\frac{\partial}{\partial u} m(u-1,T) \to -\infty$, which happens as m approaches negative critical moment s from the right, since the derivative will have opposite sign as we move in the opposite direction on the real line. Thus the asymptotic behavior of the density as $k \to \pm\infty$ depends on $u_\pm$, respectively.

A major advantage from the previous proposition is that we have coupled the asymptotic $k \to +\infty$ with $\hat{u}_+ \to u_+$ by making $\hat{u}_+$ dependent on k. Thus the integral now only has one asymptotic parameter $k \to +\infty$. The integration contour $\hat{u}_+ \pm i\mathbb{R}$ moves closer to $u_+ \pm i\mathbb{R}$ as $k \to \infty$.

Now we want to obtain local expansion of the logarithm of the moment-generating function in the vicinity of the approximate saddlepoint.

Proposition 4.10. *Let $\hat{u}_+$ be the approximate saddlepoint characterized by the equation* (4.34)*. Consider a point $u = \hat{u}_+ + iy$ in the vicinity of the saddlepoint such that $|y| < k^{-\gamma}$ with $\frac{\alpha+1}{3\alpha} < \gamma$. Then the following local expansion around the approximate saddlepoint holds*

$$m(\hat{u}_+ - 1 + iy, t) = \frac{\beta_+}{2\alpha - 1} k^{1-\frac{1}{2\alpha}} + iyk - \alpha\beta_+^{-1} k^{1+\frac{1}{2\alpha}} y^2 + \Lambda_1 + O(k^{1+\frac{1}{\alpha}-3\gamma}). \tag{4.37}$$

Remark 4.11. Note that terms in the above expansion are indeed in decreasing order due to our choice of $|y| < k^{-\gamma}$ small.

Proof. By the definition of u we have $u_+ - u = u_+ - \hat{u}_+ - iy = \beta_+ k^{-\frac{1}{2\alpha}} - iy$. By our choice of y, we have $y^3 = O(k^{-3\gamma})$. Hence using the Taylor expansion of the power laws of $u_+ - u$ at the point $u_+ - \hat{u}_+$ we obtain

$$\begin{aligned}
(u_+ - u)^{1-2\alpha} &= (u_+ - \hat{u}_+)^{1-2\alpha} + (u_+ - \hat{u}_+)^{-2\alpha}(1-2\alpha)iy \\
&\quad + \alpha(1-2\alpha)(u_+ - \hat{u}_+)^{-1-2\alpha}(iy)^2 \\
&= \beta_+^{1-2\alpha} k^{1-\frac{1}{2\alpha}} - (1-2\alpha)\beta_+^{-2\alpha} iyk \\
&\quad + \alpha(1-2\alpha)\beta_+^{-1-2\alpha} y^2 k^{1+\frac{1}{2\alpha}} + O(k^{1+\frac{1}{\alpha}-3\gamma})
\end{aligned} \tag{4.38}$$

Analogous calculation yields

$$\begin{aligned}
(u_+ - u)^{1-\alpha} &= \beta_+^{1-\alpha} k^{\frac{1}{2}-\frac{1}{2\alpha}} - (1-\alpha)\beta_+^{-\alpha} iyk^{\frac{1}{2}} \\
&\quad + \frac{\alpha}{2}(1-\alpha)\beta_+^{-1-\alpha} y^2 k^{\frac{1}{2}+\frac{1}{2\alpha}} + O(k^{\frac{1}{2}+\frac{1}{\alpha}-3\gamma})
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
(u_+ - u)^{2-2\alpha} &= \beta_+^{2-2\alpha} k^{1-\frac{1}{\alpha}} - (2-2\alpha)\beta_+^{1-2\alpha} iyk^{1-\frac{1}{2\alpha}} \\
&\quad - \frac{1}{2}(1-2\alpha)(2-2\alpha)\beta_+^{-2\alpha} y^2 k + O(k^{1+\frac{1}{2\alpha}-3\gamma})
\end{aligned} \tag{4.40}$$

By our choice of the lower bound for γ, the asymptotic terms are $o(1)$. Moreover, by our choice of y all contributions from $(u_+ - u)^{2-2\alpha}$ and $(u_+ - u)^{1-\alpha}$ are $O(k^{1+\frac{1}{\alpha}-3\gamma})$, and by our choice of γ we have $k^{1+\frac{1}{\alpha}-3\gamma} = o(1)$. Hence plugging the expansions above into (4.24) we obtain the local expansion around the approximate saddlepoint

$$\begin{aligned}
m(\hat{u}_+ - 1 + iy, t) &= \Lambda_0 \sigma_+^{1-2\alpha} \beta_+^{1-2\alpha} k^{1-\frac{1}{2\alpha}} + iyk \\
&\quad + \Lambda_0 \sigma_+^{1-2\alpha} \alpha(1-2\alpha)\beta_+^{-1-2\alpha} y^2 k^{1+\frac{1}{2\alpha}} + \Lambda_1 + O(k^{1+\frac{1}{\alpha}-3\gamma}) \\
&= \frac{\beta_+}{2\alpha-1} k^{1-\frac{1}{2\alpha}} + iyk - \alpha\beta_+^{-1} k^{1+\frac{1}{2\alpha}} y^2 + \Lambda_1 + O(k^{1+\frac{1}{\alpha}-3\gamma})
\end{aligned}$$

□

4.3.2 Saddlepoint approximation of the density

In order to apply the saddlepoint method, we need to ensure that tails of the integrand (4.25) are asymptotically smaller than the part of the integrand near approximate saddlepoint and hence can be neglected. We first consider the estimate for the tail with fixed cutoff $\Im(u) > B > 0$. Thereafter, we establish asymptotics for the tail $\Im(u) > k^{-\gamma} > 0$, where the limit of integration $k^{-\gamma}$ also depends on the asymptotic parameter k.

Lemma 4.12. *Let $B > 0$. Then*

$$\left|\int_{\hat{u}+iB}^{\hat{u}+i\infty} e^{-ku+\bar{v}\lambda I^1\psi(u-1,t)+v_0 I^{1-\alpha}\psi(u-1,t)}du\right| = O\left(\exp\left(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}}\right)\right) \tag{4.41}$$

Proof. Note that by the definition (4.34) of the approximate saddlepoint we have

$$e^{-ku} = e^{-k(\hat{u}_+ + iy)} = e^{-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}} - iyk}. \tag{4.42}$$

For $\tilde{B} \gg B$ large enough it follows from equation (4.42) and Lemma 4.7 with $\left|e^{-iyk}\right| = 1$ that

$$\begin{aligned}\left|\int_{\hat{u}+i\tilde{B}}^{\hat{u}+i\infty} e^{-ku+\bar{v}\lambda I^1\psi(u-1,t)+v_0 I^{1-\alpha}\psi(u-1,t)}du\right| &\leq C\exp\left(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}}\right)\int_{\tilde{B}}^{\infty} e^{-Cy}dy \\ &= O\left(\exp\left(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}}\right)\right).\end{aligned}$$

For the portion of the integral between B and $\tilde{B}$ since we are integrating over a finite interval we have

$$\left|\int_{\hat{u}+iB}^{\hat{u}+i\tilde{B}} e^{-ku+\bar{v}\lambda I^1\psi(u-1,t)+v_0 I^{1-\alpha}\psi(u-1,t)}du\right| = O\left(e^{-\hat{u}_+ k}\right) = O\left(\exp\left(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}}\right)\right).$$

Thus the entire integral is $O\left(\exp\left(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}}\right)\right)$ which proves (4.41).

