(*mathmagic lite 2019.04.17*)

$$I=\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{x}{n}\right)dx\right]^{\frac{1}{n}}$$

$$n\in N^+$$

$$0\le x\le\frac{\pi}{4}\quad\Rightarrow\tan\left(\frac{x}{n}\right)\ge 0$$

$$\because \text{when } u\ge 0,\quad \text{have } \tan u\ge u$$

$$\therefore \tan^n\left(\frac{x}{n}\right)=\left[\tan\left(\frac{x}{n}\right)\right]^n\ge\left(\frac{x}{n}\right)^n$$

$$\therefore \int_0^{\frac{\pi}{4}}\tan^n\left(\frac{x}{n}\right)dx\ge\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n dx$$

$$\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n dx=n\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n d\left(\frac{x}{n}\right)=\frac{n}{n+1}\left(\frac{\pi}{4n}\right)^{n+1}$$

$$\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n dx\right]^{\frac{1}{n}}=\lim_{n\to\infty} n\left[\frac{n}{n+1}\left(\frac{\pi}{4n}\right)^{n+1}\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty} n*\left(\frac{n}{n+1}*\frac{\pi}{4n}\right)^{\frac{1}{n}}*\frac{\pi}{4n}$$

$$=\frac{\pi}{4}\lim_{n\to\infty}\left(\frac{1}{n+1}*\frac{\pi}{4}\right)^{\frac{1}{n}}=\frac{\pi}{4}$$

（mathmagic lite 2019.04.17）

$$I=\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{x}{n}\right)dx\right]^{\frac{1}{n}}$$

$n\in N^+$

$$0\le x\le\frac{\pi}{4}\quad\Rightarrow\tan\left(\frac{x}{n}\right)\le\tan\left(\frac{\pi}{4n}\right)$$

$$\therefore\tan^n\left(\frac{x}{n}\right)\le\tan^n\left(\frac{\pi}{4n}\right)$$

$$\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{x}{n}\right)dx\le\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{\pi}{4n}\right)dx$$

$$\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{\pi}{4n}\right)dx=\tan^n\left(\frac{\pi}{4n}\right)\int_0^{\frac{\pi}{4}}dx=\frac{\pi}{4}\tan^n\left(\frac{\pi}{4n}\right)$$

$$\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{\pi}{4n}\right)dx\right]^{\frac{1}{n}}=\lim_{n\to\infty} n\left[\frac{\pi}{4}\tan^n\left(\frac{\pi}{4n}\right)\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty} n*\left(\frac{\pi}{4}\right)^{\frac{1}{n}}*\tan\left(\frac{\pi}{4n}\right)=\lim_{n\to\infty} n*1*\frac{\pi}{4n}=\frac{\pi}{4}$$

（*mathmagic lite 2019.04.17*）

$$I=\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{x}{n}\right)dx\right]^{\frac{1}{n}}$$

$$n\in N^+$$

$$0\le x\le\frac{\pi}{4}\quad\Rightarrow\tan\left(\frac{x}{n}\right)\le\tan\left(\frac{\pi}{4n}\right)$$

$$\therefore \tan^n\left(\frac{x}{n}\right)\le\tan^n\left(\frac{\pi}{4n}\right)$$

$$\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{x}{n}\right)dx\le\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{\pi}{4n}\right)dx$$

$$\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{\pi}{4n}\right)dx=\tan^n\left(\frac{\pi}{4n}\right)\int_0^{\frac{\pi}{4}}dx=\frac{\pi}{4}\tan^n\left(\frac{\pi}{4n}\right)$$

$$\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}}\tan^n\left(\frac{\pi}{4n}\right)dx\right]^{\frac{1}{n}}=\lim_{n\to\infty} n\left[\frac{\pi}{4}\tan^n\left(\frac{\pi}{4n}\right)\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty} n*\left(\frac{\pi}{4}\right)^{\frac{1}{n}}*\tan\left(\frac{\pi}{4n}\right)=\lim_{n\to\infty} n*1*\frac{\pi}{4n}=\frac{\pi}{4}$$

(*mathmagic lite 2019.04.17*)

$$I=\lim_{n\to\infty}\left(\int_0^{\frac{\pi}{2}}\sin^n x\,dx\right)^{\frac{1}{n}}$$

$$I=\lim_{n\to\infty}\exp\left\{\frac{1}{n}\ln\left(\int_0^{\frac{\pi}{2}}\sin^n x\,dx\right)\right\}$$

$$let\ a_n=\frac{1}{n}\ln\left(\int_0^{\frac{\pi}{2}}\sin^n x\,dx\right)$$

use wallis $\qquad a_1=0$

$$a_{2n}=\frac{1}{2n}\ln\left[\frac{(2n-1)!!}{(2n)!!}*\frac{\pi}{2}\right]$$

$$a_{2n+1}=\frac{1}{2n+1}\ln\left[\frac{(2n)!!}{(2n+1)!!}\right]$$

use stolz

$$\lim_{n\to\infty}a_{2n}=\lim_{n\to\infty}\frac{\ln\left[\frac{(2n+1)!!}{(2n+2)!!}*\frac{\pi}{2}\right]-\ln\left[\frac{(2n-1)!!}{(2n)!!}*\frac{\pi}{2}\right]}{(2n+2)-2n}$$

$$=\lim_{n\to\infty}\frac{1}{2}\ln\frac{2n+1}{2n+2}=0$$

$$\lim_{n\to\infty}a_{2n+1}=\lim_{n\to\infty}\frac{\ln\left[\frac{(2n+2)!!}{(2n+3)!!}\right]-\ln\left[\frac{(2n)!!}{(2n+1)!!}\right]}{(2n+3)-(2n+1)}$$

$$=\lim_{n\to\infty}\frac{1}{2}\ln\frac{2n+2}{2n+3}=0$$

$$\lim_{n\to\infty}a_{2n}=0,\quad \lim_{n\to\infty}a_{2n+1}=0\ \Rightarrow \lim_{n\to\infty}a_n=0\ \Rightarrow I=1$$

(mathmagic lite 2019.04.17)

$$\varphi=\frac{\sqrt{5}+1}{2}$$

$$prove: I=\int_0^{+\infty}\frac{1}{(x+\sqrt{x^2+1})^{\varphi}}dx=1$$

$$\varphi^2-\varphi-1=0$$

$$let\ t=x+\sqrt{x^2+1},\quad \frac{1}{t}=\sqrt{x^2+1}-x$$

$$2x=t-\frac{1}{t}\quad \Rightarrow dx=\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$$

$$\begin{cases} x=+\infty, t=+\infty \\ x=0, \quad t=1 \end{cases}$$

$$I=\int_1^{+\infty}\frac{1}{t^{\varphi}}*\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$$

$$=\frac{1}{2}\left(\frac{1}{1-\varphi}t^{1-\varphi}+\frac{1}{-1-\varphi}t^{-1-\varphi}\right)\Bigg|_1^{+\infty}$$

$$\because \lim_{t\longrightarrow+\infty}\left(\frac{1}{1-\varphi}t^{1-\varphi}+\frac{1}{-1-\varphi}t^{-1-\varphi}\right)=0$$

$$\therefore I=-\frac{1}{2}\left(\frac{1}{1-\varphi}+\frac{1}{-1-\varphi}\right)$$

$$=-\frac{1}{2}*\frac{-2\varphi}{\varphi^2-1}=1$$

$$\int \frac{x(x+1)}{(x+e^x+1)^2}dx$$

$$=\int \frac{(x+e^x+1)^2-(e^x+1)(x+e^x+1)-xe^x}{(x+e^x+1)^2}dx$$

$$=x-\int \frac{(x+e^x+1)'}{(x+e^x+1)}dx+\int d\frac{x+1}{(x+e^x+1)}$$

$$=x-\ln|x+e^x+1|+\frac{x+1}{x+e^x+1}+C$$

(mathmagic lite 2018.12.12)

$$I=\int\frac{x}{\sqrt{e^x+(x+2)^2}}dx$$

$$=\int\frac{x}{\sqrt{e^x}*\sqrt{1+(x+2)^2*e^{-x}}}dx$$

$$=\int\frac{(x+2-2)e^{-\frac{x}{2}}}{\sqrt{1+[(x+2)e^{-\frac{x}{2}}]^2}}dx$$

$$=-2\int\frac{(x+2)'e^{-\frac{x}{2}}+(x+2)(e^{-\frac{x}{2}})'}{\sqrt{1+[(x+2)e^{-\frac{x}{2}}]^2}}dx$$

$$=-2\int\frac{d[(x+2)e^{-\frac{x}{2}}]}{\sqrt{1+[(x+2)e^{-\frac{x}{2}}]^2}}$$

$$=-2\int\frac{du}{\sqrt{1+u^2}}$$

$$=-2\ln(u+\sqrt{u^2+1})+C$$

$$=-2\ln\left[(x+2)e^{-\frac{x}{2}}+\sqrt{1+[(x+2)e^{-\frac{x}{2}}]^2}\right]+C$$

(mathmagic lite 2019.04.19)

$$I=\lim_{n\to\infty}\frac{1}{n}\left(\frac{n}{\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}}\right)^n$$

$$=\lim_{n\to\infty}\exp\left\{-n\ln\left(1-\frac{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}}{n}\right)-\ln n\right\}$$

$$A_n=\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}=\ln n-1+\gamma+\varepsilon_n$$

$$\frac{A_n}{n}=\frac{\ln n-1+\gamma+\varepsilon_n}{n}\sim\frac{\ln n}{n}$$

$$\ln\left(1-\frac{A_n}{n}\right)=-\frac{A_n}{n}+O\left[\left(\frac{A_n}{n}\right)^2\right]=-\frac{A_n}{n}+O\left(\frac{\ln^2 n}{n^2}\right)$$

$$-n\ln\left(1-\frac{A_n}{n}\right)-\ln n=A_n+O\left(\frac{\ln^2 n}{n}\right)-\ln n$$

$$=\gamma-1+\varepsilon_n+O\left(\frac{\ln^2 n}{n}\right)$$

$$\lim_{n\to\infty}\left[-n\ln\left(1-\frac{A_n}{n}\right)-\ln n\right]=\gamma-1$$

$$\therefore I=e^{\gamma-1}$$

（*mathmagic lite 2019.04.20*）

$$I=\lim_{n\to\infty}\frac{a(a+1)(a+2)\cdots(a+n)}{b(b+1)(b+2)\cdots(b+n)}\quad(0<a<b)$$

$$\ln I=\sum_{n=0}^{\infty}\ln\left(\frac{a+n}{b+n}\right)=\sum_{n=0}^{\infty}\ln\left(1+\frac{a-b}{b+n}\right)$$

$$\because \frac{a-b}{b+n}\sim\frac{a-b}{n}$$

$$\therefore \ln\left(1+\frac{a-b}{b+n}\right)=\frac{a-b}{b+n}+O\left[\left(\frac{a-b}{b+n}\right)^2\right]$$

$$=\frac{a-b}{b+n}+O\left(\frac{1}{n^2}\right)=-\frac{b-a}{b+n}+O\left(\frac{1}{n^2}\right)$$

$$\because \sum_{n=0}^{\infty}\frac{b-a}{b+n}=+\infty,\quad O\left(\frac{1}{n^2}\right)\ \textit{is convergent}$$

$$\therefore \ln I=-\infty$$

$$\therefore I=0$$

(mathmagic lite 2019.04.20)

$$I=\lim_{n\to\infty}\left[\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right]$$

$$\text{let } A=\frac{\ln[(n+1)!]}{n+1}=\frac{\ln 1+\ln 2+\cdots+\ln(n+1)}{n+1}$$

$$B=\frac{\ln(n!)}{n}=\frac{\ln 1+\ln 2+\cdots+\ln n}{n}$$

$$A-B=\frac{n\ln(n+1)-(\ln 1+\ln 2+\cdots+\ln n)}{(n+1)n}$$

$$\because 0\le A-B\le\frac{n\ln(n+1)}{(n+1)n},\quad \lim_{n\to\infty}\frac{n\ln(n+1)}{(n+1)n}=0$$

