§15.1. 含参变量常义积分

设 $f(x,y)$ 定义在 $[a,b]\times[c,d]$. 连续.

$$I(y)=\int_a^b f(x,y)dx.$$ 关于 y. 连续? 可积? 可导?

(1) $I(y)$. $\forall y_0\in[c,d]$. 连续

$$\Leftrightarrow \lim_{y\to y_0} I(y)=\lim_{y\to y_0}\int_a^b f(x,y)dx=\int_a^b f(x,y_0)dx$$

$$=\int_a^b \lim_{y\to y_0} f(x,y)dx$$

(2). $I(y)$ 可积.

$$\int_c^d I(y)\,dy=\underbrace{\int_c^d dy\int_a^b f(x,y)dx}=\int_a^b dx\int_c^d f(x,y)dy$$

(3) $I(y)$ 可导性. 假设 $f(x,y)$ 关于 y 有连续偏导数. $\frac{\partial f}{\partial y}(x,y)$.

则 $I(y)$ 关于 y 也可导. 并且

$$\frac{dI}{dy}=\int_a^b \frac{\partial f}{\partial y}(x,y)dx.$$

(3)' $$W(y)=\int_{\psi(y)}^{\varphi(y)} f(x,y)dx.$$ f. $\varphi(y)$. $\psi(y)$, 关于 y 都可导.

$$\frac{dw}{dy}(y) = \int_{\psi(y)}^{\varphi(y)} \frac{\partial f}{\partial x}(x,y)dx + f(\varphi(y), y)\cdot\varphi'(y) - f(\psi(y), y)\cdot\psi'(y)$$

(1). $\lim\limits_{\alpha\to 0}\int_0^{1+\alpha} \frac{dx}{1+x^2+\alpha^2} := I \qquad f(x,\alpha) = \frac{1}{1+x^2+\alpha^2}$.

≤ 1

$$\int_0^1 \frac{dx}{1+x^2+\alpha^2} + \underbrace{\int_1^{1+\alpha} \frac{1}{1+x^2+\alpha^2}} = (\quad) + \boxed{\frac{1}{1+\xi_\alpha^2+\alpha^2}}\cdot\alpha$$

ξ_α 介于1与 $1+\alpha$ 之间

$$I = \lim_{\alpha\to 0}\int_0^1 \frac{dx}{1+x^2+\alpha^2} + 0 = \int_0^1 \lim_{\alpha\to 0}\frac{dx}{1+x^2+\alpha^2} = \underbrace{\int_0^1 \frac{dx}{1+x^2}}$$

$$\int_0^{1+\alpha} \frac{dx}{1+x^2+\alpha^2} = \frac{1}{1+\alpha^2}\int_0^{1+\alpha} \frac{dx}{1+\left(\frac{x}{\sqrt{1+\alpha^2}}\right)^2}$$

$$= \underbrace{\frac{1}{\sqrt{1+\alpha^2}}\int_0^{1+\alpha} \frac{d\left(\frac{x}{\sqrt{1+\alpha^2}}\right)}{1+\left(\frac{x}{\sqrt{1+\alpha^2}}\right)^2}}$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx$$

$$\underbrace{\frac{1}{1+(1+\frac{x}{n})^n} \rightrightarrows \frac{1}{1+e^x}}$$

(2) $\lim\limits_{n\to\infty}\int_0^1 \frac{dx}{1+(1+\frac{x}{n})^n}$

$$= \int_0^1 \frac{dx}{1+e^x} = \cdots$$

3. (1). $I=\int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}dx \quad (b>a>0)$

令 $F(y)=\frac{x^y}{\ln x}$ $\qquad \frac{x^b-x^a}{\ln x}=F(b)-F(a)=\int_a^b F'(y)dy$

$F'(y)=x^y$

$$I=\int_0^1\left(\sin\left(\ln\frac{1}{x}\right)\cdot\int_a^b x^y dy\right)dx=\int_0^1 dx\int_a^b \sin\left(\ln\frac{1}{x}\right)\cdot x^y dy$$

$$=\int_a^b dy\int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot x^y dx$$

$$W=\int_0^1 \sin\left(\ln\frac{1}{x}\right)x^y dx=\frac{1}{y+1}\int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot d\,x^{y+1}$$