□

We now establish the tail estimate for $k^{-\gamma}$. For our construction we additionally pick tails $|y| > k^{-\gamma}$ such that $\gamma < \frac{3}{4\alpha}$. Combining this with a bound on γ imposed in Proposition 4.10 we have for γ

$$\frac{\alpha+1}{3\alpha} < \gamma < \frac{3}{4\alpha}.$$

Lemma 4.13. *The tail can be estimated by*

$$\begin{aligned}&\left|\int_{\hat{u}_++ik^{-\gamma}}^{\hat{u}_++i\infty} \exp(-ku+m(u-1,t))du\right| \\ &\qquad = \exp\left(-ku_+ + \frac{2\alpha\beta_+}{2\alpha-1}k^{1-\frac{1}{2\alpha}} + \alpha\beta_+^{-1}k^{1+\frac{1}{2\alpha}-2\gamma} + O(k^{1+\frac{3}{2\alpha}-4\gamma})\right).\end{aligned} \tag{4.43}$$

Proof. From the expansion (4.24) around the critical moment we know that

$$m(u-1,t) = \frac{\beta_+^{2\alpha}}{2\alpha-1}(u_+-u)^{1-2\alpha} + O(1)$$

Hence in the vicinity of u_+ inside the analyticity strip, i.e. $\exists B > 0 : \Im(u) < B$ and $\Re(u) > u_+ - B$ we can estimate the moment-generating function

$$|\exp(m(u-1,t))| \leq C\exp\left(\frac{\beta_+^{2\alpha}}{2\alpha-1}(u_+-u)^{1-2\alpha}\right).$$

for some constant $C > 0$. Note that from the expansion (4.38) we can write

$$\frac{\beta_+^{2\alpha}}{2\alpha-1}\Re\left\{(u_+-(\hat{u}_++ik^{-\gamma}))^{1-2\alpha}\right\} = \frac{\beta_+}{2\alpha-1}k^{1-\frac{1}{2\alpha}} - \alpha\beta_+^{-1}k^{1+\frac{1}{2\alpha}-2\gamma} + O(k^{1+\frac{3}{2\alpha}-4\gamma}). \tag{4.44}$$

Passing to the integral we thus obtain

$$\begin{aligned}
&\left|\int_{\hat{u}_+ + ik^{-\gamma}}^{\hat{u}_+ + iB} \exp(-ku + m(u-1,t))du\right| \\
&\quad \leq C\exp(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}}) \int_{k^{-\gamma}}^{B} \exp\left(\Re\left\{\frac{\beta_+^{2\alpha}}{2\alpha-1}(u_+ - (\hat{u}_+ + iy))^{1-2\alpha}\right\}\right) dy \\
&\quad \leq C\exp\left(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}} + \Re\left\{\frac{\beta_+^{2\alpha}}{2\alpha-1}(u_+ - (\hat{u}_+ + ik^{-\gamma}))^{1-2\alpha}\right\}\right) \\
&\quad = C\exp\left(-ku_+ + \beta_+ k^{1-\frac{1}{2\alpha}} + \frac{\beta_+}{2\alpha-1}k^{1-\frac{1}{2\alpha}} - \alpha\beta_+^{-1}k^{1+\frac{1}{2\alpha}-2\gamma} + O\left(k^{1+\frac{3}{2\alpha}-4\gamma}\right)\right) \\
&\quad = C\exp\left(-ku_+ + \frac{2\alpha\beta_+}{2\alpha-1}k^{1-\frac{1}{2\alpha}} - \alpha\beta_+^{-1}k^{1+\frac{1}{2\alpha}-2\gamma} + O\left(k^{1+\frac{3}{2\alpha}-4\gamma}\right)\right).
\end{aligned} \tag{4.45}$$

where we used Hölder inequality in the second inequality and equation (4.44) right after that. Since $B > 0$ it follows from Proposition 4.7 and equation (4.45) that the portion of the integral from the fixed tail is asymptotically smaller than the portion from the local expansion:

$$\int_{\hat{u}_+ + iB}^{\hat{u}_+ + i\infty} \exp(-ku + m(u-1,t))du = O\left(\int_{\hat{u}_+ + ik^{-\gamma}}^{\hat{u}_+ + iB} \exp(-ku + m(u-1,t))du\right).$$

Thus for the entire integral asymptotically holds

$$\begin{aligned}
\left|\int_{\hat{u}_+ + ik^{-\gamma}}^{\hat{u}_+ + i\infty} \exp(-ku + m(u-1,t))du\right| &\sim \left|\int_{\hat{u}_+ + ik^{-\gamma}}^{\hat{u}_+ + iB} \exp(-ku + m(u-1,t))du\right| \\
&= \exp\left(-ku_+ + \frac{2\alpha\beta_+}{2\alpha-1}k^{1-\frac{1}{2\alpha}} - \alpha\beta_+^{-1}k^{1+\frac{1}{2\alpha}-2\gamma} + O\left(k^{1+\frac{3}{2\alpha}-4\gamma}\right)\right).
\end{aligned}$$

□

We are now ready to address the asymptotic expansion of the Rough Heston density.

Proposition 4.14. *For any $\varepsilon > 0$ as $\log x \to +\infty$ the Rough Heston density $D_T(x)$ satisfies asymptotic relation*

$$D_T(x) = A_1 x^{-A_3} \exp\left(A_2 \log(x)^{1-\frac{1}{2\alpha}}\right) (\log x)^{-\frac{1}{2}-\frac{1}{4\alpha}} \left(1 + O(\log(x)^{1-\frac{5}{4}\alpha+\varepsilon})\right) \tag{4.46}$$

$$\begin{aligned} A_1 &= \frac{\exp(\Lambda_1)}{2\pi}\sqrt{\frac{\pi\beta_+}{\alpha}} \\ A_2 &= \frac{2\beta_+}{2\alpha - 1} \\ A_3 &= s_+(T) + 1. \end{aligned}$$

Proof. We start by shifting the contour to the approximate saddlepoint $\hat{u}_+$ in order to proceed with the saddlepoint method. We obtain the density as the inverse Mellin transform

$$\begin{aligned} D_T(x) &= \frac{1}{2\pi i}\int_{\hat{u}_+-i\infty}^{\hat{u}_++i\infty} e^{-ku+\bar{v}\lambda I^1\psi(u-1,t)+v_0 I^{1-\alpha}\psi(u-1,t)} du \\ &= \frac{e^{-k\hat{u}_+}}{2\pi}\int_{-\infty}^{+\infty} e^{-iyk+m(\hat{u}_++iy-1,t)} du \\ &= \frac{e^{-k\hat{u}_+}}{2\pi}\int_{-k^{-\gamma}}^{k^{-\gamma}} e^{-iyk+m(\hat{u}_++iy-1,t)} du + \text{tails} \\ &=: I + J. \end{aligned}$$

