11月27日. 习题课.

$\Sigma \subseteq \mathbb{R}^3$ 曲面. $P.Q.R$ 定义在 Σ 上. 给定了 Σ 上的单位法向量 $\vec{n}$.（Σ 的侧）.

$$\iint_\Sigma P(x.y.z)\,dydz + Q(x.y.z)\,dzdx + R(x.y.z)\,dxdy.$$

$$\pm\sum_{i=1}^{n} P(\xi_i.\eta_i.\lambda_i)\,|\Delta_{yz} S_i|$$

Σ 在 yoz 平面的投影.

设 $\vec{x}=(1,0,0)$. $\vec{y}=(0,1,0)$. $\vec{z}=(0,0,1)$.

$$\iint_\Sigma P\,dydz + Q\,dzdx + R\,dxdy$$

$$= \iint_\Sigma \left[P\cos(\vec{n}.\vec{x}) + Q\cos(\vec{n},\vec{y}) + R\cos(\vec{n}.\vec{z})\right] dS.$$

第一型曲面积分

Σ: $z=z(x.y)$. $(x.y)\in D\subseteq\mathbb{R}^2$

当 $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ 与 $\vec{n}$ 同向. $+$

（反向, $-$）

$$=\begin{cases} \pm\iint_D \left(P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} + R\cdot(-1)\right) dxdy. \\ \pm\iint_\Omega \left(P\frac{\partial(y.z)}{\partial(u.v)} + Q\frac{\partial(z.x)}{\partial(u.v)} + R\frac{\partial(x.y)}{\partial(u.v)}\right) dudv. \end{cases}$$

二重积分

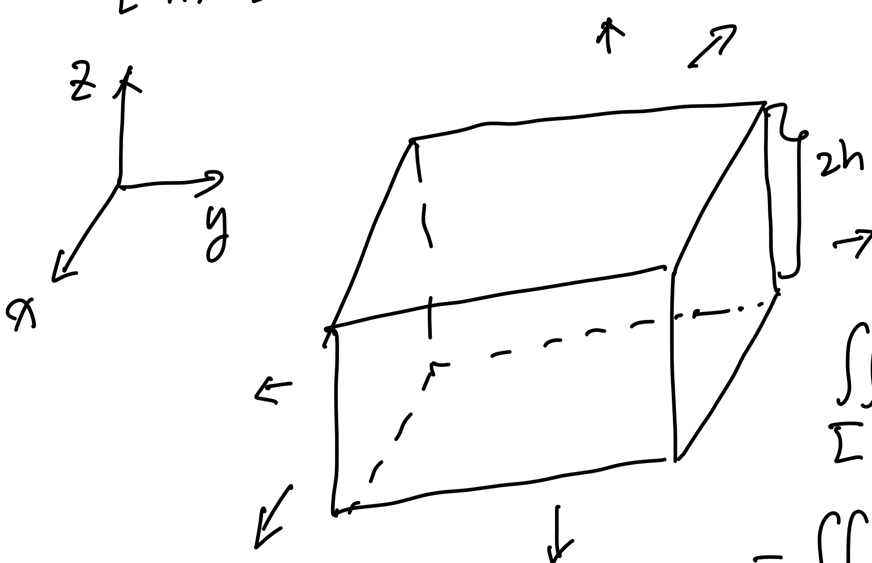
$$\Sigma:\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad (u,v) \in \Omega \subseteq \mathbb{R}^2$$

$\left(\frac{\partial(y,z)}{\partial(u,v)}, \dots \right)$ 与 $\vec{n}$ 同向取+，反向取−.

4. (1) $\displaystyle\iint_{\Sigma} (x+y)\,dy\,dz + (y+z)\,dz\,dx + (z+x)\,dx\,dy.$

Σ 为中心在原点、边长为 2h 的立方体

$[-h,h]\times[-h,h]\times[-h,h]$. 方向取外侧.



