§7 条件极值与Lagrange乘数法

(I) 无条件极值 $f(x_1,\cdots,x_n)$. 连续的各阶偏导数.

① 求驻点. (Fermat引理)

$$\frac{\partial f}{\partial x_1}=\cdots=\frac{\partial f}{\partial x_n}=0$$

② 判断驻点是否为极值点.

$$\mathrm{Hess}\, f\big|_{驻点}=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)_{n\times n}$$

正定. 极小值

负定. 极大值.

不定. 不取极值.

(II) 条件极值: 目标函数 $f(\underbrace{x_1,\cdots,x_n}_{})$. 在约束条件

$$\begin{cases} g_1(x_1,\cdots,x_n)=0 \\ \vdots \\ g_m(x_1,\cdots,x_n)=0 \end{cases}$$

变量为n个

约束条件的个数 $<n$.

如何求 f 的极值?

① 构造Lagrange函数

$$L(x_1,\cdots,x_n,\underbrace{\lambda_1,\cdots,\lambda_m}_{m\text{个变量. 对应约束条件}})=f+\lambda_1 g_1+\cdots+\lambda_m g_m$$

极值点满足的必要条件: $P=(x_1^0,\cdots,x_n^0)$

$$\begin{cases} \frac{\partial L}{\partial x_1}=0 \\ \vdots \\ \frac{\partial L}{\partial x_n}=0 \\ \frac{\partial L}{\partial \lambda_1}=g_1=0 \\ \vdots \\ \frac{\partial L}{\partial \lambda_m}=g_m=0 \end{cases}$$

解方程.

② 判断极值点的类型. 令 $\boxed{A=\left(\frac{\partial^2 L}{\partial x_i \partial x_j}\right)_{n\times n}}\Big|_P$.

A 正定. $f(P)$ 极小值

A 负定. $f(P)$ 极大值

A 不定. P 未知.

例: $f(x,y,z)=x^2+y^2-z^2$. 约束条件 $z=0$.

Lagrange 函数 $L(x,y,z,\lambda)=x^2+y^2-z^2+\lambda z$

$$\begin{cases} \frac{\partial L}{\partial x}=2x=0 \\ \frac{\partial L}{\partial y}=2y=0 \\ \frac{\partial L}{\partial z}=-2z+\lambda=0, \quad z=0 \end{cases}$$

$x=y=z=\lambda=0$

$P=(0,0,0)$

$$A=\begin{pmatrix}2&0&0\\0&2&0\\0&0&-2\end{pmatrix}$$，不定矩阵

但是 f 在 P 点取极小值.

$f(x,y,0)=x^2+y^2\geq 0=f(0,0,0)$

注：以二元函数为例. 求 $f(x,y)$ 在约束条件 $\varphi(x,y)=0$

下的极值.

(无条件极值)

$$\begin{cases} f(x,y) \text{ 的极值} \\ \varphi(x,y)=0 \end{cases} \xrightarrow[\text{数定理}]{\text{用隐函}} f(x,y(x)) \text{ 极值.}$$

如果由 $\varphi(x,y)=0$ 确定一个函数 $y=y(x)$.

由 Fermat 引理. $\dfrac{df(x,y(x))}{dx}=0$

$$\underline{\frac{\partial f}{\partial x}}+\underline{\frac{\partial f}{\partial y}}\cdot\frac{dy}{dx}=0$$

$$\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}}$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\left(-\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}}\right)=0$$

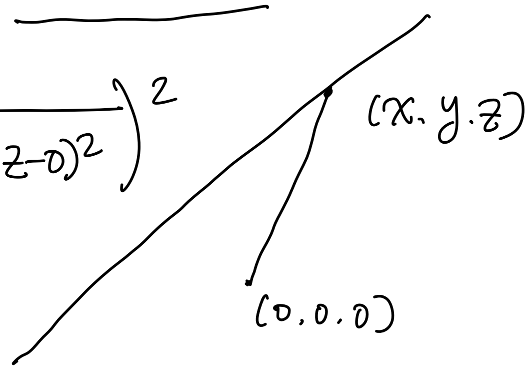
$\Uparrow$

$$\boxed{\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}}} = -\lambda \Longleftrightarrow \begin{cases} \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \end{cases}$$

构造函数 $L(x.y.\lambda) = f \cancel{-} + \lambda\varphi$.

$$\begin{cases} \frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \\ \frac{\partial L}{\partial x} = \varphi = 0 \end{cases}$$

求原点到直线 $\begin{cases} x+y+z=1 \\ x+2y+3z=6 \end{cases}$ 的最短距离.