Note that from the definition of the saddlepoint in (4.34) we have $k\hat{u}_+ = ku_+ - \beta_+ k^{1-\frac{1}{2\alpha}}$. Hence the first part can be calculated explicitly from the expansion (4.37)

$$\begin{aligned}
I &= \frac{e^{-k\hat{u}_+}}{2\pi} \int_{-k^\gamma}^{k^\gamma} e^{-iyk+m(\hat{u}_+ + iy - 1, t)} du \\
&= \frac{e^{-ku_+}}{2\pi} \exp\left(\Lambda_1 + \frac{2\beta_+}{2\alpha-1} k^{1-\frac{1}{2\alpha}}\right) \left\{ \int_{-k^\gamma}^{k^\gamma} \exp\left(-\alpha\beta_+^{-1} k^{1+\frac{1}{2\alpha}} y^2\right) dy \right\} \left(1 + O\left(k^{1+\frac{1}{\alpha}-3\gamma}\right)\right) \\
&= \frac{e^{-ku_+}}{2\pi} \sqrt{\frac{\pi\beta_+}{\alpha}} \exp\left(\Lambda_1 + \frac{2\beta_+}{2\alpha-1} k^{1-\frac{1}{2\alpha}}\right) k^{-\frac{1}{2}-\frac{1}{4\alpha}} \left(1 + O\left(k^{1+\frac{1}{\alpha}-3\gamma}\right)\right).
\end{aligned}$$

Meanwhile, the tail estimate in Lemma 4.13 yields

$$J = \frac{e^{-ku_+}}{\pi} \exp\left(\frac{2\alpha\beta_+}{2\alpha-1} k^{1-\frac{1}{2\alpha}} + 2\alpha\beta_+^{-1} k^{1+\frac{1}{2\alpha}-2\gamma} + O(k^{1+\frac{3}{2\alpha}-4\gamma})\right).$$

By our choice of the constant γ, the asymptotic terms in both I and J are $o(1)$, whereas $k^{1+\frac{1}{2\alpha}-2\gamma} \to \infty$. Hence asymptotic decay of the tails dominates the power law decay of the main part and leads to $J = o(I)$.

Taking γ as close to upper bound $\frac{3}{4\alpha}$ as desired concludes the expansion.

□

Corollary 4.15. *As indicated in the Remark 4.9, the saddlepoint method can be applied in the identical fashion to consider the left-wing behavior of the model as $k \to -\infty$. By analogy we define $\sigma_- := T^*(s_-)' \geq 0$ and $\beta_- = \left(\Lambda_0(2\alpha - 1)\sigma_-^{1-2\alpha}\right)^{\frac{1}{2\alpha}}$. We then obtain for the left-wing asymptotics as $k \to -\infty$*

$$D_T(x) = B_1 x^{-B_3} \exp\left(B_2(-\log(x))^{1-\frac{1}{2\alpha}}\right)(-\log x)^{-\frac{1}{2}-\frac{1}{4\alpha}}\left(1 + O((-\log(x))^{1-\frac{5}{4}\alpha+\varepsilon})\right) \tag{4.47}$$

$$B_1 = \frac{\exp(\Lambda_1)}{2\pi}\sqrt{\frac{\pi\beta_-}{\alpha}}$$

$$B_2 = \frac{2\beta_-}{2\alpha - 1}$$

$$B_3 = s_-(T) + 1.$$

5 Extrapolation analytics

In this section we want to reap the benefits of the Proposition 4.14 and Corollary 4.15 by showing how asymptotics of the density translates to other material quantities. After ensuring that the asymptotic expansion of the Rough Heston density (4.25) satisfies certain analytical conditions, we apply asymptotic formulae by Gulisashvili to obtain asymptotics of the call price and implied variance.

5.1 Call pricing

We aim to establish asymptotics of a general call pricing function as defined in [Gul09]. We cite a result which translates asymptotics of the density to the asymptotics of a call pricing function. The result lends itself to the Rough Heston model.

First we present some generalities on regularly varying functions from [Gul09].

Definition 5.1. A Lebesgue measurable function f is called regularly varying with index $\alpha \in \mathbb{R}$ if for every $\lambda > 0$ holds $\frac{f(\lambda x)}{f(x)} \to \lambda^\alpha$ as $x \to \infty$.

Definition 5.2. Let $g(x)$ be a function on $(0, \infty)$ such that $g(x) \downarrow 0$ as $g \to \infty$. A Lebesgue measurable function f is called slowly varying with remainder $g(x)$ if f is regularly varying with $\alpha = 0$ and $\frac{f(\lambda x)}{f(x)} - 1 = O(g(x))$ for $x \to \infty$ for all $\lambda > 1$.

In the following we want to ensure that asymptotics of the Rough Heston density (4.46) factorize into power law and slowly varying function.

Lemma 5.3. *Function* $h(x) = A_1 \exp\left(A_2 \log(x)^{1-\frac{1}{2\alpha}}\right) (\log x)^{-\frac{1}{2}-\frac{1}{4\alpha}}$ *with positive constants* A_1 *and* A_2 *is slowly varying with remainder* $g(x) = (\log x)^{-\frac{1}{2\alpha}}$.

Proof. The proof is similar to Remark 6.1 in [Gul09].

We use that $-\frac{1}{2\alpha} \in \left(-\frac{1}{2}, -\frac{1}{4}\right)$ to show that

$$
\begin{aligned}
\frac{h(\lambda x)}{h(x)} &= \left(\frac{\log \lambda x}{\log x}\right)^{-\frac{1}{2}-\frac{1}{4\alpha}} \exp\left(A_2\left((\log \lambda x)^{1-\frac{1}{2\alpha}} - (\log x)^{1-\frac{1}{2\alpha}}\right)\right) \\
&= \left(1+\frac{\log \lambda}{\log x}\right)^{-\frac{1}{2}-\frac{1}{4\alpha}} \exp\left(A_2 (\log x)^{1-\frac{1}{2\alpha}}\left(\left(1+\frac{\log \lambda}{\log x}\right)^{1-\frac{1}{2\alpha}} - 1\right)\right) \\
&= \left(1+O(\log x)^{-1}\right)\left(1+O(\log x)^{-\frac{1}{2\alpha}}\right) = 1+O(\log x)^{-\frac{1}{2\alpha}}.
\end{aligned}
$$

□

We next cite Theorem 7.1 from [Gul09] which formulates the asymptotics of the call pricing function. The key insight is that the asymptotics of the call pricing function factorize in a similar way as asymptotics of the density. The exponent in the leading order power law of the call price is larger than the exponent in the asymptotics of the density by two, which heuristically manifests the fact that the density is the second derivative of the call price. We start with a technical definition of a general call pricing function which is used in Theorem 5.5.

Definition 5.4. Let C be a strictly positive function on $[0,\infty)^2$. The function C is called a *call pricing function* if

- For every $T \geq 0$ the function $K \to C(T,K)$ is convex.
- For every $T \geq 0$ the second distributional derivative μ_T of the function $K \mapsto e^{rT}C(T,K)$ is a Borel probability measure such that
$$\int_0^\infty x d\mu_T(x) = x_0 e^{rT}.$$
- For every $K \geq 0$, the function $T \to C\left(T, e^{rT}K\right)$ is non-decreasing.
- For every $K \geq 0, C(0,K) = (x_0 - K)^+$.
- For every $T \geq 0, \lim_{K\to\infty} C(T,K) = 0$.