$$\therefore \lim_{n\to\infty}(A-B)=0 \quad \Rightarrow e^{A-B}-1\sim A-B$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e} \quad \Rightarrow e^{B}=\sqrt[n]{n!}\sim\frac{n}{e}$$

$$I=\lim_{n\to\infty}(e^{A}-e^{B})=\lim_{n\to\infty}e^{B}(e^{A-B}-1)=\lim_{n\to\infty}\frac{n}{e}(A-B)$$

$$=\frac{1}{e}\lim_{n\to\infty}\frac{n\ln(n+1)-(\ln 1+\ln 2+\cdots+\ln n)}{n+1}$$

$$I=\frac{1}{e}\lim_{n\to\infty}\frac{n\ln(n+1)-(\ln 1+\ln 2+\cdots+\ln n)}{n+1}$$

$$\because n\ln(n+1)-(\ln 1+\ln 2+\cdots+\ln n)$$

$$=\ln(n+1)-\ln 1+\ln(n+1)-\ln 2+\cdots$$

$$+\ln(n+1)-\ln n+\ln(n+1)-\ln(n+1)$$

$$=\sum_{k=1}^{n+1}\ln\left(\frac{n+1}{k}\right)=-\sum_{k=1}^{n+1}\ln\left(\frac{k}{n+1}\right)$$

$$\therefore I=\frac{1}{e}\lim_{n\to\infty}\frac{1}{n+1}*(-1)*\sum_{k=1}^{n+1}\ln\left(\frac{k}{n+1}\right)$$

$$=-\frac{1}{e}\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n+1}\ln\left(\frac{k}{n+1}\right)$$

$$=-\frac{1}{e}\int_0^1\ln x dx=-\frac{1}{e}*(-1)=\frac{1}{e}$$

(mathmagic lite 2019.04.20)

$$I=\lim_{x\to+\infty}\left[\frac{a^x-1}{x(a-1)}\right]^{\frac{1}{x}}\quad(a>0,a\neq 1)$$

$$I=\lim_{x\to+\infty}\exp\left\{\frac{1}{x}\ln\left[\frac{a^x-1}{x(a-1)}\right]\right\}$$

$$\lim_{x\to+\infty}\frac{1}{x}\ln\left[\frac{a^x-1}{x(a-1)}\right]$$

$$\overset{\frac{*}{\infty}}{=}\lim_{x\to+\infty}\frac{x(a-1)}{a^x-1}*\frac{1}{a-1}*\frac{x*\ln a*a^x-(a^x-1)}{x^2}$$

$$=\lim_{x\to+\infty}\frac{xa^x\ln a-(a^x-1)}{x(a^x-1)}=\lim_{x\to+\infty}\left[\frac{a^x\ln a}{a^x-1}-\frac{1}{x}\right]$$

$if\ 0<a<1$

$$\lim_{x\to+\infty}\frac{a^x}{a^x-1}=0\qquad\Rightarrow I=e^0=1$$

$if\ a>1$

$$\lim_{x\to+\infty}\frac{a^x}{a^x-1}=1\qquad\Rightarrow I=e^{\ln a}=a$$

（*mathmagic lite 2019.05.03*）

$f(x) \in C[a,b], g(x) \in C[a,b], (a < b)$

$\forall x \in [a,b], have\ f(x) \geq 0, g(x) > 0$

$let\ M = \max\limits_{[a,b]} f(x), \quad M > 0$

$prove: I = \lim\limits_{n \to \infty} \left[\int_a^b f^n(x) g(x) dx \right]^{\frac{1}{n}} = M$

$0 \leq f(x) \leq M \quad \Rightarrow f^n(x) g(x) \leq M^n g(x)$

$\therefore \int_a^b f^n(x) g(x) dx \leq \int_a^b M^n g(x) dx = M^n \int_a^b g(x) dx$

$\lim\limits_{n \to \infty} \left[\int_a^b g(x) dx \right]^{\frac{1}{n}} = 1 \quad \Rightarrow \lim\limits_{n \to \infty} \left[M^n \int_a^b g(x) dx \right]^{\frac{1}{n}} = M$

$let\ f(c) = M, c \in [a,b] \quad \Rightarrow \lim\limits_{x \to c^+} f(x) = M\ or\ \lim\limits_{x \to c^-} f(x) = M$

$\therefore \forall \varepsilon > 0, \exists \delta > 0, when\ x \in (c, c+\delta)\ or\ x \in (c-\delta, c)$

$have\ |f(x) - M| < \varepsilon, \quad \Rightarrow M - f(x) < \varepsilon, \quad because\ f(x) \leq M$

$\therefore \exists p, q, a \leq p < q \leq b, when\ x \in [p,q], have\ M - f(x) < \varepsilon$

$\therefore when\ x \in [p,q] \subseteq [a,b], \quad have\ f(x) > M - \varepsilon \geq 0$

$\int_p^q (M-\varepsilon)^n g(x) dx \leq \int_p^q f^n(x) g(x) dx \leq \int_a^b f^n(x) g(x) dx$

$\lim\limits_{n \to \infty} \left[\int_p^q g(x) dx \right]^{\frac{1}{n}} = 1 \quad \Rightarrow \lim\limits_{n \to \infty} \left[(M-\varepsilon)^n \int_a^b g(x) dx \right]^{\frac{1}{n}} = M - \varepsilon$

$\therefore \forall \varepsilon > 0, have\ M - \varepsilon \leq I \leq M \quad \Rightarrow I = M$

定理 7.5.1 (定积分第一中值定理) 设函数 $f(x) \in C[a,b]$, $g(x) \in R[a,b]$ 且在区间 $[a,b]$ 上不变号, 则存在 $\xi \in [a,b]$, 使得

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$$

特别地, 当 $g(x) \equiv 1$ 时, 有 $\displaystyle\int_a^b f(x)\mathrm{d}x = f(\xi)(b-a)$.

例 7.5.1 设函数 $f(x) \in C[0,1]$, 证明

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x = \frac{\pi}{2}f(0).$$

证明 由于函数 $f(x) \in C[0,1]$, 从而存在 $M>0$, 使得对于 $\forall x \in [0,1]$, 有 $|f(x)| \leqslant M$. 由定积分的性质, 对于 $\forall n \in \mathbb{N}$, 我们有

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x = \int_0^{n^{-\frac{1}{3}}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x + \int_{n^{-\frac{1}{3}}}^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x.$$

下面我们分别来估计上面等式右边中的两个定积分. 对于第二个定积分我们有

$$\begin{aligned}\left|\int_{n^{-\frac{1}{3}}}^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x\right| &\leqslant \int_{n^{-\frac{1}{3}}}^1 \left|\frac{nf(x)}{1+n^2x^2}\right|\mathrm{d}x \\ &\leqslant \frac{nM}{1+n^{1+\frac{1}{3}}} \to 0 \quad (n\to\infty).\end{aligned}$$

而在第一个定积分 $\displaystyle\int_0^{n^{-\frac{1}{3}}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x$ 中, $\dfrac{n}{1+n^2x^2}$ 在 $[0,1]$ 上不变号, 因此存在 $\xi \in [0, n^{-\frac{1}{3}}]$, 使得

$$\begin{aligned}\int_0^{n^{-\frac{1}{3}}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x &= f(\xi)\int_0^{n^{-\frac{1}{3}}} \frac{n}{1+n^2x^2}\mathrm{d}x \\ &= f(\xi)\arctan nx\Big|_0^{n^{-\frac{1}{3}}} = f(\xi)\arctan n^{\frac{2}{3}}.\end{aligned}$$

当 $n\to\infty$ 时, 由于 $\xi \in [0, n^{-\frac{1}{3}}]$, 有 $\xi \to 0$, 而 $\arctan n^{\frac{2}{3}} \to \dfrac{\pi}{2}$, 因此

$$\int_0^{n^{-\frac{1}{3}}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x \to \frac{\pi}{2}f(0).$$

最终我们有

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x &= \lim_{n\to\infty}\int_0^{n^{-\frac{1}{3}}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x + \lim_{n\to\infty}\int_{n^{-\frac{1}{3}}}^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x \\ &= \frac{\pi}{2}f(0).\end{aligned}$$

（*mathmagic lite 2019.03.23*）

$$I=\int_0^1\frac{x\ln(1-x)}{1+x^2}dx,\quad J=\int_0^1\frac{x\ln(1+x)}{1+x^2}dx$$

we know $\displaystyle\int_0^1\frac{\ln(1+x)}{x}dx=-\int_0^1\frac{\ln x}{1+x}dx=\frac{\pi^2}{12}$

$$I+J=\int_0^1\frac{x\ln(1-x^2)}{1+x^2}dx=\frac{1}{2}\int_0^1\frac{\ln(1-x^2)}{1+x^2}d(x^2)=\frac{1}{2}\int_0^1\frac{\ln(1-t)}{1+t}dt$$

let $\displaystyle x=\frac{1-t}{1+t},\ t=\frac{1-x}{1+x},\ \frac{1}{1+t}dt=\frac{1+x}{2}*\frac{-2dx}{(1+x)^2},\ 1-t=\frac{2x}{1+x}$

$$I+J=\frac{1}{2}\int_1^0\frac{\ln 2+\ln x-\ln(1+x)}{1+x}*(-1)dx$$

$$=\frac{1}{2}\int_0^1\frac{\ln 2}{1+x}dx+\frac{1}{2}\int_0^1\frac{\ln x}{1+x}dx-\frac{1}{2}\int_0^1\frac{\ln(1+x)}{1+x}dx=\frac{1}{4}\ln^2 2-\frac{\pi^2}{24}$$

$$I-J=\int_0^1\frac{x}{1+x^2}\ln\left(\frac{1-x}{1+x}\right)dx$$

let $\displaystyle t=\frac{1-x}{1+x},\ x=\frac{1-t}{1+t},\ \frac{1}{1+x^2}=\frac{(1+t)^2}{(1+t)^2+(1-t)^2}=\frac{(1+t)^2}{2(1+t^2)}$

$$\frac{x}{1+x^2}dx=\frac{1-t}{1+t}*\frac{(1+t)^2}{2(1+t^2)}*\frac{-2dt}{(1+t)^2}=\frac{(1-t)*(-1)dt}{(1+t)(1+t^2)}$$

$$I-J=\int_1^0\ln t*\frac{(1-t)*(-1)dt}{(1+t)(1+t^2)}=\int_0^1\ln t*\left(\frac{1}{1+t}-\frac{t}{1+t^2}\right)dt$$

$$=\int_0^1\frac{\ln t}{1+t}dt-\frac{1}{2}\int_0^1\ln t\,d[\ln(1+t^2)]$$

$$=-\frac{\pi^2}{12}-\frac{1}{2}\ln t\ln(1+t^2)\Big|_0^1+\frac{1}{2}\int_0^1\ln(1+t^2)*\frac{1}{t}dt$$

$$=-\frac{\pi^2}{12}+\frac{1}{2}\int_0^1\frac{t\ln(1+t^2)}{t^2}dt=-\frac{\pi^2}{12}+\frac{1}{4}\int_0^1\frac{\ln(1+t^2)}{t^2}d(t^2)$$

$$=-\frac{\pi^2}{12}+\frac{1}{4}\int_0^1\frac{\ln(1+x)}{x}dx=-\frac{\pi^2}{12}+\frac{1}{4}*\frac{\pi^2}{12}=-\frac{\pi^2}{16}$$

$$\therefore I=\frac{1}{8}\ln^2 2-\frac{5\pi^2}{96},\quad J=\frac{1}{8}\ln^2 2+\frac{\pi^2}{96}$$

$$\int_0^1\frac{\ln(1+x)}{x(1+x^2)}dx=\int_0^1\left[\frac{\ln(1+x)}{x}-\frac{x\ln(1+x)}{1+x^2}\right]dx=\frac{\pi^2}{12}-J=-\frac{1}{8}\ln^2 2+\frac{7\pi^2}{96}$$

Exercise 1.83: 设 $g(x)>0, f(x)>0$ 都是 $[a,b]$ 上连续函数，求

$$\lim_{n\to\infty}\frac{\int_a^b g(x)(f(x))^{n+1}\,\mathrm{d}x}{\int_a^b g(x)(f(x))^n\,\mathrm{d}x}$$

Solution: 令

$$D_n=\int_a^b g(x)(f(x))^n\,\mathrm{d}x, M=\max_{a\leqslant x\leqslant b}f(x)$$

注意到

$$D_{n+1}=\int_a^b g(x)(f(x))^{n+1}\,\mathrm{d}x\leqslant M\int_a^b g(x)(f(x))^n\,\mathrm{d}x=MD_n\Longrightarrow\frac{D_{n+1}}{D_n}\leqslant M$$

即 $\{\frac{D_{n+1}}{D_n}\}$ 有上界.

