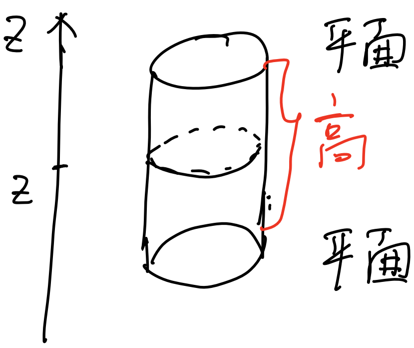
三重积分的计算

二重积分. 定义. 性质. 计算 (转化为累次积分)　X型. Y型

$$\int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy$$

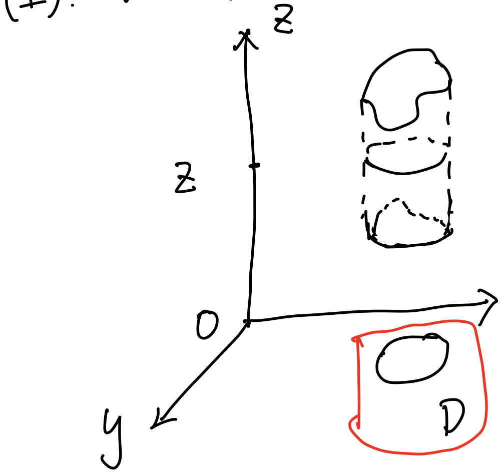
三重积分. $V \subseteq \mathbb{R}^3$ (区域)

$$\iiint_V dx\,dy\,dz$$ (V的体积).



计算. (转化为累次积分).

(I). $V \subseteq \mathbb{R}^3$ (两个曲面 "广义柱体")

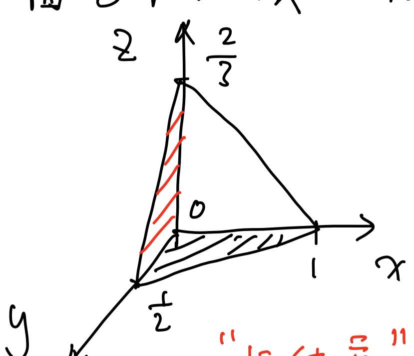


$z_2 = \psi(x,y)$

$z_1 = \varphi(x,y)$. ψ, φ 的定义域 $D \subseteq \{x,y$ 平面$\}$

D称为广义柱体的"底"

除了曲面 $\psi(x,y)$. $\varphi(x,y)$ 以外. 任取 $z \in \mathbb{R}$. $\boxed{\text{过 }(0,0,z)}$ 的平面与V的截面在 xy 平面的投影是区域 $D \subset \mathbb{R}^2$



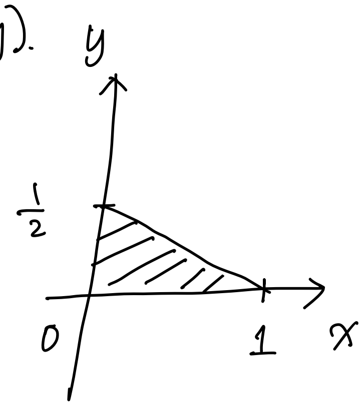
$2x+4y+3z=2$. 与坐标平面所围成的立体.

$$x + \frac{y}{\frac{1}{2}} + \frac{z}{\frac{2}{3}} = 1$$

"柱体高"为0

(a) $z_2=\psi(x,y)=\frac{1}{3}(2-2x-4y)$.

$z_1=0$



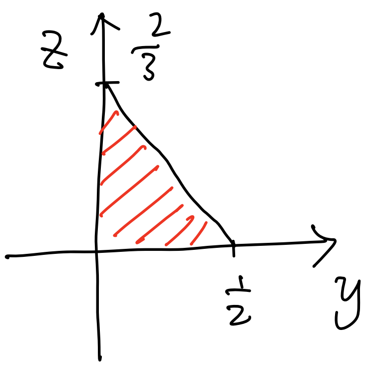
(b) 柱体投影在 yz 平面上.