Theorem 5.5. *(Gulisashvili, 2010) Let C be a call pricing function from Definition 5.4. Suppose that the spot process admits a density D_T. Assume that*

$$D_T(x) = x^\beta h(x)(1+O(\rho(x))) \qquad x\to\infty,$$

where $\beta < -2$, h is a slowly varying function with remainder g, and $\rho(x) \downarrow 0$ as $x \to \infty$. Then

$$C(K) = e^{-rT} \frac{1}{(\beta+1)(\beta+2)} K^{\beta+2} h(K)[1 + O(\rho(K)) + O(g(K))] \qquad K \to \infty.$$

In conclusion, we can apply Theorem 5.5 to the Rough Heston density and formulate asymptotics of the call pricing function.

Proposition 5.6. *Call pricing function C in the Rough Heston model satisfies*

$$C(K) = \frac{A_1}{(-A_3+1)(-A_3+2)} K^{-A_3+2} \exp(A_2(\log(K))^{1-\frac{1}{2\alpha}}) \log(K)^{-\frac{1}{2}-\frac{1}{4\alpha}} \left(1 + O\left(\log(K)^{1-\frac{5}{4}\alpha+\varepsilon}\right)\right). \tag{5.1}$$

Proof. According to Proposition 4.14 density of the Rough Heston model factors into

$$D_T(x) = x^{-A_3} h(x)(1 + O(\rho(x))$$

with $-A_3 = -(s_+ + 1)$, $h(x) = A_1 \exp\left(A_2 \log(x)^{1-\frac{1}{2\alpha}}\right) (\log x)^{-\frac{1}{2}-\frac{1}{4\alpha}}$ and $\rho(x) = \log(K)^{1-\frac{5}{4}\alpha+\varepsilon}$.

Note that $-A_3 = -(s_+ + 1) < -2$. Also due to Lemma 5.3 we know that $h(x)$ is a slowly varying function with remainder $g(x) = (\log x)^{-\frac{1}{2\alpha}}$. In addition, for $\alpha \in (1/2, 1)$ and ε small we have

$$O(\rho(K)) + O(g(K)) = O(\log x)^{-\frac{1}{2\alpha}} + O(\log x)^{1-\frac{5}{4}\alpha+\varepsilon} = O(\log x)^{1-\frac{5}{4}\alpha+\varepsilon}.$$

Hence the conditions of Theorem 5.5 are satisfied and we obtain (5.1).

□

5.2 Implied variance smile

In Section 3 we introduced the Black–Scholes model. The price of the European call option in the Black–Scholes model is given by

$$C_{BS}(K,T,\sigma) = N(d_1)\,S_0 - N(d_2)\,Ke^{-rT}$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

where r is the risk-free rate, S_0 is the current value of the spot, $K > 0$ is strike, σ is the Black–Scholes volatility, $T > 0$ is maturity and N is the cumulative distribution function of the standard normal distribution. For a given call option price $C > 0$, the Black–Scholes *implied volatility* $\sigma_{BS}(K,T)$ is a unique numerical value that matches

$$C = C_{BS}(K,T,\sigma_{BS}(K,T)). \tag{5.2}$$

In practice options are often quoted in terms of their Black–Scholes implied volatilities rather than their prices [Hag+02]. In this section we aim to obtain asymptotics of the implied variance $\sigma_{BS}^2(K,T)$ for extreme values of the strike $\log K \to \pm\infty$.

Let $\bar{K} > 0$, and let f and g be positive functions on the interval $[\bar{K},\infty)$. We will write $f(x) \approx g(x)$ for $x \to \infty$, if there exist constants $c_1 > 0, c_2 > 0$, and $K_0 > \bar{K}$ such that the inequalities $c_1 g(x) \leq f(x) \leq c_2 g(x)$ hold for all $x > K_0$.

The following statement was established in [GS09b] as Lemma 3.2.

Theorem 5.7. *Suppose ϕ and ψ are positive increasing functions with*

$$\lim_{K\to\infty}\psi(K) = \lim_{K\to\infty}\phi(K) = \infty$$

and such that

$$C(K) \approx \frac{\psi(K)}{\phi(K)}\exp\left\{-\frac{\phi(K)^2}{2}\right\}. \tag{5.3}$$

Then

$$I(K) = \frac{1}{\sqrt{T}}(\sqrt{2\log K + \phi(K)^2} - \phi(K)) + O\left(\frac{\psi(K)}{\phi(K)}\right) \qquad \textit{as } K \to \infty.$$

The following proposition establishes asymptotics of the right wing of the implied variance surface in the Rough Heston model. The proposition is proved along the lines of Lemma 10.1 in [GS09a].

Proposition 5.8. *The implied variance in the Rough Heston model satisfies*

$$\sigma_{BS}(k,T)^2 T = \left(\beta_1 k^{1/2} + \beta_2 k^{\frac{1}{2}-\frac{1}{2\alpha}} + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)\right)^2 \qquad \text{as } k \to +\infty \quad (5.4)$$

$$\beta_1 = \frac{\sqrt{2}}{\sqrt{T}}\left(\sqrt{A_3 - 1} - \sqrt{A_3 - 2}\right)$$

$$\beta_2 = \frac{A_2}{\sqrt{2T}}\left(\frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}}\right)$$

$$\beta_3 = \frac{1}{\sqrt{2T}}\left(1 + \frac{1}{2\alpha}\right)\left(\frac{1}{\sqrt{A_3 - 1}} - \frac{1}{\sqrt{A_3 - 2}}\right)$$

Proof. Let ψ be any positive increasing function on $(0, \infty)$.

From Proposition 5.6 we can write

$$C(K) \approx K^{-A_3+2} \exp\left(A_2 (\log K)^{1-\frac{1}{2\alpha}}\right) (\log K)^{-\frac{1}{2}-\frac{1}{4\alpha}} \quad (5.5)$$

We see that it holds

$$\phi(K) = \sqrt{(2A_3 - 4)\log K - 2A_2(\log K)^{1-\frac{1}{2\alpha}} + \left(1 + \frac{1}{2\alpha}\right)\log\log K - 2\log\psi(K)} \quad (5.6)$$

Observe that plugging (5.5) and (5.6) into (5.3) works. Hence also $\phi(K) \sim \sqrt{k}$, hence requirement of the lemma fulfilled.

Application of the Theorem 5.7 and the mean value theorem implies

$$
\begin{aligned}
\sigma_{BS}(k,T)\frac{\sqrt{T}}{\sqrt{2}} &= \sqrt{(A_3-1)\log K - A_2(\log K)^{1-\frac{1}{2\alpha}} + \left(1+\frac{1}{2\alpha}\right)\log\log K} \\
&\quad - \sqrt{(A_3-2)\log K - A_2(\log K)^{1-\frac{1}{2\alpha}} + \left(1+\frac{1}{2\alpha}\right)\log\log K} \\
&\quad + O\left(\frac{\psi(K)}{\sqrt{\log K}}\right) \qquad \text{as } K \to \infty.
\end{aligned}
$$

Application of the Taylor expansion $\sqrt{1-h} = 1 - \frac{1}{2}h + O\left(h^2\right)$ as $h \downarrow 0$ finalizes the proof.

