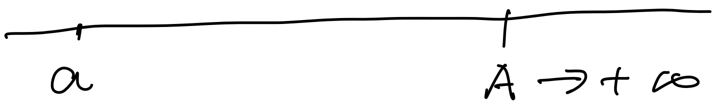
数分(III). 习题课. 第十周. 10.30.

反常(二)重积分.

$\int_a^{+\infty} f(x)dx$ 收敛 $\Longleftrightarrow$ $\lim\limits_{A\to+\infty}\int_a^A f(x)dx$ 存在

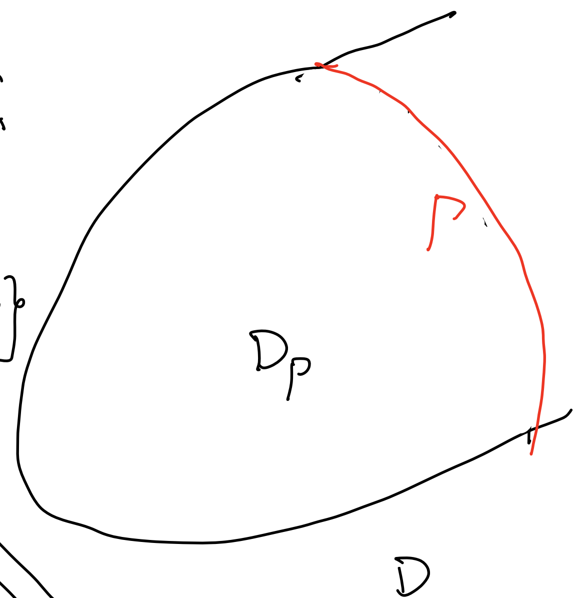


$(a, A) \to (0, +\infty)$

$\iint\limits_D f(x,y)dxdy$. (D: 无界区域) (用有界区域"逼近"无界区域).

选取曲线 Γ. 记 Γ 把 D 分割成的有界区域记为 D_Γ. 令

$d(\Gamma) = \inf\{\sqrt{x^2+y^2},\ (x,y)\in\Gamma\}$



$\lim\limits_{d(\Gamma)\to+\infty}\iint\limits_{D_\Gamma} f(x,y)dxdy$ 存在. $= \iint\limits_D f(x,y)$.

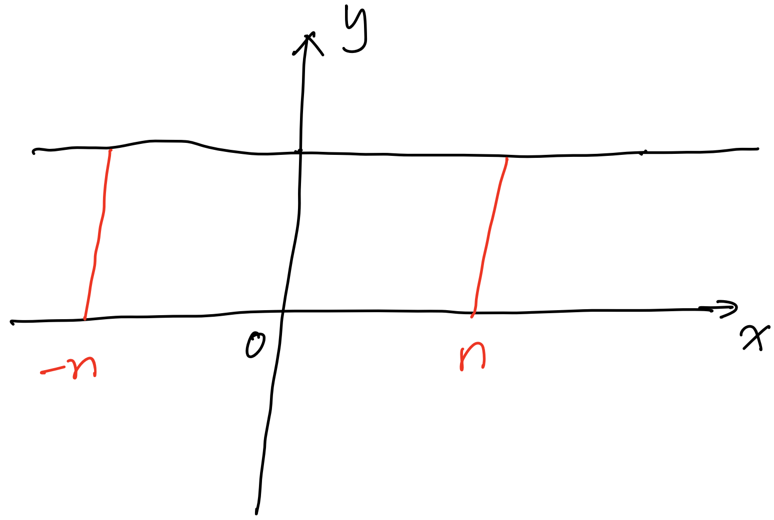
$d(\Gamma)\to+\infty$. D_Γ "趋向" D

"P的选取".

