2022.10.16 习题课（三重积分）

三重 $\longrightarrow$ 二重

(I) "广义柱体"　可积函数 $f(x,y,z)$.

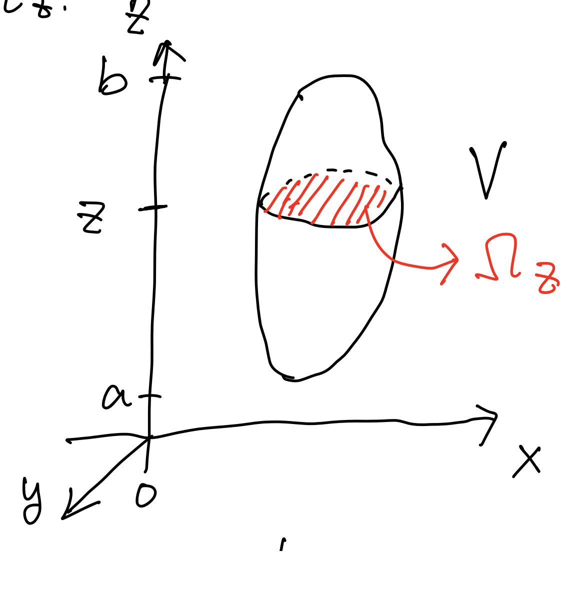
$$V=\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in D\subset\mathbb{R}^2,\ \underbrace{z_1(x,y)}_{\text{下表面}}\le z\le \underbrace{z_2(x,y)}_{\text{上表面}}\}$$

D "底".

$$\iiint_V f(x,y,z)\,dxdydz=\underbrace{\iint_D dxdy}_{\text{二重}}\underbrace{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz}_{\text{定积分}}$$

(II) "截面法"

$V=\bigcup\limits_{z\in[a,b]}\Omega_z$.　（$\Omega_z$ 是 V 与 $(0,0,z)$ 的截面）

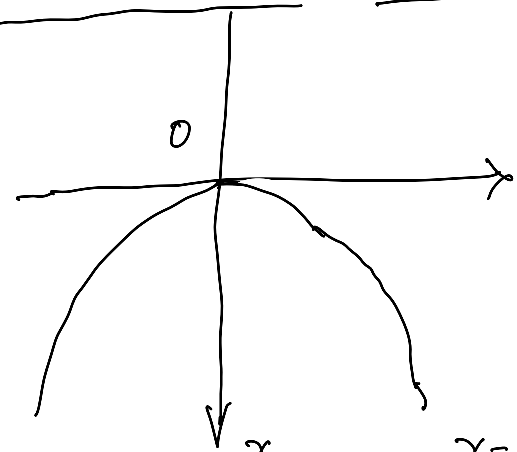


$$\iiint\limits_V f(x,y,z)\,dxdydz = \underline{\int_a^b dz}\ \underline{\iint\limits_{\Omega_z} f(x,y,z)\,dxdy}$$

课本216. 例.13.2.4.

求抛物柱面 $2y^2=x$ 与 $\underline{\frac{x}{4}+\frac{y}{2}+\frac{z}{2}=1}$，$\underline{z=0}$ 所围立体的体积.（V）