由柯西不等式, 得

$$\begin{aligned}D_{n+1}^2&=\left(\int_a^b(f(x))^{n+1}g(x)\,\mathrm{d}x\right)^2\\&=\left((f(x))^{\frac{n+2}{2}}(g(x))^{\frac{1}{2}}\cdot(f(x))^{\frac{n}{2}}(g(x))^{\frac{1}{2}}\,\mathrm{d}x\right)^2\\&\leqslant D_{n+2}D_n\Longrightarrow\frac{D_{n+1}}{D_n}\leqslant\frac{D_{n+2}}{D_{n+1}}\end{aligned}$$

即 $\{\frac{D_{n+1}}{D_n}\}$ 单调递增.

由单调有界准则, 可知 $\lim\limits_{n\to\infty}\frac{D_{n+1}}{D_n}$ 存在.

故

$$\lim_{n\to\infty}\frac{D_{n+1}}{D_n}=\lim_{n\to\infty}(D_n)^{\frac{1}{n}}=M$$

$$①\int_0^{\infty}\frac{\ln x}{(1+x^2)(1+x^3)}\mathrm{d}x=\int_0^1\frac{\ln x}{(1+x^2)(1+x^3)}\mathrm{d}x-\int_0^1\frac{x^3\ln x}{(1+x^2)(1+x^3)}\mathrm{d}x$$

$$=\int_0^1\left(\frac{x+1}{2(x^2+1)}-\frac{x^2+x-1}{2(x^3+1)}\right)\ln x\,\mathrm{d}x-\int_0^1\left(\frac{1-x}{2(x^2+1)}+\frac{x^2+x-1}{2(x^3+1)}\right)\ln x\,\mathrm{d}x$$

$$=-\frac{37\pi^2}{432}$$

$$②\frac{2n}{(n+1)\pi}\leqslant\frac{\int_0^x|\sin t|\mathrm{d}t}{x}\leqslant\frac{2(n+1)}{n\pi}\Rightarrow\lim_{n\to\infty}\left(\frac{\int_0^x|\sin t|\mathrm{d}t}{x}\right)^2=\frac{4}{\pi^2}$$

$$③\exp\left(3\sum_{k=0}^{x}\frac{1}{x+k}\right)=\exp\left(3\int_0^1\frac{1}{1+t}\mathrm{d}t\right)=8$$

$$④\mathrm{Li}_5\left(\frac{1}{x}\right)\sim\frac{1}{3125x^5}+\frac{1}{1024x^4}+\frac{1}{243x^3}+\frac{1}{32x^2}+\frac{1}{x}$$

$$\Rightarrow x^5\mathrm{Li}_5\left(\frac{1}{x}\right)-x^4-\frac{x^3}{32}\sim\frac{x^2}{243}+\frac{x}{1024}+\frac{1}{3125}$$

$$\lim_{x\to\infty}-\frac{\int_0^x\frac{\ln t\,\mathrm{d}t}{(1+t^2)(1+t^3)}\left(\int_0^x|\sin t|\mathrm{d}t\right)^2\exp\left(3\sum_{k=0}^{x}\frac{1}{x+k}\right)}{x^5\mathrm{Li}_5\left(\frac{1}{x}\right)-x^4-\frac{x^3}{32}}$$

$$=\frac{37\pi^2}{54}\lim_{x\to\infty}\frac{\left(\int_0^x|\sin t|\mathrm{d}t\right)^2}{\frac{x^2}{243}+\frac{x}{1024}+\frac{1}{3125}}=\frac{37\pi^2}{54}\lim_{x\to\infty}\frac{\left(\frac{\int_0^x|\sin t|\mathrm{d}t}{x}\right)^2}{\frac{1}{243}+\frac{1}{1024x}+\frac{1}{3125x^2}}$$

$$=\frac{37\pi^2}{54}\cdot\frac{4}{\pi^2}\cdot 243=666$$

（mathmagic lite 2019.06.07）

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (p>0,q>0)$$

$$B(p,q)=B(q,p) \quad (p>0,q>0)$$

$$B(p,q)=\int_0^{+\infty}\frac{x^{q-1}}{(1+x)^{p+q}}dx \quad (p>0,q>0)$$

$$let\ t=\frac{x}{1+x}, \quad \begin{cases} x=+\infty, t=1 \\ x=0, \quad t=0 \end{cases}$$

$$t(1+x)=x \quad \Rightarrow t+tx=x \quad \Rightarrow t=x(1-t)$$

$$x=\frac{t}{1-t} \quad \Rightarrow dx=d\left(-1+\frac{1}{1-t}\right)=\frac{1}{(1-t)^2}dt$$

$$\int_0^{+\infty}\frac{x^{q-1}}{(1+x)^{p+q}}dx=\int_0^1\frac{\left(\frac{t}{1-t}\right)^{q-1}}{\left(1+\frac{t}{1-t}\right)^{p+q}}*\frac{1}{(1-t)^2}dt$$

$$=\int_0^1\frac{t^{q-1}}{(1-t)^{q-1}}*(1-t)^{p+q}*\frac{1}{(1-t)^2}dt$$

$$=\int_0^1 t^{q-1}(1-t)^{1-q+p+q-2}dt=\int_0^1 t^{q-1}(1-t)^{p-1}dt$$

$$=B(q,p)=B(p,q)$$

（mathmagic lite 2019.06.07）

$$J(a)=\int_0^{+\infty}\frac{x^a}{(1+x^2)(1+x^3)}dx$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{x^a(x+1)}{1+x^2}dx-\frac{1}{2}\int_0^{+\infty}\frac{x^a(x^2+x-1)}{1+x^3}dx$$

$$let\ x=t^{\frac{1}{2}},dx=\frac{1}{2t^{\frac{1}{2}}}dt,\quad let\ x=u^{\frac{1}{3}},dx=\frac{1}{3u^{\frac{2}{3}}}du$$

$$J(a)=\frac{1}{2}\int_0^{+\infty}\frac{t^{\frac{a}{2}}(t^{\frac{1}{2}}+1)}{1+t}*\frac{1}{2t^{\frac{1}{2}}}dt-\frac{1}{2}\int_0^{+\infty}\frac{u^{\frac{a}{3}}(u^{\frac{2}{3}}+u^{\frac{1}{3}}-1)}{1+u}*\frac{1}{3u^{\frac{2}{3}}}du$$

$$=\frac{1}{4}\int_0^{+\infty}\frac{t^{\frac{a}{2}}+t^{\frac{a-1}{2}}}{1+t}dt-\frac{1}{6}\int_0^{+\infty}\frac{u^{\frac{a}{3}}+u^{\frac{a-1}{3}}-u^{\frac{a-2}{3}}}{1+u}du$$

$$=\frac{1}{4}B\left(1+\frac{a}{2},-\frac{a}{2}\right)+\frac{1}{4}B\left(\frac{1+a}{2},\frac{1-a}{2}\right)$$

$$-\frac{1}{6}B\left(1+\frac{a}{3},-\frac{a}{3}\right)-\frac{1}{6}B\left(\frac{2+a}{3},\frac{1-a}{3}\right)-\frac{1}{6}B\left(\frac{1+a}{3},\frac{2-a}{3}\right)$$

$$J'(a)=\int_0^{+\infty}\frac{x^a\ln x}{(1+x^2)(1+x^3)}dx$$

$$J'(0)=\int_0^{+\infty}\frac{\ln x}{(1+x^2)(1+x^3)}dx=-\frac{37\pi^2}{432}$$

第一类欧拉积分（B 函数）：

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\quad(x,y>0)$$

B 函数的其他积分表达式：

$$B(p,q)\xlongequal{t\to\cos^2\theta}2\int_0^{\frac{\pi}{2}}\cos^{2p-1}\theta\sin^{2q-1}\theta\,d\theta$$

$$B(p,q)\xlongequal{t\to\frac{t}{1+t}}\int_0^{+\infty}\frac{t^{p-1}}{(1+t)^{p+q}}dt$$

余元公式：$B(x,1-x)=\Gamma(x)\Gamma(1-x)=\dfrac{\pi}{\sin\pi x}$

第二类欧拉积分（Γ 函数）：

$$\Gamma(z)=\int_0^{+\infty}t^{z-1}e^{-t}dt$$

Γ 函数的递推公式为 $\Gamma(x+1)=x\Gamma(x)$， 对于非负整数 n，有 $\Gamma(n+1)=n!$

特殊值 $\Gamma\left(\dfrac{1}{2}\right)=2\displaystyle\int_0^{+\infty}e^{-x^2}dx=\sqrt{\pi}$

Digamma 函数是 Γ 函数的对数导数，用 $\psi(x)$ 或 $\psi^{(0)}(x)$ 表示。

$$\psi(x):=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)},$$

在计算中我们经常需要对 Γ 函数求导，也就是需要用到 $\Gamma'(x)$ 的特殊值，比如

$\Gamma'(1)=-\gamma$，$\Gamma'\left(\dfrac{1}{2}\right)=-\sqrt{\pi}(\gamma+2\ln 2)$，…… 以下给出推导过程：

利用维尔斯特拉斯公式：$\Gamma(x)=\dfrac{e^{-\gamma x}}{x}\displaystyle\prod_{k=1}^{\infty}\frac{e^{\frac{x}{k}}}{1+\frac{x}{k}}$

两边取对数：$\ln\Gamma(x)=-\gamma x-\ln x+\displaystyle\sum_{k=1}^{\infty}\left(\frac{x}{k}-\ln\frac{k+x}{k}\right)$

等式两边求导：$\psi(x)=-\gamma-\dfrac{1}{x}+\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+x}\right)$

即 $\dfrac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\dfrac{1}{x}+\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+x}\right)$

等式两边继续求导：$\dfrac{\Gamma''(x)\Gamma(x)-\Gamma'(x)^2}{\Gamma(x)^2}=\dfrac{1}{x^2}+\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+x)^2}=\zeta(2,x)$

因此 $\Gamma''(x)=\dfrac{\Gamma'(x)^2}{\Gamma(x)}+\Gamma(x)\zeta(2,x)$

据此很容易得到：$\psi(1)=-\gamma$，　$\psi\left(\dfrac{1}{2}\right)=-\gamma-2+\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{2}{2k+1}\right)=-\gamma-2\ln 2$

$\Gamma''(1)=\gamma^2+\dfrac{\pi^2}{6}$，$\Gamma''\left(\dfrac{1}{2}\right)=\sqrt{\pi}\left[(\gamma+2\ln 2)^2+\dfrac{\pi^2}{2}\right]$，$\Gamma''(2)=\gamma^2-2\gamma+\dfrac{\pi^2}{6}$

例 3.21（全国大学生 2016 年预赛题、北京市 1997 年竞赛题） 设函数 $f(x)$ 在区间 $[a,b]$ 上具有连续导数，证明：

$$\lim_{n\to\infty} n\left[\int_a^b f(x)\mathrm{d}x-\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{k(b-a)}{n}\right)\right]=\frac{b-a}{2}[f(a)-f(b)]$$

解析 对区间 $[a,b]$ 进行 n 等分，分点分别为 $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$，并记 $h=\dfrac{b-a}{n}$，则 $x_k=a+kh(k=1,2,\cdots,n)$. 因此，上式

$$\begin{aligned}
\text{左边}&=\lim_{n\to\infty} n\left[\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} f(x)\mathrm{d}x-\sum_{k=1}^{n} hf(x_k)\right]\\
&=\lim_{n\to\infty} n\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}(f(x)-f(x_k))\mathrm{d}x\\
&=\lim_{n\to\infty} n\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\frac{f(x)-f(x_k)}{x-x_k}\cdot(x-x_k)\mathrm{d}x\\
&=\lim_{n\to\infty} n\cdot\sum_{k=1}^{n}\frac{f(\xi_k)-f(x_k)}{\xi_k-x_k}\cdot\int_{x_{k-1}}^{x_k}(x-x_k)\mathrm{d}x \quad \text{（推广积分中值定理）}\\
&=\lim_{n\to\infty} n\cdot\sum_{k=1}^{n} f'(\eta_k)\left(-\frac{1}{2}\right)(x_k-x_{k-1})^2 \quad \text{（拉格朗日中值定理）}\\
&=-\frac{1}{2}\lim_{n\to\infty}(b-a)\sum_{k=1}^{n} f'(\eta_k)\cdot h=-\frac{1}{2}(b-a)\cdot\int_a^b f'(x)\mathrm{d}x\\
&=\frac{b-a}{2}[f(a)-f(b)]=\text{右边}
\end{aligned}$$

其中，$\xi_k\in(x_{k-1},x_k)$，$\eta_k\in(\xi_k,x_k)$.