□

Corollary 5.9. *Building on the results from Corollary 4.15 concerning left-wing asymptotic of the density, in the identical fashion we can translate asymptotic density expansion of the left wing in the Rough Heston to the left wing asymptotic of the implied variance. Thus the implied variance in the Rough Heston model as $k \to -\infty$ satisfies*

$$
\sigma_{BS}(k,T)^2 T = \left(\rho_1 k^{1/2} + \rho_2 k^{\frac{1}{2}-\frac{1}{2\alpha}} + \rho_3 \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)\right)^2 \tag{5.7}
$$

$$
\begin{aligned}
\rho_1 &= \frac{\sqrt{2}}{\sqrt{T}}\left(\sqrt{B_3+2} - \sqrt{B_3+1}\right) \\
\rho_2 &= \frac{B_2}{\sqrt{2T}}\left(\frac{1}{\sqrt{B_3+1}} - \frac{1}{\sqrt{B_3+2}}\right) \\
\rho_3 &= \frac{1}{\sqrt{2T}}\left(1+\frac{1}{2\alpha}\right)\left(\frac{1}{\sqrt{B_3+2}} - \frac{1}{\sqrt{B_3+1}}\right).
\end{aligned}
$$

6 Numerical illustration

In this section we verify the implied variance asymptotics derived in Proposition 5.8 by a numerical experiment. For that we fix the timeframe and calculate prices of vanilla options in the Rough Heston model at extreme strikes with transform methods. We then use prices of the vanilla instruments to calculate the Black–Scholes implied volatilities with a simple root-finding procedure. We will conclude that the asymptotic expansion (5.7) does indeed give rise to a qualitatively better approximation than the first-order asymptotics resulting from the Lee's moment formula. In this section we first describe numerical procedures involved in the pricing of the vanilla options and present the resulting plots at the end of the section.

Table 2 summarizes parameter configuration used throughout this section (unless specified otherwise). We use the values of v_0, $\bar{v}$, λ, ξ and ρ from the table below for the classical Heston model, and then roughen it by setting $\alpha = 0.9$.

Parameter	Interpretation	Value
v_0	initial variance	0.05
$\bar{v}$	long-term mean variance	0.05
λ	mean-reversion speed	0.6
ξ	volatility of volatility	0.6
ρ	spot-variance correlation	-0.8
α	roughness	0.9
S_0	spot	1
r	risk-free rate	0

Table 2: Configuration of model and market parameters of the Rough Heston model.

6.1 Numerical constituents

In the following we describe numerical procedures involved in the pricing of vanilla options. In order to ensure that our implementation of the numerical constituents is valid, we additionally perform sanity checks by considering a limiting case of the Rough Heston model with $\alpha = 0.999$. We expect the results to be close to the analytical results of classical Heston model corresponding to $\alpha = 1$.

6.1.1 Characteristic function

Fractional Riccati equation The characteristic function of the Rough Heston model results from the solution of non-linear fractional Riccati equation (3.9). Numerical methods for this type of equations are based on the fact that it can be rewritten as a Volterra integral equation, as indicated in the Proposition 3.8. We thus consider a function h, which solves the integral equation

$$h(a,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R(a, h(a,s)) ds$$

$$h(\cdot, 0) = 0$$
$$R(s,w) = \frac{1}{2}\left(s^2 - s\right) + (\rho\xi s - \lambda) w + \frac{1}{2}\xi^2 w^2.$$

In order to solve the equation, we employ fractional Adams method. In the following we present the numerical scheme as described in [ER16].

Let us write $g(a,t) = R(a, h(a,t))$. We use the equidistant time grid $(t_k)_{k\in\mathbb{N}}$ with mesh size Δ, i.e. $t_k = k\Delta$, and consider the trapezoid rule

$$\hat{g}(a,t) = \frac{t_{j+1} - t}{t_{j+1} - t_j} \hat{g}\left(a, t_j\right) + \frac{t - t_j}{t_{j+1} - t_j} \hat{g}\left(a, t_{j+1}\right), \quad t \in \left[t_j, t_{j+1}\right), \quad 0 \le j \le k$$

to estimate

$$h\left(a, t_{k+1}\right) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} \left(t_{k+1} - s\right)^{\alpha-1} g(a,s) ds \approx \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} \left(t_{k+1} - s\right)^{\alpha-1} \hat{g}(a,s) ds.$$

This leads to the numerical scheme

$$\hat{h}\left(a, t_{k+1}\right) = \sum_{0\le j\le k} a_{j,k+1} R\left(a, \hat{h}\left(a, t_j\right)\right) + a_{k+1,k+1} R\left(a, \hat{h}\left(a, t_{k+1}\right)\right)$$

with

$$
\begin{aligned}
a_{0,k+1} &= \frac{\Delta^\alpha}{\Gamma(\alpha+2)}\left(k^{\alpha+1} - (k-\alpha)(k+1)^\alpha\right) \\
a_{j,k+1} &= \frac{\Delta^\alpha}{\Gamma(\alpha+2)}\left((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}\right), \quad 1 \le j \le k \\
a_{k+1,k+1} &= \frac{\Delta^\alpha}{\Gamma(\alpha+2)}
\end{aligned}
$$

The resulting numerical scheme is implicit, since $\hat{h}(a, t_{k+1})$ appears on the both sides of the equation. Therefore, we compute the initial estimate of $\hat{h}(a, t_{k+1})$ as a Riemann sum and afterwards plug in the trapezoidal quadrature. We define this pre-estimation by

$$
\begin{aligned}
\hat{h}^P(a, t_{k+1}) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\tilde{g}(a,s)ds, \\
\tilde{g}(a,t) &= \hat{g}(a, t_j), \quad t \in [t_j, t_{j+1}), \quad 0 \le j \le k.
\end{aligned}
$$

It leads to

$$
\begin{aligned}
\hat{h}^P(a, t_{k+1}) &= \sum_{0 \le j \le k} b_{j,k+1} F\left(a, \hat{h}(a, t_j)\right), \\
b_{j,k+1} &= \frac{\Delta^\alpha}{\Gamma(\alpha+1)}\left((k-j+1)^\alpha - (k-j)^\alpha\right), \quad 0 \le j \le k.
\end{aligned}
$$

The ultimate explicit numerical scheme is thus given by

$$
\begin{aligned}
\hat{h}(a, t_{k+1}) &= \sum_{0 \le j \le k} a_{j,k+1} F\left(a, \hat{h}(a, t_j)\right) + a_{k+1,k+1} F\left(a, \hat{h}^P(a, t_j)\right) \\
\hat{h}(a, 0) &= 0.
\end{aligned}
$$

Convergence of the method is addressed in [LT09]. Specifically, for $t > 0$ and $a \in \mathbb{R}$ holds

$$
\max_{t_j \in [0,t]} \left|\hat{h}(a, t_j) - h(a, t_j)\right| = o(\Delta).
$$

Fractional integration We recall from (3.8) that the logarithm of moment-generating function in the Rough Heston model can be obtained from the solution of the fraction Riccati equation by considering

$$m(s,t) = \log \mathbb{E}\left[e^{sX_t}\right] = \bar{v}\lambda I^1\psi(s,t) + v_0 I^{1-\alpha}\psi(s,t).$$

This requires fractional integration of the numerical solution of the characteristic function. We consider the simplest method to numerically evaluate the fractional derivatives appearing in [Pod98]. Specifically, the fractional derivative can be calculated using the scheme

$$D_t^\alpha f(t) \approx \Delta_h^\alpha f(t) \Delta_h^\alpha$$

$$f(t) = \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t-jh)$$

where $[x]$ is the integer part of x. Fractional integrals with $\alpha > 0$ correspond to the negative exponents in the above scheme reading $I^\alpha = D^{-\alpha}$.