求 $f(x.y.z) = \left(\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}\right)^2$

在约束条件

$$\begin{cases} x+y+z-1=0 \\ x+2y+3z-6=0 \end{cases}$$ 的最小值.

解：① 构造 $L(x,y,z,\lambda,\mu)$

$$= x^2+y^2+z^2+\lambda(x+y+z-1)+\mu(x+2y+3z-6)$$

$$\begin{cases} \frac{\partial L}{\partial x} = 2x+\lambda+\mu=0 & ① \\ \frac{\partial L}{\partial y} = 2y+\lambda+2\mu=0 & ② \\ \frac{\partial L}{\partial z} = 2z+\lambda+3\mu=0 & ③ \\ \frac{\partial L}{\partial \lambda} = x+y+z-1=0 & ④ \\ \frac{\partial L}{\partial \mu} = \underline{x+2y+3z-6}=0 & ⑤ \end{cases}$$

5个未知量

① + ② + ③ $\quad 2(x+y+z)+3\lambda+6\mu=0$

$$\Rightarrow \underline{3\lambda+6\mu=-2}$$

① + 2×② + 3×③

$$2(x+2y+3z)+\lambda+2\lambda+3\lambda+\mu+4\mu+9\mu=0$$

$$\Rightarrow \underline{6\lambda+14\mu=-12}$$

$\lambda=\frac{22}{3}$, $\mu=-4$. . $x=-\frac{5}{3}$, $y=\frac{1}{3}$. $z=\frac{7}{3}$.

② 判断 最短距离 $d=\sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{7}{3}\right)^2}$

例：设 $a>0$，求 $f(x,y,z)=xyz$ 在约束条件

$$x+y+z=a,\ \underline{x>0,\ y>0,\ z>0}$$

下的极值．

解：$L(x,y,z,\lambda)=xyz+\lambda(x+y+z-a)$

$$\begin{cases}\dfrac{\partial L}{\partial x}=yz+\lambda=0\\[2mm] \dfrac{\partial L}{\partial y}=xz+\lambda=0\\[2mm] \dfrac{\partial L}{\partial z}=xy+\lambda=0\\[2mm] x+y+z-a=0\end{cases}\Rightarrow\begin{cases}xyz=-x\lambda\\ xyz=-y\lambda\\ xyz=-z\lambda\end{cases}\Rightarrow x=y=z$$

$$3x=a\Rightarrow x=y=z=\frac{a}{3}>0$$

$P=\left(\frac{a}{3},\frac{a}{3},\frac{a}{3}\right)$．极值点．

$$A=\begin{pmatrix}\dfrac{\partial^2 L}{\partial x^2} & \dfrac{\partial^2 L}{\partial x\partial y} & \dfrac{\partial^2 L}{\partial x\partial z}\\[2mm] \dfrac{\partial^2 L}{\partial y\partial x} & \dfrac{\partial^2 L}{\partial y^2} & \dfrac{\partial^2 L}{\partial y\partial z}\\[2mm] \dfrac{\partial^2 L}{\partial z\partial x} & \dfrac{\partial^2 L}{\partial z\partial y} & \dfrac{\partial^2 L}{\partial z^2}\end{pmatrix}=\left.\begin{pmatrix}0 & z & y\\ z & 0 & x\\ y & x & 0\end{pmatrix}\right|_P$$

$$=\begin{pmatrix}0 & \frac{a}{3} & \frac{a}{3}\\ \frac{a}{3} & 0 & \frac{a}{3}\\ \frac{a}{3} & \frac{a}{3} & 0\end{pmatrix}$$ 作业

例：求 $f(x_1,\cdots,x_n)=a_1x_1^2+\cdots+a_nx_n^2$ $(a_1,\cdots,a_n>0)$

在约束条件 $x_1+\cdots+x_n=C>0$, $x_1>0,\cdots,x_n>0$

下的最小值.