（mathmagic lite 2019.04.21）

$$a<b$$

$$h=\frac{b-a}{n},\quad x_k=a+kh=a+\frac{k(b-a)}{n}$$

$$k=0,1,2,3,\cdots,n$$

$$x_0=a,\quad x_n=b$$

$$\int_a^b f(x)dx=\int_{x0}^{x_n} f(x)dx$$

$$=\int_{x0}^{x1} f(x)dx+\int_{x1}^{x2} f(x)dx+\cdots+\int_{x_{n-1}}^{x_n} f(x)dx$$

$$=\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} f(x)dx$$

$$\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{k(b-a)}{n}\right)$$

$$=h\sum_{k=1}^{n} f(x_k)=\sum_{k=1}^{n} hf(x_k)$$

（mathmagic lite 2019.04.21）

$$x_k = a + kh$$

$$x_k - x_{k-1} = (a+kh) - [a + (k-1)h] = h$$

$$\sum_{k=1}^{n} hf(x_k) = \sum_{k=1}^{n} (x_k - x_{k-1}) * f(x_k)$$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} dx * f(x_k)$$

$$\because f(x_k) \text{ not contain } x$$

$$\therefore \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} dx * f(x_k) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x_k)\, dx$$

$$\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\, dx - \sum_{k=1}^{n} hf(x_k)$$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\, dx - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x_k)\, dx$$

$$= \sum_{k=1}^{n} \left[\int_{x_{k-1}}^{x_k} f(x)\, dx - \int_{x_{k-1}}^{x_k} f(x_k)\, dx \right]$$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [f(x) - f(x_k)]\, dx$$

（mathmagic lite 2019.04.21）

$$\int_{x_{k-1}}^{x_k}(x-x_k)dx$$

$$=\int_{x_{k-1}}^{x_k}(x-x_k)d(x-x_k)$$

$$=\frac{1}{2}(x-x_k)^2\Big|_{x_{k-1}}^{x_k}$$

$$=0-\frac{1}{2}(x_{k-1}-x_k)^2$$

$$=-\frac{1}{2}(x_k-x_{k-1})^2$$

$$n*\left(-\frac{1}{2}\right)(x_k-x_{k-1})^2=-\frac{1}{2}nh^2$$

$$=-\frac{1}{2}h*n*\frac{b-a}{n}=-\frac{1}{2}(b-a)h$$

(mathmagic lite 2019.04.21)

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^{n} f'(\eta_k) * h$$

$$\left[a, a+\frac{b-a}{n}\right],$$

$$\left[a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}\right],$$

$$\cdots,$$

$$\left[a+\frac{(n-1)(b-a)}{n}, a+\frac{n(b-a)}{n}\right]$$

$$\triangle x=\frac{b-a}{n}=h$$

$$\eta_k \in [x_{k-1}, x_k]$$

$$\therefore \lim_{n\longrightarrow\infty}\sum_{k=1}^{n} f'(\eta_k) * h=\int_a^b f'(x)dx$$

$$=f(x)\Big|_a^b=f(b)-f(a)$$

(mathmagic lite 2019.04.21)

$$c < d,\ \int_c^d (x-d)dx \neq 0$$

prove: $\int_c^d [f(x)-f(d)]dx = f'(\eta)\int_c^d (x-d)dx$

$\eta \in [c,d]$

lagrange $\Rightarrow f(x)-f(d)=(x-d)f'(\xi_x)$

ξ_x is between x and d

$\because f'(x)$ is continuous

$\therefore$ when $x \in [c,d]$, have $m \le f'(x) \le M$

$\because x-d \le 0$

$\therefore m(x-d) \ge (x-d)f'(\xi_x) \ge M(x-d)$

$$\therefore \int_c^d m(x-d)dx \ge \int_c^d (x-d)f'(\xi_x)dx \ge \int_c^d M(x-d)dx$$

$$\therefore m\int_c^d (x-d)dx \ge \int_c^d (x-d)f'(\xi_x)dx \ge M\int_c^d (x-d)dx$$

$$\because \int_c^d (x-d)dx < 0$$

$$\therefore m \le \frac{\int_c^d (x-d)f'(\xi_x)dx}{\int_c^d (x-d)dx} = \frac{\int_c^d [f(x)-f(d)]dx}{\int_c^d (x-d)dx} \le M$$

$\because f'(x)$ is continuous

$$\therefore \exists \eta \in [c,d],\ \text{have } f'(\eta) = \frac{\int_c^d [f(x)-f(d)]dx}{\int_c^d (x-d)dx}$$

we prove it.

定理 7.5.1 (定积分第一中值定理) 设函数 $f(x)\in C[a,b]$, $g(x)\in R[a,b]$ 且在区间 $[a,b]$ 上不变号, 则存在 $\xi\in[a,b]$, 使得

$$\int_a^b f(x)g(x)\mathrm{d}x=f(\xi)\int_a^b g(x)\mathrm{d}x.$$

特别地, 当 $g(x)\equiv 1$ 时, 有 $\int_a^b f(x)\mathrm{d}x=f(\xi)(b-a)$.

证明 不妨设 $g(x)\geqslant 0$, 并令 m,M 分别是函数 $f(x)$ 在区间 $[a,b]$ 上的最小值和最大值, 则对于 $\forall x\in[a,b]$, 有

$$mg(x)\leqslant f(x)g(x)\leqslant Mg(x).$$

因此有

$$m\int_a^b g(x)\mathrm{d}x\leqslant\int_a^b f(x)g(x)\mathrm{d}x\leqslant M\int_a^b g(x)\mathrm{d}x.$$

若 $\int_a^b g(x)\mathrm{d}x=0$, 由上式得 $\int_a^b f(x)g(x)\mathrm{d}x=0$, 此时任取 $\xi\in[a,b]$ 即可. 当 $\int_a^b g(x)\mathrm{d}x\neq 0$ 时, 则有 $m\leqslant\dfrac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x}\leqslant M.$

由连续函数的介值定理, 必存在 $\xi\in[a,b]$, 使得 $f(\xi)=\dfrac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x}.$

注 1 在定积分第一中值定理中, 若只假定 $f(x)\in R[a,b]$, 令

$$m=\inf_{x\in[a,b]}\{f(x)\},\quad M=\sup_{x\in[a,b]}\{f(x)\},$$

则用同样的方法可以证明存在 $\mu\in[m,M]$, 使得

$$\int_a^b f(x)g(x)\mathrm{d}x=\mu\int_a^b g(x)\mathrm{d}x.$$

注 2 若在定积分第一中值定理中假定 $f(x)$ 与 $g(x)$ 均是连续函数, 且 $g(x)$ 在 $[a,b]$ 上不变号, 则可以证明必存在 $\xi\in(a,b)$, 使得

推广形式

$$\int_a^b f(x)g(x)\mathrm{d}x=f(\xi)\int_a^b g(x)\mathrm{d}x$$

一道积分等式的证明

设 f 是一个 n 次多项式, 且满足条件 $\int_0^1 x^k f(x)\mathrm{d}x = 0, k = 1, 2, \cdots, n$, 证明:

$$\int_0^1 f^2(x)\,\mathrm{d}x = (n+1)^2\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

证明　设 $f(x) = a_0 + a_1x + \cdots + a_nx^n, a_n \neq 0$. 对任意 $t > -1$, 有

$$\begin{aligned}\int_0^1 x^t f(x)\,\mathrm{d}x &= \int_0^1 \left(a_0x^t + a_1x^{t+1} + \cdots + a_nx^{t+n}\right)\mathrm{d}x \\ &= \frac{a_0}{t+1} + \frac{a_1}{t+2} + \cdots + \frac{a_n}{t+n+1} = \frac{p(t)}{(t+1)(t+2)\cdots(t+n+1)},\end{aligned}$$

其中 $p(t)$ 是 t 的 n 次多项式. 根据题意有 $p(1) = p(2) = \cdots = p(n) = 0$, 因此 $p(t) = A(t-1)(t-2)\cdots(t-n)$. 而

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{p(0)}{(n+1)!} = \frac{A(-1)^n n!}{(n+1)!} = \frac{(-1)^n A}{n+1},$$

所以 $A = (-1)^n(n+1)\int_0^1 f(x)\mathrm{d}x$. 另一方面, 在等式 $\frac{a_0}{t+1} + \frac{a_1}{t+2} + \cdots + \frac{a_n}{t+n+1} = \frac{p(t)}{(t+1)(t+2)\cdots(t+n+1)}$ 两边乘以 $t+1$, 再令 $t = -1$ 可得

$$a_0 = \frac{p(-1)}{n!} = \frac{(-1)^n(n+1)!A}{n!} = (-1)^n(n+1)A = (n+1)^2\int_0^1 f(x)\,\mathrm{d}x,$$

因此

$$\int_0^1 f^2(x)\,\mathrm{d}x = \int_0^1 (a_0 + a_1x + \cdots + a_nx^n) f(x)\,\mathrm{d}x = a_0\int_0^1 f(x)\,\mathrm{d}x = (n+1)^2\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

□

（mathmagic lite 2018.08.26）

$$F(x)=\int_{\alpha(x)}^{\beta(x)} f(t)dt$$

$$\frac{d[F(x)]}{dx}=f[\beta(x)]*\frac{d[\beta(x)]}{dx}-f[\alpha(x)]*\frac{d[\alpha(x)]}{dx}$$

$$G(x)=\int_{\mu(x)}^{\lambda(x)} g(t,x)dt$$

$$\frac{dG(x)}{dx}=\int_{\mu(x)}^{\lambda(x)}\frac{\partial[g(t,x)]}{\partial x}dt$$

$$+g[\lambda(x),x]*\frac{d[\lambda(x)]}{dx}-g[\mu(x),x]*\frac{d[\mu(x)]}{dx}$$

（*mathmagic lite 2019.07.09*）

$\Omega: z = x^2 + y^2, x^2 + y^2 + z^2 = t^2,\ (t > 0)$

$F(t) = \iiint\limits_{\Omega} f(x^2 + y^2 + z^2)dxdydz$

$f(x)$ *is continuous*

$F'(t) = ?$

$$\begin{cases} z = x^2 + y^2 \\ x^2 + y^2 + z^2 = t^2 \end{cases} \Rightarrow z^2 + z = t^2 \Rightarrow z = \frac{-1 \pm \sqrt{4t^2 + 1}}{2}$$

$$z \geq 0 \quad \Rightarrow z = \frac{-1 + \sqrt{4t^2 + 1}}{2}, \quad let\ a^2 = \frac{-1 + \sqrt{4t^2 + 1}}{2}$$

$$\Omega: 0 \leq x^2 + y^2 \leq a^2, \quad x^2 + y^2 \leq z \leq \sqrt{t^2 - x^2 - y^2}$$

$$F(t) = \int_0^{2\pi} d\theta \int_0^a r dr \int_{r^2}^{\sqrt{t^2 - r^2}} f(r^2 + z^2)dz$$

$$= 2\pi \int_0^a \left[\int_{r^2}^{\sqrt{t^2 - r^2}} f(r^2 + z^2)dz \right] r dr$$