解：$\iiint\limits_V dxdydz.$

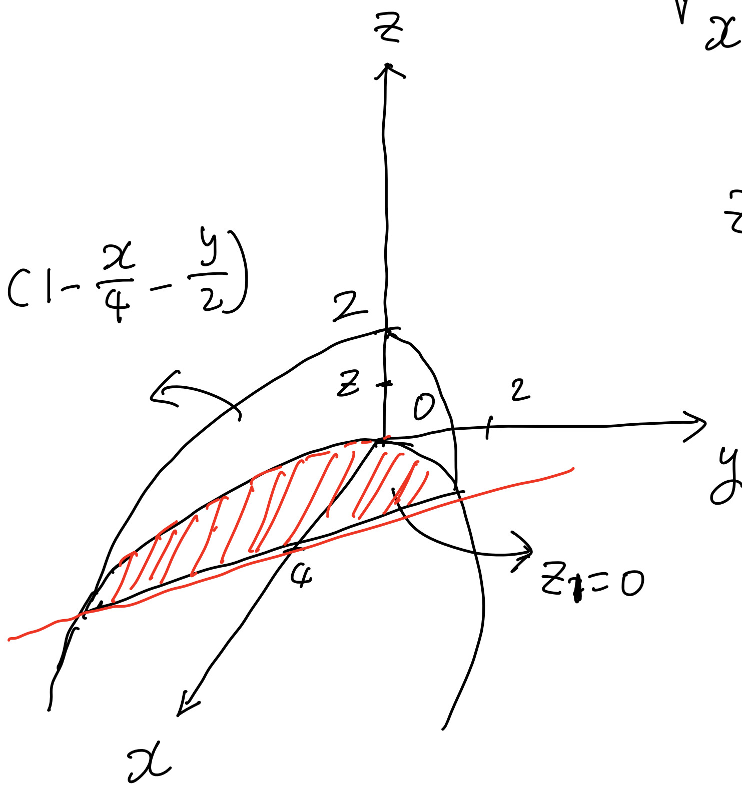


$z \in \mathbb{R}$

$x=2y^2$

$z \in [0,2]$

$z_2 = 2(1-\frac{x}{4}-\frac{y}{2})$

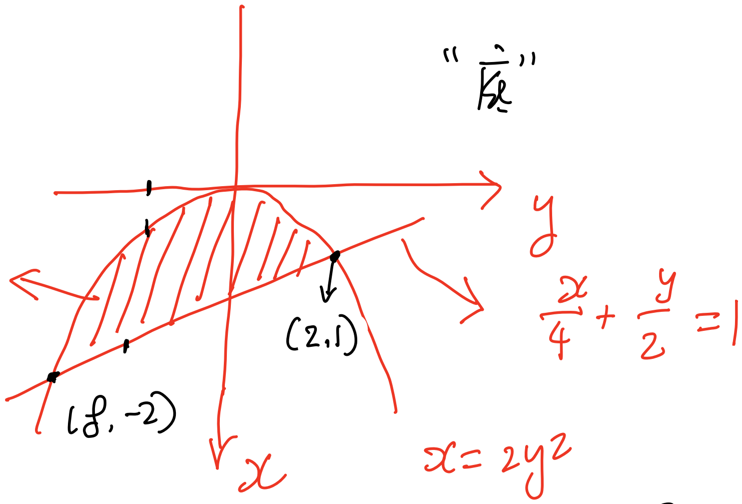


$$\begin{cases} 2y^2 = x. \\ \frac{x}{4}+\frac{y}{2} = z \end{cases}$$

$(x.y)$

D是抛物线 $2y^2=x$ 与(平面 $\frac{x}{4}+\frac{y}{2}+\frac{z}{2}=1$. $z=0$ 的交)

“底”



$$\iiint_V dxdydz=\int_0^2 dz\underline{\iint_D dxdy}.=\int_0^2 dz\int_{-2}^{1} dy\int_{2y^2}^{4(1-\frac{y}{2})} dx$$

$(y+2)(y-1)=0$

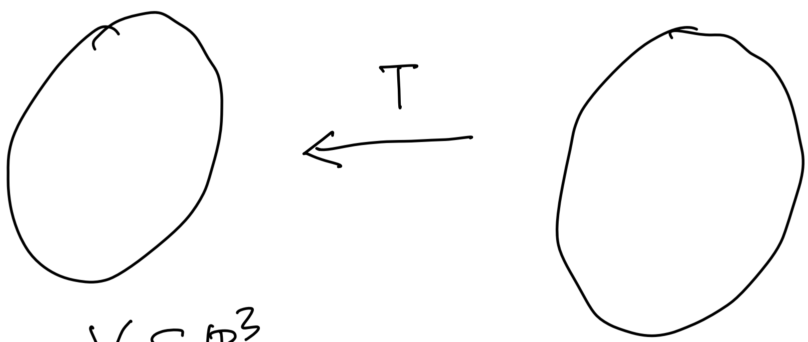
$$\begin{cases}\frac{x}{4}+\frac{y}{2}=1\\ x=2y^2\end{cases}\qquad \frac{y^2}{2}+\frac{y}{2}=1\qquad y^2+y^2-2=0$$