P243. 1. (2).

$$\iint\limits_{D} \frac{\varphi(x,y)}{(1+x^2+y^2)^p}\,dxdy. \quad \underline{D=\{(x,y) \mid 0 \le y \le 1\}}.$$

$$0 < m \le |\varphi(x,y)| \le M. \quad (x,y) \in D.$$



$$\frac{m}{(1+x^2+y^2)^p} \le \frac{|\varphi(x,y)|}{(1+x^2+y^2)^p} \le \frac{M}{(1+x^2+y^2)^p}$$

$$\frac{m}{(1+x^2+y^2)^p} \geq \frac{m}{(1+x^2+1^2)^p} \qquad \frac{M}{(1+x^2+y^2)^p} \le \frac{M}{(1+x^2)^p}$$

令 $\Gamma_n = \{x=n.\ x=-n\}$. 把有界区域 $\{-n \le x \le n,\ 0 \le y \le 1\}$

记为 D_n.

$$\underline{\lim_{n\to\infty} \iint\limits_{D_n} \frac{\varphi(x,y)}{(1+x^2+y^2)^p}\,dxdy.}$$

收敛与发散

$\downarrow$ [illegible]

$$2\int_0^n \frac{m\,dx}{(2+x^2)^p} \le \iint_{D_n} \frac{|\varphi(x,y)|}{(1+x^2+y^2)^p}\,dx\,dy \le 2\int_0^n \frac{M}{(1+x^2)^p}\,dx$$

$$\lim_{n\to\infty}\int_0^n \frac{1}{(1+x^2)^p}\,dx.$$

$p>\frac{1}{2}$ 时收敛.

$p\le\frac{1}{2}$ 发散.

4. 判断 $\displaystyle\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2)(1+y^2)} = I$ $\quad \mathbb{R}^2=(-\infty,+\infty)\times(-\infty,+\infty)$

如果 $\displaystyle\int_a^{+\infty}dx\int_c^{+\infty}f(x,y)\,dy$ 和 $\displaystyle\int_a^{+\infty}dx\int_c^{+\infty}|f(x,y)|\,dy$ 都存在.

则 $$\iint_{[a,+\infty)\times[c,+\infty)} f(x,y)\,dx\,dy = \int_a^{+\infty}dx\int_c^{+\infty}f(x,y)\,dy.$$

$$I=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}\frac{1}{(1+x^2)(1+y^2)}\,dy$$

$$=\int_{-\infty}^{+\infty}\frac{1}{(1+x^2)}\,dx\cdot\int_{-\infty}^{+\infty}\frac{1}{(1+y^2)}\,dy = \underbrace{\left(\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx\right)^2}$$

$$\left(\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\right).$$

5. 设 $F(t)=\iint\limits_{\substack{0\le x\le t\\ 0\le y\le t}} e^{-\frac{tx}{y^2}}\,dx\,dy$. 求 $F'(t)$.

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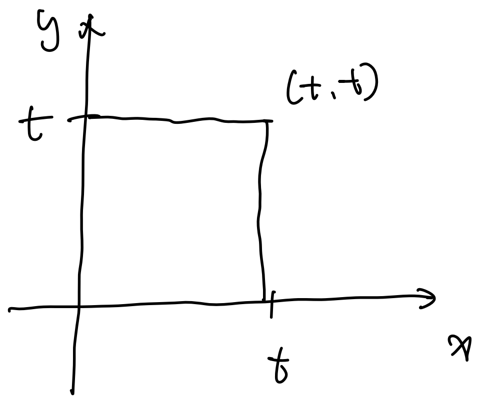
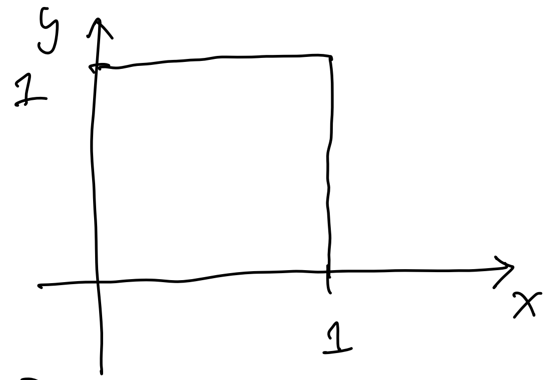
t与积分区域、也与被积函数有关. $(t\geq 0)$.