（mathmagic lite 2019.07.09）

$$a^2=\frac{-1+\sqrt{4t^2+1}}{2},\quad a^2+a^4=t^2,\quad a^2=\sqrt{t^2-a^2}$$

$$F(t)=2\pi\int_0^a\left[\int_{r^2}^{\sqrt{t^2-r^2}}f(r^2+z^2)dz\right]rdr$$

$$let\ g(r,t)=\left[\int_{r^2}^{\sqrt{t^2-r^2}}f(r^2+z^2)dz\right]*r$$

$$\frac{\partial g}{\partial t}=f(r^2+t^2-r^2)*\frac{2t}{2\sqrt{t^2-r^2}}*r=\frac{rtf(t^2)}{\sqrt{t^2-r^2}}$$

$$F'(t)$$

$$=2\pi\left[\int_{a^2}^{\sqrt{t^2-a^2}}f(a^2+z^2)dz\right]*a*\frac{da}{dt}+2\pi\int_0^a\frac{\partial g}{\partial t}dr$$

$$\because a^2=\sqrt{t^2-a^2}$$

$$\therefore F'(t)=2\pi\int_0^a\frac{\partial g}{\partial t}dr=2\pi\int_0^a\frac{rtf(t^2)}{\sqrt{t^2-r^2}}dr$$

$$=-\pi tf(t^2)\int_0^a\frac{1}{\sqrt{t^2-r^2}}d(t^2-r^2)$$

$$=-\pi tf(t^2)*2\sqrt{t^2-r^2}\Big|_0^a=-2\pi tf(t^2)*(a^2-t)$$

$$=2\pi tf(t^2)\left(t-\frac{-1+\sqrt{4t^2+1}}{2}\right)$$

（mathmagic lite 2019.07.28）

$$I=\lim_{n\longrightarrow\infty} n\int_0^{\frac{\pi}{2}} x\ln\left(1+\frac{\sin x}{x}\right)\cos^n x dx$$

$$let\ A=n\int_0^{\frac{\pi}{2}} x\ln\left(1+\frac{\sin x}{x}\right)\cos^n x dx$$

$$\because x\leq \tan x,\quad \frac{\sin x}{x}\leq 1$$

$$\therefore A\leq n\int_0^{\frac{\pi}{2}} \tan x*\ln 2*\cos^n x dx$$

$$=n\ln 2*\int_0^{\frac{\pi}{2}} \sin x\cos^{n-1} x dx$$

$$=-n\ln 2\int_0^{\frac{\pi}{2}} \cos^{n-1} x d(\cos x)=\ln 2$$

（*mathmagic lite 2019.07.28*）

$$I=\lim_{n\to\infty} n\int_0^{\frac{\pi}{2}} x\ln\left(1+\frac{\sin x}{x}\right)\cos^n x dx$$

$$let\ A=n\int_0^{\frac{\pi}{2}} x\ln\left(1+\frac{\sin x}{x}\right)\cos^n x dx$$

$$let\ f(x)=\frac{1}{x}\ln(1+x),\quad (0<x\leq 1)$$

$$\lim_{x\to 0^+} f(x)=1,\quad f(1)=\ln 2$$

$$f'(x)=\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}$$

$$g(x)=x-(1+x)\ln(1+x)$$

$$g'(x)=1-\ln(1+x)-1\leq 0\quad \Rightarrow f'(x)\leq 0$$

$$\therefore when\ 0<x\leq 1,\quad have\ f(x)\geq f(1)=\ln 2$$

$$\therefore \frac{x}{\sin x}\ln\left(1+\frac{\sin x}{x}\right)\geq \ln 2$$

$$\therefore A\geq n\int_0^{\frac{\pi}{2}} \ln 2 * \sin x * \cos^n x dx=\frac{n}{n+1}\ln 2$$

$$\lim_{n\to\infty}\frac{n}{n+1}\ln 2=\ln 2$$

(mathmagic lite 2019.08.09)

$$f(x)=\cos x,\quad B=\lim_{n\to\infty}\left[\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-An\right]$$

A and B exist, $A=?,B=?$

$$B=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-A}{\frac{1}{n}}$$

$$\frac{0}{0}\quad\Rightarrow\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-A\right]=0$$

$$\therefore A=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)dx=\sin 1$$

$$\lim_{n\to\infty}n\left[\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)dx\right]$$

$$=\frac{1-0}{2}*[f(1)-f(0)]=\frac{\cos 1-1}{2}$$

(mathmagic lite 2019.08.09)

$$I=\lim_{n\to\infty}\frac{1}{(2n-1)^{2011}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3 x\cos^2 x dx$$

$$x_n=\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3 x\cos^2 x dx$$

$$y_n=(2n-1)^{2011}$$

$$x_{n+1}-x_n=\sum_{k=0}^{n}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3 x\cos^2 x dx-\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3 x\cos^2 x dx$$

$$=\int_{2n\pi}^{(2n+1)\pi}x^{2010}\sin^3 x\cos^2 x dx$$

$$\because\ t\longrightarrow 0,\quad (1+t)^{\mu}-1\sim\mu t$$

$$\therefore y_{n+1}-y_n=(2n+1)^{2011}-(2n-1)^{2011}$$

$$=(2n-1)^{2011}*\left[\left(\frac{2n+1}{2n-1}\right)^{2011}-1\right]$$

$$\sim(2n-1)^{2011}*2011*\left(\frac{2n+1}{2n-1}-1\right)$$

$$=(2n-1)^{2010}*2011*2$$

（mathmagic lite 2019.08.09）

$$I=\lim_{n\to\infty}\frac{1}{(2n-1)^{2011}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3 x\cos^2 x dx$$

$$=\lim_{n\to\infty}\frac{x_{n+1}-x_n}{2*2011*(2n-1)^{2010}}$$

$$x_{n+1}-x_n=\int_{2n\pi}^{2n\pi+\pi}x^{2010}\sin^3 x\cos^2 x dx$$

$$\because 2n\pi\le x\le 2n\pi+\pi$$

$$\therefore \sin x\ge 0 \quad \Rightarrow \sin^3 x\cos^2 x\ge 0$$

$$let\ A=\int_{2n\pi}^{2n\pi+\pi}\sin^3 x\cos^2 x dx=\int_0^{\pi}\sin^3 x\cos^2 x dx=\frac{4}{15}$$

$$\therefore (2n\pi)^{2010}*A\le x_{n+1}-x_n\le(2n\pi+\pi)^{2010}*A$$

$$\because \lim_{n\to\infty}\frac{(2n\pi)^{2010}}{(2n-1)^{2010}}=\pi^{2010},\quad \lim_{n\to\infty}\frac{(2n\pi+\pi)^{2010}}{(2n-1)^{2010}}=\pi^{2010}$$

$$\therefore I=\pi^{2010}*\frac{1}{2*2011}*A=\frac{2}{30165}\pi^{2010}$$

(mathmagic lite 2019.08.22)

$$f(x)=x^{\frac{1}{n}}+x-r$$

$$x\geq 0,(n=1,2,3,\cdots),r>0$$

$f(x)$ *is continuous on* $[0,+\infty)$

$$f(0)=-r<0$$

$$f(r)=r^{\frac{1}{n}}>0$$

$$f'(x)=\frac{1}{n}x^{\frac{1}{n}-1}+1$$

when $x>0$, *have* $f'(x)>1$

$\therefore$ *only one* $x_n\in(0,+\infty)$, *have* $f(x_n)=0$

$$0<x_n<r$$

（mathmagic lite 2019.08.22）

$$f_n(x)=x^{\frac{1}{n}}+x-r,\quad f_{n+1}(x)=x^{\frac{1}{n+1}}+x-r$$

$$f_n(x_n)=0,\quad f_{n+1}(x_{n+1})=0,\quad 0<x_n<r$$

if $r=1$

$$\because \frac{(x_n)^{\frac{1}{n}}}{(x_n)^{\frac{1}{n+1}}}=(x_n)^{\frac{1}{n}-\frac{1}{n+1}}=(x_n)^{\frac{1}{n(n+1)}}<1$$

$$\therefore (x_n)^{\frac{1}{n}}<(x_n)^{\frac{1}{n+1}}\quad\Rightarrow f_n(x_n)<f_{n+1}(x_n)$$

$$\therefore f_{n+1}(x_n)>f_n(x_n)=0=f_{n+1}(x_{n+1})$$

$$[f_{n+1}(x)]'>0\quad\Rightarrow x_n>x_{n+1}$$

$$\therefore 0<\cdots<x_{n+1}<x_n<\cdots<x_2<x_1<1$$

$$\therefore \lim_{n\to\infty}x_n=A\quad exist,\quad A\geq 0$$

$$(x_n)^{\frac{1}{n}}+x_n-1=0\quad\Rightarrow x_n=(1-x_n)^n$$

$$if\ \lim_{n\to\infty}x_n=A>0$$

$$\lim_{n\to\infty}(1-x_n)^n=\lim_{n\to\infty}\exp\{n\ln(1-x_n)\}$$

$$\because \lim_{n\to\infty}n\ln(1-x_n)=\lim_{n\to\infty}n\ln(1-A)=-\infty$$

$$\therefore \lim_{n\to\infty}x_n=\lim_{n\to\infty}(1-x_n)^n=0$$

it is impossible, so $\lim_{n\to\infty}x_n=A=0$

（mathmagic lite 2019.08.22）

$f_n(x)=x^{\frac{1}{n}}+x-r,\quad f_{n+1}(x)=x^{\frac{1}{n+1}}+x-r$

$f_n(x_n)=0,\quad f_{n+1}(x_{n+1})=0,\quad 0<x_n<r$

if $r=2$

$f_n(1)=0 \quad\Rightarrow x_n=1 \quad\Rightarrow \lim\limits_{n\to\infty}x_n=1$

if $r>2$

$f_n(1)=1+1-r<0 \quad\Rightarrow 1<x_n<r$

$(x_n)^{\frac{1}{n}}+x_n-r=0 \quad\Rightarrow x_n=r-(x_n)^{\frac{1}{n}}$

$1<x_n<r \quad\Rightarrow 1<(x_n)^{\frac{1}{n}}<r^{\frac{1}{n}}$

$\lim\limits_{n\to\infty}r^{\frac{1}{n}}=1 \quad\Rightarrow \lim\limits_{n\to\infty}(x_n)^{\frac{1}{n}}=1 \quad\Rightarrow \lim\limits_{n\to\infty}x_n=r-1$

if $1<r<2$

$f_n(r-1)=(r-1)^{\frac{1}{n}}+(r-1)-r=(r-1)^{\frac{1}{n}}-1<0$

$f_n(1)=1+1-r>0$

$\therefore r-1<x_n<1 \quad\Rightarrow (r-1)^{\frac{1}{n}}<(x_n)^{\frac{1}{n}}<1$

$\lim\limits_{n\to\infty}(r-1)^{\frac{1}{n}}=1 \quad\Rightarrow \lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\left[r-(x_n)^{\frac{1}{n}}\right]=r-1$

if $0<r<1$

$(x_n)^{\frac{1}{n}}+x_n-r=0 \quad\Rightarrow (x_n)^{\frac{1}{n}}=r-x_n \quad\Rightarrow x_n=(r-x_n)^n$

$0<r-x_n<r \quad\Rightarrow 0<(r-x_n)^n<r^n$

$\lim\limits_{n\to\infty}r^n=0 \quad\Rightarrow \lim\limits_{n\to\infty}(r-x_n)^n=0 \quad\Rightarrow \lim\limits_{n\to\infty}x_n=0$