$x_2=\psi(y,z)=\frac{1}{2}(2-4y-3z)$

$x_1=0$

$2x+4y=2$

$x+\frac{y}{\frac{1}{2}}=1$



$4y+3z=2$

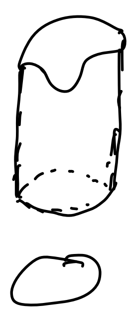
$\frac{y}{\frac{1}{2}}+\frac{z}{\frac{2}{3}}=1$

(c) 把柱体投影在 xz 平面上.

(练习)

性质: 设 $f(x,y,z)$ 在 V 上连续(可积).

$$\iiint_V f(x,y,z)\,dxdydz=\iint_D\left(\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz\right)dxdy$$



$z_2=\psi(x,y)$

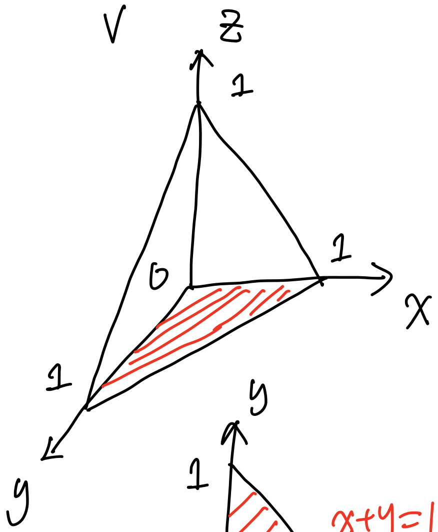
$z_1=\varphi(x,y)$

D.

$$=\iint_D dxdy\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz.$$

竖线

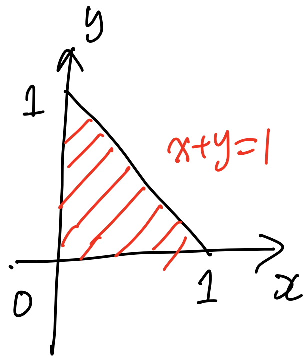
例：求 $\iiint\limits_V x\,dx\,dy\,dz$，V：由 $x+y+z=1$ 与坐标平面所围成的区域。