(0,0,0)

$$\iint_{\Sigma} (y+z)\,\underbrace{dz\,dx}$$

$$= \iint_{\Sigma_3+\Sigma_4} (y+z)\,dz\,dx$$

(0,−1,0)　(0,1,0)

六个面 $\underbrace{\text{上下}}$，左右，前后

$\Sigma_1\ \Sigma_2\quad \Sigma_3\ \Sigma_4\quad \Sigma_5\ \Sigma_6$

$$\iint_{\Sigma} (x+y)\,\underbrace{dy\,dz} = \iint_{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6} (x+y)\,dy\,dz$$

Σ在 yoz 平面的投影 e. 取积分

$$= \iint\limits_{\Sigma_5+\Sigma_6} (x+y)\,dy\,dz.$$

Σ_5: $x=h.\quad -h\le y\le h,\ -h\le z\le h\qquad \vec{n}=(1,0,0)$

$$\iint\limits_{\Sigma_5}(x+y)\,dy\,dz = \iint\limits_{\substack{-h\le y\le h\\ -h\le z\le h}}(h+y)\,dy\,dz$$

$$\iint\limits_{\Sigma_6}(x+y)\,dy\,dz = \iint\limits_{\substack{-h\le y\le h\\ -h\le z\le h}}(-h+y)(-1)\,dy\,dz \qquad \vec{n}=(-1,0,0)$$

Σ_6: $x=-h.\quad -h\le y\le h,\ -h\le z\le h$

$$\iint\limits_{\Sigma_5+\Sigma_6}(x+y)\,dy\,dz = 2h\iint\limits_{\substack{-h\le y\le h\\ -h\le z\le h}}dy\,dz = 8h^3.$$

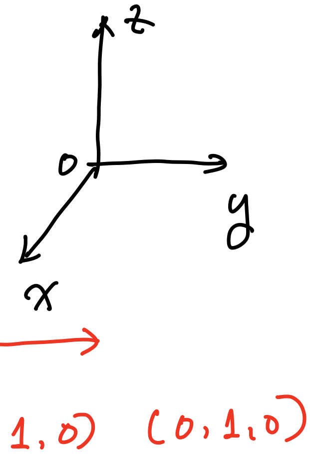
后面学）Gauss公式计算

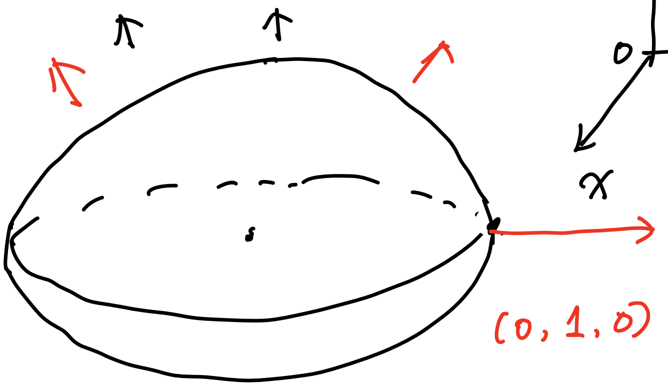
$$\iint\limits_{\Sigma}(x+y)\,dy\,dz+(y+z)\,dz\,dx+(z+x)\,dx\,dy$$

$$=\iiint\limits_{V(\text{由}\Sigma\text{围成的立体})}\left(\frac{\partial(x+y)}{\partial x}+\frac{\partial(y+z)}{\partial y}+\frac{\partial(z+x)}{\partial z}\right)dx\,dy\,dz$$

$$= \iiint_V 3 \, dxdydz = 3\cdot(2h)^3$$

(2) $\iint_\Sigma yz\,dzdx$. Σ 椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 的上半部分. 方向取上侧.





$$\boxed{\left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)}$$

(I). $z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, $(x,y)\in\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}=D$

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

$$=\left(-\frac{\frac{c}{a^2}\cdot x}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}, -\frac{\frac{c}{b^2}y}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}, -1\right)$$

因此 $\iint_{\Sigma} yz\,dz\,dx$ （从 y 轴正半轴去投影）

$$= -\iint_{\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}} y\cdot c\cdot\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\cdot\left(-\frac{c}{b^2}\cdot\frac{y}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\right)dx\,dy$$

$$= \frac{c^2}{b^2}\underbrace{\iint_{\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}} y^2\,dx\,dy}$$

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = abr$$

$$= \frac{c^2}{\cancel{b^2}}\int_0^{2\pi} d\theta\int_0^1 \cancel{b^2} r^2\sin^2\theta\cdot ab\,r\,dr$$

$$= abc^2\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^1 r^3\,dr$$

$$\begin{cases} x = a\sin\varphi\cos\theta \\ y = b\sin\varphi\sin\theta \\ c = c\cos\varphi \end{cases}\qquad \varphi\in[0,\tfrac{\pi}{2}] \qquad \theta\in[0,2\pi].$$

$$\left(\frac{\partial(y,z)}{\partial(\varphi,\theta)},\frac{\partial(z,x)}{\partial(\varphi,\theta)},\frac{\partial(x,y)}{\partial(\varphi,\theta)}\right) = \underline{\left(bc\sin^2\varphi\cos\theta,\ ac\sin^2\varphi\sin\theta,\ ab\sin\varphi\cos\varphi\right)}$$