(mathmagic lite 2019.08.22)

$$f_n(x)=x^{\frac{1}{n}}+x-r,\quad f_{n+1}(x)=x^{\frac{1}{n+1}}+x-r$$

$$f_n(x_n)=0,\quad f_{n+1}(x_{n+1})=0,\quad 0<x_n<r$$

$$if\ 0<r<1$$

$$0<x_n<r<1$$

$$\because \frac{(x_n)^{\frac{1}{n}}}{(x_n)^{\frac{1}{n+1}}}=(x_n)^{\frac{1}{n}-\frac{1}{n+1}}=(x_n)^{\frac{1}{n(n+1)}}<1$$

$$\therefore (x_n)^{\frac{1}{n}}<(x_n)^{\frac{1}{n+1}}\quad \Rightarrow f_n(x_n)<f_{n+1}(x_n)$$

$$\therefore f_{n+1}(x_n)>f_n(x_n)=0=f_{n+1}(x_{n+1})$$

$$[f_{n+1}(x)]'>0\quad \Rightarrow x_n>x_{n+1}$$

$$\therefore 0<\cdots<x_{n+1}<x_n<\cdots<x_2<x_1<r$$

$$\therefore \lim_{n\to\infty}x_n\quad exist$$

$$(x_n)^{\frac{1}{n}}+x_n-r=0\quad \Rightarrow x_n=(r-x_n)^n$$

$$0<r-x_n<r\quad \Rightarrow 0<(r-x_n)^n<r^n$$

$$\lim_{n\to\infty}r^n=0\quad \Rightarrow \lim_{n\to\infty}(r-x_n)^n=0\quad \Rightarrow \lim_{n\to\infty}x_n=0$$

(*mathmagic lite 2019.08.22*)

$f_n(x)=x^{\frac{1}{n}}+x-r,\quad f_{n+1}(x)=x^{\frac{1}{n+1}}+x-r$

$f_n(x_n)=0,\quad f_{n+1}(x_{n+1})=0,\quad 0<x_n<r$

$if\ 1<r<2$

$f_n(1)=1+1-r>0\quad\Rightarrow x_n<1$

$f_n(r-1)=(r-1)^{\frac{1}{n}}-1<0\quad\Rightarrow x_n>r-1$

$\because\dfrac{(x_n)^{\frac{1}{n}}}{(x_n)^{\frac{1}{n+1}}}=(x_n)^{\frac{1}{n}-\frac{1}{n+1}}=(x_n)^{\frac{1}{n(n+1)}}<1$

$\therefore (x_n)^{\frac{1}{n}}<(x_n)^{\frac{1}{n+1}}\qquad\Rightarrow f_n(x_n)<f_{n+1}(x_n)$

$\therefore f_{n+1}(x_n)>f_n(x_n)=0=f_{n+1}(x_{n+1})$

$[f_{n+1}(x)]'>0\quad\Rightarrow x_n>x_{n+1}$

$\therefore r-1<\cdots<x_{n+1}<x_n<\cdots<x_2<x_1<1$

$\therefore \lim\limits_{n\to\infty}x_n\quad exist$

$(x_n)^{\frac{1}{n}}+x_n-r=0\quad\Rightarrow x_n=r-(x_n)^{\frac{1}{n}}$

$r-1<x_n<1\ \Rightarrow (r-1)^{\frac{1}{n}}<(x_n)^{\frac{1}{n}}<1$

$\lim\limits_{n\to\infty}(r-1)^{\frac{1}{n}}=1\quad\Rightarrow \lim\limits_{n\to\infty}(x_n)^{\frac{1}{n}}=1\quad\Rightarrow \lim\limits_{n\to\infty}x_n=r-1$

（mathmagic lite 2019.08.22）

$f_n(x)=x^{\frac{1}{n}}+x-r,\quad f_{n+1}(x)=x^{\frac{1}{n+1}}+x-r$

$f_n(x_n)=0,\quad f_{n+1}(x_{n+1})=0,\quad 0<x_n<r$

$if\ r>2$

$f_n(1)=1+1-r<0\quad \Rightarrow x_n>1$

$f_n(r-1)=(r-1)^{\frac{1}{n}}+(r-1)-r=(r-1)^{\frac{1}{n}}-1>0$

$\therefore x_n<r-1$

$\because \dfrac{(x_n)^{\frac{1}{n}}}{(x_n)^{\frac{1}{n+1}}}=(x_n)^{\frac{1}{n}-\frac{1}{n+1}}=(x_n)^{\frac{1}{n(n+1)}}>1$

$\therefore (x_n)^{\frac{1}{n}}>(x_n)^{\frac{1}{n+1}}\quad \Rightarrow f_n(x_n)>f_{n+1}(x_n)$

$\therefore f_{n+1}(x_n)<f_n(x_n)=0=f_{n+1}(x_{n+1})$

$[f_{n+1}(x)]'>0\quad \Rightarrow x_n<x_{n+1}$

$\therefore 1<x_1<x_2<\cdots<x_n<x_{n+1}<\cdots<1-r$

$\therefore \lim\limits_{n\to\infty}x_n\quad exist$

$(x_n)^{\frac{1}{n}}+x_n-r=0\quad \Rightarrow x_n=r-(x_n)^{\frac{1}{n}}$

$1<x_n<r-1\quad \Rightarrow 1<(x_n)^{\frac{1}{n}}<(r-1)^{\frac{1}{n}}$

$\lim\limits_{n\to\infty}(r-1)^{\frac{1}{n}}=1\quad \Rightarrow \lim\limits_{n\to\infty}(x_n)^{\frac{1}{n}}=1\quad \Rightarrow \lim\limits_{n\to\infty}x_n=r-1$

(*mathmagic lite 2019.08.22*)

$$f_n(x) = x^{\frac{1}{n}} + x - r,\quad f_{n+1}(x) = x^{\frac{1}{n+1}} + x - r$$

$$f_n(x_n) = 0,\quad f_{n+1}(x_{n+1}) = 0,\quad 0 < x_n < r$$

if $r = 1$

$$(x_n)^{\frac{1}{n}} + x_n - 1 = 0 \quad \Rightarrow (x_n)^{\frac{1}{n}} = 1 - x_n$$

$$\frac{1}{n}\ln x_n = \ln(1 - x_n)$$

$$\lim_{n\to\infty}\frac{x_n}{\frac{1}{n}} = -\lim_{n\to\infty} n(-x_n) = -\lim_{n\to\infty} n\ln(1 - x_n)$$

$$= -\lim_{n\to\infty}\ln x_n = +\infty$$

$$\therefore \sum_{n=1}^{\infty} x_n \textit{ is not convergent.}$$

(*mathmagic lite 2019.08.22*)

$f_n(x) = x^{\frac{1}{n}} + x - r,\quad f_{n+1}(x) = x^{\frac{1}{n+1}} + x - r$

$f_n(x_n) = 0,\quad f_{n+1}(x_{n+1}) = 0,\quad 0 < x_n < r$

if $0 < r < 1$

$(x_n)^{\frac{1}{n}} + x_n - r = 0 \quad \Rightarrow (x_n)^{\frac{1}{n}} = r - x_n$

$x_n = (r - x_n)^n < r^n$

$\because \sum_{n=1}^{\infty} r^n$ *is convergent.*

$\therefore \sum_{n=1}^{\infty} x_n$ *is convergent.*

(mathmagic lite 2019.08.22)

$$f_n(x)=x^{\frac{1}{n}}+x-r,\quad f_{n+1}(x)=x^{\frac{1}{n+1}}+x-r$$

$$f_n(x_n)=0,\quad f_{n+1}(x_{n+1})=0,\quad 0<x_n<r$$

$$if\ r>1 \quad \Rightarrow \lim_{n\to\infty}x_n=r-1$$

$$\because (x_n)^{\frac{1}{n}}+x_n-r=0$$

$$\therefore x_n-r+1=(x_n)^{\frac{1}{n}}-1$$

$$\sim \ln\left[1+(x_n)^{\frac{1}{n}}-1\right]=\ln\left[(x_n)^{\frac{1}{n}}\right]=\frac{1}{n}\ln x_n$$

$$if\ r=2 \quad \Rightarrow x_n=1$$

$$\therefore \sum_{n=1}^{\infty}(x_n-r+1)\ is\ convergent$$

$$if\ 1<r<2 \quad \Rightarrow x_n-r+1<0$$

$$\sum_{n=1}^{\infty}(x_n-r+1)\ is\ not\ convergent$$

$$if\ r>2 \quad \Rightarrow x_n-r+1>0$$

$$\sum_{n=1}^{\infty}(x_n-r+1)\ is\ not\ convergent$$

（mathmagic lite 2019.09.16）

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}$$

$$(0<\theta<1)$$

$$let\ f(x)=e^x, x=1, x_0=0$$

$$f'(x)=e^x, f''(x)=e^x, \cdots, f^{(n)}(x)=e^x$$

$$x-x_0=1, f^{(n)}(x_0)=e^0=1$$

$$\therefore e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{e^{\theta_n}}{(n+1)!}$$

$$\therefore e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{1}{(n+1)!}+\frac{e^{\theta_n}}{(n+2)!}$$

（mathmagic lite 2019.09.16）

$$I=\lim_{n\to\infty} n\sin(2\pi n!e)=?$$

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{1}{(n+1)!}+\frac{e^{\theta_n}}{(n+2)!}$$

$$=A_n+\frac{1}{(n+1)!}+\frac{e^{\theta_n}}{(n+2)!}\qquad (0<\theta_n<1)$$

$$2\pi n!e=2\pi n!A_n+\frac{2\pi}{n+1}+\frac{2\pi e^{\theta_n}}{(n+1)(n+2)}$$

$$\because n!A_n\in Z$$

$$\therefore \sin(2\pi n!e)=\sin\left[2\pi n!A_n+\frac{2\pi}{n+1}+\frac{2\pi e^{\theta_n}}{(n+1)(n+2)}\right]$$

$$=\sin\left[\frac{2\pi}{n+1}+\frac{2\pi e^{\theta_n}}{(n+1)(n+2)}\right]$$

$$\sim\frac{2\pi}{n+1}+\frac{2\pi e^{\theta_n}}{(n+1)(n+2)}$$

$$\therefore I=\lim_{n\to\infty} n*\left[\frac{2\pi}{n+1}+\frac{2\pi e^{\theta_n}}{(n+1)(n+2)}\right]$$

$$\because 0<\theta_n<1$$

$$\therefore I=2\pi+0=2\pi$$

(*mathmagic lite 2019.09.16*)

$$I = \lim_{n\to\infty} n\sin(\pi n!e) = ?$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{e^{\theta_n}}{(n+2)!}$$

$$= A_n + \frac{1}{(n+1)!} + \frac{e^{\theta_n}}{(n+2)!} \qquad (0 < \theta_n < 1)$$

$$\pi n!e = \pi n!A_n + \frac{\pi}{n+1} + \frac{\pi e^{\theta_n}}{(n+1)(n+2)}$$

$$n!A_n = n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{(n-2)!} + \frac{n!}{(n-1)!} + \frac{n!}{n!}$$

$$= B_n * \frac{n!}{(n-2)!} + n + 1 = n(n+1)B_n + n + 1, \quad B_n \in Z$$

$$\therefore \sin(\pi n!e)$$

$$= \sin\left[n(n+1)\pi B_n + (n+1)\pi + \frac{\pi}{n+1} + \frac{\pi e^{\theta_n}}{(n+1)(n+2)}\right]$$

$$= (-1)^{n+1}\sin\left[\frac{\pi}{n+1} + \frac{\pi e^{\theta_n}}{(n+1)(n+2)}\right]$$

$$\sim (-1)^{n+1}\left[\frac{\pi}{n+1} + \frac{\pi e^{\theta_n}}{(n+1)(n+2)}\right]$$

$$\therefore I = \lim_{n\to\infty} n * (-1)^{n+1}\left[\frac{\pi}{n+1} + \frac{\pi e^{\theta_n}}{(n+1)(n+2)}\right]$$

$$\because 0 < \theta_n < 1$$

$$\therefore I = \lim_{n\to\infty}(-1)^{n+1}\pi \qquad \textit{not exist}$$

(*mathmagic lite 2019.09.26*)

$$I(m)=\int_0^{\frac{\pi}{2}}\frac{\arctan(m\tan x)}{\tan x}dx \quad (m\geq 0)$$

$$\frac{d}{dm}\left[\frac{\arctan(m\tan x)}{\tan x}\right]=\frac{1}{\tan x}*\frac{\tan x}{1+(m\tan x)^2}=\frac{1}{1+m^2\tan^2 x}$$