$z_2=\psi(x,y)=1-(x+y)$

$z_1=\varphi(x,y)\equiv 0$

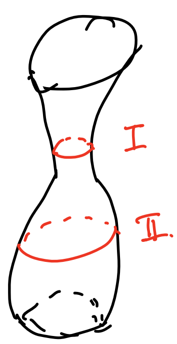
柱体在xy平面的投影



$$\iint\limits_D\left(\int_0^{1-(x+y)} x\,dz\right)dx\,dy$$

$$=\iint\limits_D x(1-x-y)\,dx\,dy=\int_0^1 dx\int_0^{1-x} x(1-x-y)\,dy$$

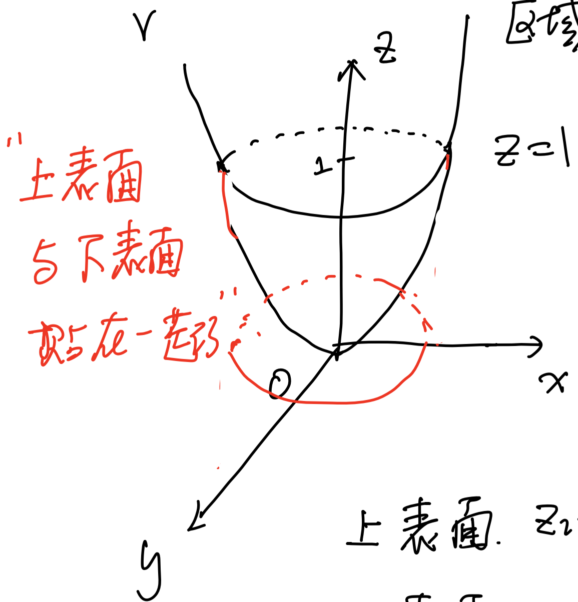
例：

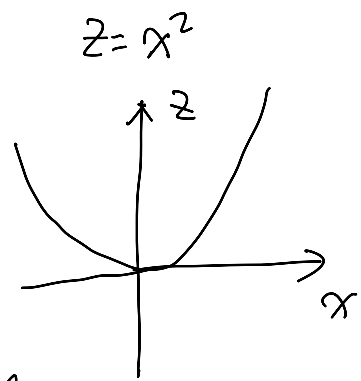


$\boxed{\text{不是广义柱体}}$

截面I与截面II的投影不一样.

例：$\iiint\limits_V z^2\,dx\,dy\,dz$ ：V 由旋转抛物面与 $z=1$ 所围成的区域. $z=x^2+y^2$.





上表面 $z_2=\psi(x,y)=1$

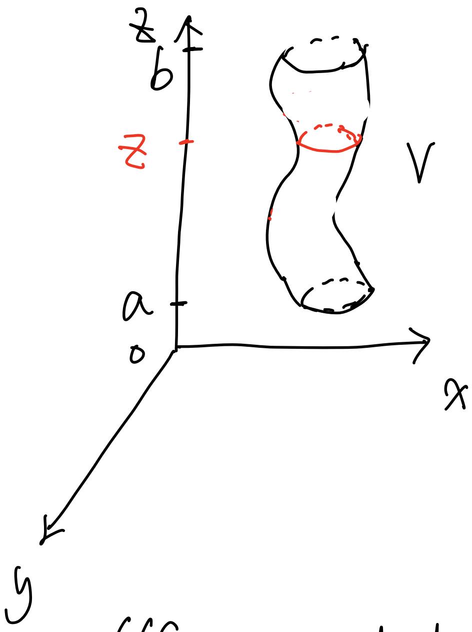
下表面 $z_1=\varphi(x,y)=x^2+y^2$

投影区域 D: $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$

$$\iiint\limits_V z^2\,dx\,dy\,dz = \iint\limits_D dx\,dy\int_{x^2+y^2}^{1} z^2\,dz$$

$$= \iint\limits_D \frac{1}{3}\left(1-(x^2+y^2)^3\right)dx\,dy \quad (\text{练习})$$

(II).



V 不是广义柱体. V的上下表面为平面

$f(x.y.z)$在V上可积

在$[a.b]$中任取z. 平面$(0,0,z)$与V的截面记为 D_z

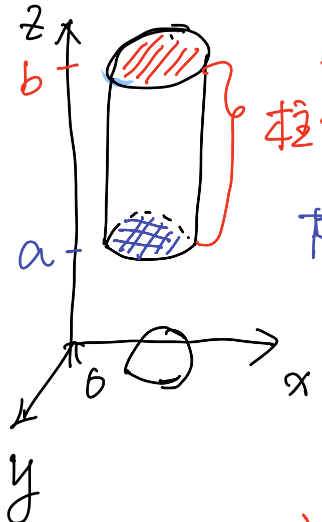
$$\iiint_V f(x.y.z)\,dx\,dy\,dz = \int_a^b \left(\iint_{D_z} f(x.y.z)\,dx\,dy \right) dz$$

$$= \int_a^b dz \iint_{D_z} f(x.y.z)\,dx\,dy$$

(I). 广义柱体: 上表面与下表面是曲面. 侧面由直线形成

(II) 上表面与下表面是平面. 侧面是曲线形成的.
(退化为点)

(I). 柱体.