$(t>0)$

解: 做变换 $\begin{cases} x=tu, & u\in[0,1] \\ y=tv. & v\in[0,1]. \end{cases}$ $\quad \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2.$

$$F(t)=\iint\limits_{[0,1]\times[0,1]} e^{-\frac{t\cdot tu}{t^2v^2}}\, t^2\, du\,dv.$$

$$=t^2 \underbrace{\iint\limits_{[0,1]\times[0,1]} e^{-\frac{u}{v^2}}\,du\,dv.}$$

当 $t>0$ 时

$$F'(t)=2t\cdot \underbrace{\iint\limits_{[0,1]\times[0,1]} e^{-\frac{u}{v^2}}\,du\,dv.}$$

$F(0)=0$

$$=\frac{2}{t}\cdot t^2 \underbrace{\iint\limits_{[0,1]\times[0,1]} e^{-\frac{u}{v^2}}\,du\,dv}$$

$t=0$. 用定义.　　　　$= \frac{2}{t}\cdot F(t)$.

$$F'(0)=\lim_{t\to 0^+}\frac{F(t)-F(0)}{t}=\lim_{t\to 0^+}\frac{F(t)-0}{t}$$

$$=\lim_{t\to 0^+} t\cdot \iint\limits_{[0,1]\times[0,1]} e^{-\frac{u}{v^2}}\,du\,dv=0$$

注：$\iint\limits_{[0,1]\times[0,1]} e^{-\frac{u}{v^2}}\,du\,dv$. 反常重积分.收敛.

复习. 变上限积分函数的导数.

$$F(x)=\int_{\varphi(x)}^{\psi(x)} h(t)\,dt.\quad F'(x).$$

6. 设函数 $f(x)$ 在 $[0,a]$ 上连续. 证明:

$$\iint\limits_{0\le y\le x\le a}\frac{f(y)}{\sqrt{(a-x)(x-y)}}\,dx\,dy=\pi\int_0^a f(x)\,dx.$$

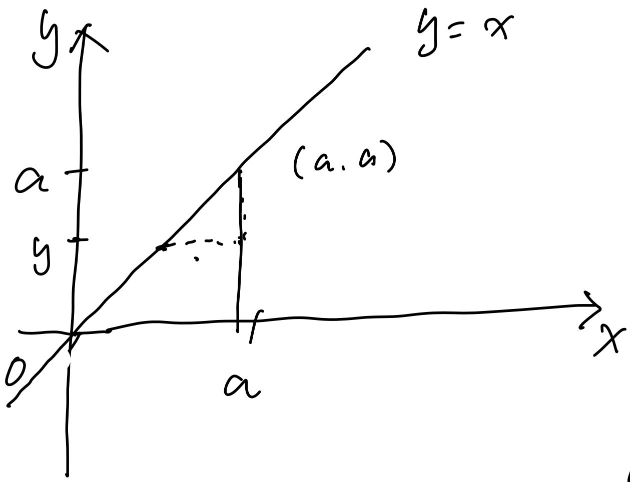
注: 反常重积分. $x=a$. $x=y$. 时. 被积函数无界.

注: 先画出积分区域. (画区域. 找区域的边界)

$0\le y\le x\le a$ 边界 $y=x$.

注：选择积分次序

把区域当成"Y"型区域



$$\iint\limits_{0\le y\le x\le a} \frac{f(y)}{\sqrt{(a-x)(x-y)}}\,dx\,dy = \int_0^a dy \int_y^a \frac{f(y)}{\sqrt{(a-x)(x-y)}}\,dx$$

$$= \int_0^a f(y)\,dy \int_y^a \frac{1}{\sqrt{(a-x)(x-y)}}\,dx$$

$$\Omega = \int_y^a \frac{1}{\sqrt{(a-x)(x-y)}}\,dx$$

积分变量 (x)

a. y 常数.

x 从 y 变化到 a.

为什么这样取.

