第二章 一阶微分方程的初等解法

§2.3.1 恰当微分方程

引例

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$ydx + xdy = 0$$

$$y = ce^{\int -\frac{1}{x}dx} = \frac{c}{x}$$

$$d(xy) = 0$$

$$xy = c$$

一、恰当方程——定义

$$\begin{tikzcd} f(x,y)dx-dy=0 \arrow[d] & f(x,y)dx=dy \arrow[l] \\ M(x,y)dx+N(x,y)dy=0 & \dfrac{dy}{dx}=f(x,y) \arrow[l, "\text{改写}"] \arrow[u] \end{tikzcd}$$

定义 若有函数$u(x,y)$，使得 $du(x,y)=M(x,y)dx+N(x,y)dy$，

则称微分方程

$$M(x,y)dx+N(x,y)dy=0 \qquad (1)$$

是恰当方程. 此时(1)的通解为$u(x,y)=c$.

如： $d(xy)=xdy+ydx=0$

$$d(x^3y+xy^2)=(3x^2y+y^2)dx+(x^3+2xy)dy=0$$

二、恰当方程——充要条件

需考虑的问题 $\boxed{M(x,y)dx+N(x,y)dy=0 \qquad (1)}$

- 方程(1)是否为恰当方程?
- 若(1)是恰当方程, 怎样求解?
- 若(1)不是恰当方程, 有无可能转化为恰当方程求解?

方程(1)为恰当方程的充要条件

定理1 设函数$M(x,y)$和$N(x,y)$在一个矩形域R内连续且具有连续的一阶偏导数,则方程

$$M(x,y)dx+N(x,y)dy=0 \qquad (1)$$

为恰当方程的充要条件是 $\dfrac{\partial M(x,y)}{\partial y}=\dfrac{\partial N(x,y)}{\partial x}. \qquad (2)$

二、恰当方程——必要条件证明

证明："必要性"　$\boxed{M(x,y)dx+N(x,y)dy=0 \qquad (1)}$

设(1)是恰当方程,则有函数$u(x,y)$,使得

$$du(x,y)=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy \ =M(x,y)dx+N(x,y)dy$$

故有 $$\frac{\partial u}{\partial x}=M(x,y), \qquad \frac{\partial u}{\partial y}=N(x,y)$$

从而 $$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x},\frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$$

由于$\frac{\partial^2 u}{\partial y\partial x}$和$\frac{\partial^2 u}{\partial x\partial y}$都是连续的,从而 $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$

故(1)是恰当方程的必要条件为 $\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}.$

二、恰当方程——充分条件证明

“充分性” $\boxed{M(x,y)dx+N(x,y)dy=0 \qquad (1)}$

若 $\dfrac{\partial M(x,y)}{\partial y}=\dfrac{\partial N(x,y)}{\partial x}$,

则需构造函数$u(x,y)$, 满足

$$du(x,y)=M(x,y)dx+N(x,y)dy, \qquad (3)$$

即应满足 $\dfrac{\partial u}{\partial x}=M(x,y), \qquad (4)$

$$\frac{\partial u}{\partial y}=N(x,y). \qquad (5)$$

从(4)出发, 把y看作参数, 解这个方程得

$$u(x,y)=\int M(x,y)dx+\varphi(y).$$

这里$\varphi(y)$是y的任意可微函数, 选择$\varphi(y)$, 使u同时满足(5).

二、恰当方程——充分条件证明

$$\boxed{\frac{\partial u}{\partial y} = N(x,y) \quad (5)}$$

$$\boxed{u(x,y) = \int M(x,y)dx + \varphi(y).}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + \frac{d\varphi(y)}{dy} = N$$

因此 $$\frac{d\varphi(y)}{dy} = N - \frac{\partial}{\partial y}\int M(x,y)dx. \qquad (6)$$

下面证明(6)的右端与x无关，即对x的偏导数恒等于零.