$= (0, ac, 0)$ $\quad a, c > 0$

$(0,0,1)$ $\quad (0,1,0)$ $\quad (1,0,0)$

$\varphi = 0.$ $\quad 1$

$\theta =$ $(\varphi, \theta) = (\frac{\pi}{2}, \frac{\pi}{2})$

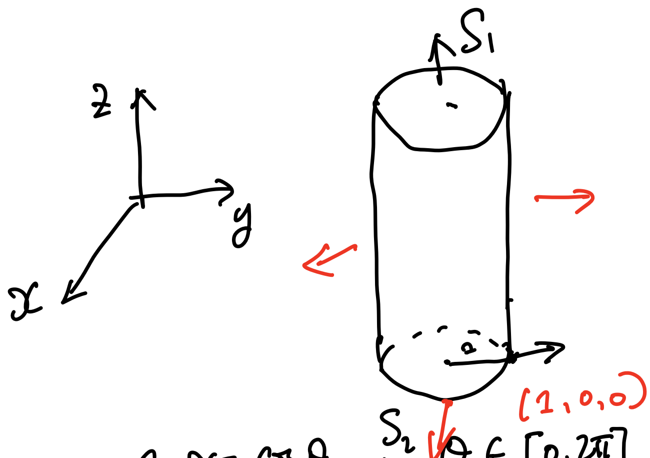
$$\iint_{\Sigma} zy\,dz\,dx = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} (b \sin\varphi \sin\theta)\cdot(c\cos\varphi)\cdot \frac{\partial(z,x)}{\partial(\varphi,\theta)}\, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} (bc \sin\varphi\cos\varphi\sin\theta \cdot ac\cdot \sin^2\varphi\sin\theta)\, d\varphi.$$

$$= abc^2 \int_0^{2\pi} \sin^2\theta\, d\theta \underbrace{\int_0^{\frac{\pi}{2}} \sin^3\varphi\cos\varphi\, d\varphi}_{\int_0^1 t^3\,dt} \qquad t = \sin\varphi$$

(3). $$\iint_{\Sigma} z\,dy\,dz + x\,dz\,dx + y\,dx\,dy = \iiint_V \left(\overset{0}{\frac{\partial z}{\partial x}} + \overset{0}{\frac{\partial x}{\partial y}} + \overset{0}{\frac{\partial y}{\partial z}}\right) dx\,dy\,dz - \iint_{S_1} (\ ) - \iint_{S_2} (\ )$$

Σ是柱面 $x^2+y^2=1$. 被平 $z=0$. 和 $z=4$ 所截部分. 方向取外侧



柱坐标系

$\theta \in [0, 2\pi]$

$(0,1,0)$

$$\Sigma: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z. \end{cases} \qquad 0 \le z \le 4$$

$$\left(\frac{\partial(y,z)}{\partial(\theta,z)},\ \frac{\partial(z,x)}{\partial(\theta,z)},\ \frac{\partial(x,y)}{\partial(\theta,z)}\right)$$

$$=\left(\begin{vmatrix}\cos\theta & 0\\ 0, & 1\end{vmatrix},\ \begin{vmatrix}0 & 1\\ -\sin\theta, & 0\end{vmatrix},\ \begin{vmatrix}-\sin\theta & 0\\ \cos\theta, & 0\end{vmatrix}\right)$$

$=(\cos\theta.\ \sin\theta,\ 0).$ 而 柱面上一点 $(1,0,0)$. 对应

$(\theta,z)=(0,0)$

目 $(1,0,0)$

$$I=\int_0^{2\pi}d\theta\int_0^4\left(\underline{z\cdot\cos\theta}+\cos\theta\cdot\sin\theta+\underline{\sin\theta\cdot 0}\right)dz$$

$$=\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\cdot\int_0^4 dz=0$$

$x^2+y^2=1.\quad 0\le z\le 4$

$$=4\,\frac{1}{2}\int_0^{2\pi}\sin 2\theta\,d\theta=0$$

$(x-\sqrt{1-y^2}=0=F(x,y,z)=0$

$\left(\frac{\partial F}{\partial x},\ \frac{\partial F}{\partial y},\ \frac{\partial F}{\partial z}\right)$