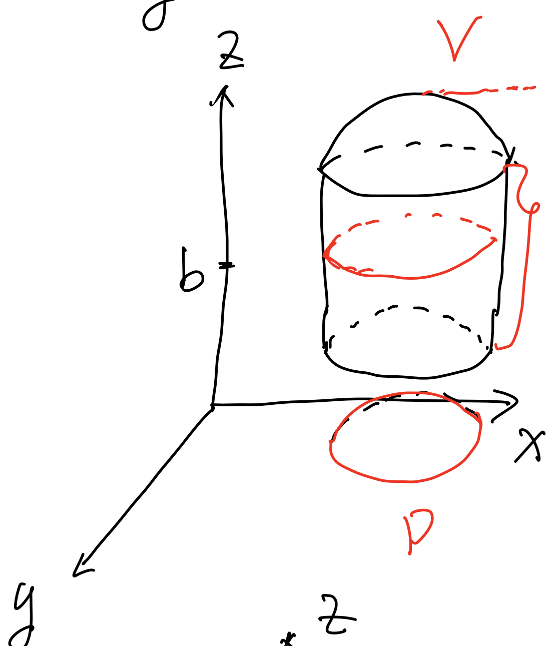


上表面 $z=b$

柱体的高

下表面 $z=a$

柱体在 xy 平面的投影记为区域 D.

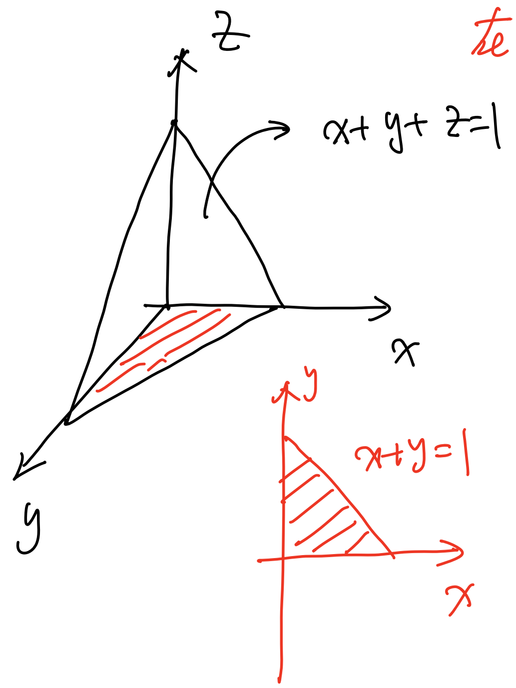


上表面. 曲面 $z=\psi(x,y)$. $(x,y)\in D$.

高

下表面 平面.

平面 $z=b$ 与 V 的截面在 xy 的投影还是 D.



锥形区域

上表面 $z=1-(x+y)$

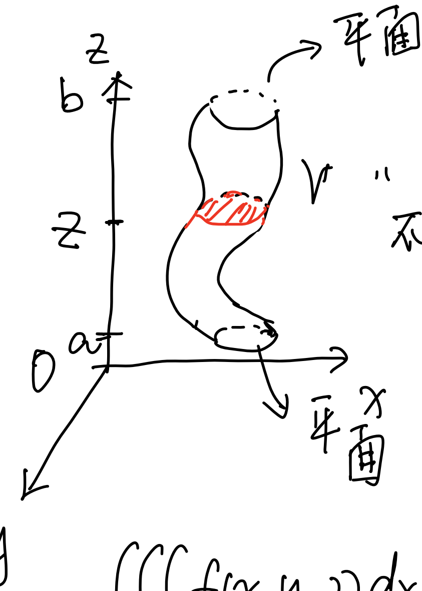
下表面 $z=0$

高"退化"为平面.

$\psi(x,y)$

$$\iiint_V f(x,y,z)\,dxdydz = \iint_D dxdy \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz$$