事实上，$$\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\int M(x,y)dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\int M(x,y)dx\right]$$

二、恰当方程——充分条件证明

$$\frac{d\varphi(y)}{dy}=N-\frac{\partial}{\partial y}\int M(x,y)dx \qquad (6) \qquad\qquad u(x,y)=\int M(x,y)dx+\varphi(y).$$

$$=\frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int M(x,y)dx\right]=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\equiv 0.$$

于是,(6)右端的确只含有y,积分得

$$\varphi(y)=\int[N-\frac{\partial}{\partial y}\int M(x,y)dx]dy,$$

故 $$u(x,y)=\int M(x,y)dx+\int[N-\frac{\partial}{\partial y}\int M(x,y)dx]dy, \qquad (7)$$

即$u(x,y)$存在,从而(1)为恰当方程.

注:若(1)为恰当方程,则其通解为

$$\int M(x,y)dx+\int[N-\frac{\partial}{\partial y}\int M(x,y)dx]dy=c, \quad c\text{为任意常数}.$$

二、恰当方程——充要条件

需考虑的问题 $\boxed{M(x,y)dx+N(x,y)dy=0 \qquad (1)}$

- 方程(1)是否为恰当方程?
- 若(1)是恰当方程, 怎样求解?
- 若(1)不是恰当方程, 有无可能转化为恰当方程求解?

方程(1)为恰当方程的充要条件

定理1 设函数$M(x,y)$和$N(x,y)$在一个矩形域R内连续且具有连续的一阶偏导数,则方程

$$M(x,y)dx+N(x,y)dy=0 \qquad (1)$$

为恰当方程的充要条件是 $\dfrac{\partial M(x,y)}{\partial y}=\dfrac{\partial N(x,y)}{\partial x}. \qquad (2)$

例题

引例 求 $ydx + xdy = 0$ 的通解.

三、恰当方程——求解问题

方法一:不定积分法

1^0 判断$M(x,y)dx+N(x,y)dy=0$是否为恰当方程,若是进入下一步.

2^0 求$u(x,y)=\int M(x,y)dx+\varphi(y)$.

3^0 由$\dfrac{\partial u}{\partial y}=N(x,y)$求$\varphi(y)$.

例题

例1 验证方程 $(e^x+y)dx+(x-2\sin y)dy=0$ 是恰当方程，并求它的通解.

解： 这里 $M(x,y)=e^x+y, N(x,y)=x-2\sin y.$

所以 $\dfrac{\partial M(x,y)}{\partial y}=1=\dfrac{\partial N(x,y)}{\partial x}$，故所给方程是恰当方程.

由于所求函数 $u(x,y)$ 满足

$$\frac{\partial u}{\partial x}=e^x+y,\quad \frac{\partial u}{\partial y}=x-2\sin y,$$

由偏导数的定义，只要将 y 看作常数，将 e^x+y 对 x 积分得

$$u(x,y)=\int(e^x+y)dx+\varphi(y)=e^x+yx+\varphi(y).$$

例题

例1 验证方程 $(e^x + y)dx + (x - 2\sin y)dy = 0$ 是恰当方程，并求它的通解.

$$u(x,y) = \int (e^x + y)dx + \varphi(y) = e^x + yx + \varphi(y)$$

$$\frac{\partial u}{\partial y} = x - 2\sin y$$

对$u(x,y)$关于y求偏导数，得$\varphi(y)$应满足的方程为

$$x + \frac{d\varphi(y)}{dy} = x - 2\sin y \quad 即 \quad \frac{d\varphi(y)}{dy} = -2\sin y$$

积分后得 $\varphi(y) = 2\cos y,$

故 $u(x,y) = e^x + yx + 2\cos y.$

从而方程的通解为 $e^x + yx + 2\cos y = c.$

练习题

练习 求方程$(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$ 的通解.

解: 这里$M(x,y)=3x^2+6xy^2, N(x,y)=6x^2y+4y^3$,

所以$\dfrac{\partial M(x,y)}{\partial y}=12xy=\dfrac{\partial N(x,y)}{\partial x}$,

故所给方程是恰当方程.

$$\frac{\partial u}{\partial x}=3x^2+6xy^2, \frac{\partial u}{\partial y}=6x^2y+4y^3,$$

则$u(x,y)=\int(3x^2+6xy^2)dx+\varphi(y)=x^3+3x^2y^2+\varphi(y)$.