解：$L(x_1,\cdots,x_n,\lambda)=a_1x_1^2+\cdots+a_nx_n^2+\lambda(x_1+\cdots+x_n-C)$

$$\begin{cases}\dfrac{\partial L}{\partial x_k}=2a_kx_k+\lambda=0, & k=1,2,\cdots,n, \quad x_k=-\dfrac{\lambda}{2a_k}\\[2ex] x_1+\cdots+x_n=C, & \lambda=-\dfrac{2C}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}\end{cases}$$

$$x_k=\frac{\frac{C}{a_k}}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}},\quad k=1,2,\cdots,n$$

$$a_1x_1^2+\cdots+a_nx_n^2\geqslant f\left(\frac{\frac{C}{a_1}}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}},\cdots,\frac{\frac{C}{a_n}}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}\right)$$

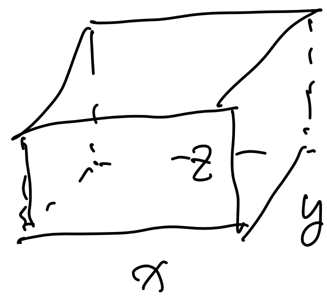
$$=\frac{C^2}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}.$$

特别地，取 $a_1=\cdots=a_n=1$.

$$x_1^2+\cdots+x_n^2\geqslant\frac{C^2}{n}=\frac{(x_1+\cdots+x_n)^2}{n},$$ Cauchy 不等式.

例：构造容积为 $a\ m^3$ 无盖长方体水箱，求长、宽、高，使得用料最省？

解：设长、宽、高为 x, y, z.



$$f(x,y,z) = xy + 2yz + 2zx$$

$$xyz = a > 0$$

$$L(x,y,z) = xy + 2yz + 2zx + \lambda(xyz - a)$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = y + 2z + \lambda yz = 0 \\ \dfrac{\partial L}{\partial y} = x + 2z + \lambda xz = 0 \\ \dfrac{\partial L}{\partial z} = 2x + 2y + \lambda xy = 0 \\ \dfrac{\partial L}{\partial \lambda} = xyz - a = 0 \quad ④ \end{cases} \qquad \begin{cases} \dfrac{1}{z} + \dfrac{2}{y} = -\lambda \quad ① \\ \dfrac{1}{z} + \dfrac{2}{x} = -\lambda \quad ② \\ \dfrac{2}{y} + \dfrac{2}{x} = -\lambda \quad ③ \end{cases}$$

① − ② $\dfrac{2}{y} - \dfrac{2}{x} = 0 \Rightarrow x = y$. $x = y = -\dfrac{4}{\lambda}$.

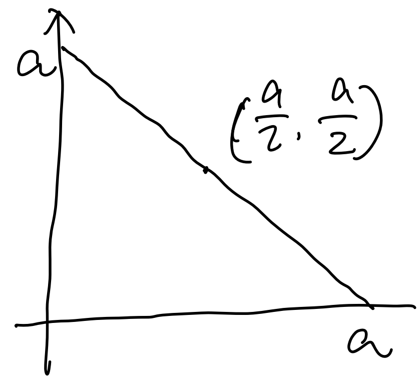
代入 ② $z = \dfrac{x}{2} = -\dfrac{2}{\lambda}$ $\left(-\dfrac{4}{\lambda}\right)^2 \cdot \left(-\dfrac{2}{\lambda}\right) = a$

$x = \sqrt[3]{2a} = y$. $z = \dfrac{\sqrt[3]{2a}}{2}$

例: 求 $f(x,y)=\frac{1}{2}(x^4+y^4)$ 在条件 $x+y=a$ 下的最小值. $x\geq 0$. $y\geq 0$. a 为常数. 并证明不等式

$$\frac{x^4+y^4}{2}\geq\left(\frac{x+y}{2}\right)^4.$$

证: $L(x,y,\lambda)=\frac{1}{2}(x^4+y^4)+\lambda(x+y-a)$



$$\begin{cases}\frac{\partial L}{\partial x}=2x^3+\lambda=0\\ \frac{\partial L}{\partial y}=2y^3+\lambda=0\\ x+y=a.\end{cases}\qquad x=y=\frac{a}{2}$$

$$f\left(\frac{a}{2},\frac{a}{2}\right)=\frac{a^4}{16}.\quad f(0,a)=f(a,0)=\frac{a^4}{2}>\frac{a^4}{16}$$

$$\frac{x^4+y^4}{2}\geq\frac{a^4}{16}=\frac{(x+y)^4}{16}=\left(\frac{x+y}{2}\right)^4$$