$D \downarrow$ 投影区域

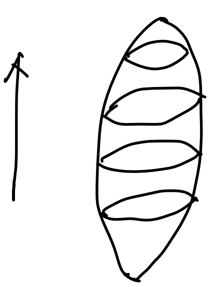


(II) "不是广义柱体"　$f(x,y,z)$ 在 V 上可积

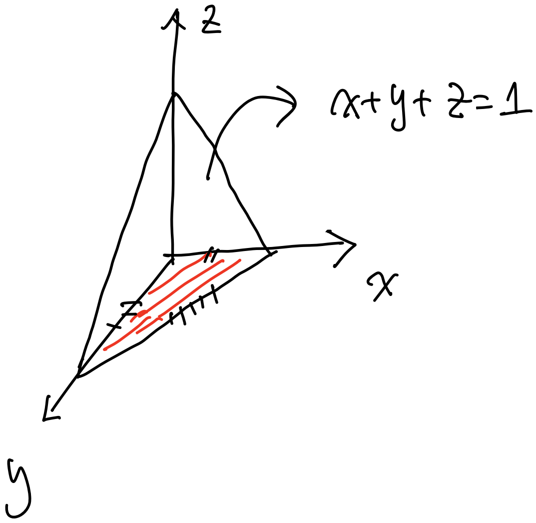
$$\iiint_V f(x,y,z)\,dxdydz = \int_a^b dz \iint_{D_z} f(x,y,z)\,dxdy$$

记 V 与平面 $(0,0,z)$ 的交为区域 D_z (截面)

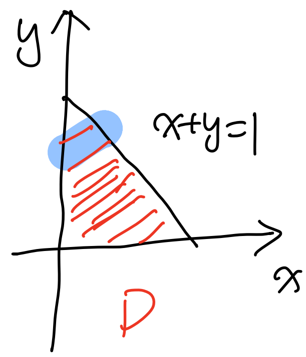
$\int dz$



例: $\iiint_V x\,dxdydz$.　V: $x+y+z=1$ 与坐标平面围成的立体.



方法(一)：把V看做广义柱形区域



上表面 $z=1-x-y.$

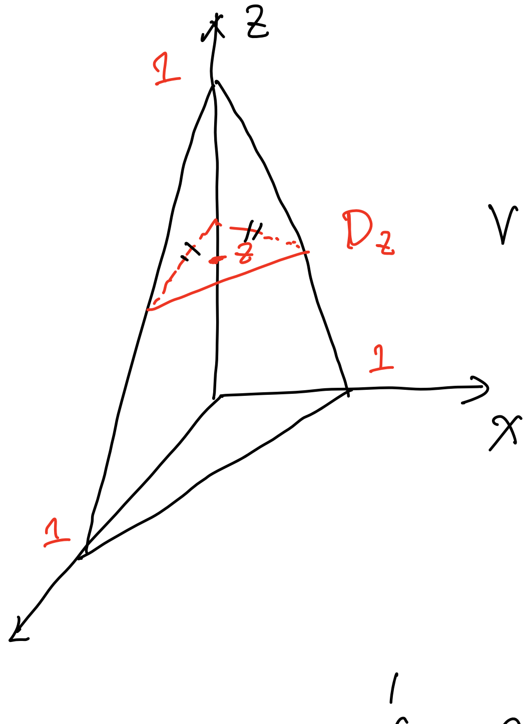
下表面 $z=0$

投影区域D与表面重合.

$$\iiint\limits_V x\,dx\,dy\,dz=\iint\limits_D dx\,dy\int_0^{1-x-y} x\,dz$$

$$=\iint\limits_D x(1-x-y)\,dx\,dy.$$

方法(二)：

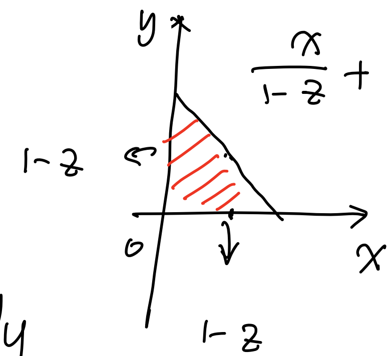


$z \in [0,1]$.