Figure 2 demonstrates how the characteristic functions for values of α close to one compare to the explicit formula of characteristic function corresponding to the classical Heston model. We see that the characteristic function of the Rough Heston model gradually approaches the characteristic function of the Heston model for $\alpha \to 1$, as anticipated.

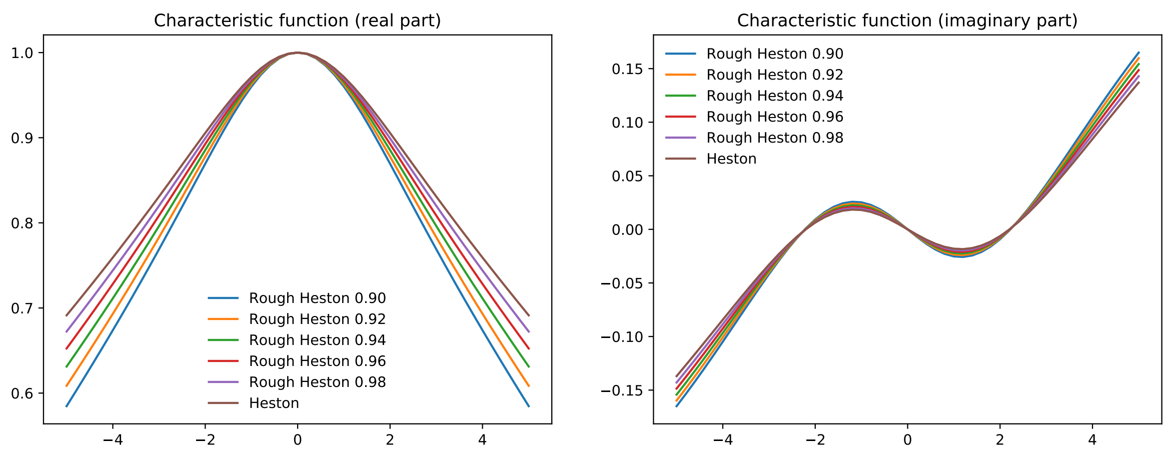

Figure 2: Real and imaginary part of the characteristic functions of the classical Heston model and Rough Heston model with different configurations of roughness parameter α.

6.1.2 Option pricing with transform methods

Recalling discussion from Section 3.2 we point out once again owning to its affine structure the Rough Heston model features semi-closed formula for the characteristic function identified by the fraction Riccati equation (3.9). Models with closed or semi-closed formula for characteristic function lend themselves to option pricing by transform methods which reduces computational time. In the following we elaborate on the pricing formulae described in [CM01].

We fix the timescale $T > 0$ corresponding to option maturity. We write $f(u) = \mathbb{E}\left[e^{iu \log S_T}\right]$ for the characteristic function of $\log S_T$. We work with the log-strike $k = \log K$ and consider the call option price $C(k) = \mathbb{E}\left[(S_T - e^k)_+\right]$.

Fourier transform of the call price We want to consider the Fourier transform of the call option price $C(k)$. However, Fourier transform of the call option price does not exist since $C(k) \to \infty$ as $k \to -\infty$. Hence we need to consider the "damped" option price. We use the following convention for the Fourier transform.

Definition 6.1. The one-dimensional Fourier transform of a Lebesgue integrable function $f \in L^1(\mathbb{R}^n)$ and its inverse are given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi.$$

Lemma 6.2. *For the option price $C(k)$ we have for any $p > 0$ the bounds*

$$\begin{aligned} C(k) &\leqslant \frac{B_0\mathbb{E}\left[\exp((p+1)\log S_T)\right]}{(p+1)\exp(pk)}\left(\frac{p}{p+1}\right)^p \\ C(k) &\leqslant B_0\mathbb{E}\left[S_T\right] \end{aligned} \tag{6.1}$$

Proof. It holds for all $s > 0$

$$s - e^k \leq \frac{s^{p+1}}{(p+1)\exp(pk)}\left(\frac{p}{p+1}\right)^p$$

since the left-hand side and right-hand side have the same value and first derivative at $s = (p+1)\exp(k)/p$ and second derivative of the right-hand side is everywhere positive. Making use of $(s-e^k)_+ \leq s-e^k$, replacing s with $\log S_T$ and taking expectations yields the first inequality. The second bound is trivial.

□

As we mentioned before, the option price does not decay as $k \to -\infty$. Hence Fourier transform of C does not exist. We therefore introduce the damping constant $\theta > 0$ in order for the Fourier transform to exist:

$$c_\theta(k) := \exp(\theta k)C(k).$$

The damped call option price can be used to calculate the actual call option price with Fourier transform.

Proposition 6.3. *For the call option price* $C(k) = \mathbb{E}\left[(S_T - e_k)_+\right]$ *we have*

$$C(k) = \frac{\exp(-\theta k)}{\pi} \int_0^\infty \Re\left[e^{-iuk}\hat{c}_\theta(u)\right] du \tag{6.2}$$

$$\hat{c}_\theta(u) = \frac{f(-u + i(\theta+1))}{\theta^2 + 2\theta i u - u^2 + \theta + iu}. \tag{6.3}$$

Proof. Bounds in (6.1) imply that $c_\theta(k)$ decays exponentially for $k \to \pm\infty$. Since $c(k)$ is also bounded, it is in L^1 and has a Fourier transform.

With Fubini theorem we obtain the Fourier transform of the damped option price

$$\begin{aligned}
\hat{c}_\theta(u) &= \int_\infty^\infty e^{iuk} c_\theta(k) dk \\
&= \int_0^\infty \int_{-\infty}^\infty (e^x - e^k)_+ e^{(\theta+ui)k} dk dx \\
&= \int_{-\infty}^\infty e^{(\theta+iu)k} \left(\int_k^\infty (e^x - e^k) f(x) dx \right) dk \\
&= \int_{-\infty}^\infty e^x f(x) \int_{-\infty}^x e^{(\theta+iu)k} dk dx - \int_{-\infty}^\infty e^k f(x) \int_{-\infty}^x e^{(\theta+iu)k} dk dx \qquad (6.4) \\
&= \int_{-\infty}^\infty \frac{e^{(\theta+iu+1)x}}{\theta+iu} f(x) dx - \int_\infty^\infty \frac{e^{(\theta+iu+1)x}}{\theta+iu+1} f(x) dx \\
&= \left(\frac{1}{\theta+iu} - \frac{1}{\theta+iu+1} \right) \mathbb{E}\left[e^{(\theta+iu+1)\log S_T}\right] \\
&= \frac{f(-u + i(\theta+1))}{\theta^2 + 2\theta i u - u^2 + \theta + iu}.
\end{aligned}$$

The damped call option price is continuous and the transform is L^1 since

$$|\hat{c}_\theta(u)| \leq \frac{|f(-(\theta+1)i)|}{(u^2+\theta^2)}.$$

Hence the option price can be recovered by the inverse Fourier transform

$$
\begin{aligned}
C(k) &= \exp(-\theta k)c(k) \\
&= \frac{\exp(-\theta k)}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\hat{c}_\theta(u)du \\
&= \frac{\exp(-\theta k)}{\pi}\int_{0}^{\infty} \Re\left[e^{-iuk}\hat{c}_\theta(u)\right]du.
\end{aligned}
$$

The last equation results from the complex conjugation.