$$I(0)=0,\ \ I'(m)=\int_0^{\frac{\pi}{2}}\frac{1}{1+m^2\tan^2 x}dx$$

$$if\ m=1 \ \ \Rightarrow I'(1)=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^2 x}dx=\int_0^{\frac{\pi}{2}}\cos^2 x dx=\frac{\pi}{4}$$

$$if\ m\neq 1 \ \ \Rightarrow I'(m)=\int_0^{\frac{\pi}{2}}\frac{1}{1+m^2\tan^2 x}*\frac{\sec^2 x}{\sec^2 x}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{1+m^2\tan^2 x}*\frac{1}{1+\tan^2 x}d(\tan x)$$

$$=\int_0^{+\infty}\frac{1}{1+m^2t^2}*\frac{1}{1+t^2}dt=\frac{1}{m^2-1}\int_0^{+\infty}\left(\frac{m^2}{1+m^2t^2}-\frac{1}{1+t^2}\right)dt$$

$$=\frac{1}{m^2-1}[m\arctan(mt)-\arctan t]\Big|_0^{+\infty}$$

$$=\frac{1}{m^2-1}*\left(m*\frac{\pi}{2}-\frac{\pi}{2}\right)=\frac{1}{m+1}*\frac{\pi}{2}$$

$$\therefore when\ m\geq 0, have\ I'(m)=\frac{1}{m+1}*\frac{\pi}{2}$$

$$I(m)=I(0)+\int_0^m I'(t)dt=0+\frac{\pi}{2}\int_0^m\frac{1}{t+1}dt$$

$$=\frac{\pi}{2}\ln(1+t)\Big|_0^m=\frac{\pi}{2}\ln(1+m)$$

关于Froullani积分　　Frullani

积分定义：设函数$f(x)$在$[0,+\infty)$有定义且连续，并且具有有穷极限$f(+\infty)$，则广义积分

$$\mathcal{I}=\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx=[f(0)-f(+\infty)]\ln\frac{b}{a}$$

证：对于$0<r<R<+\infty$，有

$$\mathcal{J}=\int_r^R\frac{f(ax)-f(bx)}{x}dx=\int_r^R\frac{f(ax)}{x}dx-\int_r^R\frac{f(bx)}{x}dx$$

$$=\int_{ar}^{aR}\frac{f(x)}{x}dx-\int_{br}^{bR}\frac{f(x)}{x}dx$$

$$=\int_{ar}^{br}\frac{f(x)}{x}dx-\int_{aR}^{bR}\frac{f(x)}{x}dx$$

由积分第一中值定理：$\mathcal{J}=f(\xi_1)\int_{ar}^{br}\frac{dx}{x}-f(\xi_2)\int_{aR}^{bR}\frac{dx}{x}=[f(\xi_1)-f(\xi_2)]\ln\frac{b}{a}$

其中$ar<\xi_1<br$，$aR<\xi_2<bR$.

令$r\to0^+$，$R\to+\infty$，即$\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx=[f(0)-f(+\infty)]\ln\frac{b}{a}$.

(1) 若极限$f(+\infty)$不存在，但存在$0<A<+\infty$，积分$\int_A^{+\infty}\frac{f(x)}{x}dx$收敛，由无穷积分收敛的柯西准则，

对$\forall\varepsilon>0$，当R充分大时，$\left|\int_{aR}^{bR}\frac{f(x)}{x}dx\right|<\varepsilon$，即$\lim\limits_{R\to+\infty}\int_{aR}^{bR}\frac{f(x)}{x}dx=0$，

故$\mathcal{I}=f(0)\ln\frac{b}{a}$；

(2) 若$f(x)$在$x=0$处不连续，但存在$0<A<+\infty$，积分$\int_0^A\frac{f(x)}{x}dx$收敛，由瑕积分收敛的充要条件，

对$\forall\varepsilon>0$，当r充分小时，$\left|\int_{ar}^{br}\frac{f(x)}{x}dx\right|<\varepsilon$，即$\lim\limits_{r\to0^+}\int_{ar}^{br}\frac{f(x)}{x}dx=0$，

故$\mathcal{I}=-f(+\infty)\ln\frac{b}{a}=f(+\infty)\ln\frac{a}{b}$.

一些广义积分即常见定积分汇总.

Dirichlet积分：$\int_0^{+\infty}\frac{\sin x}{x}dx=\int_0^{+\infty}\frac{\tan x}{x}dx=\frac{\pi}{2}$，$\int_0^{+\infty}\frac{\sin ax}{x}dx=\begin{cases}\frac{\pi}{2} & (a>0)\\ -\frac{\pi}{2} & (a<0)\end{cases}$

$\int_0^{+\infty}\sin x^2dx=\int_0^{+\infty}\cos x^2dx=\frac{\sqrt{2\pi}}{4}$　　$\int_0^{+\infty}\frac{\sin^2 ax}{x^2}dx=\int_0^{+\infty}\frac{\sin ax\cos bx}{x^2}dx=\frac{\pi}{2}a,\ (a<b)$

$\int_0^1\frac{\arcsin x}{x}dx=\frac{\pi}{2}\ln 2$　　$\int_0^{+\infty}\frac{\cos ax-\cos bx}{x}dx=\ln\left|\frac{b}{a}\right|$

$\int_0^{+\infty}\frac{\arctan ax-\arctan bx}{x}dx=\frac{\pi}{2}\ln\frac{a}{b},\ (a,b>0)$

$\int_0^{\frac{\pi}{2}}\frac{1}{a+b\cos x}dx=\frac{\arccos\frac{b}{a}}{\sqrt{a^2-b^2}},(a>b)$　　$\int_0^{\pi}\frac{1}{a+b\cos x}dx=\frac{\pi}{\sqrt{a^2-b^2}},(a>b>0)$

$\int_0^{+\infty}\frac{\cos ax}{1+x^2}dx=\begin{cases}\frac{\pi}{2}e^{-a} & (a>0)\\ \frac{\pi}{2}e^{a} & (a<0)\end{cases}$　　$\int_0^{\frac{\pi}{2}}\ln\sin x\,dx=\int_0^{\frac{\pi}{2}}\ln\cos x\,dx=-\frac{\pi}{2}\ln 2$

$\int_0^{\pi}x\ln\sin x\,dx=-\frac{\pi^2}{2}\ln 2$　　$\int_0^{\pi}\frac{\ln(1+a\cos x)}{\cos x}dx=\pi\arcsin a$

$\int_0^{\pi}\ln(a\pm b\cos x)dx=\pi\ln\frac{a+\sqrt{a^2-b^2}}{2},\ (a\geq b)$

$\int_0^{\frac{\pi}{2}}\frac{1}{a\cos^2x+b\sin^2x}dx=\frac{\pi}{2\sqrt{ab}},(a,b>0)$　　$\int_0^{\pi}\frac{a-b\cos x}{a^2+b^2-2ab\cos x}dx=\begin{cases}\frac{\pi}{a}, & (|a|>|b|)\\ \frac{\pi}{2a}, & (|a|=|b|)\\ 0, & (|a|<|b|)\end{cases}$

(mathmagic lite 2019.10.02)

(1) $\int_0^1 \frac{\ln(1+x^k)}{x}dx = \frac{\pi^2}{12k} \quad (k>0)$

(2) $\int_0^1 \frac{x^{k-1}\ln x}{1+x^k}dx = -\frac{\pi^2}{12k^2} \quad (k>0)$

(3) $I = \int_0^{+\infty} \frac{\ln x}{(1+x^2)(1+x^3)}dx = -\frac{37\pi^2}{432}$

(mathmagic lite 2019.10.02)

$$I=\int_0^1 \frac{\ln(1+x^k)}{x}dx \quad (k>0)$$

$$\because \ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x^k)^n$$

$$\therefore \frac{\ln(1+x^k)}{x}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{nk-1}$$

$$I=\int_0^1\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{nk-1}dx=\sum_{n=1}^{\infty}\int_0^1\frac{(-1)^{n-1}}{n}x^{nk-1}dx$$

$$\because \int_0^1 x^{nk-1}dx=\frac{1}{nk}x^{nk}\Big|_0^1=\frac{1}{nk}$$

$$\therefore I=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}*\frac{1}{nk}=\frac{1}{k}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

$$\because \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\quad \sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{24}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{8}$$

$$\therefore I=\frac{1}{k}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{1}{k}\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\right]$$

$$=\frac{1}{k}\left(\frac{\pi^2}{8}-\frac{\pi^2}{24}\right)=\frac{\pi^2}{12k}$$

(mathmagic lite 2019.10.02)

$$\int_0^1 \frac{\ln(1+x^k)}{x}dx = \frac{\pi^2}{12k} \quad (k>0)$$

$$prove: \int_0^1 \frac{x^{k-1}\ln x}{1+x^k}dx = -\frac{\pi^2}{12k^2}$$

$$\int_0^1 \frac{\ln(1+x^k)}{x}dx = \int_0^1 \ln(1+x^k)d(\ln x)$$

$$= \ln(1+x^k)\ln x\Big|_0^1 - \int_0^1 \ln x d[\ln(1+x^k)]$$

$$= 0 - \lim_{x\to 0^+} \ln(1+x^k)\ln x - \int_0^1 \frac{kx^{k-1}\ln x}{1+x^k}dx$$

$$= 0 - 0 - k\int_0^1 \frac{x^{k-1}\ln x}{1+x^k}dx$$

$$\therefore -k\int_0^1 \frac{x^{k-1}\ln x}{1+x^k}dx = \frac{\pi^2}{12k}$$

$$\therefore \int_0^1 \frac{x^{k-1}\ln x}{1+x^k}dx = -\frac{\pi^2}{12k^2}$$

(mathmagic lite 2019.10.02)

$$I=\int_0^{+\infty}\frac{\ln x}{(1+x^2)(1+x^3)}dx$$

$$I=\int_0^{1}\frac{\ln x}{(1+x^2)(1+x^3)}dx+\int_1^{+\infty}\frac{\ln x}{(1+x^2)(1+x^3)}dx$$

$$let\ t=\frac{1}{x}$$

$$\int_1^{+\infty}\frac{\ln x}{(1+x^2)(1+x^3)}dx=\int_1^{0}\frac{\ln\left(\frac{1}{t}\right)}{\left(1+\frac{1}{t^2}\right)\left(1+\frac{1}{t^3}\right)}*\frac{-1}{t^2}dt$$

$$=\int_0^{1}\frac{-t^3\ln t}{(t^2+1)(t^3+1)}dt=\int_0^{1}\frac{-x^3\ln x}{(1+x^2)(1+x^3)}dx$$

$$\therefore I=\int_0^{1}\frac{\ln x}{(1+x^2)(1+x^3)}dx+\int_0^{1}\frac{-x^3\ln x}{(1+x^2)(1+x^3)}dx$$

$$=\int_0^{1}\frac{(1-x^3)\ln x}{(1+x^2)(1+x^3)}dx$$

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$$I=\int_0^1\frac{(1-x^3)\ln x}{(1+x^2)(1+x^3)}dx$$

$$\frac{Ax+B}{1+x^2}+\frac{Dx^2+Ex+F}{1+x^3}$$

$$=\frac{(Ax+B)(1+x^3)+(1+x^2)(Dx^2+Ex+F)}{(1+x^2)(1+x^3)}$$

$compare\ a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$

$$\begin{cases}A+D=0\\B+E=-1\\D+F=0\\A+E=0\\B+F=1\end{cases}\Rightarrow\begin{cases}A=1\\B=0\\D=-1\\E=-1\\F=1\end{cases}$$

$$I=\int_0^1\frac{x}{1+x^2}\ln x dx+\int_0^1\frac{-x^2-x+1}{1+x^3}\ln x dx$$

$$\because\frac{-x^2-x+1}{1+x^3}=\frac{(x^2-x+1)-2x^2}{(1+x)(x^2-x+1)}=\frac{1}{1+x}-\frac{2x^2}{1+x^3}$$

$$I=\int_0^1\frac{x\ln x}{1+x^2}dx+\int_0^1\frac{\ln x}{1+x}dx-2\int_0^1\frac{x^2\ln x}{1+x^3}dx$$

$$=-\frac{\pi^2}{12*2^2}+\left(-\frac{\pi^2}{12*1^2}\right)-2\left(-\frac{\pi^2}{12*3^2}\right)=-\frac{37\pi^2}{432}$$