~~(II)~~ Σ: $\begin{cases}\Sigma_1 & x=\sqrt{1-y^2},\quad -1\le y\le 1,\quad 0\le z\le 4\\ \Sigma_2 & x=-\sqrt{1-y^2}.\quad -1\le y\le 1,\quad 0\le z\le 4\end{cases}$

$x^2+y^2=1$

Σ_1: $(2x,2y.0)$ 单位化

$$\left(\frac{2x}{2\sqrt{x^2+y^2}},\ \frac{2y}{2\sqrt{x^2+y^2}},0\right)=\left(\frac{x}{\sqrt{x^2+y^2}}.\ \frac{y}{\sqrt{x^2+y^2}}.0\right)$$

在 $(0,2,0)$ 点，方向 $(0,1,0)$

$$I=\iint_{\Sigma} z\,dy\,dz+x\,dz\,dx+y\,dx\,dy$$

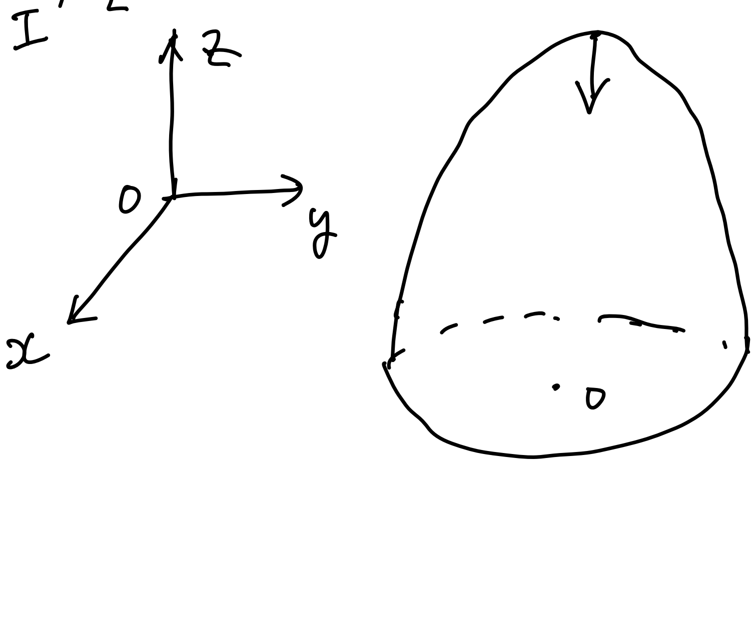
$$=\iint_{\Sigma}\left(z\cos(\vec{n},\vec{x})+x\cos(\vec{n},\vec{y})+y\cos(\vec{n},\vec{z})\right)dS$$

$$=\iint_{\Sigma}\left(z\cdot\frac{x}{\sqrt{x^2+y^2}}+x\cdot\frac{y}{\sqrt{x^2+y^2}}\right)dS$$

$$=\underbrace{\iint_{\Sigma} z\frac{x}{\sqrt{x^2+y^2}}dS}_{\Sigma\text{关于}yz\text{平面对称}}+\underbrace{\iint_{\Sigma}\frac{xy}{\sqrt{x^2+y^2}}dS}_{\Sigma\text{关于}yz\text{平面对称}}=0$$

(4) $\iint_{\Sigma} zx\,dy\,dz+3\,dx\,dy$. Σ: $z=4-x^2-y^2$. $z\geq 0$. 方向取下侧

I //



$\cdot\ z=4$

$x^2+y^2\leq 4$

$z=0$

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right) = (-2x, -2y, -1)\Big|_{(0,0,4)} = (0,0,-1)$$

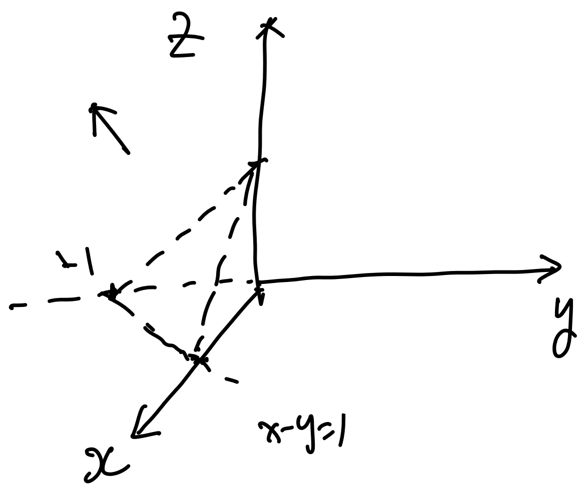
$(0,0,4)$ 对应 $\vec{n} = (0,0,-1)$

$$I = \iint\limits_{\{x^2+y^2\le 4\}} \left[(4-x^2-y^2)x\cdot(-2x) + 3\cdot(-1)\right] dx\,dy$$