"积分重现"

$$=\frac{1}{y+1}\cdot\sin\left(\ln\frac{1}{x}\right)\cdot x^{y+1}\Big|_0^1+\frac{1}{y+1}\cdot\int_0^1 x^{y+1}\cdot\frac{1}{x}\cos\left(\ln\frac{1}{x}\right)dx$$

$$=\frac{1}{y+1}\cdot\int_0^1 x^y\cos\left(\ln\frac{1}{x}\right)dx$$

$$=\frac{1}{y+1}\cdot\frac{1}{y+1}\cdot\int_0^1 \cos\left(\ln\frac{1}{x}\right)d\,x^{y+1}$$

$$=\frac{1}{(y+1)^2}\cdot\cos\left(\ln\frac{1}{x}\right)\cdot x^{y+1}\Big|_0^1-$$

$$\frac{1}{(y+1)^2}\cdot\int_0^1 x^{y+1}\,\frac{1}{x}\cdot\sin\left(\ln\frac{1}{x}\right)dx$$

$$=\frac{1}{(y+1)^2}-\frac{1}{(y+1)^2}\cdot\int_0^1 x^{y-}\sin\left(\ln\frac{1}{x}\right)dx$$

$$=\frac{1}{(y+1)^2}-\frac{1}{(y+1)^2}W$$

$$W=\frac{1}{1+(y+1)^2}$$

$$I=\int_a^b W\,dy=\int_a^b\frac{dy}{1+(y+1)^2}=\int_a^b\frac{d(y+1)}{1+(y+1)^2}$$

$$=\arctan(b+1)-\arctan(a+1)$$

$0<a<1$

(2) $$I=\int_0^{\frac{\pi}{2}}\ln\frac{1+a\sin x}{1-a\sin x}\cdot\frac{1}{\sin x}\,dx.$$

$$\left[\frac{1}{\sin x}\left(\ln(1+y\sin x)-\ln(1-y\sin x)\right)\right]'$$

$$=\frac{1}{1+y\sin x}-\frac{1}{1-y\sin x}=2\,\frac{1}{1-y^2\sin^2 x}$$

令 $F(y)=\ln\dfrac{1+y\sin x}{1-y\sin x}\cdot\dfrac{1}{\sin x}$, $F(0)=0$

$$\ln\frac{1+a\sin x}{1-a\sin x}\cdot\frac{1}{\sin x}=F(a)-F(0)=\int_0^a F'(y)\,dy=2\int_0^a\frac{dy}{1-y^2\sin^2 x}$$

$f(x,y)=\dfrac{1}{1-y^2\sin^2 x}$. 在 $[0,\frac{\pi}{2}]\times[0,a]$

$$I = 2\int_0^{\frac{\pi}{2}} dx \int_0^a \frac{dy}{1-y^2\sin^2 x} = 2\int_0^a dy \int_0^{\frac{\pi}{2}} \frac{dx}{1-y^2\sin^2 x}$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1-y^2\sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{dx}{(1-y^2)\sin^2 x + \cos^2 x}$$

$x = \tan\frac{t}{2}$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x} \frac{dx}{1-y^2+\cot^2 x}$$

$$= -\int_0^{\frac{\pi}{2}} \frac{d\cot x}{1-y^2+\cot^2 x}$$

$$= -\frac{1}{\sqrt{1-y^2}} \int_0^{\frac{\pi}{2}} \frac{d\frac{\cot x}{\sqrt{1-y^2}}}{1+\left(\frac{\cot x}{\sqrt{1-y^2}}\right)^2}$$

$$= -\frac{1}{\sqrt{1-y^2}} \cdot \arctan \frac{\cot x}{\sqrt{1-y^2}} \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{1-y^2}}$$

$$I = 2\int_0^a \frac{\pi}{2\sqrt{1-y^2}} dy = \pi \arcsin y \Big|_0^a = \pi \arcsin a.$$

4. (1). $I(y) = \int_y^{y^2} e^{-x^2 y} dx$

$$I(y)=\int_{y}^{y^2} -x^2 e^{-x^2 y}\,dx + e^{-(y^2)^2\cdot y}\cdot(y^2)' - e^{-y^2\cdot y}\cdot(y)'$$

$$= -\int_{y}^{y^2} x^2 e^{-x^2 y}\,dx + 2y\cdot e^{-y^5} - e^{-y^3}.$$

(3) $$F(t)=\int_0^{t^2} dx \int_{x-t}^{x+t} \sin(x^2+y^2-t^2)\,dy$$

$$= \int_0^{t^2} \underbrace{\left[\int_{x-t}^{x+t} \sin(x^2+y^2-t^2)\,dy\right]} dx$$

$$\triangleq \int_0^{t^2} G(x,t)\,dx \qquad G(x,t)=\int_{x-t}^{x+t} \sin(x^2+y^2-t^2)\,dy$$

$$F'(t) = \int_0^{t^2} \frac{\partial G}{\partial t}(x,t)\,dx + G(t^2,t)(2t) - 0$$

$$\frac{\partial G}{\partial t} = \int_{x-t}^{x+t} -2t\cos(x^2+y^2-t^2)\,dy + \sin(x^2+(x+t)^2-t^2)\cdot 1$$

$$- \sin(x^2+(x-t)^2-t^2)\cdot(-1).$$

5. $I(y)=\int_0^y (x+y)\cdot f(x)\,dx$. 其中 $f(x)$ 可导. 求 $I''(y)$.