$y=1$ 时. $x=2$

$y=-2$ 时. $x=8$

$$\iint_D dxdy=\int_{-2}^{1} dy\int_{2y^2}^{4(1-\frac{y}{2})} dx$$

三重积分的变量代换



可积 f:　$V \subseteq \mathbb{R}^3$　　　　Ω

(x, y, z)　　　　　　(u, v, w)

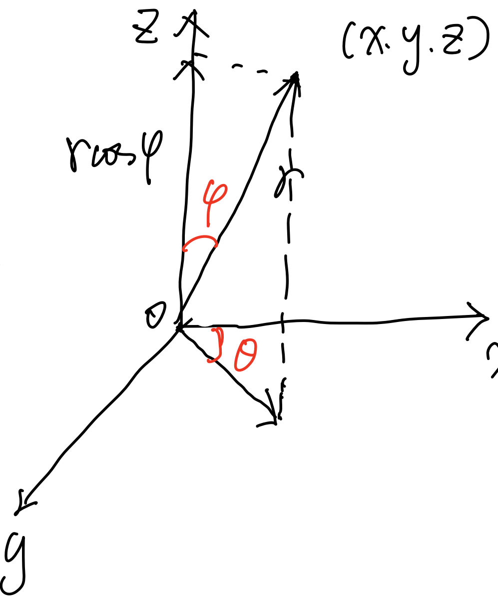
设 T 是从 Ω 到 V 的一一对应（一对一的，到上）.

$$T: \begin{cases} x = x(u, v, w). \\ y = y(u, v, w). \\ z = z(u, v, w) \end{cases} \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$$

$$\boxed{\iiint_V f(x, y, z)\,dx\,dy\,dz = \iiint_{\underset{\text{区域}}{\Omega}} \underbrace{f(x(u, v, w), y(u, v, w), z(u, v, w))}_{\text{被积函数}} \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\,dv\,dw.}$$

几种常见的坐标变换.

(1) 球坐标变换.



(a) r与z轴正半轴的夹角记为φ

(b) 把r在xy平面做投影

$r\sin\varphi$

(c) 记投影向量与x轴正半轴夹角为θ. 则该向量在xy平面内的极坐标 $(r\sin\varphi\cos\theta.\ r\sin\varphi.\sin\theta)$

故 $\boxed{(r\sin\varphi.\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)}$

$$\begin{cases} x = r\sin\varphi\cos\theta, & \varphi\in[0,\pi] \\ y = r\sin\varphi\sin\theta, & \theta\in[0,2\pi]. \\ z = r\cos\varphi & r\in[0,+\infty) \end{cases}$$

$$\frac{\partial(x.y.z)}{\partial(r.\varphi.\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi. & 0 \end{vmatrix}$$

$$= r^2\sin\varphi$$

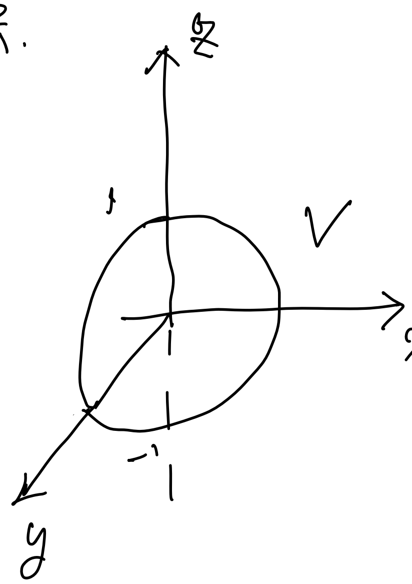
例：求 $\{x^2+y^2+z^2\le 1\}$ 的体积.