□

Remark 6.4. Note that positive values of θ facilitate integration as $k \to -\infty$, while negative values of θ ensure decay of the call price in the opposite direction $k \to +\infty$. Thus a calculation along the lines of (6.4) implies that for the negative values of θ we recover put prices with the same formula.

We also point out that while positive values of θ ensure that the call price decays for negative log-strikes and facilitate integration of (6.2) as $k \to -\infty$, they impair integrability in the opposite direction $k \to +\infty$. In order for the call price to be integrable on the entire real line, it is sufficient to satisfy the upper bound

$$
\mathbb{E}\left[S_T^{\theta+1}\right] < \infty.
$$

Optimal choice of the damping parameter θ is all the more important for the calculation of the implied volatilities for extreme strikes. Consider example of a call option which pays $(S_T - K)_+$ at maturity. Price of a call option consists of the intrinsic value corresponding to its payoff $(S_T - K)_+$ at maturity and time value of the option corresponding to potential future earnings due to volatility in the spot price. Time value of the option is sensitive to volatility (typically grows as the volatility grows) while the intrinsic value is not.

Figure 3 demonstrates a typical pattern of intrinsic and time values of a call option. The difference of the actual option price and the intrinsic value is the time value of the option.

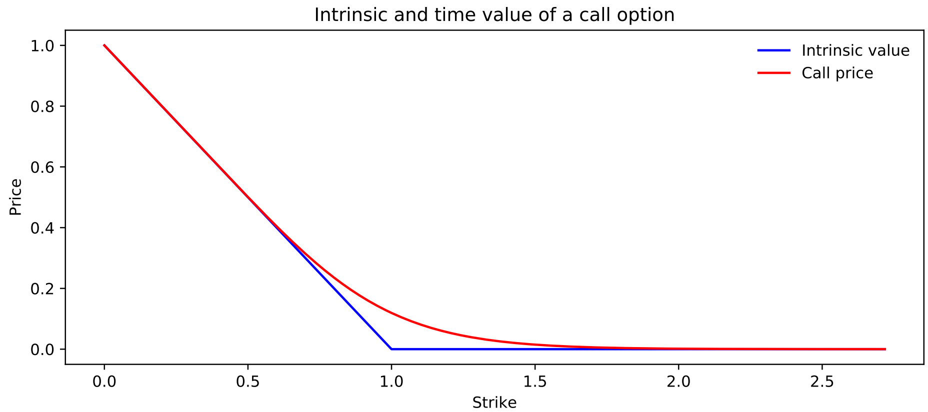

Figure 3: Intrinsic and time value of a call option struck at $K = 1$.

For deep in-the-money options we have $C \approx S_T$ and for the deep out-of-the-money options we have $C \approx 0$. Thus for deep-in-the-money options sensitivity to volatility is negligible in relative terms (large changes in volatility barely affect the call price $C \approx S_T$) and it is more challenging to imply volatilities (5.2) using the deep-in-the-money options. On the other hand, for deep out-of-the-money options the intrinsic value of the option is zero and their price reflects the solely time value of the option, whence it is easier to imply volatility. Thus deep out-of-the-money puts should be used for the calculation of the implied volatilities for negative log-strikes instead.

Accordingly, appropriate choice of the damping constant θ is imperative to avoid negative option prices at extreme strikes and ensure that a root-finding procedure solving (5.2) produces reasonable implied volatility smile.

Option pricing with fast Fourier transform In order to efficiently compute the integral (6.2), we want to rewrite it in terms of the discrete Fourier transform. It would allow us to apply the fast Fourier transform algorithm (FFT). Fast Fourier transform is a divide-and-conquer type of algorithm that elegantly uses roots of unity to reduce the calculation time from an order of $O(N^2)$ to an order of $O(N \log N)$.

Definition 6.5. For a vector $x(j)$ with $j \in \{1, \ldots, N\}$ the *discrete Fourier transform (DFT)* is given by

$$w(k)=\sum_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}(j-1)(k-1)}x(j)\quad \text{for } k=1,\ldots,N. \tag{6.5}$$

We now aim to rewrite the integral (6.2) corresponding to the option price in the shape of discrete Fourier transform (6.5). Note that the integral can be discretized

$$C_T(k)\approx\frac{\exp(-\theta k)}{\pi}\sum_{j=1}^{N}e^{-iv_jk}\psi_T\left(v_j\right)\eta$$

with the step size η and $v_j=\eta(j-1)$ for $j\in\{1,\ldots,N\}$. We consider a range of log-strikes with spacing $\lambda>0$

$$k_u=-b+\lambda(u-1)\quad \text{for } u=1,\ldots,N$$

for $b=\frac{1}{2}N\lambda$ and aim to calculate call prices

$$C_T\left(k_u\right)\approx\frac{\exp\left(-\theta k_u\right)}{\pi}\sum_{j=1}^{N}e^{-iv_j[-b+\lambda(u-1)]}\psi_T\left(v_j\right)\eta\quad \text{for } u=1,\ldots,N$$

By construction, the set of log-strikes k_u ranges from $-b$ to b. We can plug in $v_j=(j-1)\eta$ and set a constraint

$$\lambda\eta=\frac{2\pi}{N} \tag{6.6}$$

to obtain the call price in the form of discrete Fourier transform

$$\begin{aligned}C_T\left(k_u\right)&\approx\frac{\exp\left(-\theta k_u\right)}{\pi}\sum_{j=1}^{N}e^{-i\lambda\eta(j-1)(u-1)}e^{ibv_j}\psi_T\left(v_j\right)\eta\\&=\frac{\exp\left(-\theta k_u\right)}{\pi}\sum_{j=1}^{N}e^{-i\frac{2\pi}{N}(j-1)(u-1)}e^{ibv_j}\psi_T\left(v_j\right)\eta.\end{aligned}$$

In the procedure above the number of the log-strikes computed and the grid spacing are coupled by the reciprocity relation (6.6). Thus in order to increase the calculation precision one needs to incorporate more call prices. However, computational advantage created by fast Fourier transform is so significant that calculation of a larger number of call prices with fast Fourier transform algorithm is often considerably faster than calculating the sums (6.5) directly.

6.2 Implied variance asymptotics

In order to ensure that our implementation is correct, we first perform a sanity check by reconciling the classical Heston implied variance asymptotics from [Fri+10] with the asymptotics obtained in (5.4) and (5.7).

Figure 4 shows the variance smiles of the Heston model and Rough Heston model with $\alpha = 0.999$ generated by matching the implied volatility of options priced with transform methods as elaborated in Section 6.1.2. As suggested before, we use out-of-the-money call options for positive log-strikes and out-of-the-money put options for negative log-strikes to imply volatilities. We use explicit formulae from [Hes93] for the characteristic function of the Heston model. The characteristic function of the Rough Heston model was calculated by means of the fractional Adams method described in Section 6.1.