$x = y\cos^2\theta + a\sin^2\theta$.

$x = h(\theta)$. $h(\theta)$ 当 θ 在 $[0, \frac{\pi}{2}]$ 时. $h(\theta)$ 单射.

当 θ 从 $0 \to \frac{\pi}{2}$ 时 x 从 $y \to a$

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\underline{(a-y\cos^2\theta - a\sin^2\theta)}} \cdot \sqrt{(y\cos^2\theta + a\sin^2\theta - y)}} \times$$

$$(-2y\cos\theta\cdot\sin\theta + 2a\sin\theta\cdot\cos\theta)\,d\theta.$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(a-y)\cos^2\theta}} \cdot \frac{1}{\sqrt{(a-y)\sin^2\theta}}\, 2(a-y)\cdot\sin\theta\cdot\cos\theta\, d\theta$$

$$= 2\cdot\int_0^{\frac{\pi}{2}} d\theta = \pi$$

原式 $= \int_0^a f(y)\cdot\pi\, dy = \pi\int_0^a f(y)dy.$

第一型曲线积分.

(1). $\mathbb{R}^3$ 曲线 γ: $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ $t\in[a,b]$.

γ 分段光滑: $(x'(t), y'(t), z'(t)) \neq (0,0,0)$. (去掉有限个点 $z(t)$)

(2). 曲线的长度:

$$|\gamma| = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

(3). 设 $f(x,y,z)$ 在 γ 上连续(可积). 则第一型曲线积分

$$\int_{\gamma} f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt.$$

注: 在 $\mathbb{R}^2$: $L: \begin{cases} x = x(t) \\ y = y(t). \end{cases}$ $t \in [a,b]$.

$\left(\text{或} \begin{cases} x = x \\ y = y(x). \quad x \in [a,b] \end{cases} \right)$

$$\int_L f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

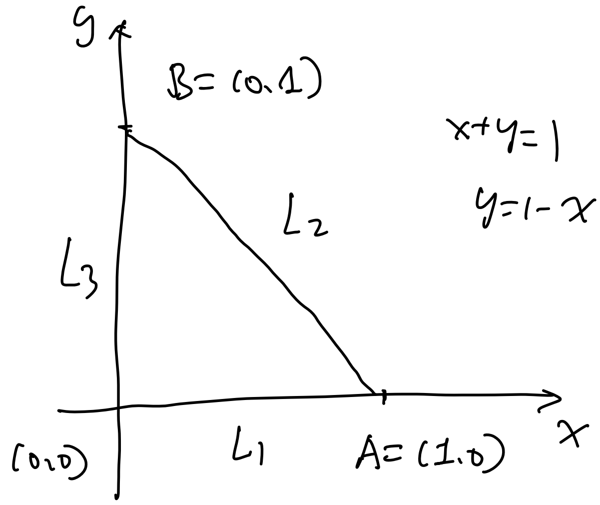
$$\int_L f(x,y)\,ds = \int_a^b f(x,y(x))\sqrt{1^2+(y'(x))^2}\,dx.$$

例: P.263. 1. (1).

记号: 重积分. $\iiint_D f(x,y,z)\,dx\,dy\,dz$

第一型 ~~曲~~ 曲线积分 $\int_L f(x,y,z)\,ds$. $\underbrace{ds}_{\text{弧长微元}}$

$\int_L (x+y)\,ds$. L是 $(0,0)$. $A=(1,0)$. $B=(0,1)$为顶点的三角形.