(5). $$I = \iint\limits_{\Sigma} (f(x,y,z)+x)\,dy\,dz + [2f(x,y,z)+y]\,dz\,dx + [f(x,y,z)+z]\,dx\,dy.$$

$f(x,y,z)$ 连续函数. Σ: $x-y+z=1$ 在第4卦限部分

方向取上侧



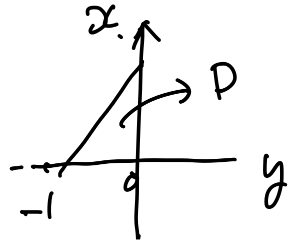
$z = 1-(x-y) = 1-x+y$

$0 \le x \le 1$

$-1 \le y \le 0$

$x+y \le 1$

$\cos\gamma > 0$



$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right) = (-1, 1, -1)$$

$(1,-1,1)$

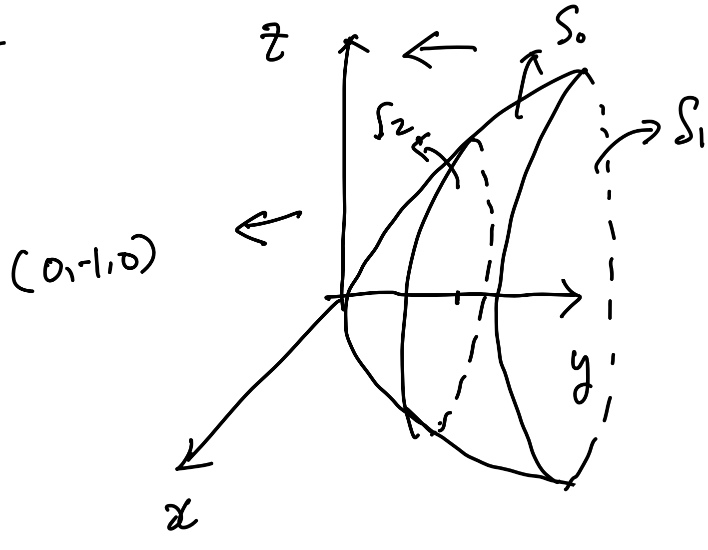
$$I=\iint_D \left[(f+x)\cdot 1+(2f+y)\cdot(-1)+(f+z)\cdot 1\right]dxdy$$

$$=\iint_D [x-y+z]\,dxdy=\iint_D (x-y+1-(x-y))\,dxdy$$

$$=\iint_D dxdy=\frac{1}{2}.$$

(7) $\displaystyle\iint_\Sigma \frac{e^{\sqrt{y}}}{\sqrt{z^2+x^2}}\,dzdx$. : Σ 是 $y=x^2+z^2$ 与 $y=1$ 与 $y=2$ 所围立体表面，方向为外侧.

I



$1\le y\le 2$

$D=\{1\le x^2+z^2\le 2\}$

$y=y(x,z)=x^2+z^2$ $\qquad$ $x^2+z^2-y=0=F(x,y,z)$

$$\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)=(2x,-1,2z)$$

$$I = \iint_{S_0} + \iint_{S_1} + \iint_{S_2} = I_0 + I_1 + I_2$$

$x = r\cos\theta$ $\quad 1 \le r \le \sqrt{2}$

$z = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$$I_0 = \iint_D \frac{e^{\sqrt{x^2+z^2}}}{\sqrt{z^2+x^2}} \cdot (-1)\, dx\, dz.$$

$$= -\int_0^{2\pi} d\theta \int_1^{\sqrt{2}} \frac{e^r}{r} \cdot (-1) \cdot r\, dr$$

$(0, 1, 0)$

S_1: $y = 2$ $\quad \{x^2+z^2 \le 2\} = D_1$

S_2: $y = 1$ $\quad \{x^2+z^2 \le 1\} = D_2$

$$I_1 = \iint_{S_1} (\ \ ) = \iint_{x^2+z^2 \le 2} \frac{e^{\sqrt{2}}}{\sqrt{z^2+x^2}} \times 1\, dz\, dx =$$

$$I_2 = \iint_{S_2} (\ ) = \iint_{x^2+z^2 \le 1} \frac{e^{\sqrt{1}}}{\sqrt{z^2+x^2}} \cdot (-1)\, dz\, dx =$$