(一)"似柱体"

$z=\sqrt{1-x^2+y^2}$ 上表面

$z=-\sqrt{1-x^2-y^2}$ 下表面

$D=\{x^2+y^2\le 1\}$



$$\iiint_V dxdydz=\iint_D dxdy\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz=2\iint_D\sqrt{1-x^2-y^2}\,dxdy$$

(二)截面法 $(0,0,z)$ 与 V 相交.

$\Omega_z=\{x^2+y^2\le 1-z^2\}$.

πR^2

$$\iiint_V dxdydz=\int_{-1}^{1}dz\iint_{\Omega_z}dxdy$$

$$=\int_{-1}^{1}\pi(1-z^2)\,dz=2\pi-\frac{1}{3}(1-(-1))\pi$$

$$=\frac{4}{3}\pi.$$

(三).变量替换法.

$$\begin{cases} x = r\sin\varphi\cos\theta & \theta\in[0,2\pi] \\ y = r\sin\varphi\sin\theta & \varphi\in[0,\pi] \\ z = r\cos\theta. & r\in[0,1] \end{cases}$$

$$\iiint_V dxdydz = \iiint 1\cdot r^2\sin\varphi\, d\varphi\, d\theta\, dr$$

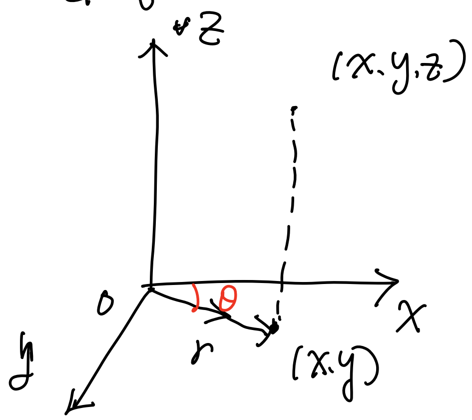
$$\underset{\varphi}{[0,\pi]}\times\underset{\theta}{[0,2\pi]}\times\underset{r}{[0,1]}$$

$$= \int_0^1 r^2 dr \int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\, d\varphi.$$

$$= \frac{1}{3}\cdot 2\pi\cdot 2 = \cancel{4}\,\frac{4}{3}\pi.$$

(四) 柱坐标、

(a) 把 $(x.y.z)$ 投影到 $x.y$ 平面

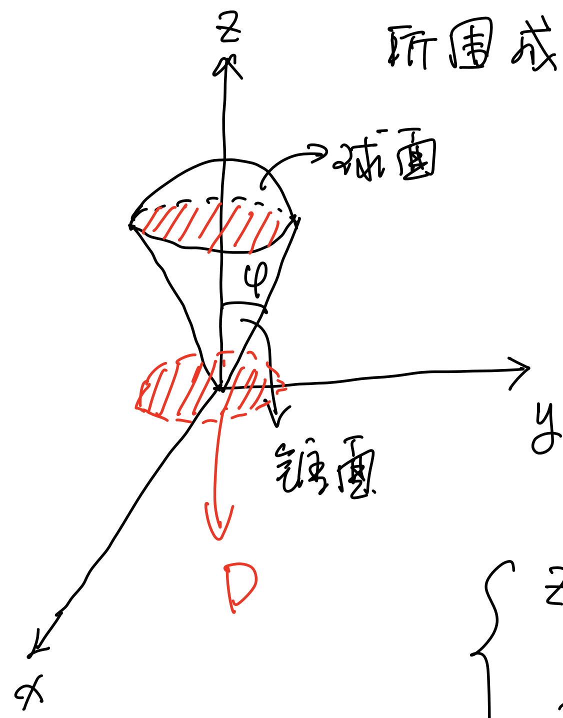


$$\begin{cases} x = r\cos\theta & \\ y = r\sin\theta & \theta\in[0.2\pi]. \quad r\geq 0 \\ z = z & z\in\mathbb{R}. \end{cases}$$