L_1: $\begin{cases} x=t, & 0\le t\le 1 \\ y=0. \end{cases}$

L_2: $\begin{cases} x=t \\ y=1-t, & t\in[0,1]. \end{cases}$

L_3: $\begin{cases} x=0. \\ y=t. & t\in[0,1]. \end{cases}$

$$\int_L (x+y)\,ds = \int_{L_1} + \int_{L_2} + \int_{L_3} (x+y)\,ds.$$

$$\int_{L_1} (x+y)\,ds = \int_0^1 (t+0)\sqrt{1^2+0^2}\,dt$$

$$\int_{L_2} (x+y)\,ds = \int_0^1 (t+(1-t))\sqrt{1^2+(-1)^2}\,dt$$

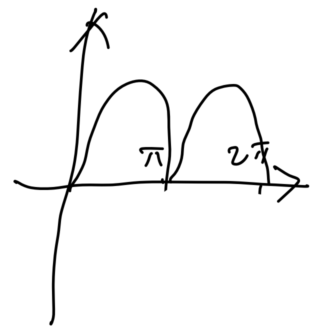
$$\int_{L_3} (x+y)\,ds = \int_0^1 (0+t)\sqrt{0^2+1^2}\,dt$$

(2) $\int_L |y|\,ds$. L为圆周. $x^2+y^2=1$.

$$L:\begin{cases} x=\sin t \\ y=\cos t \end{cases},\quad t\in[0,2\pi].$$

$$\int_L |y|\,ds=\int_0^{2\pi}|\cos t|\cdot\sqrt{(\cos t)^2+(-\sin t)^2}\,dt$$

$$=\int_0^{2\pi}|\cos t|\,dt.$$



(3). $\int_L |x|^{\frac{1}{3}}\,ds$. $L:\quad x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

先写出L的参数方程: $\begin{cases} x=a\cos^3 t \\ y=a\sin^3 t. \end{cases}\quad t\in[0,2\pi]$

$$\int_L |x|^{\frac{1}{3}}\,ds=\int_0^{2\pi}|a|^{\frac{1}{3}}|\cos t|\cdot\sqrt{[3a\cdot\cos^2 t\cdot(-\sin t)]^2+[3a\sin^2 t\cos t]^2}\,dt$$

$$=\int_0^{2\pi}|a|^{\frac{1}{3}}\cdot|\cos t|\cdot|3a|\cdot\sqrt{\sin^2 t\cos^2 t}\,dt$$

$$=3|a|^{\frac{4}{3}}\cdot\int_0^{2\pi}|\cos^2 t\sin t|\,dt=3|a|^{\frac{4}{3}}\cdot\int_0^{2\pi}\cos^2 t\cdot|\sin t|\,dt$$

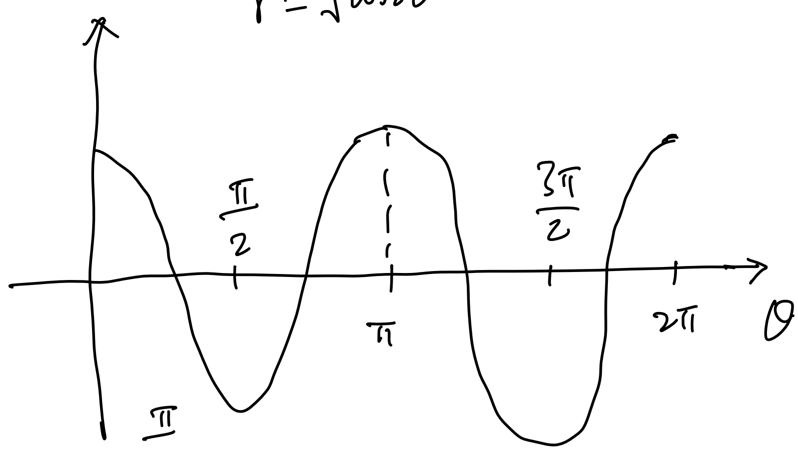
$$=3|a|^{\frac{4}{3}}\times 2\cdot\int_0^{\pi}\cos^2 t\sin t\,dt.$$

(4) $\int_L |x| ds$. $L: (x^2+y^2)^2 = x^2-y^2$. (*)

写出L的参数方程: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ 代入 (*)

$$(r^2)^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$$

即 $\underbrace{r^2 = \cos 2\theta \geq 0}_{r=\sqrt{\cos 2\theta}}$ $\quad \theta \in [0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi]$



$$\int_L |x| ds = 4\int_0^{\frac{\pi}{2}} \left|\sqrt{\cos 2\theta}\cos\theta\right| \sqrt{\left[(\sqrt{\cos 2\theta}\cdot\cos\theta)'\right]^2 + \left[(\sqrt{\cos 2\theta}\cdot\sin\theta)'\right]^2}\, d\theta$$

(6), (7). 练习.