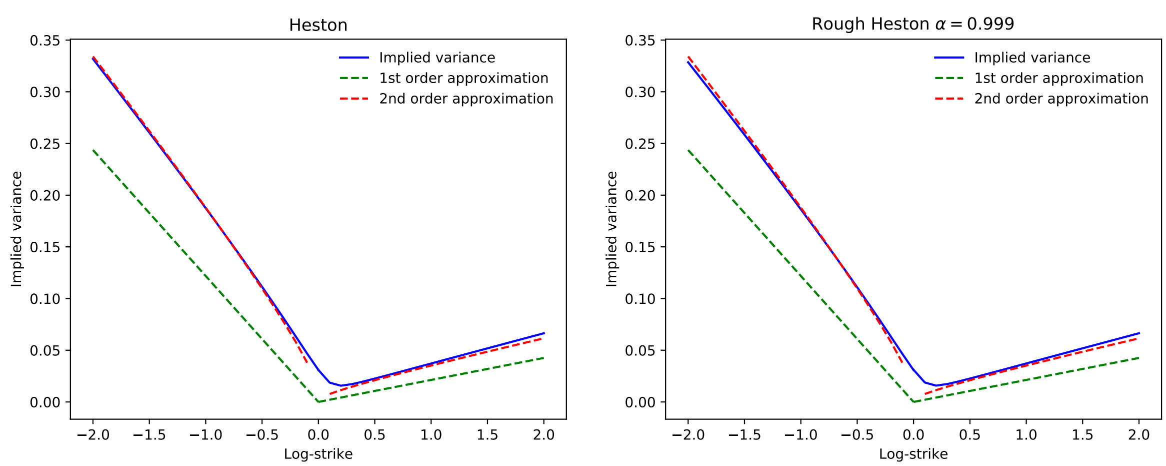

Figure 4: Variance smile, first-order asymptotic and second-order asymptotics of the classical Heston model and Rough Heston model with $\alpha = 0.999$.

The leading order approximation in the Figure 4 is given by the Lee's moment formula [Lee04a] which establishes one-to-one correspondence between tail slope and finite moments of the log-spot. The first-order implied variance asymptotic for large strikes reads

$$\sigma_{BS}(k,T)^2 T \sim \Psi\left(s_+ - 1\right) \times k$$

where $\Psi(x) = 2 - 4(\sqrt{x^2 - x} - x)$. The higher-order expansion of the left and right wing of the implied variance smile in the classical Heston model has been obtained in [Fri+10] and is given by

$$\sigma_{BS}(k,T)^2 T = \left(\beta_1^{\pm} k^{1/2} + \beta_2^{\pm} + \beta_3^{\pm} \frac{\log k}{k^{1/2}} + O\left(\frac{1}{k^{1/2}}\right)\right)^2 \qquad \text{as} \qquad k \to \pm\infty$$

where $\beta_1^{\pm}, \beta_2^{\pm}$ and $\beta_3^{\pm}$ are corresponding constants of the classical Heston model. Equivalent higher-order expansion of the implied variance smile in the Rough Heston model is given by (5.7).

It is evident from Figure 4 that the Rough Heston model with $\alpha = 0.999$ practically recovers the results of the classical Heston model (correspoding to $\alpha = 1$) obtained in [Fri+10], and we conclude that our implementation satisfies the limiting case.

Having established robustness of our implementation, we roughen the model by setting $\alpha = 0.9$ all other things equal. Figure 5 demonstrates the Rough Heston implied variance smile and corresponding smile asymptotics. We conclude that the second-order approximation provides significant improvement over the first-order asymptotics and allows for more accurate modelling of the volatility smile at extreme strikes.

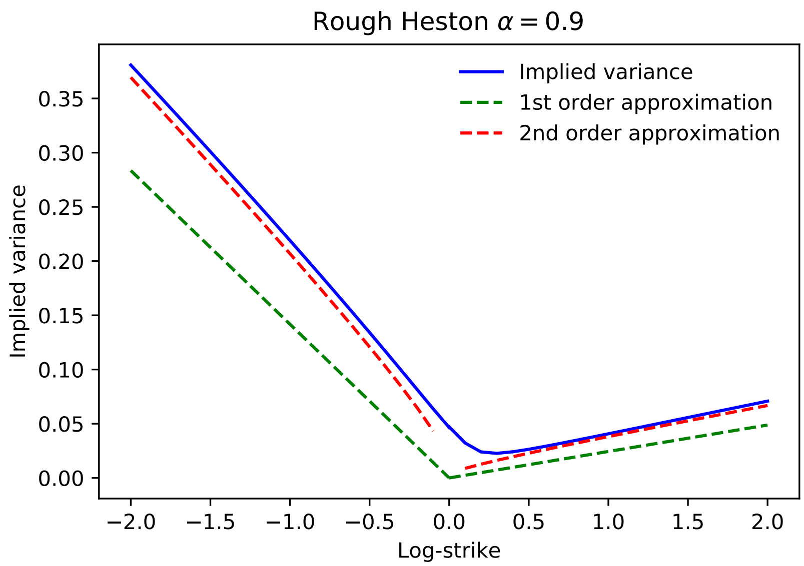

Figure 5: Variance smile, first-order asymptotic and second-order asymptotic of the Rough Heston model with $\alpha = 0.9$.

7 Conclusion

In this thesis we have obtained a refined asymptotic expansion of the marginal density in the Rough Heston model. The analysis relies on the asymptotic expansion of the Volterra integral equation near explosion. Asymptotic approximation of the marginal density is subsequently obtained with the saddlepoint method. The results feature expansion of the Rough Heston marginal density at infinity and near the origin. As anticipated, asymptotics of the density factorize into leading order power law and a slowly varying function. Constants in the asymptotic expansion of the density depend solely on model parameters, the critical moment $s_{\pm}$ and the critical slope $\sigma_{\pm}$.

The marginal density asymptotics translate naturally to higher-order asymptotics of the implied variance smile. The obtained approximations demostrate a better fit for the implied volatility smile asymptotics than the first-order approximation with Lee's moment formula. Therefore, the results allow for more accurate modelling of volatility surface at extreme strikes.

Importantly, the results generalize equivalent asymptotic expansions for the classical Heston case. In particular, with the Hurst parameter $H = \frac{1}{2}$ the results are conformal with the analysis of the classical Heston model in [Fri+10].

The verification of the assumptions imposed on the asymptotic expansion of the solution of the fractional Riccati equation listed in Remark 4.2 and assumptions concerning differentiability of the critical time in the Rough Heston model in Proposition 4.5 is left for future investigation. They would entail further examination of the blow-up behavior of the Volterra integral equation (3.10).

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Declaration of Authorship

I hereby confirm that I, Konstantins Starovoitovs, have authored this master thesis independently and without use of others than the indicated sources. Where I have consulted the published work of others, in any form (e.g. ideas, equations, figures, text, tables), this is always explicitly attributed.

July 2, 2020
Berlin

Konstantins Starovoitovs

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Hiermit erkläre ich, Konstantins Starovoitovs, dass ich die vorliegende Arbeit allein und nur unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe. Die Prüfungsordnung ist mir bekannt. Ich habe in meinem Studienfach bisher keine Masterarbeit eingereicht bzw. diese nicht endgültig nicht bestanden.

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