例: 当 $x>0$. $y>0$. $z>0$ 时. 求函数

$$f(x,y,z)=\ln x+2\ln y+3\ln z\to-\infty$$

在球面 $x^2+y^2+z^2=6R^2$ 上的最大值. 并

证明：当 $a, b, c > 0$ 时，成立不等式

$$ab^2c^3 \leq 108\left(\frac{a+b+c}{6}\right)^6.$$

证明：$L(x,y,z,\lambda) = \ln x + 2\ln y + 3\ln z + \lambda(x^2+y^2+z^2 - 6R^2)$

$$\begin{cases} \dfrac{\partial L}{\partial x} = \dfrac{1}{x} + 2\lambda x = 0 & \dfrac{1}{x^2} = -2\lambda \quad x^2 = -\dfrac{1}{2\lambda} \\ \dfrac{\partial L}{\partial y} = \dfrac{2}{y} + 2\lambda y = 0 & \dfrac{2}{y^2} = -2\lambda \quad y^2 = -\dfrac{2}{2\lambda} \\ \dfrac{\partial L}{\partial z} = \dfrac{3}{z} + 2\lambda z = 0 & \dfrac{3}{z^2} = -2\lambda \quad z^2 = -\dfrac{3}{2\lambda} \\ x^2+y^2+z^2 - 6R^2 = 0 & -\dfrac{6}{2\lambda} = 6R^2 \end{cases}$$

$x^2 = R^2$. $y^2 = 2R^2$. $z^2 = 3R^2$.

$$\ln x + 2\ln y + 3\ln z = \ln(xy^2z^3) \leq \ln(6\sqrt{3}R^6)$$

令 $a = x^2$. $b = y^2$. $c = z^2 > 0$ $\qquad R^6 = (R^2)^3 =$

$$\ln(\sqrt{a}\, b\, c^{\frac{3}{2}}) \leq \ln\left(6\sqrt{3}\cdot\left(\frac{a+b+c}{6}\right)^3\right)$$

$$ab^2c^3 \le \frac{(6\sqrt{3})^2}{108} \cdot \left(\frac{a+b+c}{6}\right)^6$$

例：抛物面 $z=x^2+y^2$ 被平面 $x+y+z=1$ 截成一椭圆，求原点到该椭圆的最长、最短距离。

解：$d=\sqrt{x^2+y^2+z^2}$，$d^2=x^2+y^2+z^2$

下的

约束条件 $\begin{cases} z=x^2+y^2 \\ x+y+z=1 \end{cases}$，最大、最小值。

$$L(x,y,z,\lambda,\mu)=x^2+y^2+z^2+\lambda(x^2+y^2-z)+\mu(x+y+z-1)$$

$$\begin{cases} \frac{\partial L}{\partial x}=2x+2\lambda x+\mu=0 & ① \\ \frac{\partial L}{\partial y}=2y+2\lambda y+\mu=0 & ② \\ \frac{\partial L}{\partial z}=2z-\lambda+\mu=0 & ③ \\ \text{— — —} \\ \text{— — —} \end{cases}$$

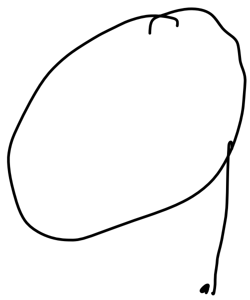
① − ②　$2(\lambda+1)(x-y)=0$，则 $\lambda=-1$ 或 $x=y$

(1) 当 $\lambda=-1$ 时，$\underline{\mu=0}$，$z=-\frac{1}{2}$ ×

(II) $\lambda \neq -1$ $\begin{cases} x = y. \\ x^2 + y^2 = z \\ x + y + z = 1 \end{cases} \Rightarrow$ $x = y = \frac{1}{2}(-1 \pm \sqrt{3})$

$z = 2x^2 = 2 \mp \sqrt{3}.$

$d^2 = 9 \mp 5\sqrt{3}.$



因为 d^2 连续函数. 椭圆. 有界闭集. 一定有最大. 最小值. $\sqrt{9+5\sqrt{3}}$. $\sqrt{9-5\sqrt{3}}$.

作业: 课本201页

1. (1). (2). 2, 6, 13. 14. 下周. 周日晚上 7:00-9:00 上习题课.