V与平面 $x+y+z=1$ 的交.(截面)

$x+y=1-z$

$\frac{x}{1-z}+\frac{y}{1-z}=1$

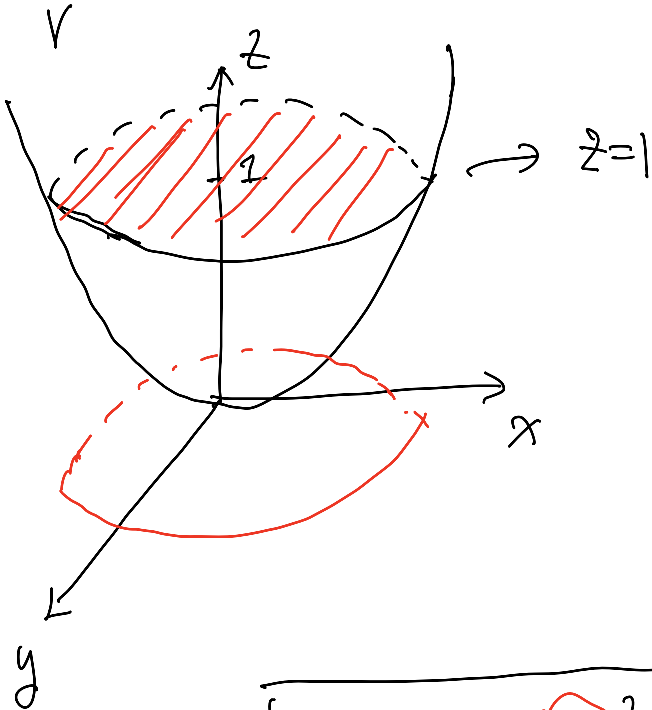


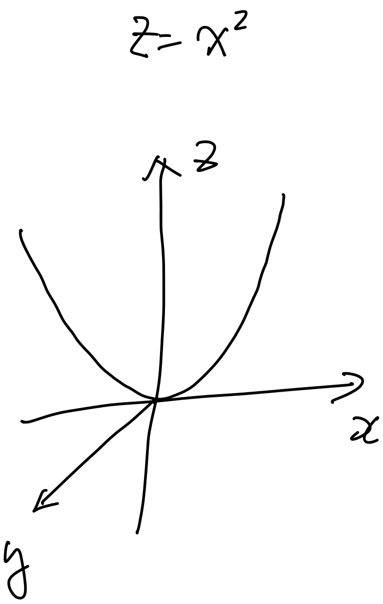
$$\iiint\limits_V x\,dx\,dy\,dz = \int_0^1 dz \iint\limits_{D_z} x\,dx\,dy$$

$$\iint\limits_{D_z} x\,dx\,dy = \int_0^{1-z} dx \int_0^{(1-z)(1-\frac{x}{1-z})} x\,dy$$

$$\iiint\limits_V x\,dx\,dy\,dz = \int_0^1 dz \int_0^{1-z} dx \int_0^{(1-z)(1-\frac{x}{1-z})} x\,dy$$

例: $\iiint\limits_V z^2\,dxdydz$:　V: $z=x^2+y^2$ 与 $z=1$ 所围成的区域





$\boxed{\{x^2+y^2\le 1,\ z=1\}\ \text{截面}}$

方法(一): 把V看做广义柱体　上表面 $z=1$

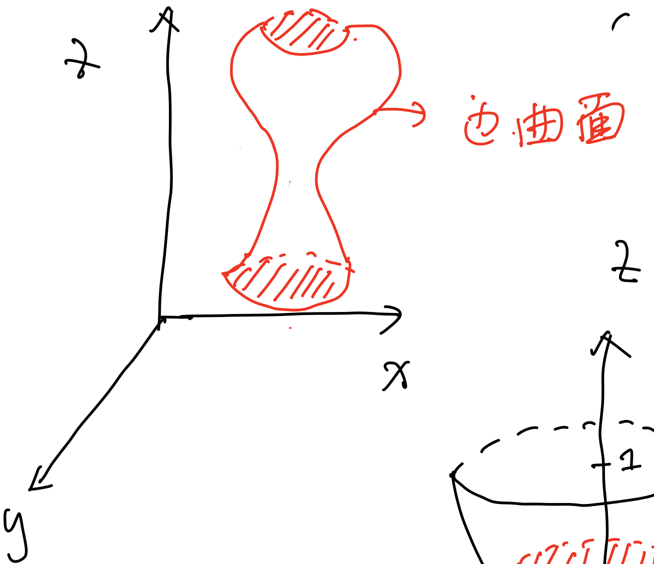
下表面 $z=x^2+y^2$

(上表面与下表面"粘在一起","高"$=0$)