期中考试: (多元函数的极值. 重积分).

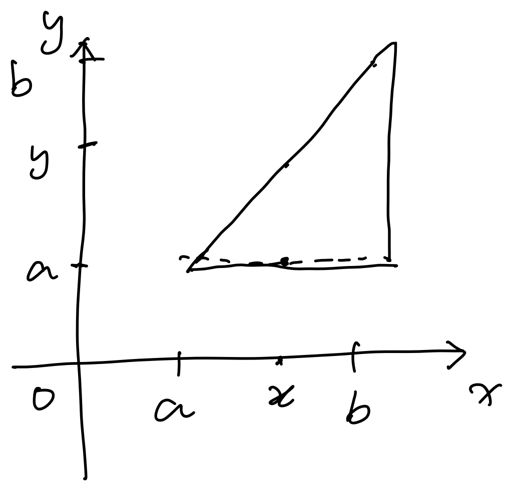
(天津)

例: 证明: $\int_a^b dx \int_a^x (x-y)^{n-2} f(y) dy = \frac{1}{n-1}\int_a^b (b-y)^{n-1} f(y) dy$.

例: 求积分 $\int_0^1 dy \int_1^y (e^{-x^2}+e^x \sin x) dx$. (人大).

"7分钟" (交换累次积分的积分次序)

$x \in [a,b]$. $y \in [a,x]$



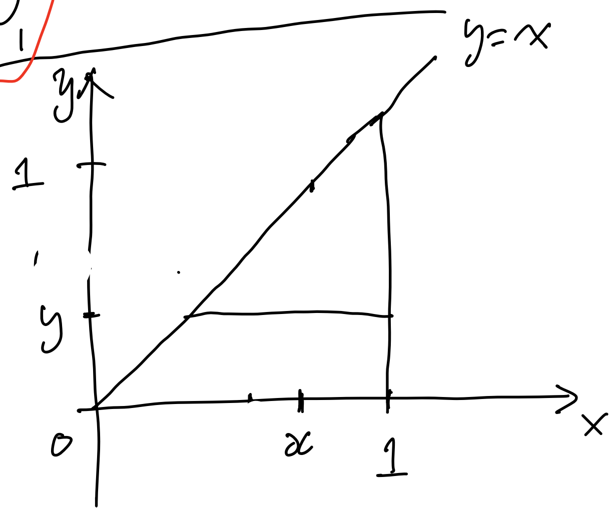
$$\int_a^b dx \int_a^x (x-y)^{n-2} f(y) dy$$

$$= \int_a^b dy \int_y^b (x-y)^{n-2} f(y) dx = \int_a^b f(y) dy \int_y^b (x-y)^{n-2} dx$$

$$= \int_a^b f(y) \left(\frac{1}{n-1} \cdot \left((b-y)^{n-1} - 0\right)\right) dy$$

$$= \frac{1}{n-1} \int_a^b f(y) \cdot (b-y)^{n-1} dy$$

(2) $I = \int_0^1 dy \int_1^y (e^{-x^2} + e^{x}\sin x)\,dx.$ 先x后y.



$0 \le y \le 1$

$$I = -\int_0^1 dy \int_y^1 (e^{-x} + e^{x}\sin x)\,dx$$

$$= -\int_0^1 dx \int_0^x (e^{-x} + e^{x}\sin x)\,dy$$

$$= -\int_0^1 x(e^{-x} + e^{x}\sin x)\,dx$$

$$= -\int_0^1 x e^{-x}\,dx - \int_0^1 x e^{x}\sin x\,dx$$

$$= \int_0^1 x\,d(e^{-x}) - \int_0^1 x\sin x\,de^{x}$$ （积分重致）

预习：第一型曲面积分.