$z=|x|$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r$$

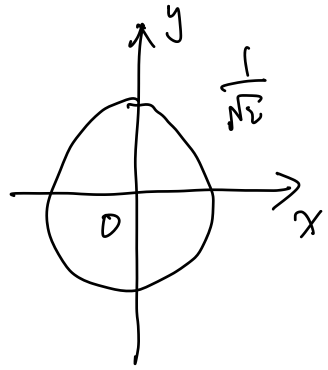
例: $\iiint\limits_{\Omega} z e^{-(x^2+y^2+z^2)}dxdydz$. Ω是锥面 $z=\sqrt{x^2+y^2}$
与球面 $x^2+y^2+z^2=1$
所围成的区域



$$\sin\varphi=\frac{\frac{1}{\sqrt{2}}}{1}=\frac{\sqrt{2}}{2}$$

$$\begin{cases} z=\sqrt{x^2+y^2} \\ x^2+y^2+z^2=1 \end{cases}$$

做法一. "广义柱体"



$$x^2+y^2+(x^2+y^2)=1$$

$$x^2+y^2=\frac{1}{2}$$

$$\sqrt{1-x^2-y^2}$$

$$I=\iiint_{\Omega} z e^{-(x^2+y^2+z^2)}dxdydz = \iint_{\{x^2+y^2\le\frac{1}{2}\}} dxdy \int_{\sqrt{x^2+y^2}} z e^{-(x^2+y^2+z^2)} dz$$

做法二：坐标变换

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi. \end{cases} \qquad \begin{aligned} &\theta\in[0,2\pi] \\ &r\in[0,1] \\ &\varphi\in[0,\tfrac{\pi}{4}]. \end{aligned}$$

$$I = \iiint_{[0,\frac{\pi}{4}]\times[0,2\pi]\times[0,1]} r\cos\varphi\, e^{-r^2} r^2\sin\varphi\, dr d\theta d\varphi.$$

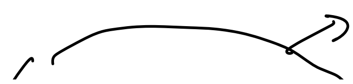
$$= 2\pi\int_0^1 r^3 e^{-r^2}dr\int_0^{\frac{\pi}{4}}\sin\varphi\cos\varphi\, d\varphi.$$

椭球

例：$\iiint_V z^2 dxdydz.$ V: $\{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\}$

广义的球坐标变换

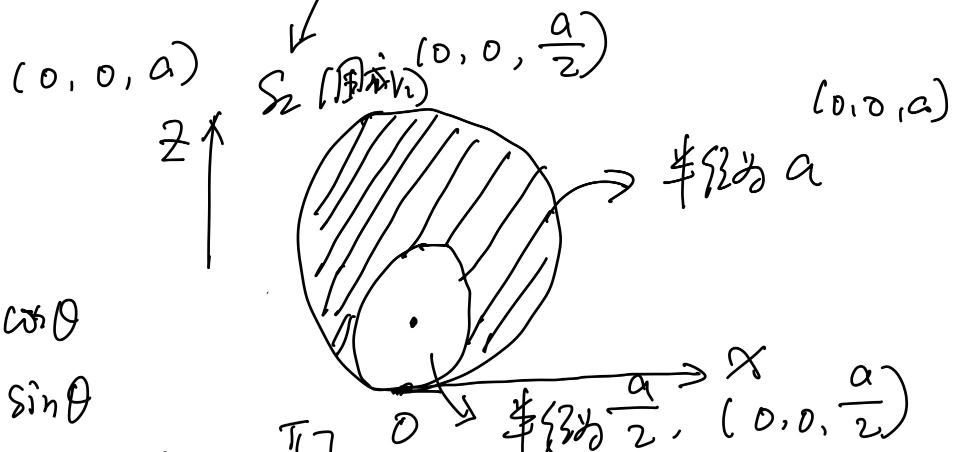
$$\begin{cases} x = ar\sin\varphi\cos\theta \\ y = br\sin\varphi\sin\theta \\ z = cr\cos\theta \end{cases} \qquad \begin{aligned} &r\in[0,1] \\ &\theta\in[0,2\pi] \\ &\varphi\in[0,\pi]. \end{aligned} \qquad \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = abcr^2\times\sin\varphi$$