(mathmagic lite 2019.10.03)

$$a_n=\frac{1^n+2^n+\cdots+(n-\tau_n-1)^n}{n^n}$$

$$\int_k^{k+1}x^n dx-k^n=\int_k^{k+1}x^n dx-k^n\int_k^{k+1}dx$$

$$=\int_k^{k+1}x^n dx-\int_k^{k+1}k^n dx=\int_k^{k+1}(x^n-k^n)dx$$

$$\because k\le x\le k+1\quad\Rightarrow x^n-k^n\ge 0$$

$$\therefore\int_k^{k+1}x^n dx-k^n\ge 0\quad\Rightarrow k^n\le\int_k^{k+1}x^n dx$$

$$\therefore a_n\le\frac{\int_1^2 x^n dx+\int_2^3 x^n dx+\cdots+\int_{n-\tau_n-1}^{n-\tau_n}x^n dx}{n^n}$$

$$=\frac{1}{n^n}\int_1^{n-\tau_n}x^n dx<\frac{1}{n^n}\int_0^{n-\tau_n}x^n dx=\frac{1}{n^n}*\frac{(n-\tau_n)^{n+1}}{n+1}$$

$$=\frac{(n-\tau_n)^n}{n^n}*\frac{n-\tau_n}{n+1}<\frac{(n-\tau_n)^n}{n^n}=\left(1-\frac{\tau_n}{n}\right)^n$$

$$=\exp\left\{n\ln\left(1-\frac{\tau_n}{n}\right)\right\}$$

$$when\ 1-x>0,\ have\ \ln(1-x)\le -x$$

$$\therefore a_n<\exp\left\{n\ln\left(1-\frac{\tau_n}{n}\right)\right\}\le\exp\left\{n*\left(-\frac{\tau_n}{n}\right)\right\}=e^{-\tau_n}$$

$$\lim_{n\to\infty}\tau_n=+\infty\quad\Rightarrow\lim_{n\to\infty}e^{-\tau_n}=0$$

$$a_n\ge 0\quad\Rightarrow\lim_{n\to\infty}a_n=0$$

(mathmagic lite 2019.10.03)

$$b_n=\frac{(n-\tau_n)^n+(n-\tau_n+1)^n+\cdots+n^n}{n^n}$$
$$=\left(\frac{n-\tau_n}{n}\right)^n+\left(\frac{n-\tau_n+1}{n}\right)^n+\cdots+\left(\frac{n-0}{n}\right)^n$$
$$=\left(1-\frac{\tau_n}{n}\right)^n+\left(1-\frac{\tau_n-1}{n}\right)^n+\cdots+\left(1-\frac{0}{n}\right)^n$$
$$=\sum_{k=0}^{\tau_n}\left(1-\frac{k}{n}\right)^n$$
$$\because\left(1-\frac{k}{n}\right)^n=\exp\left\{n\ln\left(1-\frac{k}{n}\right)\right\}\le\exp\left\{n*\left(-\frac{k}{n}\right)\right\}=e^{-k}$$
$$\therefore b_n\le\sum_{k=0}^{\tau_n}e^{-k}$$
$$let\ I_n=-b_n+\sum_{k=0}^{\tau_n}e^{-k}=\sum_{k=0}^{\tau_n}e^{-k}-\sum_{k=0}^{\tau_n}\left(1-\frac{k}{n}\right)^n$$
$$\therefore I_n\ge 0$$
$$I_n=\sum_{k=0}^{\tau_n}\left[e^{-k}-\left(1-\frac{k}{n}\right)^n\right]=\sum_{k=0}^{\tau_n}e^{-k}\left[1-e^k\left(1-\frac{k}{n}\right)^n\right]$$
$$=\sum_{k=0}^{\tau_n}e^{-k}\left[1-\exp\left\{k+n\ln\left(1-\frac{k}{n}\right)\right\}\right]$$
$$\because 1-e^x\le -x$$
$$\therefore I_n\le\sum_{k=0}^{\tau_n}e^{-k}(-1)\left[k+n\ln\left(1-\frac{k}{n}\right)\right]$$

(*mathmagic lite 2019.10.03*)

$$0 \le I_n \le \sum_{k=0}^{\tau_n} e^{-k}(-1)\left[k + n\ln\left(1-\frac{k}{n}\right)\right]$$

$$\because \text{ when } 0 \le x < \frac{1}{2}, \text{ have } x + \ln(1-x) + x^2 \ge 0$$

$$\therefore -x - \ln(1-x) \le x^2$$

$$\because \frac{k}{n} \le \frac{\tau_n}{n} \in \left[0, \frac{1}{2}\right) \quad \text{because } \lim_{n\to\infty}\frac{\tau_n^2}{n} = 0$$

$$\therefore -\frac{k}{n} - \ln\left(1-\frac{k}{n}\right) \le \left(\frac{k}{n}\right)^2$$

$$\therefore (-1)\left[k + n\ln\left(1-\frac{k}{n}\right)\right] = n\left[-\frac{k}{n} - \ln\left(1-\frac{k}{n}\right)\right]$$

$$\le n * \left(\frac{k}{n}\right)^2 = \frac{k^2}{n} \le \frac{\tau_n^2}{n}$$

$$\therefore I_n \le \sum_{k=0}^{\tau_n}\left(e^{-k} * \frac{\tau_n^2}{n}\right) \le \frac{\tau_n^2}{n}\sum_{k=0}^{\tau_n} e^{-k}$$

$$\because \sum_{k=0}^{\tau_n} e^{-k} = 1 + e^{-1} + e^{-2} + \cdots + e^{-\tau_n} = \frac{1-e^{-\tau_n-1}}{1-e^{-1}} < \frac{1}{1-e^{-1}} = \frac{e}{e-1}$$

$$\therefore I_n \le \frac{\tau_n^2}{n} * \frac{e}{e-1}$$

$$\lim_{n\to\infty}\frac{\tau_n^2}{n} = 0, \quad I_n \ge 0 \quad \Rightarrow \lim_{n\to\infty} I_n = 0$$

(*mathmagic lite 2019.10.03*)

$$let\ f(x)=x+\ln(1-x)+x^2$$

$$f'(x)=1+\frac{-1}{1-x}+2x=\frac{(1-x)-1+2x(1-x)}{1-x}$$

$$=\frac{x-2x^2}{1-x}=\frac{x(1-2x)}{1-x}$$

$$when\ 0<x<\frac{1}{2},\ have\ f'(x)>0$$

$$\therefore when\ 0<x<\frac{1}{2},\ have\ f(x)>f(0)=0$$

$$\therefore when\ 0<x<\frac{1}{2},\ have\ x+\ln(1-x)+x^2>0$$

$$\because \lim_{n\to\infty}\frac{\tau_n^2}{n}=0$$

$$\therefore \exists N>0,\ when\ n>N,\ have\ \frac{\tau_n^2}{n}<\varepsilon$$

$$we\ need\ \frac{k}{n}\leq\frac{\tau_n}{n}\in\left[0,\frac{1}{2}\right)$$

$$\frac{\tau_n}{n}<\frac{\varepsilon}{\tau_n}<\varepsilon$$

$$\therefore we\ use\ \varepsilon\leq\frac{1}{2},\ when\ n>N,\ have\ all\ \frac{k}{n}\leq\frac{\tau_n}{n}\in\left[0,\frac{1}{2}\right)$$

(*mathmagic lite 2019.10.03*)

$$\lim_{n\to\infty} I_n = 0$$

$$I_n = -b_n + \sum_{k=0}^{\tau_n} e^{-k}$$

$$\because \sum_{k=0}^{\tau_n} e^{-k} = \frac{1-e^{-\tau_n-1}}{1-e^{-1}}$$

$$\because \lim_{n\to\infty} \tau_n = +\infty \quad \Rightarrow \lim_{n\to\infty} e^{-\tau_n-1} = 0$$

$$\therefore \lim_{n\to\infty} \sum_{k=0}^{\tau_n} e^{-k} = \lim_{n\to\infty} \frac{1-e^{-\tau_n-1}}{1-e^{-1}} = \frac{1-0}{1-e^{-1}} = \frac{e}{e-1}$$

$$\therefore \lim_{n\to\infty} b_n = \lim_{n\to\infty} \sum_{k=0}^{\tau_n} e^{-k} - \lim_{n\to\infty} I_n = \frac{e}{e-1} - 0 = \frac{e}{e-1}$$

比如大家常问的题: 求证 $\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\left(\frac{k}{n}\right)^{n}=\frac{e}{e-1}$.

我们是否也考虑减去右边极限? 此时加什么项? 我们如果利用泰勒公式, 就可以得到很好的结果.

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}=\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)^{n}=\sum_{k=1}^{n}\exp\left[n\ln\left(1-\frac{k}{n}\right)\right]$$

注意到

$$e^{n\ln\left(1-\frac{k}{n}\right)}=e^{-k}\left[1-\frac{k^2}{2n}+\frac{k^3(3k-8)}{24n^2}-\frac{k^4(k^2-8k+12)}{48n^3}\right]+o(1/n^4),$$

且注意到

$$\sum_{k=0}^{\infty}e^{-k}=\frac{e}{e-1},\sum_{k=0}^{\infty}k^2e^{-k}=\frac{e(e+1)}{(e-1)^3},$$

和

$$\sum_{k=0}^{\infty}k^3(3k-8)e^{-k}=\frac{e(-e^2+2e+11)(5e+1)}{(e-1)^5},$$

$$\sum_{k=0}^{\infty}k^4(k^2-8k+12)e^{-k}=\frac{e(21+365e+502e^2-138e^3-35e^4+5e^5)}{(e-1)^7}$$

代入即可得到

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}=\frac{e}{e-1}-\frac{1}{2n}\cdot\frac{e(e+1)}{2(e-1)^3}+\frac{1}{24n^2}\cdot\frac{e(-e^2+2e+11)(5e+1)}{(e-1)^5}-\frac{1}{48n^3}\cdot\frac{e(21+365e+502e^2-138e^3-35e^4)}{(e-1)^7}$$

那么我们可以得到:

$$\lim_{n\to+\infty}n\left\{\frac{e(-e^2+2e+11)(5e+1)}{24(e-1)^5}-n\left[n\left(\frac{e}{e-1}-\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}\right)-\frac{e(e+1)}{2(e-1)^3}\right]\right\}$$
$$=\frac{e(5e^5-35e^4-138e^3+502e^2+365e+21)}{48(e-1)^7}$$

（*mathmagic lite 2019.04.17*）

$$I=\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}} \tan^n\left(\frac{x}{n}\right)dx\right]^{\frac{1}{n}}$$

$$n\in N^+$$

$$0\le x\le \frac{\pi}{4} \quad \Rightarrow \tan\left(\frac{x}{n}\right)\ge 0$$

$$\because \textit{when } u\ge 0,\quad \textit{have } \tan u\ge u$$

$$\therefore \tan^n\left(\frac{x}{n}\right)=\left[\tan\left(\frac{x}{n}\right)\right]^n\ge\left(\frac{x}{n}\right)^n$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^n\left(\frac{x}{n}\right)dx\ge\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n dx$$

$$\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n dx=n\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n d\left(\frac{x}{n}\right)=\frac{n}{n+1}\left(\frac{\pi}{4n}\right)^{n+1}$$

$$\lim_{n\to\infty} n\left[\int_0^{\frac{\pi}{4}}\left(\frac{x}{n}\right)^n dx\right]^{\frac{1}{n}}=\lim_{n\to\infty} n\left[\frac{n}{n+1}\left(\frac{\pi}{4n}\right)^{n+1}\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty} n*\left(\frac{n}{n+1}*\frac{\pi}{4n}\right)^{\frac{1}{n}}*\frac{\pi}{4n}$$

$$=\frac{\pi}{4}\lim_{n\to\infty}\left(\frac{1}{n+1}*\frac{\pi}{4}\right)^{\frac{1}{n}}=\frac{\pi}{4}$$