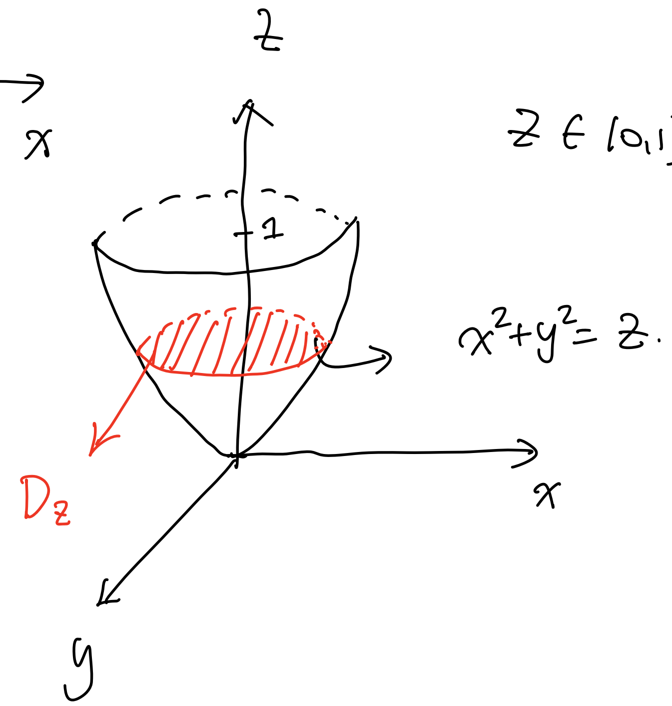
$D=\{x^2+y^2\le 1,\ z=0\}\subset\{xy\text{平面中}\}$

$$\iiint\limits_V z^2\,dxdydz=\iint\limits_D dxdy\int_{x^2+y^2}^{1} z^2\,dz$$

$$= \iint_D \frac{1}{3}\left(1-(x^2+y^2)^3\right)dxdy$$



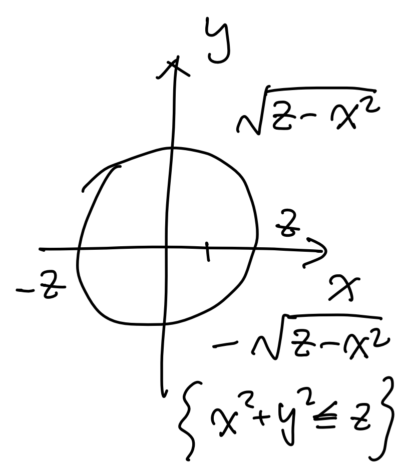
$z \in (0,1)$



方法(三) 截面

任取 $z \in (0,1)$，则

$$I = \iiint_V z^2 dxdydz = \int_0^1 dz \iint_{D_z} z^2 dxdy$$



$$D_z = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le z, \}$$

$$\iint_{D_z} z^2\,dxdy = z^2\iint_{D_z} dxdy = z^2\int_{-z}^{z} dx\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy \qquad \sqrt{z}$$

$$I = \int_0^1 dz\int_{-z}^{z} dx\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} z^2\,dy$$

$$= z^2 \boxed{\iint_{D_z} dxdy}$$

$$= z^2\pi\cdot z = \pi z^3$$

$$= \int_0^1 \pi z^3\,dz = \frac{\pi}{4}$$

截面法比"广义柱体"

适用范围更广.

课本234页: 第五题. (1). (2). (3) 小题.

有问题发QQ群.