S_1（围成V_1）

例: $I = \iiint\limits_V z\,dxdydz$. V: $x^2+y^2+z^2=2az$ 与 $(a>0)$

$x^2+y^2+z^2=az$ 所围成的立体



令

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi. \end{cases} \qquad \varphi\in[0,\frac{\pi}{2}]$$

S_1的方程. $r^2 = 2ar\cos\varphi. \Rightarrow r = 2a\cos\varphi$

S_2的方程. $r^2 = ar\cos\varphi. \Rightarrow r = a\cos\varphi$

$$I = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_{a\cos\varphi}^{2a\cos\varphi} r\cos\varphi\,(r^2\sin\varphi)\,dr$$

$$I = \iiint\limits_{V_2} z\,dxdydz - \iiint\limits_{V_1} z\,dxdydz$$

$$= \underline{\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2a\cos\varphi} r\cos\varphi\,(r^2\sin\varphi)\,dr} - \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{\varphi}} d\varphi \int_0^{a\cos\varphi} (\ )\,dr.$$

例: $\begin{cases} a_1x+b_1y+c_1z=\pm h_1 \\ a_2x+b_2y+c_2z=\pm h_2 \\ a_3x+b_3y+c_3z=\pm h_3 \end{cases}$ $\begin{matrix} h_1>0 \\ h_2>0 \\ h_3>0 \end{matrix}$ $\left|\underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}_{A}\right| \neq 0$

求体积 (六面体) → 围成 V.

$$\begin{cases} u=a_1x+b_1y+c_1z, & u\in[-h_1,h_1] \\ v=a_2x+b_2y+c_2z & v\in[-h_2,h_2] \\ w=a_3x+b_3y+c_3z & w\in[-h_3,h_3] \end{cases}$$

$$V\text{的体积}=\iiint_V dxdydz=\iiint_{[-h_1,h_1]\times[-h_2,h_2]\times[-h_3,h_3]} 1\cdot\underline{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}dudvdw$$

$$=\iiint 1\cdot\frac{1}{|A|}dudvdw$$

课本234页: 第五题. 六. 七. 八. 第十五题.

P.218页. 第六题. (8)

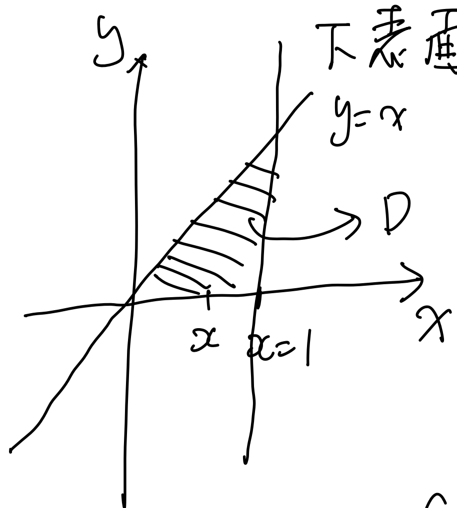
$\iiint\limits_{\Omega} xy^2z^3\,dxdydz$. Ω. $z=xy$. 平面 $y=x$. $x=1$.

和 $z=0$ 所围成的立体.

解: $z=xy$ $\{y=x,\ x=1,\ z=0\}$

Ω 是"广义柱体". 上表面. $z=xy$.

下表面. $z=0$



$$\iiint\limits_{\Omega} xy^2z^3\,dxdydz = \iint\limits_{D} dxdy\int_0^{xy} xy^2z^3\,dz$$

$$= \int_0^1 dx\int_0^x dy\int_0^{xy} xy^2z^3\,dz$$

明天第一节课. 例题.

$= \cdots$ 反常重积分.