解: $I'(y)=\int_0^y f(x)\,dx+(y+y)f(y)\cdot 1$

$I''(y)=f(y)+2f(y)\cdot 2y\cdot f'(y)$

6. $F(y)=\int_a^b f(x)|y-x|\,dx$. $(a<b)$. $f(x)$ 可微. 求 $F''(y)$.

$f(x)|y-x|$. 关于 y 在 x 点不可导的.

(1). $y\leq a$. $F(y)=\int_a^b f(x)(x-y)\,dx$. $F'(y)$. $F''(y)$.

(2) $y\geq b$. $F(y)=\int_a^b f(x)(y-x)\,dx$. $F'(y)$. $F''(y)$

(3). $a<y<b$. $F(y)=\int_a^y f(x)(y-x)\,dx+\int_y^b f(x)\cdot(x-y)\,dx$.

$F'(y)$. $F''(y)$.

8. 用积分号下求导计算

(1) $I(a)=\int_0^{\frac{\pi}{2}} \ln(a^2-\sin^2 x)\,dx \quad (a>1)$

$$I'(a) = \int_0^{\frac{\pi}{2}} \frac{2a}{a^2 - \sin^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{2a}{(a^2-1)\sin^2 x + a^2\cos^2 x} dx$$

$$\frac{1}{\sin^2 x} dx = -d\cot x \qquad = 2a \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x} \cdot \frac{dx}{a^2-1 + a^2(\cot x)^2}$$

$$= -2a \int_0^{\frac{\pi}{2}} \frac{d\cot x}{(a^2-1) + a^2(\cot x)^2}$$

$$= \frac{\sqrt{a^2-1}}{a} \cdot \left(-\frac{2a}{a^2-1}\right) \int_0^{\frac{\pi}{2}} \frac{d\,\frac{a\cot x}{\sqrt{a^2-1}}}{1 + \left(\frac{a\cot x}{\sqrt{a^2-1}}\right)^2}$$

$$= -\frac{2}{\sqrt{a^2-1}} \arctan \frac{a\cot x}{\sqrt{a^2-1}} \Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{\sqrt{a^2-1}}.$$

$$\underbrace{I(a) = \int \frac{\pi}{\sqrt{a^2-1}} da}_{} + C = \ln(a + \sqrt{a^2-1}) + C.$$

令 $a \to 1^+$. $\quad \lim_{a \to 1^+} I(a) = I(2) = \underbrace{\int_0^{\frac{\pi}{2}} \ln(1 - \sin^2 x) dx}_{}$

$\underbrace{\text{故 } C = I(1)}_{}$

(2) $I(\alpha) = \int_0^{\pi} \ln(1 - 2\alpha\cos x + \alpha^2) dx. \quad (|\alpha| < 1)$

$$I'(\alpha) = \underbrace{\int_0^{\pi} \frac{-2\cos x + 2\alpha}{1-2\alpha\cos x + \alpha^2}\, dx}$$

$$= \frac{1}{\alpha}\int_0^{\pi} \frac{1-2\alpha\cos x + \alpha^2 - 1 + \alpha^2}{1-2\alpha\cos x + \alpha^2}\, dx$$

$$= \frac{1}{\alpha}\int_0^{\pi} \left(1 + \alpha^2 \frac{dx}{1-2\alpha\cos x + \alpha^2} \right)$$

$$\underbrace{\alpha\int_0^{\pi} \frac{dx}{1-2\alpha\cos x + \alpha^2}}.$$ 万能公式. $t = \tan\frac{x}{2}$.

$x = 2\arctan t$

$$\underbrace{dx = \frac{2}{1+t^2}\, dt}$$

$$I(\alpha) = \int I'(\alpha) + C$$

$$I(0) = \int_0^{\pi} \ln 1\, dx = 0. \quad C = 0$$

(3). $\int_0^{\frac{\pi}{2}} \ln(a^2\sin^2 x + b^2\cos^2 x)\, dx.$

§15.2. 含参变量反常积分.（一致收敛）.