从而$\dfrac{\partial u}{\partial y}=6x^2y+\dfrac{d\varphi(y)}{dy}=6x^2y+4y^3$, $\quad \dfrac{d\varphi(y)}{dy}=4y^3$,

故 $\varphi(y)=y^4, u(x,y)=x^3+3x^2y^2+y^4$.

因此,方程的通解为 $x^3+3x^2y^2+y^4=c$, c是任意常数.

三、恰当方程——求解问题

方法二:分项组合法

把本身已构成全微分的项分出来, 再把余下的项凑成全微分.

应熟记一些简单二元函数的全微分.

$$ydx + xdy = d(xy), \qquad \frac{ydx - xdy}{xy} = d\left(\ln\left|\frac{x}{y}\right|\right),$$

$$\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right), \qquad \frac{ydx - xdy}{x^2 + y^2} = d\left(\arctan\frac{x}{y}\right),$$

$$\frac{-ydx + xdy}{x^2} = d\left(\frac{y}{x}\right), \qquad \frac{ydx - xdy}{x^2 - y^2} = \frac{1}{2}d\left(\ln\left|\frac{x-y}{x+y}\right|\right),$$

$$e^x(dy + ydx) = d(ye^x).$$

例题

例2 求方程$(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$的通解.

解: 这里$M(x,y)=3x^2+6xy^2, N(x,y)=6x^2y+4y^3,$

所以$$\frac{\partial M(x,y)}{\partial y}=12xy=\frac{\partial N(x,y)}{\partial x},$$

故所给方程是恰当方程. 把方程重新“分项组合”得

$$3x^2dx++4y^3dy+(6xy^2dx+6x^2ydy)=0$$

即 $$dx^3+dy^4+(3y^2dx^2+3x^2dy^2)=0$$

或写成 $$d(x^3+y^4+3x^2y^2)=0$$

故通解为: $x^3+y^4+3x^2y^2=c$, c是任意常数.

练习题

练习 求方程 $(e^x + y)dx + (x - 2\sin y)dy = 0$ 的通解.

解: 这里 $M(x, y) = e^x + y, N(x, y) = x - 2\sin y,$

$$\frac{\partial M(x, y)}{\partial y} = 1 = \frac{\partial N(x, y)}{\partial x},$$

故所给方程是恰当方程. 把方程重新“分项组合”得

$$e^x dx - 2\sin y dy + y dx + x dy = 0,$$

即 $de^x + d\left(2\cos y\right) + d(xy) = 0$ 或 $d(e^x + 2\cos y + xy) = 0,$

故方程的通解为 $e^x + 2\cos y + xy = c, c$为任意常数.

练习题：分项组合法

练习 验证方程 $(\cos x\sin x - xy^2)dx + y(1-x^2)dy = 0$ 是恰当方程，并求它满足初始条件 $y(0)=2$ 的解.

解: 这里 $M(x,y) = \cos x\sin x - xy^2$, $N(x,y) = y(1-x^2)$,

$$\frac{\partial M(x,y)}{\partial y} = -2xy = \frac{\partial N(x,y)}{\partial x},$$

故所给方程是恰当方程. 把方程重新“分项组合”得

$$\cos x\sin x dx - (xy^2 dx + x^2 y dy) + y dy = 0,$$

即 $d\left(\frac{1}{2}\sin^2 x\right) - d\left(\frac{1}{2}x^2y^2\right) + d\left(\frac{1}{2}y^2\right) = 0$ 或 $d(\sin^2 x - x^2y^2 + y^2) = 0$,

故方程的通解为 $\sin^2 x - x^2y^2 + y^2 = c$, c 为任意常数.

由初始条件 $y(0) = 2$, 得 $c = 4$,

因此所求的初值问题的解为 $\sin^2 x - x^2y^2 + y^2 = 4$.

练习题：不定积分法

练习 验证方程 $(\cos x\sin x - xy^2)dx + y(1-x^2)dy = 0$ 是恰当方程，并求它满足初始条件 $y(0)=2$ 的解.