一致收敛: $f(x,y)$ 定义 $[a,+\infty)\times[c,d]$.

$$I(y)=\int_a^{+\infty}f(x,y)\,dx.$$ 一致收敛

$\Leftrightarrow$ $\forall\ \varepsilon>0$, $\exists\ A_0\geq a$. 使得对任意的 $A\geq A_0$. 有

$$\left|\int_A^{+\infty}f(x,y)\,dx\right|\leq\varepsilon.\quad(\forall\ y\in[c,d])$$

$$\left(\sup_{y\in[c,d]}\left|\int_A^{+\infty}f(x,y)\,dx\right|\leq\varepsilon\right)$$

$\Leftrightarrow$ $\forall\ \varepsilon>0$, $\exists\ A_0\geq a$. 使得 $\forall\ A'.A''\geq A_0$. 有

$$\left|\int_{A'}^{A''}f(x,y)\,dx\right|\leq\varepsilon,\quad(\forall\ y\in[c,d])$$

$$\left(\sup_{y\in[c,d]}\left|\int_{A'}^{A''}f(x,y)\,dx\right|\leq\varepsilon\right).$$

(1). Weierstrass 判别法. $|f(x,y)| \le F(x)$. $\forall (x,y) \in [a,+\infty) \times [c,d]$

且 $\int_a^{+\infty} F(x)dx < +\infty$. 则

$I(y) = \int_a^{+\infty} f(x,y)dx$ 关于 $y \in [c,d]$ 一致收敛.

$$\int_a^{+\infty} f(x,y)\, g(x,y)dx$$

(2). Abel. $\int_a^{+\infty} f(x,y)dx$ 关于 y 一致收敛.

$g(x,y)$. 固定 y 时. 关于 x 单调. 且 $g(x,y)$ 一致有界

$\exists L>0$, 使得 $|g(x,y)| \le L$. $(x,y) \in [a,+\infty) \times [c,d]$

(3) Dirichlet. $\int_a^{A} f(x,y)dx$ 一致有界. $\exists L>0$. 使得. $\forall A \geq a$

$$\left| \int_a^{A} f(x,y)dx \right| \le L. \left(\forall\ y \in [c,d],\ \forall A \geq a\right)$$

$g(x,y)$ 固定 y 时. 关于 x 单调. 且 $\lim\limits_{x\to+\infty} g(x,y) = 0$

(对 $y \in [c,d]$ 一致收敛于 0)

反常积分. (Abel. Dirichlet)

数项级数 (Abel. Dirichlet)

函数项级数的一致收敛性 (Abel. Dirichlet)

(1). $\displaystyle\int_0^{+\infty} \frac{\cos(xy)}{x^2+y^2}dx.$ $y \geq a > 0.$

$$\left|\frac{\cos(xy)}{x^2+y^2}\right| \leq \frac{1}{x^2+a^2}. \quad \int_0^{+\infty} \frac{dx}{x^2+a^2} < +\infty$$

(2). $\displaystyle\int_0^{+\infty} \frac{\sin 2x}{x+\alpha} \cdot e^{-\alpha x} dx.$ $0 \leq \alpha \leq \alpha_0.$

$$\sin 2x \cdot \frac{e^{-\alpha x}}{x+\alpha}, \quad \forall A \geq 0. \ \forall\ 0 \leq \alpha \leq \alpha_0$$

$$\left|\int_0^A \sin 2x\, dx\right| \leq 2. \quad (\text{一致有界})$$

$\dfrac{e^{-\alpha x}}{x+\alpha}$. 固定 α. 关于 x 单调递减.

$$\left|\frac{e^{\alpha x}}{x+\alpha}\right| \leq \underbrace{\frac{1}{x+\alpha_0}} \quad \lim_{x\to+\infty} \frac{1}{\alpha_0+x} = 0$$

由 Dirichlet 判别法. 关于 $\alpha \in [0, \alpha_0]$ 一致收敛.

"反常"积分 $\begin{cases} \text{积分区域"无界区域"} \quad \underbrace{[a,+\infty)\times[c,d]} \\ [a,b]\times[c,d]\text{ 上 } f(x,y) \text{ 在某点处无界} \end{cases}$

$$\left(\underbrace{\int_1^{+\infty} \frac{1}{x^\alpha} dx}, \quad \underbrace{\int_0^1 \frac{1}{x^\alpha} dx} \right)$$

定义（无界函数的含参变量反常积分）

$f(x,y)$ 定义在 $[a,b)\times[c,d]$（$f(x,y)$ 在 $y=b$ 无界的）

$[a,b)$

$$\underbrace{I(y)=\int_a^b f(x,y)dx}.$$

"瑕积分"

$I(y)$ 在 $[c,d]$ 上一致收敛

$\Longleftrightarrow$ $\forall \varepsilon>0$, $\exists \delta>0$. 当 $0<\eta<\delta$ 时.

$$\left|\int_b^{b-\eta} f(x,y)\,dx\right| \le \varepsilon. \quad (\forall\ y\in[c,d])$$

$$\left(\underbrace{\sup_{y\in[c,d]}} \left|\int_b^{b-\eta} f(x,y)dx\right| \le \varepsilon \right)$$

$\downarrow$

一致性.

Cauchy 收敛准则. Weierstrass 判别法. Abel. Dirichlet

$I(y)$ 连续性. 可积. 可导.